University of Groningen

Global Properties of Integrable Hamiltonian Systems

Lukina, O.V.; Takens, F.; Broer, H.W.

Published in:
Regular & chaotic dynamics

DOI:
[10.1134/S1560354708060105](https://doi.org/10.1134/S1560354708060105)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2008

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Lukina, O. V., Takens, F., & Broer, H. W. (2008). Global Properties of Integrable Hamiltonian Systems. *Regular & chaotic dynamics*, 13(6), 588-630. <https://doi.org/10.1134/S1560354708060105>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Global Properties of Integrable Hamiltonian Systems

O.V. Lukina*, F. Takens**, and H.W. Broer***

*Institute for Mathematics and Computer Science, University of Groningen
P.O. Box 407, 9700 AK Groningen, The Netherlands*

Received May 31, 2008; accepted August 22, 2008

Abstract—This paper deals with Lagrangian bundles which are symplectic torus bundles that occur in integrable Hamiltonian systems. We review the theory of obstructions to triviality, in particular monodromy, as well as the ensuing classification problems which involve the Chern and Lagrange class. Our approach, which uses simple ideas from differential geometry and algebraic topology, reveals the fundamental role of the integer affine structure on the base space of these bundles. We provide a geometric proof of the classification of Lagrangian bundles with fixed integer affine structure by their Lagrange class.

MSC2000 numbers: 37J15, 37J35, 57R17, 57R20, 57R22

DOI: 10.1134/S1560354708060105

Key words: integrable Hamiltonian system, global action-angle coordinates, symplectic topology, monodromy, Lagrange class, classification of integrable systems

Contents

1	PROBLEM SETTING AND SUMMARY OF THE RESULTS	603
1.1	Setting of the Problem	603
1.2	Motivation and Terminology	603
1.3	Preliminaries on Geometry of Lagrangian Bundles	604
1.4	Summary of the Results	606
1.5	Historical Comments	606
2	THE LIOUVILLE–ARNOLD INTEGRABILITY THEOREM REVISITED	607
2.1	Local Topological Triviality of Lagrangian Bundles	608
2.2	Proof of the Liouville–Arnold Theorem	609
3	COMPLETE INVARIANTS OF LAGRANGIAN BUNDLES	618
3.1	Monodromy	618
3.2	The Chern and the Lagrange Classes of a Lagrangian Bundle	620
	APPENDIX A. BASIC FACTS OF SYMPLECTIC GEOMETRY	628
	APPENDIX B. CHARACTERISTIC CLASSES FOR CIRCLE BUNDLES	631
B.1	Bundles and Bundle Morphisms	631
B.2	Classification of Principal Circle Bundles	632
	APPENDIX C. SHEAVES AND SHEAF COHOMOLOGY	635
C.1	Sheaves and Sheaf Cohomology of an Open Cover	635
C.2	Fine Sheaves	638
C.3	Real and Integer Cohomology of a Smooth Manifold	639
	ACKNOWLEDGMENTS	643
	REFERENCES	643

* Address from November 2008: Department of Mathematics, University of Leicester, University Road, LE1 7RH, Leicester, UK; e-mail: o.v.lukina@gmail.com

** E-mail: h.w.broer@rug.nl

*** E-mail: f.takens@rug.nl

1. PROBLEM SETTING AND SUMMARY OF THE RESULTS

We continue the study by Duistermaat [1] and Nguyen [2] of global properties of Lagrangian bundles, i.e. symplectic n -torus bundles with Lagrangian fibres, as these occur in integrable Hamiltonian systems. The main interest is in obstructions to triviality and classification. We review the geometrical setting of the problem and give a summary of the results. Details will be dealt with in sections 2 and 3 and in the appendices.

1.1. Setting of the Problem

Let (M, σ) be a connected $2n$ -dimensional symplectic manifold, that is, M is a smooth manifold and σ is a closed 2-form on M such that the $2n$ -form

$$\underbrace{\sigma \wedge \cdots \wedge \sigma}_n$$

is nowhere zero. Let B be a connected n -dimensional smooth manifold, and call a locally trivial bundle $f : M \rightarrow B$ a *Lagrangian* bundle if df has maximal rank everywhere on M and the fibres of f are compact connected Lagrangian submanifolds of M , that is, for any $p \in M$ and any $X, Y \in T_p F_p$, where $F_p = f^{-1}(f(p))$,

$$\sigma(X, Y) = 0.$$

Obstructions to global triviality of Lagrangian bundles were studied in [1, 2]. In this work we give new proofs and elucidate certain aspects of the results of [1]. In particular, we stress the role of the *integer affine structure*, the definition of which follows, and explain in more detail the symplectic classification of Lagrangian bundles with a given integer affine structure.

1.2. Motivation and Terminology

Lagrangian bundles arise naturally in the study of integrable Hamiltonian systems, as we show now. Recall [3, 4] that the phase space of a conservative mechanical system is a symplectic manifold (M, σ) , and the dynamics in the system is given by a function

$$H : M \rightarrow \mathbb{R},$$

called the *Hamiltonian*, which defines a vector field X_H on M such that

$$dH = -\sigma(X_H, -).$$

The vector field X_H is called the *Hamiltonian* vector field associated to H , and we note that the time t map $X_{H,t}$ of X_H preserves the symplectic form in the sense that

$$X_{H,t}^* \sigma = \sigma.$$

Such a Hamiltonian dynamical system is denoted by (M, σ, H) . Let F be a function on M with $X_H(F) = dF(X_H) = 0$. We call such a function a *first integral* of X_H , or of H . From the definition of the Hamiltonian vector field X_H which also applies to X_F it follows that

$$\sigma(X_F, X_H) = 0,$$

which is usually expressed by saying that F and H are *in involution*. The vector field X_F defines a *symmetry* of the system (M, σ, H) , that is, for any time t

$$X_{F,t}^* \sigma = \sigma \quad \text{and} \quad H \circ X_{F,t} = H,$$

where, as before, $X_{F,t}$ denotes the time t map of X_F . Recall [4] that a Hamiltonian system is called *integrable* if there are n functions $F_1, \dots, F_n = H$ on M , such that $dF_1, \dots, dF_n$ are linearly independent everywhere on M except on a set of points of measure zero, and $F_1, \dots, F_n$ are pairwise in involution.

Now let $f : M \rightarrow B$ be a Lagrangian bundle, and $h : B \rightarrow \mathbb{R}$ be a smooth function. We can consider the function $H = h \circ f$ as the Hamiltonian on M . Then for any $h' : B \rightarrow \mathbb{R}$ the function $h' \circ f$ is a first integral of $h \circ f$, and $h \circ f$ is a first integral of $h' \circ f$. All first integrals obtained in such a way are in involution. Notice that, whenever it will cause no confusion, in what follows we will use the same symbol for a function $h : B \rightarrow \mathbb{R}$ and the corresponding function $h \circ f$ on M . Locally on B one can choose n functions $h_1, \dots, h_n$ such that $dh_1, \dots, dh_n$ are linearly independent, thus the bundle $f : M \rightarrow B$ admits locally n first integrals $h_1, \dots, h_n$, where $h_n = H$.

Example 1.1 (The spherical pendulum). [1, 5] The spherical pendulum has an associated Lagrangian bundle. To see that first recall that the spherical pendulum consists of a unit mass particle moving on the surface of the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ under a uniform vertical gravitational force. Thus the configuration space of the spherical pendulum is $\mathbb{S}^2$, and the phase space is the cotangent bundle $T^*\mathbb{S}^2$ with canonical 2-form σ . The Hamiltonian H of the pendulum is the sum of the kinetic and the potential energy of the particle. The system is invariant under the rotation with respect to the vertical axis in $\mathbb{R}^3$, and the corresponding first integral is the angular momentum J . The map (J, H) has maximal rank almost everywhere on $T^*\mathbb{S}^2$, and compactness of fibres of (J, H) follows from compactness of level sets of the Hamiltonian H . Removing from $T^*\mathbb{S}^2$ fibres containing the points where dJ and dH are linearly dependent, one obtains a Lagrangian bundle, the fibres of which are 2-tori.

If there is no danger of confusion, in what follows we refer to the bundle $f : M \rightarrow B$ by its total space M .

1.3. Preliminaries on Geometry of Lagrangian Bundles

We recall the setting of local and global geometry of Lagrangian bundles and introduce the notion of integer affine structure as this plays a fundamental role in our considerations.

1.3.1. The local geometry of Lagrangian bundles

The local geometry of Lagrangian bundles was studied in [1, 3, 6–8] and is described by the Liouville–Arnold theorem, which states that for each $p \in M$ there exists an open neighborhood $\mathcal{O} \subset M$ of the fibre $F_p = f^{-1}(f(p))$ and local *action-angle* coordinates $(x, \varphi) = (x_1, \dots, x_n, \varphi_1, \dots, \varphi_n)$ on $\mathcal{O}$ such that (x, φ) is symplectic, that is,

$$\sigma = \sum_{i=1}^n d\varphi_i \wedge dx_i,$$

where the functions φ_i take values in $\mathbb{R}/\mathbb{Z}$, and the functions x_i factor through f to smooth local coordinate functions on B . The existence of action-angle coordinates implies that $f : M \rightarrow B$ locally is a principal torus bundle (see Appendix B or [9–11]). The action of the torus in M is induced by a natural action of fibres of T^*B on fibres of M , which can be defined independent of the coordinates, and with respect to our present action-angle coordinates (x, φ) has the following form.

For each $b \in B$ an element $\sum_{i=1}^n \alpha_i dx_i(b)$ in T_b^*B , regarded as an additive group, acts on $f^{-1}(b)$ by sending the point with coordinates $(\bar{\varphi}_1, \dots, \bar{\varphi}_n)$ to the point with coordinates $(\bar{\varphi}_1 + \alpha_1, \dots, \bar{\varphi}_n + \alpha_n)$. The kernel P_b of the action is a discrete subgroup of T_b^*B generated by $dx_1(b), \dots, dx_n(b)$. The set

$$P = \bigcup_{b \in B} P_b$$

is a subset of T^*B , and the projection $P \rightarrow B$ is a locally trivial bundle with fibre $\mathbb{Z}^n$, called the bundle of period lattices. By abuse of language we will call the bundle $g : P \rightarrow B$ the *period lattice* [1]. For each $b \in B$ the n -torus T_b^*B/P_b acts freely and transitively on $f^{-1}(b)$, and for $b' \in B$ near b there is a canonical isomorphism between the tori T_b^*B/P_b and $T_{b'}^*B/P_{b'}$. Thus locally there is a smooth torus action, and $f : M \rightarrow B$ is a principal n -torus bundle locally.

We note that the canonical symplectic structure on T^*B induces a symplectic structure on the set

$$T^*B/P = \bigcup_{b \in B} T_b^*B/P_b,$$

making the projection $T^*B/P \rightarrow B$ another Lagrangian bundle. The Liouville–Arnold theorem implies that the bundles $f : M \rightarrow B$ and $T^*B/P \rightarrow B$ are locally isomorphic; globally this need

not be the case. In fact, the set of all possible Lagrangian bundles with base B and period lattice P is classified, up to symplectic bundle isomorphisms, by their Lagrange classes, see Section 1.3.2.

Transformations between local action-angle coordinates have to preserve the period lattice, which makes them very rigid. If $(\tilde{x}, \tilde{\varphi})$ is another system of action-angle coordinates then in the common domain of x and $\tilde{x}$ the differentials $(dx_1, \dots, dx_n)$ and $(d\tilde{x}_1, \dots, d\tilde{x}_n)$ generate the same period lattice. This means that $(dx_1, \dots, dx_n)$ and $(d\tilde{x}_1, \dots, d\tilde{x}_n)$ are related by a matrix in $\mathbb{GL}(n, \mathbb{Z})$, the group of invertible $(n \times n)$ matrices with integer coefficients, and the same is true for $(x_1, \dots, x_n)$ and $(\tilde{x}_1, \dots, \tilde{x}_n)$ up to a translation. A transformation in $\mathbb{R}^n$, which can be obtained as a composition of an element in $\mathbb{GL}(n, \mathbb{Z})$ and a translation, is called an *integer affine transformation*, and the group of such transformations is denoted by $\mathbb{R}^n \rtimes \mathbb{GL}(n, \mathbb{Z})$, where $\rtimes$ stands for the *semi-direct product* [12]. We see that the base manifold B necessarily has an *integer affine structure*, meaning that it admits an atlas such that the transformations between charts are integer affine transformations.

1.3.2. Global properties of Lagrangian bundles

Global properties of Lagrangian bundles were studied in [1, 2]. Obstructions to triviality of Lagrangian bundles, monodromy and the Lagrange class were introduced in [1]. Briefly, they are defined as follows.

Monodromy. Monodromy is an invariant for the period lattices P and the obstruction to the period lattice being trivial, i.e. to the existence of an isomorphism $P \rightarrow B \times \mathbb{Z}^n$, and the obstruction to the Lagrangian bundle being a principal torus bundle globally. On the one hand, this means that if the monodromy of P is non-trivial, then the Lagrangian bundle M is not a principal bundle globally and is certainly non-trivial; on the other hand, if the monodromy is trivial, the Lagrangian bundle M is a principal torus bundle globally, but it still need not be trivial. More precisely, if the monodromy is zero, then the period lattice $P \rightarrow B$ is trivial in the sense that there is a basis $\alpha_1, \dots, \alpha_n$ of P_b depending smoothly on $b \in B$. Then for each $b, b' \in B$ there is a canonical isomorphism between the groups T_b^*B/P_b and $T_{b'}^*B/P_{b'}$. Then a smooth action of $\mathbb{T}^n$ can be defined globally, and $f : M \rightarrow B$ is globally a principal torus bundle. The lack of triviality of $P \rightarrow B$ is described by the monodromy map

$$H : \pi_1(B, b_0) \rightarrow \text{Aut } P_{b_0} \cong \mathbb{GL}(n, \mathbb{Z}), \quad (1.1)$$

where $b_0 \in B$ is fixed, and $\mathbb{GL}(n, \mathbb{Z})$ is the group of automorphisms of $\mathbb{Z}^n$. The lattice P has unique path-lifting, as in the case of covering spaces [13–15], and the map H is defined using this property. Namely, for each closed loop γ based at the point $b_0 \in B$ the image $H([\gamma])$ is the automorphism of P_{b_0} which maps the initial point of any lift $\tilde{\gamma}$ of γ to its terminal point. We say that period lattices $P \rightarrow B$ and $P' \rightarrow B$ with monodromy maps H and H' respectively have the same monodromy if there exists a group isomorphism

$$\mathcal{I} : P_{b_0} \rightarrow P'_{b_0}$$

such that $H' = \mathcal{I}H\mathcal{I}^{-1}$. The monodromy map is a complete invariant of period lattices in the sense that if $P \rightarrow B$ and $P' \rightarrow B$ have the same monodromy, then there exists a diffeomorphism

$$\Phi : P \rightarrow P'$$

commuting with projections on B , and such that $\Phi|_{P_{b_0}} = \mathcal{I}$. However, the existence of such a diffeomorphism $\Phi : P \rightarrow P'$ need not imply that there is a diffeomorphism $\phi : B \rightarrow B$ such that $\phi^*P' = P$.

Example 1.2 (Integer affine structures on the circle). Consider the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with a coordinate $\bar{x} \in \mathbb{R}/\mathbb{Z}$, denote by $\pi : T^*\mathbb{S}^1 \rightarrow \mathbb{S}^1$ the projection and let (x, y) be standard symplectic coordinates on $T^*\mathbb{S}^1$ such that $x = \bar{x} \circ \pi$. Define period lattices P and P' by

$$P = \{(x, y) \in T^*\mathbb{S}^1 \mid y \in \mathbb{Z}\} \text{ and } P' = \{(x, y) \in T^*\mathbb{S}^1 \mid y \in \tfrac{1}{2}\mathbb{Z}\}.$$

The length of $\mathbb{S}^1$ measured by action coordinates in $T^*\mathbb{S}^1/P$ and $T^*\mathbb{S}^1/P'$ is 1 and 2 respectively, so these integer affine structures are non-equivalent, that is, there is no diffeomorphism $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $\phi^*P' = P$. Also the symplectic areas of $T^*\mathbb{S}^1/P$ and $T^*\mathbb{S}^1/P'$ are 1 and $\frac{1}{2}$ respectively.

Chern and Lagrange classes. The Lagrange class can be seen as measuring the difference between the bundles $f : M \rightarrow B$ and $T^*B/P \rightarrow B$. First we restrict to the case where the monodromy is trivial, which means that the monodromy map H is trivial in the sense that it maps each homotopy class of loops in B to the identity of $\mathrm{GL}(n, \mathbb{Z})$. In this case $f : M \rightarrow B$ is a principal $\mathbb{T}^n$ -bundle. Such bundles, but without symplectic structure, are classified by their Chern classes. In the case $n = 1$ the classification is given by the classical theory (Appendix B) with the Chern class taking values in the singular cohomology $H^2(B, \mathbb{Z})$; for $n > 1$ a similar invariant is introduced in the cohomology with coefficients in $\mathbb{Z}^n$. The fact that Lagrangian bundles are n -torus bundles with symplectic structure complicates the problem in two ways. First, a given $\mathbb{T}^n$ -bundle may or may not admit a symplectic form so as to make the bundle Lagrangian; second, if a $\mathbb{T}^n$ -bundle admits a symplectic form, this form is not unique. If σ and σ' are two such symplectic forms on M , then the corresponding bundles are isomorphic if there is a diffeomorphism $\Phi : M \rightarrow M$ such that $\Phi^*\sigma' = \sigma$ and $f = f \circ \Phi$. It turns out that Lagrangian bundles up to isomorphisms are described by the *Lagrange* characteristic class in the first cohomology group of B with coefficients in the sheaf of closed sections of the principal bundle T^*B/P (see Appendix C on preliminaries on sheaf cohomology). This group is related through an exact sequence with cohomology groups of B with integer and real coefficients. If the monodromy is not trivial, the same argument remains true, only instead of cohomology with coefficients in $\mathbb{Z}^n$ the Chern class is an element of the cohomology with coefficients in a sheaf of sections in the period lattice $P \rightarrow B$. This is also called the cohomology with *twisted coefficients* (see [11] for a simple example). The bundle T^*B/P is a *locally* principal bundle, and the Lagrange class is an element in the cohomology with coefficients in the sheaf of sections of the *locally* principal bundle T^*B/P .

1.4. Summary of the Results

The present work continues the study of obstructions to triviality of Lagrangian bundles by Duistermaat [1] and Nguyen [2], providing new geometric proofs and clarifying subtle aspects of the theory. The Lagrange class classifies Lagrangian bundles with *fixed* integer affine structure. Monodromy is related to the integer affine structure but does not describe the latter completely, as Example 1.2 shows. Asking for a complete classification of integer affine structures makes no sense, since this would be similar to asking for a classification of open sets in $\mathbb{R}^n$ up to translation. The problem of classification of Lagrangian bundles can be solved only partially, namely, one can classify period lattices by monodromy, and classify Lagrangian bundles with fixed integer affine structures by the Lagrange class. The role of an integer affine structure is somewhat latent in [1, 2], and is stressed in this work. We also give a systematic geometric treatment of the classification of Lagrangian bundles with fixed integer affine structure by the Chern and the Lagrange classes.

1.5. Historical Comments

The study of the local geometry of Lagrangian bundles started in the 19-th century with the work of Liouville [16], and the first geometric formulation of the problem and a geometric proof of the Liouville–Arnold theorem were given in the 20-th century by Arnold and Avez [6]. Certain assumptions of Arnold and Avez [6] were later relaxed by Markus and Meyer [7] and by Jost [17]. Some authors also mention the works by Mineur which appeared earlier than Arnold and Avez [6] but remained unknown to a wider audience (see Miranda and Nguyen [18] for a discussion). Other proofs of the Liouville–Arnold theorem can be found in Duistermaat [1], Bates and Śniatycki [8], Audin [19], Abraham and Marsden [4], and others.

Global properties of Lagrangian bundles were studied by Duistermaat in [1], who introduced monodromy and the Lagrange class as obstructions to triviality of Lagrangian bundles. The term “Lagrangian class” (we choose the name “Lagrange class” by analogy with “Chern class”) was introduced by Nguyen [2] who elucidated certain aspects of [1] and extended the theory to Lagrangian bundles with singularities. The first example of a Lagrangian bundle with non-trivial monodromy, the spherical pendulum, was found by Cushman and first mentioned by Duistermaat [1]. The interest in global properties of Lagrangian bundles increased after Cushman and Duistermaat [20], and later Child [21], Cushman and Sadovskii [22] discovered that monodromy explains certain phenomena in joint spectra of atoms. Examples of classical and quantum systems

with non-trivial monodromy were found by Cushman and Bates [5], Bates [23], Giacobbe, Cushman and Sadovskii [24], Efstathiou [25] and many others. Vũ Ngọc [26], Matveev [27] and Nguyen [28] proved that monodromy in 2-dimensional Lagrangian bundles is related to the presence of focus-focus singularities, and systems exhibiting such phenomena as fractional monodromy and bidromy were discovered in Nekhoroshev, Sadovskii and Zhilinskii [29], Sadovskii and Zhilinskii [30]. The work by Miranda and Nguyen [18], Giacobbe [31], Vũ Ngọc [32] is devoted to semi-local classification of integrable Hamiltonian systems near singular fibres, and the papers by Nekhoroshev, Sadovskii and Zhilinskii [33], Efstathiou, Cushman and Sadovskii [34] investigate fractional monodromy.

2. THE LIOUVILLE–ARNOLD INTEGRABILITY THEOREM REVISITED

In this section we give a geometric proof of the Liouville–Arnold theorem and establish the existence of the integer affine structure on the base space of a Lagrangian bundle, a precise definition of which we give below. We recall that we assume the base manifold B and fibres of the Lagrangian bundle to be (pathwise) connected [13].

Definition 2.1 (Lagrangian bundle). Let (M, σ) be a connected $2n$ -dimensional symplectic manifold, and B be a connected n -dimensional smooth manifold. Then a surjective map $f : M \rightarrow B$ is called a *Lagrangian bundle* if it has the following properties:

1. df has maximal rank everywhere on M .
2. Fibres of f are connected compact Lagrangian submanifolds of M .

Definition 2.1 does not require a priori that a Lagrangian bundle is locally trivial (see Appendix B on bundles). We prove local triviality in Section 2.1.

Theorem 2.1 (Liouville–Arnold). [1] *Let $f : M \rightarrow B$ be a Lagrangian bundle and $b \in B$. Then there exists an open neighborhood $V \subset B$ of b and coordinates $(I_1, \dots, I_n, \varphi_1, \dots, \varphi_n)$ on $f^{-1}(V)$ such that:*

1. *Each I_j factors through B , that is: there exist functions x_j on B such that $I_j = x_j \circ f$, and $dx_1, \dots, dx_n$ are linearly independent.*
2. *Each φ_j takes values in $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$,*
3. *The coordinates (I, φ) are symplectic, that is*

$$\sigma|_{f^{-1}(V)} = \sum_{j=1}^n d\varphi_j \wedge dI_j. \quad (2.1)$$

4. *The coordinates I are unique up to an integer affine transformation in $\mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{Z})$.*
5. *Given I , the coordinates φ are determined by the choice of a Lagrangian submanifold on which all φ_i are zero.*

The coordinates (I, φ) are called *action-angle* coordinates. The Lagrangian submanifold in item 5 plays the role of the “zero section” in the bundle M , and action-angle coordinates (I, φ) play the role of local trivializations of a Lagrangian bundle, making it locally a principal n -torus bundle. Our proof of Theorem 2.1 is based on the idea of Duistermaat [1] to use the natural fibrewise action of T^*B on M to construct local trivializations and is given in section 2.2.

2.1. Local Topological Triviality of Lagrangian Bundles

We show that a Lagrangian bundle $f : M \rightarrow B$ (Definition 2.1) is locally topologically trivial with smooth trivializations. Denote by $F_b = f^{-1}(b)$ the fibre at $b \in B$.

Proposition 2.1 (Local triviality of Lagrangian bundles). *A Lagrangian bundle $f : M \rightarrow B$ is locally trivial, that is, for each $b \in B$ there exists an open neighborhood $V \subset B$ and a diffeomorphism*

$$\phi_V : f^{-1}(V) \rightarrow V \times F_b \quad (2.2)$$

commuting with projections on V .

Proof. Since F_b is compact, it has a tubular neighborhood $\mathcal{W}$ in M [35] with a smooth retraction map

$$r : \mathcal{W} \rightarrow F_b,$$

that is, $r|_{F_b} = \text{id}$. We show that the map

$$(f, r) : \mathcal{W} \rightarrow f(\mathcal{W}) \times F_b : p \mapsto (f(p), r(p)),$$

which makes the following diagram commutative,

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{(f, r)} & f(\mathcal{W}) \times F_b \\ f \downarrow & & \downarrow \text{pr}_1 \\ f(\mathcal{W}) & \xrightarrow{\text{id}} & f(\mathcal{W}) \end{array}$$

is a diffeomorphism near F_b . Indeed, since df has maximal rank everywhere on M and $r|_{F_b} = \text{id}$, $d(f, r)$ has maximal rank on F_b , by the Inverse Function theorem [4] (f, r) is a local diffeomorphism near each point of F_b . This means that each $p \in F_b$ has an open neighborhood $\mathcal{O}_p \subset \mathcal{W}$ such that $(f, r)|_{\mathcal{O}_p}$ is a diffeomorphism. Then (f, r) has maximal rank on $\mathcal{O} = \bigcup_{p \in F_b} \mathcal{O}_p$, and to prove that it is a diffeomorphism near F_b we only have to show that (f, r) is injective on a neighborhood $W \subset \mathcal{O}$. This is the content of the following lemma.

Lemma 2.1 (Trivializing neighborhood of F_b). *There exists an open neighborhood $W \subset \mathcal{O}$ of F_b such that the restricted map $(f, r)|_W$ is injective.*

Proof. Suppose there is no such neighborhood, and consider a decreasing sequence $\{W_j\}_{j \in \mathbb{N}}$ of relatively compact neighborhoods of F_b such that

$$\bigcap_j W_j = F_b.$$

Then there exist sequences of points $p_j, p'_j \in W_j$ such that

$$p_j \neq p'_j \text{ and } (f, r)(p_j) = (f, r)(p'_j). \quad (2.3)$$

Passing to a subsequence, we may assume that both p_j and p'_j converge to p and p' respectively, then by continuity $(f, r)(p_j)$ converges to $(f, r)(p) = (f, r)(p') \in F_b$. Since (f, r) is injective on F_b this implies $p = p'$. Fix an open neighborhood $\mathcal{O}_p$ of p such that $(f, r)|_{\mathcal{O}_p}$ is a diffeomorphism. Then there exists a number $j \in \mathbb{N}$ such that $p_j, p'_j \in \mathcal{O}_p$, so $p_j \neq p'_j$ implies $(f, r)(p_j) \neq (f, r)(p'_j)$ which contradicts (2.3). This means that there exists a neighborhood W of F_b such that $(f, r)|_W$ is injective. $\square$

Returning to the proof of Proposition 2.1 we now choose a neighborhood $V \subset B$ of b such that $f^{-1}(V) \subset W$. Then the restriction

$$\phi_V = (f, r) : f^{-1}(V) \rightarrow V \times F_b$$

is a local trivialization of $f : M \rightarrow B$. $\square$

It should be noted that Proposition 2.1 is just the Ehresmann theorem [36].

2.2. Proof of the Liouville–Arnold Theorem

In this section we prove the Liouville–Arnold theorem 2.1, using the natural action of fibres of T^*B on fibres of M , as it was described in Section 1.3.1.

2.2.1. Action of fibres of T^*B on fibres of M

To construct the action of fibres of T^*B on fibres of M , for each $b \in B$ we will define the map

$$s_b : T_b^*B \rightarrow \text{Diff}(F_b),$$

where $F_b = f^{-1}(b)$, such that

$$s_b(\alpha + \beta) = s_b(\alpha) \circ s_b(\beta).$$

Then T_b^*B , regarded as an additive group, acts on F_b by diffeomorphisms

$$T_b^*B \times F_b \rightarrow F_b : (\alpha, p) \mapsto s_b(\alpha)(p),$$

and the action is transitive but not free. We study the isotropy group P_b of the action, identify the fibre F_b with the n -torus T_b^*B/P_b and show that there is a free smooth action of the n -torus on a small neighborhood of F_b .

Definition of the action. Fix a point $b \in B$. To obtain the map s_b we first define the map

$$S_b : T_b^*B \rightarrow \mathcal{X}(F_b), \quad (2.4)$$

where $\mathcal{X}(F_b)$ denotes the set of vector fields on the fibre F_b , and obtain s_b by taking the time 1 map of a vector field $S_b(-)$ in (2.4). To define $S_b(\alpha)$ for $\alpha \in T_b^*B$ choose a smooth function $h : B \rightarrow \mathbb{R}$ such that $dh(b) = \alpha$, and set

$$S_b(\alpha) = X_{h \circ f}|_{F_b}, \quad (2.5)$$

where $X_{h \circ f}$ is the Hamiltonian vector field associated to the function $h \circ f$ on M by

$$\sigma(X_{h \circ f}, -) = -d(h \circ f)(-), \quad (2.6)$$

and $X_{h \circ f}|_{F_b}$ denotes the restriction of the vector field to the fibre F_b . The following lemma shows that the map S_b is well-defined, i.e. the vector field $S_b(\alpha)$ is tangent to the fibre F_b , and $S_b(\alpha)$ does not depend on the choices made in the definition.

Lemma 2.2 (S_b is well-defined). *The map*

$$S_b : T_b^*B \rightarrow \mathcal{X}(F_b) : \alpha \mapsto X_{h \circ f}|_{F_b}$$

is well-defined, that is, given $\alpha \in T_b^*B$:

1. *For any choice of a function $h : B \rightarrow \mathbb{R}$ such that $dh(b) = \alpha$, the Hamiltonian vector field $X_{h \circ f}$ is tangent to fibres of M .*
2. *The restriction $X_{h \circ f}|_{F_b}$ depends only on α , and not on the choice of the function h .*

Proof. To prove the first item notice that for any vector field Y on F_b we have

$$d(h \circ f)(Y) = -\sigma(X_{h \circ f}, Y) = 0,$$

which implies that at each $p \in M$ we have $X_{h \circ f}(p) \in T_p F_{f(p)}^\sigma$, where $T_p F_{f(p)}^\sigma$ denotes the skew-orthogonal complement of $T_p F_{f(p)}$ (Appendix A). Since fibres of M are Lagrangian, $T_p F_{f(p)}^\sigma = T_p F_{f(p)}$, so $X_{h \circ f}$ is tangent to fibres of M . The fact that the restriction $X_{h \circ f}|_{F_b}$ depends only on $\alpha \in T_b^*B$ follows from the fact that the restriction of the 1-form $d(h \circ f)$ to F_b depends only on α (see (2.6)). $\square$

Remark 2.1 (Involutivity of first integrals). For any two functions $h, h' : B \rightarrow \mathbb{R}$ we have by Lemma 2.2

$$\{h' \circ f, h \circ f\} = \sigma(X_{h' \circ f}, X_{h \circ f}) = 0, \quad (2.7)$$

i.e. the functions $h \circ f$ and $h' \circ f$ are in involution. This means that any two functions on M which factor through B are first integrals of each other (see Section 1 for the definition of the first integral).

We also show that S_b is linear, and the image of S_b in $\mathcal{X}(F_b)$ consists of vector fields with commuting flows, i.e. [9, 37]

$$[S_b(\alpha), S_b(\beta)] = 0.$$

This is the content of the following lemma.

Lemma 2.3 (Linearity of S_b). *The map*

$$S_b : T_b^*B \rightarrow \mathcal{X}(F_b) : \alpha \mapsto X_{h \circ f}|_{F_b}$$

has the following properties:

1. S_b is linear,
2. If α is non-zero, then $S_b(\alpha)$ is nowhere zero on F_b ,
3. For any $\alpha, \beta \in T_b^*B$

$$[S_b(\alpha), S_b(\beta)] = 0, \quad (2.8)$$

so the flows of $S_b(\alpha)$ and $S_b(\beta)$ commute.

Proof. The items 1 and 2 follow from bilinearity of the symplectic form σ and the definition (2.6) of the vector field $X_{h \circ f}$. To prove the third item, let $\beta \in T_b^*B$ and let $h' : B \rightarrow \mathbb{R}$ be such that $dh'(b) = \beta$. Then by the properties of the Poisson bracket $\{\cdot, \cdot\}$ (Appendix A) and Remark 2.1 we have

$$[S_b(\alpha), S_b(\beta)] = [X_{h \circ f}|_{F_b}, X_{h' \circ f}|_{F_b}] = X_{\{h \circ f, h' \circ f\}}|_{F_b} = 0.$$

□

Taking the time 1 map $(S_b(-))_1$ of the vector field in (2.4) we obtain the map

$$s_b : T_b^*B \rightarrow \text{Diff}(F_b) : \alpha \mapsto (S_b(\alpha))_1. \quad (2.9)$$

The fibre F_b is compact, so any vector field $S_b(\alpha)$ is complete and the time 1 map $(S_b(\alpha))_1$ is defined for any α . To show that s_b defines an action of T_b^*B as an additive group we will prove that for any $\alpha, \beta \in T_b^*B$

$$s_b(\alpha + \beta) = s_b(\alpha) \circ s_b(\beta),$$

in the following lemma.

Lemma 2.4 (Additivity of T_b^*B -action). *For any $\alpha, \beta \in T_b^*B$ we have*

$$s_b(\alpha + \beta) = s_b(\alpha) \circ s_b(\beta), \quad (2.10)$$

where $s_b : T_b^*B \rightarrow \text{Diff}(F_b)$ is given by (2.9).

Proof. By Lemma 2.3 for any $\alpha, \beta \in T_b^*B$ the flows of $S_b(\alpha)$ and $S_b(\beta)$ commute, so (2.10) holds (see [37]). □

By Lemma 2.4 each element $\alpha \in T_b^*B$ acts on F_b by the diffeomorphism $s_b(\alpha)$, i.e. there is a map

$$T_b^*B \times F_b \rightarrow F_b : (\alpha, p) \mapsto s_b(\alpha)(p).$$

Terminology 2.1 (Action of T_b^*B on F_b). The map s_b is the action of T_b^*B on F_b , and $s_b(-)(p)$ is the map from T_b^*B to F_b , determined by $p \in F_b$; by Lemma 2.5 this map is surjective.

By the hypothesis fibres of M are connected, which implies that the action s_b is transitive. This is the content of the following lemma.

Lemma 2.5 (Transitivity of T_b^*B -action). *The action s_b is transitive, that is, for any $p, p' \in F_b$ there exists an element $\alpha \in T_b^*B$ such that*

$$s_b(\alpha)(p) = p'.$$

Proof. Fix $p \in F_b$, then the derivative of the map

$$s_b(-)(p) : T_b^*B \rightarrow F_b \quad (2.11)$$

has maximal rank, so $s_b(-)(p)$ is a local diffeomorphism. This implies that the orbit of the action through p , i.e. the image of (2.11) in F_b , is open in F_b . Suppose the action s_b is not transitive, then F_b is a union of two or more disjoint orbits which are open sets in F_b . This contradicts the assumption that F_b is connected [4, 13], hence s_b is transitive. $\square$

Remark 2.2 (Smooth dependence on point in the base). We notice that the action s_b depends smoothly on the point b in the base space B . In particular, and this will be of importance later in this section, the smooth action of fibres of T^*B can be defined on a small neighborhood of F_b . Choose $V \subset B$ open, $b \in V$, such that both bundles M and T^*B are trivial when restricted to V , and let $z : V \rightarrow M$ be a smooth local section. Then define the map

$$K : T^*(V) \rightarrow f^{-1}(V),$$

where $T^*(V)$ stands for the restriction $T^*B|V$, i.e. the set of all covectors with base point in $V \subset B$, by

$$K(b', \alpha) = s_{b'}(\alpha)(z(b')),$$

where (b', α) denotes an element $\alpha \in T_{b'}^*B$. The map K has maximal rank and is a local diffeomorphism [4] near each $\alpha \in T^*(V)$. We will use this map later in this section to construct local trivializations of the bundle M .

The isotropy subgroup of the action Since the fibre F_b is compact, and the cotangent space T_b^*B is not compact, the action s_b is not free. The isotropy group of the action is the subgroup

$$P_b = \{\alpha \in T_b^*B \mid s_b(\alpha)(p) = p\}.$$

Since the action s_b is transitive and commutative, the isotropy group P_b does not depend on the point $p \in F_b$ [3]. The following lemma proves that P_b is a lattice in T_b^*B , i.e. P_b is a discrete subgroup of T_b^*B isomorphic to $\mathbb{Z}^n$.

Lemma 2.6 (Isotropy subgroup). *The isotropy subgroup $P_b \subset T_b^*B$ is a lattice in T_b^*B , that is, there exist linearly independent elements $\gamma_1, \dots, \gamma_n \in T_b^*B$ such that*

$$P_b = \{\ell_1\gamma_1 + \dots + \ell_n\gamma_n \mid \ell_j \in \mathbb{Z}\}. \quad (2.12)$$

Proof. For each $p \in F_b$ the map $s_b(-)(p)$ is a local diffeomorphism at each point in T_b^*B , hence each $\alpha \in P_b$ has an open neighborhood in T_b^*B which contains no other point of P_b , i.e. P_b is a discrete subgroup of T_b^*B . It follows [3, 8] that there exists $k \leq n$ linearly independent elements $\gamma_1, \dots, \gamma_k \in T_b^*B$ such that for each $\alpha \in P_b$

$$\alpha = \sum_{i=1}^k \ell_i \gamma_i, \text{ where } \ell_j \in \mathbb{Z}.$$

Dividing by P_b we obtain a diffeomorphism

$$s_b(-)(p) : T_b^*B/P_b \rightarrow F_b,$$

denoted by the same symbol as the similar map of T_b^*B . Since F_b is compact, the quotient T_b^*B/P_b must be compact. This implies that $k = n$, i.e. P_b is generated by n elements. $\square$

Remark 2.3 (Identification of isotropy group with $\mathbb{Z}^n$). The isomorphism between the lattice P_b and $\mathbb{Z}^n$ is determined by the choice of a basis $\gamma_1, \dots, \gamma_n$ of the lattice. Such a basis is not unique, and any two bases of P_b are related by a linear transformation in $\mathbb{GL}(n, \mathbb{Z})$.

The action of the quotient T_b^*B/P_b on F_b , given by

$$T_b^*B/P_b \times F_b \rightarrow F_b : ([\alpha], p) \mapsto s_b(\alpha)(p),$$

is transitive and free. The map

$$s_p = s_b(-)(p) : T_b^*B/P_b \rightarrow F_b \quad (2.13)$$

is a diffeomorphism, depending on the choice of the point $p \in F_b$.

Terminology 2.2 (Action of T_b^*B/P_b on F_b). As for T_b^*B , we denote by s_b the action of T_b^*B/P_b on F_b . We also call the maps $s_b(-)(p)$ given by (2.13) *preferred identifications* of T_b^*B/P_b with F_b .

The quotient T_b^*B/P_b is isomorphic to $\mathbb{T}^n$, and the fibre F_b can also be identified with $\mathbb{T}^n$, which is done in the following lemma.

Lemma 2.7 (Identification of the quotient with the standard torus). *Given a basis $\gamma = (\gamma_1, \dots, \gamma_n)$ in P_b , there is a corresponding group isomorphism*

$$\gamma^* : T_b^*B/P_b \rightarrow \mathbb{T}^n. \quad (2.14)$$

Proof. The choice of a basis $\gamma_1, \dots, \gamma_n$ of P_b determines the isomorphism

$$\gamma^* = (\gamma_1^*, \dots, \gamma_n^*) : T_b^*B \rightarrow \mathbb{R}^n,$$

where γ_i^* , $i = 1, \dots, n$, stand for linear coordinates on T_b^*B so that $\gamma_i^*(\gamma_j) = \delta_{ij}$, and which maps P_b onto the subgroup $\mathbb{Z}^n \subset \mathbb{R}^n$. Dividing by the lattice P_b we obtain the identification

$$\gamma^* : T_b^*B/P_b \rightarrow \mathbb{T}^n, \quad (2.15)$$

of the fibre T_b^*B/P_b with the standard n -torus, which we again denote by γ^* . $\square$

Terminology 2.3 (Identification of fibres of M with standard tori). Given a point $p \in F_b$ and an isomorphism γ^* (2.14), there is an isomorphism

$$\gamma^* \circ s_p^{-1} : F_b \rightarrow \mathbb{T}^n, \quad (2.16)$$

which we call the *preferred* diffeomorphism between F_b and $\mathbb{T}^n$.

Definition 2.2 (Integer affine structure). An *integer affine structure* on a manifold F is given by an atlas, i.e. a collection of local charts covering F , such that the transition maps between these charts are transformations in the semi-direct product $\mathbb{R}^n \rtimes \mathbb{GL}(n, \mathbb{Z})$. We call the transformations in $\mathbb{R}^n \rtimes \mathbb{GL}(n, \mathbb{Z})$ *integer affine transformations*. Let F and F' be manifolds with integer affine structures. A diffeomorphism $g : F \rightarrow F'$ preserves the integer affine structure if for any $p \in F$ and any integer affine chart (V, ψ) on F' such that $g(p) \in V$, there exists an integer affine chart (U, ϕ) of F with $p \in U$ such that $\psi \circ g \circ \phi^{-1} \in \mathbb{R}^n \rtimes \mathbb{GL}(n, \mathbb{Z})$ and the same holds for its inverse. In this case integer affine structures on F and F' are called *isomorphic*.

Remark 2.4 (Integer affine structure on fibres of Lagrangian bundles). The preferred diffeomorphisms (2.16) are determined up to an integer affine transformation, where an element in $\mathbb{GL}(n, \mathbb{Z})$ corresponds to the choice of a basis in the lattice P_b (Remark 2.3), and a translation in $\mathbb{R}^n$ corresponds to the choice of a point $p \in F_b$. This means that the preferred diffeomorphisms give the fibre F_b an integer affine structure.

Smooth local torus action on fibres of M . We prove that the isotropy group P_b smoothly depends on a point $b \in B$, which implies that the quotient T_b^*B/P_b also smoothly depends on b . This implies that locally there is a smooth action of fibres of T^*B/P on M . We prove these facts in the rest of this section.

Choose a small open neighborhood $V \subset B$ of b and a smooth local section $z : V \rightarrow M$, so that (Remark 2.2) there is a local diffeomorphism

$$K : T^*(V) \rightarrow f^{-1}(V) : (b', \alpha) \mapsto s_{b'}(\alpha)(z(b')),$$

determined by z . For each $b' \in V$ the lattice $P_{b'}$ is the preimage of the point $z(b')$. Consider the set

$$P = \bigcup_{b \in B} P_b,$$

and denote by $g = \pi|_P$ the restriction of $\pi : T^*B \rightarrow B$ to P . We call the bundle $g : P \rightarrow B$ the *period lattice*. The restriction $P|_V$, for which in what follows we write $P(V)$, is the preimage of $z(V)$ under the local diffeomorphism K . It follows that P is a closed smooth n -dimensional submanifold of T^*B and the map g is a local diffeomorphism [4]. Moreover, it turns out that $g : P \rightarrow B$ is a locally trivial bundle with fibre $\mathbb{Z}^n$ (Appendix B.1), as is proven in the following lemma.

Lemma 2.8 (Local triviality of P). *The projection $g : P \rightarrow B$ is a locally trivial bundle with fibre $\mathbb{Z}^n$, that is, for $b \in B$ there exists an open neighborhood $V \subset B$ and a homeomorphism*

$$\psi_V : V \times \mathbb{Z}^n \rightarrow P(V), \quad (2.17)$$

commuting with projections on V .

Proof. Let $\gamma_1, \dots, \gamma_n$ be a basis of P_b , and $s_1, \dots, s_n$ be the sections of P through $\gamma_1, \dots, \gamma_n$ respectively. We can find an open neighborhood V of b such that for each s_i , $i = 1, \dots, n$, the restriction $g|_{s_i(V)}$ is a diffeomorphism, and such that for $b' \in V$ the elements $s_1(b'), \dots, s_n(b')$ are linearly independent. Then for any integral linear combination $s = \sum_{i=1}^n \ell_i s_i$, $\ell_i \in \mathbb{Z}$, the restriction $g|_s(V)$ is also a diffeomorphism. Define the map (2.17) by

$$\psi_V(b', \ell) = \ell_1 s_1(b') + \dots + \ell_n s_n(b'), \quad \ell = (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n.$$

This map is injective since the sections $s_1, \dots, s_n$ are linearly independent. We next prove that ψ_V is surjective, i.e. $s_1, \dots, s_n$ is a basis of sections in $P(V)$. Assume that there exists $\beta \in P(V)$ which is not in the image of ψ_V . The elements $s_i(g(\beta))$, $i = 1, \dots, n$, are in the lattice $P_{g(\beta)}$ and form a basis in $T_{g(\beta)}^*B$ over $\mathbb{R}$. Without loss of generality we may assume that β is not an integral multiple of any other element in P . Then there exists a basis $(\beta_1, \dots, \beta_n = \beta)$ of $P_{g(\beta)}$ [3], and

$$s_i(g(\beta)) = \sum_{j=1}^n m_{ij} \beta_j, \quad \text{where } m_{ij} \in \mathbb{Z}.$$

The matrix (m_{ij}) is invertible over $\mathbb{R}$ and its inverse has rational coefficients by the Cramer rule. Therefore,

$$\beta = \sum_{i=1}^n \lambda_i s_i(g(\beta)), \quad \text{where } \lambda_i \in \mathbb{Q},$$

and, since β is not in the image of ψ_V , not all λ_i are integers. Choose a continuous curve $c : [0, 1] \rightarrow V$ such that $c(0) = g(\beta)$ and $c(1) = b$. On a small neighborhood of $c(0)$ we can lift the curve $c(t)$ to a continuous curve $\tilde{c}(t)$ in P through β by a local section, so

$$\tilde{c}(t) = \sum_{i=1}^n \lambda_i(t) s_i(c(t)).$$

We note that, as before, $\lambda_i(t)$ have to be rational and continuous in t . This means that the $\lambda_i(t)$ are independent of t . From this it follows that the local lift $\tilde{c}(t) \subset P$ can be extended over the whole interval $[0, 1]$. This implies that

$$\tilde{c}(1) = \sum_{i=1}^n \lambda_i s_i(b) = \sum_{i=1}^n \lambda_i \gamma_i \in P_b,$$

which is a contradiction. It follows that ψ_V is surjective. Thus ψ_V is a homeomorphism. $\square$

Remark 2.5 (Unique path-lifting). It follows from Lemma 2.8 that, like covering spaces [13, 14], the period lattice $g : P \rightarrow B$ has *unique path-lifting*, i.e., given a continuous curve $c(t)$ in the base B and a point $\alpha \in P_{c(0)}$ there is a unique continuous curve $\tilde{c}(t)$ in P such that $\tilde{c}(0) = \alpha$ and $g \circ \tilde{c} = c$.

Now, both $T^*(V)$ and $P(V)$ are trivial, i.e. there are local bundle isomorphisms

$$V \times \mathbb{R}^n \rightarrow T^*(V), \text{ and } V \times \mathbb{Z}^n \rightarrow P(V),$$

and it follows that there is a diffeomorphism

$$T^*(V)/P(V) \rightarrow V \times \mathbb{T}^n,$$

where $T^*(V)/P(V)$ stands for the restriction $T^*B/P|_V$. This means that T^*B/P is a smooth manifold and $\tau : T^*B/P \rightarrow B$ is a locally trivial bundle with fibre $\mathbb{T}^n$. Fibres of the bundle T^*B/P are groups acting on fibres of the bundle M with the same base and, since T^*B/P is locally trivial, the action of fibres of T^*B/P on fibres of M is locally smooth.

Terminology 2.4 (Locally principal bundles T^*B/P and M). Recall (Appendix B) that a bundle $\pi : E \rightarrow B$ is called a principal bundle with group G if there is a fibre-preserving, smooth and free action of G on E , which is transitive on fibres. In our case a local identification of fibres in T^*B/P is induced by the identification of fibres in the period lattice P , which by continuity is unique. This means that there is a smooth local action of the n -torus T_b^*B/P_b on T^*B/P and M near T_b^*B/P_b and F_b respectively, so T^*B/P and M locally are principal bundles.

The map $K : T^*(V) \rightarrow f^{-1}(V)$ defined in Remark 2.2 factors on V to a diffeomorphism

$$K : T^*(V)/P(V) \rightarrow f^{-1}(V)$$

which we denote by the same symbol and which, as the map in Remark 2.2, is determined by the choice of a local section $z : V \rightarrow M$.

Terminology 2.5 (Smooth local trivializations of M). The maps K can be viewed as local bundle isomorphisms between T^*B/P and M , which are in one-to-one correspondence with smooth local sections $z : V \rightarrow M$. There is a similar situation in principal circle bundles (Appendix B): a local trivialization of a principal circle bundle $\pi : S \rightarrow B$ can be regarded as a local isomorphism between S and the trivial bundle $B \times \mathbb{T}^1 \rightarrow B$, and the trivializations are in one-to-one correspondence with local sections in S . By analogy we call K or z smooth local *trivializations* of the Lagrangian bundle $f : M \rightarrow B$.

Terminology 2.6 (Transition maps of smooth trivializations of M). In a Lagrangian bundle $f : M \rightarrow B$ let V be a simply connected neighborhood of b (such a neighborhood always exists, see Appendix C.3 or [38]), and let $z, z' : V \rightarrow M$ be smooth local section with corresponding trivializations. Then there exists a differential 1-form β on V determined up to a local section in $P(V)$ such that for each $b' \in V$

$$z'(b') = s_{b'}(\beta(b'))(z(b')).$$

Thus, like in a principal circle bundle (Appendix B), the difference between two trivializations of M is determined by a local section in the bundle T^*B/P of groups, acting on fibres of M , i.e. such sections occurs as *transition maps* of M . This resemblance to principal bundles will be used in Section 3 to define obstructions to triviality of Lagrangian bundles.

2.2.2. Local symplectic identification of T^*B/P and M

In this section we will show that local trivializations

$$K : T^*(V)/P(V) \rightarrow f^{-1}(V) \quad (2.18)$$

constructed in Section 2.2.1, can be chosen to be symplectic. For that we have to prove that the quotient manifold T^*B/P is symplectic. This is achieved in the following sequence of steps: we show that the smooth map

$$K : T^*(V) \rightarrow f^{-1}(V),$$

constructed in Remark 2.2 in Section 2.2.1, is symplectic if and only if the section $z : V \rightarrow M$, which determines the map, is Lagrangian. The necessity is straightforward: if K is symplectic, then it has to map the Lagrangian zero section $V \subset T^*(V)$ onto a Lagrangian submanifold of M ; hence K can only be symplectic if the section $z : V \rightarrow M$ is Lagrangian. We will show that z being Lagrangian is sufficient for K to be symplectic. Then the preimage $P(V)$ of the Lagrangian section $z(V)$ under K is a Lagrangian submanifold of T^*B . It will then follow that the quotient T^*B/P by the Lagrangian period lattice P has the induced symplectic structure from T^*B , and the trivialization (2.18) is symplectic if and only if $z : V \rightarrow M$ is Lagrangian.

Recall the construction of K for a given z from Remark 2.2 in Section 2.2.1, only now take z Lagrangian. For this choose local coordinates $(x_1, \dots, x_n)$ on B . Then $dx_1, \dots, dx_n$ are linearly independent, and for each b in the coordinate neighborhood the frame $dx_1(b), \dots, dx_n(b)$ is a basis of T_b^*B . The functions $a_j = x_j \circ f$ are pairwise in involution (section 2.2.1 Remark 2.1), so by the Darboux theorem (Appendix A) there exist local functions $\eta = (\eta_1, \dots, \eta_n)$ on M (not unique) such that the coordinate system (a, η) is symplectic, i.e.

$$\sigma = \sum_{i=1}^n d\eta_i \wedge da_i.$$

Now choose a local “zero section” $z : V \rightarrow M$ in such a way that

$$z(V) = \{\eta = 0\}.$$

Such a section z is Lagrangian (Appendix A), and for each dx_i , $i = 1, \dots, n$, on V we have

$$\left. \frac{\partial}{\partial \eta_i} \right|_{F_b} = S_b(dx_i(b)), \quad (2.19)$$

where $S_b : T_b^*B \rightarrow \mathcal{X}(F_b)$ is defined in Section 2.2.1. As before, the map K is given by

$$K : T^*(V) \rightarrow f^{-1}(V) : (b, \alpha) \mapsto s_b(\alpha)(z(b)),$$

where $s_b(\alpha) = (S_b(\alpha))_1$ is the time 1 map (section 2.2.1).

Remark 2.6 (Expression of K in local coordinates). We obtain the expression for K in local coordinates near the zero section $V \subset T^*(V)$. We can do this only locally since the coordinates (a, η) are local. By (2.19) and the definition of the map s_b (Section 2.2.1) we have for $b \in V$ and small λ_i , $i = 1, \dots, n$,

$$s_b\left(\sum_{i=1}^n \lambda_i dx_i\right)(a_1, \dots, a_n, 0, \dots, 0) = (a_1, \dots, a_n, \lambda_1, \dots, \lambda_n). \quad (2.20)$$

Let (x, y) be canonical symplectic coordinates on $T^*(V)$ (Appendix A), and recall that in these coordinates we have

$$y_j\left(\sum_{i=1}^n \lambda_i dx_i\right) = \lambda_j, \quad j = 1, \dots, n. \quad (2.21)$$

Then (2.20) yields that near the zero section $V \subset T^*(V)$ the map K is expressed in local coordinates by

$$a_i \circ K = x_i, \quad \eta_i \circ K = y_i,$$

which implies that, if z is Lagrangian, K preserves the symplectic structure at the points near the zero section in $T^*(V)$.

The following lemma shows that z being Lagrangian is sufficient for K to be symplectic.

Lemma 2.9 (Symplectic maps between T^*B and M). *Provided $z : V \rightarrow M$ is a Lagrangian section, the map*

$$K : T^*(V) \rightarrow f^{-1}(V) : (b, \alpha) \mapsto s_b(\alpha)(z(b)),$$

where $s_b : T_b^*B \rightarrow \text{Diff}(F_b)$ is given by (2.9), is symplectic.

Proof. By Remark 2.6 K is symplectic at points near the zero section in $T^*(V)$. It follows that K is symplectic everywhere on $T^*(V)$. To see this for $\alpha \in T_b^*B$ choose $h : B \rightarrow \mathbb{R}$ such that $dh = \alpha$. The function $h \circ \pi$ depends only on a point in the base, and the Hamiltonian vector field $X_{h \circ \pi}$ is tangent to fibres of $T^*(V)$. Moreover,

$$(X_{h \circ \pi})_1(b, 0) = (b, \alpha),$$

where $(X_{h \circ \pi})_1$ denotes the time one map. Recall from the definition of K (Section 2.2.1) that

$$K(b, \alpha) = (S_b(\alpha))_1(z(b)) = (X_{h \circ f})_1(z(b)),$$

so the following diagram is commutative.

$$\begin{array}{ccc} (b, \alpha) \in T^*(V) & \xrightarrow{K} & K(b, \alpha) \in f^{-1}(V) \\ (X_{h \circ \pi})_1 \uparrow & & \uparrow (X_{h \circ f})_1 \\ (b, 0) \in T^*(V) & \xrightarrow{K} & z(b) \in f^{-1}(V) \end{array}$$

Since K is symplectic near the points of the zero section in $T^*(V)$, and the flows of the Hamiltonian vector fields $X_{h \circ \pi}$ and $X_{h \circ f}$ preserve the symplectic form, K is symplectic near α and hence everywhere on $T^*(V)$. $\square$

We use the properties of K to prove that the period lattice P is a Lagrangian submanifold of T^*B , and the quotient T^*B/P is a symplectic manifold.

Lemma 2.10 (Induced symplectic structure). *The following holds:*

1. *The period lattice P is a Lagrangian submanifold of T^*B .*
2. *The fibrewise quotient T^*B/P by the period lattice P has an induced symplectic structure, making the projection $T^*B \rightarrow T^*B/P$ a symplectic map.*

Proof.

1. Let $z : V \rightarrow M$ be a Lagrangian section, and $K : T^*(V) \rightarrow f^{-1}(V)$ be the corresponding map. Then $P(V)$ is a preimage of $z(V)$ under K . Since $z(V)$ is Lagrangian, and K is symplectic, $P(V)$ is a Lagrangian submanifold.
2. Let $[\alpha] \in T^*B/P$, and let $\mathcal{O} \subset T^*B/P$ be a trivializing neighborhood of $[\alpha]$ in the bundle $q : T^*B \rightarrow T^*B/P$, such that $\tau(\mathcal{O}) = V$. Choose a local section $m : \mathcal{O} \rightarrow T^*B$ and define the symplectic form ω' on T^*B/P by

$$\omega' = m^*\omega,$$

where ω is the canonical form on T^*B . This definition does not depend on the choice of m by the following argument. Let $m' : \mathcal{O} \rightarrow T^*B$ be another section, then m and m' differ by a Lagrangian section $\gamma : V \rightarrow P$. Then by Lemma A.1 in Appendix A we have

$$m^*\omega = m'^*(\omega + d\gamma) = m'^*\omega.$$

This proves that T^*B/P has a well-defined symplectic structure induced from T^*B . $\square$

The following corollary is immediate.

Corollary 2.1 (Local symplectomorphism K). *Provided $z : V \rightarrow M$ is a Lagrangian section, the corresponding trivialization*

$$K : T^*(V)/P(V) \rightarrow f^{-1}(V) : (b, [\alpha]) \mapsto s_b(\alpha)(z(b)),$$

where $s_b : T_b^*B \rightarrow \text{Diff}(F_b)$ is given by (2.9), is symplectic.

Terminology 2.7 (Local symplectic trivializations of M). We call a local Lagrangian section $z : V \rightarrow M$ and the corresponding symplectic diffeomorphism K a *local symplectic trivialization* of M (compare to Terminology 2.5).

Terminology 2.8 (Transition maps of symplectic trivializations of M). Recall (Section 2.2.1 Terminology 2.6) that transition maps between smooth trivializations of M are given by smooth local sections in the bundle $\tau : T^*B/P \rightarrow B$. In the case of symplectic trivializations of M such sections have to be Lagrangian. To see that let z and z' be local Lagrangian sections of M , and let K be the trivialization defined by z . Then $K^{-1}(z'(V))$ is locally the image of a closed section $\beta : V \rightarrow T^*B$ under the quotient map $T^*B \rightarrow T^*B/P$, and hence is a Lagrangian section in T^*B/P (Lemma 2.10). We will often say a “closed section of T^*B/P ” meaning a Lagrangian section of T^*B/P . Such sections occur as transition maps of symplectic trivializations of M .

2.2.3. Adapted coordinates on B and action coordinates on M and the freedom in their construction

In this section we prove the existence of an integer affine structure on the base B , i.e. we show that B admits a system of local charts (V, x) such that the transition maps between the charts are transformations in $\mathbb{R}^n \rtimes \mathbb{GL}(n, \mathbb{Z})$. The coordinates x will be called *adapted* coordinates.

Recall from section 2.2.1 that the cotangent bundle T^*B contains the period lattice P which is a locally trivial $\mathbb{Z}^n$ -bundle (Lemma 2.8 section 2.2.1). Let $\alpha_1, \dots, \alpha_n$ be a local basis of sections in the period lattice. The sections α_i are locally defined differential 1-forms on B , hence shrinking, if necessary, to a smaller simply connected neighborhood $V \subset B$ of b , by the Poincaré lemma there exist local functions $x_1, \dots, x_n$, such that $dx_i = \alpha_i$, $i = 1, \dots, n$ (such a neighborhood always exists, see Appendix C.3 or [38]). By taking V even smaller, if necessary, we can arrange that $x_1, \dots, x_n$ are coordinates on V .

Terminology 2.9 (Adapted coordinates). Local coordinates $x_1, \dots, x_n$ on B , such that the differentials $dx_1, \dots, dx_n$ are a local basis of sections in the period lattice P , are called the *adapted coordinates*.

By Remark 2.4 in section 2.2.1, if x and x' are two choices of adapted coordinates on V , then at each $b \in V$ the bases $dx_1(b), \dots, dx_n(b)$ and $dx'_1(b), \dots, dx'_n(b)$ are related by a linear transformation $A \in \mathbb{GL}(n, \mathbb{Z})$. Since the sections dx_i are smooth and the group $\mathbb{GL}(n, \mathbb{Z})$ is discrete, this transformation is locally constant on V . This determines the transition maps between x and x' up to a translation of the origin, and the latter can be moved by a vector $\xi \in \mathbb{R}^n$. Therefore,

$$x' = Ax + \xi, \quad (\xi, A) \in \mathbb{R}^n \rtimes \mathbb{GL}(n, \mathbb{Z}),$$

and adapted coordinates give B an integer affine structure (section 1.3.1).

As in Section 2.2.2, we can define the functions $I_i = x_i \circ f$, $i = 1, \dots, n$ on M , called the *action* coordinates. It follows that for any two choices I and I' of action coordinates we have

$$I' = AI + \xi, \quad (\xi, A) \in \mathbb{R}^n \rtimes \mathbb{GL}(n, \mathbb{Z}),$$

i.e. the functions I are determined up to the same integer affine transformation in $\mathbb{R}^n \rtimes \mathbb{GL}(n, \mathbb{Z})$ as the corresponding adapted coordinates.

2.2.4. Angle coordinates and the freedom of their construction

To finish the proof of the Liouville–Arnold theorem we only have to obtain angle coordinates $\varphi = (\varphi_1, \dots, \varphi_n)$ and determine the freedom of their construction. For that let $x = (x_1, \dots, x_n)$ be adapted coordinates on B , as obtained in section 2.2.3, and let (x, y) be the corresponding canonical symplectic coordinates on $T^*(V)$ (Appendix A), i.e. consider the isomorphism

$$(x, y) : T^*(V) \rightarrow U \times \mathbb{R}^n,$$

where $U = x(V) \subset \mathbb{R}^n$ is open. The image of the period lattice P under this isomorphism is the subset $U \times \mathbb{Z}^n \subset U \times \mathbb{R}^n$, and the induced coordinates on the quotient T^*B/P are

$$(x, y) : T^*(V)/P(V) \rightarrow U \times \mathbb{T}^n,$$

where y_i , $i = 1, \dots, n$, take values in $\mathbb{T}^1$. The composition of (x, y) with the inverse of a local symplectic trivialization $K : T^*(V)/P(V) \rightarrow f^{-1}(V)$, determined by the choice of a local Lagrangian section $z : V \rightarrow M$, provides us with local symplectic coordinates

$$(I, \varphi) = (x, y) \circ K^{-1} : f^{-1}(V) \rightarrow U \times \mathbb{T}^n,$$

such that the functions $I = (I_1, \dots, I_n)$ factor through B , and the functions $\varphi = (\varphi_1, \dots, \varphi_n)$ take values in $\mathbb{T}^1$. These coordinates are called *action-angle* coordinates.

To determine the freedom in the construction of action-angle coordinates, notice that, given the adapted coordinates $x = (x_1, \dots, x_n)$, the coordinate system (x, y) on $T^*(V)$ is completely determined (Appendix A), and the only freedom in the construction of the angle coordinates $\varphi = (\varphi_1, \dots, \varphi_n)$ is to choose a “zero section” $z : V \rightarrow M$.

This finishes our proof of the Liouville–Arnold theorem 2.1.

3. COMPLETE INVARIANTS OF LAGRANGIAN BUNDLES

In this section we consider obstructions to triviality of Lagrangian bundles, as introduced by Duistermaat [1]. We review the theory of obstructions to triviality of Lagrangian bundles. A Lagrangian bundle $f : M \rightarrow B$ is trivial if it is isomorphic to a trivial principal symplectic n -torus bundle, and the obstructions to its triviality are the following.

In Section 3.1 we consider the monodromy, which, as was already explained in Section 1.3.2, is a complete invariant of period lattices P and an obstruction to the Lagrangian bundle being a principal bundle. Notice (Section 1.3.2 Example 1.2), that if two period lattices P and P' are isomorphic, this does not imply that they define isomorphic integer affine structures on the base manifold B (Section 2.2.1 Definition 2.2), so if two Lagrangian bundles M and M' have the same monodromy, their lattices P and P' are isomorphic, but may not be the same or even not related by a diffeomorphism in B , i.e. there may be no diffeomorphism $\phi : B \rightarrow B$ such that $\phi^*P' = P$. Next we fix the integer affine structure on the base manifold B , and study Lagrangian bundles $f : M \rightarrow B$ with the corresponding period lattice P . We are interested in a more general question than obstructions to triviality: given two Lagrangian bundles $f : M \rightarrow B$ and $f' : M' \rightarrow B$ with fixed integer affine structure, when are these bundles isomorphic (see Section 3.2 for the definition of isomorphism of Lagrangian bundles)? In Section 3.2 we show that two such Lagrangian bundles are smoothly isomorphic if and only if they have the same Chern class, and they are symplectically isomorphic if and only if they have the same Lagrange class. So a Lagrangian bundle with trivial monodromy is trivial if its Lagrange class is zero, which means that the bundle admits a global Lagrangian section or a global trivialization. In Section 3.2.2 we elaborate on the relation between the Chern and the Lagrange classes of Lagrangian bundles.

3.1. Monodromy

In this section we introduce monodromy. The symplectic structure does not play any role in the considerations of this section, for this reason we consider the general case, namely, locally trivial $\mathbb{Z}^n$ -bundles $g : P \rightarrow B$ over a connected base space B . We introduce a complete invariant of such bundles, called *monodromy*. Notice that, given B , not every $\mathbb{Z}^n$ -bundle P over B can be realized as a period lattice in T^*B ; for example, if T^*B is non-trivial, the period lattice can only be non-trivial. We show that the monodromy can be seen as an obstruction to triviality of Lagrangian bundles.

Monodromy of $\mathbb{Z}^n$ -bundles. Like covering spaces, the bundle P has unique path-lifting (Section 2.2.1 Remark 2.5), and the classification of $\mathbb{Z}^n$ -bundles is analogous to that of covering spaces. For this reason in our treatment of monodromy we will be brief and omit proofs, referring to [13, 14].

Using the property of unique path-lifting, given $b \in B$ we define the homomorphism

$$H : \pi_1(B, b) \rightarrow \text{Aut}(P_b) \quad (3.1)$$

of the fundamental group $\pi_1(B, b)$ into the group $\text{Aut}(P_b) \cong \text{GL}(n, \mathbb{Z})$, as follows. Choose a closed curve $c : [0, 1] \rightarrow B$ which represents the equivalence class $[c] \in \pi_1(B, b)$, and for each $\alpha \in P_b$ denote by c_α the unique lift of c with the starting point $c_\alpha(0) = \alpha$. Then define

$$H([c])(\alpha) = c_\alpha(1).$$

The homomorphism (3.1) is called the *monodromy homomorphism* or just the *monodromy*. For a proof that $H([c])$ does not depend on the choice of a representative in $[c]$ and is an automorphism of P_b we refer to [13, 14].

We notice that the homomorphism H is independent of the point b in the following sense. Let $b' \in B$ and denote by H' the monodromy homomorphism at b' . Since B is connected, the fundamental groups $\pi_1(B, b)$ and $\pi_1(B, b')$ are isomorphic [13, 14], and, due to the unique path-lifting of P , there exists a group isomorphism

$$\mathcal{C} : P_b \rightarrow P_{b'}$$

such that $H' \circ \mathcal{C} = \mathcal{C} \circ H$. The following theorem states that the monodromy is a complete invariant of $\mathbb{Z}^n$ -bundles P .

Theorem 3.1 (Monodromy theorem). *Let B be a connected manifold and $\pi_1(B, b)$ be the fundamental group of B with base point b . Let $g : P \rightarrow B$ and $g' : P' \rightarrow B$ be $\mathbb{Z}^n$ -bundles, and denote by H and H' the monodromy homomorphisms associated to P and P' respectively. Then P and P' are isomorphic, i.e. there exists a diffeomorphism*

$$\Phi : P \rightarrow P',$$

commuting with the projection maps g and g' , if and only if there exists a group isomorphism

$$\mathcal{I} : P_b \rightarrow P'_b$$

such that the following diagram commutes for each $[c] \in \pi_1(B, b)$,

$$\begin{array}{ccc} P_b & \xrightarrow{\mathcal{I}} & P'_b \\ H([c]) \uparrow & & \uparrow H'([c]) \\ P_b & \xrightarrow{\mathcal{I}} & P'_b \end{array} \quad (3.2)$$

i.e. we have $\Phi|_{P_b} = \mathcal{I}$.

Moreover, for any group morphism H (3.1) there is a corresponding $\mathbb{Z}^n$ -bundle. These statements are proved by standard methods of theory of covering spaces [13, 14].

Monodromy of period lattices. Recall (Section 2.2) that the period lattice P is a locally trivial $\mathbb{Z}^n$ -subbundle of T^*B , so the above applies to P . The monodromy homomorphism H , defined by (3.1), is trivial, if its image is the identity in $\text{Aut}(P_b)$. The lattice P is trivial, if there exists a diffeomorphism $P \rightarrow B \times \mathbb{Z}^n$ commuting with obvious projections. By Theorem 3.1 the period lattice P is trivial if and only if the monodromy homomorphism H is trivial. If P is trivial, the bundle T^*B is trivial, and it follows that also the quotient bundle T^*B/P is trivial as well, i.e. there exists an isomorphism $T^*B/P \rightarrow B \times \mathbb{T}^n$. Recall (Section 2.2) that fibres of T^*B/P act freely and transitively on fibres of the corresponding Lagrangian bundle $f : M \rightarrow B$. If T^*B/P is trivial, for each $b, b' \in B$ there is a canonical isomorphism between the fibres T_b^*B/P_b and $T_{b'}^*B/P_{b'}$, and there is the smooth action of the n -torus $\mathbb{T}^n$ on M . This means that if the period lattice P is trivial, then the Lagrangian bundle M is a principal torus bundle which, however, needs not be trivial, as

we show later in Section 3.2.2. If the monodromy is non-trivial, the bundle M is not a principal bundle and is certainly non-trivial. In this sense the monodromy is the obstruction to M being a principal bundle.

Remark 3.1 (Significance of monodromy). The monodromy can be regarded as an obstruction to the existence of a Lagrangian bundle with a given manifold B as its base space, as the following examples show.

1. Let $B = \mathbb{S}^2$. Since $\mathbb{S}^2$ is simply connected, the monodromy map has to be trivial, and also the period lattice $P \subset T^*\mathbb{S}^2$ must be trivial. But this is not possible, since $T^*\mathbb{S}^2$ is non-trivial (Hairy Ball Theorem, [9]). Thus there is no Lagrangian bundle over $\mathbb{S}^2$.
2. Let $B = \mathbb{S}^3$. The tangent bundle $T\mathbb{S}^3$, and hence the cotangent bundle $T^*\mathbb{S}^3$, is trivial. However, as we show now, there is no integer affine structure on $\mathbb{S}^3$ (and no period lattice over $\mathbb{S}^3$). Indeed, suppose such a period lattice $P \subset T^*\mathbb{S}^3$ exists. Since $\mathbb{S}^3$ is simply connected, the monodromy is trivial and hence P is a product, i.e. $P \cong \mathbb{S}^3 \times \mathbb{Z}^3$. So we can take a global non-zero section α in P . Because P is Lagrangian, the 1-form α is closed. Since the de Rham cohomology $H_{dR}^1(\mathbb{S}^3)$ is trivial [11], α is exact and there exists a smooth function $f : \mathbb{S}^3 \rightarrow \mathbb{R}$ with $df = \alpha$. But this leads to a contradiction: since $\mathbb{S}^3$ is compact, f must have a maximum on $\mathbb{S}^3$. Then df vanishes at the maximum point, while α is nowhere zero.

3.2. The Chern and the Lagrange Classes of a Lagrangian Bundle

In this section we introduce the classification of Lagrangian bundles with fixed integer affine structure on the base manifold B by the Lagrange class. This means that the period lattice P and the quotient bundle T^*B/P are fixed. Since we are going to deal with different kinds of bundles, first we review the terminology.

Recall (Section 2.2) that a Lagrangian bundle $f : M \rightarrow B$ is a locally trivial n -torus bundle, although the fibres of M do not have a group structure. The quotient bundle $\tau : T^*B/P \rightarrow B$ is a locally trivial (Lagrangian) bundle whose fibres are torus groups, acting on the corresponding fibres of $f : M \rightarrow B$. The action of fibres of T^*B/P on fibres of M is free, transitive and smooth. Local identification of fibres in T^*B/P is induced from the identification of fibres in the period lattice P , so it is unique. This means that locally on M (and T^*B/P) there is a smooth fibre-preserving and free action of an n -torus, and M (and also T^*B/P) *locally* is a principal bundle (Section 2.2 Terminology 2.4).

We can treat both M and T^*B/P as general locally principal n -torus bundles. Then (*local*) trivializations of M are local bundle isomorphisms of T^*B/P to M . These trivializations are in one-to-one correspondence with local sections $z : V \rightarrow M$ (Section 2.2.1 Terminology 2.5), and the transition maps of the trivializations are defined by local sections in T^*B/P (Section 2.6). It should be stressed here that we adopt a modified notion of local triviality; it reduces to the standard notion of local triviality only if the quotient bundle T^*B/P is trivial in the usual sense, that is, if it is diffeomorphic to the product $B \times \mathbb{T}^n$ (see also Terminology 3.1). Furthermore, if we regard M and T^*B/P as symplectic n -torus bundles, then local *symplectic* trivializations of M are local symplectic bundle isomorphisms of T^*B/P to M , which are in one-to-one correspondence with local *Lagrangian* sections in M (Section 2.2.2 Terminology 2.7), and transition maps of the trivializations are symplectic and are determined by local closed sections in T^*B/P (Section 2.2.2 Terminology 2.8).

Recall from Appendix B, that $f : M \rightarrow B$ and $f' : M' \rightarrow B$ with fixed P and, therefore, fixed T^*B/P are *isomorphic* as general locally principal torus bundles, if there exists a diffeomorphism $\Psi : M \rightarrow M'$ such that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\Psi} & M' \\ & \searrow f & \swarrow f' \\ & B & \end{array}$$

and Ψ commutes with the action of tori in T^*B/P on M . We say that Lagrangian bundles $f : M \rightarrow B$ and $f' : M' \rightarrow B$ are *isomorphic* if they are isomorphic as general locally principal n -torus bundles. Two Lagrangian bundles $f : M \rightarrow B$ and $f' : M' \rightarrow B$ are *symplectically isomorphic* if there is such a diffeomorphism Ψ , which also preserves the symplectic structure, i.e. such that $\Psi^*\sigma' = \sigma$, where σ and σ' are the symplectic forms on M and M' respectively. Note that $\tau : T^*B/P \rightarrow B$ can be also seen as a Lagrangian bundle whose fibres have a group structure. The bundle T^*B/P has a global zero Lagrangian section, so we say that T^*B/P has a *global trivialization* (in the extended sense, see Terminology 3.1 below).

Terminology 3.1 (Triviality of a Lagrangian bundle). We say that a Lagrangian bundle $f : M \rightarrow B$ admits a *global trivialization* if it admits a global section. In this case $\tau : T^*B/P \rightarrow B$ and $f : M \rightarrow B$ are isomorphic as general locally principal n -torus bundles. We say that the bundle M admits a *global symplectic trivialization* if it admits a global Lagrangian section. In this case T^*B/P and M are symplectically isomorphic.

We will define the *Chern class* of a general locally principal torus bundle, and show that it is a complete invariant of such bundles, which means at least that if two general n -torus bundles are not isomorphic, their Chern classes should be different. The notion of a complete invariant will be made more precise in Section 3.2.1. We also note that the Chern class can be seen as the obstruction to a general torus bundle to admit a global section, i.e. a general torus bundle may admit a global section only if its Chern class is trivial. For Lagrangian bundles this means that the Chern class is the obstruction to the Lagrangian bundle being topologically trivial in the sense of Terminology 3.1.

Similarly, we will define the *Lagrange class* of a Lagrangian bundle, and show that it is a complete invariant of Lagrangian bundles with fixed integer affine structure and the obstruction to a Lagrangian bundle to admit a global *Lagrangian* section. Then in the sense of Terminology 3.1 the Lagrange class is the obstruction to a Lagrangian bundle M being symplectically isomorphic to the bundle T^*B/P and the bundle M being trivial in that sense.

Notice that each symplectic n -torus bundle can be seen as a general torus bundle by forgetting the symplectic structure on M ; however, not every general n -torus bundle admits a symplectic structure, as will be shown in Section 3.2.2. Thus for a symplectic torus bundle one can define both the Lagrange and the Chern class, although in this case the Chern class is not a complete invariant, which means that not every Chern class can be realized as the Chern class of a Lagrangian bundle, and bundles, which are not symplectically isomorphic, may have the same Chern class. We will clarify the relation between the Chern and the Lagrange classes. In particular, we will point out conditions on a manifold B under which *any* n -torus bundle over B admits a symplectic structure; we will also show that if an n -torus bundle admits a symplectic structure, this structure is never unique, and often is not unique up to a symplectic bundle isomorphism.

3.2.1. The Chern class of a general n -torus bundle and the Lagrange class of a Lagrangian bundle

In this section we develop the classification of locally principal general n -torus bundles by the Chern class, and the classification of Lagrangian bundles by the Lagrange class. We do this first for a fixed good cover $\mathcal{V}$ of B , and then show that the results do not depend on the choice of the good cover $\mathcal{V}$ (see Appendix B.2).

The Chern class of a general n -torus bundle. We start with the classification of locally principal general n -torus bundles. Let $\pi : R \rightarrow B$ be a locally trivial vector bundle over B with fibre $\mathbb{R}^n$ and $g : P \rightarrow B$ be a locally trivial $\mathbb{Z}^n$ -subbundle of R . Then the fibrewise quotient $\tau : T = R/P \rightarrow B$ is a locally trivial n -torus bundle whose fibre is a *group*. Let $f : M \rightarrow B$ be an n -torus bundle, i.e. fibres of M are diffeomorphic to $\mathbb{T}^n$ but do not have a group structure. We assume that there is a free and transitive action of fibres of T on fibres of M , given by

$$\mathcal{G}_b : T_b \times F_b \rightarrow F_b,$$

where $F_b = f^{-1}(b)$ and $T_b = \tau^{-1}(b)$. Note, that locally there is the canonical identification of fibres in T , which is induced by the identification of fibres in P . This means that the smooth action of T_b is defined on a neighborhood of each F_b , and locally M is a principal bundle. If T is trivial, the action of T_b is smooth on the whole of M , and M is a principal bundle. We fix the bundles R and P , and look for invariants classifying general locally principal n -torus bundles.

To define a complete invariant of general locally principal n -torus bundles we will use the same approach as for principal circle bundles (Appendix B). As in the case of principal circle bundles, trivializations of M are determined by the choice of local “zero” sections, and the difference between trivializations is determined by a local section in the bundle of groups T whose fibres act on fibres of M (Section 2.2.1 Terminology 2.6).

Recall (Appendix C.3, [38]) that a good cover $\mathcal{V}$ of B is a locally finite open cover such that all sets of $\mathcal{V}$ and all finite intersections of sets of $\mathcal{V}$ are contractible. Any good cover is a trivializing cover of any bundle, e.g. M , R , P etc. For each $V_i \in \mathcal{V}$ let $z_i : V_i \rightarrow M$ be a local section determining a trivialization of M . As we have already mentioned, on each non-empty intersection $V_i \cap V_j$ of open sets $V_i, V_j \in \mathcal{V}$ the difference between $z_i(b)$ and $z_j(b)$, $b \in V_i \cap V_j$ determines a local section

$$\alpha_{ij} : V_i \cap V_j \rightarrow T(V_i \cap V_j), \quad (3.3)$$

called the transition map of M , such that for each $b \in V_i$

$$\mathcal{G}_b(\alpha_{ij}(b), z_j(b)) = z_i(b).$$

The transition maps are alternating,

$$\alpha_{ij} = -\alpha_{ji}, \quad (3.4a)$$

and satisfy the cocycle condition,

$$\alpha_{ij} + \alpha_{jk} = \alpha_{ik}. \quad (3.4b)$$

Denote by $\mathcal{T}^n$ the sheaf of sections of the bundle T . As in the case of principal circle bundles, (3.4) implies that the 1-cochain

$$\alpha : (ij) \mapsto \alpha_{ij} \in C^1(\mathcal{V}, \mathcal{T}^n)$$

is a 1-cocycle. We denote by

$$[\alpha] \in H^1(\mathcal{V}, \mathcal{T}^n)$$

the cohomology class of α . The proof of the following results is analogous to the proofs of Lemma B.1 and Theorem B.1 in Appendix B.

Lemma 3.1 (Independence of choice of local trivializations). *The cohomology class $[\alpha] \in H^1(\mathcal{V}, \mathcal{T}^n)$ of a general n -torus bundle $f : M \rightarrow B$ does not depend on the choice of local trivializations, that is, if $\{z_i\}$ and $\{z'_i\}$ are two choices of trivializations over a good cover $\mathcal{V}$ with transition maps α_{ij} and α'_{ij} respectively, then*

$$[\alpha] = [\alpha'] \in H^1(\mathcal{V}, \mathcal{T}^n).$$

Theorem 3.2 (Complete invariant of general n -torus bundles). *Let $\mathcal{V}$ be a good cover of a manifold B , and let P (and hence T) be fixed. Then:*

1. *For each element $[\alpha] \in H^1(\mathcal{V}, \mathcal{T}^n)$ there is a corresponding general locally principal n -torus bundle $f : M \rightarrow B$.*
2. *Two general locally principal n -torus bundles $f : M \rightarrow B$ and $f' : M' \rightarrow B$ with corresponding cohomology classes $[\alpha]$ and $[\alpha']$ respectively are isomorphic if and only if $[\alpha] = [\alpha']$.*
3. *The cohomology class $[\alpha] \in H^1(\mathcal{V}, \mathcal{T}^n)$ of the bundle $f : M \rightarrow B$ is zero if and only if M admits a global trivialization, and so is isomorphic to $T = R/P$ (Terminology 3.1).*

Terminology 3.2 (Complete invariants). By Theorem 3.2 each value of $[\alpha] \in H^1(\mathcal{V}, \mathcal{T}^n)$ corresponds to a general locally principal n -torus bundle, and each $[\alpha] \in H^1(\mathcal{V}, \mathcal{T}^n)$ determines a general locally principal n -torus bundle up to an isomorphism. This means that $[\alpha]$ is a *complete* invariant of general locally principal n -torus bundles.

Denote by $\mathcal{P}$ the sheaf of sections of $g : P \rightarrow B$. By Example C.7 in Appendix C.3 for every good cover $\mathcal{V}$ of B there is an isomorphism

$$\delta : H^1(\mathcal{V}, \mathcal{T}^n) \rightarrow H^2(\mathcal{V}, \mathcal{P}),$$

so also the cohomology class

$$\nu = \delta[\alpha] \in H^2(\mathcal{V}, \mathcal{P}) \quad (3.5)$$

is a complete invariant of general locally principal n -torus bundles. The cohomology class ν is called the *Chern class* of $f : M \rightarrow B$.

Remark 3.2 (Chern class of a general n -torus bundle with trivial P). If the bundle $g : P \rightarrow B$ is trivial, then $\tau : T \rightarrow B$ is a trivial principal n -torus bundle and $f : M \rightarrow B$ is a principal n -torus bundle. Then P is isomorphic to the direct sum of n trivial $\mathbb{Z}$ -bundles, which yields the following isomorphism on sheaf cohomology (Appendix C.3 Example C.6)

$$H^2(\mathcal{V}, \mathcal{P}) \cong H^2(\mathcal{V}, \mathcal{Z}) \oplus \cdots \oplus H^2(\mathcal{V}, \mathcal{Z}),$$

where $\mathcal{Z}$ is the sheaf of locally constant $\mathbb{Z}$ -valued functions. In its turn, $\tau : T \rightarrow B$ is isomorphic to the direct sum of n circle bundles $\tau_i : S_i \rightarrow B$, $i = 1, \dots, n$, with the Chern classes $c_i \in H^2(\mathcal{V}, \mathcal{Z})$. Thus the Chern class $\nu \in H^1(\mathcal{V}, \mathcal{P})$ of T is the n -tuple

$$(c_1, \dots, c_n) \in H^2(\mathcal{V}, \mathcal{Z}) \oplus \cdots \oplus H^2(\mathcal{V}, \mathcal{Z}).$$

The Lagrange class of a Lagrangian bundle. By an argument, similar to the one used in section 3.2.1 we introduce the Lagrange class of Lagrangian bundles, only now local trivializations and transition maps, involved in the construction, are symplectic. Let $f : M \rightarrow B$ be a Lagrangian bundle, then $R = T^*B$, the period lattice $g : P \rightarrow B$ is a $\mathbb{Z}^n$ -subbundle of T^*B , and the quotient bundle $\tau : T^*B/P \rightarrow B$ plays the role of the bundle T . The quotient bundle T^*B/P is a bundle of torus groups, whose fibres act freely and transitively on fibres of M by

$$\mathcal{G}_b : T_b^*B/P_b \times F_b \rightarrow F_b, \quad b \in B, \quad F_b = f^{-1}(b).$$

We develop the classification of Lagrangian bundles $f : M \rightarrow B$ with fixed integer affine structure on B , hence with fixed period lattice P and the quotient bundle T^*B/P .

Recall (Section 2.2.2 Terminology 2.7 and 2.8) that in a Lagrangian bundle local symplectic trivializations over open sets of a good cover $\mathcal{V}$ are determined by local *Lagrangian* sections $z_i : V_i \rightarrow M$, $V_i \in \mathcal{V}$, and on each non-empty intersection $V_i \cap V_j$ the difference between z_i and z_j defines a *closed* local section

$$\alpha_{ij} : V_i \cap V_j \rightarrow T^*B/P,$$

such that for all $b \in V_i \cap V_j$

$$\mathcal{G}(\alpha_{ij}(b), z_j(b)) = z_i(b).$$

Denote by $\widehat{\mathcal{Z}}$ the sheaf of closed sections of T^*B/P . Then the assignment

$$\alpha : (ij) \mapsto \alpha_{ij} \in C^1(\mathcal{V}, \widehat{\mathcal{Z}})$$

is a 1-cocycle with coefficients in $\widehat{\mathcal{Z}}$, and the cohomology class

$$\lambda = [\alpha] \in H^1(\mathcal{V}, \widehat{\mathcal{Z}})$$

is called the *Lagrange class* of the Lagrangian bundle $f : M \rightarrow B$. By exactly the same argument as in the case of general locally principal n -torus bundles (and for principal circle bundles, Appendix B.2 Lemma B.1) one can prove that the Lagrangian class λ is independent of the choice of local trivializations z_i . There is the following classification theorem.

Theorem 3.3 (The Lagrange class is a complete invariant). *Let $\mathcal{V}$ be a good cover of a manifold B with given integer affine structure. Then:*

1. *Each element $[\alpha] \in H^1(\mathcal{V}, \widehat{\mathcal{Z}})$ is the Lagrange class of a Lagrangian bundle $f : M \rightarrow B$.*

2. Two Lagrangian bundles $f : M \rightarrow B$ and $f' : M' \rightarrow B$ with the fixed integer affine structure on B and Lagrange classes $[\alpha]$ and $[\alpha']$ respectively are symplectically isomorphic if and only if $[\alpha] = [\alpha']$.
3. The Lagrange class of $f : M \rightarrow B$ is zero if and only if M admits a global symplectic trivialization and is symplectically isomorphic to T^*B/P (Terminology 3.1).

Similarly to principal circle bundles (see Appendix B.2), cohomology groups $H^q(\mathcal{V}, \mathcal{S})$, $q \geq 1$, where $\mathcal{S}$ may be the sheaf $\mathcal{T}^n$ or $\widehat{\mathcal{Z}}$, is independent of the good cover $\mathcal{V}$. Taking the direct limit of cohomology groups over all good covers of B , one obtains an isomorphism $H^q(\mathcal{V}, \mathcal{S}) \rightarrow H^q(B, \mathcal{S})$. The image of the cohomology class $[\alpha]$ under this isomorphism corresponds to the same bundle over B as $[\alpha]$. It follows that the Chern and the Lagrange class are independent of the good cover of B . In what follows we consider the Chern and the Lagrange classes with values in the cohomology of the manifold B .

3.2.2. Relating Chern and Lagrange classes

By forgetting the symplectic structure on M , a Lagrangian bundle $f : M \rightarrow B$ is also a general locally principal n -torus bundle, and one can define the Chern class for it. In this section we elaborate on the relation between the Chern and the Lagrange classes of a Lagrangian bundle. For background in sheaf theory we refer to Appendix C.3.

We will relate the Chern and the Lagrange classes by a diagram with exact rows which we construct now. Denote by $\mathcal{Z}^1$ the sheaf of closed sections in T^*B , and denote by $\mathcal{P}$ the sheaf of (closed) sections in the period lattice P . Then the following sequence is exact,

$$0 \longrightarrow \mathcal{P} \xrightarrow{\iota} \mathcal{Z}^1 \xrightarrow{q} \widehat{\mathcal{Z}} \longrightarrow 0,$$

where ι and q are the inclusion and the quotient sheaf homomorphisms respectively, and induces the corresponding long exact sequence of sheaf cohomology

$$\cdots \longrightarrow H^1(B, \mathcal{P}) \longrightarrow H^1(B, \mathcal{Z}^1) \xrightarrow{\iota_*} H^1(B, \widehat{\mathcal{Z}}) \xrightarrow{\tilde{\delta}_*} H^2(B, \mathcal{P}) \xrightarrow{q_*} H^2(B, \mathcal{Z}^1) \longrightarrow \cdots \quad (3.6)$$

The period lattice P can be seen as a smooth $\mathbb{Z}^n$ -subbundle of T^*B , and the symplectic bundle T^*B/P is also a smooth bundle of groups if one forgets the symplectic structure on T^*B/P . Denoting by $\widehat{\Lambda}$ the sheaf of smooth sections in T^*B/P , and by Λ^1 the sheaf of smooth sections in T^*B , i.e. differential 1-forms, we obtain the following exact sequence of sheaves

$$0 \longrightarrow \mathcal{P} \xrightarrow{\iota} \Lambda^1 \xrightarrow{q} \widehat{\Lambda} \longrightarrow 0,$$

which induces the corresponding long exact sequence on cohomology

$$\cdots \longrightarrow H^1(B, \mathcal{P}) \longrightarrow H^1(B, \Lambda^1) = 0 \xrightarrow{\iota_*} H^1(B, \widehat{\Lambda}) \xrightarrow{\delta_*} H^2(B, \mathcal{P}) \xrightarrow{q_*} H^2(B, \Lambda^1) = 0 \longrightarrow \cdots \quad (3.7)$$

where $H^q(B, \Lambda)$ vanishes for $q \geq 1$ since Λ is a fine sheaf (Appendix C.2 Example C.4). This implies that δ_* is an isomorphism. The inclusion homomorphisms of sheaves of closed sections into the sheaves of smooth sections

$$C : \mathcal{Z}^1 \rightarrow \Lambda^1 \text{ and } A : \widehat{\mathcal{Z}} \rightarrow \widehat{\Lambda}$$

yield the following commutative diagram with exact rows,

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^1(B, \mathcal{P}) & \longrightarrow & H^1(B, \mathcal{Z}^1) & \xrightarrow{\iota} & H^1(B, \widehat{\mathcal{Z}}) & \xrightarrow{\tilde{\delta}_*} H^2(B, \mathcal{P}) \xrightarrow{q_*} H^2(B, \mathcal{Z}^1) \longrightarrow \cdots \\ & \downarrow C_* & & \downarrow A_* & & \downarrow = & \downarrow C_* \\ & H^1(B, \Lambda^1) = 0 & \longrightarrow & H^1(B, \widehat{\Lambda}) & \xrightarrow[\cong]{\delta_*} & H^2(B, \mathcal{P}) & \longrightarrow H^2(B, \Lambda^1) = 0 \end{array} \quad (3.8)$$

By the de Rham theorem (Appendix C.3 Theorem C.1) we have the isomorphisms

$$H^q(B, \mathcal{Z}^1) \rightarrow H_{dR}^{q+1}(B), \quad q \geq 1, \quad (3.9)$$

where $H_{dR}^{q+1}(B)$ denotes the de Rham cohomology of B . Combining (3.8) with (3.9), we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^1(B, \mathcal{P}) & \longrightarrow & H_{dR}^2(B) & \xrightarrow{Q} & H^1(B, \widehat{\mathcal{Z}}) & \xrightarrow{\tilde{\delta}_*} & H^2(B, \mathcal{P}) & \xrightarrow{E} & H_{dR}^3(B) & \longrightarrow & \cdots \\ & & & & \downarrow C_* & & \downarrow A_* & & \downarrow = & & \downarrow C_* & & \\ & & & & H^1(B, \Lambda^1) = 0 & \longrightarrow & H^1(B, \widehat{\Lambda}) & \xrightarrow[\cong]{\delta_*} & H^2(B, \mathcal{P}) & \longrightarrow & H^2(B, \Lambda^1) = 0 & & \end{array} \quad (3.10)$$

Then

$$\nu = \tilde{\delta}_* \lambda \in H^2(B, \mathcal{P})$$

is the Chern class of the Lagrangian bundle M . Analyzing Diagram (3.10) we immediately draw the following conclusions.

Remark 3.3 (Symplectic structure on n -torus bundles).

1. The map $\tilde{\delta}_*$ might not be surjective, which implies that a general locally principal n -torus bundle over B might not admit a symplectic structure. Finding examples of this situation presents an interesting problem.
2. Since the rows in (3.10) are exact, if $H_{dR}^3(B) = 0$ then we have

$$\text{image } \tilde{\delta}_* = \ker E = H^2(\mathcal{V}, \mathcal{P}),$$

i.e. in this case every general locally principal torus bundle over B admits a symplectic structure. In particular, this means that every general locally principal torus bundle over a 2-dimensional manifold can be made Lagrangian (see Example 3.1 of torus bundles over $\mathbb{T}^2$).

3. A Lagrangian bundle M is trivial, i.e. symplectically isomorphic to T^*B/P , if and only if it admits a global section $z : B \rightarrow M$ such that $z^*\sigma = \Theta$ is exact (see Lemma A.1 in Appendix A).
4. If $H_{dR}^2(B) = H_{dR}^3(B) = 0$, then classifications by the Chern and the Lagrange classes coincide, i.e. all symplectic structures on a given general n -torus bundles are isomorphic (see Example 3.2 of torus bundles over the Klein bottle).

Theorem 3.4 (Modification of symplectic structure on M). *Let $f : M \rightarrow B$ be a Lagrangian bundle with symplectic form σ , and let $\mathcal{V}$ be a good cover with trivializations $z_i : V_i \rightarrow M$, transition maps α_{ij} and Lagrange class*

$$\lambda = [\alpha] \in H^1(\mathcal{V}, \widehat{\mathcal{Z}}).$$

Let Θ be a closed differential 2-form on B . Then the following holds:

1. *The closed 2-form*

$$\sigma' = \sigma - f^*\Theta \quad (3.11)$$

is non-degenerate and defines a symplectic structure on M .

2. *There exist trivializations $z'_i : V_i \rightarrow M$ with transition maps α'_{ij} and the Lagrange class $\lambda' = [\alpha'] \in H^1(\mathcal{V}, \widehat{\mathcal{Z}})$ such that*

$$\lambda - \lambda' = Q[\Theta].$$

Proof.

1. Let (a, η) be local symplectic coordinates near $p \in M$ such that the functions a_i factor through B . Then at each $p \in M$ the symplectic form σ defines a quadratic form on $T_p M$, represented by the matrix

$$\Sigma = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad (3.12)$$

where 0 is the $n \times n$ zero matrix, and I is $n \times n$ identity matrix. Similarly, $\sigma' = \sigma - f^* \Theta$ defines a quadratic form represented by the matrix

$$\Sigma' = \begin{pmatrix} 0 & -I \\ I & A \end{pmatrix}, \quad (3.13)$$

where A is an $n \times n$ matrix corresponding to $-f_* \Theta$. We have $\det \Sigma = \det \Sigma'$, so σ' is non-degenerate.

2. To construct trivializations z'_i , denote by $\mathcal{Z}^2$ the sheaf of closed differential 2-forms on B , and consider the following short exact sequence.

$$0 \longrightarrow \mathcal{Z}^1 \xrightarrow{\iota} \Lambda^1 \xrightarrow{d} \mathcal{Z}^2 \longrightarrow 0,$$

where, as before, $\mathcal{Z}^1$ and Λ^1 are sheaves of closed 1-forms and all 1-forms on B respectively, ι is the inclusion sheaf homomorphism, and d is the sheaf homomorphism, induced by the exterior differentiation. Since the cover $\mathcal{V}$ is good, the following sequence of cochain complexes is exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^0(\mathcal{V}, \mathcal{Z}^1) & \xrightarrow{\iota} & C^0(\mathcal{V}, \Lambda^1) & \xrightarrow{d} & C^0(\mathcal{V}, \mathcal{Z}^2) \longrightarrow 0 \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 & \longrightarrow & C^1(\mathcal{V}, \mathcal{Z}^1) & \xrightarrow{\iota} & C^1(\mathcal{V}, \Lambda^1) & \xrightarrow{d} & C^1(\mathcal{V}, \mathcal{Z}^2) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \dots & & \dots & & \dots \end{array}$$

We use this diagram to construct sections z'_i . The global 2-form Θ corresponds to a 0-cocycle in $C^0(\mathcal{V}, \mathcal{Z}^2)$. Since the sheaf homomorphism d is surjective, there exists a 0-cochain $\beta : i \mapsto \beta_i \in C^0(\mathcal{V}, \Lambda^1)$ such that for each V_i

$$d\beta_i = \Theta|_{V_i}.$$

The sections $d\beta_i$ agree on intersections, so

$$d(\delta\beta)_{ij} = \delta(d\beta)_{ij} = d\beta_i - d\beta_j = 0.$$

Thus $\delta\beta$ is a 1-cochain with coefficients in $\mathcal{Z}^1$. Define the local section $z'_i : V_i \rightarrow f^{-1}(V_i)$ by

$$z'_i(b) = K(\beta_i(b)), \quad b \in V_i,$$

where $K : T^*B|_{V_i} \rightarrow f^{-1}(V_i)$ is the symplectic map determined by the section z_i . Then z'_i are Lagrangian with respect to σ' , indeed,

$$z_i'^*(\sigma - f^* \Theta) = \Theta|_{V_i} - z_i'^* \circ f^* \Theta = \Theta|_{V_i} - \Theta|_{V_i} = 0.$$

Denote $\gamma_i = q(\beta_i)$, where $q : T^*B \rightarrow T^*B/P$. Then the sections $(\delta\gamma)_{ij}$ are closed (Section 2.2.2 Lemma 2.10), and $\delta\gamma$ is a 1-cocycle with coefficients in $\widehat{\mathcal{Z}}$. Transition maps α'_{ij} of sections z'_i satisfy

$$\alpha'_{ij} = \alpha_{ij} + (\delta\gamma)_{ij},$$

so, denoting $\lambda' = [\alpha']$ we obtain

$$\lambda - \lambda' = [\delta\gamma], \quad (3.14)$$

where by the construction $\delta\gamma = Q(\Theta)$. $\square$

Example 3.1 (Lagrangian bundles over the torus). Let $B = \mathbb{T}^2$, then $H_{dR}^3(B) = 0$, and by the exactness of rows in diagram (3.10) the map $\tilde{\delta}_*$ is surjective. This means that any general locally principal n -torus bundle over B admits a symplectic structure and so becomes Lagrangian. Let (x_1, x_2) be coordinates modulo 1 on the base B , and $(x_1, x_2, \bar{y}_1, \bar{y}_2)$ be symplectic coordinates on T^*B with the canonical form

$$\omega = \sum_{i=1}^2 d\bar{y}_i \wedge dx_i.$$

Choose $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$, and consider the period lattice in $T^*\mathbb{T}^2$, given by

$$P_\alpha = \{\bar{y}_i = \ell_i \alpha_i \mid \ell_i \in \mathbb{Z}, i = 1, 2\}. \quad (3.15)$$

The lattice P_α is trivial, which means (section 3.1) that the bundle $T^*\mathbb{T}^2/P_\alpha$ is a trivial principal bundle, i.e. there is an isomorphism $T^*\mathbb{T}^2/P_\alpha \rightarrow \mathbb{T}^2 \times \mathbb{T}^2$, commuting with the action of $\mathbb{T}^2$. The bundle $T^*\mathbb{T}^2/P_\alpha$ with symplectic form ω induced from $T^*\mathbb{T}^2$, has a global Lagrangian zero section, i.e. it has both zero Chern and Lagrange classes (Terminology 3.1). We want to classify the Lagrangian bundles which can be obtained from this bundle by modifying the symplectic structure. For this we compute the kernel of the map Q in the diagram (3.10). These Lagrangian bundles are then in one-to-one correspondence with $H_{dR}^2(\mathbb{T}^2)/\ker Q$.

To this end introduce the coordinates $(x, y) = (x_1, x_2, y_1, y_2)$ on the cotangent bundle $T^*\mathbb{T}^2$ such that P_α is mapped onto $\mathbb{T}^2 \times \mathbb{Z}^2 \subset \mathbb{T}^2 \times \mathbb{R}^2$, by

$$y_i = \frac{1}{\alpha_i} \bar{y}_i, \quad i = 1, 2.$$

Then $dy_i = \frac{1}{\alpha_i} d\bar{y}_i$, and

$$\omega = \sum_{i=1}^2 \alpha_i dy_i \wedge dx_i.$$

Note that (x, y) are not symplectic with respect to ω . According to Theorem 3.4 the action of Q is such that for $[\Theta] \in H_{dR}^2(\mathbb{T}^2)$, $Q[\Theta]$ can be represented by $T^*\mathbb{T}^2/P_\alpha$ with symplectic form $\omega + \pi^*\Theta$, where Θ is a representative of $[\Theta]$ and π is the canonical projection in $T^*\mathbb{T}^2/P_\alpha$. We note that $H_{dR}^2(\mathbb{T}^2) \cong \mathbb{R}$ and that we may assume that $\Theta = \gamma dx_1 \wedge dx_2$, where $\gamma \in \mathbb{R}$. In order to investigate for which $[\Theta]$ we get a Lagrangian bundle with Lagrange class zero, we have to find out whether there is a section z in $T^*\mathbb{T}^2/P_\alpha$ for which $z^*(\omega + \pi^*\Theta)$ is exact. The cohomology class of $z^*(\omega + \pi^*\Theta)$ depends only on the homotopy class of z . Up to homotopy, any section z is “linear” in the sense that [39]

$$z : (x_1, x_2) \mapsto (x, A(x^\top)), \quad (3.16)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{gl}(2, \mathbb{R}),$$

and juxtaposition denotes matrix multiplication. We note that z is a global section if and only if $A \in \text{gl}(2, \mathbb{Z})$. Setting $a_{12} = n_1$, $a_{21} = n_2$, we compute the pullback

$$z^*\sigma = z^*(\alpha_1 dy_1 \wedge dx_1 + \alpha_2 dy_2 \wedge dx_2) = \alpha_1 n_1 dx_2 \wedge dx_1 + \alpha_2 n_2 dx_1 \wedge dx_2 = (\alpha_2 n_2 - \alpha_1 n_1) dx_1 \wedge dx_2.$$

So $[\Theta] = [\gamma dx_1 \wedge dx_2]$ is in the kernel of Q if and only if for some $n_1, n_2 \in \mathbb{Z}$, $\gamma = \alpha_2 n_2 - \alpha_1 n_1$, i.e. $\gamma \in (\alpha_1, \alpha_2)\mathbb{Z}$, where $(\alpha_1, \alpha_2)\mathbb{Z}$ is an additive subgroup of $\mathbb{R}$ with elements of the form $\alpha_2 n_2 - \alpha_1 n_1$, $n_1, n_2 \in \mathbb{Z}$. So

$$\ker Q \cong (\alpha_1, \alpha_2)\mathbb{Z} \quad \text{and} \quad \text{image } Q \cong \mathbb{R}/(\alpha_1, \alpha_2)\mathbb{Z}.$$

Thus for a fixed P_α , $\alpha = (\alpha_1, \alpha_2)$, and zero Chern class the set of isomorphism classes of Lagrangian bundles over $\mathbb{T}^2$ is in one-to-one correspondence with the set $\mathbb{R}/(\alpha_1, \alpha_2)\mathbb{Z}$, and the Lagrange class of the bundle $(T^*\mathbb{T}^2/P_\alpha, \omega')$ with $\omega' = \omega - f^*\Theta$ equals $\lambda = Q([\Theta])$. Since Q is a homomorphism, the same holds for *any* Chern class (see Remark 3.4).

Remark 3.4 (Conclusions on Example 3.1).

1. The set of isomorphism classes of Lagrangian bundles over $\mathbb{T}^2$ with fixed Chern class is in one-to-one correspondence with $\mathbb{R}/(\alpha_1, \alpha_2)\mathbb{Z}$, where $(\alpha_1, \alpha_2)\mathbb{Z}$ is an additive subgroup of $\mathbb{R}^2$ with elements of the form $(\alpha_2 n_2 - \alpha_1 n_1)$, $n_1, n_2 \in \mathbb{Z}$.
2. We note that if the ratio α_1/α_2 is rational, the quotient $\mathbb{R}/(\alpha_1, \alpha_2)\mathbb{Z}$ has the natural topology of the circle $\mathbb{T}^1$, otherwise the topology is more complicated.

Example 3.2 (Lagrangian bundles over the Klein bottle). Let (x, y) be standard coordinates on $\mathbb{R}^2$. Under the identification

$$(x, y) \sim (x + 1, y) \quad \text{and then} \quad (x, y) \sim (1 - x, y + 1)$$

one obtains the Klein bottle. The integer affine structure on $\mathbb{R}^2$ induces an integer affine structure on the Klein bottle $\mathcal{K}$, and the period lattice P_α in $T^*\mathcal{K}$ can be defined similarly to Example 3.1, i.e. by (3.15). Recall [11] that $H_{dR}^2(\mathcal{K}) = 0$ and, since $\mathcal{K}$ is not orientable, $H_{dR}^3(\mathcal{K}) = 0$. Then it follows from diagram (3.10) that the map $\tilde{\delta}^*$ is an isomorphism, which implies that there is an isomorphism between the sets of Chern and Lagrange classes of bundles over $\mathcal{K}$. In other words, all symplectic structures on a general locally principal torus bundle over $\mathcal{K}$ are isomorphic.

This completes our treatment of global invariants of Lagrangian bundles. As we already mentioned in Section 1, non-trivial monodromy have been found in many examples of real-world integrable systems. It is interesting also to know possible mechanical interpretations of other parts of the theory, i.e. integer affine structure, the Chern and the Lagrange class.

APPENDIX A. BASIC FACTS OF SYMPLECTIC GEOMETRY

We recall some definitions and basic results of symplectic geometry [3, 4, 40, 41].

Definitions and sign conventions. A *symplectic manifold* is a pair (M, σ) where M is a $2n$ -dimensional smooth manifold and σ is a closed 2-form on M such that the $2n$ -form

$$\underbrace{\sigma \wedge \cdots \wedge \sigma}_n$$

is nowhere zero. Denote by $\mathcal{X}(M)$ the set of vector fields on M . Using σ , to each vector field $X \in \mathcal{X}(M)$ one can associate a unique differential 1-form $\iota_X \sigma$ by

$$\iota_X \sigma = -\sigma(X, -).$$

The vector field X is called *symplectic* if $\iota_X \sigma$ is closed; X is called *Hamiltonian* if $\iota_X \sigma$ is exact, that is, if there exists a smooth function $g : M \rightarrow \mathbb{R}$ such that

$$dg = -\sigma(X, -).$$

In this case X is often denoted by X_g , and the function g is called the *Hamiltonian* of X_g . We notice that locally, by the Poincaré lemma [11], every symplectic vector field is Hamiltonian. The

Poisson bracket $\{-, -\}$ of two smooth functions $g, h : M \rightarrow \mathbb{R}$ on a symplectic manifold (M, σ) is defined by

$$\{g, h\} = \sigma(X_g, X_h),$$

where X_g, X_h are the Hamiltonian vector fields associated to g and h . We note that, up to sign, the Poisson bracket $\{g, h\}$ is a Lie derivative of g (resp. h) along the Hamiltonian vector field X_h (resp. X_g), that is,

$$\{g, h\} = -X_h(g) = -\mathcal{L}_{X_h}(g) = X_g(h) = \mathcal{L}_{X_g}(h).$$

For the Poisson bracket we have the *Jacobi identity*

$$\{\{g, h\}, k\} + \{\{h, k\}, g\} + \{\{k, g\}, h\} = 0,$$

so the space of smooth functions on M is a Lie algebra with respect to the Poisson bracket. It follows that the Lie bracket of two Hamiltonian vector fields is again a Hamiltonian vector field,

$$[X_g, X_h] = X_{\{g, h\}},$$

and the set of the Hamiltonian vector fields on M is a Lie subalgebra of $\mathcal{X}(M)$.

Symplectic coordinates on a smooth manifold.

Theorem A.1 (Darboux). [3, 40, 41] *Let (M, σ) be a symplectic manifold of dimension $2n$. Then for each $p \in M$ there exists an open neighborhood $V \subset M$ and a coordinate system*

$$(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n) : V \rightarrow \mathbb{R}^{2n}$$

such that σ takes the canonical form

$$\sigma = \sum_{j=1}^n dy_j \wedge dx_j. \quad (\text{A.1})$$

Sketch of the proof. We include the sketch of a proof of the Darboux theorem which shows that any system of local functions near $p \in M$, such that their differentials are linearly independent and the conditions (A.2) below are satisfied, can be completed to a symplectic coordinate system on a possibly smaller neighborhood of p , however, in a non-unique way. In the proof we use the Frobenius theorem [42] which says that if M is a smooth manifold of dimension m and $Z_1, \dots, Z_r$, where $r \leq m$, are linearly independent vector fields on an open set $U \subset M$ with vanishing Lie brackets, i.e.

$$[Z_j, Z_k] = 0,$$

then for each point $p \in U$ there exists an open neighborhood V of p and coordinate functions $(x_1, \dots, x_m) : V \rightarrow \mathbb{R}^m$ such that

$$Z_j = \frac{\partial}{\partial x_j}, \quad j = 1, \dots, r.$$

Symplectic coordinates (x, y) will be constructed by induction on the number ℓ (resp. m) of functions x (resp. y). Since x and y are interchangeable, we give the argument only for the case when we add a function $x_{\ell+1}$. The argument for y_{m+1} is the same.

If $\ell = m = 0$ then as x_1 we choose any function whose differential does not vanish near p . Let $(x_1, \dots, x_\ell, y_1, \dots, y_m)$ be functions near p such that $dx_1, \dots, dx_\ell, dy_1, \dots, dy_m$ are linearly independent, and satisfy the conditions

$$X_j(y_k) = \delta_{jk}, \quad X_j(x_j) = Y_k(y_{\hat{k}}) = 0, \quad (\text{A.2})$$

where $X_{x_j} = X_j$ (resp. $Y_{y_k} = Y_k$) are the Hamiltonian vector fields associated to x_j (resp. y_k) and $j, \hat{j} = 1, \dots, \ell$, $k, \hat{k} = 1, \dots, m$. It follows that $X_1, \dots, X_\ell, Y_1, \dots, Y_m$ commute, i.e. have zero Lie brackets. So by the Frobenius theorem on a possibly smaller neighborhood of p there exists a coordinate system

$$(\chi_1, \dots, \chi_\ell, \xi_1, \dots, \xi_m, \eta_1, \dots, \eta_{2n-\ell-m})$$

such that $X_j = \partial/\partial\chi_j$ and $Y_k = \partial/\partial\xi_k$. If $\ell + 1 > m$ we choose

$$x_{\ell+1} = g(\eta_1, \dots, \eta_{2n-\ell-m})$$

and, if $\ell + 1 \leq m$,

$$x_{\ell+1} = -\xi_{\ell+1} + g(\eta_1, \dots, \eta_{2n-\ell-m}),$$

where g is an arbitrary function such that $dg \neq 0$. Thus $x_{\ell+1}$ is determined up to a function of the remaining coordinates $\eta_1, \dots, \eta_{2n-\ell-m}$. It is easy to see that $(x_1, \dots, x_{\ell+1}, y_1, \dots, y_m)$ satisfy (A.2) and their differentials are linearly independent. The induction procedure is continued till $\ell = m = n$, when the coordinate system $(x_1, \dots, x_n, y_1, \dots, y_n)$ near p is complete. $\square$

The coordinates (x, y) on M satisfying (A.1) are called *symplectic* or *Darboux coordinates*.

Recall [41] that a submanifold W of a symplectic manifold M is called *Lagrangian* if for all $p \in W$ the tangent space $T_p W$ coincides with its *symplectic complement* in $T_p M$, that is, $T_p W = T_p W^\sigma$, where

$$T_p W^\sigma = \{Y \in T_p M \mid \sigma_p(X, Y) = 0 \text{ for all } X \in T_p W\}.$$

We note that if W is Lagrangian then $\dim W = \frac{1}{2} \dim M$, and σ vanishes on W , that is, for any vector fields X_1, X_2 on W

$$\sigma(X_1, X_2) = 0.$$

For example, if (x, y) are symplectic coordinates on an open set $V \subset M$, then the subset of V , defined by $y_1 = \dots = y_n = 0$, is a Lagrangian submanifold of M .

Symplectic coordinates on a cotangent bundle. Let N be a smooth n -dimensional manifold, T^*N be its cotangent bundle, and let $\pi : T^*N \rightarrow N$ be the projection. The *tautological* 1-form θ on N is defined by

$$\theta_\alpha(X) = \alpha(\pi_*(X)),$$

for any $\alpha \in T^*N$ and any $X \in T_\alpha(T^*N)$, and the natural symplectic structure on T^*N is given by the *canonical form* $\omega = d\theta$. Given local coordinates $x = (x_1, \dots, x_n)$ near $p \in N$, any $\alpha \in T^*N$ can be expressed in the basis $dx_1, \dots, dx_n$, i.e.

$$\alpha = \sum_{i=1}^n \alpha_i dx_i,$$

and one can define local coordinate functions $y_1, \dots, y_n$ by

$$y_i(\alpha) = \alpha_i, \quad i = 1, \dots, n.$$

Then

$$\theta = \sum_{i=1}^n y_i dx_i \quad \text{and} \quad \omega = \sum_{i=1}^n dy_i \wedge dx_i,$$

and the coordinate system (x, y) is symplectic with respect to the canonical form ω . The zero section of T^*N is locally given by equations $y_i = 0$, $i = 1, \dots, n$, and we observe that the zero section and the fibres of T^*N are Lagrangian submanifolds.

Lemma A.1 (Lagrangian sections of T^*N). [41] *Let θ be a tautological 1-form on T^*N and consider a section $s_\alpha : U \rightarrow T^*N : p \mapsto \alpha(p)$ over $U \subset N$. Then the following holds.*

1. $s_\alpha^* \theta = \alpha$.
2. *The following statements are equivalent.*
 - (a) $d\alpha = 0$.
 - (b) $s_\alpha(U)$ is a Lagrangian submanifold of T^*N .

(c) The following map is a symplectomorphism:

$$\psi : T^*N \rightarrow T^*N : \beta \mapsto \beta + s_\alpha(\pi(\beta)).$$

If $s_\alpha(U)$ is a Lagrangian submanifold of T^*N , then s_α is called a *Lagrangian section*. By Lemma A.1 s_α is Lagrangian if and only if α is a closed 1-form, so s_α is also called a *closed section*.

APPENDIX B. CHARACTERISTIC CLASSES FOR CIRCLE BUNDLES

We recall the definition of a bundle and bundle morphism [11], and the classification of principal circle bundles by the Chern class.

B.1. Bundles and Bundle Morphisms

Let E , B and F be smooth manifolds, and recall [11] that a surjective map $\pi : E \rightarrow B$ is called a *bundle* with fibre F if it is *locally trivial*, that is, there exists an open cover $\mathcal{V} = \{V_i\}$ of B and diffeomorphisms

$$\phi_i : \pi^{-1}(V_i) \rightarrow V_i \times F,$$

called *trivializations*, such that the following diagram commutes for each $V_i \in \mathcal{V}$.

$$\begin{array}{ccc} \pi^{-1}(V_i) & \xrightarrow{\phi_i} & V_i \times F \\ \pi \searrow & & \swarrow \text{pr}_1 \\ & V_i & \end{array}$$

The manifolds E and B are called the *total* and the *base* manifolds (or spaces) respectively; F is called the *fibre* and π is called the *projection*. For each $b \in B$ the set $\pi^{-1}(b)$ is diffeomorphic to F and is called the fibre at b . We will often refer to the bundle by its total space E . A smooth map

$$s : V \rightarrow E,$$

where $V \subset B$ is open, such that $\pi \circ s = \text{id}$, is called a (local) *section* of E . For each non-empty intersection $V_i \cap V_j$ and each $b \in V_i \cap V_j$ there is a diffeomorphism

$$\alpha_{ij}(b) = \phi_i \circ \phi_j^{-1}|_{\{b\} \times F} \in \text{Diff}(F) \quad (\text{B.1})$$

of F , and the function $\alpha_{ij} : V_i \cap V_j \rightarrow \text{Diff}(F)$ is called the *transition function*. Transition functions are smooth and satisfy the *cocycle condition*, that is,

$$\alpha_{ij} \circ \alpha_{jk} = \alpha_{ik}. \quad (\text{B.2})$$

A *morphism* between two bundles $\pi : E \rightarrow B$ and $\pi' : E' \rightarrow B$ is a smooth map

$$\Phi : E \rightarrow E'$$

such that the following diagram is commutative,

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ \pi \searrow & & \swarrow \pi' \\ & B & \end{array} \quad (\text{B.3})$$

that is, for any $p \in E$ we have $\pi' \circ \Phi(p) = \pi(p)$. If, moreover, Φ is a diffeomorphism, it defines an *isomorphism* between the two bundles. The bundles are *locally isomorphic* if they have diffeomorphic fibres. For example, any bundle $\pi : E \rightarrow B$ with fibre F is locally isomorphic to the trivial bundle $B \times F$.

A bundle $\pi : E \rightarrow B$ is called a *vector bundle* if for any $b \in B$ the fibre $\pi^{-1}(b)$ is a vector space and $\alpha_{ij}(b)$ are linear maps. In other words, we can take $F = \mathbb{R}^n$, and transition functions α_{ij} taking

values in $\mathrm{GL}(n, \mathbb{R})$. Well-known examples of vector bundles are the tangent and the cotangent bundles of a smooth manifold B , denoted by TB and T^*B respectively [4, 9, 11].

Recall [9, 11, 43] that a Lie group G is a smooth manifold with group structure such that the group operations, namely multiplication and inverse, are smooth. Examples of Lie groups are $\mathbb{R}^n$ and the circle $\mathbb{T}^1$ with additive group structure, and $\mathrm{GL}(n, \mathbb{R})$ with matrix multiplication as a group operation. Notice that we denote the circle by $\mathbb{S}^1$ when we are interested in its manifold structure, and by $\mathbb{T}^1$ when we want to stress that it has a group structure. Recall [9, 40] that an action of G on E is free if $g \cdot p = p$ for some $p \in E$ implies $g = \mathrm{id}_G$. A bundle $\pi : E \rightarrow B$ is called a *principal* bundle with group G if there is a fibre-preserving, smooth and free action of G on E , which is transitive on fibres. This means that fibres of E are diffeomorphic to G , and local trivializations are given by

$$\phi_i : V_i \times G \rightarrow \pi^{-1}(V_i) : (b, g) \mapsto g \circ s_i(b), \quad (\text{B.4})$$

where $s_i : V_i \rightarrow E$ is a smooth local section of E . Thus every local section determines a local trivialization of E and vice versa. This means that in the case of a principal bundle the terms “trivializations” and “local sections” are interchangeable. For each $b \in V_i \cap V_j$, where $V_i \cap V_j \neq \emptyset$, the difference between $s_i(b)$ and $s_j(b)$ is given by an element of G , so the transition functions α_{ij} take values in G . A simple example of a principal bundle is the torus $\mathbb{T}^2$ which can be thought of as a principal $\mathbb{T}^1$ -bundle over $\mathbb{S}^1$.

We require morphisms between bundles E and E' to preserve additional structure. Thus, a map Φ is a morphism between vector bundles E and E' if it is a linear map on fibres. Similarly, a morphism of principal bundles has to commute with the group action.

Remark B.1 (Bundles with symplectic structure on the total space). If the manifold E is symplectic, one may wish to take the symplectic structure in consideration in the definition of a bundle. However, the notion of a symplectic bundle cannot be defined in the general case, since the symplectic structure on the product $V \times F$, $V \subset B$ is not defined. There are some exceptions: the cotangent bundle T^*B with the canonical form ω , and Lagrangian bundles, where the symplectic structure on the product is given by the Liouville–Arnold theorem. In both cases these bundles have Lagrangian fibres.

B.2. Classification of Principal Circle Bundles

Let S and B be smooth manifolds, and $\tau : S \rightarrow B$ be a principal $\mathbb{T}^1$ -bundle with a free action of $\mathbb{T}^1$ on S denoted by

$$\mathcal{G} : \mathbb{T}^1 \times S \rightarrow S.$$

Non-triviality of S is measured by a characteristic class in the sheaf cohomology of the base manifold B , which we introduce in this section. For an introduction to sheaf theory we refer to Appendix C.

Characteristic class of principal circle bundles. Let $\mathcal{V} = \{V_i\}_{i \in I}$ be a good cover of B (Appendix C, [38]), that is, an intersection of any finite number of elements in $\mathcal{V}$ is open and contractible. Recall from Appendix B.1 that trivializations over open sets of $\mathcal{V}$ are given by local sections $s_i : V_i \rightarrow S$, and the transition functions of S take values in $\mathbb{T}^1$,

$$\alpha_{ij} : V_i \cap V_j \rightarrow \mathbb{T}^1,$$

that is, at each point b of a non-empty intersection $V_j \cap V_i$

$$\mathcal{G}(\alpha_{ij}(b), s_j(b)) = s_i(b).$$

The function α_{ij} can be regarded as a section of the trivial bundle $B \times \mathbb{T}^1 \rightarrow B$, and hence is an element of the sheaf $\mathcal{S}$ of sections of $B \times \mathbb{T}^1 \rightarrow B$ (see Appendix C on sheaf theory). Define a cochain $\alpha \in C^1(\mathcal{V}, \mathcal{S})$ by

$$\alpha : (ij) \mapsto \alpha_{ij}.$$

We notice that the transition functions are alternating,

$$\alpha_{ij} = -\alpha_{ji}.$$

Together with the cocycle condition (B.2), written in the additive form, this yields

$$(\delta^1 \alpha)_{ijk} = \alpha_{jk} - \alpha_{ik} + \alpha_{ij} = 0,$$

where we use the same notation for α_{jk} and the restriction of α_{jk} to $V_i \cap V_j \cap V_k$. As we will see in Appendix C, this means that α is a 1-cocycle and it represents the cohomology class

$$[\alpha] \in H^1(\mathcal{V}, \mathcal{T}).$$

We show that $[\alpha]$ is a complete invariant of principal circle bundles. We do this first for a fixed good cover $\mathcal{V}$ of B , and then show that our construction is independent of the good cover.

Lemma B.1 (Independence of choice of local trivializations). *The cohomology class $[\alpha] \in H^1(\mathcal{V}, \mathcal{T})$ of a principal circle bundle S does not depend on the choice of local trivializations, that is, if $\{s_i\}$ and $\{s'_i\}$ are two choices of trivializations over an open cover $\mathcal{V}$ with transition maps α_{ij} and α'_{ij} respectively, then*

$$[\alpha] = [\alpha'] \in H^1(\mathcal{V}, \mathcal{T}).$$

Proof. We show that α and α' differ by a coboundary. Indeed, for each $b \in V_i$ the values $s_i(b)$ and $s'_i(b)$ differ by an element in $\mathbb{T}^1$, that is,

$$\mathcal{G}(\beta_i(b), s_i(b)) = s'_i(b)$$

for some smooth function

$$\beta_i : V_i \rightarrow \mathbb{T}^1.$$

The assignment

$$\beta : i \mapsto \beta_i$$

is a 0-chain with coefficients in $\mathcal{T}$, and for each $(ij) \in I \times I$ we have

$$\alpha'_{ij} = \alpha_{ij} + \beta_j - \beta_i = \alpha_{ij} + (\delta^0 \beta)_{ij},$$

where we use the same notation for β_i and its restriction to $V_i \cap V_j$. Thus $[\alpha'] = [\alpha]$. $\square$

We show that $[\alpha] \in H^1(\mathcal{V}, \mathcal{T})$ is a complete invariant of principal circle bundles over a manifold B with a trivializing good cover $\mathcal{V}$.

Remark B.2 (Trivializations over good covers). Note that any good cover $\mathcal{V}$ is a trivializing cover, since all open sets in $\mathcal{V}$ are contractible.

Theorem B.1 (Complete invariant of principal circle bundles). *Let $\mathcal{V}$ be a good cover of a smooth manifold B . Then*

1. *each element $[\alpha] \in H^1(\mathcal{V}, \mathcal{T})$ corresponds to a principal circle bundle over B ,*
2. *two principal circle bundles $\tau : S \rightarrow B$ and $\tau' : S' \rightarrow B$ with corresponding cohomology classes $[\alpha]$ and $[\alpha']$ respectively are isomorphic if and only if $[\alpha] = [\alpha']$.*

Proof.

1. Consider a disjoint union $\coprod_i V_i \times \mathbb{T}^1$, let $\alpha : (ij) \mapsto \alpha_{ij}$ be a representative of $[\alpha]$, and define

$$S = \left(\coprod_i V_i \times \mathbb{T}^1 \right) / \left\{ (b, \beta)_i \sim (b, \alpha_{ij}(b) + \beta)_j \right\},$$

where $b \in V_i \cap V_j$, the sign $\sim$ denotes the equivalence relation, $(b, \beta)_i \in V_i \times \mathbb{T}^1$ and $(b, \alpha_{ij}(b) + \beta)_j \in V_j \times \mathbb{T}^1$. The projection $\text{pr}_1 : S \rightarrow B$ is a principal circle bundle.

2. Suppose $\Phi : S \rightarrow S'$ is a bundle isomorphism, and let $s_i : V_i \rightarrow S$ be trivializations over $V_i \in \mathcal{V}$ with transition functions α_{ij} . Define trivializations on S' by

$$s'_i = \Phi \circ s_i.$$

It is easy to see that s'_i have the same transition functions, i.e. $\alpha_{ij} = \alpha'_{ij}$, so that $[\alpha] = [\alpha']$. Conversely, assume $[\alpha] = [\alpha']$. Then by lemma B.1 S and S' admit local trivializations ϕ_i and ϕ'_i with the same transition functions, and one can define an isomorphism $\Phi : S \rightarrow S'$ by

$$\Phi(p) = \phi'^{-1}_i \circ \phi_i(p),$$

where i is an index such that $\pi(p) \in V_i$. $\square$

Remark B.3 (Trivial circle bundle). By Theorem B.1 a principal circle bundle S is trivial if and only if $[\alpha] = 0$. In this sense $[\alpha]$ measures non-triviality of S .

Characteristic classes under refining maps of good covers. We show that the cohomology class $[\alpha] \in H^1(\mathcal{V}, \mathcal{T})$ is independent of the choice of the good cover.

Every two good covers $\mathcal{V}$ and $\mathcal{W}$ have common good refinement $\mathcal{U}$ [11, 44]. For $\mathcal{V}$ and $\mathcal{U}$, choose a refining map $k : J \rightarrow I$ so that $U_j \subset V_{k(j)}$ for all $j \in J$. We will show that this map induces an isomorphism on the cohomology of good covers with coefficients in $\mathcal{T}$, independent on the choice of k , and the image of $[\alpha]$ under this map corresponds to the same principal circle bundle. Taking direct limit of cohomology over all good covers of B (see Appendix C.3) we obtain a complete invariant of principal circle bundles which does not depend on the good cover.

To see that recall [11, 44] that the refining map k defines a homomorphism

$$k^* : C^q(\mathcal{V}, \mathcal{T}) \rightarrow C^q(\mathcal{U}, \mathcal{T})$$

on the cochain complexes by the formula

$$(k^* f)(j_0, \dots, j_q) = \rho_{U_{j_0} \cap \dots \cap U_{j_q}}^{V_{k(j_0)} \cap \dots \cap V_{k(j_q)}} f(k(j_0), \dots, k(j_q)),$$

where $\rho_{U_{j_0} \cap \dots \cap U_{j_q}}^{V_{k(j_0)} \cap \dots \cap V_{k(j_q)}}$ is the restriction map in the sheaf $\mathcal{T}$. For each $q \geq 1$ the map k^* commutes with the coboundary homomorphisms δ on the covers $\mathcal{V}$ and $\mathcal{U}$. The homomorphism k^* induces a homomorphism on cohomology

$$\kappa : H^1(\mathcal{V}, \mathcal{T}) \rightarrow H^1(\mathcal{U}, \mathcal{T}), \quad (\text{B.5})$$

which is independent of the choice of the refining map k [44]. Both $\mathcal{V}$ and $\mathcal{U}$ are good covers, hence the elements of their first cohomology with coefficients in $\mathcal{T}$ are in one-to-one correspondence with isomorphism classes of principal circle bundles by Theorem B.1. Let $z_i : V_i \rightarrow S$ be local trivializations corresponding to the cocycle α . For each $j \in J$, the corresponding trivialization is obtained as the restriction $z_{k(j)}|_{U_j}$, and the corresponding cocycle is $k^*\alpha$. Hence α and $k^*\alpha$ correspond to the same principal circle bundle. It follows that $[\alpha]$ and $\kappa[\alpha]$ also correspond to the same principal circle bundle, and the map (B.5) is an isomorphism. Taking the direct limit of cohomology groups with coefficients in $\mathcal{T}$ over all good covers of B , one obtains the cohomology $H^q(B, \mathcal{T})$ of the smooth manifold B (see Appendix C.3), and the isomorphism $H^q(\mathcal{V}, \mathcal{T}) \rightarrow H^q(B, \mathcal{T})$, $q \geq 1$. The image $\mu \in H^1(B, \mathcal{T})$ of $[\alpha]$ under this isomorphism is a complete invariant of principal circle bundles which does not depend on the choice of the good cover.

Remark B.4 (Chern class of principal circle bundles). By Example C.7 in Appendix C.3 for each good cover $\mathcal{V}$ of B there is an isomorphism

$$\delta : H^1(\mathcal{V}, \mathcal{T}) \rightarrow H^2(\mathcal{V}, \mathcal{Z}),$$

where $\mathcal{Z}$ is the sheaf of locally constant $\mathbb{Z}$ -valued functions on B . We define

$$\nu = \delta[\alpha] \in H^2(\mathcal{V}, \mathcal{Z}), \quad (\text{B.6})$$

and also call ν the Chern class of the bundle S . An analogue of Theorem B.1 holds for the Chern class ν , i.e. ν is a complete invariant of principal circle bundles.

APPENDIX C. SHEAVES AND SHEAF COHOMOLOGY

This section contains a brief introduction to sheaf theory [11, 44, 45]. In particular, sheaf cohomology of an open cover and fine sheaves are introduced; the de Rham theorem is proven as a first application of sheaf theory.

C.1. Sheaves and Sheaf Cohomology of an Open Cover

We give the definition and examples of sheaves.

Definition C.1 (Abelian sheaf). [44] An *Abelian sheaf* $\mathcal{S}$ on a topological space B is an assignment to each open set $V \subset B$ of an Abelian group $\mathcal{S}(V)$

$$\mathcal{S} : V \mapsto \mathcal{S}(V),$$

and, to each inclusion $U \subset V$, of a homomorphism of Abelian groups ρ_U^V

$$\rho_U^V : \mathcal{S}(V) \rightarrow \mathcal{S}(U),$$

with the following properties.

1. $\mathcal{S}(\emptyset) = \{0\}$.
2. $\rho_V^V = \text{id}_V$.
3. $\rho_W^U \circ \rho_U^V = \rho_W^V$ for all inclusions $W \subset U \subset V$ of open sets.
4. For all open sets $V, U, W \subset B$ such that $V = U \cup W$ and all elements $s_1, s_2 \in \mathcal{S}(V)$ satisfying

$$\rho_U^V(s_1) = \rho_U^V(s_2) \quad \text{and} \quad \rho_W^V(s_1) = \rho_W^V(s_2)$$

we have $s_1 = s_2$.

5. For all open sets $V, U, W \subset B$ such that $V = U \cup W$ and all elements $s_1 \in \mathcal{S}(U)$ and $s_2 \in \mathcal{S}(W)$ satisfying

$$\rho_{U \cap W}^U(s_1) = \rho_{U \cap W}^W(s_2)$$

there exists an element $s \in \mathcal{S}(V)$ such that

$$\rho_U^V(s) = s_1 \quad \text{and} \quad \rho_W^V(s) = s_2.$$

In other words, if s_1 and s_2 agree on overlaps they come from an element of $\mathcal{S}(V)$.

The homomorphisms ρ_U^V are called the *restriction maps* of $\mathcal{S}$.

Example C.1 (Sheaves of sections). Let $\pi : S \rightarrow B$ be a smooth bundle where fibres are Abelian groups. For example, S may be any vector bundle.

1. Define the sheaf $\mathcal{S}$ by

$$\mathcal{S} : V \mapsto \Gamma(V, S),$$

where $\Gamma(V, S)$ is the set of all smooth sections of S over V . For each $U \subset V$ define the homomorphism ρ_U^V as the restriction

$$\rho_U^V : \mathcal{S}(V) \rightarrow \mathcal{S}(U) : s \mapsto s|_U.$$

The sheaf $\mathcal{S}$ is called a *sheaf of sections*, and an element $s \in \mathcal{S}(V)$ is called a *section* of $\mathcal{S}$.

2. In item 1 let $S = B \times \mathbb{R}$, the trivial bundle over B . The sections of this bundle are smooth $\mathbb{R}$ -valued functions, and the corresponding sheaf Λ^0 is called the *sheaf of smooth functions* on B .

3. In item 1 let $S = T^*B$, the cotangent bundle over B . The sections are differential 1-forms on B , and the corresponding sheaf is the sheaf Λ^1 of differential 1-forms on B . Restricting to closed sections of T^*B one obtains the sheaf $\mathcal{Z}^1$ of closed differential 1-forms on B .

Let $\mathcal{S}$ be an Abelian sheaf on B and $\mathcal{V} = \{V_i\}_{i \in I}$ be an open cover of B . A q -cochain f is a function which associates to each $(q+1)$ -tuple $(i_0, \dots, i_q)$ of indices in I an element $f(i_0, \dots, i_q)$ of $\mathcal{S}(V_{i_0} \cap \dots \cap V_{i_q})$ alternating in these indices, that is,

$$f(i_0, \dots, i_\ell, i_{\ell+1}, \dots, i_q) = -f(i_0, \dots, i_{\ell+1}, i_\ell, \dots, i_q).$$

The set of all q -cochains, denoted by $C^q(\mathcal{V}, \mathcal{S})$, forms a group. A *coboundary homomorphism* $\delta^q : C^q(\mathcal{V}, \mathcal{S}) \rightarrow C^{q+1}(\mathcal{V}, \mathcal{S})$ is defined by the formula

$$(\delta^q f)(i_0, \dots, i_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \rho_{V_{i_0} \cap \dots \cap \hat{V}_{i_i} \cap \dots \cap V_{i_{q+1}}} (f(i_0, \dots, \hat{i}_i, \dots, i_{q+1})), \quad (\text{C.1})$$

where the “hat” over a symbol means that the symbol is to be omitted. A straightforward calculation shows that

$$\delta^{q+1} \circ \delta^q = 0.$$

In what follows, if there is no danger of confusion, we omit the superscript and write δ for δ^q . The collection $\{C^q(\mathcal{V}, -), \delta\}$ is called the *cochain complex*. An element of $\ker \delta^q$ is called a q -cocycle. The set of q -cocycles form a subgroup of $C^q(\mathcal{V}, \mathcal{S})$, and the group

$$H^q(\mathcal{V}, \mathcal{S}) = \ker \delta^q / \text{image } \delta^{q-1}, \quad q \geq 1, \quad H^0(\mathcal{V}, \mathcal{S}) = \ker \delta^0,$$

is called the q -th *cohomology group* of the open cover $\mathcal{V}$ with coefficients in the sheaf $\mathcal{S}$.

Lemma C.1 (Zero cohomology of a sheaf). *The zero cohomology group $H^0(\mathcal{V}, \mathcal{S})$ does not depend on the open cover $\mathcal{V}$.*

Proof. Denote by $\Gamma(B, \mathcal{S})$ the set of global sections in $\mathcal{S}$, which by definition is

$$\Gamma(B, \mathcal{S}) = \mathcal{S}(B).$$

Let $s \in \Gamma(B, \mathcal{S})$ and for each $V_i \in \mathcal{V}$ denote by $s_i = s|_{V_i}$ the restriction. Then by the definition of a sheaf $s_i = s_j$ on $V_i \cap V_j$. Define a 0-cochain $f \in C^0(\mathcal{V}, \mathcal{S})$ by $f(i) = s_i$, then

$$(\delta f)(ij) = f(i) - f(j) = 0, \quad (\text{C.2})$$

so $f \in \ker \delta = H^0(\mathcal{V}, \mathcal{S})$. Conversely, let $f = [f] \in H^0(\mathcal{V}, \mathcal{S}) \subset C^0(\mathcal{V}, \mathcal{S})$, then (C.2) holds, and f determines a global section of $\mathcal{S}$. Thus for any open cover $\mathcal{V}$ there is an isomorphism

$$H^0(\mathcal{V}, \mathcal{S}) \cong \Gamma(B, \mathcal{S}),$$

and $H^0(\mathcal{V}, \mathcal{S})$ does not depend on a cover of B . □

We define the 0-th cohomology of B with coefficients in $\mathcal{S}$ by

$$H^0(B, \mathcal{S}) = \Gamma(B, \mathcal{S}).$$

Definition C.2 (Sheaf homomorphism). [45] Let $\mathcal{S}$ and $\mathcal{S}'$ be Abelian sheaves on B . A *sheaf homomorphism*

$$h : \mathcal{S} \rightarrow \mathcal{S}'$$

is given by a collection of group homomorphisms

$$h_V : \mathcal{S}(V) \rightarrow \mathcal{S}'(V),$$

where $V \subset B$ is open, such that the following diagram commutes for each inclusion of open sets $U \subset V$.

$$\begin{array}{ccc} \mathcal{S}(V) & \xrightarrow{h_V} & \mathcal{S}'(V) \\ \rho_U^V \downarrow & & \downarrow \rho_U'^V \\ \mathcal{S}(U) & \xrightarrow{h_U} & \mathcal{S}'(U) \end{array} \quad (\text{C.3})$$

Given a sheaf homomorphism $h : \mathcal{S} \rightarrow \mathcal{S}'$, for each $V \subset B$ the set

$$\ker h_V = \{s \in \mathcal{S}(V) \mid h_V(s) = 0\}$$

is a subgroup of $\mathcal{S}(V)$. A sheaf homomorphism h is called a *monomorphism* if for each $V \subset B$ the homomorphism h_V is injective, that is, $\ker h_V = \{0\}$.

Remark C.1 (Kernel of h as sheaf on B). Given a sheaf homomorphism $h : \mathcal{S} \rightarrow \mathcal{S}'$, the assignment

$$\mathcal{K} : V \mapsto \mathcal{K}(V) = \ker h_V$$

with restriction maps $\varrho_U^V = \rho_U^V|_{\mathcal{K}(V)}$ is a sheaf over B [45].

Example C.2 (Sheaf of locally constant functions). Consider the sheaf homomorphism

$$d^0 : \Lambda^0 \rightarrow \mathcal{Z}^1,$$

where Λ^0 and $\mathcal{Z}^1$ are the sheaves of smooth functions and closed differential 1-forms on B respectively (Example C.1), and d^0 is the homomorphism induced by the exterior differentiation. Define

$$\mathcal{R} : V \mapsto \ker d_V^0.$$

For any open connected $V \subset B$ the set $\ker d_V^0$ contains $\mathbb{R}$ -valued functions, constant on V , and the sheaf $\mathcal{R}$ is called the sheaf of locally constant $\mathbb{R}$ -valued functions.

Remark C.2 (Assignment image h). Unlike in the case of the kernel of a sheaf homomorphism the assignment

$$\text{image } h : V \mapsto \text{image } h_V$$

is not a sheaf. Indeed, let $B = \mathbb{R}^2 \setminus \{0\}$ and consider the sheaf homomorphism

$$d^0 : \Lambda^0 \rightarrow \mathcal{Z}^1,$$

where d^0 , Λ^0 and $\mathcal{Z}^1$ are like in Example C.2. Let $V_1, V_2 \subset B$ be open simply connected subsets such that $V_1 \cup V_2 = B$. Then consider a closed differential 1-form on B , given by

$$\alpha = \frac{1}{x^2 + y^2}(xdy - ydx).$$

Its restrictions to V_1 and V_2 are exact by the Poincaré lemma, hence

$$\alpha_1 \in \text{image } d_{V_1}^0 \quad \text{and} \quad \alpha_2 \in \text{image } d_{V_2}^0,$$

but there is no smooth function g on $V_1 \cup V_2$ such that $dg = \alpha$, which means that the assignment

$$\text{image } d^0 : V_i \mapsto \text{image } d_{V_i}^0, \quad i = 1, 2,$$

does not satisfy the last axiom of definition C.1, and so it is not a sheaf.

A sheaf homomorphism $h : \mathcal{S} \rightarrow \mathcal{S}'$ is called an *epimorphism* if for each $b \in B$ there exists an open neighborhood $V \subset B$ of b such that the group homomorphism

$$h_V : \mathcal{S}(V) \rightarrow \mathcal{S}'(V)$$

is surjective.

Example C.3 (Surjective homomorphism d^0). The sheaf homomorphism

$$d^0 : \Lambda^0 \rightarrow \mathcal{Z}^1$$

studied in Example C.2 and Remark C.2 is surjective. Indeed, every point $b \in B$ has a simply connected neighborhood $V \subset B$, so by the Poincaré lemma every $\alpha \in \mathcal{Z}^1(V)$ is exact, and

$$d_V^0 : \Lambda^0(V) \rightarrow \mathcal{Z}^1(V)$$

is surjective.

C.2. Fine Sheaves

We define a fine sheaf $\mathcal{F}$ on a smooth manifold B , and prove that for any locally finite open cover $\mathcal{V}$ of B and any integer $q \geq 1$ the cohomology group $H^q(\mathcal{V}, \mathcal{F}) = 0$.

Recall [11, 35, 44] that a topological space B is paracompact if it is Hausdorff and every open cover of B has a locally finite refinement. Every finite dimensional smooth manifold B is paracompact, and every locally finite open cover $\mathcal{V} = \{V_i\}_{i \in I}$ of a smooth manifold B has a subordinate smooth *partition of unity* [35, 44], that is, there exists a family $\{\psi_i\}_{i \in I}$ of non-negative smooth functions on B such that

1. for each $i \in I$ the support of ψ_i is contained in V_i ,
2. each $b \in B$ has a neighborhood which meets only a finite number of supports of ψ_i ,
3. for each $b \in B$ we have $\sum_{i \in I} \psi_i(b) = 1$.

All fine sheaves we will encounter are sheaves of smooth sections of vectors bundles, so we restrict to vector bundles in the following definition of a fine sheaf.

Definition C.3 (Fine sheaf of sections of vector bundle). [44] A sheaf $\mathcal{F}$ of smooth sections of a vector bundle E is called *fine* if for any smooth function $f : B \rightarrow \mathbb{R}$ and any element $s \in \mathcal{F}(V)$, where $V \subset B$ is open, $\bar{f}s$, where $\bar{f} = f|_V$, again is an element of $\mathcal{F}(V)$.

Example C.4 (Fine sheaves of sections). Consider the sheaves defined in Example C.1.

1. The sheaves Λ^0 and Λ^1 of smooth $\mathbb{R}$ -valued functions and differential 1-forms on B are fine.
2. The sheaf $\mathcal{R}$ of locally constant $\mathbb{R}$ -valued functions is not fine. Indeed, multiplication of a locally constant function by a smooth function need not produce a locally constant function.
3. Similarly, the sheaf $\mathcal{Z}^1$ of closed sections of T^*B is not fine, since the result of the multiplication of a closed local section in $T^*B|_V$, $V \subset B$ is open, by a smooth function need not be a closed section in $T^*B|_V$.

Since every open cover of a smooth manifold has a locally finite subcover, all covers are assumed to be locally finite. We compute cohomology with coefficients in a fine sheaf.

Lemma C.2 (Cohomology of a fine sheaf of sections). [44] *Let $\mathcal{F}$ be a fine sheaf of sections on a smooth manifold B and $\mathcal{V}$ be an open cover of B . Then*

$$H^q(\mathcal{V}, \mathcal{F}) = 0 \quad \text{for } q \geq 1.$$

Proof. For each $q \geq 1$ we define a homotopy operator

$$k^q : C^q(\mathcal{V}, \mathcal{F}) \rightarrow C^{q-1}(\mathcal{V}, \mathcal{F})$$

as follows. Let $\{\psi_i\}_{i \in I}$ be a smooth partition of unity subordinate to $\mathcal{V}$ and define

$$(k^q f)(i_0, \dots, i_{q-1}) = \sum_{i \in I} \bar{f}(i, i_0, \dots, i_{q-1}), \quad (\text{C.4})$$

where $\bar{f}(i, i_0, \dots, i_{q-1})$ is equal to $\psi_i f(i, i_0, \dots, i_{q-1})$ on $V_i \cap V_{i_0} \cap \dots \cap V_{i_{q-1}}$ and vanishes outside this set. The sum can be formed because $\mathcal{V}$ is locally finite. Using (C.4) and formula (C.1) for the coboundary homomorphism

$$\delta^q : C^q(\mathcal{V}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{V}, \mathcal{F})$$

one computes that $\delta^{q-1}k^q + k^{q+1}\delta^q = \text{id}$ for $q \geq 1$. Hence $H^q(\mathcal{V}, \mathcal{S})$ vanishes for $q \geq 1$. $\square$

C.3. Real and Integer Cohomology of a Smooth Manifold

As a first application of sheaf theory we state the de Rham theorem which computes cohomology of a smooth manifold with coefficients in the sheaf $\mathcal{R}$ of locally constant $\mathbb{R}$ -valued functions. We quote a similar result [11] for the cohomology with coefficients in the sheaf of locally constant $\mathbb{Z}$ -valued functions, and the *Leray theorem* [46] which allows to compute cohomology with coefficients in certain sheaves.

Before proceeding we introduce a *good cover* of a smooth manifold B .

Definition C.4 (Good cover). A good cover is a locally finite open cover $\mathcal{V}$ of B such that all finite intersections of elements in $\mathcal{V}$ are contractible.

Lemma C.3 (Existence of a good cover). [38] *Any cover $\mathcal{V}$ of a manifold B has a good refinement.*

Proof. Let ρ be a complete Riemannian metric on B , i.e. every Cauchy sequence converges. Recall [38] that every point $b \in B$ has arbitrarily small *geodesically convex* neighborhoods V , i.e. any two points in V can be joined by a unique geodesic in V . The intersection of any geodesically convex neighborhoods is again geodesically convex. Any open cover $\mathcal{V}$ has a refinement $\mathcal{U}$ by geodesically convex neighborhoods. We show that such a refinement can be chosen to be also locally finite.

Choose $b_0 \in B$ and define

$$K_j = \{b \mid \rho(b, b_0) \leq j\},$$

so K_j is compact and $K_{j-1} \subset \text{int } K_j$. Denote by $C(K_j)$ the complement of K_j in B . Then for each j there exists a number $r(j) > 0$ such that

$$\rho(K_{j-1}, \overline{C(K_j)}) > 2r(j).$$

Consider the following sequence of compact sets

$$K_1, \overline{K_2 - K_1}, \overline{K_3 - K_2}, \dots, \overline{K_{i+1} - K_i}, \dots$$

Cover each $\overline{K_{i+1} - K_i}$ by geodesically convex neighborhoods of radius $\varepsilon_i \leq r(i)$ and ε_i so small that each neighborhood is contained in V_i for some $i \in I$. Since each $\overline{K_{i+1} - K_i}$ is compact, it is covered by a finite number of such neighborhoods. This shows that the refinement $\mathcal{U}$ can be chosen locally finite. $\square$

Since every open cover of a manifold has a good refinement, we may assume all covers in our consideration to be good.

De Rham theorem. Denote by Λ^q the sheaf of differential q -forms on B and let

$$d : \Lambda^q \rightarrow \Lambda^{q+1} \tag{C.5}$$

be the homomorphism induced by the exterior differentiation. For each $q \geq 0$ denote by $\mathcal{Z}^q$ the sheaf given by the kernel of d (Appendix C.1 Remark C.1). Elements of $\mathcal{Z}^q$ are closed differential q -forms, and $\mathcal{Z}^0 = \mathcal{R}$, where $\mathcal{R}$ is the sheaf of locally constant functions on B (Appendix C.1 Example C.2). Define the q -th de Rham cohomology group by [44]

$$H_{\text{dR}}^q(B) = \Gamma(B, \mathcal{Z}^q) / d(\Gamma(B, \Lambda^{q-1})) \text{ for } q \geq 1,$$

and

$$H_{\text{dR}}^0(B) = \Gamma(B, \mathcal{Z}^0)$$

Theorem C.1 (De Rham). [44] *The sheaf cohomology with coefficients in $\mathcal{R}$ does not depend on the good cover $\mathcal{V}$ of B , and there is an isomorphism*

$$H_{\text{dR}}^q(B) \cong H^q(\mathcal{V}, \mathcal{R}) \text{ for } q \geq 0.$$

Proof. Let $\mathcal{V}$ be a good cover of B , then the Poincaré lemma holds on any finite intersection of open sets in $\mathcal{V}$, so the following sequence of sheaves is exact

$$0 \longrightarrow \mathcal{Z}^q \xrightarrow{\iota} \Lambda^q \xrightarrow{d} \mathcal{Z}^{q+1} \longrightarrow 0, \quad (\text{C.6})$$

where $\iota: \mathcal{Z}^q \rightarrow \Lambda^q$ is the inclusion homomorphism. Then the following sequence of cochain complexes is exact,

$$\begin{array}{ccccccc} & \cdots & & \cdots & & \cdots & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & C^{k-1}(\mathcal{V}, \mathcal{Z}^q) & \xrightarrow{\iota} & C^{k-1}(\mathcal{V}, \Lambda^q) & \xrightarrow{d} & C^{k-1}(\mathcal{V}, \mathcal{Z}^{q+1}) \longrightarrow 0 \\ & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 & \longrightarrow & C^k(\mathcal{V}, \mathcal{Z}^q) & \xrightarrow{\iota} & C^k(\mathcal{V}, \Lambda^q) & \xrightarrow{d} & C^k(\mathcal{V}, \mathcal{Z}^{q+1}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & \cdots & & \cdots & & \cdots & \end{array}$$

and gives rise to the long exact sequence of cohomology groups [11, 13],

$$0 \longrightarrow H^0(\mathcal{V}, \mathcal{Z}^q) \xrightarrow{\iota_*} H^0(\mathcal{V}, \Lambda^q) \xrightarrow{d_*} H^0(\mathcal{V}, \mathcal{Z}^{q+1}) \xrightarrow{\delta_*} H^1(\mathcal{V}, \mathcal{Z}^q) \longrightarrow \dots \quad (\text{C.7})$$

where ι_* , d_* , δ_* are homomorphisms on cohomology induced by the inclusion, exterior differentiation and coboundary homomorphisms respectively. Since Λ^q is a fine sheaf, by lemma C.2 its cohomology vanishes for $k \geq 1$ and (C.7) falls apart into

$$0 \longrightarrow H^0(\mathcal{V}, \mathcal{Z}^q) \xrightarrow{\iota_*} H^0(\mathcal{V}, \Lambda^q) \xrightarrow{d_*} H^0(\mathcal{V}, \mathcal{Z}^{q+1}) \xrightarrow{\delta_*} H^1(\mathcal{V}, \mathcal{Z}^q) \longrightarrow 0 \quad (\text{C.8})$$

and, for $k \geq 1$,

$$0 \longrightarrow H^k(\mathcal{V}, \mathcal{Z}^{q+1}) \xrightarrow{\delta_*} H^{k+1}(\mathcal{V}, \mathcal{Z}^q) \longrightarrow 0. \quad (\text{C.9})$$

In (C.8) set $q = 0$ and recall (section C.1 Lemma C.1) that $H^0(\mathcal{V}, -)$ does not depend on the cover and is isomorphic to the set of global sections $\Gamma(B, -)$. Then

$$H^0(B, \mathcal{R}) = \Gamma(B, \mathcal{Z}^0) = H_{\text{dR}}^0(B).$$

Applying (C.9) consecutively we obtain

$$H^q(\mathcal{V}, \mathcal{R}) = H^q(\mathcal{V}, \mathcal{Z}^0) \cong H^{q-1}(\mathcal{V}, \mathcal{Z}^1) \cong \dots \cong H^1(\mathcal{V}, \mathcal{Z}^{q-1}),$$

and by the exactness of (C.8) we have

$$H^1(\mathcal{V}, \mathcal{Z}^{q-1}) \cong H^0(\mathcal{V}, \mathcal{Z}^q) / d_*(H^0(\mathcal{V}, \Lambda^{q-1})).$$

Applying again Lemma C.1 for zero-dimensional cohomology we write

$$H^q(\mathcal{V}, \mathcal{R}) \cong \Gamma(B, \mathcal{Z}^q) / d(\Gamma(B, \Lambda^{q-1})) = H_{\text{dR}}^q(B), \text{ for } q \geq 1.$$

□

It follows from Theorem C.1 that $H^q(\mathcal{V}, \mathcal{R})$ is independent of the good cover $\mathcal{V}$.

Remark C.3 (Singular cohomology with real coefficients). Notice [11] that the de Rham cohomology of a manifold B is isomorphic to the singular cohomology of B with real coefficients, i.e.

$$H_{\text{dR}}^q(B) \cong H^q(B, \mathbb{R}), \quad q \geq 0.$$

The Leray theorem. In Theorem C.1 we expressed the cohomology $H^q(\mathcal{V}, \mathcal{R})$ of the good cover $\mathcal{V}$ of B with coefficient in the sheaf $\mathcal{R}$ of locally constant $\mathbb{R}$ -valued functions in terms of the de Rham cohomology, which is independent of the choice of the good cover $\mathcal{V}$. However, such a representation is not readily available for an arbitrary sheaf $\mathcal{S}$, and now we give a general definition of a sheaf cohomology of a smooth manifold B .

Let $\mathcal{S}$ be a sheaf on B , and let $\mathcal{V}$ be an open cover of B . Recall [11] that an open cover $\mathcal{U}$ of B is called a *refinement* of $\mathcal{V}$ if each $U_i \in \mathcal{U}$ is contained in some $V_j \in \mathcal{V}$. Choose a refining map $k : J \rightarrow I$ so that $U_j \subset V_{k(j)}$ for all $j \in J$. The refining map k induces a homomorphism

$$k^* : C^q(\mathcal{V}, \mathcal{S}) \rightarrow C^q(\mathcal{U}, \mathcal{S})$$

on the cochain complexes, which commutes with the coboundary operators δ on $\mathcal{V}$ and $\mathcal{U}$ and induces a homomorphism

$$\kappa_{\mathcal{U}}^{\mathcal{V}} : H^q(\mathcal{V}, \mathcal{S}) \rightarrow H^q(\mathcal{U}, \mathcal{S}), \quad q \geq 0, \quad (\text{C.10})$$

independent of the choice of the refining map $k : J \rightarrow I$ [11, 44]. Given open covers $\mathcal{V}$ and $\mathcal{W}$ of B , we say that elements

$$v \in H^q(\mathcal{V}, \mathcal{Z}) \quad \text{and} \quad w \in H^q(\mathcal{W}, \mathcal{Z})$$

are equivalent and write $v \sim w$, if there exists a common refinement $\mathcal{U}$ of $\mathcal{V}$ and $\mathcal{W}$ such that

$$\kappa_{\mathcal{U}}^{\mathcal{V}}(v) = \kappa_{\mathcal{U}}^{\mathcal{W}}(w).$$

We define

$$H^q(B, \mathcal{S}) = \coprod_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{S}) / \sim, \quad q \geq 0, \quad (\text{C.11})$$

where $\mathcal{U}$ runs through all open covers of B . Notice that since the map k^* on the cochain complexes is the restriction (Appendix B.2), for $q = 0$ the definition (C.11) is compatible with the definition of $H^0(B, \mathcal{S})$ given in Appendix C.1.

Remark C.4 (Restriction to good covers in the limit of cohomology groups). Notice that any open cover $\mathcal{V}$ of a manifold B has a good refinement [11], so in (C.11) one can restrict to only good covers of B .

For $q \geq 1$ there is a natural homomorphism

$$\kappa : H^q(\mathcal{V}, \mathcal{S}) \rightarrow H^q(B, \mathcal{S}), \quad q \geq 1, \quad (\text{C.12})$$

given by

$$\kappa(v) = \coprod_{\mathcal{U}} \kappa_{\mathcal{U}}^{\mathcal{V}}(v) / \sim,$$

where $\mathcal{U}$ runs through all refinements of $\mathcal{V}$. We are interested in cases when (C.12) is an isomorphism.

Theorem C.2 (Leray theorem). [46] *Let $\mathcal{S}$ be a sheaf of Abelian groups on B , and $\mathcal{V}$ be an open cover of B such that for any finite intersection $U = V_{i_0} \cap \cdots \cap V_{i_z}$, $z \in \mathbb{N}$*

$$H^q(U, \mathcal{S}) = 0, \quad q \geq 1.$$

Then the cohomology $H^q(\mathcal{V}, \mathcal{S})$ is independent of the cover, that is, (C.12) is an isomorphism.

Example C.5 (Cohomology with coefficients in $\mathcal{Z}$). A result similar to the de Rham theorem can be obtained for the cohomology of a manifold B (in fact, B being a triangularizable space is sufficient) with integer coefficients [11], i.e. one can show that, if $\mathcal{V}$ is good, for $q \geq 1$ the cohomology $H^q(\mathcal{V}, \mathcal{Z})$ with coefficients in the sheaf $\mathcal{Z}$ of locally constant $\mathbb{Z}$ -valued functions is isomorphic to the singular cohomology $H^q(B, \mathbb{Z})$ and hence independent of the good cover $\mathcal{V}$. To

see that notice that the $\mathcal{Z}$ -cohomology of a contractible open set U is isomorphic to the *singular cohomology* [11, 14] of $\mathbb{R}^n$, which is given by

$$H^q(\mathbb{R}^n, \mathbb{Z}) = \begin{cases} 0, & q \geq 1 \\ \mathbb{Z}, & q = 0. \end{cases}$$

If $\mathcal{V}$ is good, any finite intersection $U = V_{i_0} \cap \dots \cap V_{i_z}$ of open sets in $\mathcal{V}$ is contractible, hence the cohomology groups $H^q(U, \mathcal{Z})$ vanish for $q \geq 1$. Then by the Leray theorem there is an isomorphism

$$H^q(\mathcal{V}, \mathcal{Z}) \rightarrow H^q(B, \mathbb{Z}), \quad q \geq 1.$$

Restricting to only good covers in (C.11), one obtains that there is an isomorphisms for $q \geq 1$

$$H^q(B, \mathcal{Z}) \rightarrow H^q(B, \mathbb{Z}).$$

Remark C.5 (Čech cohomology). One can give the following alternative prove of Theorem C.1 and Example C.5. We note that by definition the sheaf cohomology with coefficients in $\mathcal{Z}$ and $\mathcal{R}$ is the integer and real Čech cohomology respectively. Hence it satisfies Eilenberg–Steenrod axioms [47], and is isomorphic to singular cohomology of B with integer and real coefficients respectively.

Example C.6 (Cohomology with coefficients in $\mathcal{P}$). Let $P \rightarrow B$ be a locally trivial bundle with fibre $\mathbb{Z}^n$ and transition maps taking values in $\mathbb{GL}(n, \mathbb{Z})$, and denote by $\mathcal{P}$ the sheaf of sections of this bundle. Let $\mathcal{V}$ be a good cover of B . We prove that for $q \geq 0$ the map

$$H^q(\mathcal{V}, \mathcal{P}) \rightarrow H^q(B, \mathcal{P}) \tag{C.13}$$

is an isomorphism. For $q = 0$ this is true because of Lemma C.1 in Appendix C.1. Next, suppose $g : P \rightarrow B$ is trivial, i.e. P is isomorphic to the product $B \times \mathbb{Z}^n$, P can be represented as a direct sum of n trivial bundles $B \times \mathbb{Z}$, which implies that there is an isomorphism

$$H^q(\mathcal{V}, \mathcal{Z}^n) \cong \underbrace{H^q(\mathcal{V}, \mathcal{Z}) \oplus \dots \oplus H^q(\mathcal{V}, \mathcal{Z})}_n \cong \underbrace{H^q(B, \mathbb{Z}) \oplus \dots \oplus H^q(B, \mathbb{Z})}_n,$$

i.e. $H^q(\mathcal{V}, \mathcal{Z}^n)$ is independent of the good cover $\mathcal{V}$. In other words, for $\mathcal{P} = \mathcal{Z}^n$ the map (C.13) is an isomorphism.

Finally, suppose the bundle $g : P \rightarrow B$ is non-trivial. We note that the sheaf $\mathcal{P}$ is locally isomorphic to the sheaf $\mathcal{Z}^n$, which implies that for any open contractible set $U \subset B$ there is an isomorphism

$$H^q(U, \mathcal{P}) \rightarrow H^q(U, \mathcal{Z}^n),$$

where $H^q(U, \mathcal{Z}^n)$ vanishes. Since $\mathcal{V}$ is a good cover, all finite intersection $U = V_{i_0} \cap \dots \cap V_{i_z}$ are contractible, $H^q(\mathcal{V}, \mathcal{P})$ vanishes for $q \geq 1$, and by the Leray theorem C.2 the map (C.13) is an isomorphism for any sheaf $\mathcal{P}$ of sections of $g : P \rightarrow B$.

Example C.7 (Cohomology with coefficients in $\mathcal{T}^n$). Let $\pi : R \rightarrow B$ be a vector bundle with fibre $\mathbb{R}^n$, and $P \rightarrow B$ be a locally trivial $\mathbb{Z}^n$ -subbundle of R . Then the fibrewise quotient $\tau : T = R/P \rightarrow B$ is a locally trivial bundle with fibre $\mathbb{T}^n$. Denote by $\mathcal{T}^n$ and $\mathcal{P}$ the sheaves of sections of T and P respectively. Notice that $\mathcal{P}$ is exactly the sheaf considered in Example C.6. We show that there is an isomorphism

$$H^q(\mathcal{V}, \mathcal{T}^n) \rightarrow H^{q+1}(\mathcal{V}, \mathcal{P}), \quad q \geq 1, \tag{C.14}$$

and, since $H^{q+1}(\mathcal{V}, \mathcal{P})$ is independent of the good cover $\mathcal{V}$, the cohomology $H^q(\mathcal{V}, \mathcal{T}^n)$ is independent of the good cover $\mathcal{V}$.

Denote by $\mathcal{R}^n$ the sheaf of sections of R . Then the following sequence of sheaves is exact

$$0 \longrightarrow \mathcal{P} \xrightarrow{i} \mathcal{R}^n \xrightarrow{q} \mathcal{T}^n \longrightarrow 0.$$

Since $\mathcal{R}^n$ is a fine sheaf, the induced long exact sequence fall apart, and for $q \geq 1$ we have

$$0 \longrightarrow H^q(\mathcal{V}, \mathcal{T}^n) \xrightarrow{\delta_*} H^{q+1}(\mathcal{V}, \mathcal{P}) \longrightarrow 0,$$

and the map δ_* is exactly the isomorphism C.14. The independence of $H^q(\mathcal{V}, \mathcal{T}^n)$ of the good cover follows.

ACKNOWLEDGMENTS

The authors would like to thank J.J. Duistermaat and R.H. Cushman for useful discussions.

REFERENCES

1. Duistermaat, J. J., On Global Action-Angle Coordinates, *Comm. Pure Appl. Math.*, 1980, vol. 33, no. 6, pp. 687–706.
2. Zung, Nguyen Tien, Symplectic Topology of Integrable Hamiltonian Systems. II. Topological Classification, *Compos. Math.*, 2003, vol. 138, no. 2, pp. 125–156.
3. Arnold, V. I., *Mathematical Methods of Classical Mechanics*, New York: Springer, 1989.
4. Abraham, R. and Marsden, J. E., *Foundations of Mechanics*, Reading, Mass.: Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, 1978.
5. Cushman, R. H. and Bates, L. M., *Global Aspects of Classical Integrable Systems*, Basel: Birkhäuser, 1997.
6. Arnold, V. I. and Avez, A., *Ergodic Problems of Classical Mechanics*, New York–Amsterdam: W. A. Benjamin, Inc., 1968.
7. Markus, L. and Meyer, K. R., Generic Hamiltonian Dynamical Systems are Neither Integrable Nor Ergodic, *Mem. Amer. Math. Soc.*, 1974, vol. 144, iv+52 pp.
8. Bates, L. and Śniatycki, J., On Action-Angle Variables, *Arch. Ration. Mech. Anal.*, 1992, vol. 120, pp. 337–343.
9. Spivak, M., *A Comprehensive Introduction to Differential Geometry. Vol. 1*, Waltham, Mass.: M. Spivak, Brandeis Univ., Waltham, Mass., 1970.
10. Husemoller, D., *Fibre bundles*, New York: McGraw-Hill, 1966.
11. Bott, R. and Tu, L. W., *Differential Forms in Algebraic Topology*, New York: Springer, 1982.
12. Scott, W. R., *Group Theory*, Englewood Cliffs, NJ: Prentice-Hall Inc., 1964.
13. Godbillon, C., *Éléments de topologie algébrique*, Paris: Hermann, 1971.
14. Massey, W. S., *A Basic Course in Algebraic Topology*, New York: Springer, 1991.
15. Massey, W. S., *Algebraic Topology: An Introduction*, New York: Harcourt, Brace & World, Inc., 1967.
16. Liouville, J., Note sur l'intégration des équations différentielles de la dynamique, *J. Math. Pures Appl.*, 1855, vol. 20, pp. 137–138.
17. Jost, R., Winkel- und Wirkungsvariable für allgemeine mechanische Systeme, *Helvetica Phys. Acta*, 1968, vol. 41, pp. 965–968.
18. Miranda, E. and Zung, Nguyen Tien, A Note on Equivariant Normal Forms of Poisson Structures, *Math. Res. Lett.*, 2006, vol. 13, no. 5–6, pp. 1001–1012.
19. Audin, M., *Les systèmes hamiltoniens et leur intégrabilité*, Paris: Soc. Math. de France, 2001.
20. Cushman, R. and Duistermaat, J. J., The Quantum Mechanical Spherical Pendulum, *Bull. Amer. Math. Soc. (N.S.)*, 1988, vol. 19, no. 2, pp. 475–479.
21. Child, M. S., Quantum States in a Champagne Bottle, *J. Phys. A*, 1998, vol. 31, no. 2, pp. 657–670.
22. Cushman, R. H. and Sadovskii, D. A., Monodromy in Perturbed Kepler Systems: Hydrogen Atom in Crossed Fields, *Europhys. Lett.*, 1999, vol. 47, no. 1, pp. 1–7.
23. Bates, L. M., Monodromy in the Champagne Bottle, *Z. Angew. Math. Phys.*, 1991, vol. 42, no. 6, pp. 837–847.
24. Giacobbe, A., Cushman, R. H., Sadovskii, D. A., and Zhilinskiĭ, B. I., Monodromy of the Quantum $1 : 1 : 2$ Resonant Swing Spring, *J. Math. Phys.*, 2004, vol. 45, no. 12, pp. 5076–5100.
25. Efsthathiou, K., *Metamorphoses of Hamiltonian Systems with Symmetries*, Berlin: Springer, 2005.
26. Vũ Ngọc, S., Quantum Monodromy in Integrable Systems, *Comm. Math. Phys.*, 1999, vol. 203, no. 2, pp. 465–479.
27. Matveev, V. S., Integrable Hamiltonian Systems with Two Degrees of Freedom. Topological Structure of Saturated Neighborhoods of Points of Focus-Focus and Saddle-Saddle Types, *Mat. Sb.*, 1996, vol. 187, no. 4, pp. 29–58.
28. Zung, Nguyen Tien, A Note on Focus-Focus Singularities, *Differential Geom. Appl.*, 1997, vol. 7, no. 2, pp. 123–130.
29. Nekhoroshev, N. N., Sadovskii, D. A., and Zhilinskii, B. I., Fractional Monodromy of Resonant Classical and Quantum Oscillators, *C. R. Math. Acad. Sci. Paris*, 2002, vol. 335, no. 11, pp. 985–988.
30. Sadovskii, D. A. and Zhilinskiĭ, B. I., Hamiltonian Systems with Detuned $1 : 1 : 2$ Resonance: Manifestation of Bidromy, *Ann. Physics*, 2007, vol. 322, no. 1, pp. 164–200.
31. Giacobbe, A., Fractional Monodromy: Parallel Transport of Homology Cycles, to appear in *Differential Geom. Appl.*
32. Vũ Ngọc, S., On Semi-Global Invariants for Focus-Focus Singularities, *Topology*, 2003, vol. 42, no. 2, pp. 365–380.
33. Nekhoroshev, N. N., Sadovskii, D. A., and Zhilinskiĭ, B. I., Fractional Hamiltonian Monodromy, *Ann. Henri Poincaré*, 2006, vol. 7, no. 6, pp. 1099–1211.

34. Efstathiou, K., Cushman, R. H., and Sadowski, D. A., Fractional Monodromy in the $1 : -2$ Resonance, *Adv. Math.*, 2007, vol. 209, no. 1, pp. 241–273.
35. Hirsch, M. W., *Differential Topology*, New York: Springer, 1976.
36. Ehresmann, C., Les connexions infinitésimales dans un espace fibré différentiable, in *Colloque de topologie (espaces fibrés)*, Bruxelles, 1950, Liège: Georges Thone, 1951, pp. 29–55.
37. Olver, P. J., *Applications of Lie Groups to Differential Equations*, New York: Springer, 1986.
38. Helgason, S., *Differential Geometry, Lie Groups, and Symmetric Spaces*, New York: Academic Press Inc. [Harcourt Brace Jovanovich Publishers], 1978.
39. Farb, B. and Margalit, D., *A Primer on Mapping Class Groups*, a working draft, version 2.95, <http://www.math.utah.edu/margalit/primer/>, 2007.
40. Marsden, J. E. and Ratiu, T. S., *Introduction to Mechanics and Symmetry*, New York: Springer, 1999.
41. McDuff, D. and Salamon, D., *Introduction to Symplectic Topology*, New York: The Clarendon Press, Oxford Univ. Press, 1998.
42. Boothby, W. M., *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Orlando, FL: Academic Press Inc., 1986.
43. Tu, L. W., *An Introduction to Manifolds*, New York: Springer, 2008.
44. Hirzebruch, F., *Topological Methods in Algebraic Geometry*, Berlin–New York: Springer, 1966.
45. Tennison, B. R., *Sheaf Theory*, Cambridge, England: Cambridge Univ. Press, 1975.
46. Godement, R., *Topologie algébrique et théorie des faisceaux*, Paris: Hermann, 1958.
47. Eilenberg, S. and Steenrod, N., *Foundations of Algebraic Topology*, Princeton, NJ: Princeton Univ. Press, 1952.