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Normal linear stability of quasi-periodic tori ☆

H.W. Broer *, J. Hoo, V. Naudot

*Department of Mathematics and Computing Science, University of Groningen,
PO Box 800, 9700 AV Groningen, The Netherlands*

Abstract

We consider families of dynamical systems having invariant tori that carry quasi-periodic motions. Our interest is the persistence of such tori under small, nearly-integrable perturbations. This persistence problem is studied in the dissipative, the Hamiltonian and the reversible setting, as part of a more general KAM theory for classes of structure preserving dynamical systems. This concerns the parametrized KAM theory as initiated by Moser [J.K. Moser, On the theory of quasiperiodic motions, SIAM Rev. 8 (2) (1966)145–172; J.K. Moser, Convergent series expansions for quasi-periodic motions, Math. Ann. 169 (1967) 136–176] and further developed in [G.B. Huitema, Unfoldings of quasi-periodic tori, PhD thesis, University of Groningen, 1988; H.W. Broer, G.B. Huitema, F. Takens, Unfoldings of quasi-periodic tori, Mem. Amer. Math. Soc. 83 (421) (1990) 1–82; H.W. Broer, G.B. Huitema, Unfoldings of quasi-periodic tori in reversible systems, J. Dynam. Differential Equations 7 (1) (1995) 191–212]. The corresponding nondegeneracy condition involves certain (trans-)versality conditions on the normal linear, leading, part at the invariant tori. We show that as a consequence, a Cantor family of Diophantine tori with positive Hausdorff measure is persistent under nearly-integrable perturbations. This result extends the above references since presently the case of multiple Floquet exponents is included. Our leading example is the normal $1:-1$ resonance, which occurs a lot in applications, both Hamiltonian and reversible. As an illustration of this we briefly describe the Lagrange top coupled to an oscillator.
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Keywords: KAM theory; Nearly-integrable system; Perturbation theory; Quasi-periodic stability; (Uni-)versal unfolding; Linear centralizer unfolding; Normal linear part; Quasi-periodic bifurcations; Hamiltonian and reversible Hopf bifurcation; Lagrange top

1. Introduction

We consider dynamical systems, in particular, vector fields with invariant tori that carry quasi-periodic dynamics. These systems may depend on parameters and also may respect a given structure such as a symplectic 2-form, a volume form, or a symmetry, including reversibility, or combinations of these. Starting with an integrable family, the persistence of such tori is investigated, when the perturbation is not necessarily integrable. This is a part of KAM theory, where it turns out that persistence is closely related to the notion of (trans-)versality with respect to the normal linear (leading) part at the tori. In [26,55,66,67,70,71,78,79,84] this persistence problem was addressed for the case where all Floquet exponents are simple. According to [23,26,55], the 'majority' of invariant tori of an integrable family of vector fields is persistent under small perturbation, whenever this family satisfies the following conditions:

- The Floquet exponents (or normal eigenvalues) are simple;
- The Floquet matrix is nonsingular;
- The family depends on parameters in a generic way, only affecting the normal linear (leading) part;
- The internal and normal frequencies satisfy Diophantine conditions.

These various concepts will be explained below. The main purpose of the present paper is to generalize the result of [23,26,55], by extending it to the case of multiple Floquet exponents. We begin by briefly reviewing its setting, which is based on [70,71,78,79] and on [31,52]. Also see [17,66,67,84] and [25,27].

Our phase space will be a smooth, finite-dimensional manifold M and the parameter space $P \subset \mathbb{R}^p$ as an open and connected domain. Let $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ denote the standard n-torus group. We assume that $\mathbb{T}^n$ acts on M by a smooth free action, that is, there is a smooth map $\Phi : \mathbb{T}^n \times M \to M$, such that for all $x \in M$,

- $\Phi(0, x) = x$;
- $\Phi(\theta_1, \Phi(\theta_2, x)) = \Phi(\theta_1 + \theta_2, x)$, for all $\theta_1, \theta_2 \in \mathbb{T}^n$;
- The induced map $\Phi(\cdot, x) : \mathbb{T}^n \to M$ is injective, for all $x \in M$.

The latter condition means that the action Φ is free. A vector field on M is called *integrable* or $\mathbb{T}^n$-*symmetric* if it is equivariant with respect to the $\mathbb{T}^n$-action. Consider a smooth family $X = X_\mu(x)$ of $\mathbb{T}^n$-symmetric vector fields on M, where $\mu \in P$, with an invariant submanifold $T \subseteq M \times P$ which is assumed to have the form

$$T = \bigcup_{\mu \in P} T_\mu \times \{\mu\} \subseteq M \times P,$$

where $T_\mu \subseteq M$ is an X_μ-invariant $\mathbb{T}^n$-orbit for each μ. Since the action is free, each T_μ is diffeomorphic to $\mathbb{T}^n$, carrying conditionally periodic (or parallel) dynamics. It is known that in the Hamiltonian and reversible setting, the invariant tori generically occur as continua. However, in Sections 1.2 and 1.3, we shall see that one can always reduce to the above situation, where each parameter value μ corresponds to exactly one $\mathbb{T}^n$-orbit T_μ. This can be achieved by introducing extra distinguished parameters that serve to single out individual invariant tori and introduce local coordinates around these, compare with [55], [26, Section 5a] and [23, Section 2.1].

In the sequel we study the persistence of the invariant tori T_μ when the integrable family X is perturbed into a not necessarily integrable one, usually referred to as *nearly-integrable*. The present setting is axiomatic, thereby including both the Lie algebra approach of [26,55,71] and the reversible case of [23], also compare with [31,52]. See [27] for a survey, also compare with [25,92]. We briefly discuss a few consequences of this approach. The ensuing parametrized KAM theory has many applications, in various symplectic, volume preserving and equivariant (including reversing) settings, or in combinations of these.

- The nondegeneracy condition as used in [23,26], which is based on generic unfolding of the normal linear part, often is referred to as Broer–Huitema–Takens (or BHT) nondegeneracy, which contains the classical Kolmogorov nondegeneracy condition as a special case.
- Although BHT nondegeneracy may need quite a lot of parameters, it leads to transparent proofs of KAM persistence results for individual systems, for instance in the Hamiltonian isotropic case under Rüssmann nondegeneracy conditions using Herman's method. Here the geometric and algebraic properties of the Diophantine conditions are exploited, for a description see [25,27,92]. Here it should be mentioned that in the cases where the integrable invariant tori occur in continua, many of the parameters are distinguished, which means that these can be 'compensated' by phase space variables.
- On the other hand the parametrized KAM theory allows to develop typical models for quasi-periodic bifurcation in all the various settings, e.g., compare with [13,25,44]. For an extensive survey with many applications, we refer to [20]. Typicality here means that the persistence covers an open subset in the C^k-topology, for k large, including the cases C^∞ and real-analytic. For details see below.
- As said earlier, the main contribution of the present paper is generalization of the parametrized KAM theory to the case with multiple Floquet exponents. The present results include the cases of [23,26] and the Lie algebra proof of [26,55] is largely followed, where the in the linearization used for Newtonian iteration, the linear versal unfolding theory of matrices [3,39,40] is being used. The case of zero Floquet exponents is not covered here, but is dealt with in [97]; for case studies in this respect, also see, e.g., [13,21,22,43,45,96].

In the presentation we shall treat three of the settings in detail, namely the dissipative, the Hamiltonian and the reversible setting, taking as a leading example the normal $1 : -1$ resonance, which in the reversible case coincides with the normal $1 : 1$ resonance. The corresponding nonlinear theory leads to quasi-periodic versions of the Hamiltonian and reversible Hopf bifurcations, compare with [16,19,31,52].

For simplicity we stick to the real analytic setting, meaning that the manifold M, all families of vector fields $X = X_\mu(x)$, $x \in M$, $\mu \in P \subset \mathbb{R}^p$ are assumed real analytic. The corresponding topology is the compact-open topology on complex-analytic extensions, for details see below. We note that the theory has extensions to the case of C^k, for $k \in \mathbb{N}$ sufficiently large [78], also compare with [25,26].

Notation

We here collect a few notations needed later on. For a real-analytic manifold N, by $\mathcal{X}(N)$ and $\mathcal{X}_P(N)$ we denote the sets of all real-analytic vector fields on N and the set of all real-analytic p-parameter families of vector fields on N, parametrized over P, respectively. For $m \in \mathbb{N}$, the set of all real $m \times m$ matrices is denoted by $\mathfrak{gl}(m, \mathbb{R})$ and the set of invertible matrices in $\mathfrak{gl}(m, \mathbb{R})$

by $GL(m, \mathbb{R})$. We write I_m for the identity matrix in $GL(m, \mathbb{R})$. For vector fields $X, Y \in \mathcal{X}(N)$, the Lie bracket $[X, Y]$ is defined by $[X, Y](\phi) = Y(X(\phi)) - X(Y(\phi))$, for all smooth functions $\phi : N \to \mathbb{R}$, which is in accordance with [1,95]. Throughout we use $| \cdot |$ for maximum norm of vectors $u \in \mathbb{R}^n$ or $\mathbb{C}^n$ (i.e., $|u| = \max |u_j|$) and similarly for the ℓ_1-norm of integer vectors $u \in \mathbb{Z}^n$ (i.e., $|u| = \sum |u_j|$).

1.1. Normal linear stability: Nearly-integrable dissipative case

Since the $\mathbb{T}^n$-action on M was assumed free, for all $\mu \in U \subset P$, where U is an open domain, a full neighborhood of the $\mathbb{T}^n$-orbit T_μ is diffeomorphic to $\mathbb{T}^n \times \mathbb{R}^m$, for $m \in \mathbb{Z}_{\geqslant 0}$. From now on we take $m > 0$. Moreover, such a neighborhood can be given coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{T}^n \times \mathbb{R}^m$, so with the angular variables x_j $(j = 1, \ldots, n)$ counted modulo $2\pi\mathbb{Z}$, such that

$$T_\mu = \big\{(x, y, \mu) \in \mathbb{T}^n \times \mathbb{R}^m \times U : y = 0\big\},$$

while the vector fields $\partial/\partial x_j$ $(j = 1, \ldots, n)$ are the infinitesimal generators of the $\mathbb{T}^n$-action, see [26,55]. Integrability now implies that, for each μ, the vector field X_μ is independent of the angles x. Hence, the family $X = X_\mu(x, y)$ can be locally written as

$$\begin{cases} \dot{x} = \omega(\mu) + O(|y|), \\ \dot{y} = \Omega(\mu)y + O(|y|^2), \end{cases}$$

where $\omega(\mu) = (\omega_1(\mu), \ldots, \omega_n(\mu)) \in \mathbb{R}^n$ are the *internal frequencies* of the invariant torus T_μ and where $\Omega(\mu) \in \mathfrak{gl}(n, \mathbb{R})$ is the *Floquet* matrix. The eigenvalues of $\Omega(\mu)$ are called the *Floquet exponents* of the torus T_μ. In vectorial shorthand notation, we write

$$X_\mu(x, y) = \big[\omega(\mu) + O(|y|)\big]\frac{\partial}{\partial x} + \big[\Omega(\mu)y + O(|y|^2)\big]\frac{\partial}{\partial y}. \tag{1}$$

Definition 1.1. The family $X = X_\mu(x, y)$ as in (1) is nondegenerate at the invariant torus T_0, if the following is satisfied:

(a) The Floquet matrix $\Omega_0 = \Omega(0)$ is invertible;
(b) The local product map $\omega \times \Omega : (\mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathfrak{gl}(m, \mathbb{R}), (\omega(0), \Omega(0)))$ has the following property. The first component ω is submersive, while the second component Ω is a versal unfolding of Ω_0 in $\mathfrak{gl}(m, \mathbb{R})$, in the sense of [3–5,40].

Remarks 1.2.

1. By the Inverse Function Theorem, the invertibility of Ω_0 implies that, for each μ in a neighborhood of 0, the vector field X_μ locally has exactly one single invariant torus T_μ. Again by the Inverse Function Theorem, up to a μ-dependent translation, we have $T_\mu \cong \mathbb{T}^n \times \{0\}$. From now on we shall assume this simplification to have taken place.
2. The nondegeneracy condition means that the normal linear, leading part

$$NX_\mu(x, y) = \omega(\mu)\frac{\partial}{\partial x} + \Omega(\mu)y\frac{\partial}{\partial y}$$

of X is transversal to the conjugacy class of $N X_0$ in the space of normally affine vector fields. Such a generic condition plays a role in persistence results in the following sense. At the level of affine conjugacies and affine vector fields, the transversality condition provides the persistence of the tori $\{y = 0\}$ by the unfolding theorem [3–5,40]. Without the invertibility of Ω_0, this transversality property in general will be lost. As a result, the persistence of X-invariant tori is not guaranteed. For instance, consider $m = 1$ and

$$X_\mu(x, y) = \omega(\mu)\frac{\partial}{\partial x} + \left[\varepsilon\mu y + y^2\right]\frac{\partial}{\partial y},$$

where μ is in a small neighborhood of $0 \in \mathbb{R}$ and $\varepsilon > 0$. Observe that the family X has $V = \{y = 0\} \cup \{y = -\varepsilon\mu\}$ as invariant tori. The Floquet matrices $\Omega(\mu) = \varepsilon\mu$ form a versal unfolding of the zero matrix $\Omega_0 = 0$. But V is not persistent under perturbations of X. Indeed, for any small value $\varepsilon > 0$, a perturbed family $\widetilde{X}$ of the form

$$\widetilde{X}_\mu(x, y) = \omega(\mu)\frac{\partial}{\partial x} + \left[\varepsilon\mu y + y^2 + \varepsilon\left(\frac{\varepsilon\mu^2}{4} + 1\right)\right]\frac{\partial}{\partial y},$$

has no (relative) equilibria. Hence, the versality of the matrix unfolding in this case does not give persistence of V. We plan to come back to this problem in the future. For partial solutions see [13,96,97].

For the persistence of invariant tori, we restrict to a special class of quasi-periodic dynamics: we impose Diophantine conditions on the frequencies of the unperturbed integrable systems as follows.

Diophantine conditions. We denote by $\omega^N(\mu)$ the array of the positive imaginary parts $\omega_1^N(\mu), \ldots, \omega_r^N(\mu)$ of eigenvalues of $\Omega(\mu)$. These positive parts are referred to as the *normal frequencies* of the torus T_μ. In contrast to the case of simple eigenvalues, see [26,55], here the number r of normal frequencies in general may depend on μ. Let $s \leqslant m/2$ be the maximal number of these frequencies for $\mu \in \mathcal{U}$. We introduce the *generalized frequency map* $\mathcal{F}: P \to \mathbb{R}^n \times \mathbb{R}^s$ defined by

$$\mathcal{F}(\mu) = \left(\omega(\mu), \omega^N(\mu), \mathbf{0}\right),$$

where $\mathbf{0}$ denotes the zero vector of $\mathbb{R}^{s-r}$. We abbreviate $\tilde{\omega}^N(\mu) = (\omega^N(\mu), \mathbf{0}) \in \mathbb{R}^s$, and write $\mathcal{F}(\mu) = (\omega(\mu), \omega^N(\mu))$ when $s = r$. We say that the frequency vectors $(\omega(\mu), \omega^N(\mu))$ are (τ, γ)-*Diophantine* if the following holds. For a constant $\tau > n - 1$ and a 'parameter' $\gamma > 0$, we require that

$$\left|\langle\omega(\mu), k\rangle + \langle\tilde{\omega}^N(\mu), \ell\rangle\right| \geqslant \gamma|k|^{-\tau}, \tag{2}$$

for all $k \in \mathbb{Z}^m \setminus \{0\}$ and for all $\ell \in \mathbb{Z}^s$, with $|\ell| \leqslant 2$.

Remark 1.3. By $(\mathbb{R}^n \times \mathbb{R}^s)_{\tau,\gamma}$ we denote the set of all $(\omega, \tilde{\omega}^N) \in \mathbb{R}^n \times \mathbb{R}^s$ subject to the Diophantine conditions (2). This set is a nowhere dense, uncountable union of closed half lines. It intersects the unit sphere $\mathbb{S}^{n+s-1} \subset \mathbb{R}^n \times \mathbb{R}^s$ in a closed set, which by the Cantor–Bendixson theorem

[48] is the union of a perfect set and a countable set. The complement of $(\mathbb{R}^n \times \mathbb{R}^s)_{\tau,\gamma} \cap \mathbb{S}^{n+s-1}$ in $\mathbb{S}^{n+s-1}$ contains the dense resonant web which causes the perfect set to be totally disconnected and hence a Cantor set. In $\mathbb{S}^{n+s-1}$ the measure of this Cantor set tends to full Lebesgue measure as $\gamma \downarrow 0$. Colloquially we call the set of frequencies $(\omega(\mu), \tilde{\omega}^N(\mu))$ satisfying (2) (and also the corresponding set of parameters $\mu \in P$) a 'Cantor set,' i.e., a foliation of manifolds (with boundary) over a Cantor set.

For any subset $U \subseteq P$, we introduce the 'Cantor set' $\Gamma_{\tau,\gamma}(U)$ in U defined by

$$\Gamma_{\tau,\gamma}(U) = \mathcal{F}^{-1}\big(\big(\mathbb{R}^n \times \mathbb{R}^r \big)_{\tau,\gamma}\big) \cap U.$$

This set $\Gamma_{\tau,\gamma}(U)$ has positive Lebesgue measure [26,37,55,71,78,79]. Also we need the subset $\Gamma_{\tau,\gamma}(U') \subset \Gamma_{\tau,\gamma}(U)$, where the set U' is defined by

$$U' = \big\{ \mu \in U : \operatorname{dist}\big(\mathcal{F}(\mu), \partial\mathcal{F}(U)\big) \geqslant \gamma \big\}. \tag{3}$$

Note that if U is an open neighborhood of $0 \in P$, then the subset $U' \subset U$ is still an open neighborhood of 0 for $\gamma > 0$ sufficiently small.

Now we are ready to announce our main KAM theorem in the dissipative setting, which reads

Theorem 1.4 (*Normal linear stability: nearly-integrable dissipative case*). *Let* $X \in \mathcal{X}_P(M)$ *be a* p-*parameter real-analytic family of integrable vector fields given by* (1). *Assume that* X *is nondegenerate at the invariant central torus* $T_0 = \{(x, y, \mu) : y = 0, \ \mu = 0\}$. *Then, for* $\gamma > 0$ *sufficiently small and for any real-analytic family* $\widetilde{X} \in \mathcal{X}_P(M)$ *sufficiently close to* X *in the compact-open topology on complex analytic extensions, there exists a domain* U *around* $0 \in P$ *and a map*

$$\Phi : M \times U \to M \times P,$$

defined near the invariant torus T_0, *such that,*

  i. Φ *is a* C^∞-*near-identity diffeomorphism onto its image;*
 ii. *The image of the Diophantine tori* $V = \bigcup_{\mu \in \Gamma_{\tau,\gamma}(U')}(T_\mu \times \{\mu\})$ *under* Φ *is* $\widetilde{X}$-*invariant, and for the restriction* $\widehat{\Phi} = \Phi|_V$ *we have*

$$\widehat{\Phi}_*(X) = \widetilde{X},$$

   that is, $\widehat{\Phi}$ *conjugates* X *to* $\widetilde{X}$;
iii. *The restriction* $\Phi|_V$ *preserves the normal linear part of* X.

The conclusion of Theorem 1.4 is referred to as *normal linear stability* of the family X at $(y, \mu) = (0, 0)$ in the space $\mathcal{X}_P(M)$. We call the Diophantine torus union V (as well as its diffeomorphic image $\Phi(V)$) a *Cantor family* of invariant n-tori, since it is parametrized over a 'Cantor set' in a Whitney smooth sense [78,99,102,103], also compare with [25,26].

Remarks 1.5.

1. Suppose that the family X has a holomorphic extension to a complex domain $\mathcal{N}$. A *compact-open neighborhood* of the family X consists of families in $\mathcal{X}_P(M)$, which are close to X in the supremum norm on all compacta in $\mathcal{N}$. We will specify such a neighborhood in the proof of Theorem 1.6, see Appendix B.1.2.

2. It turns out that in the allowed perturbation size of $\widetilde{X} - X$ the positive Diophantine 'parameter' γ shows up in a linear way, compare with formula (67) in Appendix B.1.2. Measure theoretically the amount of $\widetilde{X}$-invariant n-tori is larger for smaller γ, so the 'game' in applications of Theorem 1.4 (as well as its companion Theorems 1.6 and 1.8), is to take $\gamma > 0$ as small as the perturbation size allows. In fact, γ can be chosen as a function of the distinguished and external parameters of X in an appropriate way, tending to 0 at a boundary component of the domain.[1]

   As a result for the perturbation $\widetilde{X}$ sets of *density points of quasi-periodicity* can be obtained, in the sense of measure theory. Compare with, e.g., [25,26]. In certain cases one also speaks of *exponential condensation*, for a discussion of these subjects and many references, see [27]. At the end of Section 5 we come back to this.

3. In the proof of Theorem 1.4 and all its counterparts, we use a Hölder condition on the spectra $\mathrm{Spec}\,\Omega(\mu)$ of the Floquet matrices $\Omega(\mu)$. In fact, since the family $X = X_\mu(x, y)$ is real analytic in all variables, the map $\Omega : U \to \mathfrak{gl}(m, \mathbb{R})$ has a holomorphic extension to a complex domain

$$U + r_0 = \left\{ \tilde{\mu} \in \mathbb{C}^p : \exists \mu \in U \text{ such that } |\mu - \tilde{\mu}| \leqslant r_0 \right\} \subset \mathbb{C}^p \tag{4}$$

   for a certain value $r_0 > 0$. Then for any $\tilde{\mu} \in U + r_0$ and $\mu \in U$ and for any $\tilde{\lambda} \in \mathrm{Spec}\,\Omega(\tilde{\mu})$, there exists an eigenvalue $\lambda \in \mathrm{Spec}\,\Omega(\mu)$ such that

$$|\mathrm{Im}\,\lambda - \mathrm{Im}\,\tilde{\lambda}| \leqslant L|\mu - \tilde{\mu}|^{1/m}. \tag{5}$$

   See Appendix A, Theorem A.1. A full proof is included as well, which is closely related to [41] and based on an application of the Rouché lemma to the characteristic polynomial of $\Omega(\mu)$.

4. The present KAM theory allows generalizations to the world of C^k-systems endowed with the C^k-topology [50], for $k \in \mathbb{N}$ sufficiently large. Compare with [78] and for discussions on this subject also see, e.g., [25,26,55,72,90]. To give an idea, for $k > 4\tau + 2$, the conjugation is at least of class $C^{k-2\tau}$. Therefore in the C^∞-case, no losses of differentiability occur and the conjugations also are of class C^∞. For an elaborate discussion and more references, compare with [27].

1.2. Normal linear stability: Nearly-integrable symplectic case

   In the symplectic setting, the even-dimensional manifold M (i.e., where $n + m$ is even) is endowed with a symplectic 2-form σ [1,7]. The infinitesimal generators of the free $\mathbb{T}^n$-action

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[1] In the Lagrangian KAM theory near an elliptic Hamiltonian equilibrium, γ is taken as a positive power of the distance to the equilibrium, which here acts as a distinguished parameter, leading to a 'small twist' version, compare with, e.g., [78].

are assumed to be Hamiltonian with respect to σ in a full neighborhood of the central n-torus T, which is a $\mathbb{T}^n$-orbit.[2] Then, the $\mathbb{T}^n$-orbit T is isotropic in the sense that the restriction $\sigma|_T \equiv 0$ [1,98]. By the generalized Darboux theorem [1,74], near the submanifold T, there exist local coordinates $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q}$ such that T is given by

$$T = \big\{(x, y, z)\colon (y, z) = (0, 0)\big\},$$

where σ takes the local form

$$\sigma = \sum_{i=1}^n dx_i \wedge dy_i + \sum_{j=1}^q dz_j \wedge dz_{q+j} = dx \wedge dy + dz^2. \tag{6}$$

Moreover, the $\mathbb{T}^n$-action is infinitesimally generated by the vector fields $\partial/\partial x_j$ $(j = 1, \ldots, n)$. Let us first consider a single integrable Hamiltonian vector field $X = X_H$ on (M, σ) with $H : M \to \mathbb{R}$ as the corresponding Hamiltonian function and with T as invariant submanifold. Then, the local form of X is given by

$$
\begin{aligned}
X(x, y, z) &= \sum_{i=1}^n \frac{\partial H}{\partial y_i} \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial H}{\partial x_i} \frac{\partial}{\partial y_i} + \sum_{i=1}^q \frac{\partial H}{\partial z_i} \frac{\partial}{\partial z_{q+i}} - \sum_{j=1}^q \frac{\partial H}{\partial z_{q+j}} \frac{\partial}{\partial z_j} \\
&= \frac{\partial H}{\partial u} \frac{\partial}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial}{\partial u} + \frac{\partial H}{\partial z_-} \frac{\partial}{\partial z_+} - \frac{\partial H}{\partial z_+} \frac{\partial}{\partial z_-},
\end{aligned}
$$

in shorthand notation, where $z_+ = (z_1, \ldots, z_q)$ and $z_- = (z_{q+1}, \ldots, z_{2q})$.

Since that the Hamiltonian H is constant along the $\mathbb{T}^n$-orbit T, it follows that $\partial H/\partial x \equiv 0$. Hence, in the local coordinates (x, y, z), the integrable Hamiltonian vector field X has the form

$$X(x, y, z) = f(y, z) \frac{\partial}{\partial x} + h(y, z) \frac{\partial}{\partial z},$$

where $h(0, 0) = 0$ and where $h(y, z)\frac{\partial}{\partial z}$ is Hamiltonian with respect to the 2-form $dz^2 = \sum_{j=1}^q dz_j \wedge dz_{q+j}$. Under the assumption that $\det(\partial h/\partial z)(0, 0) \neq 0$, by the Implicit Function Theorem it follows that the equation $h(y, z) = 0$ has a continuum $z = z(y)$, with $z(0) = 0$, of solutions, locally parametrized by y. This zero-set corresponds to a continuum X-invariant tori all carrying conditionally periodic (or parallel) dynamics with frequency vector $f(y, z(y))$. After a suitable translation, this continuum, locally coincides with $\{z = 0\}$. From now on we assume to be in this normalized situation. It follows that $\partial h/\partial z(y, 0) \in \mathfrak{sp}(2q, \mathbb{R})$ for all small values of y. Recall that for $A \in \mathfrak{gl}(2q, \mathbb{R})$ one has that $A \in \mathfrak{sp}(2q, \mathbb{R})$ if and only if $\mathrm{J}_{2q} A$ is symmetric, where J_{2q} denotes the standard symplectic $2q \times 2q$-matrix, that is,

$$\mathrm{J}_{2q} = \begin{pmatrix} 0 & \mathrm{I}_q \\ -\mathrm{I}_q & 0 \end{pmatrix} \tag{7}$$

compare with, e.g., [1,7,30].

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[2] If the corresponding $\mathbb{T}^n$-bundle has nontrivial monodromy, the actions in general are not globally Hamiltonian.

Next, we introduce the multi-parameter $\mu \in \mathbb{R}^p$ by considering a (real analytic) family $X = X_\mu(x, y, z)$ of the form

$$X_\mu(x, y, z) = f(y, z, \mu)\frac{\partial}{\partial x} + h(y, z, \mu)\frac{\partial}{\partial z}, \tag{8}$$

where now $h(0, 0, \mu) \equiv 0$. Proceeding as above for the case of individual vector fields and assuming that $\det(\partial h/\partial z)(0, 0, 0) \neq 0$, we again obtain a continuum of X-invariant n-tori normalized to $\{z = 0\}$, locally parametrized by both y and μ. This means that $h(y, 0, \mu) \equiv 0$, where $\partial h/\partial z(y, 0, \mu) \in \mathfrak{sp}(2q, \mathbb{R})$ for each (y, μ).

However, as in the dissipative case, see Section 1.1, we prefer to work with one n-torus per parameter value. To achieve this we define a distinguished parameter ν, that varies over a neighborhood of $0 \in \mathbb{R}^n$, introducing a localized coordinate $y_{\text{loc}} = y - \nu$. For this *localization procedure* compare with [26,55]. This leads to the n-torus family

$$\begin{aligned} T_{\text{loc}} &= \left\{(x, y, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \mathbb{R}^p \times \mathbb{R}^n \mid y = \nu,\ z = 0\right\} \\ &= \left\{(x, y_{\text{loc}}, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \mathbb{R}^p \times \mathbb{R}^n \mid y_{\text{loc}} = 0,\ z = 0\right\}, \end{aligned} \tag{9}$$

where (μ, ν) varies over a neighborhood of $(0, 0) \in \mathbb{R}^p \times \mathbb{R}^n$, which is invariant for the extended integrable family $X_{\mu,\nu}(x, y_{\text{loc}}, z) = X_\mu(x, y_{\text{loc}} + \nu, z)$. We so have

$$X_{\mu,\nu}(x, y_{\text{loc}}, z) = f(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial x} + h(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial z}, \tag{10}$$

observing that $\sigma = dx \wedge dy_{\text{loc}} + \sum_{j=1}^{q} dz_j \wedge dz_{q+j}$. As before we abbreviate $\omega(\mu, \nu) = f(\nu, 0, \mu)$ and $\Omega(\mu, \nu) = (\partial h/\partial z)(\nu, 0, \mu)$.

Now, an analogue of Theorem 1.4 can be formulated for the extended integrable family $X = X_{\mu,\nu}(x, u, z)$ as in (10), regarding the invariant tori T_{loc} as in (9). To this end, we need to adapt the nondegeneracy condition for the present setting: we say that the (extended) family $X = X_{\mu,\nu}$ is nondegenerate at the torus $T_{\text{loc}}^{0,0} = T_{\text{loc}} \cap \{(\mu, \nu) = (0, 0)\}$, if the following holds:

(a) The Floquet matrix $\Omega_0 = \Omega(0, 0)$ is invertible;
(b) The local product map $\omega \times \Omega : (\mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathfrak{sp}(2q, \mathbb{R}), (\omega(0, 0), \Omega(0, 0)))$ has the following property. The first component ω is submersive, while the second component Ω is a versal unfolding of $\Omega(0, 0)$ in $\mathfrak{sp}(2q, \mathbb{R})$, in the sense of [3,5,39,40].

Let $\mathcal{X}_{P_{\text{loc}}}^\sigma(M)$ be the set of real-analytic families of Hamiltonian vector fields of (M, σ), locally parametrized by $(\mu, \nu) \in P_{\text{loc}} = \mathbb{R}^p \times \mathbb{R}^n$. Then, we have the following Hamiltonian counterpart of Theorem 1.4.

Theorem 1.6 (*Normal linear stability: nearly-integrable symplectic case*). *Let $X \in \mathcal{X}_{P_{\text{loc}}}^\sigma(M)$ be the real-analytic $(p + n)$-parameter family of Hamiltonian vector fields given by (10). Assume that X is nondegenerate at the invariant torus $T_{\text{loc}}^{0,0} = \{(x, y_{\text{loc}}, z, \mu, \nu): y_{\text{loc}} = 0,\ (\mu, \nu) = (0, 0)\}$. Then, for $\gamma > 0$ sufficiently small and for any $\widetilde{X} \in \mathcal{X}_{P_{\text{loc}}}^\sigma(M)$ sufficiently close to X in*

the compact-open topology on complex analytic extensions, there exists a domain U around $(0, 0) \in P_{\mathrm{loc}}$ and a map

$$\Phi : M \times U \to M \times P_{\mathrm{loc}},$$

defined near the torus $T_{\mathrm{loc}}^{0,0}$, such that

  i. *Φ is a C^∞-near-identity symplectic diffeomorphism onto its image;*
  ii. *The image of the Diophantine tori $V = \bigcup_{(\mu,\nu) \in \Gamma_{\tau,\gamma}(U')} (T_{\mathrm{loc}}^{\mu,\nu} \times \{(\mu,\nu)\})$ under Φ is $\widetilde{X}$-invariant, and for the restriction $\widehat{\Phi} = \Phi|_V$ we have*

$$\widehat{\Phi}_*(X) = \widetilde{X},$$

  that is, $\widehat{\Phi}$ conjugates X to $\widetilde{X}$;
  iii. *The restriction $\Phi|_V$ preserves the (symplectic) normal linear part of X.*

Remarks 1.7.

1. Theorem 1.6 for the extended family $X = X_{\mu,\nu}$ as in (10) induces a similar stability result for the non-extended case (i.e., for the family $X = X_\mu$ given by (8)) via the projection $M \times P_{\mathrm{loc}} \to M \times P$ given by

$$(x, y_{\mathrm{loc}}, z, \mu, \nu) \mapsto (x, y_{\mathrm{loc}} + \nu, z, \mu).$$

   However, in the non-extended case, the induced conjugacy between the integrable family $X = X_\mu(x, y, z)$ and its perturbation $\widetilde{X} = \widetilde{X}_\mu(x, y, z)$ in general is not symplectic, see [26, 55].
2. The symplectic normal linear part $N^\sigma X$ of the extended family $X = X_{\mu,\nu}$ at V given by (10) is of the form

$$N^\sigma X(x, y, z) = \omega(\nu, \mu) \frac{\partial}{\partial x} + \Omega(\nu, \mu) z \frac{\partial}{\partial z}. \tag{11}$$

   The situation is more involved here than in the dissipative case. For a discussion compare with [26, Section 6b]; also see Section 3.3 and Appendix C.
3. For the conclusions (ii) and (iii) of Theorem 1.6, it is sufficient to show that $\Phi^* \widetilde{X} = NX + O(|y|, |z|) \frac{\partial}{\partial x} + O(|y|, |z|) \frac{\partial}{\partial y} + O(|y|, |z|^2) \frac{\partial}{\partial z}$. Indeed, first we observe that this relation implies that the Cantor family V of the X-invariant Diophantine n-tori is invariant under $\Phi^* \widetilde{X}$. It follows that the image $\widetilde{V} = \Phi(V)$ is a Cantor family of the $\widetilde{X}$-invariant tori. Secondly, the above relation also implies that the symplectic normal linear part of $N^\sigma \widetilde{X}$ at $\widetilde{V}$ has the same form as $N^\sigma X$ at V, see (11).

A leading example is the case where the Floquet matrix $\Omega_0 = \Omega(0, 0)$ has a double pair of purely imaginary eigenvalues with a nontrivial nilpotent part, i.e., when Ω_0 is in generic (or non-semisimple) $1 : -1$ resonance. This case allows a direct application of Theorem 1.6. This generically involves a Hamiltonian Hopf bifurcation [64]. For a nonlinear treatment of the quasi-periodic Hamiltonian Hopf bifurcation see [19]. The generic $1 : -1$ resonance is an interesting phenomenon, since it occurs in many classical mechanical systems including the Lagrange top

[34,35], the restricted three body problem [1,28], etc., but also in celestial mechanics and astronomy [33,76].

1.3. Normal linear stability: Nearly-integrable reversible case

We now formulate the corresponding reversible KAM theory, which runs similar to the symplectic case of the previous section, and which is completely covered by the approach of this paper. For other proofs and further details compare with [17,23,31], also see [16,32].

Let the manifold M again be the phase space. Now we are given an involution $G : M \to M$, i.e., such that $G^2 = \mathrm{Id}_M$, which is assumed to commute with the free $\mathbb{T}^n$-action. A vector field X on M is said to be G-reversible whenever

$$G_*(X) = -X, \tag{12}$$

which expresses that G takes integral curves of X to integral curves of X, reversing the time-direction. Given an integrable, reversible vector field X on M, as before let T be a $\mathbb{T}^n$-orbit, which then also is an X-invariant n-torus with $G(T) = T$. By Bochner's theorem [14,68] we then find local coordinates $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q}$, such that $T = \{(x, y, z): (y, z) = (0, 0)\}$ and G takes the local form $(x, y, z) \mapsto (-x, y, Rz)$. Here $R \in GL(2q, \mathbb{R})$ is a linear involution, taking the eigenvalue $+1$ with multiplicity q, which implies that the fixed point manifold of G has dimension $m + q$; this is also expressed by saying that G is of type $(n + q, m + q)$. For a discussion compare with [25, Section 1.3.2]. Thus the G-reversible vector field X obtains the format

$$X(x, y, z) = \sum_j f_j(x, y, z) \frac{\partial}{\partial x_j} + \sum_k g_k(x, y, z) \frac{\partial}{\partial y_k} + \sum_\ell h_\ell(x.y, z) \frac{\partial}{\partial z_\ell}$$

$$= f(x, y, z) \frac{\partial}{\partial x} + g(x, y, z) \frac{\partial}{\partial y} + h(x, y, z) \frac{\partial}{\partial z}, \tag{13}$$

compare with (8), where now (12) translates to

$$f(-x, y, Rz) \equiv f(x, y, z),$$
$$g(-x, y, Rz) \equiv -g(x, y, z),$$
$$h(-x, y, Rz) \equiv -Rh(x, y, z). \tag{14}$$

It directly follows from (14) that in the integrable case $g(y, z_0) \equiv 0$, for any z_0 with $Rz_0 = z_0$. This means that for any $y_0 \in \mathbb{R}^m$, with $h(y_0, z_0) = 0$, the n-torus $\mathbb{T}^n \times \{y_0\} \times \{z_0\}$ is X-invariant. Observe that this torus also is G-invariant and an orbit of the above $\mathbb{T}^n$-action, while the dynamics on this torus is conditionally periodic with frequency vector $\omega = f(y_0, z_0)$. For simplicity restricting to the above case $z_0 = 0$, we again consider the equation $h(y, z) = 0$ for (y, z) near $(y_0, 0)$. Under the generic condition $\det(\partial h / \partial z)(0, 0) \neq 0$, the Implicit Function Theorem provides us with a continuum $z = z(y)$ of solutions where $z(0) = 0$. Translating as in the symplectic case puts normalizes this continuum locally to $\{z = 0\}$.

Once more we turn to a *p*-parameter (real-analytic) family of integrable reversible vector fields

$$X_\mu(x, y, z) = f(y, z, \mu)\frac{\partial}{\partial x} + g(y, z, \mu)\frac{\partial}{\partial y} + h(y, z, \mu)\frac{\partial}{\partial z}, \tag{15}$$

where $h(y_0, 0, \mu_0) = 0$, investigating invariant *n*-tori of the form

$$T_{y_0,\mu_0} = \mathbb{T}^n \times \{y_0\} \times \{0\}$$

for their behaviour under nearly-integrable perturbations of the family (15). As in the symplectic case we normalize in a parameter dependent way, assuming that $\det(\partial h/\partial z(y_0, 0, \mu_0)) \neq 0$, so obtaining a continuum of *X*-invariant *n*-tori $\{z = 0\}$, locally parametrized by (y, μ). This means that $h(y, 0, \mu) = 0$, while $\Omega(\mu, \nu) = \partial h/\partial z(y, 0, \mu)$ is (infinitesimally) *R*-reversible in the sense that $\Omega R = -R\Omega$. The space of all such reversible matrices is denoted by $\mathfrak{gl}_{-R}(2q, \mathbb{R})$. From here simplify by setting $(y_0, \mu_0) = (0, 0)$.

We next localize as in the symplectic case, introducing the distinguished parameter $\nu \in \mathbb{R}^m$ and putting $y_{\text{loc}} = y - \nu$, compare with [26,55]. This leads to the *n*-torus family

$$\begin{aligned}
T_{\text{loc}} &= \left\{(x, y, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q} \times \mathbb{R}^m \times \mathbb{R}^p \,\middle|\, y = \nu,\ z = 0\right\} \\
&= \left\{(x, y_{\text{loc}}, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q} \times \mathbb{R}^m \times \mathbb{R}^p \,\middle|\, y_{\text{loc}} = 0,\ z = 0\right\}, \tag{16}
\end{aligned}$$

where (μ, ν) varies over a neighborhood of $(0, 0) \in \mathbb{R}^p \times \mathbb{R}^m$, which is invariant for the extended integrable family $X_{\mu,\nu}(x, y_{\text{loc}}, z) = X_\mu(x, y_{\text{loc}} + \nu, z)$. We so have

$$\begin{aligned}
&X_{\mu,\nu}(x, y_{\text{loc}}, z) \\
&= f(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial x} + g(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial y_{\text{loc}}} + h(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial z}. \tag{17}
\end{aligned}$$

In the localized coordinates *G* gets the form $(x, y_{\text{loc}}, z) \mapsto (-x, y_{\text{loc}}, Rz)$. Also the *n*-torus action directly carries over. Again we abbreviate $\omega(\mu, \nu) = f(\nu, 0, \mu)$ and $\Omega(\nu, 0, \mu) = \partial h/\partial z(y, 0, \mu)$.

The present analogue of Theorems 1.4 and 1.6 now can be given for the extended integrable family $X = X_{\mu,\nu}(x, y, z)$ as in (17), with respect to the invariant tori T_{loc} as in (16). The corresponding nondegeneracy condition is a variation on its symplectic counterpart. In particular we have to consider the adjoint action $GL_R(2q, \mathbb{R}) \times gl_{-R}(2q, \mathbb{R}) \to \mathfrak{gl}_{-R}(2q, \mathbb{R})$, $(A, \Omega) \mapsto A\Omega A^{-1}$, where $GL_R(2q, \mathbb{R}) \subset GL(2q, \mathbb{R})$ is the Lie-subgroup of *R*-equivariant matrices, which is a well defined, cf. [23]. The family *X* is nondegenerate at the *n*-torus $T_{\text{loc}}^{0,0} = T_{\text{loc}} \cap \{(\mu, \nu) = (0, 0)\}$, if the following holds for the product map

$$(\mu, \nu) \in \mathbb{R}^q \times \mathbb{R}^m \to \big(\omega(\mu, \nu), \Omega(\mu, \nu)\big) \in \mathbb{R}^n \times \mathfrak{gl}_{-R}(2q, \mathbb{R}):$$

(a) The Floquet matrix $\Omega_0 = \Omega(0, 0)$ is invertible;
(b) The first component $(\mu, \nu) \mapsto \omega(\mu, \nu)$ is submersive at $(\mu, \nu) = (0, 0)$. Meanwhile, for the second component $(\mu, \nu) \mapsto \Omega(\mu, \nu)$ we further require the unfolding $(\mu, \nu) \mapsto \Omega(\mu, \nu)$ to be versal at $(\mu, \nu) = (0, 0)$, i.e., transversal to the $GL_R(2q, \mathbb{R})$-orbit of $\Omega(0, 0)$, compare with [3,16,23,26,27,31,40].

Let $\mathcal{X}_{P_{\text{loc}}}^{-G}(M)$ be the space of all real-analytic families of G-reversible vector fields on $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q} = \{x, y_{\text{loc}}, z\}$, parametrized by $(\mu, \nu) \in P_{\text{loc}} = \mathbb{R}^m \times \mathbb{R}^p$, varying over a neighborhood of $(0, 0)$.

Theorem 1.8 *(Normal linear stability: nearly-integrable reversible case).* [17,31] *Let* $X \in \mathcal{X}_{P_{\text{loc}}}^{-G}(M)$ *be a real-analytic* $(q + m)$*-parameter family of* G*-reversible vector fields given by* (17). *Assume that* X *is nondegenerate at the invariant torus* $T_{\text{loc}}^{0,0} = \{(x, y_{\text{loc}}, \mu, \nu) \mid y_{\text{loc}} = 0,$ $z = 0$ *and* $(\mu, \nu) = (0, 0)\}$. *Then, for* $\gamma > 0$ *sufficiently small and for any* $\widetilde{X} \in \mathcal{X}_{P_{\text{loc}}}^{-G}(M)$ *sufficiently close to* X *in the compact-open topology on complex analytic extensions, there exists a domain* U *around* $(0, 0) \in P_{\text{loc}}$ *and a map*

$$\Phi : M \times U \to M \times P_{\text{loc}},$$

defined near the torus $T_{\text{loc}}^{0,0}$, *such that*

  i. *The map* Φ *is a* C^∞ *near-identity diffeomorphism onto its image*;
 ii. *The image of the* X*-invariant Diophantine tori*

$$V = \bigcup_{(\mu, \nu) \in \Gamma_{\tau, \gamma}(U')} T_{\text{loc}}^{\mu, \nu} \times \left\{ (\mu, \nu) \right\}$$

   under Φ *is* $\widetilde{X}$*-invariant and for the restriction* $\widehat{\Phi} = \Phi|_V$ *we have*

$$\widehat{\Phi}_*(X) = \widetilde{X},$$

   that is, $\widehat{\Phi}$ *conjugates* X *to* $\widetilde{X}$;
iii. *The restriction* $\Phi|_V$ *is* G*-equivariant and preserves the normal linear part of* X.

The proof of Theorem 1.8 is a variation on the proof given in Appendix B, where both similarity and differences are comparable with those between [23] and [26]. For further details on Theorem 1.8 see [17,31].

Remarks 1.9.

1. As in the dissipative Theorems 1.4, the conclusion of Theorem 1.8 is called normal linear stability. It induces a similar conclusion for the nonextended family $X = X_\mu(x, y, z)$ as in (15), namely via the projection $M \times P_{\text{loc}} \to M \times P$ given by

$$(x, y_{\text{loc}}, z, \mu, \nu) \mapsto (x, y_{\text{loc}} + \nu, z, \mu).$$

2. As in the symplectic case also here we consider the example where the Floquet matrix $\Omega_0 = \Omega(0, 0)$ has a double pair of purely imaginary eigenvalues with nontrivial nilpotent part, where we can restrict to the case $q = 2$. We say that Ω_0 is in $1 : 1$ resonance, which in the reversible case happens to coincide with the $1 : -1$ resonance. In the equilibrium or periodic case the nonlinear theory of this generically involves a reversible Hopf bifurcation [57,58]. In the presently setting of invariant tori, we are dealing with a quasi-periodic reversible Hopf bifurcation, which is studied in detail in [16,31].

3. It is instructive to compare the present reversible case with the symplectic case. That the linear details are already quite different has been shown in [53,54]. Nonlinear aspects of the present situation have been studied by [16,19], also see [15,57,58,65].

   In the case $q = 2$ the geometries, both in the linear and the nonlinear case are the same. This statement involves normal form reductions to $\mathbb{R}^4$, where a ramified circle bundle plays a role. Also the mathematics involves singularity theory. As a byproduct, this moreover shows that in the reversible supercritical Hopf case, there is nontrival monodromy. However, it turns out that the unreduced dynamics is slightly different, due to an additional drift in the fibre direction in the reversible case [16].

   One may conclude that in qualitative terms the quasi-periodic Hopf bifurcation in the reversible and Hamiltonian cases are comparable to a large extent. For $q \geqslant 3$ Knobloch and Vanderbauwhede [57,58] state similar conclusions, although the mathematical treatments of both cases may differ. Also compare with [15,65].

4. Generally speaking reversible KAM theory, as this starts with Moser [69], to a great extent is parallel to its Hamiltonian counterpart, see, e.g., [6,8,10,16,17,23–25,70,71,73,78,85,86, 88,89,92,93]. In the case of reversible diffeomorphisms, however, some special effects show up [80]. For general references on reversible dynamical systems, see [60,82]. For a similar discussion also see [27].

1.4. Plan of the paper

We briefly overview the plan for the remainder of this paper. In Section 2, we revisit the persistence of equilibria of vector fields in terms of linear stability. Here the concept of (uni-)versal matrix unfolding or deformation plays a central role. This study is related to the normal linear stability of Theorems 1.4, 1.6 and 1.8, which can be most clearly seen in case the perturbations are assumed to be integrable as well. These are the contents of Section 3. Indeed, in that case linear stability almost directly applies to the relative equilibria obtained by factoring out the n-torus symmetry. This means that everything is proven by the Inverse Function Theorem and no KAM theory is needed. Also the surviving n-tori occur in continua, depending smoothly on the parameters, for this approach compare with [26,55]. Next, Section 4 is devoted to a case study on the generic (or nonsemisimple) $1 : -1$ resonance, involving a corollary of Theorem 1.6. The generic nonlinear theory then leads to a quasi-periodic Hamiltonian Hopf bifurcation, which we sketch in Section 5, illustrated on the Lagrange top coupled with a quasi-periodic oscillator near gyroscopic stabilization. This is a typical application of the theory, for details we refer to [19].

Theorems 1.4, 1.6 and 1.8 all are within the axiomatic approach of Section 2. Among these three, the symplectic case of Theorem 1.6 undoubtedly is the most involved, which is why we have chosen to include a full proof of this in Appendix B. Proofs of Theorems 1.8 and 1.4 can be obtained by verbatim transcriptions, though surely the latter case could be further simplified. For a similar proof in the reversible case we refer to [17,31]. In Appendix A we deal with the Hölder condition that plays a technical role in Appendix B. Finally, in Appendix C we briefly describe the normal linear theory in the symplectic case, compare with [26, Section 6b].

2. Linear stability of equilibria

Persistence of equilibria of vector fields is closely related to normal linear stability. Compare with, e.g., the conclusions of Theorems 1.4, 1.6 and 1.8 are all formulated in terms of smooth local conjugacies near a nondegenerate invariant torus. In view of this we now express the persis-

tence of equilibria in terms of smooth local conjugacies. Existence of such a conjugacy implies similarity of the linear parts at the equilibria. Indeed, let X_1 and X_2 be two vector fields on $\mathbb{R}^m$ locally conjugated by the diffeomorphism ϕ. Suppose that $X_1(p) = 0$ and $q = \phi(p)$, then $X_2(q) = 0$ and $D_p X_1$ is similar to $D_q X_2$ as follows: $(D_p \phi) D_p X_1 (D_p \phi)^{-1} = D_q X_2$. Similarity between linear parts of vector fields at equilibria defines an equivalence relation on the germs of such equilibria. The linear part (up to similarity) thus becomes an intrinsic object.

Our aim in this section is to give sufficient conditions for linear stability of equilibria. This involves the concept of versal matrix unfoldings [3–5,40], for which reason we first revisit the main ideas of (trans-)versality. This program is carried out, keeping a suitable preservation of structure into account, like a volume or a symplectic form, a symmetry or a reversing symmetry. For particular studies in this direction compare with, e.g., [39,51,87].

2.1. Stability in terms of versal unfoldings

Let $G \neq \{I_m\}$ be a Lie subgroup of $GL(m, \mathbb{R})$ and $V \neq \{0\}$ a linear subspace of $\mathfrak{gl}(m, \mathbb{R})$. Suppose that V is invariant under G, i.e., that for any $Q \in G$ and $\Omega \in V$, one has $Q \Omega Q^{-1} \in V$. Then the map $\Psi : G \times V \to V$ given by $\Psi(Q, \Omega) = Q \Omega Q^{-1}$, is a group action on V, called the G-action on V. The subset

$$O(\Omega, G) = \left\{ Q \Omega Q^{-1} \colon Q \in G \right\} \subseteq V$$

is called the G-orbit of Ω. Suppose that the orbit $O(\Omega, G)$ is a smooth submanifold, then the co-dimension of Ω, denoted by $\operatorname{cod} \Omega$, is defined as the co-dimension in V of the submanifold $O(\Omega, G) \subseteq V$. Each orbit $O(\Omega, G)$ is a smooth submanifold, provided, for instance, that the group G is semi-algebraic as a subset of $\mathfrak{gl}(m, \mathbb{R})$, see [11,40]. Well-known examples are: $GL(m, \mathbb{R})$, $SL(m, \mathbb{R})$, $SP(m, \mathbb{R})$ and $GL_{+R}(m, \mathbb{R})$,[3] where $R \in GL(m, \mathbb{R})$ is a linear involution.

Definition 2.1. The pair (G, V) as above, where V is G-invariant, is a *linear structure* on $\mathbb{R}^m$, if the following holds:

A1. G is a semi-algebraic subset of $\mathfrak{gl}(m, \mathbb{R})$;
A2. If $\Omega \in V$, so is the transpose Ω^T of Ω with respect to the standard inner product of $\mathbb{R}^m$.

Compare these conditions with set-ups of [26,55,71]. Condition (A1) ensures that the all orbits are smooth submanifolds; condition (A2) is used to compute the complements of the tangent spaces of the orbits, needed for constructing versal unfoldings, in a systematic way. Since these complements can be determined in other ways, condition (A2) is not necessary for the versal unfolding stability to be developed below.

Example 2.2. The pairs formed by $GL(m, \mathbb{R})$, $SL(m, \mathbb{R})$, $SP(m, \mathbb{R})$ and $GL_{+R}(m, \mathbb{R})$ together with their Lie algebras are linear structures. The same is true for the pair $(GL_{+R}(m, \mathbb{R}),$ $\mathfrak{gl}_{-R}(m, \mathbb{R}))$, where $\mathfrak{gl}_{-R}(m, \mathbb{R})$ denotes the set of infinitesimally R-reversible matrices, i.e., $\mathfrak{gl}_{-R}(m, \mathbb{R}) = \{A \in \mathfrak{gl}(m, \mathbb{R}) \colon AR = -RA\}$. Observe that $\mathfrak{gl}_{-R}(m, \mathbb{R})$ is not the Lie algebra of $GL_{+R}(m, \mathbb{R})$, compare with [17,23,70,71].

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[3] $GL_{+R}(m, \mathbb{R})$ denotes the group of R-equivariant matrices in $GL(m, \mathbb{R})$.

From now on, we assume that (G, V) is a linear structure and let $\Pi_p(V)$ be the set of all smooth p-parameter families on V, that is, the set of germs $\Omega : \mu \in \mathbb{R}^p \mapsto \Omega(\mu) \in V$ at $\mu = 0$. These germs are also called matrix unfoldings or deformations of $\Omega_0 = \Omega(0)$. We recall from [3,5,40] the definition of versality:

Definition 2.3. An unfolding $\Omega \in \Pi_p(V)$ of Ω_0 is *versal* with respect to the G-action, if for any $B \in \Pi_q(V)$ with $B(0) = \Omega_0$, there exists a smooth local map $(\rho, Q) : (\mathbb{R}^q, 0) \to (\mathbb{R}^p, 0) \times (G, I_m)$ such that

$$B(\mu) = Q(\mu) \Omega\big(\rho(\mu)\big) Q^{-1}(\mu). \tag{18}$$

Again by [3,5,40], the family Ω is versal at $\mu = 0$ if and only if it is transversal at Ω_0, that is, if

$$D_0 \Omega\big(\mathbb{R}^p\big) + T_{\Omega_0}\big(O(\Omega_0, G)\big) = V, \tag{19}$$

where $D_0\Omega$ is the derivative of Ω at $\mu = 0$ and $T_{\Omega_0}(O(\Omega_0, G))$ is the tangent space of the submanifold $O(\Omega_0, G)$ at Ω_0. By (19), the minimal number of parameters, needed for versality of the family Ω, equals the co-dimension $\operatorname{cod} \Omega_0$ (i.e., $p \geqslant \operatorname{cod} \Omega_0$). A p-parameter versal matrix unfolding of Ω_0 is called *universal* if $p = \operatorname{cod} \Omega_0$.

Remark 2.4. By invariance of eigenvalues under similarity, the co-dimension of any single matrix within $(GL(m, \mathbb{R}), \mathfrak{gl}(m, \mathbb{R}))$, seen as a constant unfolding, is nonzero. In particular, constant unfoldings can never be versal in $(GL(m, \mathbb{R}), \mathfrak{gl}(m, \mathbb{R}))$, or, in other words: individual matrices never are structurally stable.

To fulfill condition (19), we need to know the tangent space $T_{\Omega_0}(O(\Omega_0, G))$ explicitly. To this end, we consider the map $\Psi_{\Omega_0} : G \to O(\Omega_0, G) \subseteq V$ given by $\Psi_{\Omega_0}(Q) = Q\Omega_0 Q^{-1}$. Let $\mathbf{g}$ be the Lie algebra of G. For any $B \in \mathbf{g}$, the map $t \mapsto \exp(tB)$ is a smooth curve through the identity I_m in the group G, where $\exp : \mathbf{g} \to G$ denotes the exponential map [1,30,77,95]. Then, the map $t \mapsto \Psi_{\Omega_0}(\exp(tB))$ defines a smooth curve in the orbit $O(\Omega_0, G)$ passing through Ω_0. Since for any $B \in \mathbf{g}$, we have

$$\frac{d}{dt}\bigg|_{t=0} \Psi_{\Omega_0}\big(\exp(tB)\big) = \Omega_0 B - B\Omega_0 = [\Omega_0, B] = \operatorname{ad}\Omega_0(B), \tag{20}$$

and since $\exp$ is a local diffeomorphism near $0 \in \mathbf{g}$, it follows that

$$T_{\Omega_0}\big(O(\Omega_0, G)\big) = \operatorname{ad}\Omega_0(\mathbf{g}). \tag{21}$$

A special kind of universal unfoldings is given by the *linear centralizer unfoldings* (LCU's).

Definition 2.5. A linear unfolding $\Omega \in \Pi_p(V)$ of Ω_0 is called an LCU of Ω_0 if $p = \operatorname{cod} \Omega_0$ and $D_0\Omega(\mathbb{R}^p) = \ker \operatorname{ad} \Omega_0^T \subseteq V$.

The following is of importance for the linear stability results on equilibria of vector fields to be discussed in Section 2.2.

Theorem 2.6 (*Versal unfolding stability*). *Let $\Omega \in \Pi_p(V)$ be versal within the linear structure (G, V). Then, there exists a neighborhood $\mathcal{V}$ of Ω in $\Pi_p(V)$ in the C^1-topology, and for all $B \in \mathcal{V}$, we have smooth germs (ρ, Q), such that*

  i. $B(\mu) = Q(\mu)\Omega(\rho(\mu))Q(\mu)^{-1}$, *for all small μ;*
 ii. *The reparametrization ρ is C^1-near the identity map.*

Remark 2.7. If $B(0) = \Omega(0)$, then conclusion (i) directly follows from Definition 2.3. However, in Theorem 2.6, we do not require that $B(0) = \Omega(0)$.

In the following we give a proof of Theorem 2.6. For $\Omega_0 \in V$, let

$$S(\Omega_0) = \{P \in G \colon P\Omega_0 = \Omega_0 P\} \subseteq G,$$

be the isotropy subgroup of G of Ω_0, compare with [1,49,95]. It is known that $S(\Omega_0) \subset G$ is a Lie subgroup and that $T_{\mathrm{I}_m} S(\Omega_0) = \ker \mathrm{ad}\, \Omega_0$, in particular, we have $\dim S(\Omega_0) = \dim G - \dim O(\Omega_0, G)$.

Lemma 2.8. *Given a linear structure (G, V), suppose that $\Omega \in \Pi_p(V)$ is a universal unfolding of Ω_0. Let H be a smooth submanifold of G, transversal to the isotropy group $S(\Omega_0)$ at the identity $\mathrm{I}_m \in G$. Suppose that $\dim H = \mathrm{cod}\, S(\Omega_0)$, then the map $F \colon \mathbb{R}^p \times H \to V$ defined by*

$$F(\mu, P) = P\Omega(\mu)P^{-1},$$

is a local C^∞-diffeomorphism near $(0, \mathrm{I}_m) \in \mathbb{R}^p \times H$.

Proof. By assumption, $T_{\mathrm{I}_m} H \oplus T_{\mathrm{I}_m} S(\Omega_0) = T_{\mathrm{I}_m} G$. Since $T_{\mathrm{I}_m} S(\Omega_0) = \ker \mathrm{ad}\, \Omega_0$, we have

$$\mathrm{ad}\, \Omega_0(T_{\mathrm{I}_m} H) = \mathrm{ad}\, \Omega_0(T_{\mathrm{I}_m} G). \tag{22}$$

By universality of Ω, $p = \mathrm{cod}\, \Omega_0$. Since $\dim H = \dim O(\Omega_0, G) = \dim V - \mathrm{cod}\, \Omega_0$, we have $\dim(\mathbb{R}^p \times H) = \dim V$. The derivative $D_{(0,\mathrm{I}_m)}F$ of F at $(\mu, P) = (0, \mathrm{I}_m)$ is given by

$$D_{(0,\mathrm{I}_m)}F(\hat{\mu}, \widehat{P}) = D_0\Omega(\hat{\mu}) + [\widehat{P}, \Omega_0],$$

for $(\hat{\mu}, \widehat{P}) \in \mathbb{R}^p \times T_{\mathrm{I}_m} H$. Now by (uni-)versality, $D_{(0,\mathrm{I}_m)}F$ has maximal rank. Since $\dim(\mathbb{R}^p \times H) = \dim V$, it follows that F is a local diffeomorphism. $\quad\square$

Proof of Theorem 2.6. We first prove the claim for the case where Ω is an universal unfolding. We take a submanifold H as given in Lemma 2.8. Then, the map $F \colon \mathbb{R}^p \times H \to V$ given by

$$F(\mu, P) = P A(\mu) P^{-1}$$

is a local diffeomorphism near $(\mu, P) = (0, \mathrm{I}_m)$. Suppose that the matrix $B(0)$ is sufficiently close to $\Omega_0 = \Omega(0)$. Then, by the Inverse Function Theorem, for each sufficiently small μ, there exists a unique pair $(\rho(\mu), Q(\mu)) \in \mathbb{R}^p \times G$ such that

$$B(\mu) = F\big(\rho(\mu), Q(\mu)\big) = Q(\mu)\Omega\big(\rho(\mu)\big)Q(\mu)^{-1}. \tag{23}$$

Moreover, the pair $(\rho(\mu), Q(\mu))$ depends smoothly on μ and $(\rho(0), Q(0)) = (0, I_m)$. Hence, the local product map $\rho \times Q$ defined by

$$\rho \times Q : \mu \in \mathbb{R}^p \rightarrow (\rho(\mu), Q(\mu)) \in \mathbb{R}^p \times G \tag{24}$$

gives the required germs for part (i).

Now we turn to part (ii). Let $F^{-1} : V \rightarrow \mathbb{R}^p \times G$ be the local inverse of F near Ω_0. Then by (23), for any $B \in \Pi_p(V)$ sufficiently close to Ω and for any small μ, we have

$$(F^{-1} \circ B)(\mu) = F^{-1}(B(\mu)) = (\rho(\mu), Q(\mu)). \tag{25}$$

Let $\pi : \mathbb{R}^p \times H \rightarrow \mathbb{R}^p$ be the natural projection. Then, by (25), the change of parameter ρ is given by

$$\rho(\mu) = \pi \circ F^{-1} \circ B(\mu).$$

Observe that $(\pi \circ F^{-1} \circ \Omega)$ locally is the identity map. Hence, when $B \in \Pi_p(V)$ is sufficiently C^1-close to Ω, the derivative of the map $\rho = \pi \circ F^{-1} \circ B$ is near the identity in a neighborhood of $0 \in \mathbb{R}^p$. Hence, ρ is C^1-near the identity.

Finally, the case where Ω is versal, but not necessarily universal, can be reduced to the above case. Indeed, by versality and the Inverse Function Theorem [1,40], there exists a local reparametrization $\mu \in \mathbb{R}^p \mapsto (\nu, \kappa) \in \mathbb{R}^c \times \mathbb{R}^{p-c}$ such that Ω restricted to the ν-direction is an universal unfolding of Ω_0. Rewrite $B(\mu) = B(\nu, \kappa)$. Then, we can apply the above arguments to the ν-direction leaving the κ-direction unchanged. $\square$

Remark 2.9. In [9] Arnol'd proposes problem 1970-1 as "Construct all versal unfoldings of endomorphisms (of vector spaces and groups)." Sevryuk comments on this problem in [91], by discussing it from a general view point, with many references. The present section can be seen as a contribution to this problem, in the footsteps of [26,52,55,71] and motivated by the normal linear stability problem in KAM theory. For another general contribution see Hoveijn [53].

2.2. Linear stability of equilibria: Dissipative case

Here we formulate our first linear stability theorem for equilibria in the dissipative setting. Analogous to (1), we consider a p-parameter family $X \in \mathcal{X}_P(\mathbb{R}^m)$ of the form

$$X_\mu(y) = \left[\Omega(\mu)y + O(|y|^2)\right]\frac{\partial}{\partial y}, \tag{26}$$

where $\mu \in P$ and $\Omega(\mu) \in \mathfrak{gl}(m, \mathbb{R})$. We say that X is nondegenerate at $(y, \mu) = (0, 0)$, if the matrix $\Omega_0 = \Omega(0)$ is invertible and the unfolding Ω of Ω_0 is versal within the linear structure $(GL(m, \mathbb{R}), \mathfrak{gl}(m, \mathbb{R}))$, compare with Definition 1.1. Our present concern is with the persistence of the zero set $\mathcal{Z}(X)$ of the family X, where

$$\mathcal{Z}(X) = \{y = 0, \ \mu \text{ in a full neighborhood of } 0 \in P\} \subset \mathbb{R}^m \times P,$$

and the linear behavior of the family X.

Theorem 2.10 *(Linear stability: dissipative case). Let the family X be of the form* (26). *If X is nondegenerate at* $(y, \mu) = (0, 0)$, *then, for any perturbation* $\widetilde{X} \in \mathcal{X}_P(\mathbb{R}^m)$ *with* $X - \widetilde{X}$ *sufficient small in the C^2-topology the following holds. There exists a local diffeomorphism*

$$\Phi : \mathbb{R}^m \times P \to \mathbb{R}^m \times P$$

of the form $\Phi(y, \mu) = (\phi_\mu(y), \rho(\mu))$, *defined near* $(y, \mu) = (0, 0)$, *such that*

i. *Φ is C^1-near the identity map;*
ii. *The image $\Phi(\mathcal{Z}(X))$ is a zero set of the family $\widetilde{X}$, that is $\Phi(\mathcal{Z}(X)) = \mathcal{Z}(\widetilde{X})$;*
iii. *Φ preserves the linear behaviour of X.*

The conclusion of Theorem 2.10 is referred to as the *linear stability* of the family X at the equilibrium $(y, \mu) = (0, 0)$.

Remarks 2.11.

1. Observe that no single vector field X_0 is linearly stable. Indeed, a small perturbation of X_0 in general leads to a change in eigenvalues of the linear part of X_0, compare with Remark 2.4.
2. By the Inverse Function Theorem, the zero set $\mathcal{Z}(\widetilde{X})$ of the perturbation $\widetilde{X}$ is a smooth graph $(y(\mu), \mu)$ in $\mathbb{R}^m \times P$.

Proof of Theorem 2.10. By the Inverse Function Theorem, there is a smooth map $\mu \in P \mapsto y(\mu) \in \mathbb{R}^m$, such that, $\widetilde{X}_\mu(y(\mu)) = 0$ for each small μ. After the translation $y \mapsto y - y(\mu)$, we may assume that $\widetilde{X}_\mu(y) = [B(\mu)y + O(|y|^2)]\frac{\partial}{\partial y}$. By assumption, the unfolding $B \in \Pi_p(V)$ is C^1-close to Ω. By Theorem 2.6, there exists a μ-dependent transformation $\psi_\mu : y \mapsto Q(\mu)y$ with $Q(\mu) \in GL(m, \mathbb{R})$ and a reparametrization $\rho : P \to P$, that is C^1-near the identity, such that

$$\psi_{\rho(\mu)}^* \widetilde{X}_{\rho(\mu)}(y) = \left[\Omega(\mu)y + O(|y|^2)\right]\frac{\partial}{\partial y}. \tag{27}$$

Let $\phi_\mu(y) = y(\mu) + Q(\mu)y$ and $\Phi(y, \mu) = (\phi_{\rho(\mu)}(y), \rho(\mu))$. By construction, the map Φ is a C^1-near-identity-diffeomorphism, which shows (i). Moreover, by (27), Φ preserves the linear part of the family X, which justifies conclusion (iii). Finally, to show (ii), we notice that for all small μ

$$\widetilde{X}_{\rho(\mu)}\left(\phi_{\rho(\mu)}(0)\right) = \widetilde{X}_{\rho(\mu)}\left(y(\rho(\mu))\right) = 0, \tag{28}$$

which implies that Φ maps zeros of the family X to those of $\widetilde{X}$. $\quad\square$

2.3. Linear stability of equilibria: Symplectic case

Next we consider linear stability of Hamiltonian equilibria in the standard symplectic space $\mathbb{R}^{2q} = \{z_1, \ldots, z_{2q}\}$ endowed with the canonical 2-form

$$\sigma = \sum_{i=1}^{q} dz_i \wedge dz_{q+i} = dz^2.$$

We denote by $\mathcal{X}^\sigma(\mathbb{R}^{2q})$ the set of all Hamiltonian vector fields on $(\mathbb{R}^{2q}, \sigma)$ and $\mathcal{X}_P^\sigma(\mathbb{R}^{2q})$ the set of all p-parameter families of such Hamiltonian vector fields, with parameter space P. Let $Y \in \mathcal{X}^\sigma(\mathbb{R}^{2q})$ with $Y(0) = 0$ be given, then Y has the form $Y(z) = [\Omega_0 z + O(|z|^2)]\frac{\partial}{\partial z}$, where $\Omega_0 \in \mathfrak{sp}(2q, \mathbb{R})$. As an analogue to the family (26) in the dissipative setting, we here consider a Hamiltonian family $X = X_\mu \in \mathcal{X}_P^\sigma(\mathbb{R}^{2q})$ of the form:

$$X_\mu(z) = \left[\Omega(\mu)z + O(|z|^2)\right]\frac{\partial}{\partial z}, \tag{29}$$

where $\Omega(\mu) \in \mathfrak{sp}(2q, \mathbb{R})$ for each value μ. Similar to the dissipative case, the Hamiltonian family X given by (29) is linearly stable at $(z, \mu) = (0, 0)$, provided that X is nondegenerate at $(z, \mu) = (0, 0)$ within the linear structure $(SP(2q, \mathbb{R}), \mathfrak{sp}(2q, \mathbb{R}))$. Nondegeneracy means here that $\Omega_0 = \Omega(0)$ is nonsingular, while the unfolding $\Omega = \Omega(\mu)$ is versal at $\mu = 0$ within $(SP(2q, \mathbb{R}), \mathfrak{sp}(2q, \mathbb{R}))$. More precisely, we have

Theorem 2.12 (*Linear stability: symplectic case*). *Let the Hamiltonian family $X \in \mathcal{X}_P^\sigma(\mathbb{R}^m)$ be given by* (29). *Assume that X is nondegenerate at $(z, \mu) = (0, 0)$. Then, for any perturbation $\widetilde{X} \in \mathcal{X}_P^\sigma(\mathbb{R}^m)$ with $X - \widetilde{X}$ sufficiently small in the C^2-topology, there exists a local map*

$$\Phi : (z, \mu) \in \mathbb{R}^{2q} \times P \mapsto \left(\phi_\mu(z), \rho(\mu)\right) \in \mathbb{R}^{2q} \times P$$

defined near $(z, \mu) = (0, 0)$, such that,

  i. *Φ is C^1-near the identity map;*
 ii. *The image $\Phi(\mathcal{Z}(X))$ is a zero set of the family $\widetilde{X}$, that is $\Phi(\mathcal{Z}(X)) = \mathcal{Z}(\widetilde{X})$;*
iii. *Φ is symplectic and preserves the linear behavior.*

The proof of Theorem 2.13 is completely analogous to that of Theorem 2.10.

2.4. Linear stability of equilibria: Reversible case

The reversible linear stability theorem runs similar to its symplectic analogue of the previous subsection. Indeed, we now consider $\mathbb{R}^{2q}$ with the linear involution $R \in GL(2q, \mathbb{R})$ as before, see Section 1.3, so where the eigenvalue 1 occurs with multiplicity q. In this case we denote by $\mathcal{X}^{-R}(\mathbb{R}^{2q})$ the space of all R-reversible vector fields on $\mathbb{R}^{2q}$, i.e., with $R_*X = -X$, while the space of R-reversible families parametrized over $P \in \mathbb{R}^q$ is denoted $\mathcal{X}_P^{-R}(\mathbb{R}^{2q})$. Any $Y \in \mathcal{X}^{-R}(\mathbb{R}^{2q})$ with $Y(0) = 0$, has the form $Y(z) = [\Omega_0 z + O(|z|^2)]\frac{\partial}{\partial z}$, with $\Omega_0 \in \mathfrak{gl}_{-R}(2q, \mathbb{R})$. As an analogue to (26) and (29), we now consider a reversible family $X \in \mathcal{X}_P^{-R}(\mathbb{R}^{2q})$, of the form

$$X_\mu(z) = \left[\Omega(\mu)z + O(|z|^2)\right]\frac{\partial}{\partial z}, \tag{30}$$

with $\Omega(\mu) \in \mathfrak{gl}_{-R}(2q, \mathbb{R})$, for all μ. Similar to the above two cases, the reversible family (30) is linearly stable at $(z, \mu) = (0, 0)$, provided that X is nondegenerate at $(z, \mu) = (0, 0)$ within the linear structure $(GL_R(2q, \mathbb{R}), \mathfrak{gl}_{-R}(2q, \mathbb{R}))$, see Section 2.1, Example 2.2. Here nondegeneracy means that $\Omega_0 = \Omega(0)$ is nonsingular, while the unfolding $\Omega = \Omega(\mu)$ is versal at $\mu = 0$ within $(GL_R(2q, \mathbb{R}), \mathfrak{gl}_{-R}(2q, \mathbb{R}))$. To be precise, we have

Theorem 2.13 *(Linear stability: reversible case). Let the reversible family $X \in \mathcal{X}_P^{-R}(\mathbb{R}^{2q})$ be given by (30). Assume that X is nondegenerate at $(z, \mu) = (0, 0)$. Then, for any perturbation $\widetilde{X} \in \mathcal{X}_P^{-R}(R^{2q})$ with $X - \widetilde{X}$ sufficiently small in the C^2-topology, there exists a local map*

$$\Phi : (z, \mu) \in \mathbb{R}^{2q} \times P \mapsto \big(\phi_\mu(z), \rho(\mu)\big) \in \mathbb{R}^{2q} \times P$$

defined near $(z, \mu) = (0, 0)$, such that,

  i. *Φ is C^1-near the identity map;*
 ii. *The image $\Phi(\mathcal{Z}(X))$ is a zero set of the family $\widetilde{X}$, that is $\Phi(\mathcal{Z}(X)) = \mathcal{Z}(\widetilde{X})$;*
iii. *Φ is R-equivariant and preserves the linear behaviour.*

The proof of Theorem 2.13 again is completely analogous to that of Theorem 2.10.

2.5. Examples of linearly stable systems in resonant cases

In this section, we give a few examples of linearly stable families of Hamiltonian vector fields, which are relevant for applications of Theorem 1.6. Our interest is with local Hamiltonian systems the linear parts of which are in $1 : -1$ resonance, compare with [53,56,64]. We also shall describe this resonance in the reversible setting, compare with [17,31].

2.5.1. Preliminaries

Consider the standard symplectic space $\mathbb{R}^4 = \{z_1, z_2, z_3, z_4\}$ with symplectic 2-form $\sigma = dz_1 \wedge dz_3 + dz_2 \wedge dz_4$. To study linear stability of Hamiltonian equilibria, by Theorem 2.13, it is sufficient to consider linear Hamiltonian systems. Let Y be a Hamiltonian vector field given by

$$Y(z) = \Omega_0 z \frac{\partial}{\partial z} \in \mathcal{X}^\sigma(\mathbb{R}^4), \tag{31}$$

where $\Omega_0 \in \mathfrak{sp}(4, \mathbb{R})$. We say that the linear system (31) is in $1 : -1$ *resonance*, if the matrix Ω_0 has a double pair of purely imaginary eigenvalues, say $\pm i\lambda_0$, and if Ω_0 can be transformed (by a symplectic linear map) to the following real normal form:

$$N_\varepsilon = \begin{pmatrix} 0 & -\lambda_0 & 0 & 0 \\ \lambda_0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & -\lambda_0 \\ 0 & \varepsilon & \lambda_0 & 0 \end{pmatrix}, \tag{32}$$

where $\lambda_0 \neq 0$ and $\varepsilon = 0, \pm 1$. Depending on the value of ε, we distinguish between two cases: the *semisimple* ($\varepsilon = 0$) and the *non-semisimple* or *generic* case ($\varepsilon = \pm 1$). For computation of real normal forms of Ω_0, such as N_ε, we refer to [29,101].

Let us recall certain notations from Section 2.1. The co-dimension of Ω_0 is given by $\mathrm{cod}\,\Omega_0 = \dim \mathfrak{sp}(4, \mathbb{R}) - \dim O(\Omega_0)$, where $O(\Omega_0)$ denotes the $SP(4, \mathbb{R})$-orbit of Ω_0. A p-parameter linear unfolding $\Omega(\mu)$ is an LCU of Ω_0 if the derivative $D_0\Omega : \mathbb{R}^p \to \ker \mathrm{ad}\,\Omega_0^T$ is an isomorphism. In particular, for LCU's one always has $p = \mathrm{cod}\,\Omega_0 = \dim \ker \mathrm{ad}\,\Omega_0^T$.

2.5.2. Hamiltonian $1 : -1$ resonance

Consider the linear Hamiltonian vector field Y as in (31). In the $1 : -1$ resonant case, we can replace the linear part Ω_0 by the real normal form N_ε as in (32). As said before, the normal forms $N_{\pm 1}$ correspond to the generic (or nonsemisimple) case and N_0 to the semisimple case.

Proposition 2.14. *Depending on the value ε, we have the following.*

  i. *For $\varepsilon = \pm 1$ (i.e., the generic case), an LCU of N_ε as in (32) is given by the matrix family*

$$\Omega = \Omega(\mu) = \begin{pmatrix} 0 & -\lambda_0 + \mu_1 & \mu_2 & 0 \\ \lambda_0 - \mu_1 & 0 & 0 & \mu_2 \\ \varepsilon & 0 & 0 & -\lambda_0 + \mu_1 \\ 0 & \varepsilon & \lambda_0 - \mu_1 & 0 \end{pmatrix}, \tag{33}$$

  where $\mu_1, \mu_2 \in \mathbb{R}$.
  ii. *For $\varepsilon = 0$ (i.e., the semisimple case), an LCU Ω of N_ε as in (32) takes the form*

$$\Omega = \Omega(\mu) = \begin{pmatrix} \mu_3 & -\lambda_0 + \mu_1 & \mu_2 & 0 \\ \lambda_0 - \mu_1 & \mu_3 & 0 & \mu_2 \\ \mu_4 & 0 & -\mu_3 & -\lambda_0 + \mu_1 \\ 0 & \mu_4 & \lambda_0 - \mu_1 & -\mu_3 \end{pmatrix}, \tag{34}$$

  where $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{R}$.

In particular, in the generic case we have $\mathrm{cod}\, N_{\pm 1} = 2$, while in the semisimple case, we have $\mathrm{cod}\, N_0 = 4$.

The eigenvalues of $\Omega(\mu)$ as in (33) and (34) are of the form

$$\pm\big(i(\lambda_0 - \mu_1) \pm \sqrt{\nu}\,\big),$$

where $\nu = \mu_2$ for the generic case and $\nu = \mu_3^2 + \mu_2\mu_4$ for the semisimple case. In both cases, the eigenvalues behave as follows (see Fig. 1):

(a) Elliptic case ($\nu < 0$): $\Omega(\mu)$ has two pairs of purely imaginary eigenvalues $\pm i(\lambda_0 - \mu_1 \pm \sqrt{-\nu}\,)$.
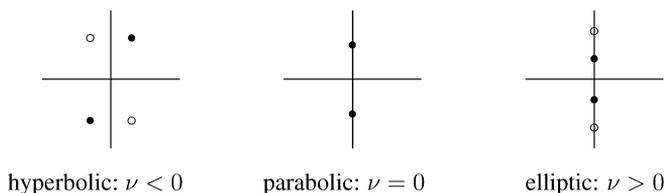

hyperbolic: $\nu < 0$ parabolic: $\nu = 0$ elliptic: $\nu > 0$

Fig. 1. Eigenvalues in both generic and semisimple $1 : -1$ resonance.

(b) Parabolic[4] case ($\nu = 0$): $\Omega(\mu)$ has a pair of double purely imaginary eigenvalues, namely $i(\lambda_0 - \mu_1), -i(\lambda_0 - \mu_1)$.

(c) Hyperbolic case ($\nu > 0$): $\Omega(\mu)$ has four complex eigenvalues with nonzero real parts, namely, $\pm(i(\lambda_0 - \mu_1) \pm \sqrt{\nu})$.

2.5.3. Reversible $1:1$ resonance

We briefly discuss the strong resonance in the reversible setting, see [16,17,31,53], which is similar to the Hamiltonian one, also compare the remarks and references at the end of Section 1.3. In both settings, the semisimple case requires (at least) four parameters to ensure the linear stability, while the generic case needs two parameters. In contrast to the Hamiltonian setting, we here do not have to distinguish between the $1:-1$ and the $1:1$ resonance, for details see [17,53].

Next we construct LCU's for the reversible $1:1$ resonance. Letting R be a linear involution on $\mathbb{R}^4 = \{z_1, z_2, z_3, z_4\}$, we denote by $\mathcal{X}_{-R}(\mathbb{R}^4)$ the set of all R-reversible vector fields on $\mathbb{R}^4$, i.e.,

$$\mathcal{X}_{-R}(\mathbb{R}^4) = \{X \in \mathcal{X}(\mathbb{R}^4): R_* X = -X\}.$$

We consider the linear system $Y(z) = \Omega_0 z \frac{\partial}{\partial z}$, where $\Omega_0 \in \mathfrak{gl}(4, \mathbb{R})$. Then, Y is R-reversible if and only if Ω_0 is a R-reversible matrix, i.e., $\Omega_0 \in \mathfrak{gl}_{-R}(4, \mathbb{R})$. Reversible $1:1$ resonance reflects the fact that Ω_0 has a double pair of purely imaginary eigenvalues, say $\pm i\lambda_0$. In this resonant case, following [17], one can normalize the involution R into $R = \text{diag}\{1, -1, 1, -1\}$, while the linear part Ω_0 into

$$\Omega_0 = \begin{pmatrix} \alpha J_2 & \varepsilon J_2 \\ O & \alpha J_2 \end{pmatrix}, \tag{35}$$

where $\varepsilon = 0$ in the semisimple case and $\varepsilon = 1$ in the nonsemisimple case; the matrix J_2 is given by

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a fixed choice of the involution $R = \text{diag}\{1, -1, 1, -1\}$, the matrix Ω_0 as in (35) has two different LCU's $\Omega(\mu)$ depending on the value ε, compare with Proposition 2.14.

i. The generic case ($\varepsilon = 1$):

$$\Omega(\mu) = \begin{pmatrix} (\lambda_0 + \mu_1)J_2 & J_2 \\ \mu_2 J_2 & (\lambda_0 + \mu_1)J_2 \end{pmatrix};$$

ii. The semisimple case ($\varepsilon = 0$):

$$\Omega(\mu) = \begin{pmatrix} (\lambda_0 + \mu_1)J_2 & \mu_2 J_2 \\ \mu_3 J_2 & (\lambda_0 + \mu_4)J_2 \end{pmatrix}.$$

The eigenvalue configurations again are as in Fig. 1.

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[4] A (4×4)-matrix is called parabolic if it has a double pair of purely imaginary eigenvalues.

3. Normal linear stability of invariant tori: The integrable case

Returning to the setting of Section 1, we start with an $(n + m)$-dimensional phase space M, a p-dimensional parameter space P and a free $\mathbb{T}^n$-action on M. We are given a p-parameter family X of integrable (i.e., $\mathbb{T}^n$-symmetric) vector fields on M with an X-invariant submanifold T of the form $T = \bigcup_{\mu \in U} T_\mu \times \{\mu\} \subset M \times P$, where U is a domain containing $0 \in P \subset \mathbb{R}^p$ and the submanifolds $T_\mu \subseteq M$ are X_μ-invariant $\mathbb{T}^n$-orbits. Recall that each orbit T_μ is diffeomorphic to the standard n-torus $\mathbb{T}^n$ carrying conditionally periodic (or parallel) dynamics, see Section 1.1. In the present section, we study persistence of the invariant torus family T of the integrable family X, in cases where the perturbations are integrable as well. This persistence is very similar to the linear stability of vector fields at equilibria (as discussed in Section 2) and is a direct consequence of the Inverse Function Theorem, compare with [26,55]. As before, the persistence is formulated both for the dissipative, the Hamiltonian and the reversible settings. Following [26, 55], one can simplify the perturbation problem by transferring it to the so-called normal bundle of T, compare with a rescaling argument given in [70].

3.1. Transfer to normal bundle

Here we briefly revisit the ideas of the simplification of our perturbation problems, as explained in [26,55]. For the sake of completeness, we recall the definition of the normal bundle: for a submanifold V of M, the *normal bundle* NV of V is the quotient $NV = T_V M / TV$ of tangent bundles, the fibers of which are the vector spaces $T_x M / T_x V$ for $x \in V$, compare, e.g., [50]. For the $\mathbb{T}^n$-orbits $T_\mu \subseteq M$, by Section 1.1, we have the identification $NT_\mu \cong \mathbb{T}^n \times \mathbb{R}^m$ for each $\mu \in \mathcal{U}$. The vector field X_μ induces a normal linear vector field NX_μ on NT_μ as follows. For each $\varepsilon > 0$, we define a scaling operator

$$\mathcal{D}_\varepsilon : (x, y) \in \left(NT_\mu \cong \mathbb{T}^n \times \mathbb{R}^m\right) \mapsto (x, \varepsilon y) \in M, \tag{36}$$

in a neighborhood of the zero-section of NT_μ. Now the normal linear part NX_μ of X_μ is defined as $NX_\mu = \lim_{\varepsilon \downarrow 0} \mathcal{D}_\varepsilon^*(X_\mu)$. Hence, for the family X as in (1), the corresponding normal linear part NX in the coordinates $(x, y) \in \mathbb{T}^n \times \mathbb{R}^m$ obtains the form:

$$NX_\mu(x, y) = \omega(\mu)\frac{\partial}{\partial x} + \Omega(\mu)y\frac{\partial}{\partial y}. \tag{37}$$

To show normal linear stability for the family X in the set $\mathcal{X}_P(M)$, see the conclusions of Theorem 1.4, it is sufficient to verify this stability for the normal linear part NX in the space $\mathcal{X}_P(\mathbb{T}^n \times \mathbb{R}^m)$. Indeed, suppose that NX is normally linearly stable. Then, there is a neighborhood $\mathcal{V} \subset \mathcal{X}_P(\mathbb{T}^n \times \mathbb{R}^m)$ of NX such that for any family $\widehat{X} \in \mathcal{V}$, there is a smooth conjugacy which sends $\widehat{X}$ to a family with NX as the normal linear part. If $\varepsilon > 0$ is sufficiently small, then we can find an open neighborhood $\mathcal{W} \subset \mathcal{X}_P(M)$ of X such that the open set $\mathcal{D}_\varepsilon^* \mathcal{W}$ is contained in $\mathcal{V}$, i.e., for any $\widetilde{X} \in \mathcal{W}$, we have $\mathcal{D}_\varepsilon^* \widetilde{X} \in \mathcal{V}$. This implies that the family $\mathcal{D}_\varepsilon^* X \in \mathcal{V}$, and therefore the family X, is normally linearly stable, compare with Fig. 2.
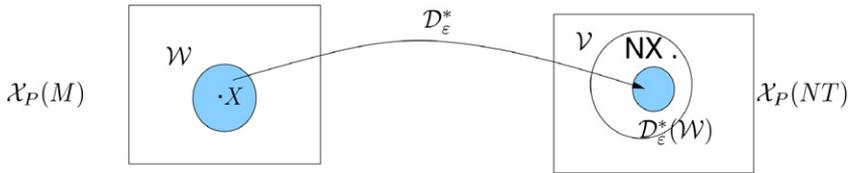
Fig. 2. Transfer of the perturbation problem to the normal bundle.

3.2. Normal linear stability: Integrable dissipative case

In the dissipative case, by transferring the perturbation problem to the normal bundle, we may replace M by $\mathbb{T}^n \times \mathbb{R}^m$ and the family (1) by its normal linear part. In the following, we simply put $M = \mathbb{T}^n \times \mathbb{R}^m$ and the p-parameter integrable family $X \in \mathcal{X}_P(M)$ as

$$X_\mu(x, y) = \omega(\mu)\frac{\partial}{\partial x} + \Omega(\mu)y\frac{\partial}{\partial y} \tag{38}$$

in vectorial notation. In the case where $X = X_\mu(x, y)$ is nondegenerate at the invariant torus $T_0 = \{(x, y, \mu): (y, \mu) = (0, 0)\}$, the following lemma provides a simplification of the parameter dependence.

Lemma 3.1. *Let the p-parameter family $Y \in \mathcal{X}_P(M)$ be given by*

$$Y_\mu(x, y) = \big[\omega(\mu) + O(|y|)\big]\frac{\partial}{\partial x} + \big[\Omega(\mu)y + O(|y|^2)\big]\frac{\partial}{\partial y}, \tag{39}$$

where $\mu \in P$. Assume that Y is nondegenerate at the torus $(y, \mu) = (0, 0)$ and that $\operatorname{cod}\Omega(0) = c$. Then, there exists a local reparametrization

$$\mu \in \mathbb{R}^p \mapsto (\alpha, \beta, \lambda) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{p-n-c}$$

such that the family Y (after reparametrization) obtains the form

$$Y_{\alpha, \beta, \lambda}(x, y) = \big[(\omega(0) + \alpha) + O(|y|)\big]\frac{\partial}{\partial x} + \big[B(\beta)y + O(|y|^2)\big]\frac{\partial}{\partial y}, \tag{40}$$

where $B(\beta)$ is an LCU of $\Omega_0 = \Omega(0)$.

Proof. Let $\widetilde{\Omega} : (\mathbb{R}^c, 0) \mapsto (\mathfrak{gl}(m, \mathbb{R}), \Omega_0)$ be an LCU of Ω_0. By versality, we have $\Omega(\mu) = Q(\mu)\widetilde{\Omega}(\rho(\mu))Q^{-1}(\mu)$, where $Q : (\mathbb{R}^p, 0) \mapsto (GL(m, \mathbb{R}), I_m)$ and $\rho : (\mathbb{R}^p, 0) \mapsto (\mathbb{R}^c, 0)$, compare with Definition 2.3. Due to universality of $\widetilde{\Omega}$ and versality of Ω, the map $\rho : \mathbb{R}^p \to \mathbb{R}^c$ is a submersion at $\mu = 0$. Let $\tilde{\pi} : \mathfrak{gl}(m, \mathbb{R}) \to \mathbb{R}^p$ be the local projection given by $\tilde{\pi}(\Omega(\mu)) = \mu$. Define the map $\pi = \rho \circ \tilde{\pi} : (\mathfrak{gl}(m, \mathbb{R}), \Omega_0) \mapsto (\mathbb{R}^c, 0)$. Notice that $\pi \circ \Omega = \rho$. Hence, $\pi \circ \Omega : \mathbb{R}^p \to \mathbb{R}^c$ is submersive at $\mu = 0$. By nondegeneracy, the germ

$$\big(\omega - \omega(0), \pi \circ \Omega\big) : (\mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^c, (0, 0))$$

is submersive at $\mu = 0$. By the Inverse Function Theorem [1,40], there exists a local diffeomorphism $\phi : \mathbb{R}^p \to \mathbb{R}^p$ with $\phi(0) = 0$ such that

$$\big(\omega - \omega(0), \pi \circ \Omega\big)\big(\phi(\mu)\big) = (\alpha, \beta) \in \mathbb{R}^n \times \mathbb{R}^c.$$

This implies that

$$Y_{\phi(\mu)}(x, y) = \big[\big(\omega(0) + \alpha\big) + O\big(|y|\big)\big]\frac{\partial}{\partial x} + \big[\pi^{-1}(\beta)y + O\big(|y|^2\big)\big]\frac{\partial}{\partial y},$$

where $B(\beta) = \pi^{-1}(\beta)$ is any universal unfolding of Ω_0. In particular, we can choose B as an LCU of Ω_0. $\square$

Now we formulate a normal linear stability for the integrable family X given by (38), regarding persistence of the X-invariant n-torus family

$$T = \big\{(x, y, \mu) \in \mathbb{T}^n \times \mathbb{R}^m \times P : y = 0\big\} \subset M \times P$$

under integrable perturbations. This integrable version of Theorem 1.4 is a consequence of the Inverse Function Theorem, also see Theorem 3.1 in [26].

Theorem 3.2 (*Normal linear stability: integrable dissipative case*). *Let $X \in \mathcal{X}_P(M)$ be the family of the vector fields given by (38). Assume that X is nondegenerate at the torus $T_0 = \{(x, y, \mu) : (y, \mu) = (0, 0)\}$. Then, for any integrable family $\widetilde{X} \in \mathcal{X}_P(M)$ that is C^2-close to X, there exists a local diffeomorphism*

$$\Phi : M \times P \to M \times P$$

of the form $\Phi(x, y, \mu) = (x, \phi_\mu(y), \rho(\mu))$ defined near the torus T_0 such that

  i. *Φ is C^1-near the identity map;*
 ii. *The image of the torus family T under Φ is $\widetilde{X}$-invariant and for the restriction of $\widehat{\Phi} = \Phi|T$ one has*

$$\widehat{\Phi}_* X = \widetilde{X},$$

   that is $\widehat{\Phi}$ conjugates X to $\widetilde{X}$;
iii. *Φ preserves the normal linear behavior.*

Proof. Suppose that the perturbation $\widetilde{X} = \widetilde{X}_\mu(y)$ is given by

$$\widetilde{X}_\mu(y) = f(y, \mu)\frac{\partial}{\partial x} + g(y, \mu)\frac{\partial}{\partial y}$$

and that the required map ϕ_μ is of the form $\phi_\mu(y) = q(\mu) + Q(\mu)y$. The map $q(\mu)$ can be obtained by an application of the Inverse Function Theorem. Indeed, since the family X is nondegenerate at the torus T_0, the Inverse Function Theorem gives a smooth μ-dependent map $q(\mu)$

such that $g(q(\mu), \mu) \equiv 0$. This implies that $\{y = q(\mu)\}$ is an invariant torus family of the perturbation $\widetilde{X}$. Next we need to find the map Q and a suitable reparametrization ρ. By Lemma 3.1, there exists a reparametrization $\mu \in \mathbb{R}^p \mapsto (\alpha(\mu), \beta(\mu), \lambda(\mu)) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{p-n-c}$ such that X (after reparametrization) takes the form:

$$X_{\alpha, \beta, \lambda}(x, y) = \left[\omega(0) + \alpha\right]\frac{\partial}{\partial x} + \widetilde{\Omega}(\beta)y\frac{\partial}{\partial y},$$

where $\widetilde{\Omega}(\beta)$ is an LCU of Ω_0. The reparametrization ρ will be chosen of the form $\rho(\alpha, \beta, \lambda) = (\rho_1, \rho_2, \lambda) \in \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{p-n-c}$, where ρ_1 and ρ_2 depend on (α, β, λ), respectively. Now the map ρ_1 is determined by the parameter shift in the α-direction, preserving the internal frequencies of the unperturbed family X. The maps ρ_2 in the β-direction and Q can be obtained by an application of Theorem 2.6, also see the proof of Theorem 2.10. $\quad\square$

3.3. Normal linear stability: Integrable symplectic case

Summarizing from Section 1.2, we are given the $(2n + 2q)$-dimensional symplectic manifold (M, σ) with a free (locally Hamiltonian) $\mathbb{T}^n$-action. By the generalized Darboux theorem [1], near each $\mathbb{T}^n$-orbit, there are local coordinates $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q}$ such that the orbit is given by the set $\{(x, y, z): (y, z) = (0, 0)\}$ and the 2-form $\sigma = \sum_{i=1}^{n} dx_i \wedge dy_i + \sum_{j=1}^{q} dz_j \wedge dz_{j+q} = dx \wedge dy + dz^2$. For a p-parameter family $X \in \mathcal{X}_P^\sigma(M)$ of integrable (i.e., $\mathbb{T}^n$-symmetric) Hamiltonian vector fields with $\mathbb{T}^n$-orbits as invariant submanifolds, we locally can write

$$X_\mu(x, y, z) = f(y, z, \mu)\frac{\partial}{\partial x} + h(y, z, \mu)\frac{\partial}{\partial z},$$

where $h(0, 0, \mu) \equiv 0$. Assuming that $\det(\partial h/\partial z)(0, 0, 0) \neq 0$, we again obtain a continuum of X-invariant n-tori, normalized to $\{z = 0\}$, locally parametrized by both y and μ. This means that $h(y, 0, \mu) \equiv 0$, where $\partial h/\partial z(y, 0, \mu) \in \mathfrak{sp}(2q, \mathbb{R})$ for each (y, μ). Moreover, we use the localized setting with the distinguished parameter ν, varying over a neighborhood of $0 \in \mathbb{R}^n$ and the localized coordinate $y_{\text{loc}} = y - \nu$, again see [26,55]. This leads to the n-torus family T_{loc} as in (9)

$$T_{\text{loc}} = \left\{(x, y_{\text{loc}}, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \mathbb{R}^p \times \mathbb{R}^n \mid y_{\text{loc}} = 0, \ z = 0\right\},$$

where (μ, ν) varies over a neighborhood of $(0, 0) \in P \subset \mathbb{R}^p \times \mathbb{R}^n$, which is invariant for the extended integrable family $X_{\mu, \nu}(x, y_{\text{loc}}, z) = X_\mu(x, y_{\text{loc}} + \nu, z)$, which has the form (10)

$$X_{\mu, \nu}(x, y_{\text{loc}}, z) = f(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial x} + h(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial z}.$$

We investigate the normal linear stability of this (extended) family $X = X_{\mu, \nu}(x, y_{\text{loc}}, z)$, regarding the persistence of the invariant tori family $T_{\text{loc}} = \{(y_{\text{loc}}, z) = (0, 0)\}$ under an integrable Hamiltonian perturbation. Non-degeneracy for the family X means: the Floquet matrix $\Omega(\mu, \nu) = \frac{\partial h}{\partial z}(\nu, 0, \mu)$ has maximal rank at $(\mu, \nu) = (0, 0)$; the map $\omega: \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$ given by $\omega(\mu, \nu) = f(\nu, 0, \mu)$ is a submersion at $(\mu, \nu) = (0, 0)$, while 'simultaneously' the matrix family $\Omega(\mu, \nu)$ is a versal unfolding of $\Omega(0, 0)$ within the linear structure $(SP(2q, \mathbb{R}), \mathfrak{sp}(2q, \mathbb{R}))$.

For each fixed parameter value (μ, ν), we denote by $T_{\text{loc}}^{\mu, \nu}$ the n-torus $T_{\text{loc}}^{\mu, \nu} = T_{\text{loc}} \cap \{(\mu, \nu)\}$. Similar to the dissipative case, we use the normal bundle NT_{loc} of T_{loc}, where $NT_{\text{loc}}^{\mu, \nu} \cong \mathbb{T}^n \times$

$\mathbb{R}^n \times \mathbb{R}^{2q}$ for each (μ, ν), to simplify the problem, see Section 3.1. However, the scaling operator $\mathcal{D}_\varepsilon$ which defines normal linearization, slightly differs:

$$\mathcal{D}_\varepsilon(x, y_{\text{loc}}, z) = (x, \varepsilon^2 y_{\text{loc}}, \varepsilon z), \quad \text{for } \varepsilon > 0, \tag{41}$$

see [26, Section 2a, p. 9]. Since $\mathcal{D}_\varepsilon^* \sigma = \varepsilon^2 N(\sigma)$, where $N(\sigma) = dx \wedge dy_{\text{loc}} + dz^2$ is the standard 2-form on the normal bundle NT_{loc}, the pull-back $\mathcal{D}_\varepsilon^*$ is a symplectic diffeomorphism for any $\varepsilon > 0$. The normal linear part NX of the family X is given by

$$NX_{\mu,\nu}(x, y_{\text{loc}}, z) = \lim_{\varepsilon \downarrow 0} \mathcal{D}_\varepsilon^* X_{\mu,\nu} = \omega(\mu, \nu) \frac{\partial}{\partial x} + \Omega(\mu, \nu) z \frac{\partial}{\partial z}, \tag{42}$$

which is Hamiltonian with respect to $N(\sigma)$.

Remark 3.3. For a nonintegrable Hamiltonian vector field $Y = Y(x, y, z)$ on (M, σ), the scaling (41) does not induce a normal linearization as the form (42). Indeed, in the non-integrable case, the flow in the y_{loc}-direction in general is not constant, giving rise to a quadratic term in z for the vector field $\lim_{\varepsilon \downarrow 0} \mathcal{D}_\varepsilon^* Y$. To overcome this problem, one can use the (smaller) *symplectic normal bundle* of T, for a discussion see [26, Section 6b, p. 30] or Appendix C.

Similar to the dissipative case, we now may assume that the phase space has the form $M = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} = \{x, y, z\}$ with symplectic 2-form $\sigma = dx \wedge dy + dz^2$. The (extended) family $X = X_{\mu,\nu}(x, y, z)$ may be taken as $X_{\mu,\nu}(x, y, z) = \omega(\mu, \nu) \frac{\partial}{\partial x} + \Omega(\mu, \nu) z \frac{\partial}{\partial z}$. Since any integrable Hamiltonian vector field $Y \in \mathcal{X}^\sigma(M)$ takes the form

$$Y(x, y, z) = f(y, z) \frac{\partial}{\partial x} + g(y, z) \frac{\partial}{\partial z},$$

the present symplectic situation is the same as that of the dissipative case, except that Floquet matrices now belong to the matrix space $\mathfrak{sp}(2q, \mathbb{R})$. Analogous to Theorem 3.2, we have the integrable version of Theorem 1.6.

Theorem 3.4 (*Normal linear stability: integrable symplectic case*). *Let $X \in \mathcal{X}^\sigma_{P_{\text{loc}}}(M)$ be the family of Hamiltonian vector fields given by* (10). *Assume that X is nondegenerate at the invariant torus $T_{\text{loc}}^{0,0} = T_{\text{loc}} \cap \{(\mu, \nu) = (0, 0)\}$. Then, for any integrable family $\widetilde{X} \in \mathcal{X}^\sigma_{P_{\text{loc}}}(M)$ that is C^2-close to X, there exists a local diffeomorphism*

$$\Phi : M \times P_{\text{loc}} \to M \times P_{\text{loc}}$$

of the form $\Phi(x, y, z, \mu) = (x, y, \phi_\mu(z), \rho(\mu))$ defined near the torus $T_{\text{loc}}^{0,0}$ such that

  i. *Φ is a C^1-near the identity map;*
 ii. *The image of the torus family T under Φ is $\widetilde{X}$-invariant and for the restriction of $\widehat{\Phi} = \Phi|T$ one has*

$$\widehat{\Phi}_* X = \widetilde{X},$$

   that is $\widehat{\Phi}$ conjugates X to $\widetilde{X}$;
iii. *Φ is symplectic and preserves the normal linear behavior.*

3.4. Normal linear stability: Integrable reversible case

We briefly recall elements from Section 1.3, where on the phase space M a free $\mathbb{T}^n$-action is given, as well as an involution G, that commute. Using Bochner's theorem as before [14,68], also compare with [25], we may assume that the phase space is $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q} = \{x, y, z\}$, where G has the form $G(x, y, z) = (-x, y, Rz)$. Here $R \in GL(2q, \mathbb{R})$ is a linear evolution, where we assume that the eigenvalue occurs with mutiplicity p. Also a free A vector field X on M G-reversible whenever

$$G_*(X) = -X.$$

We recall that integrability of X amounts to equivariance under the $\mathbb{T}^n$-action. We consider the space $\mathcal{X}_P^{-G}(\mathbb{R}^{2q})$ of all G-reversible families of vector fields, parametrized over a domain $P \subset \mathbb{R}^p$. For an integrable family $X \in \mathcal{X}_P^{-G}(\mathbb{R}^{2q})$ we consider invariant tori, that are both G-invariant and orbits under the n-torus action, for their persistence under small perturbation. In the present section we apply only integrable perturbations.

We recall that under generic circumstances such invariant tori come in continua, locally parametrized by (y, μ), say in a neighborhood of (y_0, μ_0), that can safely be put at $(0, 0)$, and that by a G-equivariant change of coordinates coincide with the submanifold $\{z = 0\}$. This leads to the normalized format (15)

$$X_\mu(x, y, z) = f(y, z, \mu)\frac{\partial}{\partial x} + g(y, z, \mu)\frac{\partial}{\partial y} + h(y, z, \mu)\frac{\partial}{\partial z},$$

with $g(y, 0, \mu) = 0$ and $h(y, 0, \mu) = 0$. Introducing again the localizing parameter ν and coordinate $y_{\text{loc}} = y - \nu$, we obtain an extended integrable family $X_{\mu,\nu}(x, y_{\text{loc}}, z) = X_\mu(x, y_{\text{loc}} + \nu, z)$ of the form (17)

$$
\begin{aligned}
&X_{\mu,\nu}(x, y_{\text{loc}}, z) \\
&= f(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial x} + g(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial y_{\text{loc}}} + h(y_{\text{loc}} + \nu, z, \mu)\frac{\partial}{\partial z}
\end{aligned}
$$

with invariant tori (16)

$$T_{\text{loc}} = \left\{ (x, y_{\text{loc}}, z, \mu, \nu) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q} \times \mathbb{R}^q \mid y_{\text{loc}} = 0, \ z = 0 \right\},$$

where (μ, ν) varies over a neighborhood of $(0, 0) \in P_{\text{loc}} \subset \mathbb{R}^q \times \mathbb{R}^m$. Both the reversor G and the commuting n-torus action directly carry over to this new setting.

One last reduction is a scaling operator, that reduces the perturbation problem to the normal bundle NT_{loc}. Similar to the dissipative and the symplectic case, we have that $NT_{\text{loc}} \cong \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2q}$, where now a scaling operator

$$\mathcal{D}_\varepsilon(x, y, z) = (x, \varepsilon y, \varepsilon^2 z), \quad \text{for } \varepsilon > 0 \tag{43}$$

has to be applied, see [23]. The pull back vector field $\mathcal{D}_\varepsilon^* X_{\mu,\nu}$ is well defined for all $\varepsilon > 0$, and so is the limit

$$NX_{\mu,\nu}(x, y_{\text{loc}}, z) = \lim_{\varepsilon \downarrow 0} \mathcal{D}_\varepsilon^* X_{\mu,\nu} = \omega(\mu, \nu)\frac{\partial}{\partial x} + \Omega(\mu, \nu)z\frac{\partial}{\partial z}, \tag{44}$$

with $\omega(\mu, \nu) = f(\nu, 0, \mu)$ and $\Omega(\mu, \nu) = h_z(\nu, 0, \mu)$. This is the normal linear part, properly defined on the normal bundle NT_{loc}, to which both the reversor G and the commuting n-torus action again directly carry over.

We recall the nondegeneracy condition imposed on the product map $\omega \times \Omega : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^n \times gl_{-R}(2q, \mathbb{R})$. The matrix $\Omega(0, 0)$ has maximal rank, while 'simultaneously' the map ω has to be a submersion at $(\mu, \nu) = (0, 0)$, the component $\Omega(\mu, \nu)$ is a versal unfolding of $\Omega(0, 0)$ within the linear structure $(GL_R(2q, \mathbb{R}), gl_{-R}(2q, \mathbb{R}))$, see Section 2.1, Example 2.2 as well as Section 2.4. Analogously to Theorems 3.2 and 3.4, we now formulate the integrable version of Theorem 1.8.

Theorem 3.5 *(Normal linear stability: integrable reversible case). Let the reversible family $X \in \mathcal{X}_P^{-G}(M)$ be given by* (17). *Assume that X is nondegenerate at the invariant n-torus $T_{\text{loc}}^{0,0} = T_{\text{loc}} \cap \{(\mu, \nu) = (0, 0)\}$. Then for any integrable family $X \in \mathcal{X}_P^{-G}(M)$ that is sufficiently C^2 close to X, there exists a local diffeomorphism*

$$\Phi : M \times P_{\text{loc}} \mapsto \big(\phi_\mu(z), \rho(\mu)\big) \in M \times P_{\text{loc}}$$

of the form $(x, y_{\text{loc}}, z, \mu, \nu) \mapsto (x, y_{\text{loc}}, \Phi_\mu(z), \rho(\mu), \nu)$, defined near the torus $T_{\text{loc}}^{0,0}$, such that,

  i. *Φ is C^1-near the identity map;*
 ii. *The image of the tori T_{loc} under Φ is $\widetilde{X}$-invariant and for the restriction $\widehat{\Phi}|_{T_{\text{loc}}}$ we have*

$$\widehat{\Phi}_*(X) = \widetilde{X},$$

    that is, $\widehat{\Phi}$ conjugates X to $\widetilde{X}$;
iii. *Φ is G-equivariant and preserves the normal linear behaviour.*

4. Case study: Generic $1 : -1$ resonance

Normal linear stability of vector fields with invariant tori for the nearly-integrable case is discussed in Sections 1.1 (the dissipative setting), 1.2 (the symplectic setting) and 1.3 (the reversible setting). As we already pointed out in Section 1.2, a special case of Theorem 1.6 occurs when the Floquet matrix of the central torus is in nonsemisimple or generic $1 : -1$ resonance. This generically involves a quasi-periodic Hamiltonian Hopf bifurcation [19,64]. The quasi-periodic reversible Hopf bifurcation is dealt with in [16] and will not be discussed here. The Hamiltonian $1 : -1$ resonance is of interest, since it is observed in many mechanical systems including the Lagrange top [34,35], the double spherical pendulum [63], the restricted three body problem [64] and the 3D Hénon–Heiles family as this models the movement of galaxies [38,46,47].

Let us briefly recall the setting from Section 1.2. According to Section 3.3, we take the phase space $M = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^4 = \{x, y, z\}$ with the symplectic 2-form $\sigma = dx \wedge dy + dz^2$. We consider the integrable Hamiltonian family $X = X_{\mu, \nu}$ of the form

$$X_{\mu, \nu}(x, y, z) = \omega(\mu, \nu)\frac{\partial}{\partial x} + \Omega(\mu, \nu)z\frac{\partial}{\partial z},$$

where $\Omega(\mu, \nu) \in \mathfrak{sp}(4, \mathbb{R})$, with the invariant tori $\{(x, y, z): (y, z) = (0, 0)\}$. To simplify notation, we include ν into a general μ (i.e., we use μ instead of (μ, ν)). The Floquet matrix

$\Omega_0 = \Omega(0)$ is assumed to be in generic $1 : -1$ resonance, that is, it has a double pair of purely imaginary eigenvalues with a nontrivial nilpotent part. Then, the matrix Ω_0 is symplectically conjugate to the real normal form

$$\begin{pmatrix} 0 & -\lambda_0 & 0 & 0 \\ \lambda_0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & -\lambda_0 \\ 0 & \varepsilon & \lambda_0 & 0 \end{pmatrix}, \tag{45}$$

where $\lambda_0 \neq 0$ and $\varepsilon = \pm 1$, see Section 2.5. As before, we assume that X is nondegenerate at the torus $(z, \mu) = (0, 0)$, that is, the map $\omega : \mathbb{R}^p \to \mathbb{R}^n$ is submersive at $\mu = 0$, while Ω is versal at $\mu = 0$ within the linear structure $(SP(4, \mathbb{R}), \mathfrak{sp}(4, \mathbb{R}))$, see Section 2.1. We recall that by Proposition 2.14 the co-dimension of Ω_0 is equal to 2. By Lemma 3.1, after a suitable reparametrization $\mu \in \mathbb{R}^p \to (\tilde{\omega}, \tilde{\mu}, \tilde{\nu}) \in \mathbb{R}^n \times \mathbb{R}^2 \times \mathbb{R}^{p-n-2}$, the family X we get the following:

$$X_{\tilde{\omega}, \tilde{\mu}, \tilde{\nu}}(x, y, z) = \tilde{\omega} \frac{\partial}{\partial x} + \widetilde{\Omega}(\tilde{\mu}) z \frac{\partial}{\partial z}, \tag{46}$$

where $\widetilde{\Omega}$ is an LCU of Ω_0 within $(SP(4, \mathbb{R}), \mathfrak{sp}(4, \mathbb{R}))$. In what follows, for simplicity we suppress the parameter $\tilde{\nu}$, since it can easily be incorporated again in our discussion below. Also in (46) we drop all tildes. Furthermore, again by Proposition 2.14, the LCU $\Omega(\mu)$ can be taken of the form

$$\Omega(\mu) = \begin{pmatrix} 0 & -\lambda_0 - \mu_1 & -\mu_2 & 0 \\ \lambda_0 + \mu_1 & 0 & 0 & -\mu_2 \\ \varepsilon & 0 & 0 & -\lambda_0 - \mu_1 \\ 0 & \varepsilon & \lambda_0 + \mu_1 & 0 \end{pmatrix}. \tag{47}$$

The eigenvalues of $\Omega(\mu)$ read

$$\pm i (\lambda_0 + \mu_1 \pm \sqrt{\varepsilon \mu_2}).$$

In what follows, let us stick to the integrable family X as in (46), where $\Omega(\mu)$ is given by the LCU (47). With these choices, the normal frequency vector $\omega^N(\mu)$ of the family X is given by

$$-\varepsilon \mu_2 \leqslant 0: \quad \omega^N(\mu) = (\lambda_0 + \mu_1, \lambda_0 + \mu_1); \tag{48}$$

$$-\varepsilon \mu_2 > 0: \quad \omega^N(\mu) = (\lambda_0 + \mu_1 + \sqrt{\varepsilon \mu_2}, \lambda_0 + \mu_1 - \sqrt{\varepsilon \mu_2}). \tag{49}$$

Let $\mathcal{U}$ be a convex domain around the origin in the parameter space $\mathbb{R}^n \times \mathbb{R}^2$. Denote $\mathcal{U}^- = \{(\omega, \mu) \in \mathcal{U} : \varepsilon \mu_2 \leqslant 0\}$ and $\mathcal{U}^+ = \{(\omega, \mu) \in \mathcal{U} : \varepsilon \mu_2 > 0\}$. Following Section 1.1, the (generalized) frequency map $\mathcal{F} : \mathcal{U} \to \mathbb{R}^n \times \mathbb{R}^2$ here is given by

$$\mathcal{F}(\omega, \mu) = \begin{cases} (\omega, \lambda_0 + \mu_1, \lambda_0 + \mu_1), & \text{for } \varepsilon \mu_2 \leqslant 0, \\ (\omega, \lambda_0 + \mu_1 + \sqrt{\varepsilon \mu_2}, \lambda_0 + \mu_1 - \sqrt{\varepsilon \mu_2}), & \text{for } \varepsilon \mu_2 > 0. \end{cases} \tag{50}$$

Recall from Remark 1.3 that the set $(\mathbb{R}^n \times \mathbb{R}^2)_{\tau,\gamma}$ consists of all pairs $(\omega, \omega^N) \in \mathbb{R}^n \times \mathbb{R}^2$ satisfying the (τ, γ)-Diophantine conditions (2). The 'Cantor set' $\Gamma_{\tau,\gamma}(\mathcal{U})$ is the union $\Gamma_{\tau,\gamma}(\mathcal{U}) = \Gamma_{\tau,\gamma}(\mathcal{U}^-) \cup \Gamma_{\tau,\gamma}(\mathcal{U}^+)$, where

$$\Gamma_{\tau,\gamma}(\mathcal{U}^-) = \left\{ (\omega, \mu) \in \mathcal{U}^-\colon \mathcal{F}(\omega, \mu) \in \left(\mathbb{R}^n \times \mathbb{R}^2\right)_{\tau,\gamma} \right\},$$

$$\Gamma_{\tau,\gamma}(\mathcal{U}^+) = \left\{ (\omega, \mu) \in \mathcal{U}^+\colon \mathcal{F}(\omega, \mu) \in \left(\mathbb{R}^n \times \mathbb{R}^2\right)_{\tau,\gamma} \right\}.$$

Remark 4.1. Recall that for $(\omega, \mu) \in \mathbb{R}^n \times \mathbb{R}^2$ we have $(\omega, \mu) \in \Gamma_{\tau,\gamma}(\mathcal{U}^+)$ if and only if

$$\left| \langle \omega, k \rangle + \ell_1(\lambda_0 + \mu_1) + \ell_2\sqrt{|\mu_2|} \right| \geqslant \gamma |k|^{-\tau}, \tag{51}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ and for integers $|\ell_1|, |\ell_2| \leqslant 2$, where $(\ell_1 \pm \ell_2)$ is even. On the other hand, the set $\Gamma_{\tau,\gamma}(\mathcal{U}^-)$ consists of $(\omega, \mu) \in \mathbb{R}^n \times \mathbb{R}^2$ satisfying

$$\left| \langle \omega, k \rangle + \ell(\lambda_0 + \mu_1) \right| \geqslant \gamma |k|^{-\tau}, \tag{52}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$ and for all $|\ell| = 0, 1, 2$.

Let us briefly describe the geometric of the 'Cantor sets' $\Gamma_{\tau,\gamma}(\mathcal{U}^-)$ and $\Gamma_{\tau,\gamma}(\mathcal{U}^+)$ for $\varepsilon = 1$. In $\Gamma_{\tau,\gamma}(\mathcal{U}^+)$ there no Cantor gaps occur in the μ_2-direction, since the parameter μ_2 does not occur in the normal frequencies, see (48). The intersections of planes $\{\mu_2 = \text{const}\}$ with $\Gamma_{\tau,\gamma}(\mathcal{U}^-)$ project to subsets of the (ω, μ_1)-plane satisfying the Diophantine conditions (52). By Remark 1.3, this subset of (ω, μ_1) is a Cantor family of closed half lines, compare with Fig. 3(a). Taking the (continuous) μ_2-direction into account, it follows that $\Gamma_{\tau,\gamma}(\mathcal{U}^-)$ is a Cantor family of closed half planes of the form $\{(t\omega, t(\lambda_0 + \mu_1))\colon t \geqslant 1\}$, where evidently all takes place inside the set $\mathcal{U}^-$. Regarding the set $\Gamma_{\tau,\gamma}(\mathcal{U}^+)$ we observe that if $(\omega, \lambda_0 + \mu_1, \mu_2)$ satisfies the Diophantine conditions (51), so do all $(t\omega, t(\lambda_0 + \mu_1), t^2\mu_2)$ for $t \geqslant 1$. Therefore, the set $\Gamma_{\tau,\gamma}(\mathcal{U}^+)$ is a Cantor family of closed half parabolæ of the form $\{(t\omega, t(\lambda_0 + \mu_1), t^2\mu_2)\colon t \geqslant 1\}$, see Fig. 3(b).

As a special case of Theorem 1.6, we now have the following KAM stability statement for the generic (or nonsemisimple) $1 : -1$ resonance.

Corollary 4.2 (*Normal linear stability: generic $1 : -1$ resonant case*). *Let X be a p-parameter real-analytic family of Hamiltonian vector fields defined on the phase space $M = \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^4 = \{x, y, z\}$ with the symplectic 2-form $\sigma = dx \wedge dy + dz^2$, where $X_\mu(x, y, z) = \omega(\mu)\frac{\partial}{\partial x} + \Omega(\mu)z\frac{\partial}{\partial z}$. Assume that*

- *X is nondegenerate at the n-torus $(y, z, \mu) = (0, 0, 0)$;*
- *The matrix $\Omega(0) \in \mathfrak{sp}(4, \mathbb{R})$ is in generic $1 : -1$ resonance.*

Then, for $\gamma > 0$ sufficiently small and for any $\widetilde{X} \in \mathcal{X}^\sigma_{\mathbb{R}^p}(M)$ sufficiently close to X in the compact-open topology on complex analytic extensions, there exists a domain U around $0 \in \mathbb{R}^p$ and a map

$$\Phi : M \times U \to M \times \mathbb{R}^p,$$

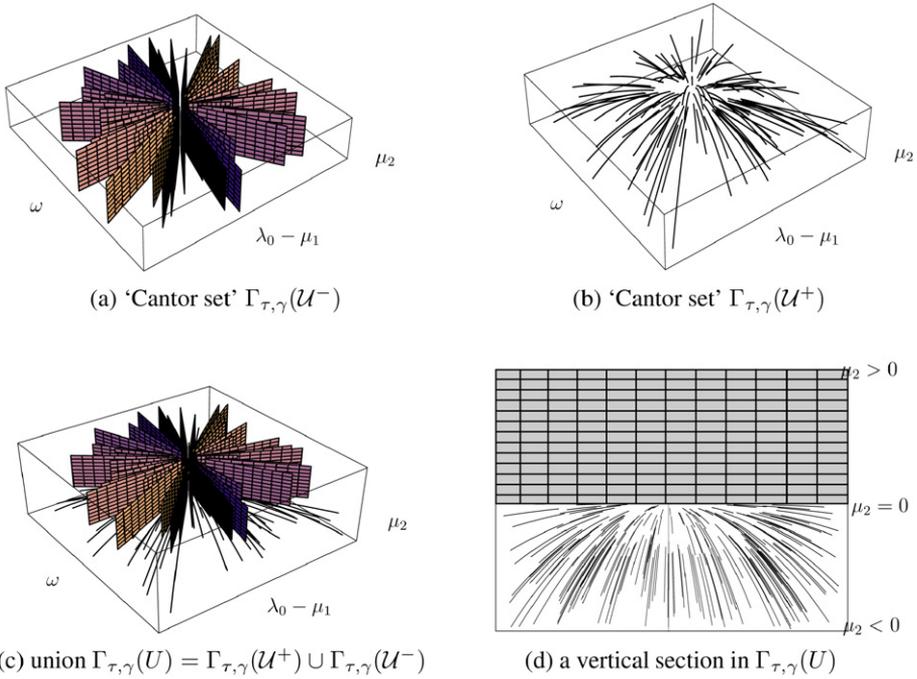*defined near the torus $(y, z, \mu) = (0, 0, 0)$ such that*

(a) 'Cantor set' $\Gamma_{\tau,\gamma}(\mathcal{U}^-)$

(b) 'Cantor set' $\Gamma_{\tau,\gamma}(\mathcal{U}^+)$

(c) union $\Gamma_{\tau,\gamma}(U) = \Gamma_{\tau,\gamma}(\mathcal{U}^+) \cup \Gamma_{\tau,\gamma}(\mathcal{U}^-)$

(d) a vertical section in $\Gamma_{\tau,\gamma}(U)$

Fig. 3. Sketch of the 'Cantor sets' $\Gamma_{\tau,\gamma}(\mathcal{U}^+)$ and $\Gamma_{\tau,\gamma}(\mathcal{U}^-)$ corresponding to $\mu_2 \geqslant 0$ and $\mu_2 < 0$, respectively, in the (ω, μ_1, μ_2)-space. The union $\Gamma_{\tau,\gamma}(U) = \Gamma_{\tau,\gamma}(\mathcal{U}^-) \cup \Gamma_{\tau,\gamma}(\mathcal{U}^+)$ is sketched in (c). The half planes in (b) and (c) give continua of invariant m-tori. In (d) a section of the 'Cantor set' $\Gamma_{\tau,\gamma}(U)$, along the μ_2-axis, is singled out: the grey region above corresponds to a half plane given in (c): this is a continuum of n-tori.

  i. Φ is a C^∞-near-identity diffeomorphism onto its image;

 ii. *The image of the Diophantine tori*

$$V = \bigcup_{\mu \in \Gamma_{\tau,\gamma}(U')} \mathbb{T}^n \times \{y = 0\} \times \{z = 0\} \times \{\mu\}$$

under Φ *is* $\widetilde{X}$*-invariant, and for the restriction* $\widehat{\Phi}|_{T_{\mathrm{loc}}}$ *we have*

$$\widehat{\Phi}_*(X) = \widetilde{X},$$

that is, $\widehat{\Phi}$ *conjugates* X *to* $\widetilde{X}$;

iii. *The restriction* $\Phi|_V$ *is symplectic and preserves the* (*symplectic*) *normal linear part of* X.

5. Application: Nearly-integrable perturbations of the Lagrange top

In the present section, we illustrate an application of Corollary 4.2 to the Lagrange top coupled with a quasi-periodic oscillator. We consider a local perturbation problem concerning the persistence of the quasi-periodic invariant tori associated with the upward spinning of the top and the quasi-periodic motion of the forcing. Our main interest is with the situation where the Lagrangian top becomes gyroscopically stabilized by Hamiltonian Hopf bifurcation [35]. For a detailed discussion on this problem see [19]. The Lagrange top is an axially symmetric rigid

body, situated in a constant gravitational field pointing downwards vertically, with its base point fixed in space, see, e.g., [34,42,61]. The configuration space of the top is the 3-dimensional Lie group $SO(3)$ of all (orientation-preserving) rotations of $\mathbb{R}^3$ and the phase space is the tangent bundle $TSO(3) \cong SO(3) \times \mathfrak{so}(3)$. The Lagrange top is described by a Hamiltonian system in the phase space $M = (SO(3) \times \mathfrak{so}(3), \sigma)$, where the symplectic form σ is inherited from the canonical 2-form of the bundle $T^*SO(3)$, compare with [1,34,81]. The Hamiltonian is defined as the total energy of the system, i.e., as the sum of the kinetic and potential energy of the top. The Lagrange top has two rotational symmetries: the rotations about the body-symmetric (or figure) axis and about the vertical axis. Let $\mathcal{S} \subset SO(3)$ denote the subgroup of rotations preserving the vertical axis. Then, for a suitable choice of the fixed space-coordinate system, the two symmetries correspond to a symplectic right action Φ^r and to a symplectic left action Φ^ℓ of the Lie subgroup $\mathcal{S}$ on M. By the Noether Theorem [1,7], these Hamiltonian symmetries give rise to integrals $\mathcal{M}^r$ and $\mathcal{M}^\ell$ of the Hamiltonian H: the angular momenta along the figure axis and along the vertical axis, respectively. These integrals induce the so-called energy–momentum map $\mathcal{EM} = (H, \mathcal{M}^r, \mathcal{M}^\ell) : M \to \mathbb{R}^3$. The inverse images of the map $\mathcal{EM}$ give a singular foliation of the phase space consisting of X_H-invariant tori, see [34], which provides a qualitative picture of the dynamics of the top.

Next, we consider a perturbation problem where the Lagrange top is weakly coupled to a quasi-periodic oscillator, e.g., the base point of the top is coupled to a vibrating table-surface by a massless spring, see Fig. 4. As said before, in this case the spring constant ε acts as a perturbation. More generally, we consider the perturbed Hamiltonian H_ε of the form

$$H_\varepsilon = H + G + \varepsilon F, \tag{53}$$

where H and G are the Hamiltonians of the Lagrange top and of the oscillator, respectively. The function F depends on the coupling between the top and the oscillator. We assume that the oscillator is Liouville-integrable and has $n \geqslant 1$ frequencies. Gyroscopic stabilization corresponds to a normally resonant invariant torus in the phase space. Near this torus the new phase space $M \times \mathbb{T}^n \times \mathbb{R}^n$ has a singular foliation consisting of invariant $(n+1)$-, $(n+2)$- and $(n+3)$-tori. This local foliation gives a stratification by tori in the (a, b, h)-space, where (a, b, h) are the values of $(\mathcal{M}^r, \mathcal{M}^\ell, H)$, as sketched in Fig. 5. Our present concern is with the fate of these invariant $(n+1)$-tori for small but nonzero ε. In particular, we address this persistence problem in the case when the Lagrange top is close to gyroscopic stabilization. We speak of gyroscopic stabilization when the unstable spinning motion of a vertical top becomes stable as the angular-
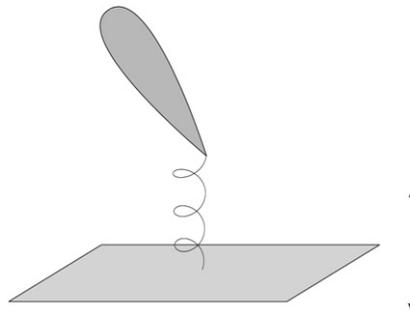

Fig. 4. The Lagrange top coupled with a vibrating table-surface by a spring.
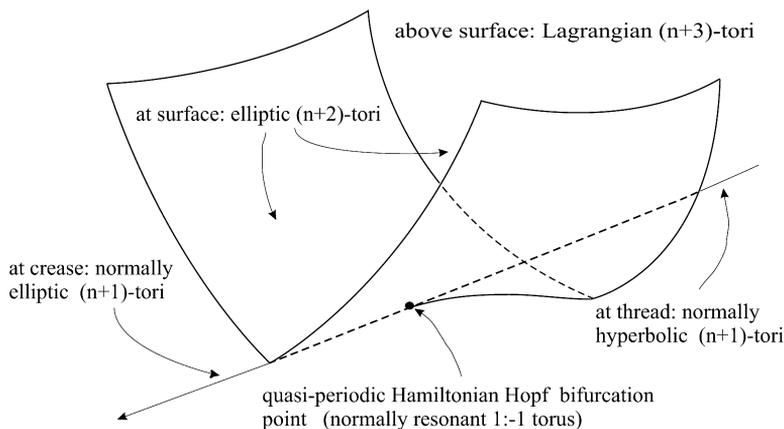
Fig. 5. Local stratification by invariant tori in a suitable parameter space, near gyroscopic stabilization, of the Lagrange top coupled with a quasi-periodic oscillator.

momentum value a increases and passes through a critical value a_0. This physical phenomenon is explained by a generic Hamiltonian Hopf bifurcation, see [34,35,64].

Before addressing the perturbation problem, let us first consider the local dynamics of the (unperturbed) Lagrange top close to gyroscopic stabilization. To this end, one applies a regular reduction [1,62] by the right symmetry Φ^r to the three-degrees-of-freedom Hamiltonian H as follows. For a fixed value a, we deduce from H a two-degrees-of-freedom Hamiltonian H_a on the four-dimensional orbit space $M_a = (\mathcal{M}^r)^{-1}(a)/\mathcal{S}$ under the Φ^r-action. This space M_a can be identified with the four-dimensional submanifold $\mathcal{R}_a = \{(u, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \colon u \cdot u = 1,\ u \cdot v = a\}$ with the induced symplectic form ω_a given by

$$\omega_a(u, v)\big((x, y), (p, q)\big) = \hat{x} \cdot \hat{q} - \hat{y} \cdot \hat{p} + v \cdot (\hat{v} \times \hat{p}), \tag{54}$$

where $(x, y) = (\hat{x} \times u, \hat{x} \times v + \hat{y}) \in T_{(u,v)}\mathcal{R}_a$, $(p, q) = (\hat{p} \times u, \hat{p} \times v + \hat{q}) \in T_{(u,v)}\mathcal{R}_a$. Here $\cdot$ and $\times$ denote the standard inner and cross product of $\mathbb{R}^3$, respectively. The reduced Hamiltonian $H_a \colon \mathcal{R}_a \to \mathbb{R}$ thereby obtains the form

$$H_a(u, v) = \frac{1}{2} v \cdot v + c u_3 + \frac{1}{2}(\kappa - 1)a^2,$$

where $\cdot$ denotes the standard inner product and $c, \kappa > 0$. Observe that $p_a = (0, 0, 1, 0, 0, a) \in \mathcal{R}_a$ is an isolated (relative) equilibrium, corresponding to a periodic solution of the full Hamiltonian system.

In suitable local Darboux coordinates $z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$ near the critical point p_a, the reduced Hamiltonian H_a reads

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_2 + O\big(|z|^4\big),$$

where $\mathcal{H}_0 = H_a(p_a)$ is the constant part and $\mathcal{H}_2$ is the quadratic terms given by

$$\mathcal{H}_2 = \frac{1}{8}\big(a^2 - 4c\big)\big(z_1^2 + z_2^2\big) + \frac{1}{2}\big(z_3^2 + z_4^2\big) + \frac{1}{2}a(z_2 z_3 - z_1 z_4),$$

see [34]. From this quadratic part, we see that gyroscopic stabilization takes place as the parameter $\mu_2 = \frac{1}{4}a^2 - c$ passes through the critical value $\mu_2 = 0$. Re-including the rotation about the figure axis in the reduced local model $\mathcal{H}$, we recover a full local Hamiltonian model H_{loc} for motions near the gyroscopic stabilization as follows. Let q_1 and p_1 denote the rotation and the angular momentum about the figure axis, respectively. Following [75], the variables $(q_1, p_1) \in \mathbb{S}^1 \times \mathbb{R}$, together with the local coordinates $(z_1, z_2, z_3, z_4) \in \mathbb{R}^4$, provide local symplectic coordinates in a neighborhood of the singularity p_a, such that, the symplectic 2-form is given by $dq_1 \wedge dp_1 + dz_1 \wedge dz_3 + dz_2 \wedge dz_4$ and the Hamiltonian H of the Lagrange top locally takes the form:

$$H(q_1, p_1, z) = \left(\frac{1}{2}\kappa(a + p_1)^2 + c\right) + \frac{1}{8}\left((a + p_1)^2 - 4c\right)\left(z_1^2 + z_2^2\right)$$

$$+ \frac{1}{2}(a + p_1)(z_2 z_3 - z_1 z_4) + \frac{1}{2}\left(z_3^2 + z_4^2\right) + O\left(|z|^4\right). \tag{55}$$

Now let us turn back to our perturbation problem (53) for $\varepsilon \neq 0$. Since our interest is with perturbations of the Lagrange top near gyroscopic stabilization, we use the local model (55) for the unperturbed Hamiltonian of the Lagrangian top. Then, the perturbed Hamiltonian system H_ε reads

$$H_\varepsilon(p_1, z, \xi, \eta) = H(q_1, p_1, z) + G(\eta) + \varepsilon F(q_1, p_1, z, \xi, \eta, \varepsilon), \tag{56}$$

where $(q_1, p_1, z) \in \mathbb{S} \times \mathbb{R} \times \mathbb{R}^4$ and where $(\xi, \eta) \in \mathbb{T}^n \times \mathbb{R}^n$ denote angle-action coordinates of the oscillator. The Hamiltonian vector field[5] X^ε associated to H_ε is given by

$$X^\varepsilon: \begin{cases} \dot{q}_1 = \kappa(a + p_1) + O(|z|^2) + O(\varepsilon), \\ \dot{p}_1 = O(\varepsilon), \\ \dot{\xi}_1 = \sigma_1(\eta) + O(\varepsilon), \\ \dot{\eta}_1 = O(\varepsilon), \\ \vdots \\ \dot{\xi}_n = \sigma_n(\eta) + O(\varepsilon), \\ \dot{\eta}_n = O(\varepsilon), \\ \dot{z} = \Omega_{a,c} z + O(|p_1|, |z|^3) + O(\varepsilon), \end{cases} \tag{57}$$

where $\sigma(\theta)$ gives the frequencies of the quasi-periodic motion of the forcing restricted to the invariant torus $\eta = \theta$, and where the Floquet matrix $\Omega_{a,c} \in \mathfrak{sp}(4, \mathbb{R})$ is of the form

$$\Omega_{a,c} = \begin{pmatrix} 0 & \mu_1 & 1 & 0 \\ -\mu_1 & 0 & 0 & 1 \\ -\mu_2 & 0 & 0 & \mu_1 \\ 0 & -\mu_2 & -\mu_1 & 0 \end{pmatrix}, \tag{58}$$

---

[5] With respect to the symplectic form $dq_1 \wedge dp_1 + d\xi \wedge d\eta + dz^2$.

where $\mu_1 = \frac{1}{2}a$ and $\mu_2 = \frac{1}{4}a^2 - c$. Near the invariant quasi-periodic n-torus $\{\eta = \theta\}$ of the forcing, we introduce the local coordinate $\tilde{\eta} = \eta - \theta$. Our goal finally is to study the persistence of the invariant $(n + 1)$-torus $\{p_1 = 0, \ \tilde{\eta} = 0, \ z = 0\}$. To this end, we let

$$x = (q_1, \xi) = (q_1, \xi_1, \ldots, \xi_n) \in \mathbb{T}^1 \times \mathbb{T}^n, \tag{59}$$

$$y = (p_1, \tilde{\eta}) = (p_1, \tilde{\eta}_1, \ldots, \tilde{\eta}_n) \in \mathbb{R}^1 \times \mathbb{R}^n. \tag{60}$$

Also, we write

$$\omega = \omega(a, \theta) = \big(\kappa a, \sigma_1(\theta), \ldots, \sigma_n(\theta)\big) \in \mathbb{R}^{n+1} \tag{61}$$

for the internal frequencies of the invariant $(n + 1)$-torus

$$\big\{(x, y, z) \in \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4 \colon (y, z) = (0, 0)\big\}$$

of the unperturbed Hamiltonian system X^0. In the new coordinates (x, y, z), the perturbed Hamiltonian $X^\varepsilon = X^\varepsilon(x, y, z)$ reads

$$
\begin{aligned}
X^\varepsilon(x, y, z) = {} & \big[\omega(a, \theta) + O\big(|y|, |z|^2\big) + O(\varepsilon)\big]\frac{\partial}{\partial x} + O(\varepsilon)\frac{\partial}{\partial y} \\
& + \big[\Omega_{a,c}z + O\big(|y|, |z|^2\big) + O(\varepsilon)\big]\frac{\partial}{\partial z},
\end{aligned}
\tag{62}
$$

where the matrix $\Omega_{a,c}$ is given by (58). By considering the conserved quantities (a, θ) and the physical quantities (c, κ) as parameters, the unperturbed system X^0 becomes an $(n + 3)$-parameter family of vector fields on $\mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4$ parameterized by

$$\nu = (\theta, a, c, \kappa) \in \mathbb{R}^{n+1} \times \mathbb{R}^2.$$

Given all this, application of Corollary 4.2 gives the following stability theorem regarding the invariant $(n + 1)$-tori $\{(y, z) = (0, 0)\}$:

Theorem 5.1 (*Forced Lagrange top: persistence of Diophantine $(n + 1)$-tori*). *Let $\nu_0 = (\theta_0, \pm 2\sqrt{c_0}, c_0, \kappa_0) \in \mathbb{R}^{n+3}$, where θ_0, c_0, κ_0 are fixed with $c_0, \kappa_0 > 0$. Assume that the map*

$$\eta \in \mathbb{R}^n \mapsto \big(\sigma_1(\eta), \ldots, \sigma_n(\eta)\big) \in \mathbb{R}^n \tag{63}$$

is a local diffeomorphism at $\eta = \theta_0$. Then, for sufficiently small $|\varepsilon|$, there exists a neighborhood U of ν_0 in the parameter space $\mathbb{R}^{n+3}$, and a local map

$$\Phi : \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4 \times U \to \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \times \mathbb{R}^4 \times \mathbb{R}^{n+3}$$

defined near the torus $\mathbb{T}^{n+1} \times \{y = 0\} \times \{z = 0\} \times \{\nu_0\}$, such that,

   i. *Φ is a C^∞-smooth diffeomorphism onto its image and is C^∞-near the identity map*;
  ii. *The image $\widetilde{V}$ of invariant Diophantine $(n+1)$-tori*

$$V = \bigcup_{\nu \in \Gamma_{\tau,\gamma}(U')} \left(\mathbb{T}^{n+1} \times \{y = 0\} \times \{z = 0\} \times \{\nu\} \right)$$

   of the unperturbed system X^0 under the map Φ is X^ε-invariant, and the restriction of Φ to V induces a conjugacy between X^0 and X^ε;
 iii. *The normal linear part of the perturbed family X^ε on $\widetilde{V}$ is (symplectically) conjugate to $NX^0 = \omega \frac{\partial}{\partial x} + \Omega z \frac{\partial}{\partial z}$, where $\nu = (\theta, a, c, \kappa)$ is restricted to the 'Cantor set' $\Gamma_{\tau,\gamma}(U')$.*

   Theorem 5.1 roughly says that the quasi-periodic motions of a weakly forced Lagrange top can be predicted, based on the uncoupled motion of the top and the forcing. More precisely, when $\varepsilon = 0$, we have a family of invariant $(n+1)$-tori carrying quasi-periodic flow generated by the vertical upwards spinning of the top and the quasi-periodic motion of the forcing in the phase space. Theorem 5.1 says that the 'majority' of these quasi-periodic invariant $(n+1)$-tori survives when ε is sufficiently small. Moreover, by conclusion (iii), the local dynamics of the unperturbed system X^0, near the surviving invariant $(n+1)$-tori, also is preserved.

   For completeness, we mention that invariant $(n+1)$-tori of X^0 in the phase space are 'surrounded' by invariant elliptic $(n+2)$- and Lagrangian $(n+3)$-dimensional tori. The local Hamiltonian H for the Lagrange top, see (55), undergoes a generic Hamiltonian Hopf bifurcation at $\mu_2 = 0$. Following [19,64], the projection of this foliation by tori into the (a, b, h)-space, near the Hamiltonian Hopf bifurcation point, is a piece of swallowtail catastrophe set [35], see Fig. 5. Now the crease (i.e., the singular part of the surface) and the thread correspond to invariant $(n+1)$-tori; the smooth part of the surface is associated with the elliptic $(n+2)$-tori; the open region above the surface gives rise to invariant Lagrangian $(n+3)$-tori. Persistence of these higher-dimensional isotropic tori can be obtained by using 'standard' KAM theory [2,26,55,71,78], for a detailed treatment of this see [18,19].

Remarks 5.2.

   1. The total union of perturbed $(n+3)$-, $(n+2)$- and $(n+1)$-tori has been called a Cantor stratification, see [27] and references given there. The relative measure of the parameter regimes with invariant $(n+3)$-, $(n+2)$- and n-tori tends to full measure when approaching the 'strata' of co-dimension 1, 2 and 3, respectively, that here occur as (secondary) bifurcation sets. Here we take the Hausdorff measure of the appropriate dimension. Compare with Fig. 5. This can be shown by repeated normalization (averaging) and 'playing' with the Diophantine parameter $\gamma > 0$ in an appropriate way, compare with Remark 1.5(2). In [25] such phenomena have been described in terms of *density points* of j-quasi-periodicity $j = n+1, n+2, n+3$. Exploiting the real analyticity, it even can be shown that the full relative measure is attained in an exponentially fast way, in which case the phenomenon also is described in terms of *exponential condensation*. For a discussion of these notions and many references compare with [27].
   2. The total number of parameters needed for the occurrence of a generic quasi-periodic Hamiltonian Hopf bifurcation is $n + 3 = (n+1) + 2$, which in this example exactly is the number of degrees of freedom.

On the one hand one may wish to view the *n* forcing frequencies as external parameters. On the other hand they may be 'compensated' by action variables conjugate tot the corresponding angles; this would be a typical example of distinguished parameters. In both set-ups one needs three extra parameters, which might be provided by the triple (a, b, h). In any case, in individual systems of at least 5 degrees of freedom, the quasi-periodic Hamiltonian Hopf bifurcation occurs as a typical phenomenon.

Acknowledgments

Appendix A. Hölder condition on spectra of Floquet matrices

In the proofs of the main Theorems 1.4, 1.6 and 1.8 we use a Hölder condition on the spectrum Spec $\Omega(\mu)$ of the unfolding $\mu \in \mathbb{R}^p \mapsto \Omega(\mu) \in \mathfrak{gl}(m, \mathbb{R})$, as this occurs in the leading, normal linear part

$$N_\mu(X) = \omega(\mu)\frac{\partial}{\partial x} + \Omega(\mu)y\frac{\partial}{\partial y}$$

of (1). Also compare with the symplectic and reversible counterparts in Sections 1.2 and 1.3, where in both cases m just has to be replaced by $2q$.

Since the family $X = X_\mu(x, y)$, $x \in \mathbb{T}^n$, $y \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$ of vector fields is real analytic in all variables, the map $\mu \mapsto \Omega(\mu)$ has a holomorphic extension to a domain $G \subset \mathbb{C}^p$.

Theorem A.1 (*Hölder condition on Floquet spectrum*). *For any $\tilde{\mu} \in G$ and $\mu \in G \cap \mathbb{R}^p$ and for any $\tilde{\lambda} \in \operatorname{Spec} \Omega(\tilde{\mu})$, there exists an eigenvalue $\lambda \in \operatorname{Spec} \Omega(\mu)$ such that*

$$|\lambda - \tilde{\lambda}| \leqslant L|\mu - \tilde{\mu}|^{1/m}, \tag{A.1}$$

for a positive constant L.

In the KAM proof in Appendix B, we shall use the weaker statement that

$$|\operatorname{Im}\lambda - \operatorname{Im}\tilde{\lambda}| \leqslant L|\mu - \tilde{\mu}|^{1/m},$$

see (5). A proof of Theorem A.1 is given now. It is closely related to [41] and based on application of the Rouché lemma to the characteristic polynomial of $\Omega(\mu)$.

We start with the statement of the Rouché lemma, that is quite standard in complex analysis.

Lemma A.2. (E. Rouché, 1862) *Let $G \subset \mathbb{C}$ be a domain bounded by a closed curve ∂G (not necessarily connected), and let F, $f : \overline{G} \to \mathbb{C}$ be two holomorphic functions. If $|F(z)| > |f(z) - F(z)|$ for every $z \in \partial G$ then the functions F and f possess the same number of zeros inside G (counting multiplicities).*

Theorem A.3. *Let G be a domain in $\mathbb{C}$ and let p be a polynomial of degree n with leading coefficient $s \neq 0$. Suppose that all the roots of p lie inside G at distances no less than $a^{1/n}$ from ∂G, where $a > 0$ is a certain number. Let $f : \overline{G} \to \mathbb{C}$ be a holomorphic function such that $|f(z) - p(z)| < a|s|$ for every $z \in \overline{G}$. Then for each root z_0 of the polynomial p, there exists a zero z_0' of the function f such that $|z_0' - z_0| < (2n - 1)a^{1/n}$.*

Proof. Denote by $D \subset G$ the union of open disks of radius $a^{1/n}$ centered at the roots of the polynomial p, and let D_0 be the connected component of D containing z_0. Each point z' of the curve ∂D_0 lies at a distance of $a^{1/n}$ from one of the roots of the polynomial p and at distances no less than $a^{1/n}$ from the remaining roots. Thus, $|p(z')| \geqslant a|s|$. According to Lemma A.2, the number of the zeros of the function f inside D_0 is equal to the number of the roots of the polynomial p inside D_0. In particular, this number is positive. Let z_0' be one of the zeros of the function f inside D_0.

The point z_0' belongs to the same connected component of the set D as z_0 does. Hence, there exists a sequence of roots $z_0, z_1, \ldots, z_k$ of the polynomial p such that $|z_1 - z_0| < 2a^{1/n}$, $|z_2 - z_1| < 2a^{1/n}, \ldots, |z_k - z_{k-1}| < 2a^{1/n}$, and $|z_0' - z_k| < a^{1/n}$. Since $k \leqslant n - 1$, one obtains $|z_0' - z_0| < (2k + 1)a^{1/n} \leqslant (2n - 1)a^{1/n}$. $\square$

Theorem A.4. *Let K be a compact metric space, and let*

$$p_t(z) = z^n + q_1(t)z^{n-1} + \cdots + q_{n-1}(t)z + q_n(t)$$

be a polynomial in z whose coefficients $q_j : K \to \mathbb{C}$ $(1 \leqslant j \leqslant n)$ are Lipschitz continuous. Then there exists a number $C > 0$ such that for any $t_1, t_2 \in K$ $(t_1 \neq t_2)$ and for each root z_0 of the polynomial p_{t_1}, there is a root z_0' of the polynomial p_{t_2} for which $|z_0' - z_0| < C[\rho(t_1, t_2)]^{1/n}$, where $\rho(t_1, t_2)$ denotes the distance between t_1 and t_2 in K.

Proof. Let G be a domain in $\mathbb{C}$ such that for any $t \in K$, all the roots of the polynomial p_t lie inside G at distances no less than 1 from ∂G. There exists a number $c > 0$ such that for any $t_1, t_2 \in K$ $(t_1 \neq t_2)$ and for each $z \in \overline{G}$, the inequality $|p_{t_1}(z) - p_{t_2}(z)| < c\rho(t_1, t_2)$ holds. Obviously, it suffices to verify the conclusion of the theorem for $t_1, t_2 \in K$ such that $\rho(t_1, t_2) \leqslant c^{-1}$.

Thus, suppose that $t_1, t_2 \in K$ $(t_1 \neq t_2)$ and $\rho(t_1, t_2) \leqslant c^{-1}$. Set $a = c\rho(t_1, t_2) \leqslant 1$. All the roots of the polynomial p_{t_1} lie inside G at distances no less than $1 \geqslant a^{1/n}$ from ∂G. On the other hand, $|p_{t_2}(z) - p_{t_1}(z)| < c\rho(t_1, t_2) = a$ for every $z \in \overline{G}$. According to Theorem A.3, for each root z_0 of the polynomial p_{t_1} there is a root z_0' of the polynomial p_{t_2} such that

$$|z_0' - z_0| < (2n - 1)a^{1/n} = (2n - 1)\big[c\rho(t_1, t_2)\big]^{1/n}.   \quad \square$$

Theorem A.1 is a direct corollary of Theorem A.4, applied to the characteristic polynomial of Ω.

Appendix B. Proof of Theorem 1.6

We now present a proof of Theorem 1.6 in the Hamiltonian setting. A direct transcription gives proofs of the reversible and dissipative counterpart Theorems 1.8 and 1.4. For further details in the reversible case also see [17,31].

The section will be split into four parts. In the first part we sketch the idea behind the proof and its frame. In the second part, for a Newton-like iteration process, we define a sequence of approximations $\{\Phi_j\}_{j \geqslant 0}$ of the desired map Φ in Theorem 1.6. Here the linear versal unfolding theory as developed in Section 2 will be used; indeed, the linear centralizer unfoldings (the LCU's) enable us to solve the so-called homological equations, used for the definition of the $\{\Phi_j\}_{j \geqslant 0}$. The maps Φ_j are defined on complex domains and will be constructed in such a way that the Inverse Approximation Lemma [78,102,103] applies, yielding that the limit Φ_∞ is a Whitney-smooth map.[6] The third part of this section concerns with the control of errors per iteration step, where in view of the Inverse Approximation Lemma, we have to make sure that the differences $|\Phi_j - \Phi_{j+1}|$ decrease in an exponential way. In this part, the Diophantine conditions on frequencies come into play. In the fourth and final part we discuss the convergence of the iteration process, where a proper choice will be made for the complex domains, introduced in the second part. Our proof closely follows that of [26,55,71,79].

B.1. Preliminaries

B.1.1. Reduction to a special case

As a starting point for the proof for Theorem 1.6, we consider the symplectic manifold $(M, \sigma) = (\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} = \{x, y, z\}, dx \wedge dy + dz^2)$, assuming that the integrable family $X = X_{\omega,\mu} \in \mathcal{X}_P^\sigma(M)$ is given by

$$X_{\omega,\mu}(x, y, z) = \omega \frac{\partial}{\partial x} + \Omega(\mu) z \frac{\partial}{\partial z}, \tag{B.1}$$

where $\Omega(\mu)$ is a LCU of $\Omega_0 = \Omega(0)$ and where $(\omega, \mu) \in P \subset \mathbb{R}^n \times \mathbb{R}^c$ (as an open subset) with $c = \mathrm{cod}\,\Omega_0$. In view of Section 3.1 and Lemma 3.1, these special choices do not limit the generality of our proof for Theorem 1.6. Furthermore, we observe that it is sufficient to prove Theorem 1.6 for $\gamma = \gamma_0$, for a positive constant γ_0. This can be realized by rescaling the time t to $\frac{\gamma_0}{\gamma} t$ and by stretching the parameters (ω, μ) to $(\frac{\gamma_0}{\gamma}\omega, \frac{\gamma_0}{\gamma}\mu)$, compare with [26,55,71,78]. We will come back to this later on.

B.1.2. Compact-open neighborhoods

Here we specify the compact-open neighborhoods of the $(n + c)$-parameter integrable family $X \in \mathcal{X}_P^\sigma(M)$ given by (B.1). To this end, for any subset $I \subseteq \mathbb{R}^k$ and $\rho > 0$, we introduce the complex 'strip' $I + \rho$ defined as

$$I + \rho = \big\{ z \in \mathbb{C}^k \colon \exists x \in I \text{ such that } |x_j - z_j| \leqslant \rho, \text{ for all } 0 \leqslant j \leqslant k \big\}.$$

Similarly we introduce $\mathbb{T}^n + \kappa \subset \mathbb{C}^n/(2\pi\mathbb{Z})^n$, where $\kappa > 0$. Since the family X is real analytic, it has a holomorphic extension to a compact neighborhood $\mathcal{N}$ of the form

$$\mathcal{N} = \big(\mathbb{T}^n + \kappa\big) \times \Delta_y \times \Delta_z \times (U + \rho) \subset \mathbb{C}^n/(2\pi\mathbb{Z})^n \times \mathbb{C}^n \times \mathbb{C}^4 \times \mathbb{C}^{n+c} \tag{B.2}$$

with κ and ρ small positive constants. Here the set $\Delta_y \times \Delta_z$ is a compact neighborhood of $(y, z) = (0, 0) \in \mathbb{C}^n \times \mathbb{C}^{2q}$ and U a compact neighborhood of $\mu = 0 \in P \subset \mathbb{R}^n \times \mathbb{R}^c$ (as an

---

[6] For background on Whitney-smoothness, see [94,100]

open subset). Now a compact-open neighborhood $\mathcal{A}$ of X in $\mathcal{X}_P^\sigma(M)$ is specified as follows. For a positive constant δ given by the proof below and the parameter $\gamma > 0$ of the Diophantine conditions (2), the set $\mathcal{A} = \mathcal{A}_{\gamma,\delta}$ consists of families of the forms

$$(\omega + f)\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + \left(\Omega(\mu)z + h\right)\frac{\partial}{\partial z} \in \mathcal{X}_P^\sigma(M),$$

where the functions f, g and h of (x, y, z, ω, μ) have holomorphic extensions to the compact domain $\mathcal{N}$, such that,

$$|f|_\mathcal{N} < \gamma\delta, \quad |g|_\mathcal{N} < \gamma\delta^2 \quad \text{and} \quad |h|_\mathcal{N} < \gamma\delta^2. \tag{B.3}$$

Here $|\cdot|_\mathcal{N}$ denotes the supremum norm on the complex domain $\mathcal{N}$. We shall show that Theorem 1.6 holds for perturbations $\widetilde{X}$ of X in such a neighborhood $\mathcal{A}$ of X.

B.1.3. Taylor- and Fourier-truncation

Suppose that the Hamiltonian vector field $Y = Y(x, y, z) \in \mathcal{X}^\sigma(M)$ has the form

$$Y(x, y, z) = f(x, y, z)\frac{\partial}{\partial x} + g(x, y, z)\frac{\partial}{\partial y} + h(x, y, z)\frac{\partial}{\partial z}.$$

For simplicity we here suppress the parameter. Associated to Y, we introduce the following Taylor- and Fourier-truncations. For arbitrary non-negative integer d,

$$Y_{\text{lin}}(x, y, z) = f(x, 0, 0)\frac{\partial}{\partial x} + \left[g(x, 0, 0) + \frac{\partial g}{\partial y}(x, 0, 0)y + \frac{\partial g}{\partial z}(x, 0, 0)z\right.$$

$$\left. + \frac{1}{2}\frac{\partial^2 g}{\partial z^2}(x, 0, 0)z^2\right]\frac{\partial}{\partial y} + \left[h(x, 0, 0) + \frac{\partial h}{\partial z}(x, 0, 0)z\right]\frac{\partial}{\partial z},$$

$$(Y)_d(x, y, z) = \sum_{|k|\leqslant d} e^{i\langle x, k\rangle}\left(f_k(y, z)\frac{\partial}{\partial x} + g_k(y, z)\frac{\partial}{\partial y} + h_k(y, z)\frac{\partial}{\partial z}\right), \tag{B.4}$$

where

$$\frac{\partial^2 g}{\partial z^2}z^2 = \left(\left\langle\frac{\partial^2 g_1}{\partial z^2}z, z\right\rangle, \dots, \left\langle\frac{\partial^2 g_n}{\partial z^2}z, z\right\rangle\right) \in \mathbb{R}^n$$

and where f_k, g_k and h_k are the kth Fourier coefficients of the f, g and h with respect to the angle variable x. Here $\langle\cdot,\cdot\rangle$ denotes the standard inner product of $\mathbb{R}^n$. We denote by $\mathcal{X}_{\text{lin}}^\sigma$ and $\mathcal{X}_d^\sigma$ the sets of these truncations. One directly shows that $Y_{\text{lin}}, (Y)_d \in \mathcal{X}^\sigma(M)$, whenever $Y \in \mathcal{X}^\sigma(M)$, compare with [26,55].

Remark B.1. In the reversible case the similar property is that $Y_{\text{lin}}, (Y)_d \in \mathcal{X}^{-G}(M)$, whenever $Y \in \mathcal{X}^{-G}(M)$, compare [23]. For this use of notation see Section 3.4. In general this property is another axiom of the theory [26,55], which amounts to homogeneity of the structure that has to preserved (symplectic or volume form, symmetry, etc.) with respect to well-chosen coordinates.

B.2. Idea behind the proof

Here we briefly discuss the idea of the proof. The unperturbed integrable family $X = X_{\omega,\mu}(y, z)$ is given by

$$X_{\omega,\mu}(y, z) = \omega \frac{\partial}{\partial x} + \Omega(\mu) z \frac{\partial}{\partial z},$$

where $\Omega(\mu) \in \mathfrak{sp}(2q, \mathbb{R})$ is a LCU of Ω_0 and where $(\omega, \mu) \in \mathbb{R}^n \times \mathbb{R}^c$ with $c = \operatorname{cod} \Omega_0$, see Section 2.1. Let the family $\widetilde{X} \in \mathcal{X}_P^\sigma(M)$ be nonintegrable small perturbation of the family X. To prove Theorem 1.6, we need to find a C^∞-near-identity symplectic map $\Phi : M \times P \to M \times P$, such that,

$$\Phi^* \widetilde{X} = X + \left[\tilde{f} \frac{\partial}{\partial x} + \tilde{g} \frac{\partial}{\partial x} + \tilde{h} \frac{\partial}{\partial z} \right], \tag{B.5}$$

where $\tilde{f} = O(|y|, |z|)$, $\tilde{g} = O(|y|, |z|)$ and $\tilde{h} = O(|y|, |z|^2)$. Indeed, the conjugacy relation (B.5) implies that the perturbation $\widetilde{X}$ possesses a family of invariant tori, which is the image $\widetilde{V}$ of the X-invariant tori $(y, z) = (0, 0)$ under the map Φ, compare with conclusion (ii) of Theorem 1.6. The family of $\widetilde{X}$-invariant tori is, however, parametrized over a 'Cantor set' of parameters, determined by the Diophantine conditions (2). Moreover, the relation (B.5) also means that normal linear part of the perturbation $\widetilde{X}$ on the invariant tori $\widetilde{V}$ is conjugate to X, compare with conclusion (iii) of Theorem 1.6.

So our task is to construct a map Φ satisfying (B.5). Suppose that the map $\Phi = \Phi(x, y, z, \omega, \mu)$ is of the form

$$\Phi(x, y, z, \omega, \mu) = \big(x + U(x, \omega, \mu), y + V(x, y, z, \omega, \mu), z + W(x, z, \omega, \mu),$$
$$\omega + \Lambda_1(\omega, \mu), \mu + \Lambda_2(\omega, \mu) \big), \tag{B.6}$$

and that the perturbation $\widetilde{X}$, in coordinates $(\xi, \eta, \zeta, \sigma, \nu) = \Phi^{-1}(x, y, z, \omega, \mu)$, can be written as

$$\widetilde{X}_{\sigma,\nu}(\xi, \eta, \zeta) = \sigma + f(\xi, \eta, \zeta, \sigma, \nu) \frac{\partial}{\partial x} + g(\xi, \eta, \zeta, \sigma, \nu) \frac{\partial}{\partial \eta}$$
$$+ \big[\Omega(\nu) \zeta + h(\xi, \eta, \zeta, \sigma, \nu) \big] \frac{\partial}{\partial \zeta}. \tag{B.7}$$

The conjugacy relation (B.5) implies that

$$\begin{cases} \frac{\partial U}{\partial x}(\omega + \tilde{f}) + \tilde{f} = \Lambda_1 + f, \\ (1 + \frac{\partial V}{\partial y})\tilde{g} + \frac{\partial V}{\partial x}(\omega + \tilde{f}) + \frac{\partial V}{\partial z}(\Omega(\mu)z + \tilde{h}) = g, \\ (1 + \frac{\partial W}{\partial z})(\Omega(\mu)z + \tilde{h}) + \frac{\partial W}{\partial x}(\omega + \tilde{f}) = h + \Omega(\mu + \Lambda_2)(z + W), \end{cases} \tag{B.8}$$

where everything is expressed in the coordinates (x, y, z, ω, μ). This nonlinear equation is to be solved in U, V, W, Λ_1 and Λ_2 by an Newtonian iteration process, proposed by [2,59]. The corresponding Newtonian linearization of Eq. (B.8) is of the form

$$\begin{cases} \frac{\partial U}{\partial z}\omega = \Lambda_1 + f(x,0,0,\omega,\mu), \\[2mm] \frac{\partial V}{\partial x}\omega + \frac{\partial V}{\partial z}\Omega(\mu)z = g(x,0,0,\omega,\mu) + \frac{\partial g}{\partial y}(x,0,0,\omega,\mu)y \\[2mm] \qquad\qquad\qquad + \frac{\partial g}{\partial z}(x,0,0,\omega,\mu) + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(x,0,0,\omega,\mu)z^2, \\[2mm] \frac{\partial W}{\partial x}\omega + \frac{\partial W}{\partial z}\Omega(\mu)z - \Omega(\mu)W = h(x,0,0,\omega,\mu) + \frac{\partial h}{\partial z}(x,0,0,\omega,\mu)z + \Omega(\Lambda_2)z. \end{cases} \tag{B.9}$$

For more details on this kind of Newtonian linearization see, e.g., [2,71,83].

Remark B.2. In order to prove Theorem 1.6, by the conjugacy relation (B.8), one can replace the family X by any family $c_0 X = c_0 \omega \frac{\partial}{\partial x} + c_0 \Omega(\mu)z\frac{\partial}{\partial z}$ with a nonzero constant c_0. Indeed, suppose that $c_0 X$ satisfies (B.8). Then, after replacing $\tilde{f}$ by $c_0\tilde{f}$, etc., the family X fulfills (B.8) as well. For the present proof, we use the scaled families $c_0 X$ and $c_0 \widetilde{X}$ as the unperturbed family X and perturbation $\widetilde{X}$, respectively, where $c_0 = \frac{\gamma_0}{\gamma}$ with γ_0 a positive constant. Under such circumstances, the 'parameter' γ of the Diophantine conditions (2) is fixed to the constant γ_0. For simplicity, in what follows we will suppress c_0 in the proof and incorporate it again when necessary.

The idea of the proof is to inductively construct a sequence $\{\Phi_j\}_{j\geqslant 0}$ of approximations of Φ based on the Newtonian linearizations (B.9). The solution Φ for (B.5) will be obtained as a Whitney-smooth limit of this sequence by applying the Inverse Approximation Theorem [25,26,78,102,103]. All approximations Φ_j will be real analytic in the $\mathbb{T}^n$-direction and be C^∞-near-identity symplectic diffeomorphisms. To describe the construction of the sequence $\{\Phi_j\}$, we introduce the following notation. For any $j \geqslant 0$, whenever Φ_j is well defined, we denote by $(x_j, y_j, z_j, \omega_j, \mu_j) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \mathbb{R}^n \times \mathbb{R}^c$ the coordinates such that $\Phi_j(x_j, y_j, z_j, \omega_j, \mu_j) = (x, y, z, \omega, \mu)$. We initialize the sequence $\{\Phi_j\}$ by setting $\Phi_0 = \mathrm{Id}$. Assume that all Φ_j $(j \geqslant 1)$ are normally affine, that is, of the form

$$\Phi_j(x_j, y_j, z_j, \omega_j, \mu_j) = \big(x_j + \widetilde{U}^j(x_j,\omega_j,\mu_j),\, y_j + \widetilde{V}^j(x_j,y_j,z_j,\omega_j,\mu_j),$$
$$z_j + \widetilde{W}^j(x_j,z_j,\omega_j,\mu_j,\omega_j) + \widetilde{\Lambda}_1(\omega_j,\mu_j),\, \mu_j + \widetilde{\Lambda}_2(\omega_j,\mu_j)\big),$$

where

$$\widetilde{V}^j(x_j,y_j,z_j,\omega_j,\mu_j) = \widetilde{V}_0^j(x_j,\omega_j,\mu_j) + \widetilde{V}_1^j(x_j,\omega_j,\mu_j)y_j$$
$$+ \widetilde{V}_2^j(x_j,\omega_j,\mu_j)z + \frac{1}{2}\widetilde{V}_3^j(x_j,\omega_j,\mu_j)z_j^2, \tag{B.10}$$

$$\widetilde{W}^j(x_j,z_j,\omega_j,\mu_j) = \widetilde{W}_0^j(x_j,\omega_j,\mu_j) + \widetilde{W}_1^j(x_j,\omega_j,\mu_j)z_j. \tag{B.11}$$

Next we introduce a sequence of perturbations $\{\widetilde{X}_j\}_{j\geqslant 0}$ of X, where $\widetilde{X}_j = \Phi_j^*\widetilde{X}$. By construction, the families $\widetilde{X}_j = \widetilde{X}_{j(\omega_j,\mu_j)}(x_j,y_j,z_j)$ are of the form

$$\widetilde{X}_{j(\omega_j,\mu_j)}(x_j,y_j,z_j) = \Big[\omega_j + f^j(x_j,y_j,z_j,\omega_j,\mu_j)\Big]\frac{\partial}{\partial x_j} + g^j(x_j,y_j,z_j,\omega_j,\mu_j)\frac{\partial}{\partial y_j}$$
$$+ \Big[\Omega(\mu_j)z_j + h^j(x_j,y_j,z_j,\omega_j,\mu_j)\Big]\frac{\partial}{\partial z_j}. \tag{B.12}$$

In view of the Inverse Approximation Theorem, we must take care that both Φ_j and $\widetilde{X}_j$ have holomorphic extensions to a complex neighborhood $D_j^e \subseteq \mathcal{N}$ of the Cantor family $\mathbb{T}^n \times \{0\} \times \{0\} \times \Gamma_\gamma(U')$ such that the complex neighborhoods D_j^e shrinking for $j \to \infty$ in an appropriate way. Application of the Inverse Approximation Theorem then yields the Whitney-smooth limits Φ_∞ and $\widetilde{X}_\infty$, both defined on the closed set $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \Gamma_\gamma(U')$. Furthermore, we have to show that the vector field $\widetilde{X}_\infty = \Phi_\infty^* \widetilde{X}$ is of the form

$$\widetilde{X}_{\infty(\omega,\mu)}(x, y, z) = \left[\omega + O(|y|, |z|)\right]\frac{\partial}{\partial x} + O(|y|, |z|)\frac{\partial}{\partial y}$$

$$+ \left[\Omega(\mu)z + O(|y|, |z|^2)\right]\frac{\partial}{\partial z}.$$

Finally, an application of the Whitney Extension Theorem [94,99] to Φ_∞ provides us the desired map Φ, see [25,26,55,78]. Including the details, these steps lead us to a complete proof of Theorem 1.6.

Remark B.3. In the following, for simplicity we suppress the occurrence of parameters whenever they are not essential for our arguments.

B.3. Construction of the sequence $\{\Phi_j\}$

In this part we construct the sequence $\{\Phi_j\}_{j \geqslant 0}$ as mentioned above by a Newtonian iteration process. We first define the iteration relation by putting $\Phi_{j+1} = \Phi_j \circ \Psi_j$, where $\Psi_j : D_{j+1}^e \to D_j^e$. It follows that $\Phi_{j+1} = \Psi_0 \circ \cdots \circ \Psi_j$ and that $\widetilde{X}_{j+1} = \Psi_j^* \widetilde{X}_j$. Notice that $\Phi_0 = \mathrm{Id}$ and $\widetilde{X}_0 = \widetilde{X}$. Our aim is to construct the sequence $\{\Psi_j\}$ by using linearizations of the type (B.9). In the following, we simplify notation as follows. As long as we are concerned with one single iteration step, we suppress the occurrence of the index j. For instance, we write f for f_j, etc. The index $(j + 1)$ will be replaced by the plus-sign. So we replace the expressions f_{j+1}, D_{j+1}^e by f^+, D_+^e, etc. Moreover we abbreviate $(\xi, \eta, \zeta, \sigma, \nu) = (x_{j+1}, y_{j+1}, z_{j+1}, \omega_{j+1}, \mu_{j+1})$. Now let $\Psi : D_+^e \to D^e$ be of the form

$$\Psi(\xi, \eta, \zeta, \sigma, \nu) = \left([\exp \overline{\Psi}_{\sigma,\nu}](\xi, \eta, \zeta), \sigma + \Lambda_1(\sigma, \nu), \nu + \Lambda_2(\sigma, \nu)\right), \tag{B.13}$$

where the vector field $\overline{\Psi}_{\sigma,\nu} \in \mathcal{X}_{\mathrm{lin}}^\sigma$ is to be determined. Assume that the time-1 map $\exp \overline{\Psi}_{\sigma,\nu}$ of $\overline{\Psi}_{\sigma,\nu}$ has the form

$$\exp \overline{\Psi}_{\sigma,\nu}(\xi, \eta, \zeta) = \left(\xi + U(\xi, \eta, \zeta, \sigma, \nu), \eta + V(\xi, \eta, \zeta, \sigma, \nu),\right.$$

$$\left. \zeta + W(\xi, \eta, \zeta, \sigma, \nu)\right), \tag{B.14}$$

where the maps V and W are as in (B.10) and (B.11), respectively. Observe that thus Ψ as defined by (B.13) is normally affine and symplectic. The conjugacy relations $\widetilde{X}_+ = \Psi^* \widetilde{X}$ are comparable with (B.8). Let $\overline{\Psi}$ be of the form $\overline{\Psi} = \overline{U}\frac{\partial}{\partial x} + \overline{V}\frac{\partial}{\partial y} + \overline{W}\frac{\partial}{\partial y}$. The unknowns $\overline{U}, \overline{V}$ and $\overline{W}$ will be determined by the following linear system, compare with (B.9):

$$
\begin{cases}
\frac{\partial \overline{U}}{\partial \xi} \sigma = \Lambda_1 + \{f(\xi, 0, 0, \sigma, \nu)\}_d, \\[2mm]
\frac{\partial \overline{V}}{\partial \xi} \sigma + \frac{\partial \overline{V}}{\partial \zeta} \Omega(\nu)\zeta = \{g(\xi, 0, 0, \sigma, \nu) + \frac{\partial g}{\partial \eta}(\xi, 0, 0, \sigma, \nu)\eta \\[2mm]
\qquad\qquad\qquad\quad + \frac{\partial g}{\partial \zeta}(\xi, 0, 0, \sigma, \nu) + \frac{1}{2}\frac{\partial^2 g}{\partial \eta^2}(\xi, 0, 0, \sigma, \nu)\zeta^2\}_d, \\[2mm]
\frac{\partial \overline{W}}{\partial \xi} \sigma + \frac{\partial \overline{W}}{\partial \zeta} \Omega(\nu)\zeta - \Omega(\nu)\overline{W} = \{h(\xi, 0, 0, \sigma, \nu) + \frac{\partial h}{\partial \zeta}(\xi, 0, 0, \sigma, \nu)\zeta\}_d + \Omega(\Lambda_2)\zeta,
\end{cases}
\tag{B.15}
$$

where $\{f\}_d$ represents the Fourier-truncations of f (with respect to the ξ-variable) up to the order d, etc. The truncation order d will be chosen appropriately later on. Notice that the left side of (B.15) consists of the components of the vector field $\mathrm{ad}\,X(\overline{\Psi})$, where $X = X_{\sigma,\nu}(\xi, \eta, \zeta) = \sigma \frac{\partial}{\partial \xi} + \Omega(\nu)\zeta \frac{\partial}{\partial \zeta}$, compare with (B.1). Therefore, we can rewrite (B.15) as follows

$$
\mathrm{ad}\,X(\overline{\Psi}) = S + N, \tag{B.16}
$$

usually called the *homological equation*, where

$$
S_{\sigma,\nu}(\xi, \eta, \zeta) = \Big\{f(\xi, 0, 0, \sigma, \nu)\Big\}_d \frac{\partial}{\partial \xi} + \Big\{g(\xi, 0, 0, \sigma, \nu) + \frac{\partial g}{\partial \eta}(\xi, 0, 0, \sigma, \nu)\eta
$$
$$
+ \frac{\partial g}{\partial \zeta}(\xi, 0, 0, \sigma, \nu)\zeta + \frac{1}{2}\frac{\partial^2 g}{\partial \zeta^2}(\xi, 0, 0, \sigma, \nu)\zeta^2\Big\}_d \frac{\partial}{\partial \eta}
$$
$$
+ \Big\{h(\xi, 0, 0, \sigma, \nu) + \frac{\partial h}{\partial \zeta}(\xi, 0, 0, \sigma, \nu)\zeta\Big\}_d \frac{\partial}{\partial \zeta},
$$
$$
N_{\sigma,\nu}(\xi, \eta, \zeta) = \Lambda_1(\sigma, \nu)\frac{\partial}{\partial \xi} + \Omega\big(\Lambda_2(\sigma, \nu)\big)\zeta \frac{\partial}{\partial \zeta}.
$$

Observe that S is determined by $\widetilde{X}$, see (B.12). Our concern is to find solutions $\overline{\Psi}$ and N for the homological equation (B.16) for a given $\widetilde{X} \in \mathcal{X}_P^\sigma(M)$ and hence for a given S. Suppose that we can solve the unknowns $\overline{\Psi}$ and N in (B.16) for any given $\widetilde{X}$, then, by (B.13), we obtain the map Ψ. Since $\widetilde{X}_+ = \Psi^*\widetilde{X}$, the perturbation $\widetilde{X}_+$ and hence S_+ are determined. So, for the moment assuming solvability of the equation, (B.16), the vector field S_+ in turn gives the vector fields $\overline{\Psi}_+$ and N_+. In this way we obtain the sequence $\{\Psi_j\}_{j \geqslant 0}$ and hence the sequence $\{\Phi_j\}_{j \geqslant 0}$. Therefore we see that the construction of the sequence of maps Φ_j is reduced to solving the homological equation (B.16).

Solvability of the homological equation

Here we study the solvability of (B.16) in the unknowns $\overline{\Psi}$ and N for a given $S \in \mathcal{X}_P^\sigma(M)$, which is determined by $\widetilde{X}$. We consider a fixed value of $(\sigma, \nu) \in \Gamma_\gamma(U')$. Observe that $S \in \mathcal{X}_{\mathrm{lin}}^\sigma \cap \mathcal{X}_d^\sigma$. Our present aim is to find $\overline{\Psi}, N \in \mathcal{X}_{\mathrm{lin}}^\sigma$ such that (B.15) holds. We write

$$
\overline{\Psi}(\xi, \eta, \zeta) = \bar{u}(\xi)\frac{\partial}{\partial \xi} + \bar{v}(\xi, \eta, \zeta)\frac{\partial}{\partial \eta} + \bar{w}(\xi, \zeta)\frac{\partial}{\partial \zeta},
$$

where the maps $\bar{v}$ and $\bar{w}$ are of the form $\bar{v}(\xi, \eta, \zeta) = \bar{v}_0(\xi) + \bar{v}_1(\xi)\eta + v_2(\xi)\zeta + \frac{1}{2}\bar{v}_3(\xi)\zeta^2$ and $\bar{w}(\xi, \zeta) = w_0(\xi) + w_1(\xi)\zeta$. Thus we have

$$\text{ad } X_{\sigma,v}(\overline{\Psi})(\xi,\eta,\zeta) = \left(\frac{\partial \bar{u}}{\partial \xi}\sigma\right)\frac{\partial}{\partial \xi} + \left(\frac{\partial \bar{v}}{\partial \xi}\sigma + \frac{\partial \bar{v}}{\partial \zeta}\Omega(v)\zeta\right)\frac{\partial}{\partial \eta}$$
$$+ \left(\frac{\partial \bar{w}}{\partial \xi}\sigma + \frac{\partial \bar{w}}{\partial \zeta}\Omega(v)\zeta - \Omega(v)\bar{w}\right)\frac{\partial}{\partial \zeta},$$

which implies that linear map $\text{ad } X_{\sigma,v}$ leaves the linear spaces $\mathcal{X}^\sigma_{\text{lin}}$ and $\mathcal{X}^\sigma_d$ invariant. Denoting by $\mathcal{X}^\sigma_{\text{lin},d}$ the intersection of the linear spaces $\mathcal{X}^\sigma_{\text{lin}}$ and $\mathcal{X}^\sigma_d$, we have

Proposition B.4. *Consider the linear map* $\text{ad } X_{\sigma,v} : \mathcal{X}^\sigma_{\text{lin},d} \to \mathcal{X}^\sigma_{\text{lin},d}$, *where* $(\sigma,v) \in \Gamma_\gamma(U')$. *Then, for any* $d \in \mathbb{N} \cup \{0\}$,

$$\mathcal{X}^\sigma_{\text{lin},d} = \text{im ad } X_{\sigma,v} \oplus \ker \text{ad } X^T_{\sigma,v}, \tag{B.17}$$

where $X^T_{\sigma,v}(\xi,\eta,\zeta) = -\sigma\frac{\partial}{\partial \xi} + \Omega^T(v)\zeta\frac{\partial}{\partial \zeta}$.[7]

Proof. Consider the vector field $\overline{\Psi}$ as given above. By comparing the Fourier coefficients of $\overline{\Psi}$ and of $\text{ad } X_{\sigma,v}\overline{\Psi}$, we see that the eigenvalues of $\text{ad } X^T_{\sigma,v}(\overline{\Psi})$ are of the form

$$i\langle\sigma,k\rangle, \quad i\langle\sigma,k\rangle - \lambda_j(v), \quad i\langle\sigma,k\rangle - \lambda_i(v) + \lambda_j(v),$$

where $\lambda_j(v)$ $(1 \leqslant j \leqslant 2q)$ are the eigenvalues of $\Omega(\mu)$. Since $(\sigma,v) \in \Gamma'_\gamma(U)$ satisfies the Diophantine conditions, it follows that for $k \neq 0$ none of the eigenvalues is equal to zero. Hence, only the zeroth Fourier coefficients of $\overline{\Psi}$ belong to $\ker \text{ad } X^T_{\sigma,v}$, which implies that

$$\ker \text{ad } X^T_{\sigma,v} = \left\{ a\frac{\partial}{\partial \xi} + A\zeta\frac{\partial}{\partial \zeta} : a \in \mathbb{R}^n \text{ and } A \in \ker \text{ad } \Omega^T(v) \right\}. \tag{B.18}$$

On the other hand, the linear subspace $\text{im ad } X_{\sigma,v}$ consists of elements in the forms: $\{u(\xi)\frac{\partial}{\partial \xi} + v(\xi,\eta,\zeta)\frac{\partial}{\partial \eta} + w_0(\xi) + w_1(\xi)\zeta\frac{\partial}{\partial \zeta}\}_d$, where $\{u\}_0 = 0$, $\{w_1\}_0 \in \text{im ad } \Omega(v)$. $\quad\square$

Remark B.5. It can be shown, with help of the Riesz Representation Theorem, cf. [36], that the linear map $\text{ad } X^T_{\sigma,v}$ is the transpose of $\text{ad } X_{\sigma,v}$ with respect to a suitable inner product of the space $\mathcal{X}^\sigma_{\text{lin},d}$. This once again establishes the decomposition (B.17).

Let us return to the homological equation (B.16). The following lemma tells us how (B.16) is solvable in $\overline{\Psi}$ and N for any given $\widetilde{X} \in \mathcal{X}^\sigma_P(M)$.

Lemma B.6. *For any given* $S \in \mathcal{X}^\sigma_{\text{lin},d}$ *and for each* $(\sigma,v) \in \Gamma_\gamma(U')$, *there exist* (*unique*) *real-analytic vector fields* $\overline{\Psi}_{\omega,\mu} \in \mathcal{X}^\sigma_{\text{lin},d}$ *and* $N_{\sigma,v} \in \ker \text{ad } X^T_{\sigma,v}$ *satisfying* (B.16).

Proof. By Proposition B.4, we have that $\mathcal{X}^\sigma_{\text{lin},d} = \text{im ad } X_0 \oplus \ker \text{ad } X^T_0$. Hence, the vector field S has the unique decomposition $S = S^1 + S^0$, where $S^1 \in \text{im ad } X_0$ and $S^0 \in \ker \text{ad } X^T_0$. Take a subspace $V \subset \mathcal{X}^\sigma_{\text{lin},d}$ such that $V \oplus \ker \text{ad } X_0 = \mathcal{X}^\sigma_{\text{lin},d}$. Then, the restriction $\text{ad } X_0|_V : V \to \text{im ad } X_0$

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[7] Here $\Omega(\mu)^T$ denotes the usual matrix transpose of $\Omega(\mu)$.

is an isomorphism. By the Inverse Function Theorem, for each small (σ, v), there exists $\overline{\Psi}_{\sigma,v} \in V$ such that $\mathrm{ad}\, X_{\sigma,v}(\overline{\Psi}_{\sigma,v}) = S^1_{\sigma,v}$. It remains to find the unknown N satisfying $N + S^0 = 0$, where $N = N_{\sigma,v}$ is of the form

$$N_{\sigma,v}(\xi, \eta, \zeta) = \Lambda_1(\sigma, v)\frac{\partial}{\partial \xi} + \Omega\big(\Lambda_2(\sigma, v)\big)\zeta\frac{\partial}{\partial \zeta}.$$

By (B.18), the vector field S^0 is the zeroth Fourier truncation of S and is of the form

$$S^0_{\sigma,v} = a(\sigma, v)\frac{\partial}{\partial \xi} + A(\sigma, v)\zeta\frac{\partial}{\partial \zeta}$$

for $a \in \mathbb{R}^n$ and $A \in \ker \mathrm{ad}\, \Omega^T_0$. Hence, the vector field $N_{\sigma,v}$ is determined by the following equations

$$\Lambda_1(\sigma, v) = -a(\sigma, v) \quad \text{and} \quad \Omega\big(\Lambda_2(\sigma, v)\big) = -A(\sigma, v).$$

Since Ω is an LCU of Ω_0, the latter equation admits the solution $\Lambda_2(\sigma, v) = \Omega^{-1}(-A(\sigma, v))$. $\quad \square$

We can also construct the vector field $\overline{\Psi}_{\sigma,v} \in \mathcal{X}^\sigma_{\mathrm{lin},d}$ explicitly from the homological equation (B.15) as follows. Suppose that

$$\overline{\Psi}_{\sigma,v}(\xi, \eta, \zeta) = \bar{u}\frac{\partial}{\partial \xi} + \left[\bar{v}_0 + \bar{v}_1\eta + \bar{v}_2\zeta + \frac{1}{2}\bar{v}_3\zeta^2\right]\frac{\partial}{\partial \eta} + [\bar{w}_0 + \bar{w}_1\zeta]\frac{\partial}{\partial \zeta}, \qquad (B.19)$$

where the functions $\bar{u}$, $\bar{v}_j$ ($j = 0, \ldots, 3$), $\bar{w}_0$ and $\bar{w}_1$ depend on ξ and on the multi-parameter (σ, v). Since $\overline{\Psi}_{\sigma,v} \in \mathcal{X}^\sigma_{\mathrm{lin}}$, the functions $\bar{v}_j$ ($j = 1, 2, 3$) are determined by $\bar{u}$, $\bar{w}_0$ and $\bar{w}_1$. Indeed, we have

$$\bar{v}_1 = -\left(\frac{\partial \bar{u}}{\partial \xi}\right)^T, \quad \bar{v}_2 = \left(\frac{\partial \bar{w}_0}{\partial \xi}\right)^T J_{2q} \quad \text{and} \quad \bar{v}_3 = \left(\frac{\partial \bar{w}_1}{\partial \xi}\right)^T J_{2q}.$$

Hence, in order to find $\overline{\Psi}$ explicitly, we only need to construct $\bar{u}$, $\bar{v}_0$, $\bar{w}_0$ and $\bar{w}_1$. By comparing the Fourier coefficients of the functions in (B.15), we obtain

$$\bar{u} = \sum_{0 < |k| \leqslant d} \frac{f_k}{i\langle \sigma, k\rangle} e^{i\langle \xi, k\rangle},$$

$$\bar{v}_0 = \sum_{0 < |k| \leqslant d} \frac{g_k}{i\langle \sigma, k\rangle} e^{i\langle \xi, k\rangle},$$

$$\bar{w}_0 = -\Omega(v)^{-1}\big(g_0(\sigma, v)\big) + \sum_{0 < |k| \leqslant d} \big(i\langle \sigma, k\rangle - \Omega(v)\big)^{-1} h_k e^{i\langle \xi, k\rangle},$$

$$\bar{w}_1 = \{\bar{w}_1\}_0 + \sum_{0 < |k| \leqslant d} \big(i\langle \sigma, k\rangle - \mathrm{ad}\, \Omega(v)\big)^{-1}\left\{\frac{\partial h}{\partial \zeta}\right\}_k e^{i\langle \xi, k\rangle}, \qquad (B.20)$$

where $\operatorname{ad}\Omega(\mu)\{\bar{w}_1\}_0 =$ the restriction of $\{-\frac{\partial h}{\partial \zeta}\}_0 \in \mathfrak{sp}(2q, \mathbb{R})$ to the subspace $\operatorname{im}\operatorname{ad}\Omega(\mu)$, according to the splitting $\mathfrak{sp}(2q, \mathbb{R}) = \ker\operatorname{ad}\Omega^T(\mu) \oplus \operatorname{im}\operatorname{ad}\Omega(\mu)$.

B.4. Estimates of errors per iteration step

In this appendix, we specify the complex domains $D^e = D_j^e$ and the choice of the order $d = d_j$ of the Fourier-truncations, see Appendix B.3. Based on these choices, we estimate the errors $|f^+|_{D_+^e}$, $|g^+|_{D_+^e}$, $|h^+|_{D_+^e}$ and $|\Phi^+ - \Phi|_{D_+^e}$ in terms of $|f|_{D^e}$, $|g|_{D^e}$, $|h|_{D^e}$. Here we take care that the sequence of maps $\{\Phi_j\}$ satisfies the conditions of the Inverse Approximation Lemma. Basic tools to be used in this section for estimates are the Cauchy Integral Formula, Gronwall's inequality and the Mean Value Theorem.

Specification of complex domains and truncation orders
We first recall the Hölder condition (5)

$$|\operatorname{Im}\lambda - \operatorname{Im}\tilde{\lambda}| \leqslant L|\mu - \tilde{\mu}|^\theta,$$

where now $\theta = 1/2q$, as this follows from Theorem A.1. Then let $\{s_j\}$, $\{\rho_j\}$ and $\{\varepsilon_j\}$ $(j \geqslant 0)$ be geometric sequences of positive numbers with ratios smaller then $\frac{1}{2}$. We define the sequence $\{r_j\}_{j\geqslant 0}$ by

$$r_j = s_j^{\frac{2\tau}{\theta}+2}, \tag{B.21}$$

where the constant $\theta \in (0, 1]$ comes from the above Hölder condition. We specify the following complex domains

$$D_j = \left(\mathbb{T}^n + \frac{\kappa}{2} + s_j\right) \times \left(\Gamma'_\gamma(U) + r_j\right),$$

$$D_j^e = \left(\mathbb{T}^n + \frac{\kappa}{2} + s_j\right) \times (\Delta_y)_j \times (\Delta_z)_j \times \Gamma'_\gamma(U),$$

where $(\Delta_y)_j = \{y \in \mathbb{C}^n: |y| \leqslant \rho_j\}$ and $(\Delta_z)_j = \{z \in \mathbb{C}^n: |z| \leqslant \varepsilon_j\}$. We require that $0 < s_0 < \min\{\frac{\kappa}{2}, \frac{1}{2n+1}\}$ and that both ρ_0 and ε_0 are sufficiently small. These conditions ensure that $D_j^e \subseteq \mathcal{N}$ for all $j \geqslant 0$, where $\mathcal{N}$ is a holomorphic extension domain of the unperturbed real analytic family X, see (B.2). The sequences $\{s_j\}$, $\{\rho_j\}$ and $\{\varepsilon_j\}$ later on will be chosen in such a way that the iteration process converges. From now on, we apply the plus-sign convention again. In order to estimate the errors $|f^+|_{D_+^e}$, etc., we introduce the intermediate sets $D_*, D_{**}, D_*^e, D_{**}^e$ and the positive numbers $s_* = \frac{1}{2}(s + s_+)$ and $s_{**} = \frac{1}{3}(2s + s_+)$, compare with [26,55]. The numbers $\rho_*, \rho_{**}, \varepsilon_*, \varepsilon_{**}, r_*$ and r_{**} are defined accordingly. The complex domain D_* is defined as

$$D_* = \left(\mathbb{T}^n + \frac{\kappa}{2} + s_*\right) \times \left(\Gamma_\gamma(U') + r_*\right),$$

and the sets D_{**}, D_*^e and D_{**}^e are defined similarly. Notice that $D_+^e \subseteq D_*^e \subseteq D_{**}^e \subseteq D^e$.

We return to the vector field $\overline{\Psi}$ as in (B.16), which is solved for a truncation order d, specified as

$$d = \mathrm{int}(s^{-2}), \tag{B.22}$$

that is, the integral part of s^{-2}. By choosing a sufficiently small s_0, we may assume that the eigenvalues $\lambda_1, \ldots, \lambda_{2q}$ of the matrix $\Omega(\mu)$ satisfy the Hölder condition (5) on the complex domain $U + r_0$ and hence on all domains $U + r_j$ for $j \geqslant 1$. Referring to Remark B.2, we now fix the parameter $\gamma = \gamma_0 = 2(L+1)$ for the Diophantine conditions (2), where L is the constant given by Hölder condition (5).

Lemma B.7. *Suppose that* $(\omega, \mu) \in \Gamma_{\tau, \gamma}(U') + r$. *Then, for all* $k \in \mathbb{Z}^n$ *with* $0 < |k| \leqslant d$ *and any eigenvalues* $\alpha(\mu), \beta(\mu)$ *of* $\Omega(\mu)$, *we have*

$$\big| i \langle k, \omega \rangle \big|, \big| i \langle k, \omega \rangle - \alpha(\mu) \big|, \big| i \langle k, \omega \rangle + \alpha(\mu) - \beta(\mu) \big| \geqslant |k|^{-\tau}. \tag{B.23}$$

Proof. Let $I = |i \langle k, \omega \rangle + \ell_1 \alpha(\mu) + \ell_2 \beta(\mu)|$, where $\ell_1, \ell_2 = 0, \pm 1$. To prove Lemma B.7, we need to show that $I \geqslant |k|^{-\tau}$. By assumption, there is a pair $(\tilde{\omega}, \tilde{\mu}) \in \Gamma_{\tau, \gamma}(U') \subset \mathbb{R}^n \times \mathbb{R}^c$ such that $|\tilde{\omega} - \omega| \leqslant r$ and $|\tilde{\mu} - \mu| \leqslant r$. Now

$$
\begin{aligned}
I &\geqslant \big| \langle k, \mathrm{Re}\,\omega \rangle + \ell_1 \mathrm{Im}\,\alpha(\mu) + \ell_2 \mathrm{Im}\,\beta(\mu) \big| \\
&= \big| \langle k, \tilde{\omega} \rangle + \ell_1 \mathrm{Im}\,\tilde{\alpha}(\tilde{\mu}) + \ell_2 \mathrm{Im}\,\tilde{\beta}(\tilde{\mu}) + \langle k, \mathrm{Re}\,\omega - \tilde{\omega} \rangle \\
&\quad + \ell_1 \mathrm{Im}\big(\alpha(\mu) - \tilde{\alpha}(\tilde{\mu})\big) + \ell_2 \mathrm{Im}\big(\beta(\mu) - \tilde{\beta}(\tilde{\mu})\big) \big| \\
&\geqslant \big| \big| \langle k, \tilde{\omega} \rangle + \ell_1 \mathrm{Im}\,\tilde{\alpha}(\tilde{\mu}) + \ell_2 \mathrm{Im}\,\tilde{\beta}(\tilde{\mu}) \big| - |k| |\omega - \tilde{\omega}| \\
&\quad - \big| \mathrm{Im}\,\alpha(\mu) - \mathrm{Im}\,\tilde{\alpha}(\tilde{\mu}) \big| - \big| \mathrm{Im}\,\beta(\mu) - \mathrm{Im}\,\tilde{\beta}(\tilde{\mu}) \big| \big|.
\end{aligned}
$$

Here $\tilde{\alpha}(\tilde{\mu})$ and $\tilde{\beta}(\tilde{\mu})$ are two eigenvalues of $\Omega(\tilde{\mu})$. Now by the Diophantine conditions (2) and the Hölder condition (5), these eigenvalues can be chosen such that

$$I \geqslant \big| \gamma |k|^{-\tau} - |k| r - 2L r^{\theta} \big|.$$

We claim that both $|k| r$ and r^{θ} are bounded from above by $|k|^{-\tau}$. Indeed, we have that $|k| r \leqslant s^{\frac{2}{\theta}\tau} \leqslant s^{2\tau} \leqslant |k|^{-\tau}$ and that $r^{\theta} = s^{2\tau + 2\theta} \leqslant |k|^{-\tau}$. Hence, $I \geqslant |\gamma - 2L - 1| |k|^{-\tau} = |k|^{-\tau}$. $\quad\square$

Now we are ready to perform the estimates for the vector field $\overline{\Psi}$ and some of its derivatives on the domain D_{**}. Recall that $\overline{\Psi}(\xi, \eta, \zeta) = \bar{u}(\xi) \frac{\partial}{\partial \xi} + \bar{v}(\xi, \eta, \zeta) \frac{\partial}{\partial \eta} + \bar{w}(\xi, \zeta) \frac{\partial}{\partial \zeta}$, where $\bar{v} = \bar{v}_0(\xi) + \bar{v}_1(\xi)\eta + \bar{v}_2(\xi)\zeta + \frac{1}{2}\bar{v}_3(\xi)\zeta^2$ and where $\bar{w} = +\bar{w}_0(\xi) + \bar{w}_1(\xi)\zeta$. The explicit forms of the components of $\overline{\Psi}$ are given by (B.20). From now on, constants which appear in our estimations will only depend on n, q, τ and κ, to be denoted by C_0, C_1 etc. Whenever there is no need to remembered these constants, we will simply put a dot sign in the corresponding inequality.

Proposition B.8. *The vector field* $\overline{\Psi}$ *is real analytic in* ξ *on the domain* D_{**}. *Moreover, there exists a constant* $C_0 = C_0(n, q, \tau, \kappa)$, *such that the following estimates hold.*

   i. $s^{2\tau+1}|\bar{u}|_{D_{**}}, s^{2\tau+2}|\frac{\partial\bar{u}}{\partial\xi}|_{D_{**}} \leqslant C_0|f|_{D^e}$;

   ii. $s^{2\tau+1}|\bar{v}_0|_{D_{**}}, s^{2\tau+2}|\frac{\partial\bar{v}_0}{\partial\xi}|_{D_{**}} \leqslant C_0|g|_{D^e}$;

   iii. $\rho s^{2\tau+1}|\bar{v}_1|_{D_{**}}, \rho s^{2\tau+2}|\frac{\partial\bar{v}_1}{\partial\xi}|_{D_{**}} \leqslant C_0|g|_{D^e}$;

   iv. $\varepsilon s^{2\tau+1}|\bar{v}_2|_{D_{**}}, \varepsilon s^{2\tau+2}|\frac{\partial\bar{v}_2}{\partial\xi}|_{D_{**}} \leqslant C_0|g|_{D^e}$;

   v. $\varepsilon^2 s^{2\tau+1}|\bar{v}_3|_{D_{**}}, \varepsilon^2 s^{2\tau+2}|\frac{\partial\bar{v}_3}{\partial\xi}|_{D_{**}} \leqslant 2C_0|g|_{D^e}$;

   vi. $s^{2\tau+1}|\bar{w}_0|_{D_{**}}, s^{2\tau+2}|\frac{\partial\bar{w}_0}{\partial\xi}|_{D_{**}} \leqslant C_0|h|_{D^e}$;

   vii. $\varepsilon s^{2\tau+1}|\bar{w}_1|_{D_{**}}, \varepsilon s^{2\tau+2}|\frac{\partial\bar{w}_1}{\partial\xi}|_{D_{**}} \leqslant C_0|h|_{D^e}$.

Proof. We only prove item (i), since all other items can be shown similarly. Our proof follows [25,26,55]. By (B.20), we have that

$$|\bar{u}|_{D_{**}} \leqslant \sum_{|k|\neq 0} \frac{|f_k|}{|\langle\sigma,k\rangle|}\big|e^{i\langle\sigma,\xi\rangle}\big| \leqslant |f|_{D^e} \sum_{|k|\neq 0} |k|^\tau e^{|k|(|\mathrm{im}\,\xi|-\frac{\kappa}{2}-s)},$$

where we used the Paley–Wiener lemma, see [25]. It remains to estimate the infinite summation at the right side. By assumption, $|\mathrm{im}\,\xi| \leqslant \frac{\kappa}{2} + s_{**}$ and $s_{**} < \frac{5}{6}s$, it follows that $\sum_{|k|\neq 0} |k|^\tau e^{|k|(|\mathrm{im}\,\xi|-\frac{\kappa}{2}-s)} \leqslant \sum_{|k|\neq 0} |k|^\tau e^{-\frac{1}{6}s|k|}$. But the latter is bounded from above by $.s^{2\tau+1}$, which gives the first estimate of item (i). By the Cauchy Integral Formula, for $(\tilde{x},\tilde{\sigma}) \in D_{**}$, we have

$$\frac{\partial\bar{u}(\tilde{x},\tilde{\sigma})}{\partial\tilde{x}_k} = \frac{1}{2i\pi} \oint \frac{\bar{u}(\tilde{x},\tilde{\sigma})}{(z-\tilde{x}_k)^2}\,dz, \quad \text{for } 1 \leqslant k \leqslant n,$$

where the integration contour is the circle centered at $\tilde{x}_k$ with the radius $s - s_{**}$. By definition, $s - s_{**} \geqslant \frac{1}{6}s$, we have that

$$\left|\frac{\partial\bar{u}(\tilde{x},\tilde{\sigma})}{\partial\tilde{x}}\right| \leqslant 6s^{-1}|\bar{u}|_{D^e},$$

which shows the second estimate of item (i), again compare with [25].   □

As a direct consequence of Proposition B.8, we have

Corollary B.9. *Let $\bar{v}$ and $\bar{w}$ be as above. Then,*

$$s^{2\tau+1}|\bar{v}|_{D_{**}^e} \leqslant 5C_0|g|_D^e \quad and \quad s^{2\tau+1}|\bar{w}|_{D_{**}^e} \leqslant 2C_0|h|_D^e.$$

Estimates on $|f^+|_{D_+^e}, |g^+|_{D_+^e}$ and $|h^+|_{D_+^e}$

Here we estimate the errors $|f^+|_{D_+^e}, |g^+|_{D_+^e}, |h^+|_{D_+^e}$ and also the transformation $\Psi = \exp\overline{\Psi}$ of the form (B.13) and some of its derivatives. These quantities will be given in terms of $|f|_{D^e}, |g|_{D^e}$ and $|h|_{D^e}$. Most of proofs given below are close to [26,55]. For $t \in \mathbb{R}$ and $(\xi,\eta,\zeta) \in D_*^e$, we define $(x(t,\xi), y(t,\xi,\eta,\zeta), z(t,\xi,\zeta)) = \exp t\overline{\Psi}(\xi,\eta,\zeta)$, where $\overline{\Psi}$ is of the form (B.19). We write

$$x(t, \xi) = \xi + u(t, \xi),$$

$$y(t, \xi, \eta, \zeta) = \eta + v_0(t, \xi) + v_1(t, \xi)\eta + v_2(t, \xi)\zeta + \frac{1}{2}v_3(t, \xi)\zeta^2,$$

$$z(t, \xi, \zeta) = \zeta + w_0(t, \xi) + w_1(t, \xi)\zeta. \tag{B.24}$$

Recall from Appendix B.3 that the time-1 map $\exp \overline{\Psi}$ has the form

$$\exp \overline{\Psi}(\xi, \eta, \zeta) = \big(\xi + U(\xi), \eta + V(\xi, \eta, \zeta), \zeta + W(\xi, \zeta)\big),$$

see (B.14). Hence, we have $U(\xi) = x(1, \xi) - \xi$, $V(\xi, \eta, \zeta) = y(1, \xi, \eta, \zeta) - \eta$ and $W(\xi, \zeta) = z(1, \xi, \zeta) - \zeta$.

Lemma B.10. *Suppose that*

$$|f|_{D^e} \leqslant \frac{1}{12C_0}s^{2\tau+2}, \quad |g|_{D^e} \leqslant \frac{1}{48C_0}\rho s^{2\tau+1}, \quad and \quad |h|_{D^e} \leqslant \frac{1}{48C_0}\varepsilon s^{2\tau+1},$$

where C_0 is the same constant as from Proposition B.8. Then, for $(\xi, \eta, \zeta) \in D_*^e$ and for $t \in [0, 1]$, we have that $(x(t, \xi, \eta, \zeta), y(t, \xi, \eta, \zeta), z(t, \xi, \eta, \zeta)) \in D_{**}^e$. Moreover, there exists a constant $C_1 > C_0$ such that the following estimates hold.

   i. $s^{2\tau+1}|U|_{D_*^e} \leqslant C_1|f|_{D^e}$;
  ii. $s^{2\tau+1}|V|_{D_*^e} \leqslant C_1|g|_{D^e}$;
 iii. $s^{2\tau+1}|W|_{D_*^e} \leqslant C_1|h|_{D^e}$.

Proof. First we notice that the maps $(x(t, \xi), y(t, \xi, \eta, \zeta), z(t, \xi, \zeta))$ satisfy the following differential equations,

$$\dot{x} = \bar{u}(x), \tag{B.25}$$

$$\dot{y} = \bar{v}_0(x) + \bar{v}_1(x)y + \bar{v}_2(x)z + \frac{1}{2}\bar{v}_3(x)z^2, \tag{B.26}$$

$$\dot{z} = \bar{w}_0(x) + \bar{w}_1(x)z. \tag{B.27}$$

Let $(\xi, \eta, \zeta) \in D_*^e$ and $t \in [0, 1]$. By (B.25), we have $|x(t, \xi) - \xi| \leqslant |\bar{u}|_{D_{**}} \leqslant \frac{1}{12}s \leqslant s_{**} - s_*$. Hence, for all $t \in [0, 1]$ and $\xi \in \mathbb{T}^n + \frac{\kappa}{2} + s_*$, we have $x(t, \xi) \in \mathbb{T}^n + \frac{\kappa}{2} + s_{**}$. The same approach gives $y(t, \xi, \eta, \zeta) \in \Delta_y^{**}$ and $z(t, \xi, \zeta) \in \Delta_z^{**}$.

Next we estimate $|U|_{D_*^e}$, $|V|_{D_*^e}$ and $|W|_{D_*^e}$. For all $t \in [0, 1]$, one has $|u(t, x)| \leqslant |\bar{u}|_{D_{**}}$, which implies that $s^{2\tau+1}|U|_{D_*^e} \leqslant s^{2\tau+1}|\bar{u}|_{D_{**}} \leqslant C_0|f|_{D^e}$. We use Gronwall's inequality to estimate $|V|_{D_*^e}$ and $|W|_{D_*^e}$. By (B.27) and (B.24) we have $\dot{w}_0 = \bar{w}_0(x(t, \xi)) + \bar{w}_1(x(t, \xi))w_0$ and $\dot{w}_1 = \bar{w}_0(x(t, \xi)) + \bar{w}_1(x(t, \xi))w_1$. By the first differential equation,

$$|w_0| \leqslant \big|\bar{w}_0(t, \xi)\big|_{D_{**}} + \int_0^t |\bar{w}_1|_{D_{**}}\big|w_0(s, \xi)\big| \, ds.$$

By Gronwall's inequality $|w_0(t,\xi)| \leqslant |\bar{w}_0|_{D_{**}} e^{|\bar{w}_1|_{D_{**}}} \leqslant 2|\bar{w}_0|_{D_{**}}$. Similarly, we have $|w_1(t,\xi)| \leqslant 2|\bar{w}_1|_{D_{**}}$ for all $t \in [0,1]$. In particular,

$$s^{2\tau+1}|W|_{D_*^e} \leqslant 2s^{2\tau+1}\big(|\bar{w}_0|_{D_{**}} + \varepsilon|\bar{w}_1|_{D_{**}}\big) \leqslant 4C_0|f|_{D^e},$$

which proves (iii).

Finally, we show the inequality (ii). By (B.26) and (B.24), the map $v_0(t,\xi)$ satisfies the differential equation

$$\dot{v}_0 = \bar{v}_0\big(x(t,\xi)\big) + \bar{v}_2\big(x(t,\xi)\big)w_0 + \frac{1}{2}\bar{v}_3\big(x(t,\xi)\big)w_0^2 + \bar{v}_1\big(x(t,\xi)\big)v_0.$$

Again by Gronwall's inequality we have

$$\big|v_0(t,\xi)\big| \leqslant e^{|\bar{v}_1|_{D_{**}}}\left(|\bar{v}_0|_{D_{**}} + |\bar{v}_2|_{D_{**}}\big|w_0(t,\xi)\big| + \frac{1}{2}|\bar{v}_3|_{D_{**}}\big|w_0(t,\xi)\big|^2\right)$$

$$\leqslant 2\big(|\bar{v}_0|_{D_{**}} + \varepsilon|\bar{v}_2|_{D_{**}} + \varepsilon^2|\bar{v}_3|_{D_{**}}\big),$$

which implies that $s^{2\tau+1}|v_0(t,\xi)| \leqslant 6C_0|g|_{D^e}$. Similarly, one can show that

$$\rho s^{2\tau+1}\big|v_1(t,\xi)\big|, \varepsilon s^{2\tau+1}\big|v_2(t,\xi)\big|, \varepsilon^2 s^{2\tau+1}\big|v_3(t,\xi)\big| \leqslant 6C_0|g|_{D^e}.$$

Therefore, $s^{2\tau+1}|V|_{D_*^e} \leqslant 21C_0|g|_{D^e}$. $\quad\square$

Taking the parameter (σ, ν) into account, the map $\Psi : D_*^e \mapsto D_{**}^e$ is written in the form $\Psi(\xi, \eta, \zeta, \sigma, \nu) = (\xi + U(\xi), \eta + V(\xi, \eta, \zeta), \zeta + W(\xi, \zeta), \sigma + \Lambda_1(\sigma, \nu), \nu + \Lambda_2(\sigma, \nu))$. We observe that from the construction of Λ_1 and Λ_2 (see the proof of Lemma B.6), we have

$$|\Lambda_1|_{D_*} \leqslant C_1|f|_{D^e} \quad \text{and} \quad \varepsilon|\Lambda_2|_{D_*} \leqslant C_1|h|_{D^e}. \tag{B.28}$$

As a consequence of Lemma B.10 and the Cauchy Integral Formula, we have the following.

Corollary B.11. *Under the assumptions of Lemma* B.10, *we have*

i. $|\Psi - \mathrm{Id}|_{D_+^e} \leqslant C_2 \max\{s^{-2\tau-1}|f|_{D^e}, s^{-2\tau-1}|g|_{D^e}, \varepsilon^{-1}s^{-2\tau-1}|h|_{D^e}\}$;
ii. $|D(\Psi - \mathrm{Id})|_{D_+^e}) \leqslant C_2 \max\{s^{-(2+\frac{2}{\theta})\tau-3}|f|_{D^e}, s^{-(2+\frac{2}{\theta})\tau-3}|g|_{D^e}, \rho^{-1}s^{-2\tau-1}|g|_{D^e},$
$\varepsilon^{-1}s^{-2\tau-1}|g|_{D^e}, s^{-(2+\frac{2}{\theta})\tau-3}|h|_{D^e}, \varepsilon^{-1}s^{-\frac{2}{\theta}\tau-2}|h|_{D^e}\}$, *where* $D(\Psi - \mathrm{Id})$ *is the derivative of* $(\Psi - \mathrm{Id})$.

Next we estimate the terms $|f^+|_{D_+^e}$, $|g^+|_{D_+^e}$ and $|h^+|_{D_+^e}$. To this end, we use the iteration relation $\widetilde{X}_+ = \Psi^*\widetilde{X}$ and the linearization (B.15), which gives the following equations

$$\begin{cases} (1 + \frac{\partial U}{\partial \xi})f^+ = R_1, \\ \frac{\partial V}{\partial \xi}f^+ + (1 + \frac{\partial V}{\partial \eta})g^+ + \frac{\partial V}{\partial \zeta}h^+ = R_2, \\ \frac{\partial W}{\partial \xi}f^+ + (1 + \frac{\partial W}{\partial \zeta})h^+ = R_3, \end{cases} \tag{B.29}$$

where

$$
\begin{aligned}
R_1 = {} & f(\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2) \\
& - \big\{ f(\xi, 0, 0, \sigma, \nu) \big\}_d + D_1(\bar{u} - U)(\xi, \sigma, \nu),
\end{aligned}
$$

$$
\begin{aligned}
R_2 = {} & g(\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2) - \bigg\{ g(\xi, 0, 0, \sigma, \nu) \\
& + \frac{\partial g}{\partial \eta}(\xi, 0, 0, \sigma, \nu)\eta + \frac{\partial g}{\partial \zeta}(\xi, 0, 0, \sigma, \nu)\zeta + \frac{1}{2}\frac{\partial^2 g}{\partial \zeta^2}(\xi, 0, 0, \sigma, \nu)\zeta^2 \bigg\} \\
& \times D_2(\bar{v} - V)(\xi, \eta, \zeta, \sigma, \nu),
\end{aligned}
$$

$$
\begin{aligned}
R_3 = {} & h(\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, \nu + \Lambda_2) - \bigg\{ h(\xi, 0, 0, \sigma, \nu) \\
& + \frac{\partial h}{\partial \zeta}(\xi, 0, 0, \sigma, \nu)\zeta \bigg\}_d + D_3(\bar{w} - W)(\xi, \zeta, \sigma, \nu).
\end{aligned}
$$

Here D_1, D_2 and D_3 are the following linear operators. For suitable maps f_1, f_2 and f_3, $D_1 f_1(\xi, \sigma, \nu) = \frac{\partial f_1}{\partial \xi}\sigma$, $D_2 f_2(\xi, \eta, \zeta, \sigma, \nu) = \frac{\partial f_2}{\partial \xi}\sigma + \frac{\partial f_2}{\partial \zeta}\Omega(\nu)\zeta$ and $D_3 f_3(\xi, \zeta, \sigma, \nu) = \frac{\partial f_3}{\partial \xi}\sigma + \frac{\partial f_3}{\partial \zeta}\Omega(\nu)\zeta - \Omega(\nu) f_3$.

Lemma B.12. *Under the assumptions of Proposition* B.8, *we have*

i. $|D_1\bar{u}|_{D_*^e}, |D_1 U|_{D_*^e} \leqslant .|f|_{D^e}$ *and* $|D_1(\bar{u} - U)|_{D_*^e} \leqslant .s^{-2\tau-2}|f|^2_{D^e}$;

ii. $|D_2\bar{v}|_{D_*^e}, |D_2 V|_{D_*^e} \leqslant .|g|_{D^e}$ *and* $|D_2(\bar{v} - V)|_{D_*^e} \leqslant .s^{-2\tau-1}|g|_{D^e} \max\{s^{-1}|f|_{D^e}, \rho^{-1}|g|_{D^e}, \varepsilon^{-1}|h|_{D^e}\}$;

iii. $|D_3\bar{w}|_{D_*^e}, |D_3 W|_{D_*^e} \leqslant .|h|_{D^e}$ *and* $|D_3(\bar{w} - W)|_{D_*^e} \leqslant .s^{-2\tau-1}|h|_{D^e} \max\{s^{-1}|f|_{D^e}, \varepsilon^{-1}|h|_{D^e}\}$.

Proof. For a proof of item (i) see [25,26]. We only prove item (iii), since item (ii) can be shown in the same way. Since

$$
D_3\bar{w} = \bigg\{ h(\xi, 0, 0, \sigma, \nu) + \frac{\partial h}{\partial \zeta}(\xi, 0, 0, \sigma, \nu) \bigg\}_d,
$$

it follows that $|D_2\bar{w}|_{D_*^e} \leqslant .|h|_{D^e}$. Observe that

$$
\begin{aligned}
\frac{d}{dt} w(t, \xi, \zeta) = {} & (D_3\bar{w})\big(x(t, \xi), z(t, \xi, \zeta)\big) + \frac{\partial \bar{w}}{\partial x}(D_1 u)(t, \xi) \\
& + \frac{\partial \bar{w}}{\partial z}(D_3 w)(t, \xi, \zeta).
\end{aligned} \tag{B.30}
$$

By Gronwall's inequality, for all $t \in [0, 1]$, we have

$$
\big| w(t, \xi, \eta, \zeta) \big| \leqslant . \max\bigg\{ |D_3\bar{w}|_{D_*^e}, \bigg| \frac{\partial \bar{w}}{\partial x} \bigg|_{**} |D_1 u|_{D_*^e} \bigg\} e^{|\frac{\partial \bar{w}}{\partial z}|_{D_{**}^e}} \leqslant .|h|_{D^e}.
$$

Since $W(\xi, \zeta) = w(1, \xi, \zeta)$, it follows that $|W|_{D_*^e} \leqslant .|h|_{D^e}$. In order to estimate $|D_2(\bar{w} - W)|_{D_*^e}$, we write

$$D_3(\bar{w} - W)(\xi, \zeta) = \int_0^1 D_3\big(\bar{w}\big(x(t, \xi), z(t, \xi, \zeta) - \bar{w}(\xi, \zeta)\big)\big)$$

$$+ \int_0^1 \frac{\partial \bar{w}}{\partial x}\big(x(t, \xi), z(t, \xi, \zeta)\big)(D_1 U)(t, \xi)\, dt$$

$$+ \int_0^1 \frac{\partial \bar{w}}{\partial z}\big(x(t, \xi), z(t, \xi, \zeta)\big)(D_3 W)(t, \xi, \zeta)\, dt,$$

where we used (B.30). Now by estimating each of the integrals, we obtain the desired estimate for $|D_3(\bar{w} - W)|_{D_*^e}$. $\quad\square$

Proposition B.13. *Assume that*

$$|f|_{D^e} \leqslant \frac{1}{12C_1} s^{2\tau+2}, \quad |g|_{D^e} \leqslant \frac{1}{48C_1}\rho s^{2\tau+1} \quad and \quad |h|_{D^e} \leqslant \frac{1}{48C_1}\varepsilon s^{2\tau+1}.$$

Then $\Psi(D_+^e) \subseteq D_*^e$. Moreover, if we assume that $ds > 2n$, then the following holds for a constant $C_3 \geqslant C_1$.

i. $|f^+|_{D_+^e} \leqslant C_3 |f|_{D^e}(\mathcal{M} + \max\{\frac{\rho_+}{\rho}, \frac{\varepsilon_+}{\varepsilon}, d^n e^{-\frac{ds}{2}}\});$
ii. $|g^+|_{D_+^e} \leqslant C_3 |g|_{D^e}(\mathcal{M} + \max\{(\frac{\rho_+}{\rho})^3, (\frac{\varepsilon_+}{\varepsilon})^3, d^n e^{-\frac{ds}{2}}\} + \mathcal{N}_1);$
iii. $|h^+|_{D_+^e} \leqslant C_3 |h|_{D^e}(\mathcal{M} + \max\{(\frac{\rho_+}{\rho})^2, (\frac{\varepsilon_+}{\varepsilon})^2, d^n e^{-\frac{ds}{2}}\} + \mathcal{N}_2),$

where $\mathcal{M} = \max\{s^{-4\tau-4}|f|_{D^e}, \rho^{-1}s^{-2\tau-1}|g|_{D^e}, \varepsilon^{-1}s^{-4\tau-4}|h|_{D^e}\}$ and where $\mathcal{N}_1 = s^{-2\tau-1} \times (s^{-1}|f^+|_{D_+^e} + \varepsilon^{-1}|h^+|_{D_+^e})$ and $\mathcal{N}_2 = s^{-2\tau-2}|f^+|_{D_+^e}).$

Proof. Since $s_* - s_+ > \frac{s}{4}$ etc., the first claim follows from the fact that $|U|_{D_+^e}$, $|V|_{D_+^e}$, $|W|_{D_+^e}$, $|\Lambda_1|_{D_+^e}$ and $|\Lambda_2|_{D_+^e}$ are smaller then $\frac{s}{4}, \frac{\rho}{4}, \frac{\varepsilon}{4}, \frac{r}{4}$ and $\frac{r}{4}$, respectively. Let $ds > 2n$. We show (iii). The other two inequalities can be shown similarly. By (B.29), we have $|h^+|_{D_+^e} \leqslant |R_3|_{D_+^e} + |\frac{\partial W}{\partial \xi}|_{D_*^e}|f^+|_{D_+^e}$. We notice that

$$|R_3|_{D_+^e} \leqslant \big|h(\xi + U, \eta + V, \zeta + W, \sigma + \Lambda_1, v + \Lambda_2) - h(\xi, \eta, \zeta, \sigma, v)\big|_{D_*^e}$$

$$+ \left|h(\xi, \eta, \zeta, \sigma, v) - h(\xi, \eta, \zeta, \sigma, v) - \frac{\partial h}{\partial \eta}(\xi, 0, 0, \sigma, v)\eta\right|_{D_+^e}$$

$$+ \big|h(\xi, 0, 0, \sigma, v) - \{h(\xi, 0, 0, \sigma, v)\}_d\big|_{D_+^e}$$

$$+ \left|\frac{\partial h}{\partial \eta}(\xi, 0, 0, \sigma, v)\eta - \left\{\frac{\partial h}{\partial \eta}(\xi, 0, 0, \sigma, v)\eta\right\}_d\right|_{D_+^e} + \big|D_3(\bar{w} - W)\big|_{D_*^e}.$$

By the Mean Value Theorem, the sum of the first two terms are bounded from above by

$$. |h|_{D^e} \left(\mathcal{M} + \max\left\{ \left(\frac{\varepsilon_+}{\varepsilon}\right)^2, \left(\frac{\rho_+}{\rho}\right)^2 \right\} \right).$$

It can be shown, see [25, p. 150], that the third and fourth terms are upper bounded by $. d^n e^{-\frac{ds}{2}} |h|_{D^e}$. These estimates together with estimates from Lemmas B.12 and B.10 give the desired inequality (iii). $\quad\square$

Finally we give estimate $|\widetilde{U} - \widetilde{U}^+|$, $|\widetilde{V} - \widetilde{V}^+|$ and $|\widetilde{W} - \widetilde{W}^+|$. To this end, we denote by $|\Psi_j|_{1,D_j^e}$ the C^1-norm on the domain D_j^e, i.e.,

$$|\Phi_j|_{1,D_j^e} = \max\left\{ |\Phi_j|_{D_j^e}, |D\Phi_j|_{D_j^e} \right\}.$$

By the Mean Value Theorem and the iteration relation $\Phi_+ = \Phi \circ \Psi$, it follows

Proposition B.14. *Under the assumptions of Proposition* B.13, *we have*

  i. $|\Phi_j|_{1,D_+^e} \leqslant |\Phi_j|_{1,D_j^e} \max\{1 + |\psi|_{D_+^e}, 1 + |D\psi|_{D_+^e}\};$
  ii. $|\widetilde{U} - \widetilde{U}^+|_{D_+}, |\widetilde{V} - \widetilde{V}^+|_{D_+}, |\widetilde{W} - \widetilde{W}^+|_{D_+}, |\widetilde{\Lambda}_1 - \widetilde{\Lambda}_1^+|_{D_+}, |\widetilde{\Lambda}_2 - \widetilde{\Lambda}_2^+|_{D_+} \leqslant |\Phi|_{1,D_+^e} |\psi|_{D_+^e}.$

B.5. Convergence of iteration process

In order to ensure that the iteration process converges, we need to specify the sequences $\{s_j\}, \{\rho_j\}$ and $\{\varepsilon_j\}$ in a proper way. We first define a sequence $\{\delta_j\}_{j \geqslant 0}$ of positive numbers by the relation $\delta_{j+1} = \delta_j^{p+1}$, where $p > 0$. Next we choose the sequences $\{s_j\}, \{\rho_j\}$ and $\{\varepsilon_j\}$ as follows.

$$s_j = s_0 a^j, \quad \rho_j = \delta_j^q \quad \text{and} \quad \varepsilon_j = \delta_j^q \quad (j \geqslant 0),$$

where $a \in (0, \frac{1}{2})$ and $q > 0$. In the following, we fix the values p and q in such a way that $q \in (0, 2)$ and $p \in (0, 1 - \frac{q}{2})$. Furthermore we assume that

$$0 < s_0 < \min\left\{ \frac{\kappa}{2}, \frac{1}{2n+1} \right\}. \tag{B.31}$$

Recall that the truncation orders $d_j = \text{int}(s^{-2})$, see (B.22). Now the exponential sequence $\{\delta_j\}$ will be used to control the errors $|f_j|_{D_j^e}, |g_j|_{D_j^e}$ and $|h_j|_{D_j^e}$. The following proposition says that we can fix the initial values δ_0, s_0 in such a way that the maps Φ_j converge to a Whitney-smooth map $\Phi = \Phi_\infty$, for $j \to \infty$.

Proposition B.15. *For a sufficiently small* $\delta_0 > 0$. *There exists* $s_0 \in (0, \min\{\frac{\kappa}{2}, \frac{1}{2n+1}\})$ *such that for all* $j \geqslant 0$, *the following holds*:

  i. *The assumptions of Proposition* B.13 *are satisfied*;
  ii. $|f^j|_{D_j^e} \leqslant \delta_j, |g^j|_{D_j^e} \leqslant \delta_j^2, |h^j|_{D_j^e} \leqslant \delta_j^2;$

iii. *There exists a number* $b \in (0, 1)$ *such that on the domain* D_{j+1},

$$\left|\widetilde{U} - \widetilde{U}^+\right|, \left|\widetilde{V} - \widetilde{V}^+\right|, \left|\widetilde{W} - \widetilde{W}^+\right|, \left|\widetilde{A}_1 - \widetilde{A}_1^+\right|, \left|\widetilde{A}_2 - \widetilde{A}_2^+\right| \leqslant C_4 \delta_j^b,$$

and that $\left|\frac{\partial h^j}{\partial z}\right|_{D_*^e} \leqslant C_4 \delta_j^b$.

Proposition B.15 will be proved with help of the lemmas given below, compare with the proof of [12, Proposition 5.9].

Remark B.16. We should keep in mind that the present proof for Theorem 1.6 is based on the scaled families $\frac{\gamma_0}{\gamma} X$ and $\frac{\gamma_0}{\gamma} \widetilde{X}$, respectively, as the perturbed and unperturbed family, compare with Remark B.2. For the unscaled families X and $\widetilde{X}$, we have to multiply the right-hand of the inequalities from item (ii) by the factor γ/γ_0.

Lemma B.17. *Assume that* $\delta_0 \in (0, 1)$ *satisfy the following conditions*

a. $\delta_0^{2-2p-q} \leqslant \frac{1}{144C_3} s_0^{4\tau+4}$, $\delta_0^{p(2-2p-q)} \leqslant a^{4\tau+4}$ *and* $\delta_0^{p(q-1)} \leqslant \frac{1}{144C_3}$;
b. $d_j^n e^{-\frac{1}{2}d_j s_j} \leqslant \frac{1}{4C_3} \delta_j^{p+1}$, *for all* $j \geqslant 0$.

Then, for all $j \geqslant 0$,

i. $\delta_j \leqslant \frac{1}{12C_1} s_j^{2\tau+2}$, $\delta_j^2 \leqslant \frac{1}{48C_1} \varepsilon_j s_j^{2\tau+1}$;
ii. $\delta_j^{2-2p-q} \leqslant \frac{1}{144C_3} s_j^{4\tau+4}$;
iii. $|f^j|_{D_j^e} \leqslant \delta_j$, $|g^j|_{D_j^e} \leqslant \delta_j^2$, $|h^j|_{D_j^e} \leqslant \delta_j^2$.

Remark B.18. Suppose that s_0, δ_0 can be chosen such that they satisfy the assumptions of Lemma B.17, then conclusions (i) and (iii) imply that the assumptions of Proposition B.13, except for the condition that $d_j s_j > 2n$, are satisfied for all j. However, the inequality $d_j s_j > 2n$ is implied by the assumption (B.31).

Proof of Lemma B.17. We apply induction on the index j.

i. Notice that conclusion (i) holds for $j = 0$. Now suppose that it also holds for the jth step. Then,

$$\delta_{j+1} = \delta_j \delta_j^p \leqslant \frac{1}{12C_1} s_j^{2\tau+2} \delta_0^p \leqslant \frac{1}{12C_1} s_{j+1}^{2\tau+2} a^{-2\tau-2} \delta_0^p \leqslant \frac{1}{12C_1} s_{j+1}^{2\tau+2},$$

and

$$\delta_{j+1}^2 = \delta_j^2 \delta_j^{2p} \leqslant \frac{1}{48C_1} \varepsilon_j s_j^{2\tau+1} \delta_j^{2p} \leqslant \frac{1}{48C_1} \varepsilon_{j+1} s_{j+1}^{2\tau+1} \delta_0^{2p-pq} a^{-2\tau-2}.$$

Since $\delta_0^{2p-pq} a^{-2\tau-2} \leqslant 1$, the claim follows.

ii. Observe that (ii) holds for $j = 0$. Supposing now that conclusion (ii) is true for $j \geqslant 0$, we then have

$$\delta_{j+1}^{2-2p-q} = \delta_j^{2-2p-q} \delta_j^{p(2-2p-q)} \leqslant \frac{1}{144C_3} s_j^{4\tau+4} \delta_j^{p(2-2p-q)}$$

$$\leqslant \frac{1}{144C_3} s_{j+1}^{4\tau+4} a^{-4\tau-4} \delta_0^{p(2-2p-q)} \leqslant \frac{1}{144C_3} s_{j+1}^{4\tau+4}.$$

Hence, we conclude that (ii) holds for all $j \geqslant 0$.

iii. Since all three the estimates of part (iii) are similar, we only show the first. Suppose that conclusion (iii) holds for the jth step. Then, by (i) and the assumption that $s_0 \leqslant \frac{1}{2n+1}$, the assumptions in Proposition B.13 and hence the claims of Proposition B.13 hold. In particular, we have

$$\left| f^{j+1} \right|_{D_{j+1}^e} \leqslant C_3 \left| f^j \right|_{D_j^e} \left(\mathcal{M} + \max\left\{ \frac{\varepsilon_{j+1}}{\varepsilon_j}, d_j^n e^{-\frac{1}{2}d_j s_j} \right\} \right).$$

Observe that $\mathcal{M} \leqslant \delta_j^{2-q} s_j^{-4\tau-4}$ for all j. Hence,

$$\left| f^j \right|_{D_j^e} \mathcal{M} \leqslant \delta_j^{3-q} s_j^{-4\tau-4} \leqslant \delta_{j+1} \delta_j^{2-2p-q} s_j^{-4\tau-4} \leqslant \frac{1}{144C_3} \delta_{j+1}.$$

Notice that $\frac{\varepsilon_{j+1}}{\varepsilon_j} = \delta_j^{pq}$. It follows that

$$\left| f^j \right|_{D_j^e} \frac{\varepsilon_{j+1}}{\varepsilon_j} \leqslant \delta_{j+1} \delta_0^{pq-p} \leqslant \frac{1}{144C_3} \delta_{j+1}.$$

The above estimates together with the assumption that $d_j^n e^{-\frac{1}{2}d_j s_j} \leqslant \frac{1}{4C_3} \delta_j^{p+1}$, yield the desired inequality $\left| f^{j+1} \right|_{D_{j+1}^e} \leqslant \delta_{j+1}$. $\quad \square$

Lemma B.19. *For a sufficiently small value δ_0, we find a positive value $s_0 \in (0, \min\{\frac{\kappa}{2}, \frac{1}{2n+1}\})$ such that assumptions* (a) *and* (b) *of Lemma* B.17 *are satisfied.*

Proof. By [25, p. 153], condition (b) of Lemma B.17 is fulfilled if

$$\frac{1}{s_0} \geqslant c_1 \log \frac{1}{s_0} + c_2 \log \frac{1}{\delta_0} + c_3$$

for suitable positive values c_1, c_2 and c_3. In order to prove Lemma B.19, we only need to find small δ_0, s_0 such that the following inequalities hold simultaneously

$$\begin{cases} \frac{1}{s_0} \geqslant c_1 \log \frac{1}{s_0} + c_2 \log \frac{1}{\delta_0} + c_3, \\ \delta_0^{2-2p-q} \leqslant c_4 s_0^{4\tau+4}, \\ \delta_0^{p(2-2p-q)} \leqslant a^{4\tau+4}, \\ \delta_0^{p(q-1)} \leqslant c_5, \end{cases} \tag{B.32}$$

where c_i $(i = 1, \ldots, 5)$ are positive constants. The latter two inequalities of (B.32) are easily satisfied when δ_0 is sufficiently small. Chose a constant $K \in (0, \frac{2-2p-q}{4\tau+4})$ and let $s_0 = c_4^{\frac{-1}{4\tau+4}} \delta_0^K$. Then, the second inequality of (B.32) also holds. Observe that $s_0 \to 0$, when $\delta_0 \to 0$. The first inequality of (B.32) becomes a special form of the following inequality

$$\frac{1}{s_0} - \tilde{c}_1 \log \frac{1}{s_0} - \tilde{c}_2 \geqslant 0, \tag{B.33}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are constants. Since the function $f(x) = x - \tilde{c}_1 \log x - \tilde{c}_2$, defined on $\mathbb{R}$, is increasing for sufficiently large x, it follows that (B.33) holds for a sufficiently small s_0, which is ensured by choosing a sufficiently small value δ_0. $\quad\square$

Recall from Remark B.2 that for our proof of the main Theorem 1.6, we used the scaled families $\frac{\gamma_0}{\gamma} X$ and $\frac{\gamma_0}{\gamma} \widetilde{X}$ as the unperturbed and perturbed family, respectively. Our proof for Theorem 1.6 is now concluded by setting $\delta = \delta_0$ and by letting the perturbation $\widetilde{X}$ be in the real-analytic neighborhood $\mathcal{A}_{\gamma/\gamma_0, \sigma}$ of the unperturbed family X, where the set $\mathcal{A}_{\gamma/\gamma_0, \sigma}$ is defined as in Appendix B.1.2. Indeed, by applying the Inverse Approximation Lemma to the approximations Φ_j on the domains D_j, we obtain the limit Φ_∞ as a Whitney-smooth map on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times \Gamma_{\tau, \gamma}(U')$, which is real analytic in the x_∞-direction. By construction, this map Φ_∞ is normally affine and symplectic. Next we consider the limit $\widetilde{X}_\infty = \Phi_\infty^* \widetilde{X}$. By Proposition B.15, it follows that f_∞, g_∞ and h_∞ vanish at the Diophantine tori $V = \{y_\infty = 0, \ z_\infty = 0\}$, where h_∞ is quadratic in z_∞. This implies that the image $\Phi_\infty(V)$ is $\widetilde{X}$-invariant and that

$$\widetilde{X}_{\infty(\tilde{\omega}, \tilde{\mu})}(\tilde{x}, \tilde{y}, \tilde{z}) = \left[\tilde{\omega} + O\big(|\tilde{y}|, |\tilde{z}|\big)\right]\frac{\partial}{\partial \tilde{x}} + O\big(|\tilde{y}|, |\tilde{z}|\big)\frac{\partial}{\partial \tilde{y}}$$

$$+ \left[\Omega(\tilde{\mu})\tilde{z} + O\big(|\tilde{y}|, |\tilde{z}^2|\big)\right]\frac{\partial}{\partial \tilde{z}},$$

where $\tilde{x} = x_\infty$ etc. and where the parameter $(\tilde{\omega}, \tilde{\mu})$ satisfies the Diophantine conditions. Finally, by the Whitney Extension Theorem we can extend the limit Φ_∞ to a smooth map Φ defined on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q} \times U$ such that Φ satisfies conclusions of Theorem 1.6.

Appendix C. Normal linearization on the symplectic normal bundle

In Section 3, we defined normal linearization of a given vector field with an invariant submanifold in the dissipative case by means of a scaling operator $\mathcal{D}_\varepsilon$, compare with (36). This procedures induces a vector field—the normal linear part—on the normal bundle of the invariant manifold. Normal linearization simplifies the perturbation problem: via the scaling $\mathcal{D}_\varepsilon$, normal linear stability of a given integrable family is implied by that of the corresponding normal linear part, see Section 3. However, the normal linearization in the dissipative setting and in the reversible setting, as explained in 3, is less satisfactory for the symplectic setting, since the normal linear part of a nonintegrable Hamiltonian vector field contains a quadratic term, compare with Remark 3.3. To see this, we consider the symplectic phase space (M, σ) endowed with an free $\mathbb{T}^n$-action, as introduced in Section 1.2, and a nonintegrable (i.e., non-$\mathbb{T}^n$-symmetric) vector field Y with the $\mathbb{T}^n$-orbit V as an invariant submanifold. By the generalized Darboux Theorem [1], there exist local coordinates $(x, y, z) = (x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_{2q}) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^{2q}$, in which the

submanifold V is given by $V = \{(x, y, z): (y, z) = (0, 0)\}$ and the 2-form $\sigma = dx \wedge dy + dz^2$. Suppose that Y takes, in local coordinates (x, y, z), the shape

$$Y(x, y, z) = f(x, y, z)\frac{\partial}{\partial x} + g(x, y, z)\frac{\partial}{\partial y} + h(x, y, z)\frac{\partial}{\partial z}.$$

Since V is Y-invariant, we have that $g(x, 0, 0) \equiv 0$ and $h(x, 0, 0) \equiv 0$. Using the scaling $\mathcal{D}_\varepsilon(x, y, z) = (x, \varepsilon^2 y, \varepsilon z)$, we obtain the normal linear part NY defined as

$$NY = \lim_{\varepsilon \downarrow 0} \mathcal{D}_\varepsilon Y = f(x, 0, 0)\frac{\partial}{\partial x} + \left[g_y(x, 0, 0)y + \frac{1}{2}G(x, 0, z) \right]\frac{\partial}{\partial y}$$

$$+ h_z(x, 0, 0)z\frac{\partial}{\partial z}, \tag{C.1}$$

where $g_y = \frac{\partial g}{\partial y}$, $h_z = \frac{\partial h}{\partial z}$ and $G(x, 0, z) = (\langle \frac{\partial^2 g_1}{\partial z^2}z, z \rangle, \ldots, \langle \frac{\partial^2 g_n}{\partial z^2}z, z \rangle)$. We see that the vector field NY contains, indeed, the nonlinear term G being quadratic in z. To avoid such a nonlinear term, we use the (smaller) symplectic normal bundle [98] instead of the ordinary one, see [26, Section 6b] to define (symplectic) normal linear part, compare with conclusion (iii) of Theorem 1.6.

We recall from [26,98] the definition of the symplectic normal bundle and the notion of symplectic normal linearization. Consider the symplectic manifold (M, σ) being our phase space and an isotropic submanifold $V \subset M$. Let $Y = Y(x, y, z)$ be a Hamiltonian vector field on (M, σ) with V as an invariant submanifold. We define a normal linearization of such a vector field by considering the (smaller) symplectic normal bundle, see [26,98]. Indeed, Let $(TV)^\perp$ be the σ-orthogonal complement of TV, that is,

$$(TV)^\perp = \{ u \in M: \sigma(u, v) = 0, \text{ for all } v \in V \}.$$

By [1,98], the quotient $N_\sigma V = (TV)^\perp / TV$ is a symplectic vector bundle over V, called the *symplectic normal bundle* of V in M. Notice that $\dim(TV)^\perp = n + 2q$ and $\dim TV_x = n$, it follows that the fiber $(T_x V)^\perp / T_x V$ $(x \in V)$ considered as a vector space is isomorphic to $\mathbb{R}^{2q} / T_x V$. The base of the bundle $N_\sigma V$ is the submanifold $V \cong \mathbb{T}^n$, we have a trivialization $N_\sigma V \cong \mathbb{T}^n \times \mathbb{R}^{2q}$. Since the flow of Y leaves the submanifold V invariant, it induces a smooth flow on $N_\sigma V$ and therefore a vector field $N_\sigma Y$ on it, called a (symplectic) normal linearization of Y. Following [26], we state that this vector field $N^\sigma Y$ obtains the form

$$N^\sigma Y(x, y, z) = f(x, 0, 0)\frac{\partial}{\partial x} + \left[\frac{\partial h}{\partial z}(x, 0, 0)z \right]\frac{\partial}{\partial z}, \tag{C.2}$$

where $\frac{\partial h}{\partial z}(x, 0, 0) \in \mathfrak{sp}(2q, \mathbb{R})$. Now preservation of the normal linear part in Theorem 1.6 is expressed in terms of the symplectic normal linearization.

A question one now may ask is what is the relation between the ordinary normal linear part NY and the symplectic normal linear part $N^\sigma Y$. By a direct computation, we obtain the following

$$g_y(x, 0, 0) = -\left(\frac{\partial f(x, 0, 0)}{\partial x} \right)^T \quad \text{and} \quad \frac{\partial^2 G(x, 0, z)}{\partial z^2} = \frac{\partial h_z(x, 0, 0)}{\partial x} J_{2q},$$

where J_{2q} is the symplectic matrix given by (7). This shows that the vector field NY is completely determined by the $N^\sigma Y$. In other words, the symplectic normal linear part $N^\sigma Y$ contains all data of the vector field Y with respect to the ordinary normal bundle $NV = T_V M / T V$.

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