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Hoenders, Bernhard J.

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The Information Content of Aberrated- and Non Aberrated Images Generated by Coherently- or Incoherently Illuminated Objects

(Invited Paper)

Bernhard J. Hoenders

Centre for Theoretical Physics and Zernike Institute for Advanced Materials,
University of Groningen, Nijenborg 4, 9747 AG Groningen,
The Netherlands
Email: b.j.hoenders@rug.nl

I. INTRODUCTION

The information content of an image is in the case of optical signals analysed in seemingly two very different ways:

a) Using coherent mode expansions of the fields, (image-object functions).

b) Using the so-called Whittaker-Shannon-Kotel'nikov-Cauchy sampling theorem.

These two approaches appear to be quite different, but they both rely on the particular choice made for the set of base functions by means of which the pertinent wave functions (fields) are characterised. The coherent modes arise in the bilinear expansion of the particular propagation (kernel, Green) function into a set of (bi)orthogonal functions known as the Schmidt functions. Then, e.g., both the image- and object wave function are expanded into these modes. The number of “easily” detectable expansion coefficients are interpreted as the Number of Degrees of Freedom (NDF).

The approach b) is based on the Whittaker-Shannon-Kotel'nikov sampling theorem, (though already known by Cauchy!), which is also known as the cardinal series: Any bandlimited function (like the image wave function of an ideal imaging device), with bandwidth a can be written as an interpolating series:

$$f(x) \equiv \int_{-a}^{+a} \exp(ixx_0)g(x_0)dx_0 = \sum_{-\infty}^{+\infty} f\left(\frac{n\pi}{a}\right) \frac{\sin(ax - n\pi)}{(ax - n\pi)} \quad (1)$$

and is therefore characterised by its values at the sampling points $n\pi/a$. The set of base functions are the sinc functions $\{\text{sinc}(ax - n\pi)\}$, $n = \pm 0, \pm 1 \pm 2, \dots \pm \infty$. Then, the values of $f(\frac{n\pi}{a})$ which are “easily” measured are interpreted as the NDF.

The approach a) choses the coherent modes (Schmidt functions) as the proper set of base functions. This appears to be an excellent choice as base function for the analysis of physical problems anyway, because Schmidt has shown that

the **most efficient** bilinear expansion of a propagation (Green) function is by these functions. Unfortunately the coherent mode approach suffers from two disadvantages:

a) The coherent modes lack a physical interpretation, e.g.: What is the physical interpretation of the coherent modes generated by the cross-spectral density function, the propagation kernels $\text{sinc}(x-x')$, $\text{sinc}^2(x-x')$, (occurring in ideal imaging), or even more realistic, the propagation kernel in the presence of **aberrations** ?

b) They are only known *analytically* for apertures with circular- or rectangular shape and ideal imaging. No (semi)analytical results are known for apertures with different shapes and for the case of aberrated images. A numerical analysis is then required in order to get insight into their properties.

In contrast to all this, the analysis of the NDF using the Sampling theorem leads to an easy physical interpretation of the mathematical analysis for the calculation of the NDF: It only involves an analysis of the distribution of the sampling points. Moreover, the calculation of the NDF using the Sampling theorem can be easily generalised for the case of images suffering from an arbitrary number of aberrations limited by apertures with arbitrary shape and provides a constructive proof of the Gabor conjecture in terms of explicitly known sampling functions and sampling points.

II. NUMBER OF DEGREES OF FREEDOM OF (ABERRATED) IMAGES

The number of degrees of freedom is loosely defined as the number of independent parameters characterising the image which can be “easily” determined. This concept is made more explicit by following conjecture, (Gabor):

“Assume that the object area, large compared with the square of the wavelength, is limited by a black screen. Assume also that there is

a similar limitation in the aperture plane at a great distance from the object plane (see figure). Then, in the domain limited by these two black screens there exist S independent solutions of the wave equation $\nabla^2 u + k^2 u = 0$, that is to say, solutions with $u = 0$ immediately behind the black screens and S is given by the formula: $S = \lambda^{-2} \int \int \int \int dx dy d(\cos \alpha_x) d(\cos \alpha_y)$, x and y are the coordinates in the object plane, and $\cos \alpha_x = k_x$ and $\cos \alpha_y = k_y$ are the direction cosines of the geometrical optical rays leaving the object plane. Any progressive wave through the object area and through the aperture can be expanded in terms of these S eigensolutions, with no more than S complex coefficients”.

F denotes the focal distance, OP the object plane, H the principal plane of the lens, A the aperture plane and K_x and K_y the coordinates of the Fourier plane (focal plane).

Two heuristical proofs of this theorem use in the case of a rectangular aperture either a) the sampling theorem or b) the coherent mode (eigenfunction) expansion of the object- and image function. The number of degrees of freedom are than either identified as (using the Sampling Theorem) the number of sampling points situated within the aperture of the image, or, expanding the image wave function into the coherent- (Schmidt) modes, as the the number of expansion coefficients of the image which are not to “small”.

But, if we try to apply these two methods to apertures with different shapes it appears that the coherent mode expansion only works for apertures with a rectangular or circular shape, but that other shapes require fully numerical methods. No (semi-) analytical theory exists.

The sampling method has been generalised to the case of apertures with circular and elliptical shapes, however with moderate success: The estimate of the number of degrees of freedom were still approximately 10% deviating from the Shannon value! Apertures with other shapes leads to the same amount of deviations of the Shannon number.

The (heuristical) calculation of the NDF for an aperture of arbitrary shape is indicated below.

Along each line within the support (band) of the bandlimited function, (aperture of the focal plane), we require that the density of the sampling points is that which occurs in the one-dimensional Whittaker-Shannon-Cauchy-Kotelnikov theorem. The longer the portion of the line within the aperture, the more dense the distribution of the sampling points! By this principle we can show, just counting, the validity of Gabor's conjecture!

The sampling points are the dots (pearls) at the two lines in the figure. We would like to stress the point that the recipe for the determination of the distribution of the sampling points

is constructive! Two other problems are: What is the NDF of an aberrated image, and how does partial coherence influence this number? Both problems can be analysed using methods a) and b), but, a generalisation of the sampling theorem for these cases leads to explicit analytical expressions which are easily interpreted, in contrast to the coherent mode expansion method.

If x_0, y_0 and x, y denote the coordinates in the focal, resp. the image plane, the image wavefunction $\psi(x, y)$ is related to the wave function $\phi(x_0, y_0)$ in the focal plane by, (see 1):

$$\psi(x, y) \equiv \int_{-b}^{+b} \int_{-a}^{+a} \exp(ixx_0 + yy_0 + iS(x, y, x_0, y_0)) \times \phi(x_0, y_0) dx_0 dy_0, \quad (2)$$

if:

$$S(x, y, x_0, y_0) = \frac{1}{4}AR^2 + \frac{1}{4}B\rho^2 + C\kappa^2 + \frac{1}{2}DR\rho + ER\kappa + F\rho\kappa + eR\sigma + c\kappa\sigma + f\rho\sigma, \quad (3)$$

where $R = x_0^2 + y_0^2$, $\rho = x^2 + y^2$, $\kappa = x_0x + y_0y$, $\sigma = x_0y - y_0x$, and S is known as the aberration eikonal. The coefficients A, B denote spherical aberration, F, f denote isotropical- and anisotropical coma, C, D and c , denote the field curvature, isotropical- and anisotropical astigmatism, resp., and E denotes the distortion.

As a typical examples for the calculation of the NDF in the presence of aberrations we consider imaging in the presence of either coma, ($F \neq 0$), or spherical aberration, ($B \neq 0$), or field curvature, $C \neq 0$. coma is easily dealt with employing a coordinate transformation $x' = x + Fx(x^2 + y^2)$ and $y' = y + Fy(x^2 + y^2)$. The relation between ψ and ϕ is then that of Fourier transforms and the number of degrees of freedom can be calculated in the conventional way, *however now using the new coordinates x', y'* . As the coordinate transformation entails higher order powers of x, y , the total number of sampling points lying within a certain interval $[0, N]$ is now not proportional to N , but rather proportional to N^3 . **The NDF is therefore much larger as in the non-aberrated case**, because more sampling points are located within the aperture of the image plane.

The spherical aberration does not influence the NDF because the sampling theorem still applies if we multiply the r.h.s. of 1 by $(\exp(i\frac{1}{4}AR^2 + i\frac{1}{4}B\rho^2))$.

The field curvature C and the distortion E are the most interesting aberrations because we then have to generalise the sampling theorem 1 for the calculation of the NDF. The generalised sampling theorem shows that sampling points are then distributed as N^2 . Hence, both field curvature and distortion increase the NDF!

The “Sampling method” applies as well to partially coherent fields. The generalised sampling theorem then shows that **the degree of partial coherence does not change the NDF (information content) of an image!!**.

III. DISCUSSION

The calculation of the information content of an ideal image rests on two different mathematical approaches: The expansion of the field into either the a) coherent modes, or b) into the sampling functions occurring at the r.h.s. of 1. The coherent mode approach leads to analytical results for a few special cases only, (rectangular- circular apertures), whereas the sampling method can be generalised in such a way that the distribution of the sampling points is known *explicitly* for all possible shapes of the apertures, thus obtaining optimal sampling!

Such a generalisation leading to *explicitly* known sampling points and functions is also possible in the case of partially coherent imaging and in the presence of aberrations and enables an “easy” calculation of the NDF. The “Sampling” method therefore appears to be *superior* compared to the “Coherent Mode expansion” method.