Fault Detection Filter for Discrete-Time Markov Jump Lur’e Systems

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Abstract—We present the design of $\mathcal{H}_\infty$ Fault Detection Filter (FDF) for Discrete-time Markov Jump Lur’e Systems with bounded sector condition based on the use of Linear Matrix Inequality (LMI). A numerical example is presented to illustrate the effectiveness of the proposed approach.

I. INTRODUCTION

An inherent characteristic of any engineering systems is that they are always subjected to the occurrence of faults which, in most cases, limit their lifetime. Consequently there have been continuous research activities that focus on the detection and active prevention of such problems [1], [2]. The importance of which has received renewed attention in recent years with the interest of many industries on predictive maintenance and prescriptive maintenance. In these cases, the costly regular preventive maintenance procedure is being reconsidered by re-planning such maintenance in a timely manner that requires the technology on the detection of the presence of faults and on the monitoring thereof.

Among many different methods for solving the aforementioned problems, the Fault Detection and Isolation (FDI) stands out as a reliable approach due to the incorporation of prior mechanistic knowledge on the dynamical model of the control systems (i.e., the plant and controller) [3], which is deemed to be crucial by many industries where safety-critical systems are deployed and all physical insights must be taken into account in any decision making processes. The FDI approach uses a Fault Detection Filter (FDF) to generate a residue signal, which is subsequently used to determine the presence of fault [4], [5].

In any control systems, a fault may occur in various places of the closed-loop systems, for example, it can emerge as an actuator, sensor or communication network fault, or it appears as a structural problem [6], [7], [8]. The ability to properly model all of these different aspects allows us to specifically distinguish between its nominal behavior and a faulty one; decreasing the occurrence of false alarms.

In this paper, we focus on the design of Fault Detection Filter (FDF) for Lur’e systems, where the presence of nonlinear elements in the closed-loop systems is represented in the feedback loop. Such nonlinear elements capture many practical situations in engineering systems, such as, the presence of dead-zones, saturation, hysteresis and many others [9], [10]. Linearization approach in this case is only valid when the closed-loop systems operate close to the nominal operating point. The design of FDF based on the linear approximation of the system can therefore lead to unreliable and imprecise detection. Correspondingly, we consider the design of FDF for the discrete-time Markov Jump Lur’e System as studied in [11]. This modeling framework allows us to capture both abrupt changes in the systems as well as the presence of nonlinear static element in the feedback loop. Using this framework, our proposed FDF design method can distinguish (actuator, sensor or structural) fault from a network problem or from a nonlinear behavior.

There are a number of related works in the literature. In [12], the authors study the fault estimation problem for a class of switched nonlinear systems of neutral type with nonlinearity satisfying global Lipschitz condition. The paper [13] presents the design of an observer-based asynchronous fault detection filter under a nonlinear Markov jump systems framework with conic-type nonlinearities. The paper [14] uses a geometric approach for discrete-time systems to design an FDI procedure, and the paper [15] investigates the problem of distributed $\mathcal{H}_\infty$ filtering for a class of discrete-time Markov jump Lur’e systems over sensor networks with stochastic switching topologies. The work in [16] focuses on the design of an event triggered robust fault detection for nonlinear uncertain MJS under the assumption of time-varying delays. The work in [17] proposes the design of FDF for nonlinear switched stochastic systems in the Takagi-Sugeno fuzzy systems.

As far as the authors are aware of, the FDF design for Markov Jump Lur’e systems with bounded sector nonlinearity is still not well-studied in literature and is the main contribution of this paper. Particularly, we provide a design method for solving guaranteed cost $\mathcal{H}_\infty$ FDF problem based on Linear Matrix Inequalities (LMI). In the classical discrete-time domain Lur’e systems with bounded sector condition assumption, it is common to assume the slope-restriction and monotonic behavior [18], which we will relax in this paper.

This work is organized as follows. Section II gives the preliminaries and some relevant background information of the underlying systems. Section III presents the FDF problem formulation and our main contributions. Section IV provides a numerical example to illustrate the efficacy of our proposed
A. Notations

The real n-dimensional Euclidian space is denoted by \( \mathbb{R}^n \). The transpose of a matrix is indicated by the symbol \( (\cdot)^T \), the symbol \( I \) denotes the identity matrix. A positive semi-definite and positive definite matrix \( P \) is denoted by \( P \geq 0 \) and \( P > 0 \), respectively. The symmetric sum is represented by the operator \( \text{Tr}(\cdot) \). The operator \( \text{diag}(\cdot) \) represents a diagonal matrix. The trace of a square matrix is denoted by \( \text{Tr}(\cdot) \). A symmetric block in matrix is denoted by \( \text{Tr}(\cdot) \).

On the probabilistic space \( (\Theta, \mathcal{F}, \mathbb{P}) \), the set of signals \( w(k) \in \mathbb{R}^n \), such that \( w(k) = \mathcal{F}_k \) measurable, for all \( k \in \mathbb{N}^+ \) and \( \|w\|_2 < \infty \) is indicated by \( L^2 \). The expected value operator is represented by \( \mathbb{E}(\cdot) \), the conditional expected operator, by \( \mathbb{E}(\cdot \mid \cdot) \). The space-state of the Markov chain is denoted by \( \mathbb{K} = \{1, 2, \ldots, N\} \). The transition matrix is described as \( \mathbb{P} = [\rho_{ij}] \), and must obey the following statements: 

1. \( \rho_{ij} \geq 0 \); ii. \( \rho_{ij} = \mathbb{P}(\theta(k + 1) = j|\theta(k) = i) \); and iii. \( \sum_{j=1}^{N} \rho_{ij} = 1 \), for all \( i \in \mathbb{K} \).

The expected value of \( X_j \) is denoted by \( \mathbb{E}_i(X) = \sum_{j=1}^{N} \rho_{ij}X_j \).

II. Preliminaries

Consider a discrete-time Markov jump Lur’e system \( \mathcal{G} \) given by

\[
\begin{align*}
    x(k + 1) &= A_{\theta(k)}x(k) + G_{\theta(k)}\varphi_{\theta(k)}(p(k)) + J_{\theta(k)}w(k), \\
    p(k) &= C_{\theta(k)}x(k), \\
    z(k) &= C_{z\theta(k)}x(k) + H_{\theta(k)}\varphi_{\theta(k)}(p(k)) + D_{\theta(k)}w(k),
\end{align*}
\]

(1)

where vectors \( x(k) \in \mathbb{R}^{n_x} \), \( p(k) \in \mathbb{R}^{n_p} \), \( z(k) \in \mathbb{R}^{n_z} \), and \( w(k) \in \mathbb{R}^{n_w} \), represent the system states, the output connected to the nonlinear element, the system output and the exogenous input, respectively. Consider that \( \{w(k)\}_{k \in \mathbb{N}^+} \in L^2 \) and the matrix \( C_{zi} \) has full column rank; hence there exists \( d > 0 \) such that \( C_{zi}^T C_{zi} > dI \) for all \( i \in \mathbb{K} \). The term \( \varphi_{\theta(k)}(p(k)) \) is considered to be a mode-dependent memoryless non-linearity. Observe that all the terms in (1) are dependent on the index \( \theta(k) \), which represents the Markov chain jump parameter [19].

The \( \mathcal{N} \) non-linearities \( \varphi_{\cdot}(\cdot) \) are characterized by the following assumptions:

- Assumption I: \( \varphi_{i}(0) = 0 \)
- Assumption II: for each non-linearity there exist positive definite matrices \( \Omega_i \in \mathbb{R}^{n_p \times n_p} \) for all \( p \in \mathbb{R}^{n_p} \), \( i = 1, \ldots, n_p \), such that

\[
    \varphi_{i}(0)\Omega_i p = \Omega_i p = \varphi_{i}(0)\Omega_i p \leq 0.
\]

(2)

Accordingly, as described in [9], the non-linearities \( \varphi_{\theta(k)}(\cdot) \) satisfy their respective cone bounded sector conditions and are decentralized, which can be expressed as

\[
    \text{SC}(\varphi_{\cdot}(\cdot), p, \Lambda_i) := \varphi_{i}(p)\Lambda_i[p] \leq 0,
\]

(3)

where \( \Lambda_i \in \text{diag}(\lambda_{i1}, \ldots, \lambda_{in_p}) \in \mathbb{R}^{n_p \times n_p} \) are diagonal positive semidefinite matrices. From (3) one has that (2) holds for all \( i \in \mathbb{K} \) and for all \( p \in \mathbb{R}^{n_p} \). As a byproduct of (2) the inequality (3) holds for

\[
    [\Omega_i p](\varphi_{i}(p) - \Omega_i p) \leq 0,
\]

(4)

which implies that

\[
    0 \leq \varphi_{i}(p)\Lambda_i \varphi_{i}(p) \leq \varphi_{i}(p)\Lambda_i \Omega_i p \leq p'\Omega_i \Lambda_i \Omega_i p,
\]

(5)

for all \( p \in \mathbb{R}^{n_p} \).

Definition 1: [11] For the system given by (1) with \( w = 0 \), the equilibrium point at the origin is called \textit{stochastically stable} if for every initial state \( (x(0), \theta(0)) \),

\[
    \|x\|^2 \leq \sum_{k=0}^{\infty} \mathbb{E}||x(k)||^2 < \infty
\]

(6)

A. Candidate Lyapunov function

Let us define the following candidate Lyapunov function for system (1)

\[
    V : \{i, x\} \rightarrow \mathbb{R}, \\
    (i, x) \rightarrow x'P_i x + 2(\varphi_{i}(C_i x))\Delta_i \Omega_i C_i x,
\]

(7)

where, for all \( i \in \mathbb{K} \), the matrix \( P_i \in \mathbb{R}^{n_x \times n_x} \) is symmetric positive definite and the diagonal matrix \( \Delta_i \in \mathbb{R}^{n_p \times n_p} \) is positive definite. Observe that inequality (5) allows us to define a lower bound, as in \( \underline{v}_i(x) = x'P_i x \), and an upper bound, \( \underline{v}_i(x) = x'(P_i + 2C_i'\Omega_i \Delta_i \Omega_i C_i)x \), for the function \( V \) as follows

\[
    \underline{v}_i(x) \leq V(i, x) \leq \bar{v}_i(x), \forall i \in \mathbb{K}.
\]

(8)

Thus the above candidate Lyapunov function \( V(i, x) \) satisfies the following properties:

- \( V(i, x) \geq 0, \forall x \in \mathbb{R}^{n_x}, \forall i \in \mathbb{K} \);
- \( V(i, x) = 0 \) if and only if \( x = 0 \), \( \forall i \in \mathbb{K} \); and
- \( V(i, x) \) is radially unbounded, \( \forall i \in \mathbb{K} \).

Correspondingly, under the Assumptions I and II, we can get rid of the nonlinearity slope condition as commonly adopted in the various results for the discrete-time domain Lur’e system [20], [11], [21]. Throughout the rest of the paper, we set for simplicity \( V(k) := V(\theta(k), x(k)) \).

B. LMI constraint for the guaranteed cost \( \mathcal{H}_\infty \)

Assuming that the system (1) is MSS and \( x_0 = 0 \). Its \( \mathcal{H}_\infty \) norm [22] is then given by

\[
    \|\mathcal{G}\|_\infty = \sup_{\mathcal{F}_2, \theta_0 \in \mathbb{K}} \frac{\|z\|_2}{\|w\|_2}.
\]

(9)

An upper bound \( \gamma > 0 \) for the \( \mathcal{H}_\infty \) norm can be acquired by using the following lemma which is based on the stochastic stability constraints presented in [11, Theorem 5].

Lemma 1: Consider that the Assumptions I and II are satisfied. System (1) is stochastic stable and the norm constraint \( \|\mathcal{G}\|_\infty \leq \gamma \) holds if there exist symmetric \( P_i > 0 \) and...
diagonal positive semidefinite matrices $T_i, W_i, \Delta_i$ such that the constraint (10) is satisfied for all $i \in \mathbb{K}$

$$
\begin{bmatrix}
    (W_i - \Delta_i) \Omega_i C_i & 2T_i & \cdots & \cdots & \cdots & \cdots \\
    \Omega_i (E_i(W)) - \Omega_i C_i & T_i & \cdots & \cdots & \cdots & \cdots \\
    \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
    0 & \cdots & \cdots & \cdots & 2E_i(W) & \cdots \\
    0 & 0 & \cdots & \cdots & \cdots & 0 \\
    \tilde{P} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix} \geq 0.
$$

(10)

where $\tilde{P} = (E_i(W) - \Omega_i \Delta_i) \Omega_i C_i$. $\tilde{P} = J_i C_i' \Omega_i (E_i(W) - \Omega_i \Delta_i)$.

Proof: Let us show that if there are matrices $P_i > 0$ such that (10) is satisfied then $\|G\|_{\infty} < \gamma$. Pre- and post-multiplying (10) by $diag(I, I, I, I, I, E_i(P)^{-1})$ and applying Schur complement in (10), we get that

$$
\begin{bmatrix}
    A'_i(E_i(P)) A_i - P_i + C_i' C_i \\
    \tilde{G}_i(E_i(W)) A_i + H_i H_i - 2T_i \\
    \tilde{G}_i(E_i(W)) C_i - 2E_i(W) \\
    J_i(E_i(P)) A_i - D_i C_i \\
    J_i(E_i(P)) C_i + D_i' H_i \\
    \tilde{P} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix} \geq 0.
$$

(11)

where $\tilde{P} = G_i'(E_i(W)) A_i + \Delta_i \Omega_i C_i - T_i \Omega_i C_i + H_i' C_i$. $\tilde{P} = J_i C_i' \Omega_i (E_i(W) - \Omega_i \Delta_i)$. $\tilde{P} = J_i(\tilde{G}_i(E_i(P)) P_i + D_i' D_i - \gamma^2$. Pre- and post-multiplying (11) by $[x(k)^T \varphi_i(p(k))]\varphi_i[p(k + 1)]w(k)]$, and following a routine computation, we obtain

$$
x(k + 1)^T \tilde{G}_i(x(k)(x(k + 1) + 1)^T \tilde{G}_i(x(k + 1)

+ 2\varphi_i(p(k + 1))^T \Delta_i \varphi_i(p(k + 1))(x(k + 1)

+ x(k)^T P_i x(k) + 2\varphi_i(p(k))^T \Delta_i \varphi_i(p(k))(x(k + 1)

- 2SC(\varphi_i(p(k)), p(k), T_i) + x(k)^T \tilde{G}_i(x(k) - \gamma^2 w(k)^T w(k) \leq 0.

(12)

Considering that the $\sigma$-field $\mathcal{F}_k$ is generated by the variables $\{x(l), w(l), \theta(l); l = 0, \cdots, k\}$ we get that $\mathbb{E}(x(k + 1)|\mathcal{F}_k) = \mathbb{E}(x(k + 1)|P_i x(k + 1)|\mathcal{G}_i)$. Hence $\mathbb{E}(x(k + 1)|\mathcal{F}_k) = \mathbb{E}(x(k + 1)|P_i x(k + 1)|\mathcal{G}_i)$. In what follows, we recall that SC(,) \leq 0 as in (5). From (12), and summing over $k$ from 0 to $T$, we get

$$
\sum_{k=0}^{T} \mathbb{E}

\begin{bmatrix}
    x(k + 1)^T P_i x(k + 1)

+ 2\varphi_i(p(k + 1))^T \Delta_i \varphi_i(p(k + 1))(x(k + 1)

- x(k)^T P_i x(k) \\
- 2\varphi_i(p(k))^T \Delta_i \varphi_i(p(k))(x(k + 1)

- 2SC(\varphi_i(p(k)), p(k), T_i) + x(k)^T \tilde{G}_i(x(k) - \gamma^2 w(k)^T w(k) \leq 0.

(13)

It follows then that

$$
\mathbb{E}(V(T + 1)) - \mathbb{E}(V(T)) + \sum_{k=0}^{T} \mathbb{E}(\|z(k)^2\|) \\
- \gamma \sum_{k=0}^{T} \mathbb{E}(\|w(k)^2\|) \leq 0.
$$

(13)

Considering $w(k) = 0$ and recalling that $C_{z_i} C_{z_i} > 0$ we obtain from (13) that $\sum_{k=0}^{T} \mathbb{E}((x(k)^2) \leq \frac{1}{\gamma^2} \mathbb{E}(V(0))$ and taking the limit as $T \to \infty$ yields the stochastic stability property. When $x_0 = 0$, it follows from (13) that $\sum_{k=0}^{T} \mathbb{E}((x(k)^2) - \gamma \sum_{k=0}^{T} \mathbb{E}(\|w(k)^2\|) \leq 0$. By taking the limit $T \to \infty$, we obtain the desired result.

III. FAULT DETECTION FILTER DESIGN

Our proposed scheme of Fault Detection Filter is presented in Fig. 1. The Markov Jump Lur’e systems, which are

subjected to faults, are described by

$$
\begin{align*}
G : & \quad x(k + 1) = A_{\theta(k)} x(k) + B_{\theta(k)} u(k) + G_{\theta(k)} \varphi_0(p(k)) + D_{\theta(k)} w(k) + E_{\theta(k)} f(k) \\
& \quad y(k) = C_{\theta(k)} x(k) + H_{\theta(k)} \varphi_0(p(k)) \\
& \quad f(k) \in \mathbb{E}_2, \forall k \in \mathbb{N}^+. \text{We assume that a control law that stabilizes system} \\
& \quad (14)
\end{align*}
$$

where $x(k) \in \mathbb{R}_{\theta}^n$ represents the system states, $u(k) \in \mathbb{R}_{\theta}^n$ is the control input, $p(k) \in \mathbb{R}_{\theta}^n$ is the output connected to the nonlinear element, $w(k) \in \mathbb{R}_{\theta}^n$ denotes the exogenous input, $y(k) \in \mathbb{R}_{\theta}^n$ is the measurement signal, and $f(k) \in \mathbb{R}_{\theta}^n$ denotes the fault signal. We assume that $u(k), f(k) \in \mathbb{L}_2$. We assume that a control law that stabilizes system

$$
u(k) = K_{\theta(k)} x(k) + R_{\theta(k)} \varphi_0(p(k)),
$$

which can be designed using various control design methods, such as the ones presented in [21]. As described before, we aim to design a FDF given by

$$
\begin{align*}
\eta(k + 1) &= A_{\eta(k)} \eta(k) + M_{\eta(k)} u(k) + B_{\eta(k)} y(k) + E_{\eta(k)} \varphi_0(p(k)) \\
r(k) &= C_{\eta(k)} \eta(k) + D_{\eta(k)} y(k) \\
\eta(0) &= \eta_0
\end{align*}
$$

(16)

where $\eta(k) \in \mathbb{R}_{\theta}^n$ represents the filter states, and $r(k) \in \mathbb{R}_{\theta}^n$ denotes the residue signal. Considering that $r_e(k) = r(k) - f(k), we get the augmented system given by

$$
\begin{align*}
\tilde{x}(k + 1) &= \tilde{A}_{\theta(k)} \tilde{x}(k) + \tilde{C}_{\theta(k)} \varphi_0(p(k)) + \tilde{J}_{\theta(k)} \tilde{w}(k) \\
r_e(k) &= \tilde{C}_{\theta(k)} \varphi_0(p(k)) + \tilde{H}_{\theta(k)} \varphi_0(p(k)) + \tilde{D}_{\theta(k)} \tilde{w}(k)
\end{align*}
$$

(17)
where \( \tilde{x}(k) = [x(k) \eta(k)] \), \( \tilde{\varphi}_{\theta}(p(k)) = \varphi_{\theta}(p(k)) \), \( \tilde{w}(k) = [w(k) f(k)] \) and the augmented matrices are

\[
\begin{align*}
\tilde{A}_i &= \begin{bmatrix} A_i + B_i K_i & 0 \\
M_{q_i} R + C_{q_i} N_{q_i} & 0 \end{bmatrix}, & \tilde{G}_i &= \begin{bmatrix} B_i R_i + G_i \\
M_{q_i} R + C_{q_i} \end{bmatrix}, \\
\tilde{J}_i &= \begin{bmatrix} J_i \\
D_{q_i} D_i + B_i E_i \end{bmatrix}, & \tilde{C}_{z_i} &= \begin{bmatrix} D_{q_i} C_i + C_{q_i} \end{bmatrix}, \\
\tilde{D}_i &= \begin{bmatrix} D_{q_i} D_i + D_{z_i} E_i \end{bmatrix}, & \tilde{H}_i &= \begin{bmatrix} D_{q_i} H_i \end{bmatrix}, & \tilde{C}_i &= \begin{bmatrix} C_i \ 0 \end{bmatrix},
\end{align*}
\]

which satisfies the upper bound \( \gamma = 0.92 \).

A. DFD design for MJ Lur’e systems with guaranteed cost \( \mathcal{H}_\infty \)

**Theorem 1:** Consider that both Assumptions I and II are satisfied. There exists a filter in (16) such that (17) is stochastic stable and \( \|G_{\text{aug}}\|_\infty \leq \gamma \) if there exist symmetric positive matrices \( Z_i, X_i \), matrices with appropriate size \( O_{q_i}, \Gamma_i, \Gamma_i, \Gamma_i, \Gamma_i, \Gamma_i, \Gamma_i \), and diagonal positive semidefinite matrices \( T_i, W_i, \Delta_i \in \mathbb{R}^{n_i \times n_i} \) such that the LMI constraints (20) are satisfied for all \( i \in \mathbb{K} \)

\[
\begin{align*}
\Pi_1 &= (E_i(W) - E_i(\Delta))\Omega_i C_i, & \Pi_2 &= (E_i(W) - E_i(\Delta))\Omega_i C_i, \\
\Pi &= J_i^T C_i \Omega_i (E_i(W) - E_i(\Delta)), & \Pi &= J_i^T C_i \Omega_i (E_i(W) - E_i(\Delta)),
\end{align*}
\]

If a feasible solution is obtained, then a suitable FDF is given by \( A_{q_i} = E_i(Z - X)^{-1} O_{q_i}, B_{q_i} = E_i(Z - X)^{-1} \Gamma_i, M_{q_i} = E_i(Z - X)^{-1} \Gamma_i, C_{q_i} = E_i(Z - X)^{-1} \Gamma_i, C_{q_i}, \) and \( D_{q_i}. \)

**Proof:** Firstly, we introduce variable substitutions \( O_{q_i} = E_i(Z - X) A_{q_i}, \Gamma_i = E_i(Z - X) B_{q_i}, \Gamma_i = E_i(Z - X) C_{q_i}, \) and \( Y_i = E_i(Z - X) C_{q_i} \) in (20). Now consider the structure, extracted from (23), for \( P_i, E_i(W), \) as

\[
\begin{align*}
P_i &= \begin{bmatrix} X_i & U_i & X_i \end{bmatrix}, & P_i^{-1} &= \begin{bmatrix} Y_i & V_i & Y_i \end{bmatrix}, \\
E_i(W) &= \begin{bmatrix} E_i(X_i) & E_i(U_i) \\
E_i(U_i)' & E_i(W) \end{bmatrix}, & E_i(W)^{-1} &= \begin{bmatrix} R_{1_i} & R_{2_i} \\
R_{2_i} & R_{3_i} \end{bmatrix}.
\end{align*}
\]

We define the matrices \( \alpha_i \) and \( \sigma_i \) as

\[
\begin{align*}
\alpha_i &= \begin{bmatrix} I & \ 0 \\
V_i & Y_i^{-1} \end{bmatrix}, & \sigma_i &= \begin{bmatrix} R_{1_i} E_i(X_i) \\
0 \ 0 \end{bmatrix}.
\end{align*}
\]

From (21) we get that \( U_i = Z_i - X_i, \ V_i = V_i, \ V_i = Z_i^{-1}, \) as well as \( R_{1_i}^{-1} = E_i(Z) \) in (24). Therefore, we can rewrite the following matrices

\[
\begin{align*}
\alpha_i^T \varphi_{\theta}(p) \alpha_i &= \begin{bmatrix} Z_i & Z_i \end{bmatrix}, & \sigma_i^T \varphi_{\theta}(p) \sigma_i &= \begin{bmatrix} E_i(Z) & E_i(Z) \end{bmatrix}, \\
(E_i(W) - E_i(\Delta))O_i C_i &= \begin{bmatrix} E_i(W) - E_i(\Delta) \Omega_i C_i, \ E_i(W) - E_i(\Delta) \Omega_i C_i \end{bmatrix}, \\
(E_i(W) - E_i(\Delta))O_i A_i &= \begin{bmatrix} E_i(W) - E_i(\Delta) \Omega_i C_i, \ E_i(W) - E_i(\Delta) \Omega_i C_i \end{bmatrix}, \\
C_{z_i} &= \begin{bmatrix} D_{q_i} C_{z_i} + C_{q_i} \end{bmatrix}, & \sigma_i &= \begin{bmatrix} R_{1_i} E_i(X_i) \\
0 \ 0 \end{bmatrix}, \quad R_{1_i}^{-1} = E_i(Z).
\end{align*}
\]

We assume the nonlinearity given by \( \varphi(p) = \Omega_i(p^3), i \in [1, 2] \). The noise signal is assumed to be a white noise, with zero mean and standard deviation of 0.1. The FDF obtained using Theorem 1 is given by

\[
\begin{align*}
\hat{A}_{q_1} &= \begin{bmatrix} -0.0097 -0.1416 \\
0.0091 0.0012 \end{bmatrix}, & \hat{A}_{q_2} &= \begin{bmatrix} -0.0257 -0.2720 \\
0.0007 0.0374 \end{bmatrix}, \\
B_{q_1} &= \begin{bmatrix} 1.0522 105.2372 \\
0.0165 0.0713 \end{bmatrix}, & B_{q_2} &= \begin{bmatrix} 2.0240 2.0240 \\
0.0172 0.0172 \end{bmatrix}, \\
M_{q_1} &= \begin{bmatrix} 9.6749 -0.0034 \\
-0.0034 9.7246 \end{bmatrix}, & M_{q_2} &= \begin{bmatrix} 106.1402 9.7246 \\
-0.0034 9.7246 \end{bmatrix}, \\
L_{q_1} &= \begin{bmatrix} 0.0362 \end{bmatrix}, & L_{q_2} &= \begin{bmatrix} 0.0362 \end{bmatrix}, \\
C_{q_1} &= \begin{bmatrix} 10^{-5} \end{bmatrix}, & C_{q_2} &= \begin{bmatrix} 10^{-5} \end{bmatrix}, \\
D_{q_1} &= \begin{bmatrix} -0.1711 -0.1893 \\
0.0170 -0.1342 \end{bmatrix}, & D_{q_2} &= \begin{bmatrix} 10^{-5} \end{bmatrix}, \\
\end{align*}
\]

which satisfies the upper bound \( \gamma = 0.92 \).
A. Evaluation function and threshold

Another important step in the fault detection procedure is the definition of the evaluation function, denoted by Eval(k), and the threshold TH. The reliability and good performance of a fault detection process is decisively connected to a proper choice of Eval(k), and TH. In this simulation we choose a well-know setup, as in [25], [26], [27], [28]. The evaluation function Eval(k) is given, for some positive integer L, by

\[
\text{Eval}(k) = \sum_{i=k-L}^{k} r(i)r(i). \tag{26}
\]

For the sake of simplicity we consider that the threshold is given by \( \text{TH} = 2 \). We consider that a fault occurred when the evaluation function Eval(k) surpasses the threshold TH.

Remark: There are several ways to define the evaluation function Eval(k) and the threshold TH. However, since it is not the focus of this paper to describe this part of the FDI process thoroughly, we refer interested readers to [26], [29], [25].

B. Simulation setup

Observing the matrices of the system (25), we are considering that the faults represent problems at the actuator. The specific fault signal is simulated by the actuator performance drops by 10\% starting at \( t = 125s \). A Monte Carlo simulation with 300 iterations was performed, and the results are presented in Figs. 3 - 4, which represent the residue signal and the evaluation function, respectively.

In Fig. 3, it can be observed that the FDF designed using Theorem 1 properly reacted to the fault signal as designed. Regarding the residue signal without fault in Fig. 3, when there is no fault signal the residue is close to zero for the entire simulation. It is not completely zero due to the presence of the noise signal \( w(k) \) and to the switching behavior from Markov Jump Systems.

Fig. 4 presents the evaluation function that is represented by the mean and standard deviation. It can be seen from this figure that the designed FDF is able to detect the fault in all cases within the range of \([127-132]\)s. It shows that the designed FDF provides a satisfactory level of reliability. The above simulation results show that the proposed method can provide a feasible solution for the fault detection problem.

V. CONCLUSION

In this paper, we solve a fault detection filter problem for discrete-time Markov Jump Lur’e systems with bounded sector nonlinearity. We present a design of \( \mathcal{H}_\infty \) FDF for such systems via LMI constraints. This particular formulation allows us to relax assumptions on the slope of nonlinearities as commonly considered in previous related works. Our current on-going extension of the work is on the inclusion of an asynchronous observation of the Markov chain as considered previously in [30], [31].

REFERENCES

(a) Evaluation function obtained using the FDF designed via Theorem 1.

(b) The mean of the evaluation function for the simulation without fault.

(c) The mean of the evaluation function for all Theorem 1, simulation without fault, and the Threshold TH.

Fig. 4: The mean and standard deviation of the evaluation function.


