Stability Analysis of Gradient-Based Distributed Formation Control With Heterogeneous Sensing Mechanism: The Three Robot Case

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Abstract—This article focuses on the stability analysis of a formation shape displayed by a team of mobile robots that uses a heterogeneous sensing mechanism. For the setups consisting of three robots, we show that the use of heterogeneous gradient-based control laws can give rise to undesired invariant sets where a distorted formation shape is possibly moving at a constant velocity. We guarantee local asymptotic stability for the correct and desired formation shape. For the setup with one distance and two bearing robots, we identify the conditions such that an incorrect moving formation is locally attractive.

Index Terms—Formation control, gradient-based control design, heterogeneous sensing.

I. INTRODUCTION

Over the years, a rich body of work has been developed on realizing a formation shape by a team of mobile robots. The use and active maintenance of a common type of constraint (distance, bearing, angle, and relative position) between two neighboring robots have been the basis for achieving robust formation shape [1]–[5]. However, neighboring robots controlling constraints using unreliable sensors lead to unstable formations [6], [7]. If a sensor failure occurs, then one solution might be to withdraw the unreliable information and consider a heterogeneous sensing setting for a pair of neighboring robots. For instance, in the case of a partial failure of a LIDAR sensor, which can normally provide relative position information, we may still measure bearing information with nonaccurate distance information. In this case, it is possible to define heterogeneous constraints on the same edge that still define the same shape (e.g., one robot controls relative position while the other one controls bearing). However, it remains an open problem whether the application of the local gradient-based control law based on the (heterogeneous) information available to each robot can still maintain the formation. Note that communication between robots to recover full information might not be possible by design. Indeed, the aforementioned works on formation with homogeneous information require only local sensing without information exchange between robots. The answer to this problem can open the way to the design of more robust strategies since distance-based/bearing-based controllers are more robust to nonaccurate bearing/distance measurements [8], [9]. For example, instead of just controlling relative positions, the bearing measurement becomes less sensitive when the robots move in a large formation shape, or the robots carrying multiple distance and bearing sensors can control the most relevant constraint depending on the accuracy or reliability of the equipped sensor for a given situation (e.g., far versus near, wide-angle versus small-angle, etc.). Intuitively, the gradient-based control law will steer each robot to the direction that minimizes the local potential function and reaches the desired constraints. However, as different types of potential function may be defined for the same edge due to the heterogeneous sensing mechanisms between the robots, the direction that is taken by each robot may not coincide anymore with the minimization of the combined potential functions.

In this article, we consider the formation stabilization problem in which the desired formation shape is specified by a mixed set of distance and bearing constraints. In [9], the authors divide the edges of formation graph into two sets where one set is associated to the distance constraints while the other one corresponds to the bearing constraints. Consequently, there are nodes that are involved in both types of edges, in which case the robots will be equipped with both types of sensor systems. In contrast to [9], we consider instead two disjoint sets of nodes where one set uses distance information while the other one employs bearing information. Analogous to the previous case, there are edges that are defined by both distance and bearing constraints. The presence of multiple constraints in these edges may lead to some robustness issues when each pair of nodes employs different control laws associated to these different constraints. In this article, we study the robustness of formation keeping in a heterogeneous network where minimal number of sensor systems for formation keeping are deployed per node. Particularly, each robot within the team has the task of maintaining a subset of either the distance or bearing constraints. For this particular work, we focus on teams consisting of three robots. Using standard gradient-based control laws specific to the constraints each robot has to maintain, we analyze the stability property, particularly, the local asymptotic stability of the desired and incorrect formation shapes. It is of interest to study the applicability of these control laws without modifying their local potential functions to incorporate the different constraints on the edges since it allows us to design distributed control laws that are completely dependent on the available local information to the robot and are independent of the eventual deployment of the robot in the formation.

The rest of this article is organized as follows. In Section II, preliminary material and problem formulation are presented. In Sections III and IV, we show that the deployment of heterogeneous gradient-based control laws can result in incorrect formation shapes, possibly moving at a constant velocity. Numerical results are given in Sections V. Finally, Section VI concludes this article.

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II. PRELIMINARIES

A. Graph Theory

A directed graph (in short, digraph) \( G \) is a pair \((V, E)\), where \( V = \{1, 2, \ldots, n\} \) is the vertex set and \( E \subseteq V \times V \) is the edge set. For \( i, j \in V \), the ordered pair \((i, j)\) represents an edge pointing from \( i \) to \( j \). We assume that \( G \) does not have self-loops, i.e., \((i, i) \notin E\) for all \( i \in V \) and \( \text{card}(E) = m \). The set of neighbors of vertex \( i \) is denoted by \( N_i = \{ j \in V \mid (i, j) \in E \} \). The digraph \( G \) is bipartite if the vertex set \( V \) can be partitioned into two subsets \( V_1 \) and \( V_2 \) with \( V_1 \cap V_2 = \emptyset \) and the edge set is \( E \subseteq (V_1 \times V_2) \cup (V_2 \times V_1) \). We assume \( \text{card}(V_1) = n_1 \) and hence \( \text{card}(V_2) = n_2 = n - n_1 \). For a complete bipartite digraph, \( \mathcal{E} = (V_1 \times V_2) \cup (V_2 \times V_1) \) and \( \text{card}(E) = 2n_1n_2 \). Fig. 1 depicts complete bipartite digraphs for \( n = 2 \) and \( 3 \) vertices and a bipartite digraph for \( n = 4 \) vertices.

B. Formations and Gradient-Based Control Laws

We consider a team consisting of \( n \) robots in which \( R_i \) is the label assigned to robot \( i \). The robots are moving in the plane according to the single integrator dynamics, i.e.,

\[
p_i = u_i, \quad i \in \{1, \ldots, n\}
\]

where \( p_i \in \mathbb{R}^2 \) (a point in the plane) and \( u_i \in \mathbb{R}^2 \) represent the position of and the control input for \( R_i \), respectively. For convenience, all spatial variables are given relative to a global coordinate frame \( \Sigma^g \).

The group dynamics is obtained as \( \dot{p} = u \) with the stacked vectors \( p = [p_1^T \ldots p_n^T]^T \in \mathbb{R}^{2n} \) representing the team configuration and \( u = [u_1^T \ldots u_n^T]^T \in \mathbb{R}^{2n} \) being the collective input. The interactions among the robots are described by a fixed graph \( \mathcal{G}(V, E) \) with \( V \) representing the team of robots and \( E \) containing the neighboring relationships. We embed \( \mathcal{G} \) into the plane by assigning to each robot \( i \in V \), a point \( p_i \in \mathbb{R}^2 \). The pair \( \mathcal{F}_p = (\mathcal{G}, p) \) denotes a framework (or equivalently a formation). We assume \( p_1 \neq p_i \) if \( i \neq 1 \), i.e., two robots cannot be at the same position. We introduce the following notation prior to providing details on the distance-based and bearing-only formation control approaches. For points \( p_i \) and \( p_j \), we define, relative to \( \Sigma^g \), the relative position as \( z_{ij} = p_j - p_i \in \mathbb{R}^2 \), the distance as \( d_{ij} = \|z_{ij}\| \in \mathbb{R}_{>0} \), and the relative bearing as \( g_{ij} = \frac{z_{ij}}{\|z_{ij}\|} \in \mathbb{R}^2 \). It follows \( z_{ji} = -z_{ij} \), \( d_{ji} = d_{ij} \) and \( g_{ji} = -g_{ij} \).

1) Distance-Based Formation Control: In distance-based formation control, a desired formation is characterized by a set of inter-robot distance constraints. Assume that the desired distance between a robot pair \((i, j)\) is \( d_{ij}^* \) and let \( d_{ij}(t) \) be the current distance at time \( t \). We define the distance error signal as \( e_{ij}(t) = d_{ij}(t) - d_{ij}^* \). A distance-based potential function for \((i, j)\) takes the form \( V_{ij}(e_{ij}) = \frac{1}{2} e_{ij}^2 \). It has a minimum at the desired distance \( d_{ij}^* \), i.e., \( V_{ij}(e_{ij}) \geq 0 \) and \( V_{ij}(e_{ij}) = 0 \iff d_{ij} = d_{ij}^* \). In this case, the corresponding gradient-based control law for maintaining distance \( d_{ij}^* \) for the robot pair \((i, j)\) is

\[
u_{id} = e_{ij} z_{ij},
\]

for \( z_{ij} \) is the measurement that \( R_i \) obtains from its neighbor \( j \in \mathcal{N}_i \). Thus, the distance-based formation control law for robot \( R_i \) in (1) is given by

\[
u_{id} = \sum_{j \in \mathcal{N}_i} e_{ij} z_{ij}.
\]

It is well studied in literature (e.g., [10]) that the abovementioned control law guarantees the local exponential stability of the desired formation shape when the desired shape is infinitesimally rigid.

2) Bearing-Only Formation Control: In bearing-only formation control, the desired formation is characterized by a set of inter-robot bearing constraints. Consider the \( i \)-th robot (with label \( R_i \)) in this setup. Robot \( R_i \) is able to obtain the bearing measurement \( g_{ij}(t) \) from its neighbors \( j \in \mathcal{N}_i \) and its goal is to achieve desired bearings \( g_{ij}^* \) with all neighbors \( j \in \mathcal{N}_i \). In this case, the bearing error signal for a robot pair \((i, j)\) can be defined by \( e_{ij}(t) = g_{ij}(t) - g_{ij}^* \). The corresponding potential function is \( V_{ij}(e_{ij}) = \frac{1}{2} d_{ij}(e_{ij})^2 \). Note that \( V_{ij}(e_{ij}) \geq 0 \) and it is only zero when \( d_{ij} = 0 \) or \( e_{ij} = 0 \iff g_{ij} = g_{ij}^* \). (In the forthcoming analysis, we will show that \( d_{ij} = 0 \), where robots \( R_i \) and \( R_j \) are at the same position, is not a viable option.) It can be verified that \( u_{ib} = e_{ij} \) is the gradient-based control law derived from \( V_{ij}(e_{ij}) \) for the robot pair \((i, j)\). The bearing-only formation control law for \( R_i \) in (1) is then given by

\[
u_{ib} = \sum_{j \in \mathcal{N}_i} e_{ij}.
\]

In [4], it has been shown that the above control law ensures the global asymptotic stability of the desired formation shape provided that the formation shape is infinitesimally bearing rigid.

C. Cubic Equations

Lemma 1: Consider the reduced cubic equation \( y^3 + cy + d = 0 \) with coefficients \( c < 0 \) and \( d > 0 \). The discriminant is \( \Delta := -4c^3 - 27d^2 \geq 0 \). Then, two positive real roots exist with these values

\[
y_{p1} = 2\sqrt[3]{r_{v}} \left(\frac{1}{3} \phi_{v}\right) \in \{1, \sqrt[3]{3}\} \sqrt[3]{r_{v}}
\]

\[
y_{p2} = 2\sqrt[3]{r_{v}} \left(\frac{1}{3} \phi_{v} - 120^\circ\right) \in (0, 1] \sqrt[3]{r_{v}}
\]

where \( r_v = \sqrt{-\frac{c}{3}} \) and \( \phi_v = \tan^{-1}\left(\frac{2}{3}\sqrt{-\frac{27d}{r_v}}\right) \in (90^\circ, 180^\circ) \).

When \( \Delta = 0 \), the two positive real roots are equal and have value

\[
y_{p1} = y_{p2} = \sqrt[3]{r_{v}} = \sqrt[3]{\frac{4}{3}}.
\]

The proof of Lemma 1 can be found in [11].

D. Problem Formulation

As discussed in Section I, we study the setup in which the robots possess heterogeneous sensing, and each robot, depending on its own local information, maintains the prescribed distance or bearing with its neighbors using the aforementioned distance-based or bearing-only formation control law. Thus, in the current setup, each robot fulfills either a distance task or a bearing task. As before, consider a pair of robots with labels \( R_i \) and \( R_j \). In case \( R_i \) is assigned a distance task, its goal is to maintain a desired distance \( d_{ij}^* \) with \( R_j \). To attain this goal, it makes use of the distance-based control law \( u_{id} = e_{ij} z_{ij} \), with \( z_{ij} \) being the obtained relative position measurement relative to a local coordinate frame \( \Sigma^g \). Note that \( \Sigma^g \) is not necessarily aligned with \( \Sigma^r \) or \( \Sigma^f \). On the other hand, when \( R_i \) is assigned a bearing task, its goal is to maintain a desired bearing \( g_{ij}^* \) with \( R_j \). It reaches this goal by employing the bearing-only control law \( u_{ib} = e_{ij} \) based on
the obtained relative bearing measurement \( g_{ij} \) relative to \( \Sigma_j \), which is aligned with \( \Sigma_i \). For the interconnection topology, we assume that each robot has only neighbors of the opposing category, i.e., a distance robot can only have edges with bearing robot(s) and vice versa. As a result, the team of \( n \) robots can be partitioned into two sets, namely, the set of distance robots \( D \) and the set of bearing robots \( B \), with \( D \neq \emptyset \) and \( B \neq \emptyset \). The edge set is given by \( \mathcal{E} \subseteq (D \times B) \cup (B \times D) \); the underlying graph structure is that of a bipartite digraph.

In the current article, we focus on the case in which the team of \( n = 3 \) robots has a complete bipartite digraph topology, i.e., the edge set is \( \mathcal{E} = (D \times B) \cup (B \times D) \). We distinguish two feasible robot setups, namely, the one distance and two bearing \((1D2B)\) or the one bearing and two distance \((1B2D)\) setup; see Fig. 2 for an illustration of these setups. For these setups, we are interested in studying the stability of the formation system employing the distance-based formation control law in (2) for the distance robot(s) and the bearing-only formation control law in (3) for the bearing robot(s). In this case, we do not modify the standard gradient-based control law for the different tasks. Consequently, we analyze whether

1. the equilibrium set contains undesired shape and/or group motion;
2. the desired shape is (exponentially) stable;
3. the undesired shape and/or group motion (if any) is attractive.

The first and last questions are motivated by the robustness issues of the distance- and displacement-based controllers as studied in [6], [7], [12], and [13] where a disagreement between neighboring robots about desired values or measurements can lead to an undesired group motion and deformation of the formation shape. Since we are considering heterogeneous sensing mechanisms with corresponding heterogeneous potential functions, it is of interest whether such undesired behavior can coexist. Such knowledge on the effect of heterogeneity in the control law can potentially be useful to design simultaneous formation and motion controller, as pursued recently in [14].

### III. (1D2B) ROBOT SETUP

In this section, we consider the case of three robots with the partition \( D = \{1\} \) and \( B = \{2, 3\} \); the \((1D2B)\) setup is shown in Fig. 2. Utilizing gradient-based control laws for each distance or bearing task, we obtain the following closed-loop dynamics:

\[
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\dot{p}_3
\end{bmatrix} =
\begin{bmatrix}
K_d e_{12d} z_{12} + K_d e_{13d} z_{13} \\
K_e e_{21b} \\
K_e e_{31b}
\end{bmatrix}
\]

\[
\begin{bmatrix}
z_{12} \\
z_{13}
\end{bmatrix} =
\begin{bmatrix}
K_b e_{12b} + K_d e_{12d} z_{12} + K_d e_{13d} z_{13} \\
K_b e_{13b} + K_d e_{12d} z_{12} + K_d e_{13d} z_{13}
\end{bmatrix}
\]

(6)

For a triangle, \( z_{12} + z_{23} - z_{13} = 0 \) holds. Hence, the dynamics related to link \( z_{23} \) evaluates to \( \dot{z}_{23} = -K_b (e_{12b} - e_{12b}) \).

In the following sections, we rigorously analyze the closed-loop formation system (5) and link dynamics (6).

### A. Equilibrium Configurations

**Proposition 1** \((1D2B)\) Equilibrium Configurations: The equilibrium configurations corresponding to the closed-loop formation system (5) belong to the set

\[
\mathcal{S}_p = \{ p \in \mathbb{R}^6 | e = 0 \}
\]

(7)

where \( e = [e_{12d}, e_{13d}, e_{12b}, e_{13b}]^T \in \mathbb{R}^6 \).

**Proof:** Setting the left-hand side (LHS) of each equation in (5) to the zero vector, we immediately obtain that the bearing constraints for robots \( R_2 \) and \( R_3 \) are satisfied since \( e_{12b} = e_{13b} = 0 \). This implies that \( d_{21} = d_{12} \neq 0 \) and \( d_{31} = d_{13} \neq 0 \). It remains to solve for \( \dot{p}_1 = 0 \). With the gathered insights, we obtain \( e_{12d} = e_{13d} = 0 \) when \( e_{12d} = e_{13d} = 0 \). Since \( g_{12}^* \neq 0 \) (the robots are colinear), the expression is satisfied when \( e_{12d} = 0 = e_{13d} = 0 \). Because \( d_{12} \neq 0 \) and also \( d_{13} \neq 0 \), we require \( e_{12d} = 0 \) and \( e_{13d} = 0 \) to hold.

### B. Moving Configurations

During the analysis of the \((1D1B)\) setup (see [11] for details), we observed that robots may move with a common velocity \( w \) while the predefined constraints are not met. For the \((1D2B)\) setup, we explore whether conditions exist such that the formation may move with a common velocity \( w \).

**Proposition 2** \((1D2B)\) Moving Configurations: The closed-loop formation system (5) moves with a constant velocity \( w = K_b \sum b_{w} \) with \( b_{w} = g_{12} + g_{13} \) when the error vector \( e \) satisfies

\[
e = -\left[ \frac{1}{d_{12}} R_{d_{d_{2}}} \right] \left[ \frac{1}{d_{13}} R_{d_{d_{3}}} \right] b_{w}^{T} \left[ b_{w}^{T} \right]^T.
\]

(8)

**Proof:** First, we solve for \( \dot{z} = 0 \). Since \( \dot{z}_{12} = \dot{z}_{23} = \dot{z}_{13} \), it follows \( \dot{z}_{23} = 0 \). This expression evaluates to \( g_{12} - g_{13} = g_{12} - g_{12} \). Define \( b_{d_{2d}} = g_{12} - g_{13} \) and let \( \angle g_{12} = \alpha \) be the angle enclosed by vector \( g_{12} \) and the positive \( x \)-axis of \( \Sigma_2 \). Similarly, let \( \angle g_{13} = \beta \). We can rewrite \( b_{d_{2d}} \) as

\[
b_{d_{2d}} = 2 \cos \left( \frac{1}{2} (\alpha - \beta) \right) \left[ \cos \left( \frac{1}{2} (\alpha + \beta) \right) \right] \left[ \sin \left( \frac{1}{2} (\alpha + \beta) \right) \right] \]

(9)

where \( \beta = \alpha + \pi \) (mod \( 2\pi \)). The expression \( \dot{z}_{23} = 0 \) can be transformed to the following set of angle constraints, namely:

\[
\begin{align*}
\alpha + \beta &= \alpha^* + \beta^* \\
\alpha - \beta &= \alpha^* - \beta^* & \iff & \alpha = \alpha^* \\
\beta &= \beta^* + \pi \\
\end{align*}
\]

(10)

and

\[
\begin{align*}
\alpha + \beta &= \alpha^* + \beta^* \\
-(\alpha - \beta) &= \alpha^* - \beta^* & \iff & \alpha = \beta^* + \pi \]

(11)

From (10), we obtain \((g_{12}, g_{13}) = (g_{12}^*, g_{13}^*)\) corresponding to the equilibrium configurations in \( \mathcal{S}_p \) while the solution in (11) corresponds to \((g_{12}, g_{13}) = (-g_{12}, -g_{12})\). Subsequently, we obtain \( e_{12b} = e_{13b} = \)}
\[-(g_1^* + g_1^*) = b_{\text{sum}}; \text{it is sufficient to consider one of the equations in } (6). \text{This leads to } (−K_ef_{13}d_{13})g_1^* + (−K_fd_{12}d_{12})g_1^* = K_bg_1^* + K_pg_1^*. \text{For it to hold, we require } −K_fd_{13}d_{13} = K_b \iff e_{13} = −\frac{1}{d_{13}^2}R_{13} \text{ and } −K_fd_{12}d_{12} = K_p \iff e_{12} = −\frac{1}{d_{12}^2}R_{12} \text{ with the gain ratio } R_{13} = \frac{K_p}{K_b}. \text{Collecting the error constraints, we obtain } (8). \text{By an immediate substitution, we obtain for the dynamics of the bearing robot } \tilde{R}_2, \tilde{p}_2 = K_b \lambda^*_{12}. \]

**Remark 1.** The signed area for a triangle can be obtained using the expression \(S_N = \frac{1}{2} |x_1y_2 - x_2y_1| \geq 0 \), \(i \neq j \text{satisfies } x, p, y \). \(R_{13}^* \) has \(\lambda^*_{12} \) and \(\lambda^*_{13} \). \(S_{12}^* \) is the signed area of the moving formation shape. \(S_{13}^* = d_{12}^*d_{13}^*g_1^* \geq 0 \), \(i \neq j \) and \(d_{12}^*d_{13}^*g_1^* \geq 0 \). \(g_1^* \approx 0 \) and \(e_1^* \approx 0 \). When \(\alpha > \beta \), \(g_1^* \neq 0 \). If \(\alpha = 0 \), then \(g_1^* \neq 0 \) and \(b_{\text{sum}} = 2g_1^* \). Finally, \(\alpha = \beta = \pi \) implies \(g_1^* = 0 \). \(S_{12}^* \approx 0 \) and \(S_{13}^* \approx 0 \), the last two mentioned cases do not occur; therefore, the magnitude of \(w \) is \(0 < |w| < 2K_b \).

**C. Local Stability Analysis of the Equilibrium and Moving Formations**

\(\lambda^*_{12} \geq \tilde{d}^* \) and \(\lambda^*_{13} \geq \lambda^*_{12} \). In this case, both the equilibrium configurations in (7) and moving configurations in (12) satisfy \(z = \Theta_0 \) and are feasible. We are interested in determining the local stability around these formations. To this end, we consider the linearization of the \(z\)-dynamics (6); this results in the Jacobian matrix \(A \in R^{4 \times 4} \) as

\[
A = \begin{bmatrix} K_bA_{12} + K_dA_{12} & K_d^2A_{13} & K_d^2A_{13} - K_dA_{12} & K_bA_{12} - K_dA_{12} \\ K_dA_{12} & K_bA_{12} - K_dA_{12} & K_d^2A_{13} & K_d^2A_{13} - K_dA_{12} \end{bmatrix}
\]

where \(A_{12} = e_{12}I_2 + 2e_{12}z_2^*e_{12}^* \) and \(A_{13b} = \frac{1}{d_{12}}P_{g_1}, \text{with } P_{g_1} = I_2 - g_1g_1^* \) and \(ij \in \{12, 13\} \).

We first consider the stability analysis around the equilibrium configurations.

**Theorem 1:** Consider a team of three robots arranged in the (1D2B) setup with closed-loop dynamics given by (5). Assume that the desired distances satisfy \(d_{12}^* \geq \tilde{d} \) and \(d_{13}^* \geq \tilde{d} \). We can develop the following results in the Jacobian matrix (15) at the equilibrium configurations in (7).

\[
A_E = \begin{bmatrix} x^* & 0 \\ 0 & p^* \end{bmatrix} \otimes I_2 - \begin{bmatrix} m^*g_1^*g_1^* & q^*g_1^*g_1^* \\ q^*g_1^*g_1^* & n^*g_1^*g_1^* \end{bmatrix}
\]

where we define the variables

\[
x = K_bd_{12}^*, \quad y = 2K_bd_{12}^* - m \quad \text{and } \quad p = K_bd_{13}^*, \quad q = K_bd_{13}^* - n = q - p
\]

and the matrices

\[
g_1^*g_1^* = \begin{bmatrix} a^2b & ab^2 \\ ab & b^2 \end{bmatrix}, \quad g_1^*g_1^* = \begin{bmatrix} c^2 & cd \\ cd & d^2 \end{bmatrix}
\]

The starred version for \(x, y, q, m, \) and \(n \) is used here since we have \(d_{12}^* = d_{12}^* \) and \(d_{13}^* = d_{13}^* \). The characteristic polynomial \(\chi_\lambda(\lambda) \) corresponding to matrix \(A_E \) is obtained as

\[
\chi_\lambda(\lambda) = (\lambda + x^*) (\lambda + p^*) \ldots
\]

\[
(\lambda^2 + (y^* + q^*) \lambda + y^*q^* \sin^2 \theta^*)
\]

where \(\sin \theta = g_1^*g_1^*0^*_{13} \). The roots of (19) are

\[
\lambda_1 = -x^*, \quad \lambda_2 = -p^*
\]

\[
\lambda_3, 4 = -\frac{1}{2} (y^* + q^*) \pm \frac{1}{2} \sqrt{(y^* + q^*)^2 - 4y^*q^* \sin^2 \theta^*}
\]

\[
\text{Authorised use limited to: University of Groningen. Downloaded on December 15, 2022 at 09:27:18 UTC from IEEE Xplore. Restrictions apply.}
\]
It can be verified that $0 < 4y^* q^* \sin^2 \theta^* \leq (y^* + q^*)^2$; all roots are real. Moreover, $- (y^* + q^*)^2 + (y^* + q^*)^2 - 4y^* q^* \sin^2 \theta^* < 0$ and we conclude all $\lambda$s are negative; matrix $A_M$ is Hurwitz. This implies that the link trajectories asymptotically converge to the desired relative positions $z^*$ as $t \to \infty$. It also means that the robots accomplish their individual tasks since $z_{ij}^* = d_{ij}^* g_{ij}$, so $p(t) \to S_p$ when $p(0)$ is close to the desired formation shape.

We continue with determining the stability of the moving formations in the set $T_p$. Based on the characterization in (8), we obtain the Jacobian matrix

$$
A_M = \begin{bmatrix} 0 & 1 \\ -x & 0 \end{bmatrix} \otimes I_2 - \begin{bmatrix} m g_{12}^T g_{12}^T \\ n g_{12}^T g_{12}^T \end{bmatrix}
$$

(21)

where the variables are defined as in (17) and (18). The corresponding characteristic polynomial $\chi_M(\lambda)$ is the quartic polynomial

$$
\chi_M(\lambda) = \lambda^4 + c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda + c_4
$$

with the coefficients

$$
c_1 = m + n, \quad c_2 = q y \sin^2 \theta^* - px, \quad c_3 = pxmn \sin^2 \theta^* \\
c_4 = xmn (q y \sin^2 \theta^* - p) + mn (y \sin^2 \theta^* - x).
$$

(22)

Recall that depending on the value of $d_{ij}^*$ and $d_{ij}^*$, we can obtain more than one feasible combination $(d_{12a}, d_{13a})$ for the moving configurations. Under certain conditions, we have the following result on the eigenvalues of the matrix $A_M$.

**Lemma 2:** Assume that the desired distances satisfy $d_{ij}^* > \hat{d}$ and $d_{ij}^* > \hat{d}$ and the desired bearing vectors are not perpendicular, i.e., $g_{ij}^T \not= g_{ij}^T$. Consider the combination $(d_{12a}, d_{13a}) = (y_{pi}(d_{12}^*), y_{pi}(d_{13}^*))$ with $y_{pi}$ given in Lemma 1. Then, all eigenvalues of the matrix $A_M$ have a real negative part if the following inequality holds:

$$
cos^2 \theta^* < \frac{mn \left( (m - ny)^2 + m(n + m)(x + p) \right)}{(m^2 q + n^2 y) (m y q + n y p)}.
$$

(23)

**Proof:** Assuming that the bearing vectors are not perpendicular, we obtain $0 < \sin^2 \theta^* < 1$. Also, since $d_{13a} = y_{pi}(d_{13})$ and $d_{13a} = y_{pi}(d_{13}^*)$, and $d_{13}^* > \hat{d}$ and $d_{13}^* > \hat{d}$, we verify that $m > 0$ and $n > 0$. We employ the Routh–Hurwitz stability criterion. The first column of the Routh–Hurwitz table, which is the column of interest, contains the following values:

$$
\begin{bmatrix} 1 & c_1 \\ c_1 c_2 - c_3 & (c_1 c_2 - c_3) c_3 - c_1^2 c_4 \\ c_1 c_2 - c_3 & c_1 c_2 \end{bmatrix}
$$

(24)

For all roots $\lambda$ to have negative real parts, all values in (25) need to be positive. With $m > 0$ and $n > 0$, the coefficients $c_1$ and $c_4$ are positive. It remains to show that the third and fourth entries in (25) are positive. In fact, it is sufficient to show that the numerators are both positive. They evaluate to

$$
c_1 c_2 - c_3 = \sin^2 \theta^* (m^2 q + n^2 y) > 0 \\
(c_1 c_2 - c_3) c_3 - c_1^2 c_4 = \sin^2 \theta^* \left( (m - ny)^2 + m(n + m)(x + p) \right) mn \\
- (m^2 q + n^2 y) (m x q + n y p) \cos^2 \theta^*.
$$

(25)

Provided (24) holds, it follows that the entries in (25) are all positive, we conclude all eigenvalues of the matrix $A_M$ have negative real parts.

**Remark 2:** The implication of Lemma 2 is that under certain conditions on the distance and bearing constraints, a subset of the moving set $T_p$ is locally asymptotically stable. Hence, initializing the robots close to the conditions for the moving formation is not desirable. An illustration of this behavior is shown in Fig. 4(b).

**Lemma 2** also holds when the desired bearing vectors are perpendicular, i.e., $g_{ij}^T = g_{ij}^T \iff \sin^2 \theta^* = 1$. In this case, the coefficients in (23) and also all entries in (25) are positive; therefore, the matrix $A_M$ will only have eigenvalues with negative real parts.

A full characterization of the remaining cases can be found in [11]. In almost all cases, the matrix $A_M$ is not Hurwitz.

### IV. (1B2D) ROBOT SETUP

In this section, the formation setup with one bearing and two distance robots (1B2D) is considered. Without loss of generality, we assume that robot $R_1$ is the bearing robot while robots $R_2$ and $R_3$ are distance robots. The right graph in Fig. 2 depicts the interconnection structure from which the closed-loop dynamics is obtained as

$$
\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \end{bmatrix} = \begin{bmatrix} K_d \delta_{12} + K_b \delta_{13b} \\ K_d \delta_{21d} \delta_{21} \\ K_d \delta_{31d} \delta_{31} \end{bmatrix}.
$$

(26)

The corresponding link dynamics evaluates to

$$
\begin{bmatrix} \dot{z}_{12} \\ \dot{z}_{13} \end{bmatrix} = \begin{bmatrix} K_d \delta_{13d} \delta_{12} + K_b \delta_{12d} + K_b \delta_{13b} \\ K_d \delta_{13d} \delta_{13} + K_b \delta_{13d} \delta_{13} + K_b \delta_{13d} \delta_{13} \end{bmatrix}.
$$

(27)

Furthermore, the dynamics of the link $z_{23}$ is found to be $\dot{z}_{23} = -K_d (\delta_{13d} \delta_{12} + \delta_{13d} \delta_{13})$. In the following, we follow similar steps to those described in Section III for analyzing (27) and (28), focusing on equilibrium configurations, possible moving formations, and their (local) stability analysis.

#### A. Equilibrium Configurations

**Proposition 3** ((1B2D) Equilibrium Configurations): The equilibrium configurations corresponding to the closed-loop formation system

Fig. 3. Robot ordering for the moving configurations in the (1B2D) setup; the black arrow is the bearing vector $g_{ij}^T$. From left to right, we have ordering I to IV. Despite the different colors, both $R_2$ and $R_3$ are distance robots.

Fig. 4. Robot trajectories for the (1B2D) setup; (•, •, •) = (R1, R2, R3), o represents the initial and × is the final position. On the left panel, we have an initial configuration (dashed lines) where robots converge to the correct formation shape (solid lines) while the right panel illustrates an instance of convergence to the moving configuration with velocity $w = K_d \delta_{12d} \delta_{12}$. 

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(27) belong to $S_p^C \cup S_p^C$, where

$$S_p^C = \{ p \in \mathbb{R}^6 \mid e = \emptyset_p \} \quad \text{and} \quad S_p^C = \{ p \in \mathbb{R}^6 \mid e = [0 \ 0 \ -b_{\text{diff}}^T \ b_{\text{diff}}^T]^T \} \quad (29)$$

with $e = [e_{124}, e_{13}, e_{12b}, e_{13b}] \in \mathbb{R}^6$ and $b_{\text{diff}} = g_{11}^T - g_{13}^T$.

Proof: Setting the LHS of each equation of (27) to the zero vector, we obtain for robot $R_2$ that $-K_{e_{124}}e_{12} = 0 \iff e_{124} = 0 \land z_{12} = 0$ and similarly, we have $-K_{e_{13}}e_{13} = 0 \iff e_{13} = 0 \land z_{13} = 0$ for $R_3$. The expression for $R_1$ evaluates to $g_{12} + g_{13} = g_{12}^T + g_{13}$. Defining $L_2 = \alpha, L_3 = \beta$ as before, and recalling the RHS of (14), we can write the following set of angle constraints, namely:

$$\begin{align*}
\alpha + \beta &= \alpha^* + \beta^* \quad \iff \quad \alpha = \alpha^* \\
\alpha - \beta &= \alpha^* - \beta^* \quad \iff \quad \beta = \beta^*. 
\end{align*} \quad (30)$$

and

$$\begin{align*}
\alpha + \beta &= \alpha^* + \beta^* \quad \iff \quad \alpha = \beta^* \\
-(\alpha - \beta) &= \alpha^* - \beta^* \quad \iff \quad \beta = \alpha^*. 
\end{align*} \quad (31)$$

Equation (30) translates to $(g_{12}, g_{13}) = (g_{12}^*, g_{13}^*)$, implying robot $R_1$ satisfies its bearing tasks while (31) translates to the flipped formation shape with bearings satisfying $(g_{12}, g_{13}) = (g_{13}, g_{12}^*)$. It follows the bearing error signals are $e_{12} = -e_{13} = -b_{\text{diff}}$. With both $g_{12}$ and $g_{13}$ defined, we obtain $d_{12} \neq 0$ and $d_{13} \neq 0$; hence, $z_{12} = 0 \land z_{13} = 0$ are both infeasible. Robots $R_2$ and $R_3$ will stop moving when $d_{12} = 0 \land d_{13} = 0$ holds, i.e., when they accomplished their individual task irrespective of $R_1$.

It can be verified that the signed area of the flipped formation satisfies $S_{12} = -S_{12}$.

B. Moving Configurations

Proposition 4 ([1D2B]) Moving Configurations: The moving configurations occur when the robots are colinear, i.e., $g_{12} = \pm g_{13}$ and oriented in the direction of $b_{\text{sum}} = g_{12}^T + g_{13}^T$.

Proof: Expanding (28) yields

$$K_{e_{124}}g_{12} + K_{e_{13}}g_{13} = K_{b_{\text{sum}}}b_{\text{sum}}$$

$$K_{b_{\text{sum}}}g_{12} + (K_{e_{13}}d_{12} + K_{e_{13}}d_{13}) + K_{g_{13}}g_{13} = K_{b_{\text{sum}}}b_{\text{sum}}. \quad (32)$$

Solving for $z_{12} = 0$, we obtain $e_{12}d_{12}g_{12} = e_{13}d_{13}g_{13}$. Two vectors are equal when they have the same magnitude and direction or opposite signs in both the magnitude and direction. Hence, we distinguish the cases $g_{12} = g_{13} \land e_{12}d_{12} = e_{13}d_{13}d_{13}$ and $g_{12} = -g_{13} \land e_{12}d_{12} = -e_{13}d_{13}d_{13}$. Since $g_{12} = \pm g_{13}$, we conclude the robots are colinear. Substituting this in (32), we obtain expressions of the form $h_{12} = K_{b_{\text{sum}}}g_{12}g_{13}b_{\text{sum}}$ where $h = K_{e_{124}}d_{12} + 2K_{b}$ when $g_{12} = g_{13}$ and $h = K_{e_{124}}d_{12}$ when $g_{12} = -g_{13}$. From this, we infer $g_{12} = \pm g_{13}$, implying the orientation of the formation is in the direction of $b_{\text{sum}}$.

In light of Proposition 4, we can obtain four different ordering of the robots, as depicted in Fig. 3. To provide a full characterization of the moving configurations, it remains to obtain the inter-robot distances for the different ordering. We first derive expressions for the distance error corresponding to the different ordering from the general expression $h_{12} = K_{b_{\text{sum}}}g_{12}g_{13}b_{\text{sum}}$. Define $e_{12} = \frac{1}{2}d_{12}R_2$ and $e_{13} = \frac{1}{2}d_{13}R_3$. For the different robot orderings in Fig. 3, we have the following for $s$ and $t$:

1. $g_{12} = g_{13} = g_{\text{sum}}, s = t = -2 + d_{12}^s$.
2. $g_{12} = g_{13} = -g_{\text{sum}}, s = t = -2 - d_{12}^s$.
3. $g_{12} = -g_{13} = g_{\text{sum}}, s = -t = d_{12}^s$.
4. $g_{12} = -g_{13} = -g_{\text{sum}}, s = t = -d_{12}^s$.

When expanded, we obtain the cubic expression in Lemma 1 with coefficients $c = -d_{12}^s$ and $d = -sR_2$ when solving for feasible distance $d_{12}$ while $c = -d_{13}^s$ and $d = -tR_3$ when we are considering distance $d_{13}$. Since $d_{13}^s \in [0, 2]$, it follows that the value for $s$ and $t$ can be positive or negative and, hence, also the coefficient $d$ of the cubic equation. In turn, this may impose a condition on the desired distances $d_{12}$ and $d_{13}$ for obtaining positive values for $d_{12}$ and $d_{13}$, as discussed in Section II-C. In particular, we can verify that coefficient $d$ has range $d \in (-2, 4)R_3$. Taking $d = 4R_3$, we obtain that all four robot orderings in Fig. 3 can occur when the desired distances satisfy $d_{ij}^s \geq \sqrt{3} \frac{\sqrt{d_{ij}^r}}{2R_3}$.

In Section IV-C, we will show that the colinear moving formations are unstable.

C. Local Stability Analysis of the Equilibrium and Moving Formations

We have characterized the equilibrium configurations and the moving configurations. It is of interest to study the local stability property of these different sets. Similar to the stability analysis for the (1D2B) setup, we will use Lyapunov’s indirect method. The Jacobian matrix corresponding to the $z$-dynamics (28) results in

$$A = -\begin{bmatrix} K_{b}A_{12b} + K_{d}A_{12d} & K_{b}A_{13b} \\ K_{b}A_{12b} & K_{b}A_{13b} + K_{d}A_{13d} \end{bmatrix} \quad (33)$$

with $A_{12d}$ and $A_{13b}$ as defined earlier.

We first consider equilibrium configurations (29).

Lemma 3: The Jacobian matrix $A_{E}$ at the equilibrium configurations in $S_p^C \cup S_p^C$ is Hurwitz.

Proof: For the correct and desired equilibrium configurations in $S_p^C$, the Jacobian matrix (33) evaluates to

$$A_{E} = -\begin{bmatrix} x^* & 0 \\ 0 & p^* \end{bmatrix} \quad (34)$$

where $x, y, p, q, m, n, b_{\text{diff}}$, and the bearing matrices are previously defined in (17) and (18). Also, for the flipped equilibrium configurations in $S_p^C$,

$$A_{E} = -\begin{bmatrix} x^* & 0 \\ 0 & (p^* + x^*) \end{bmatrix} \quad (35)$$

The characteristic polynomial $\chi_{\lambda}(\lambda)$ corresponding to the Jacobian matrices $A_{E}$ and $A_{E}$ is the same, namely

$$\chi_{\lambda}(\lambda) = (\lambda + q^*) (\lambda + y^*) \quad \lambda^2 + (p^* + x^*) \lambda + p^* x^* \sin^2 \theta^* \quad (36)$$

The roots of (36) are

$$\lambda_1 = -q^*, \quad \lambda_2 = -y^* \quad \lambda_3, 4 = -\frac{1}{2} (p^* + x^*) \pm \sqrt{\left(\frac{1}{2} (p^* + x^*) \right)^2 - 4 p^* x^* \sin^2 \theta^*}. \quad (37)$$

We can verify that $0 < 4 p^* x^* \sin^2 \theta^* \leq (p^* + x^*)^2$. This implies that all $\lambda$s are real. Also, $-\lambda_1 = \lambda_4$ and $-\lambda_2 = \lambda_3$. Hence, all roots are real negative.

This leads to the following main result.

Theorem 2: Consider a team of three robots arranged in the (1D2B) setup with closed-loop dynamics given by (27). Given an initial configuration $p(0)$ that is close to the desired formation shape, the robot trajectories asymptotically converge to a point $\tilde{p} \in S_p^C$.

Proof: Following Lemma 3, we obtain that link trajectories locally asymptotically converge to the desired relative positions $z^*$ when they
are initialized in the neighborhood of it. With $z_{ij}^* = d_{ij}^* g_{ij}^*$, it follows that the robots also converge to a point $\hat{p} \in S^r$.

Employing Lyapunov’s indirect method to the moving colinear formations yields the following statement.

**Theorem 3:** Let $p \in \mathbb{R}^r$ be a configuration yielding a colinear formation as depicted in Fig. 3. Then, $p$ is unstable.

**Proof:** We first obtain the matrix $A_M$ and the corresponding characteristic polynomial $\chi_M(\lambda)$. With $e_{12M} = \frac{1}{d_{12}} R_{bd} \quad e_{13M} = \frac{1}{d_{13}} R_{bd}$ and bearing vectors $g_{12M}$ and $g_{13M}$ oriented to $g_{sum}$, $A_M$ takes the following form:

$$A_M = \begin{bmatrix} (s + 1) x & 0 \\ 0 & (t + 1) p \end{bmatrix}$$

where the variables $x, y, p, q, m,$ and $n$ are defined in (17). The characteristic polynomial $\chi_M(\lambda)$ is

$$\chi_M(\lambda) = (\lambda + pt + q) \left(\lambda + px + y\right) \left(\lambda^2 + B\lambda + C\right)$$

where the coefficients are $B = (p(t + 1) + x(s + 1))$ and $C = px(2t + 1)(s + 1)$. We explore the nature of the roots, hereby focusing on the coefficients of the quadratic polynomial.

For the different orderings, we obtain the following:

i) $\lambda = (p(t + 1) + x(s + 1)) \quad C = px(2t + 1)(s + 1) \quad 0 < 0$

ii) $\lambda = (p(t + 1) + x(s + 1)) \quad C = px(2t + 1)(s + 1) \quad 0 < 0$

iii) $\lambda = (p(t + 1) + x(s + 1)) \quad C = px(2t + 1)(s + 1) \quad 0 < 0$

iv) $\lambda = (p(t + 1) + x(s + 1)) \quad C = px(2t + 1)(s + 1) \quad 0 < 0$

We infer that the quadratic polynomial in (39) contains at least a root with positive real part since for each ordering, either $B$ or $C$ is negative. This implies that matrix $A_M$ is not Hurwitz; therefore, the colinear formations are unstable.

**Remark 3:** The bearing-only control law proposed in [17] can be obtained from the current control law by $P_{g_{ij}} e_{ij} = - P_{g_{ij}} g_{ij}^*$. After following the steps as we have carried out for the gradient-based bearing control law, we infer that the closed-loop dynamics for the (1D2B) and (1B2D) robot setups contain only equilibrium configurations and no moving configurations. The incorrect equilibrium configurations, in which either one or both of the bearing vectors are incorrect, are found to be unstable after linearization; the desired equilibrium is almost globally stable.

### V. NUMERICAL EXAMPLE

We consider two triangular formation shapes with the same distances $d_{12}$ and $d_{13}$ but different value for the internal angle $\theta^*$ (Note: $\theta^* = \cos^{-1}(g_{12}^* g_{13}^*)$). In particular, shape $T_1$ has bearing vectors such that the internal angle is $\theta_{T_1}^* = 15^\circ$ while for shape $T_2$, we take $\theta_{T_2} = 45^\circ$. We set the gain ratio $R_{bd}$ to a value 4. Taking the different sets into consideration, the threshold distance such that moving formations (stable or unstable) exists is $\tilde{d} = 2\sqrt{3} \approx 3.4641$. We set the desired distances to $d_{12}^* = d_{13}^* = 4$ and assume $\theta_{T_1}^* = 0^\circ$. Thus, shape $T_1$ and $T_2$ has the following desired constraints:

$$T_1: \quad d_{12}^* = d_{13}^* = 4; \quad \angle g_{12}^* = 0^\circ, \quad \angle g_{13}^* = 15^\circ;$$

$$T_2: \quad d_{12}^* = d_{13}^* = 4; \quad \angle g_{12}^* = 0^\circ, \quad \angle g_{13}^* = 45^\circ.$$ (40)

For shape $T_1$, the moving formation for the (1D2B) setup is unstable, since $\cos^2(15^\circ) = 0.9330 > 0.9321$. Hence, the constraint in (24) is violated. For shape $T_2$, we obtain $\cos^2(15^\circ) = 0.9 < 0.9321$ satisfying constraint (24).

In the current example, we first intentionally set the gain ratio $R_{bd}$ and then obtain desired distances $d_{ij}^*$, in order to show the existence and local asymptotic stability of moving formations in the (1D2B) setup.

### A. (1D2B) Simulation Results

For the three robots in the (1D2B) setup, we focus on the formation shape $T_2$. The Jacobian matrix $A_M$ for the moving formation with distances $d_{12} = d_{13} \approx 3.8686$ is checked to be Hurwitz. Therefore, employing the closed-loop dynamics (5) can, depending on the initial configuration $p(0)$, lead to robot trajectories moving with a constant velocity. In Fig. 4(b), we show such an outcome for a specific $p(0)$. Fig. 4(a) depicts an initial $p(0)$ leading to convergence to the correct shape.

### B. (1B2D) Simulation Results

For the three robots in the (1B2D) setup, we focus on the formation shape $T_1$. There are two equilibrium formations, namely, the correct and desired formation and the flipped formation satisfying only the distance constraints but not the bearing constraints. Fig. 5(a) depicts an initial configuration $p(0)$, which converges to this flipped formation. Notice that the signed area corresponding to $p(0)$ is positive (counter-clockwise cyclic ordering of the robots) while the flipped formation has a negative signed area (clockwise cyclic ordering of the robots). Fig. 5(b) depicts an initial configuration $p(0)$ leading to the robots to move with a constant velocity when employing the closed-loop dynamics (27). When perturbed, it will converge either to the correct or the flipped formation shape.

### VI. CONCLUSION

In the current article, we have considered the formation shape problem for teams of three robots partitioned into two categories, namely: 1) distance and 2) bearing robots. Our aim was to employ gradient-based control laws in a heterogeneous setting and provide a systematic study on the stability of the possible formation shapes that arise as a result. We have shown that under certain conditions on the distance and bearing error signals, we obtain distorted formation shapes moving with a constant velocity $w$. For the (1B2D) robot setup, these undesired formation shapes are unstable while for the (1D2B) robot setup, we derive conditions such that one of the distorted moving formation shape is locally asymptotically stable. When the gains $K_d$ and $K_b$ are chosen such that the required desired distances $d_{ij}^*$, are smaller than a threshold distance $\tilde{d}(K_d, K_b)$, then moving formation shapes do not exist. Depending on the setup considered, this may lead to global asymptotic stability of the desired formation shape.

We note that the moving configurations in the (1B2D) setup and the flipped equilibrium configuration in the (1B2D) setup both have a signed area that has an opposite sign compared to the signed area of the desired formation shape. Hence, the use of signed constraints.
as introduced in [15] and [18] is a possible future direction. For the (1D2B) setup, the inclusion of the signed area constraint in [15] does not increase the sensing load of the distance robot while it can have the potential of mitigating the existence of distorted formation shapes.

REFERENCES