The Delaunay Tessellation Field Estimator†

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ABSTRACT — We introduce the Delaunay Tessellation Field Estimator (DTFE). Its purpose is rendering a volume-covering reconstruction of a density field from a set of discrete data points sampling this field. Reconstructing density or intensity fields from a set of irregularly sampled data is a recurring key issue in operations on astronomical data sets, both in an observational context as well as in the context of numerical simulations. The DTFE is based on the stochastic geometric concept of the Delaunay tessellation generated by the point set. We shortly describe the method and illustrate its virtues by means of an application to an N-body simulation of cosmic structure formation. The DTFE is a fully adaptive method: automatically it probes high density regions at maximum possible resolution, while low density regions are recovered as moderately varying regions devoid of the often irritating shot-noise effects. Of equal importance is its capability to sharply and undilutedly recover anisotropic density features like filaments and walls. The prominence of such features at a range of resolution levels within a hierarchical clustering scenario as the example of the standard CDM scenario is shown to be impressively recovered by the DTFE.

2.1 Introduction

Astronomical observations, physical experiments as well as computer simulations often involve discrete data sets supposed to represent a fair sample of an underlying smooth and continuous field. Conventional methods are usually plagued by one or more artefacts. Firstly, they often involve estimates at a restricted and discrete set of locations – usually defined by a grid – instead of a volume-covering field reconstruction. A problem of a more fundamental nature is that the resulting estimates are implicitly mass-weighted averages, whose comparison with often volume-weighted analytical quantities is far from trivial. For most practical purposes the disadvantage of almost all conventional methods is their insensitivity and inflexibility to the sampling point process. This leads to a far from optimal performance in both high and low density regions, which often is dealt with by rather artificial and ad hoc means.

In particular in situations of highly non-uniform distributions conventional methods tend to conceal various interesting and relevant aspects present in the data. The cosmic matter distribution exhibits conspicuous features like filaments and walls, extended along one or two directions while compact in the other(s). In addition, the density fields display structure of varying contrasts over a large range of scales. Ideally sampled by the data points, appropriate field reconstructions should be set solely and automatically by the point distribution itself. The commonly used methods, involving artificial filtering through for instance grid-size or other smoothing kernels (e.g. Gaussian filters), often fail to achieve an optimal result.

Here we describe a new and fully self-adaptive method based on the Delaunay triangulation of the given point process. After a short description of the fundamentals of our tessellation procedure, we show its convincing performance on the result of an N-body simulation of structure formation, whose particle distribution is supposed to reflect the underlying cosmic density field. A detailed specification of the method, together with an extensive quantitative and statistical evaluation of its performance is presented in Chapters 3, 4 and 8.

2.2 The Delaunay Tessellation Field Estimator

Given a set of field values sampled at a discrete number of locations along one dimension we are familiar with various prescriptions for reconstructing the field over the full spatial domain. The most straightforward way involves the partition of space into bins centered on the sampling points. The field is then assumed to have the – constant – value equal to the one at the sampling point. Evidently, this yields a field with unphysical discontinuities at the boundaries of the bins. A first-order improvement concerns the linear interpolation between the sampling points, leading to a fully continuous field.

In more than one dimension the equivalent spatial intervals of the one-dimensional bins are well-known in stochastic geometry. A point process defines a Voronoi tessellation by dividing space into a unique and volume-covering network of mutually disjunct convex polyhedral cells, each of which comprises that part of multi-dimensional space closer to the defining point than to any of the other (see van de Weygaert 1991 and references therein). These Voronoi cells (see Fig. 2.1) are the multi-dimensional generalization of the one-dimensional bins in which the zeroth-order method approximates the field value to be constant. The natural extension to a multi-dimensional linear interpolation interval then immediately implies the corresponding Delaunay tessellation (Delone 1934). In two dimensions this tessellation consists of a volume-covering tiling of space into triangles (see Fig. 2.1, in three dimensions...
these are tetrahedra) whose vertices are formed by three specific points in the dataset. The three points are uniquely selected such that their circumscribing circle does not contain any of the other data points. The Voronoi and Delaunay tessellation are intimately related, being each others dual in that the centre of each Delaunay triangle’s circumscribing circle is a vertex of the Voronoi cells of each of the three defining points, and conversely each Voronoi cell nucleus a Delaunay vertex (see Fig. 2.1). The favourable properties of the Delaunay tessellation are in fact well-known and have been applied in, amongst others, surface rendering applications such as geographical mapping and various computer imaging algorithms.

Consider a set of $N$ discrete data points in a finite region of $D$-dimensional space. Having at one’s disposal the field values at each of the $(D + 1)$ Delaunay vertices $r_0, r_1, \ldots, r_D$ at each location $r$ in the interior of a $D$-dimensional Delaunay tetrahedron, the linear interpolation field value is defined by

$$\widetilde{f}(r) = f(r_0) + \overline{\nabla f}_{\text{Del}} \cdot (r - r_0), \quad (2.1)$$

in which $\overline{\nabla f}_{\text{Del}}$ is the estimated constant field gradient within the tetrahedron. Given the $(D + 1)$ field values $f(r_0), f(r_1), \ldots, f(r_D)$, the value of the $D$ components of $\overline{\nabla f}_{\text{Del}}$ can be straightforwardly computed by evaluating Eqn. 2.1 for each of the $D$ points $r_1, \ldots, r_D$. This multi-dimensional procedure of linear interpolation has been described by Bernardeau & van de Weygaert (1996) in the context of defining procedures for volume-weighted estimates of cosmic velocity fields. While they explicitly demonstrated that the zeroth-order Voronoi estimator is the asymptotic limit for volume-weighted field reconstructions from discretely sampled field values, they showed the superior performance of the first-order Delaunay estimator in reproducing analytical predictions.

The one factor complicating a trivial and direct implementation of the above procedure in the case of density and intensity field estimates is the fact that the number density of data points itself is the measure of the underlying density field value. Contrary to the case of

Figure 2.1 — A set of 20 points with their Voronoi (left-hand frame: solid lines) and Delaunay (right-hand frame: solid lines) tesselations. Left-hand frame: the shaded region indicates the Voronoi cell corresponding to the point located just below the center. Right-hand frame: the shaded region is the ‘contiguous Voronoi cell’ of the same point as in the left-hand frame.
velocity fields we therefore cannot start with directly available field estimates at each data point. Instead, we need to define appropriate estimates from the point set itself. Most suggestive would be to base the estimate of the density field at the location \( r_i \) of each point on the inverse of the volume of its Voronoi cell \( V_i \), \( \tilde{\rho}(r_i) = m/V(V_i) \). Note that in this we take every data point to represent an equal amount of mass \( m \). The resulting field estimates are then intended as input for the above Delaunay interpolation procedure. However, one can demonstrate that integration over the resulting density field would yield a different mass than the one represented by the set of sampling points (see Chapter 3 for a demonstration). Instead, mass conservation is naturally guaranteed when the density estimate is based on the inverse of the volume of the ‘contiguous’ Voronoi cell \( W_i \) of each data point, \( \tilde{\rho}(r_i) \propto m/V(W_i) \). The contiguous Voronoi cell of point \( i \) is the cell consisting of the union of all \( N_{T,i} \) Delaunay tetrahedra \( T_{ji} \) containing point \( i \) as one of its vertices,

\[
W_i = \bigcup_{j=1}^{N_{T,i}} T_{ji} .
\]  

(2.2)

Its volume is the sum of the volumes of each of the \( N_{T,i} \) Delaunay tetrahedra,

\[
V(W_i) = \sum_{j=1}^{N_{T,i}} V(T_{ji}) .
\]  

(2.3)

Fig. 2.1 (right-hand frame) depicts an illustration of such a cell. Properly normalizing the mass contained in the reconstructed density field, taking into account the fact that each Delaunay tetrahedron is invoked in the density estimate at \( (D+1) \) locations, we find at each data point the following density estimate,

\[
\tilde{\rho}(x_i) = \frac{(D+1)m}{V(W_i)} .
\]  

(2.4)

Having computed these density estimates, we subsequently proceed to determine the complete volume-covering density field reconstruction through the linear interpolation procedure outlined in Eqn. 2.1. Hereafter we will refer to this density field reconstruction procedure as the Delaunay Tessellation Field Estimator (DTFE).

### 2.3 Analysis of a cosmological \( N \)-body simulation

Cosmological \( N \)-body simulations provide an ideal template for illustrating the virtues of our method. They contain a wide variety of objects, with diverse morphologies, a large reach of densities and spanning over a vast range of scales. They display low density regions, sparsely filled with particles, as well as highly dense and compact clumps, represented by a large number of particles. Moderate density regions typically include strongly anisotropic structures such as filaments and walls.

Each of these features has their own individual characteristics and often these may only be sufficiently highlighted by some specifically designed analysis tool. Conventional methods are usually only tuned for uncovering one or a few aspects of the full array of properties. Contrary to artificial tailor-made methods, which may be insensitive to unsuspected but intrinsically important structural elements, the DTFE is uniquely defined and fully self-adaptive. Its
Figure 2.2 — Comparison of the performance of the DTFE with a conventional grid-based TSC method in analyzing a cosmological $N$-body simulation. Left-hand column: the particle distribution in a $10h^{-1}$ Mpc wide central slice through the simulation box. Central column: the corresponding DTFE density field reconstruction. Right-hand column: the TSC density field reconstruction.
outstanding performance is clearly illustrated by Fig. 2.2. Here we have analyzed an N-body simulation of structure formation in a standard CDM scenario ($\Omega_0 = 1$, $H_0 = 50$ km/s/Mpc). It shows the resulting distribution of $128^3$ particles in a cubic simulation volume of $100h^{-1}$ Mpc, at a cosmic epoch at which $\sigma(R_{TH} = 8h^{-1}$Mpc) = 1. The figure depicts a $10h^{-1}$Mpc thick slice through the center of the box. The left-hand column shows the particle distribution in a sequence of frames at increasingly fine resolution. Specifically we zoomed in on the richest cluster in the region. The right-hand column shows the corresponding density field reconstruction on the basis of the grid-based Triangular-Shaped Clouds (TSC) method, here evaluated on a $518^2$ grid. For the TSC method, one of the most frequently applied algorithms, we refer to the description in Hockney & Eastwood (1988). A comparison with other, more elaborate methods which have been developed to deal with the various aspects that we mentioned, of which SPH-based methods are the most common, are presented in Chapters 3, 4 and 5.

A comparison of the left-hand and right-hand columns with the central column, i.e. the DTFE density field reconstruction, reveals the striking improvement rendered by our new procedure. Going down from the top to the bottom in the central column, we observe seemingly comparable levels of resolved detail. The self-adaptive skills of the Delaunay reconstruction evidently succeed in outlining the full hierarchy of structure present in the particle distribution at every spatial scale represented in the simulation. The contrast with the achievements of the fixed grid-based TSC method in the right-hand column is striking, in particular when focus tunes in on the finer structures. The central cluster appears to be a mere featureless blob! In addition, low density regions are rendered as slowly varying regions at moderately low values. This realistic conduct should be set o against the erratic behaviour of the TSC reconstructions, plagued by annoying shot-noise effects.

Fig. 2.2 also bears witness to another virtue of the DTFE. It evidently succeeds in reproducing sharp, edgy and clumpy filamentary and wall-like features. Automatically it resolves the fine details of their anisotropic geometry, seemlessly coupling sharp contrasts along one or two compact directions with the mildly varying density values along the extended direction(s). Moreover, it also manages to deal succesfully with the substructures residing within these structures. The well-known poor operation of e.g. the TSC method is clearly borne out by the central right-hand frame. Its fixed and inflexible filtering characteristics blur the finer aspects of anisotropic structures. Such methods are therefore unsuited for an objective and unbiased scrutiny of the foam-like geometry which so pre-eminently figures in both the observed galaxy distribution as well as in the matter distribution in most viable models of structure formation.

Not only qualitatively, but also quantitatively our method turns out to compare favourably with respect to conventional methods. We have carefully scrutinized our method by means of an array of quantitative tests. A full discussion is presented in Chapters 3, 4 and 5. Here we mention the fact that the method recovers the density distribution function over many orders of magnitude. The grid-based methods, on the other hand, only approach the appropriate distribution in an asymptotic fashion and yield reliable estimates of the distribution function over a mere restricted range of density values. Very importantly, on the basis of the continuous density field reconstruction of the DTFE, we obtain an estimate of the density autocorrelation function that closely adheres to the (discrete) two-point correlation function directly determined from the point distribution. Further quantitative assessments are also presented
in Chapter 4. Finally, we should also consider the computational requirements of the various methods. Given a particle distribution, the basic action of computing the corresponding Delaunay tessellation ($O(N \log N)$, van de Weygaert 1991) and the subsequent interpolation steps ($O(N)$) are considerably less CPU intensive than the TSC method ($O(N^2)$). In the case of Fig. 2.1 the DTFE is about a factor of 10 faster. In the present implementation the bottleneck of the DTFE is its memory requirement, which is about a factor of 10 larger than that of the TSC procedure.

The preceding is ample testimony of the promise of tessellation-based methods for the aim of continuous field reconstruction. The presented method, following up on earlier work by Bernardeau & van de Weygaert (1996), may be seen as a first step towards yet more advanced tessellation methods. One suggested improvement will be a second-order method rendering a continuously differentiable field reconstruction, which would dispose of the rather conspicuous triangular patches that form an inherent property of the linear procedure with discontinuous gradients. In particular, we may refer to similar attempts to deal with related problems, along the lines of natural neighbour interpolation (Sibson 1981), such as implemented in the field of geophysics (Sambridge et al. 1995, Braun & Sambridge 1995) and in engineering mechanics (Sukumar 1998). As multi-dimensional discrete data sets are a major source of astrophysical information, we wish to promote tessellation-based methods as a natural instrument for astronomical data analysis.

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