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iISS gain of dissipative systems[†]

Bayu Jayawardhana^{*}, Andrew R. Teel^{*}, Eugene P. Ryan[#]

Abstract—For a class of dissipative nonlinear systems, it is shown that an associated dissipation inequality determines an iISS gain. The result can be used to show the convergence of the state trajectory whenever the input signal has bounded energy where the energy function is related to the dissipation inequality.

I. INTRODUCTION

We consider nonlinear systems $\mathbf{P}$, with input u and output y , of the general form

$$\left. \begin{aligned} \dot{x} &= f(x, u), \quad x(0) = x^0, \quad f(0, 0) = 0, \quad y = h(x), \\ f : \mathbb{R}^n \times \mathbb{R}^m &\rightarrow \mathbb{R}^n, \quad h : \mathbb{R}^n \rightarrow \mathbb{R}^l \text{ locally Lipschitz.} \end{aligned} \right\} \quad (1)$$

The concept of *integral input-to-state stability* (iISS) of $\mathbf{P}$ is introduced by Angeli *et al.* in [1] and is defined by the following property: there exist $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that, for every locally bounded input $u \in L^\infty_{loc}$ and every initial state $x^0 \in \mathbb{R}^n$, (1) has a unique solution x defined on $[0, \infty)$ and

$$\|x(t)\| \leq \beta(\|x^0\|, t) + \alpha\left(\int_0^t \gamma(\|u(s)\|) ds\right) \quad \forall t \geq 0. \quad (2)$$

The function γ is called an *iISS gain* of $\mathbf{P}$. If the input u is such that $\int_0^\infty \gamma(\|u(s)\|) ds < \infty$, then the iISS estimate in (2) also implies the converging-state property: $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

It is shown in [1, Theorem 1 and Remark II.3] that, if the equilibrium solution $x = 0$ (corresponding to $u = 0$ and $x^0 = 0$) of $\mathbf{P}$ is globally asymptotically stable (GAS) and there exists a continuously differentiable function $H : \mathbb{R}^n \rightarrow [0, \infty)$ such that, for some $\sigma \in \mathcal{K}$,

$$\left. \begin{aligned} \lim_{\|x\| \rightarrow \infty} H(x) &= \infty \quad \text{and} \\ \langle \nabla H(x), f(x, u) \rangle &\leq \sigma(\|u\|) \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \end{aligned} \right\} \quad (3)$$

then $\mathbf{P}$ is iISS and so (2) holds for every input signal $u \in L^\infty_{loc}$ and for every initial state $x^0 \in \mathbb{R}^n$. However, given a locally Lipschitz function f and a function σ such that (3) holds, it is not clear in [1] whether $\gamma(s) \leq \sigma(s)$ holds for all $s \in [0, \infty)$ (see also the proof of (4 $\Rightarrow$ 2) in [1, Theorem 1]): the latter inequality does hold if f in [1] is also assumed to satisfy

$\|f(0, v)\| \leq c\sigma(\|v\|)$ for all $v \in \mathbb{R}^m$, with $c > 0$ (see also the proof of [1, Proposition 2.5]).

Under certain conditions on f , it is shown in Jayawardhana [4] that, if the equilibrium $x = 0$ of $\mathbf{P}$ (with zero input $u = 0$) is GAS and there exists a continuously differentiable function H such that (3) holds with $\sigma(s) = s^p$ for all $s \geq 0$, then the state $x(t)$ converges to zero as $t \rightarrow \infty$ for every initial state x^0 and for every L^p input u . Note that the argument in [4] utilizes infinite-dimensional systems theory.

Related work on the state convergence property with L^p inputs can be found in Ryan [7]. Under certain assumptions on f , it is shown in [7] that if, for a given L^p input u and an initial state x^0 , there exists a solution x defined on $[0, \infty)$ and x has non-empty ω -limit set, then $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Motivated by the state convergence result from [4], we show that (2) holds with $\gamma = \sigma$ under a mild assumption on f which is weaker than those posited in [4], [7] and is also weaker than the condition that, for some $c > 0$, $\|f(0, v)\| \leq c\sigma(\|v\|)$ for all $v \in \mathbb{R}^m$. More precisely, we assume that the locally Lipschitz function f and σ are such that, for every compact set $K \subset \mathbb{R}^n$, there exists $c > 0$ such that $\|f(\xi, v)\| \leq c(1 + \sigma(\|v\|))$ for all $(\xi, v) \in K \times \mathbb{R}^m$. In Section V, we will discuss the difference of our conditions on f with those in [1], [4] and [7].

Using the fact that $\gamma = \sigma$, we can recover the state convergence results of [4] via the iISS approach of the present paper.

This estimate of iISS gain allows one to analyze the stability of interconnected nonlinear systems that contain dissipative systems via a small-gain type argument (see, for example, Ito & Jiang [3]).

II. PRELIMINARIES

Notation. Throughout, $\mathbb{R}_+ = [0, \infty)$. For a finite-dimensional vector $x \in \mathbb{R}^n$, we define its norm $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$. For any finite-dimensional vector space $\mathcal{V}$ endowed with a norm $\|\cdot\|$, the space $L^p(\mathbb{R}_+, \mathcal{V})$, $p \in [1, \infty)$, consists of all measurable functions $f : \mathbb{R}_+ \rightarrow \mathcal{V}$ such that $\int_0^\infty \|f(t)\|^p dt < \infty$. The L^p -norm of a function $f \in L^p(\mathbb{R}_+, \mathcal{V})$ is given by $\|f\|_{L^p} = (\int_0^\infty \|f(t)\|^p dt)^{\frac{1}{p}}$. For $f \in L^p(\mathbb{R}_+, \mathcal{V})$ and $T > 0$, f_T denotes the concatenation of the functions $f|_{[0, T]}$ and 0 given by

$$f_T(t) := \begin{cases} f(t), & t \in [0, T] \\ 0, & t \in (T, \infty). \end{cases}$$

The space $L^p_{loc}(\mathbb{R}_+, \mathcal{V})$ consists of all measurable functions $f : \mathbb{R}_+ \rightarrow \mathcal{V}$ such that $f_T \in L^p(\mathbb{R}_+, \mathcal{V})$ for all $T > 0$. The space of measurable essentially bounded functions $\mathbb{R}_+ \rightarrow \mathcal{V}$

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is denoted by $L^\infty(\mathbb{R}_+, \mathcal{V})$; the space of measurable *locally* essentially bounded functions $L_{loc}^\infty(\mathbb{R}_+, \mathcal{V})$ is the space of measurable functions $f : \mathbb{R}_+ \rightarrow \mathcal{V}$ with the property that $f_T \in L^\infty(\mathbb{R}_+, \mathcal{V})$ for all $T > 0$.

The space $C(\mathbb{R}^l, \mathbb{R}^p)$ (respectively $C^1(\mathbb{R}^l, \mathbb{R}^p)$) consists of all continuous (respectively continuously differentiable) functions $f : \mathbb{R}^l \rightarrow \mathbb{R}^p$. For any $\varepsilon \geq 0$, we write

$$\mathbf{B}_\varepsilon := \{x \in \mathbb{R}^n \mid \|x\| \leq \varepsilon\}.$$

The set $\mathcal{K}$ consists of all continuous strictly-increasing functions $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\alpha(0) = 0$. The set $\mathcal{K}_\infty$ consists of all functions $\alpha \in \mathcal{K}$ such that $\alpha(s) \rightarrow \infty$ as $s \rightarrow \infty$. The set $\mathcal{KL}$ consists of all functions $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\beta(\cdot, t) \in \mathcal{K}$ and $\beta(s, \cdot)$ converges to zero. The function $H : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called *proper* if for all $c \geq 0$, the (sub-level) set $\{x \mid H(x) \leq c\}$ is compact or it is equivalent to say that $H(x) \rightarrow \infty$ whenever $\|x\| \rightarrow \infty$.

Definition 2.1: Let $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$. A *solution* of (1) is an absolutely continuous function $x : [0, \omega) \rightarrow \mathbb{R}^n$, $\omega > 0$, such that

$$x(t) - x(0) = \int_0^t f(x(\tau), u(\tau)) d\tau \quad \forall t \in [0, \omega).$$

A solution is *maximal* if it has no right extension that is also a solution.

The following is a consequence of the standard theory of ordinary differential equations (see, e.g. [8]).

Proposition 2.2: For each $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and $x^0 \in \mathbb{R}^n$, the initial-value problem (1) has unique maximal solution $x : [0, \omega) \rightarrow \mathbb{R}^n$.

For ease of reference, we reiterate the following.

Definition 2.3: [1] The system $\mathbf{P}$ given by (1) is called *integral input-to-state stable* (iISS) if there exist functions $\beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$ such that, for every $x(0) \in \mathbb{R}^n$ and for every $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$, the unique maximal solution $x : [0, \omega) \rightarrow \mathbb{R}^n$ of (1) is such that $\omega = \infty$ and

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \alpha\left(\int_0^t \gamma(\|u(s)\|) ds\right) \quad \forall t \geq 0. \quad (4)$$

Definition 2.4: [1] A continuously differentiable function $U : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called an *iISS-Lyapunov function* for system $\mathbf{P}$ given by (1) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ and a continuous positive-definite function α_3 , such that

$$\alpha_1(\|\xi\|) \leq U(\xi) \leq \alpha_2(\|\xi\|) \quad \forall \xi \in \mathbb{R}^n, \quad (5)$$

and

$$\begin{aligned} & \langle \nabla U(\xi), f(\xi, v) \rangle \\ & \leq -\alpha_3(\|\xi\|) + \sigma(\|v\|) \quad \forall (\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m. \end{aligned} \quad (6)$$

For later convenience, in the following proposition we assemble some results, which are consequences of [1].

Proposition 2.5: The system $\mathbf{P}$, given by (1), is iISS if, and only if, it admits an iISS-Lyapunov function. Furthermore, if U is an iISS-Lyapunov function for $\mathbf{P}$ with $\sigma \in \mathcal{K}$

as in (6), then

(a) (4) holds with $\gamma = \sigma$, and

(b) for each $x^0 \in \mathbb{R}^n$ and measurable $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ such that $\int_0^\infty \sigma(\|u(t)\|) dt < \infty$, the unique global solution $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ of (1) has the converging state property: $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

III. MAIN RESULT

In what follows, we will have occasion to impose the following technical assumption on f and σ .

(A) For every compact set $K \subset \mathbb{R}^n$, there exist $c > 0$ such that

$$\|f(\xi, v)\| \leq c(1 + \sigma(\|v\|)) \quad \forall (\xi, v) \in K \times \mathbb{R}^m. \quad (7)$$

Theorem 3.1: Consider the system $\mathbf{P}$ as in (1). Suppose that the zero state of $\mathbf{P}$, with $u = 0$, is GAS and there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ and $H \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\left. \begin{aligned} \alpha_1(\|\xi\|) &\leq H(\xi) \leq \alpha_2(\|\xi\|) \quad \forall \xi \in \mathbb{R}^n \\ \langle \nabla H(\xi), f(\xi, v) \rangle &\leq \sigma(\|v\|) \quad \forall (\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m, \end{aligned} \right\} \quad (8)$$

If f and σ satisfy (A) then there exist a continuous positive-definite function $\alpha_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, functions $\alpha_4, \alpha_5 \in \mathcal{K}_\infty$ and $U \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\begin{aligned} \langle \nabla U(\xi), f(\xi, v) \rangle &\leq -\alpha_3(\|\xi\|) + \sigma(\|v\|) \\ &\quad \forall (\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m \end{aligned} \quad (9)$$

and

$$\alpha_4(\|\xi\|) \leq U(\xi) \leq \alpha_5(\|\xi\|) \quad \forall \xi \in \mathbb{R}^n, \quad (10)$$

(and so U is an iISS Lyapunov function for $\mathbf{P}$).

Moreover, for every initial state $x^0 \in \mathbb{R}^n$ and for every measurable input $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ such that $\int_0^\infty \sigma(\|u(\tau)\|) d\tau < \infty$, the unique maximal solution $x : [0, \omega) \rightarrow \mathbb{R}^n$ is such that the following hold:

- (a) $\omega = \infty$;
- (b) x satisfies (4) with $\gamma = \sigma$;
- (c) $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

We preface the proof of Theorem 3.1 with three technical lemmas.

Lemma 3.2: Let $f \in C(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$. Then, for every compact set K , there exists a function $\rho_K \in \mathcal{K}_\infty$ such that

$$\|f(\xi, v) - f(\xi, 0)\| \leq \rho_K(\|v\|) \quad \forall (\xi, v) \in K \times \mathbb{R}^m. \quad (11)$$

PROOF. Let $K \subset \mathbb{R}^n$ be compact and define $\tilde{\rho}_K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$\tilde{\rho}_K(a) := \max \{ \|f(\xi, v) - f(\xi, 0)\| \mid \xi \in K, v \in \mathbf{B}_a \}. \quad (12)$$

Clearly, $\tilde{\rho}_K$ is non-decreasing and so, *a fortiori*, is measurable (in fact, it can be shown that $\tilde{\rho}_K$ is upper semicontinuous).

Therefore, the function $\rho_K : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is well defined by

$$\rho_K(0) := 0, \quad \rho_K(a) := a + \frac{1}{a} \int_a^{2a} \tilde{\rho}_K(\tau) d\tau \quad \forall a > 0.$$

It is readily verified that $\rho_K \in \mathcal{K}_\infty$. Moreover, $\rho_K(a) \geq \tilde{\rho}_K(a)$ for all $a \in \mathbb{R}_+$ and so (11) holds. $\square$

Lemma 3.3: Assume that $f \in C(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $\sigma \in \mathcal{K}$ satisfy **(A)**. Let $w \in \mathbb{R}^n \rightarrow \mathbb{R}_+$ be continuous and such that, for some $\alpha \in \mathcal{K}_\infty$,

$$\alpha(\|\xi\|) \leq w(\xi) \quad \forall \xi \in \mathbb{R}^n. \quad (13)$$

Then, for every continuous function $\theta : (0, \infty) \rightarrow (0, \infty)$, there exist a continuous function $\delta : (0, \infty) \rightarrow (0, \infty)$ such that

$$\|f(\xi, v) - f(\xi, 0)\| \leq \theta(w(\xi)) + \delta(w(\xi))\sigma(\|v\|), \quad \forall \xi \in \mathbb{R}^n \setminus \{0\} \quad \forall v \in \mathbb{R}^m. \quad (14)$$

PROOF. Since f satisfies **(A)**, it is readily verified that, for every compact set $K \subset \mathbb{R}^n$, there exists $c > 0$ such that

$$\|f(\xi, v) - f(\xi, 0)\| \leq c(1 + \sigma(\|v\|)) \quad \forall (\xi, v) \in K \times \mathbb{R}^m.$$

This implies that there exists a strictly increasing sequence (c_k) in $\mathbb{N}$ such that

$$\|f(\xi, v) - f(\xi, 0)\| \leq c_k(1 + \sigma(\|v\|)) \quad \forall (\xi, v) \in \mathbf{B}_k \times \mathbb{R}^m.$$

Let $b : [0, \infty) \rightarrow (0, \infty)$ be the continuous function that linearly interpolates the points c_k , $k \in \mathbb{N}$, that is,

$$b(\lambda) := c_k + (c_{k+1} - c_k)(\lambda + 1 - k) \quad \forall \lambda \in [k - 1, k] \quad \forall k \in \mathbb{N}.$$

Then,

$$\|f(\xi, v) - f(\xi, 0)\| \leq b(\|\xi\|)(1 + \sigma(\|v\|)) \quad \forall (\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m. \quad (15)$$

By Lemma 3.2, there exists $\rho_1 \in \mathcal{K}_\infty$ such that

$$\|f(\xi, v) - f(\xi, 0)\| \leq \rho_1(\|v\|) \quad \forall (\xi, v) \in \mathbf{B}_1 \times \mathbb{R}^m. \quad (16)$$

Let $\theta : (0, \infty) \rightarrow (0, \infty)$ be continuous. Denote by $\chi_1 \in \mathcal{K}_\infty$ the inverse of the function $\rho_1 \in \mathcal{K}_\infty$ and write $\tilde{b} = b \circ \alpha^{-1}$. Define the continuous function $\delta_1 : (0, 1] \rightarrow (0, \infty)$,

$$a \mapsto \delta_1(a) := \tilde{b}(a) + \frac{\|\tilde{b}(a) - \theta(a)\|}{\sigma(\chi_1(\theta(a)))}.$$

If $\xi \in \mathbf{B}_1 \setminus \{0\}$ and $\|v\| \leq \chi_1(\theta(w(\xi)))$ then $\rho_1(\|v\|) \leq \theta(w(\xi))$ and so, by (16),

$$\|f(\xi, v) - f(\xi, 0)\| \leq \theta(w(\xi)) \leq \theta(w(\xi)) + \delta_1(w(\xi))\sigma(\|v\|).$$

If $\xi \in \mathbf{B}_1 \setminus \{0\}$ and $\|v\| > \chi_1(\theta(w(\xi)))$ then, by (13) and (15),

$$\begin{aligned} \|f(\xi, v) - f(\xi, 0)\| &\leq b(\|\xi\|)(1 + \sigma(\|v\|)) \\ &\leq \tilde{b}(w(\xi))(1 + \sigma(\|v\|)) \\ &\leq \theta(w(\xi)) + \|\tilde{b}(w(\xi)) - \theta(w(\xi))\| \\ &\quad + \tilde{b}(w(\xi))\sigma(\|v\|) \\ &\leq \theta(w(\xi)) + \delta_1(w(\xi))\sigma(\|v\|). \end{aligned}$$

This establishes that

$$\|f(\xi, v) - f(\xi, 0)\| \leq \theta(w(\xi)) + \delta_1(w(\xi))\sigma(\|v\|) \quad \forall (\xi, v) \in (\mathbf{B}_1 \setminus \{0\}) \times \mathbb{R}^m. \quad (17)$$

For every $k \in \mathbb{N}$, $k \geq 2$, let C_k denote the compact set

$$C_k := \{\xi \in \mathbb{R}^n \mid 1 \leq w(\xi) \leq k\}.$$

By Lemma 3.2, for every $k \geq 2$, there exists a function $\rho_k \in \mathcal{K}_\infty$ such that

$$\|f(\xi, v) - f(\xi, 0)\| \leq \rho_k(\|v\|) \quad \forall (\xi, v) \in C_k \times \mathbb{R}^m.$$

For every $k \geq 2$, let $\chi_k \in \mathcal{K}_\infty$ denote the inverse of $\rho_k \in \mathcal{K}_\infty$ and define the continuous function $\delta_k : [1, k] \rightarrow (0, \infty)$ by

$$\delta_k(a) := \tilde{b}(a) + \frac{\|\tilde{b}(a) - \theta(a)\|}{\sigma(\chi_k(\theta(a)))}.$$

Then an argument analogous to that leading to (17) gives

$$\|f(\xi, v) - f(\xi, 0)\| \leq \theta(w(\xi)) + \delta_k(w(\xi))\sigma(\|v\|) \quad \forall (\xi, v) \in C_k \times \mathbb{R}^m, \quad k = 2, 3, \dots \quad (18)$$

Now, define

$$\delta_1^* := \delta_1(1), \quad \delta_k^* := \max\{\max_{a \in [1, k]} \delta_k(a), \delta_{k-1}^*\}, \quad k = 2, 3, \dots$$

The sequence $(\delta_k^*)_{k \in \mathbb{N}}$ so constructed is non-decreasing. Finally, define the function $\delta : (0, \infty) \rightarrow (0, \infty)$ as follows

$$\delta(a) := \begin{cases} \delta_1(a) + \delta_2^* - \delta_1^*, & a \in (0, 1] \\ \delta_{k+1}^* + (\delta_{k+2}^* - \delta_{k+1}^*)(a - k), & a \in (k, k+1], k \in \mathbb{N} \end{cases}$$

The function δ is continuous, with the properties

$$\begin{aligned} \delta(a) &\geq \delta_1 \quad \forall a \in (0, 1], \quad \text{and} \\ \delta(a) &\geq \delta_k(a) \quad \forall a \in [1, k], \quad k = 2, 3, \dots \end{aligned}$$

In view of (17) and (18), it follows that (14) holds. This completes the proof. $\square$

Lemma 3.4: Let $\sigma \in \mathcal{K}$. Consider the system **P** as in (1) and assume that f and σ satisfy **(A)**. If the zero state of **P** (with $u = 0$) is GAS, then, for every $\varepsilon > 0$, there exists a continuous positive-definite function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $W \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that $W(0) = 0$, $W(x) > 0$ for $x \neq 0$ and

$$\begin{aligned} \langle W(\xi), f(\xi, v) \rangle &\leq -\alpha(\|\xi\|) + \varepsilon\sigma(\|v\|) \quad \forall (\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m. \quad (19) \end{aligned}$$

Remark 3.5: We remark that the function W in Lemma 3.4 is not necessarily proper (that is, its sub-level sets are not necessarily compact).

PROOF. The GAS property implies that there exist a smooth $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $\nabla V(0) = 0$ and functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that

$$\left. \begin{aligned} \alpha_1(\|\xi\|) &\leq V(\xi) \leq \alpha_2(\|\xi\|) \\ \langle \nabla V(\xi), f(\xi, 0) \rangle &\leq -\alpha_3(\|\xi\|) \end{aligned} \right\} \quad \forall \xi \in \mathbb{R}^n \quad (20)$$

(see, for example, [5]). Define $\tilde{\alpha}_4 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$\tilde{\alpha}_4(a) = \max\{\|\nabla V(\xi)\| \mid \xi \in \mathbb{R}^n, V(\xi) \leq a\} \quad \forall a \in \mathbb{R}_+.$$

The function $\tilde{\alpha}_4$ is non-decreasing and so we may define a continuous function $\alpha_4 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$\alpha_4(0) = 0, \quad \alpha_4(a) = \frac{1}{a} \int_a^{2a} \tilde{\alpha}_4(\tau) d\tau \quad \forall a > 0.$$

Moreover, α_4 is non-decreasing with $\alpha_4(a) \geq \tilde{\alpha}_4(a)$ for all $a \in \mathbb{R}_+$ and

$$\|\nabla V(\xi)\| \leq \alpha_4(V(\xi)) \quad \forall \xi \in \mathbb{R}^n.$$

Now define the continuous function $\theta : (0, \infty) \rightarrow (0, \infty)$ by

$$\theta(a) = \min \left\{ a, \frac{\alpha_3(\alpha_2^{-1}(a))}{2\alpha_4(a)} \right\} \quad \forall a \in (0, \infty),$$

in which case, we have

$$\begin{aligned} \|\nabla V(\xi)\| \theta(V(\xi)) &\leq \alpha_4(V(\xi)) \theta(V(\xi)) \\ &\leq \frac{1}{2} \alpha_3(\alpha_2^{-1}(V(\xi))) \\ &\leq \frac{1}{2} \alpha_3(\|\xi\|) \quad \forall \xi \in \mathbb{R}^n. \end{aligned} \quad (21)$$

By Lemma 3.3, there exists a continuous function $\delta : (0, \infty) \rightarrow (0, \infty)$ such that

$$\|f(\xi, v) - f(\xi, 0)\| \leq \theta(V(\xi)) + \delta(V(\xi)) \sigma(\|v\|) \quad \forall (\xi, v) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^m. \quad (22)$$

Let $\varepsilon > 0$ and define a continuous function $\kappa \in \mathcal{K}$ by

$$\kappa(0) = 0, \quad \kappa(a) = \min \left\{ a, \frac{\varepsilon}{\alpha_4(a) \delta(a)} \right\} \quad \forall a \in (0, \infty).$$

It follows that

$$\begin{aligned} \kappa(V(\xi)) \|\nabla V(\xi)\| \delta(V(\xi)) &< \frac{\varepsilon \|\nabla V(\xi)\| \delta(V(\xi))}{\alpha_4(V(\xi)) \delta(V(\xi))} \\ &\leq \varepsilon \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \quad (23)$$

Define $W \in C^1(\mathbb{R}^n \rightarrow \mathbb{R}_+)$ by

$$W(\xi) = \int_0^{V(\xi)} \kappa(\tau) d\tau.$$

Since $V(0) = 0$, $\nabla V(0) = 0$ and $\kappa(0) = 0$, it follows that $W(0) = 0$ and $\nabla W(0) = 0$. Using the positive definiteness of κ and V , we conclude that $W(\xi) > 0$ for all $\xi \neq 0$. Invoking (20), (21), (22) and (23), we have

$$\begin{aligned} \langle \nabla W(\xi), f(\xi, v) \rangle &= \kappa(V(\xi)) \langle \nabla V(\xi), f(\xi, v) - f(\xi, 0) + f(\xi, 0) \rangle \\ &\leq \kappa(V(\xi)) \|\nabla V(\xi)\| \|f(\xi, v) - f(\xi, 0)\| \\ &\quad - \kappa(V(\xi)) \alpha_3(\|\xi\|) \\ &\leq \kappa(V(\xi)) \|\nabla V(\xi)\| [\theta(V(\xi)) + \delta(V(\xi)) \sigma(\|v\|)] \\ &\quad - \kappa(V(\xi)) \alpha_3(\|\xi\|) \\ &\leq -\frac{1}{2} \kappa(V(\xi)) \alpha_3(\|\xi\|) + \varepsilon \sigma(\|v\|) \\ &\quad \forall (\xi, v) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R}^m. \end{aligned} \quad (24)$$

Since $\langle \nabla W(0), f(0, v) \rangle = 0$ for all $v \in \mathbb{R}^m$, we conclude that (24) holds for all $(\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m$. The proof is completed by setting

$$\alpha(a) := \frac{1}{2} \min \{ \kappa(V(\eta)) \alpha_3(\|\eta\|) \mid \|\eta\| \leq a, \eta \in \mathbb{R}^n \}.$$

□

PROOF OF THEOREM 3.1. Let $\varepsilon > 0$ be arbitrary. By Lemma 3.4, there exists a continuous positive-definite function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $W \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that $W(0) = 0$, $W(x) > 0$ for $x \neq 0$ and (19) holds.

Define $U : \mathbb{R}^n \rightarrow \mathbb{R}_+$ by

$$U(\xi) = \frac{1}{1+\varepsilon} (W(\xi) + H(\xi)) \quad \forall \xi \in \mathbb{R}^n.$$

Setting $\alpha_3 = \frac{1}{1+\varepsilon} \alpha$ and invoking (19) together with the second of inequalities (8), we may conclude that (9) holds. For each $a \in \mathbb{R}_+$ define $\tilde{\alpha}_4(a) := \min\{W(\xi) \mid \|\xi\| \leq a\}$ and $\tilde{\alpha}_5(a) := \max\{W(\xi) \mid \|\xi\| \leq a\}$. Then $\alpha_4 := \frac{\alpha_1 + \alpha_4}{1+\varepsilon}$ and $\alpha_5 := \frac{\alpha_2 + \alpha_5}{1+\varepsilon}$ are $\mathcal{K}_\infty$ functions and are such that (10) holds. Therefore, U is an *iISS* Lyapunov function for **P**.

Assertions (a), (b) and (c) now follow from Proposition 2.5. □

For a measurable function $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$, assume that the functional

$$u \mapsto \int_0^\infty \sigma(\|u(\tau)\|) d\tau$$

defines the *energy* of a locally essentially bounded input signal u . Then Theorem 3.1 provides sufficient conditions for convergence to zero of the state of system (1) with bounded-energy input u ; moreover, if **P** satisfies the hypotheses of the theorem and if $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ then for every initial state x^0 , there exists a unique solution $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ of (1) and it satisfies the estimate (4) for all $t \geq 0$ with $\gamma = \sigma$.

The system **P** in (1) is said to be *zero-state detectable* if the following holds: if $u = 0$ and the unique maximal solution x of (1) exists on $\mathbb{R}_+$ and is such that $y(t) = 0$ for all $t \geq 0$, then $\lim_{t \rightarrow \infty} x(t) = 0$ (see also [2, Definition 10.7.3]).

Proposition 3.6: Consider the system **P** as in (1). Assume further that **P** is zero-state detectable and that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha, \sigma \in \mathcal{K}$ and $H \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\left. \begin{aligned} \alpha_1(\|\xi\|) &\leq H(\xi) \leq \alpha_2(\|\xi\|) \quad \forall \xi \in \mathbb{R}^n, \\ \langle \nabla H(\xi), f(\xi, v) \rangle &\leq -\alpha(\|h(\xi)\|) + \sigma(\|v\|) \\ &\quad \forall (\xi, v) \in \mathbb{R}^n \times \mathbb{R}^m. \end{aligned} \right\} \quad (25)$$

If f and σ satisfy **(A)** then there exists an *iISS* Lyapunov function $U \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ such that (9) and (10) holds.

Moreover, for every initial state $x(0) \in \mathbb{R}^n$ and for every measurable input $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R}^m)$ such that $\int_0^\infty \sigma(\|u(\tau)\|) d\tau < \infty$, the unique maximal solution $x : [0, \omega) \rightarrow \mathbb{R}_+$ of (1) is such that the following hold:

(a) $\omega = \infty$;

- (b) x satisfies (4) with $\gamma = \sigma$;
(c) $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

PROOF. The proof of the proposition follows from Theorem 3.1 once we have shown that (with $u = 0$) the zero state is GAS. By (25), we may infer that the zero state is a stable equilibrium of $\dot{x} = f(x, 0)$ and, for each x^0 , the unique maximal solution x of the initial-value problem $\dot{x} = f(x, 0)$, $x(0) = x^0$, exists on $\mathbb{R}_+$. It remains to show that the zero state is globally attractive. Let $x^0 \in \mathbb{R}^n$ be arbitrary, let $u = 0$ and let $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ be the unique maximal solution of the initial-value problem $\dot{x} = f(x, 0)$, $x(0) = x^0$. Then,

$$\frac{d}{dt}H(x(t)) \leq -\alpha(\|h(x(t))\|) \leq 0 \quad \forall t \in [0, \omega), \quad (26)$$

By the zero-state-detectability property, $\{0\}$ is the only invariant subset of $\Sigma := \{\xi \in \mathbb{R}^n \mid \alpha(\|h(\xi)\|) = 0\}$ and so, by the LaSalle invariance principle [6], we may conclude that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence, the zero state (with $u = 0$) is GAS. $\square$

IV. EXAMPLE

Let the system $\mathbf{P}$ be described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x, u) \\ &= \begin{bmatrix} -x_1(1 + 1/(u^p + 1) + u^p) - x_2^{p-1} + u \\ x_1^{p-1} \end{bmatrix}, \end{aligned} \quad (27)$$

$$y = h(x) := x_1, \quad (28)$$

where $p \in \{2n \mid n \in \mathbb{N}\}$.

Defining $\xi \mapsto H(\xi_1, \xi_2) := \xi_1^p + \xi_2^p$ and invoking Young's inequality, we have

$$\begin{aligned} \langle \nabla H(\xi), f(\xi, v) \rangle &= -p\xi_1^{p-1}(1 + 1/(v^p + 1) + v^p) + p\xi_1^{p-1}v, \\ &\leq -p\xi_1^p + (p-1)\xi_1^p + v^p \\ &= -|h(\xi)|^p + |v|^p. \end{aligned}$$

It is readily verified that, for every compact set $K \subset \mathbb{R}^2$, there exist $c > 0$ such that $\|f(\xi, v)\| \leq c(1 + \|v\|^p)$ for all $(\xi, v) \in K \times \mathbb{R}$, for example,

$$c = \max_{\xi \in K} \{5 + 4|\xi_1|^{(p-1)} + |\xi_2|^{p-1}\}$$

suffices. Hence (A) holds. The system $\mathbf{P}$ is also zero-state detectable.

Therefore, $\mathbf{P}$ satisfies the hypotheses of Proposition 3.6, with $\alpha(s) = s^p$ and $\sigma(s) = s^p$ for all $s \geq 0$, and so, for every input $u \in L^p(\mathbb{R}_+, \mathbb{R})$ and for every initial state $x^0 \in \mathbb{R}^2$, the solution x is defined on $\mathbb{R}_+$, satisfies (4) with $\gamma(s) = s^p$ for all $s \geq 0$ and $\lim_{t \rightarrow \infty} x(t) = 0$.

For simulation purpose, we take $p = 2$ and choose the function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ given by

$$u(t) = \begin{cases} n & n - \frac{1}{n^4} < t \leq n, \quad n \in \mathbb{N} \\ 0 & \text{elsewhere,} \end{cases} \quad (29)$$

so that $u \in L_{loc}^\infty(\mathbb{R}_+, \mathbb{R})$, $\int_0^\infty u^2(\tau) d\tau < \infty$ and $\limsup_{t \in \mathbb{R}_+} |u(t)| = \infty$. Using Matlab, we simulate $\mathbf{P}$ with u

Fig. 1. The state trajectory of the plant $\mathbf{P}$ as in (27) with $p = 2$, an L^2 input u as in (29) and initial states $x_1(0) = 1, x_2(0) = 1$. (a) The plot of x_1 . (b) The plot of x_2 .

as in (29) and with initial state components $x_1(0) = 1 = x_2(0)$. Figure 1 shows the convergence to zero of the state components $x_1(t)$ and $x_2(t)$ as $t \rightarrow \infty$.

V. DISCUSSION

In view of recent results on L^p -input state-convergence, we briefly discuss the various assumptions on f used in [1], [4], [7] and compare these with assumption (A) of the present paper. We focus on the case wherein $\sigma(s) = s^p$ for all $s \geq 0$ for $p \in [1, \infty)$. It is common to assume that $f \in C(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $f(0, 0) = 0$.

The article [4] assumes the following on f :

- (A1) For every compact set $K \subset \mathbb{R}^n$, there exist $c_1, c_2 > 0$ such that

$$\|f(\xi_1, v) - f(\xi_2, v)\| \leq (c_1 + c_2\|v\|^p)\|\xi_1 - \xi_2\| \quad \forall \xi_1, \xi_2 \in K, \forall v \in \mathbb{R}^m. \quad (30)$$

- (A2) For each fixed $\xi \in \mathbb{R}^n$, there exist $c_3, c_4 > 0$ such that

$$\|f(\xi, v)\| \leq c_3 + c_4\|v\|^p \quad \forall v \in \mathbb{R}^m. \quad (31)$$

It can be shown that (A1) and (A2) imply (A). Indeed, let $K \subset \mathbb{R}^n$ be compact and let $\xi \in K$. By Assumption (A2),

there exist $c_3, c_4 > 0$ such that (31) holds. Using Assumption **(A1)**, there exist $c_1, c_2 > 0$ such that (30) holds and we have

$$\begin{aligned} \|f(\eta, v)\| &\leq \|f(\eta, v) - f(\xi, v)\| + \|f(\xi, v)\| \\ &\leq (c_1 + c_2\|v\|^p)\|\eta - \xi\| + (c_3 + c_4\|v\|^p) \\ &\leq c(1 + \|v\|^p) \quad \forall (\eta, v) \in K \times \mathbb{R}^m, \end{aligned}$$

where $c = \max_{\eta \in K} \{c_1\|\eta - \xi\| + c_3 + c_2\|\eta - \xi\| + c_4\}$. However, it is easy to see that **(A)** does not imply **(A1)** and **(A2)**.

In [7], the function f satisfies the following assumptions (with the state space $\mathbb{X} = \mathbb{R}^n$):

(A3) For every compact set $K \subset \mathbb{R}^n$, there exists $k > 0$ such that

$$\|f(\xi, v) - f(\xi, 0)\| \leq k\|v\| \quad \forall (\xi, v) \in K \times \mathbb{R}^m. \quad (32)$$

(A4) $f(\cdot, 0)$ is locally Lipschitz.

Note that f in [7] is not necessarily locally Lipschitz on both arguments.

We will show that **(A3)** and **(A4)** imply **(A)**. Let $K \subset \mathbb{R}^n$ be compact. By **(A4)**, there exists $L > 0$ such that $\|f(\xi, 0)\| \leq L\|\xi\|$ for all $\xi \in K$. By **(A3)**, there exists $k > 0$ such that (32) holds. Therefore, we have

$$\begin{aligned} \|f(\xi, v)\| &\leq \|f(\xi, v) - f(\xi, 0)\| + \|f(\xi, 0)\| \\ &\leq k + k\|v\|^p + L\|\xi\| \\ &\leq c(1 + \|v\|^p) \quad \forall (\xi, v) \in K \times \mathbb{R}^m, \end{aligned}$$

where $c = \max_{\xi \in K} \{2k + L\|\xi\|\}$. On the other hand, it can be checked that **(A)** does not imply **(A3)**.

As mentioned in the Introduction, by imposing that there exist $c > 0$ and $p \geq 1$ such that $\|f(0, v)\| \leq c\|v\|^p$ for all $v \in \mathbb{R}^m$ and by using the technique to prove the main result in [1], we can conclude that $\gamma(s) = s^p$ holds for all $s \in [0, \infty)$. However, this assumption excludes many well-studied nonlinear systems. For example, for affine nonlinear systems, the function $f(x, u) := \tilde{f}(x) + g(x)u$ with continuous $\tilde{f}$ and g , does not satisfy this assumption for $p > 1$.

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