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Discrete approximations to vector spin models

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Abstract

We strengthen a result from Külske and Opoku (2008 *Electron. J. Probab.* **13** 1307–44) on the existence of effective interactions for discretized continuous-spin models. We also point out that such an interaction cannot exist at very low temperatures. Moreover, we compare two ways of discretizing continuous-spin models, and show that except for very low temperatures, they behave similarly in two dimensions. We also discuss some possibilities in higher dimensions.

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1. Introduction

If we try to approximate a continuous-spin vector model such as the classical XY-model by a discrete approximation, whether for computational or theoretical purposes⁴, we can in principle pursue two routes.

- (1) We can consider the Gibbs measure for the original continuous-spin model and discretize the spin by dividing the single-spin space into a large but finite number q of sets (intervals). By identifying all spins in such a set, we obtain a measure on a discrete spin system. This measure can, but does not have to, be a Gibbs measure for some effective discrete spin interaction.
- (2) Alternatively, we can write down the same expression for the Hamiltonian of the discrete spin system as we have for the XY-model, and consider the appropriate Gibbs measure(s) for this discrete spin interaction. For the XY-model (the plane rotor), such models are called clock models, and their study goes back to Potts [24].

Here we present some results on discrete approximations of type 1, and compare them with what is known of discrete approximations of type 2. We will see that except for very low

⁴ We remind the reader of Mark Kac's famous dictum: 'Be wise, discretise!'.

temperatures, in two dimensions both approximations have quite similar properties, and we also speculate on possible scenarios in higher dimensions.

We note that a type-1 discretization is appropriate for measuring purposes, and describes, for example, round-off errors, whereas a type-2 discretization is often used in computer simulations.

Generalizations to discretizations of more general continuous compact single-spin spaces are immediate, as follows from the analysis of [18, 21].

Stated differently, and more formally, we can apply a local discretization map $T : \mathbb{S}^1 \mapsto \{1, \dots, q\}$, mapping a continuous local spin variable and taking values on the circle $\mathbb{S}^1$, to its discretized image, before or after performing the Gibbsian modification with interaction Φ relative to the product measure α .

We then want to compare the images $T\mu$ of the Gibbs measures $\mu \in \mathcal{G}_{\Phi, \alpha}$ of the initial model with the *a priori* measure α , and interaction Φ with the Gibbs measures $\mu' \in \mathcal{G}_{\Phi', T\alpha}$, where $T\alpha$ is the product of the *a priori* measures under the local coarse-graining T , and $\mathcal{G}_{\Phi', T\alpha}$ are the Gibbs measures obtained from the specification which has the old interaction simply taken in the coarse-grained variables.

We note that conceptually such a question can be studied even more generally without making any assumptions about the first and/or second image-spin measure being discrete.

A first important question to be asked is whether and when

$$T\mathcal{G}_{\Phi, \alpha} = \mathcal{G}_{\Phi', T\alpha}$$

for some Φ' .

We remind the reader that if there is a Φ' such that $T\mu \in \mathcal{G}_{\Phi', T\alpha}$ for $\mu \in \mathcal{G}_{\Phi, \alpha}$, then $T\mathcal{G}_{\Phi, \alpha} \subset \mathcal{G}_{\Phi', T\alpha}$ [4].

We are not aware that the equality between the number of transformed Gibbs measures and the number of Gibbs measures for the transformed interaction—even if one exists—always holds, although we do not know of any counterexamples. We also note that even under the assumption of equality of these two sets, one can have a different number of extremal Gibbs measures in the original and the transformed set, if different measures are mapped to the same set, as e.g. occurs in fuzzy Potts models.

We can say that if the original system has a unique Gibbs measure, the same is true for the transformed measure. Indeed it follows from [12] (see proposition 7.11) that $|\mathcal{G}_{\Phi, \alpha}| = 1$ if for some $\mu \in \mathcal{G}_{\Phi, \alpha}$ the following mixing property holds.

For any cylinder event A in Ω ,

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{B \in \mathcal{F}_{\Lambda^c} : \mu(B) > 0} |\mu(A|B) - \mu(A)| = 0, \quad (1)$$

where $\mathcal{F}_{\Lambda^c}$ is the σ -algebra of events that depend on all spins outside the finite subset Λ of $\mathbb{Z}^d$. As the above statement is obviously also true if one only considers events A and B which are discrete spin events, once we know that the transformed measure is again a Gibbs measure and $|\mathcal{G}_{\Phi, \alpha}| = 1$, it is necessarily the case that $|T\mathcal{G}_{\Phi, \alpha}| = |\mathcal{G}_{\Phi', T\alpha}| = 1$.

We can also ask questions of closeness on the level of interactions, namely, what is the distance between the original and transformed interaction $d(\Phi, \Phi')$? Furthermore, what is a good notion for the distance d here?

There remains the problem that the spins, and hence the interactions, live on different spaces, one discrete, one continuous. If we compare the two discretizations, we at least have the advantage that the corresponding interactions will live on the same space.

What can be said about the closeness of the measures $\mu \in \mathcal{G}_{\Phi, \alpha}$ to $\tilde{\mu} \in \mathcal{G}_{\Phi', T\alpha}$? This question is subtle, since we could look here for closeness on local observables (in the weak topology), on the level of long-range characteristics such as the decay of correlations, on the

level of the phase diagram in parameter space, etc. Here we will describe the two discretizations as close if the two models have a similar phase diagram and/or similar correlation decay in their Gibbs measures.

We note that the proof concerning the locality properties in [18] for type-1 discretizations makes essential use of the Dobrushin uniqueness theorem, even though we need not be in the uniqueness regime and in fact are allowed to be in a phase-transition region when we discretize. This will also be the case here.

Discretizations can be viewed as single-site coarse-grainings, similarly to the ‘fuzzification’ or amalgamation of discrete spin systems as treated in e.g. [2, 13, 14, 31], but now the ‘fuzzification’ goes from a continuous ‘alphabet’ to a discrete one.

2. Gibbsianness of discrete approximations of the XY lattice model

2.1. Notation and definitions

We will consider lattice spin systems with a single-spin space Ω_0 , on a lattice $\mathbb{Z}^d$, and a configuration space $\Omega = \Omega_0^{\mathbb{Z}^d}$. We will mainly consider the XY model for which Ω_0 is the circle $\mathbb{S}^1$, and discrete approximations thereof, in which $\mathbb{S}^1$ is divided into q equal arcs of length $\frac{2\pi}{q}$. We will indicate the spin variables at site i (which will always be elements of the unit circle) by $\sigma_i, \omega_i, \eta_i$, and similarly spin configurations in a volume Λ by $\sigma_\Lambda, \omega_\Lambda, \eta_\Lambda$.

We will consider Gibbs measures, which are defined for (here translation-invariant) absolutely summable interactions Φ ($\sum_{A: 0 \in A} \|\Phi_A\| < \infty$) via the DLR equations, expressing that given an external configuration η_{Λ^c} , the probability density of configurations in a volume Λ is given by the Gibbs expression

$$\frac{d\mu_{\Lambda}^{\eta_{\Lambda^c}}(\sigma_{\Lambda})}{d\alpha_{\Lambda}} = \frac{\exp(-H_{\Lambda}(\sigma_{\Lambda}\eta_{\Lambda^c}))}{Z_{\Lambda}^{\eta_{\Lambda^c}}}, \quad \text{where } H_{\Lambda}(\sigma_{\Lambda}\eta_{\Lambda^c}) = \sum_{A: A \cap \Lambda \neq \emptyset} \beta \Phi_A(\sigma_{\Lambda}\eta_{\Lambda^c}), \quad (2)$$

and α_{Λ} is the product of α over the sites in Λ . This should hold for all volumes Λ , internal configurations σ_{Λ} and external configurations η_{Λ^c} . The corresponding collection of conditional probabilities (everywhere instead of almost everywhere with respect to the Gibbs measure defined) forms a ‘specification’; see e.g. [4, 7, 12]. In the standard nearest-neighbor models (the plane rotor or XY model), as well as in the clock models, where the spins take discrete values, we have

$$-H_{\Lambda}(\sigma_{\Lambda}\eta_{\Lambda^c}) = \beta \sum_{\langle i,j \rangle \in \Lambda} \sigma_i \cdot \sigma_j + \beta \sum_{\langle i \in \Lambda, j \in \Lambda^c \rangle} \sigma_i \cdot \eta_j. \quad (3)$$

A probability measure μ on Ω is said to be non-Gibbsian if there is no absolutely summable interaction Φ for which (2) holds. Non-Gibbsian measures may arise from transforming Gibbs measures. Stochastic time-evolution of Gibbs measures, restrictions to sublattices such as those in renormalization group theory, and coarse-grainings (fuzzifications) of single-state spaces, are among the transformations that have been studied in the literature. Non-Gibbsianness of a measure means that one cannot associate an effective Hamiltonian or an effective temperature with the system described by that measure.

2.2. Conservation of Gibbsianness under local transformations: fine discretizations

One of the main results of [18] (see also [21]), concerns conditions under which a discretization of a Gibbs measure is again Gibbsian. These results were obtained as corollaries to a theorem on the preservation of Gibbsianness which also holds for much more general types of local

transforms like time evolutions. So it is worthwhile to specifically reconsider the local transformations.

The condition for preservation of Gibbsianness is temperature-dependent, and the main example we want to discuss here is the discretization of the circle into q equal arcs. At inverse temperature β the result implies that for q , dependent on β being large enough, the discretized measure is a Gibbs measure.

To be more precise, suppose for each $l \in S' := \{1, 2, \dots, q\}$ we denote by $\mathbb{S}_l^1$ the l th arc of the circle $\mathbb{S}^1$ cut out by the discretization operator T . Then, $T\alpha(l) = \alpha(\mathbb{S}_l^1)$.

The constrained system for any $\mu \in \mathcal{G}_{\Phi, \alpha}$ will be the family of probability measures $\{\mu[\eta]; \eta \in (S')^{\mathbb{Z}^d}\}$ on Ω , with $\mu[\eta]$ being the restriction of μ to $T^{-1}(\{\eta\})$.

Given $\mu \in \mathcal{G}_{\Phi, \alpha}$, one of the main results in [18], theorem 2.5, is that the discretized measure $T\mu$ is Gibbs if

$$\sup_{i \in \mathbb{Z}^d} \sum_{j \in \mathbb{Z}^d \setminus \{i\}} \bar{C}_{ij} < 1, \quad (4)$$

where

$$2\bar{C}_{ij} = \begin{cases} \sup_{\substack{\eta_j, \bar{\eta}_j \in \mathbb{S}^1: \\ l \in S'}} \int_{\mathbb{S}_l^1} \alpha(d\sigma_i) \left| \frac{e^{\beta \sigma_i \eta_j}}{\int_{\mathbb{S}_l^1} \alpha(d\bar{\sigma}_i) e^{\beta \bar{\sigma}_i \eta_j}} - \frac{e^{\beta \sigma_i \bar{\eta}_j}}{\int_{\mathbb{S}_l^1} \alpha(d\bar{\sigma}_i) e^{\beta \bar{\sigma}_i \bar{\eta}_j}} \right|, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

This condition means that the constrained system must be in the Dobrushin uniqueness regime *uniformly in the chosen constraint*.

While looking for good upper bounds for the right-hand side, we can at not much additional cost revisit the more general situation and give an improvement to the criterion from [18] for Gibbsianness for local discretizations.

We put ourselves in a slightly more general context than that of the discretizations in [18], and we will take the local spin space S to be a general measurable space. No *a priori* metric is given; it will be produced by the Hamiltonian itself. As in [18], let a decomposition be given of the form $S = \bigcup_{s' \in S'} S_{s'}$. Here S' may be a finite or infinite set. Put $T(s) := s'$ for $S_{s'} \ni s$. This defines a deterministic transformation on S , called the *fuzzy map* (the discretization).

Now we deviate from [18]. Let G be the vertex set of a general graph and define a family of metrics $(d_{ij})_{j \in G \setminus \{i\}}$ on the local spin space at the site $i \in G$ by

$$d_{ij}(\sigma_i, \tau_i) := \sup_{\substack{\zeta, \bar{\zeta} \\ \zeta_j = \bar{\zeta}_j; T(\zeta_j) = T(\bar{\zeta}_j)}} |H_i(\sigma_i \zeta_{i^c}) - H_i(\sigma_i \bar{\zeta}_{i^c}) - (H_i(\tau_i \zeta_{i^c}) - H_i(\tau_i \bar{\zeta}_{i^c}))|, \quad (6)$$

where for any $i \in G$, $i^c = G \setminus \{i\}$. It is important here (as well as in the formula specific to the rotors above) that the supremum is taken over spins $\zeta, \bar{\zeta}$ which are constrained to take the same coarse-grained image at j . We are allowed to do this since we are analyzing the constrained system. In this way, the metric at the site i also depends on the size of the coarse-graining at j . The metric measures how strongly a variation at the site j can maximally change the difference in interaction energy between local spins σ_i, τ_i .

Our criterion of the fineness of the decomposition will involve the corresponding j -diameter, namely the quantity $\text{diam}_{ij}(A) = \sup_{s, t \in A} d_{ij}(s, t)$ where A runs over the sets in the decomposition.

Theorem 2.1. *Let μ be a Gibbs measure of the specification with the Gibbsian potential Φ with an arbitrary a priori measure α , on a graph with vertex set G . Let T denote the local coarse-graining map, where we assume that $\alpha(S_{s'}) > 0$ for all labels $s' \in S'$.*

Suppose that

$$\sup_{i \in G} \sum_{j \in i^c} \sup_{s'} \text{diam}_{ij}(S_{s'}) < 4. \quad (7)$$

Then the transformed measure $T(\mu)$ is Gibbs for a specification γ' with an absolutely summable discrete spin interaction Φ' .

In all cases, this is an improvement over the criterion of [18] (which we do not repeat in detail, because it requires the introduction of an additional structure which we do not need here).

It is also an improvement over what a direct application of the high-temperature version found in Georgii [12] would give for our constrained system. That would only give a bound in terms of the right-hand side of the inequality of the form

$$\text{diam}_{ij}(S_{\sigma'}) \leq 2 \sup_{\sigma_i, \tau_i, \zeta, \bar{\zeta}} |H_i(\sigma_i \zeta_{i^c}) - H_i(\tau_i \bar{\zeta}_{i^c})|. \quad (8)$$

Here we discuss the application to rotor models. Consider first the rotor model on a circle $\mathbb{S}^1$. We have for nearest neighbors i and j using the Cauchy–Schwartz inequality

$$\begin{aligned} d_{ij}(\sigma_i, \tau_i) &= \beta \sup_{\zeta_j, \bar{\zeta}_j; T(\zeta_j) = T(\bar{\zeta}_j)} |(\sigma_i - \tau_i) \cdot (\zeta_j - \bar{\zeta}_j)| \\ &\leq \beta \|\sigma_i - \tau_i\|_2 2 \sin \frac{\pi}{q} \end{aligned} \quad (9)$$

and so $\text{diam}_{ij} S_{\sigma'} = \beta \times (2 \sin \frac{\pi}{q})^2$. This gives the criterion

$$2d\beta \left(\sin \frac{\pi}{q} \right)^2 < 1 \quad (10)$$

for Gibbsianness of the coarse-grained model. Note that the standard estimate (8) would give a worse condition without the square.

For a local spin space which is a d -dimensional sphere not much changes. The formula for the metric d_{ij} stays the same. Let us assume that ψ is one-half of the maximal angle under which a set $S_{\sigma'}$ appears as seen from the origin. This quantity is a measure of fineness of the discretization. Then, going through the same steps, we obtain as a criterion for Gibbsianness that

$$2d\beta (\sin \psi)^2 < 1. \quad (11)$$

Proof of the theorem. The proof follows as in [18], by estimating $\bar{C}_{ij}$. This constant is a bound on the Dobrushin interaction matrix of the initial model conditional on the transformed spins, uniformly in the values of the transformed spins. In particular, for each site $i \in G$, $\bar{C}_{ij}$ is a uniform upper bound on the variational distance between the ‘first-layer models’ in $\{i\}$, in which σ_i is constrained to take values in $S_{\sigma'}$ for some fixed prescription of partitions given by the image spins σ'_i , w.r.t. external configurations that coincide everywhere except at site j .

More precisely, we take two conditioning configurations in the original (first-layer) model $\zeta, \eta \in \Omega$ with $\zeta_{j^c} = \eta_{j^c}$ and denote by $u_0(\sigma_i) = -H_i(\sigma_i \zeta_{i^c})$ and $u_1(\sigma_i) = -H_i(\sigma_i \eta_{i^c})$ the corresponding values of the single-site Hamiltonians anchored at i . Defining $u_t = tu_1 + (1-t)u_0$, $h_t = e^{u_t} 1_{S_{\sigma'_i}} / \alpha(e^{u_t} 1_{S_{\sigma'_i}})$ and $\lambda_t(d\sigma_i) = h_t(\sigma_i) \alpha(d\sigma_i)$, with $t \in [0, 1]$, we note that $\lambda_0(d\sigma_i) = \gamma_i(d\sigma_i | \zeta)$ and $\lambda_1(d\sigma_i) = \gamma_i(d\sigma_i | \eta)$, where γ_i are the single-site parts of the conditional distributions (kernels) of the initial model obtained via (2) after replacing S with $S_{\sigma'_i}$. Note however that the constraining configuration σ' no longer appears in the notation for the sake of simplicity.

Now we treat the estimate which, for coarse-grainings, improves that from [18] (in which, however, general transformations beyond coarse-grainings were also treated), and also (8)

from Georgii [12]. For the first step of the proof, we obtain the following bound for the total variational norm of $\lambda_0 - \lambda_1$:

$$\begin{aligned} 2\|\lambda_0 - \lambda_1\| &= \int \alpha(d\sigma_i) |h_1(\sigma_i) - h_0(\sigma_i)| = \int \alpha(d\sigma_i) \left| \int_0^1 dt \frac{d}{dt} h_t(\sigma_i) \right| \\ &\leq \int_0^1 dt \int \lambda_t(d\sigma_i) \left| H_i(\sigma_i \zeta_{i^c}) - H_i(\sigma_i \eta_{i^c}) - \int \lambda_t(d\tau_i) (H_i(\tau_i \zeta_{i^c}) - H_i(\tau_i \eta_{i^c})) \right|. \end{aligned} \quad (12)$$

The simple but essential next estimate will be uniform in the image measure of λ_t , under $\sigma_i \mapsto H_i(\sigma_i \zeta_{i^c}) - H_i(\sigma_i \eta_{i^c}) =: f^{\zeta, \eta}(\sigma_i)$. Namely, by further making use of the notion of the j -diameter of the set $S_{\sigma'_i}$ to bound the variation of the energy terms, we find

$$\begin{aligned} &\int \lambda_t(d\sigma_i) \left| H_i(\sigma_i \zeta_{i^c}) - H_i(\sigma_i \eta_{i^c}) - \int \lambda_t(d\tau_i) (H_i(\tau_i \zeta_{i^c}) - H_i(\tau_i \eta_{i^c})) \right| \\ &\leq \int \lambda_t(d\sigma_i) \int \lambda_t(d\tau_i) |f^{\zeta, \eta}(\sigma_i) - f^{\zeta, \eta}(\tau_i)| \\ &= \int f^{\zeta, \eta}(\lambda_t)(dx) \int f^{\zeta, \eta}(\lambda_t)(dy) |x - y| \\ &\leq \sup_{\lambda} \int_{-D}^D \lambda(dx) \int_{-D}^D \lambda(dy) |x - y|, \end{aligned} \quad (13)$$

where $D = \text{diam}_{ij} S_{\sigma'_i}/2$ and the supremum is over the probability measures λ on the interval $[-D, D]$. For this supremum, we use the following lemma (after scaling with D). $\square$

Lemma 2.2. *For all probability measures ρ on $[-1, 1]$, we have*

$$Q(\rho) := \int \rho(dx) \int \rho(dy) |x - y| \leq 1 \text{ with equality for } \rho_0 = \frac{1}{2}(\delta_1 + \delta_{-1}).$$

Note the improvement over the simple upper bound 2. Observe also that the upper bound on $2\|\lambda_0 - \lambda_1\|$ (12) obtained via (13) is independent of η , $\zeta \in \Omega$ and constraint σ'_i . Therefore, (13) provides a uniform upper bound on $2\bar{C}_{ij}$. By scaling up the interval in the lemma with a factor D and putting together our previous estimates we get $2\bar{C}_{ij} \leq \sup_{s'} \text{diam}_{ij} S_{\sigma'_i}/2$, and hence $\sum_{j: j \neq i} \bar{C}_{ij} \leq \frac{1}{4} \sum_{j \in G \setminus i} \sup_{s'} \text{diam}_{ij}(S_{s'})$. The rest of the proof follows from the definition of the Dobrushin constant. This proves the desired continuity properties of conditional probabilities as a function of the conditioning.

From here, the proof of the existence of an exponentially decaying potential uses resummed potentials obtained by summing the potentials obtained from the Moebius inversion formula over blocks [16, 17].

For the sake of completeness we also give an elementary proof of the lemma.

Proof of the lemma. By density arguments, we can approximate any ρ by convex combinations of finitely many Dirac measures of the form $\sum_{i=1}^n p_i \delta_{x_i}$ where $x_i \leq x_{i+1}$.

Let us look at Q as a function of the l th location, keeping the other locations fixed, and keeping the p_i 's fixed, $x_l \mapsto Q(\sum_{i=1}^n p_i \delta_{x_i})$, where x_l is constrained to be greater than or equal to its left neighbor x_{l-1} and less than or equal to its right neighbor x_{l+1} . This function is linear. Hence, the function takes its maximum when x_l becomes equal to one of its neighbors. This shows that the maximum of Q over the set of combinations of n Dirac measures is dominated by that over combinations of $n - 1$ Dirac measures. Iterating this argument, we see that the maximum of Q over all probability measures is reached for a linear combination of two Dirac measures $p\delta_x + (1-p)\delta_y$. Noting finally that the max over $Q(p\delta_x + (1-p)\delta_y) = 2p(1-p)|x - y|$ is reached for ρ_0 , the proof is complete. $\square$

As far as the bound on $\sum_{j \in \mathbb{Z}^d \setminus \{i\}} \bar{C}_{ij}$ is concerned, the above result is an improvement over theorem 2.9 of [18] which, however, was formulated in a much more general situation. Indeed, the latter gave rise to the bound $\sum_{j \in \mathbb{Z}^d \setminus \{i\}} \bar{C}_{ij} \leq \frac{4d\pi\beta}{q} e^\beta$. The more general set-up of [18] also allowed us to treat (partially) stochastic single-site maps, such as infinite-temperature stochastic dynamics. The estimates on the Dobrushin constant used there were of the ‘high-field’ type, whereas here we make use of a ‘high-temperature’ version.

Once the refinement is large enough (q very large at a fixed temperature), the effective interaction has as its dominant term the nearest-neighbor interaction of the clock model.

We note that the discretized model inherits various properties from the original XY-model. In particular, if the correlation functions decay slowly, as they do in two dimensions at low temperature when one is in a Kosterlitz–Thouless phase, this remains true after the discretization. The characteristic spin patterns with bound vortices which occur in the Kosterlitz–Thouless phase clearly persist also at fine discretizations.

We give a formal proof of slow decay for the slightly changed coarse-graining procedure where the discretized spin is given as a conditional expected value over the coarse-graining sets. This is more convenient for the purpose of the proof and should be essentially equivalent.

Proposition 2.3. *Suppose that μ is a Gibbs measure for a model on a graph G with local state space $S = S^q$ and let a finite decomposition $S = \bigcup_{s' \in S'} S_{s'}$ with the associated coarse-graining map T be given as above. Denote by $\mathcal{F}$ the sigma-algebra generated by the infinite-volume spin configurations which are constant over the coarse-grained sets and put $\hat{\sigma}_i = \mu(\sigma_i | \mathcal{F})$ for the coarse-grained spin.*

Suppose the discretization is fine enough such that the condition from theorem 1 holds. Then, the two-point correlation functions of the coarse-grained spins $\hat{\sigma}_i$ have power law decay if and only if the two-point correlation functions of the original spin σ_i do so.

Proof. Considering a correlation function of the original continuous model and adding and subtracting conditional expectations gives

$$\begin{aligned} & \mu[(\sigma_i - \mu(\sigma_i)) \cdot (\sigma_j - \mu(\sigma_j))] \\ &= \mu[(\sigma_i - \mu(\sigma_i | \mathcal{F}) + \mu(\sigma_i | \mathcal{F}) - \mu(\sigma_i)) \cdot (\sigma_j - \mu(\sigma_j | \mathcal{F}) + \mu(\sigma_j | \mathcal{F}) - \mu(\sigma_j))] \\ &= \mu(\mu[(\sigma_i - \mu(\sigma_i | \mathcal{F})) \cdot (\sigma_j - \mu(\sigma_j | \mathcal{F})) | \mathcal{F}]) \\ & \quad + \mu[(\mu(\sigma_i | \mathcal{F}) - \mu(\sigma_i)) \cdot (\mu(\sigma_j | \mathcal{F}) - \mu(\sigma_j))] \end{aligned} \quad (14)$$

as the cross terms vanish. Conditioning on the sigma-algebra $\mathcal{F}$ puts us in the uniform Dobrushin-uniqueness regime, by virtue of our first theorem, and so we have

$$|\mu[(\sigma_i - \mu(\sigma_i | \mathcal{F})) \cdot (\sigma_j - \mu(\sigma_j | \mathcal{F})) | \mathcal{F}]| \leq K e^{-\gamma|i-j|} \quad (15)$$

uniformly in the choice of the conditioning, with a decay constant depending on the discretization width. Hence, there is power law behavior for the two-point functions of $\hat{\sigma}_i$, written in the last line in (14) if and only if there is power law behavior for the term in the first line. $\square$

The continuous symmetry of the original model is also inherited. This means that the discretization image of a rotated continuous spin Gibbs measure is again a Gibbs measure for the same discretized interaction. Moreover, if the original model displays a broken symmetry, which happens in three or more dimensions for the XY model, there is a continuum of discretized pure Gibbs measures arising in this way. Discretization images of pure measures are pure, since they inherit all ergodic properties for observables because only a smaller subset of observables needs to be considered. That a continuum of discretized measures results,

follows immediately from the fact that the direction of the magnetization can take a continuum of values, and that they occur with different probabilities. In other words, there is a probability density on these directions which cannot be constant in a magnetized state.

The continuum of Gibbs measures obtained in this way, however, are not all related to a broken symmetry of the discrete spin model.

We have thus proved the following theorem.

Theorem 2.4. *For each $d \geq 3$, there is a q_0 such that for $q \geq q_0$ there is an interaction Φ' with a discrete clock rotation invariance such that there are uncountably many translation-invariant ergodic states in the set of Gibbs measures $\mathcal{G}_{\Phi'}$ (taken with uniform a priori measures).*

This argument provides an independent rigorous route to the existence of an intermediate enhanced-symmetry Kosterlitz–Thouless phase in a discrete spin model, combining our general criteria for preservation of Gibbsianness under local coarse-grainings with properties of the original continuous-spin model.

3. Comparing the discretizations

3.1. Results at low temperatures

At high temperatures in the paramagnetic regime, everything is well behaved, but not of great physical interest. We will therefore discuss what happens in subcritical-temperature regimes.

It is a remarkable fact that the standard nearest-neighbor large- q clock model in two dimensions has the property that there is a Kosterlitz–Thouless phase with slow decay and an enhanced continuous symmetry occurs at an intermediate temperature regime [8, 9]. On symmetry enhancement, see also [20].

The values for which this occurs are such that q should be large enough for a given low temperature. As we have just seen, the discretized XY-model can be described by a summable interaction—in which the nearest-neighbor terms are the dominant ones—in just such an intermediate regime.

On the other hand, the nearest-neighbor clock model at fixed q and at very low temperatures ($\beta \geq O(q^2)$) will have q ordered phases, that is, q different Gibbs measures, similar to the q ground states, all with exponential correlation decay. This follows directly from a Pirogov–Sinai argument.

For the type-1 discretization, there is also a change once the temperature is low enough, but now the change is going from a Gibbsian to a non-Gibbsian regime.

Let us sketch how such an argument would proceed. We remind the reader that for a measure to be a Gibbs measure, each spin configuration needs to be a point where all conditional probabilities are continuous in the product topology as a function of the boundary condition, which translates into there being a summable (not too long range) interaction. It is thus sufficient for a proof on non-Gibbsianness to show that there is a single configuration which is a point of—essential—discontinuity. Conditional on this configuration, one has to show that the system has a first-order phase transition [4].

The main thing to show is this phase transition for the constrained system. Let us take q even. Indeed, if we take an alternating configuration for the discretized spin, this implies that alternatingly the spin is either in the most northern (on sites in one sublattice) or the most southern interval (on the other sublattice) of size $\frac{2\pi}{q}$. We argue that such a configuration is a point of essential discontinuity for a conditional probability of the discretized measure. Namely, conditional on this, the original spins (which are forced by the constraint on which we condition to be almost opposite, but by their interaction prefer to be pointing in the same

direction) will have two ground states, one pointing alternatingly north-west, south-west, and the other one alternatingly north-east and south-east. The deviations in either the western or the eastern, direction are of order $O(\frac{1}{q})$, which means that the energy gap between the two ground states is of order $O(\frac{1}{q^2})$. Therefore, at sufficiently low temperatures ($\beta \geq O(q^2)$), there will be two different Gibbs measures for the constrained model, and this will imply the non-Gibbsianness of the discretized measure. The details of the argument, in which the main step is proving the existence of a phase transition in a kind of alternating uniaxial field, can be worked out in a straightforward manner along the lines of [5, 6, 25], see also [3], where proofs for the same type of transition have been published.

Thus, the analogy between the two discretizations breaks down only in this very low temperature regime. The measures then are no longer close on the level of local observables. Thus, in this regime, one finds very different behavior, with one discretization resulting in a non-Gibbsian measure, and the other in q different Gibbs measures.

In higher dimensions, for the XY model there is a continuum of Gibbs measures [11] at low temperatures, which, as indicated above, are mapped to a continuum of different Gibbs measures for the discrete spins in an intermediate regime.

3.2. Analogies and conjectures

It would be interesting to see if the restoration of continuous symmetries which occurs for the two-dimensional clock model would have a higher-dimensional analog, i.e. in some intermediate-temperature regime there might exist a continuum of Gibbs measures, even for the nearest-neighbor clock model.

Conjecture 1. *There exists a continuum of massless Gibbs measures in an intermediate regime below the critical temperature for large- q clock models in three and more dimensions.*

We conjecture that the intermediate phase studied in [15, 23, 26, 27, 29, 30] might be of this type. In the terminology of Ueno *et al* [30], we would have a continuum of ‘incompletely ordered phases’, where the order can be in the two spin directions $n, n+1$, where $n \bmod q \in \{1, \dots, q\}$, with continuously varying weights of these directions. Although there seems to be some doubt whether an intermediate phase exists at all in a region inbetween the q ordered, ground-state-like, phases and the high-temperature paramagnetic phase, the numerical results for the nearest-neighbor clock model appear to be inconclusive thus far. It therefore seems worthwhile to investigate if an ‘enhanced-broken-symmetry phase’ as described above, which can be obtained by discretizing a continuous-spin model, could also occur for the nearest-neighbor clock model.

The breakdown of the analogy at very low temperatures holds for the same reason as in two dimensions. The only property we used in our sketched proof was the bipartiteness of the lattice.

As we have just seen, the transition between the Gibbsian behavior and the non-Gibbsian behavior occurs at $\beta \geq O(q^2)$ for type-1 discretizations.

Conjecture 2. *The intermediate regime which is conjectured to exist in conjecture 1 will occur once $\beta_c \lesssim \beta \leq O(q^2)$.*

The analysis of [8] for type-2 discretizations similarly appears to provide a transition value at $\beta = O(q^2)$ for the transition. For numerical results, showing these asymptotics in more detail, see [28].

A heuristic reason for this behavior is that the model in the scaled variables q times spin-angles with discretization width 1 approximates a discrete Gaussian model at effective inverse temperature β/q^2 ; if this parameter is below the value for the roughening transition (which is rigorously known to take place in the discrete Gaussian), the model behaves like a massless Gaussian, while above it behaves like a massive model in the Peierls regime. Compare also theorem C on p 40 of [10]. There it is mentioned as a conjecture that at fixed $q \geq 5$, the threshold values in temperature between the low temperature regime and intermediate regime on the one hand, and the intermediate and high temperature regime on the other hand, should be sharp and different. See also [1] for some numerical support for this, and the recent work [22] for analytical arguments to support this.

4. Conclusions

We have shown how to compare two different ways of discretizing spin models, either starting from the Gibbs measures, for which we have a controlled approximation in a temperature-dependent regime, or starting from the interactions. We extended the regime in which we have such a controlled approximation of the discretized Gibbs measure, and also found that it cannot be extended to very low temperatures. Thus, the results are essentially optimal. In the two-dimensional XY model, both discretizations display the same Kosterlitz–Thouless phase in an intermediate-temperature regime.

For the higher-dimensional case, we have suggested the possibility of an enhanced-continuous-symmetry-breaking phase occurring at a region of intermediate temperatures in discrete d -dimensional clock models for d at least 3.

After we posted the first version of our paper, independently of us, the existence of an intermediate massless regime for large- q clock models, as well as some of its putative properties and some non-equilibrium analogs, was conjectured in [19].

References

- [1] Brito A F, Redinz J A and Plascak J A 2010 Two-dimensional XY and clock models studied via the dynamics generated by rough surfaces *Phys. Rev. E* **81** 031130
- [2] Chazottes J-R and Ugalde E 2011 On the preservation of Gibbsianness under symbol amalgamation *Entropy of Hidden Markov Processes and Connections to Dynamical Systems* ed B Marcus, K Petersen and T Weissman (LMS Lecture Notes vol 385) (Cambridge: Cambridge University Press) pp 72–97
- [3] Crawford N 2011 On random field induced ordering in the classical XY model *J. Stat. Phys.* **142** 11–42
- [4] van Enter A C D, Fernández R and Sokal A D 1993 Regularity properties and pathologies of position-space renormalization-group transformations: scope and limitations of Gibbsian theory *J. Stat. Phys.* **72** 879–1167
- [5] van Enter A C D, Külske C, Opoku A A and Ruszel W M 2010 Gibbs–non-Gibbs properties for n -vector lattice and mean-field models *Braz. J. Probab. Stat.* **24** 226–55
- [6] van Enter A C D and Ruszel W M 2009 Gibbsianness vs. non-Gibbsianness of time-evolved planar rotor models *Stoch. Proc. Appl.* **119** 1866–88
- [7] Fernández R 2005 Gibbsianness and non-Gibbsianness in lattice random fields *Les Houches, LXXXIII* (Amsterdam: Elsevier) pp 731–99
- [8] Fröhlich J and Spencer T 1981 The Kosterlitz–Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas *Commun. Math. Phys.* **81** 527–602
- [9] Fröhlich J and Spencer T 1982 Massless phases and symmetry restoration in Abelian Gauge symmetries and spin systems *Commun. Math. Phys.* **83** 411–54
- [10] Fröhlich J and Spencer T 1983 The Berezinskii–Kosterlitz–Thouless transition *Scaling and Self-Similarity in Physics* ed J Fröhlich (*Progress in Physics*) (Basel: Birkhäuser)
- [11] Fröhlich J, Simon B and Spencer T 1976 Infrared bounds, phase transitions and continuous symmetry breaking *Commun. Math. Phys.* **50** 79–95

- [12] Georgii H-O 1988 *Gibbs Measures and Phase Transitions (de Gruyter Studies in Mathematics vol 9)* (Berlin: de Gruyter)
- [13] Häggström O 2003 Is the fuzzy Potts model Gibbsian? *Ann. Inst. H Poincaré B* **39** 891–917
- [14] Häggström O and Külske C 2004 Gibbs properties of the fuzzy Potts model on trees and in mean field *Markov Proc. Rel. Fields* **10** 477–506
- [15] Heilmann R K, Wang J S and Swendsen R B 1996 Rotationally symmetric ordered phase in the three-state antiferromagnetic Potts model *Phys. Rev. B* **53** 2210
- [16] Kozlov O K 1974 Gibbs description of a system of random variables *Problem Inform. Transm.* **10** 258–65
- [17] Külske C, Le Ny A and Redig F 2004 Relative entropy and variational properties of generalized Gibbsian measures *Ann. Probab.* **32** 1691–726
- [18] Külske C and Opoku A A 2008 The posterior metric and the goodness of Gibbsianness for transforms of Gibbs measures *Electron. J. Probab.* **13** 1307–44
- [19] Maes C and Shlosman S B 2011 Rotating states in driven clock- and XY-models *J. Stat. Phys.* **144** 1238–44
- [20] Newman C M and Schulman L S 1982 Asymptotic symmetry: enhancement and stability *Phys. Rev. B* **26** 3910–4
- [21] Opoku A A 2009 On Gibbs properties of transforms of lattice and mean-field systems *Groningen Thesis*
- [22] Ortiz G, Cobanera E and Nussinov Z 2012 Dualities and the phase diagram of the p-clock model *Nucl. Phys. B* **854** 780–814
- [23] Oshikawa M 2000 Ordered phase and scaling in Z_n models and the three-state antiferromagnetic Potts model in three dimensions *Phys. Rev. B* **61** 3430–4
- [24] Potts R B 1952 Some generalized order-disorder transformations *Math. Proc. Camb. Phil. Soc.* **48** 106–9
- [25] Ruszel W M 2010 Gibbs and non-Gibbs aspects of continuous spin models *Groningen Thesis*
- [26] Todoroki N, Ueno Y and Miyashita S 2002 Ordered phase and phase transitions in the three-dimensional generalized six-state clock model *Phys. Rev. B* **66** 214405
- [27] Scholten P D and Irakliotis L J 1993 Critical behavior of the q-state clock model in three dimensions *Phys. Rev. B* **48** 1291–4
- [28] Tomita Y and Okabe Y 2002 Probability-changing cluster algorithm for two-dimensional XY and clock models *Phys. Rev. B* **65** 184405
- [29] Ueno Y 1995 Description of ordering and phase transition in terms of local connectivity: proof of a novel type of percolated state in the general clock model *J. Stat. Phys.* **80** 843–70
- [30] Ueno Y and Kasono K 1993 Incompletely ordered phases and phase transitions in the three-dimensional general clock model *Phys. Rev. B* **48** 16471
- [31] Verbitskiy E A 2010 Variational principle for fuzzy Gibbs measures *Moscow Math. J.* **10** 811–29