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Notch Filters for Port-Hamiltonian Systems

D. A. Dirks, J. M. A. Scherpen, A. J. van der Schaft, and M. Steinbuch

Abstract—Many powerful tools exist for control design in the frequency domain, but are theoretically only justified for linear systems. On the other hand, nonlinear control deals with control design methodologies that are theoretically justified for a larger and more realistic class of systems, but primarily dealing with stability and to a lesser extent with performance. In this technical note a standard linear notch filter is modeled in the port-Hamiltonian (PH) framework, thereby proving that the notch filter is a passive system. The notch filter can then be interconnected with any other (nonlinear) PH system, while preserving the overall passivity property. By doing so, we can combine a frequency-based control method to improve performance, the notch filter, with the nonlinear control methodology of passivity-based control.

Index Terms—Control design, nonlinear control systems, nonlinear dynamical systems.

I. INTRODUCTION

Network modeling of lumped-parameter physical systems naturally leads to a geometrically defined class of systems, i.e., port-Hamiltonian (PH) systems [15], [20]. The PH modeling framework describes a large class of (nonlinear) systems including passive mechanical systems, electrical systems, electromechanical systems, mechanical systems with nonholonomic constraints, thermal systems and distributed parameter systems with boundary control [16]. The popularity of PH systems can be largely accredited to its application for analysis and control design of multi-input multi-output (MIMO) physical systems, as shown in [3], [6], [16]–[18], [20] and many others. Control can be realized by energy-shaping and damping injection.

Most nonlinear control methods, including the PH framework, however have the disadvantage of not including any frequency information in the modeling and control design. In industrial applications the frequency-based control methods are preferred because of the importance of the system performance in the frequency domain. In the frequency domain transfer functions, Nyquist/Bode plots, PID controllers, lag-lead compensators and filters are some examples of powerful tools for analysis and control [5]. However, such methods are theoretically only justified for linear systems, while physical systems can be linear and nonlinear. With the modern technological advances systems are becoming more complex, with higher demands, and linearization alone for control design of nonlinear systems is not always enough. However, as mentioned before, performance in the frequency domain is often ignored. The literature on including frequency meth-

ods into nonlinear control is rather limited. For some nonlinear systems the *describing function analysis* [23] method can be used to approximately analyze and predict nonlinear behavior. In [12] a dissipativity framework is presented for power system stabilizer design. The framework is based on including frequency dependent weights into the dissipation inequality. Frequency can then be included into the study of internal stability and disturbance attenuation of power systems. In [19], a frequency domain perspective on feedforward friction compensation of nonlinear single-input single-output (SISO) systems is provided.

The main contribution of this technical note is to include frequency filtering into nonlinear control design by describing a standard notch filter in the PH framework. In a similar way we also present a PH description of the inverse notch filter. In control applications a notch filter is often used for control of systems with lightly damped flexible modes (resonance frequencies) [5]. A notch filter is a filter that passes signals of all frequencies except those in a stop band centered around a center frequency. Furthermore, by a specific choice for the notch parameters it is possible to realize an *inverse* notch filter. An inverse notch filter does the opposite of a notch filter, it only passes signals which are centered around a center frequency. Inverse notch filters are usually used for disturbance attenuation since high feedback gain is applied at the disturbance frequency [24].

Since the notch filter is passive (implied by the PH structure), the passive interconnection between PH plant and notch is also passive and has the PH structure. We can then combine nonlinear control methodologies, like passivity based-control (PBC) [14], [20] with a notch filter, which is tuned to reduce (or increase in the case of an inverse notch) the feedback gains at a specific frequency. Compared to [12] we deal with a larger class of systems and we describe the dynamics of the notch filter which is tuned to have a specific frequency behavior. In [10], the internal model principle [4] is generalized to nonlinear systems. Compared to [10], we deal with the specific class of PH systems. More results on nonlinear output regulation and nonlinear internal models can be found in, e.g., [1], [2], [11], [21], and [22], however limited to SISO systems. The results of this technical note can be applied to nonlinear MIMO systems.

Section II briefly summarizes the PH modeling framework and PBC. In Section III we present the PH models for both a notch filter and the inverse notch filter. An inverse notch filter is simply a normal notch filter with a different choice of parameters. However, in the PH framework this is not so straightforward. For this reason a different PH model is presented for the inverse notch filter. Furthermore, in Section IV the disturbance attenuation properties of the inverse notch filter for nonlinear systems are analyzed. In Section V PBC is applied for control of a nonlinear mechanical system, while the inverse notch filter is used for disturbance attenuation. Final remarks are then given in Section VI.

II. PORT-HAMILTONIAN MODELING AND CONTROL

In the PH framework [15], a nonlinear system is described by

$$\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)(u + d), \quad y = g^T(x) \frac{\partial H}{\partial x}(x) \quad (1)$$

with $x \in \mathbb{R}^n$, $J(x) = -J^T(x) \in \mathbb{R}^{n \times n}$ the interconnection matrix, $R(x) = R^T(x) \in \mathbb{R}^{n \times n}$ positive-semidefinite, damping matrix, the Hamiltonian $H(x) \geq 0$, input u , input disturbance d and output y ,

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with $u, d, y \in \mathbb{R}^m$, $m \leq n$. Many systems described in the form of (1) satisfy the energy-balance

$$\frac{dH}{dt} = (u + d)^\top y - \frac{\partial H}{\partial x}^\top(x) R(x) \frac{\partial H}{\partial x}(x) \leq (u + d)^\top y \quad (2)$$

and are called *passive* systems [20], [25]. Otherwise, PBC can be applied to make a system passive with respect to a storage function which has the desired equilibrium point. PBC is a control methodology for nonlinear MIMO systems that achieves stabilization of a system by passivation of the closed-loop dynamics. Briefly summarized and assuming $d = 0$, by PBC a smooth state-feedback law $u = \alpha(x) + v$ is found which transforms (1) into

$$\dot{x} = [J_d(x) - R_d(x)] \frac{\partial H_d}{\partial x}(x) + g(x)v, \quad \bar{y} = g^\top(x) \frac{\partial H_d}{\partial x}(x) \quad (3)$$

with $H_d(x)$ the new Hamiltonian (storage function) having strict minimum at the desired equilibrium point x_d . Notice in (3) that besides shaping the Hamiltonian it is also possible to change (if necessary) the interconnection and damping matrices, from $J(x)$ into $J_d(x)$ and $R(x)$ into $R_d(x)$ respectively. The function $H_d(x)$ is characterized by the partial differential equation (PDE)

$$g^\perp(x) \left([J_d(x) - R_d(x)] \frac{\partial H_d}{\partial x}(x) - [J(x) - R(x)] \frac{\partial H}{\partial x}(x) \right) = 0 \quad (4)$$

where $g^\perp(x)$ is a full rank left annihilator of $g(x)$, i.e., $g^\perp(x)g(x) = 0$. When $R_d = 0$ and system (3) is zero-state detectable the new input $v = -C(x)\bar{y}$, with $C(x)$ a positive definite matrix, realizes asymptotic stability of (3). For more cases of PBC we refer to e.g., [6], [8], [14], [17], [18], [20].

It is known that the interconnection of two PH systems is again PH. By having a PH description of a notch filter, we can now interconnect it with another PH system and/or combine it with PBC. We are then able to also include frequency filtering to nonlinear control and preserve the properties and advantages of the nonlinear control method.

III. NOTCH FILTERS

In this section, we present the PH models for the notch and inverse notch filters.

A. Port-Hamiltonian Modeling of the Notch Filter

A notch filter is in general described by the transfer function

$$T(s) = \frac{s^2 + 2\beta_1\omega_0 s + \omega_0^2}{s^2 + 2\beta_2\omega_0 s + \omega_0^2} \quad (5)$$

where $\beta_1 > 0, \beta_2 > 0$ are the constant notch parameters which determine the depth and width of the notch and ω_0 is the center frequency. Fig. 1 shows the Bode plot for two notch examples.

The notch filter (5) can be described in state-space form by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta_2\omega_0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_z$$

$$y_z = [0 \quad 2\omega_0(\beta_1 - \beta_2)] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + u_z \quad (6)$$

where $z = (z_1, z_2)^\top$ are the system states, $u_z \in \mathbb{R}$ the notch filter input and $y_z \in \mathbb{R}$ the notch filter output. Notice that the notch filter is a system with feedthrough.

In [3], [7], and [13] a more general description of PH systems is given which includes a direct feedthrough channel. A PH system with feedthrough is described by

$$\dot{z} = [J_z(z) - R_z(z)] \frac{\partial H_z}{\partial z}(z) + [g_z(z) - P_z(z)] u_z$$

$$y_z = [g_z(z) + P_z(z)]^\top \frac{\partial H_z}{\partial z}(z) + [M_z(z) + S_z(z)] u_z \quad (7)$$

Fig. 1. Bode plots for $\omega_0 = 10$ rad/s.

with state vector $z \in \mathbb{R}^k$ and similar to (1) the interconnection matrix $J_z(z)$, damping matrix $R_z(z)$, Hamiltonian $H_z(z)$, input vector u_z and output vector y_z . PH systems with feedthrough have a matrix $M_z = -M_z^\top$, positive semidefinite matrix $S_z(z)$ and matrix $P_z(z)$. As shown in [3], [13], PH systems with feedthrough (7) have the power-balance equation given by

$$\dot{H}_z = u_z^\top y_z - \begin{bmatrix} \frac{\partial H_z}{\partial z} \\ u_z \end{bmatrix}^\top \begin{bmatrix} R_z(z) & P_z(z) \\ P_z^\top(z) & S_z(z) \end{bmatrix} \begin{bmatrix} \frac{\partial H_z}{\partial z} \\ u_z \end{bmatrix}. \quad (8)$$

The condition for passivity is then given by

$$\begin{bmatrix} R_z(z) & P_z(z) \\ P_z^\top(z) & S_z(z) \end{bmatrix} \geq 0 \quad (9)$$

for all z and admissible inputs u_z . Notice that $R_z(z)$, $S_z(z)$ and $P_z(z)$ determine a kind of dissipation structure of (7). Although it is possible to describe (6) in the form (7), we choose to apply the coordinate transformation

$$\xi_1 = \omega_0 z_1, \quad \xi_2 = 2(\beta_2 - \beta_1) z_2. \quad (10)$$

The notch filter (6) is then described in the PH form (7) by

$$\dot{\xi} = [J_\xi - R_\xi] \frac{\partial H_\xi}{\partial \xi}(\xi) + [g_\xi - P_\xi] u_\xi$$

$$y_\xi = [g_\xi + P_\xi]^\top \frac{\partial H_\xi}{\partial \xi}(\xi) + [M_\xi + S_\xi] u_\xi \quad (11)$$

with $\xi = (\xi_1, \xi_2)^\top$, Hamiltonian

$$H_\xi(\xi) = (\beta_2 - \beta_1)\omega_0 \xi_1^2 + \frac{\omega_0}{4(\beta_2 - \beta_1)} \xi_2^2 \quad (12)$$

and

$$J_\xi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad R_\xi = \begin{bmatrix} 0 & 0 \\ 0 & 4\beta_2(\beta_2 - \beta_1) \end{bmatrix} \quad g_\xi = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$P_\xi = \begin{bmatrix} 0 \\ 2(\beta_1 - \beta_2) \end{bmatrix} \quad M_\xi = 0 \quad S_\xi = 1. \quad (13)$$

The coordinate transformation is applied such that interconnection matrix J_ξ agrees with the interconnection matrix for standard mechanical systems [20]. The interconnection structure for standard mechanical systems is described by

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad (14)$$

with identity matrix $I \in \mathbb{R}^{j \times j}$, for j degrees of freedom. Having the same interconnection matrix as for standard mechanical systems provides a physical interpretation for the PH notch filter. However, this choice is not restrictive, i.e., it can be interconnected with any physical system. The passivity condition for (11) is then similar to (9), i.e.,

$$\begin{bmatrix} R_\xi & P_\xi \\ P_\xi^\top & S_\xi \end{bmatrix} \geq 0. \quad (15)$$

For a standard notch filter $\beta_2 > \beta_1$, such that the Hamiltonian is positive definite and all the principal minors of the matrix (15) can be verified to be non-negative. Non-negativity of all the principal minors implies that (15) always holds and proves passivity of the notch filter.

B. Port-Hamiltonian Modeling of the Inverse Notch Filter

In Section I, we also mentioned another kind of notch filter, namely the inverse notch filter. The inverse notch filter is actually the standard notch filter (5) but with $\beta_1 > \beta_2$. The result is a filter that only passes and amplifies signals containing the center frequency ω_0 . However, when $\beta_1 > \beta_2$ the PH formulation of the notch filter described above by (12) and (13) is not anymore adequate. Since $\beta_1 > \beta_2$ the Hamiltonian (12) becomes negative definite and the matrix R negative semidefinite. In other words, while for (5) and (6) the choice of the notch parameters has no implications for the system description, the PH formulation of a notch filter is not anymore PH when $\beta_1 > \beta_2$, i.e., we need a different PH formulation. For the same reasons as in the previous subsection, we apply for the inverse notch filter the coordinate transformation

$$\tau_1 = \omega_0 z_1, \quad \tau_2 = 2(\beta_1 - \beta_2)z_2. \quad (16)$$

Here we again have the PH structure (7), but with the parameters $J_\tau, R_\tau, g_\tau, P_\tau, M_\tau$ and S_τ given by

$$\begin{aligned} J_\tau &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & R_\tau &= \begin{bmatrix} 0 & 0 \\ 0 & 4\beta_2(\beta_1 - \beta_2) \end{bmatrix} & P_\tau &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ g_\tau &= \begin{bmatrix} 0 \\ 2(\beta_1 - \beta_2) \end{bmatrix} & M_\tau &= 0 & S_\tau &= 1 \end{aligned} \quad (17)$$

$\tau = (\tau_1, \tau_2)^\top$, input u_τ , output y_τ and Hamiltonian

$$H_\tau(\tau) = (\beta_1 - \beta_2)\omega_0\tau_1^2 + \frac{\omega_0}{4(\beta_1 - \beta_2)}\tau_2^2. \quad (18)$$

The passivity condition

$$\begin{bmatrix} R_\tau & P_\tau \\ P_\tau^\top & S_\tau \end{bmatrix} \geq 0 \quad (19)$$

is satisfied since $P_\tau = 0$ and $\beta_1 > \beta_2$ in R_τ .

IV. DISTURBANCE ATTENUATION OF THE INVERSE NOTCH FILTER

A. Disturbance Attenuation

In this section, we apply the inverse notch filter ($\beta_1 > \beta_2$) for disturbance attenuation in PH systems. It is shown that if the inverse notch is tuned to have a center frequency equal (or close) to the disturbance frequency, that the steady-state effect of the disturbance is attenuated. As described in Section II, we exploit the fact that a passive interconnection between PH plant and inverse notch is also passive. The input of the inverse notch then depends on the passive output of the plant, and is tuned to attenuate part of the plant output with a specific frequency. Since we deal with nonlinear systems, first it is necessary to show that a periodic input for the plant also results in a periodic output. For this purpose we refer to the L_2 -interpretation of nonlinear systems for periodic inputs from [9]. In [9] it is described under which

conditions a nonlinear system with a periodic input yields a steady-state output with same period as the input. Recall from Section II that, if system (3) with $x_d = 0$ is zero-state detectable, that $v = -C(x)\bar{y}$ asymptotically stabilizes (3) in $x = 0$. Assume furthermore that the linearization of the resulting asymptotically stable PH system at $x = 0$ has all its eigenvalues in the left-half plane. According to [9] a periodic input with period T then yields a steady-state output with period T for the PH system (3).

To show the disturbance attenuation properties of the inverse notch filter, we analyze the L_2 -gain of the closed-loop system (26) with respect to an input disturbance d . For a nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)d \\ y &= h(x) \end{aligned} \quad (20)$$

the L_2 -gain bound $\gamma > 0$ is found if for γ there exists a smooth non-negative solution $\mathcal{W}(x)$ to the Hamilton-Jacobi inequality (HJI) [20]

$$\frac{\partial \mathcal{W}^\top}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial \mathcal{W}^\top}{\partial x} g(x)g^\top(x) \frac{\partial \mathcal{W}}{\partial x} + \frac{1}{2} h^\top(x)h(x) \leq 0 \quad (21)$$

The idea behind the application of inverse notch filters is that one can realize high gain feedback only for a specific frequency. We show later on that this corresponds to having a high damping at the specific frequency, and a lower damping at all the other frequencies.

Assume that the linearization of system (3) at $x = x_d$ has all eigenvalues in the left-half plane. In the previous subsection we explained that a periodic input with period T for system (3) then yields a steady-state output with period T . Let $\bar{y}_i$ be the output i of the output vector $\bar{y}$, with $i = 1, \dots, m$. For a nonlinear system with a periodic input with frequency ω_d and having also a periodic output, output i is described by the disturbance frequency ω_d and its higher harmonics, i.e.,

$$\bar{y}_i = \sum_{j=1}^{\infty} A_{i,j} \sin(j\omega_d t + \varphi_{i,j}) \quad (22)$$

where $A_{i,j}$ is the amplitude and $\varphi_{i,j}$ the phase of harmonic j for output i respectively. Notice that output (22) satisfies

$$|\bar{y}_i| \leq \sum_{j=1}^{\infty} A_{i,j} = \bar{A}_i. \quad (23)$$

with $\bar{A}_i$ a positive constant. It is then possible to describe the contribution of each harmonic to the output (23) by

$$A_{i,j} = \alpha_{i,j} \bar{A}_i, \quad \sum_{j=1}^{\infty} \alpha_{i,j} = 1 \quad (24)$$

with $\alpha_{i,j}$ a positive constant. Since we deal with MIMO nonlinear systems a periodic input with period T on input i can result in a periodic output with period T for all outputs of the output vector $\bar{y}$, or a set of the outputs. Notice then that each degree of freedom has to be interconnected with an inverse notch filter, in order to attenuate the influence of the input disturbance on each output. For one input disturbance frequency, we then have m inverse notch filters. Let $\hat{u}_\tau = (u_{\tau,1}, \dots, u_{\tau,m})^\top$ be the vector with the inputs and $\hat{y}_\tau = (y_{\tau,1}, \dots, y_{\tau,m})^\top$ the vector with the outputs for the $i = 1, \dots, m$ inverse notch filters. Let $\hat{\tau} = (\hat{\tau}_1, \dots, \hat{\tau}_m)^\top$ be the vector with the states of all m inverse notch filters, and where $\hat{\tau}_i = (\tau_{i,1}, \tau_{i,2})^\top$ describes the states of inverse notch filter i . Consider system (3) with $x_d = 0$, and assume its linearization to have all eigenvalues in the left half-plane. A passive interconnection between (3) and the m inverse notch filters is realized via the interconnection law

$$v = -K\hat{y}_\tau, \quad \hat{u}_\tau = K\bar{y} \quad (25)$$

with $K = \text{diag}\{k_1, \dots, k_m\} > 0$ constant. The resulting closed-loop system is then given by¹

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{\tau}} \end{bmatrix} = \begin{bmatrix} J_d - R_d - gK^2g^\top & -gK\hat{g}_\tau^\top \\ \hat{g}_\tau K g^\top & \hat{J}_\tau - \hat{R}_\tau \end{bmatrix} \begin{bmatrix} \frac{\partial H_d}{\partial x} \\ \frac{\partial \hat{H}_\tau}{\partial \hat{\tau}} \end{bmatrix} + \begin{bmatrix} g \\ 0 \end{bmatrix} d$$

$$\bar{y} = g^\top \frac{\partial H_d}{\partial x} \quad (26)$$

where $\hat{J}_\tau = \text{diag}\{J_{\tau,1}, \dots, J_{\tau,m}\}$ is the interconnection matrix, $\hat{R}_\tau = \text{diag}\{R_{\tau,1}, \dots, R_{\tau,m}\}$ is the damping matrix and $\hat{g}_\tau = \text{diag}\{g_{\tau,1}, \dots, g_{\tau,m}\}$ is the input matrix for the set of m inverse notch filters. The Hamiltonian $\hat{H}_\tau$ is given by

$$\hat{H}_\tau = \sum_{i=1}^m H_{\tau,i} \quad (27)$$

with $H_{\tau,i}$ the Hamiltonian for inverse notch filter i . The disturbance attenuation properties of (26) are described in the following proposition.

Proposition 1: Consider the closed-loop system (26) with $x_d = 0$. Let the input disturbance vector $d = (d_1, \dots, d_m)^\top$ be described by $d_i = c_i \sin(\omega_d t + \phi_i)$, where ω_d is the disturbance frequency, c_i is the amplitude and ϕ_i is the phase for disturbance inputs $i = 1, \dots, m$. Assume that all m inverse notch filters have the same parameters, with $\lambda = \beta_1/\beta_2$. System (26) then has a steady-state L_2 -gain bound γ which satisfies

$$\gamma^2 I \geq (2\hat{K} - I)^{-1}, \quad \hat{K} > \frac{1}{2}I \quad (28)$$

with $\hat{K} = (\lambda - 1)\bar{\alpha}^\top K^2 \bar{\alpha} + K^2$ when $\omega = \omega_d$, where $\bar{\alpha} = (\alpha_{1,1}, \dots, \alpha_{m,1})$. When the difference between ω_0 and ω_d is large, $\hat{K} \approx K^2$.

Proof: Consider the closed-loop system (26) and function $\mathcal{W} = H(x) + \hat{H}_\tau(\hat{\tau})$ fulfilling the inequality (21). Then for system (26), we obtain

$$\begin{aligned} & -\frac{\partial H_d}{\partial x}^\top R_d(x) \frac{\partial H_d}{\partial x} - \frac{\partial \hat{H}_\tau}{\partial \hat{\tau}}^\top \hat{R}_\tau \frac{\partial \hat{H}_\tau}{\partial \hat{\tau}} - \bar{y}^\top K^2 \bar{y} \\ & + \frac{1}{2} \frac{1}{\gamma^2} \bar{y}^\top \bar{y} + \frac{1}{2} \bar{y}^\top \bar{y} \leq 0. \end{aligned} \quad (29)$$

The inequality (29) includes the states $\tau_{i,2}$ of the $i = 1, \dots, m$ inverse notch filters, since

$$-\frac{\partial \hat{H}_\tau}{\partial \hat{\tau}}^\top \hat{R}_\tau \frac{\partial \hat{H}_\tau}{\partial \hat{\tau}} = -\frac{\beta_2 \omega_0^2}{\beta_1 - \beta_2} \sum_{i=1}^m \tau_{i,2}^2. \quad (30)$$

The inverse notch is a linear system, so it is straightforward for inverse notch i that

$$\ddot{\tau}_{i,2} + 2\beta_2 \omega_0 \dot{\tau}_{i,2} + \omega_0^2 \tau_{i,2} = 2(\beta_1 - \beta_2) \dot{u}_{\tau,i}. \quad (31)$$

Since the input disturbance is sinusoidal the disturbance d can also be assumed to be generated from a linear exosystem with all its eigenvalues on the imaginary axis. From the previous subsection we then know that the resulting (steady-state) output $\bar{y}$ of the plant is periodic with frequency ω_d . From (25) we know that $u_{\tau,i} = k_i \bar{y}_i$, then (31) with (22) gives the well-known differential equation

$$\begin{aligned} & \ddot{\tau}_{i,2} + 2\beta_2 \omega_0 \dot{\tau}_{i,2} + \omega_0^2 \tau_{i,2} \\ & = \sum_{j=1}^{\infty} 2(\beta_1 - \beta_2) k_i A_{i,j} j \omega_d \cos(j \omega_d t + \varphi_{i,j}) \end{aligned} \quad (32)$$

which for $\omega_0 = \omega_d$ has the particular solution

$$\begin{aligned} \tau_{i,2}(t) &= \sum_{j=1}^{\infty} \frac{2(\beta_1 - \beta_2) k_i j A_{i,j} \omega_d \cos(j \omega_d t + \varphi_{i,j} + \Phi(j \omega_d))}{[(\omega_0^2 - j^2 \omega_d^2)^2 + 4\beta_2^2 \omega_0^2 j^2 \omega_d^2]^{\frac{1}{2}}} \\ &= \sum_{j=1}^{\infty} \frac{2(\beta_1 - \beta_2) k_i j A_{i,j}}{\sqrt{(1 - j^2)^2 + 4\beta_2^2 j^2 \omega_d}} \cos(j \omega_d t + \varphi_{i,j} + \Phi(j \omega_d)) \end{aligned} \quad (33)$$

with $\Phi(j \omega_d)$ the polar angle of the point $(\omega_0^2 - j^2 \omega_d^2, -2\beta_2 \omega_0 j \omega_d)$. We are here interested in the steady-state behavior of (32), so in (33) we leave out the homogeneous part of the solution (which converges to zero). Notice that when $\omega_0 = \omega_d$ we can define a bound on (30) based on only the first harmonic, i.e.,

$$-\frac{\beta_2 \omega_0^2}{\beta_1 - \beta_2} \tau_{i,2}^2 \leq -\frac{\beta_1 - \beta_2}{\beta_2} k_i^2 A_{i,1}^2 \leq -(\lambda - 1) k_i^2 \alpha_{i,1}^2 \bar{A}_i^2. \quad (34)$$

From (25) with (22), (23), and (34), and since $\hat{R}_\tau$ is diagonal, we can write for (29)

$$-(\lambda - 1)(\bar{\alpha} \bar{A})^\top K^2 (\bar{\alpha} \bar{A}) - \bar{A}^\top K^2 \bar{A} + \frac{1}{2} \frac{1}{\gamma^2} \bar{A}^\top \bar{A} + \frac{1}{2} \bar{A}^\top \bar{A} \leq 0 \quad (35)$$

where $\bar{A} = (\bar{A}_1, \dots, \bar{A}_m)$. Now study (35), then with (28) and (34) the inequality (29) corresponds to the HJI inequality (21) for system (26), and it can be seen that γ becomes smaller (since $\lambda > 1$ for $\beta_1 > \beta_2$) when the disturbance frequency ω_d is equal to the notch frequency ω_0 . When the difference between ω_d and ω_0 is large the denominator in (33) becomes much larger than the numerator, and so the $\tau_{i,2}$ state has no significant influence on the HJI (29), i.e., $\hat{K} \approx K^2$ in (28). $\square$

The improved disturbance attenuation, with $\omega_0 = \omega_d$, is realized by an increase in the feedback gain by the factor $\lambda = \beta_1/\beta_2$ at the specific frequency ω_d . A higher feedback gain corresponds to injecting more damping into the system. The HJI (21) implies that the disturbance attenuation of system (3) is also improved by the simple feedback $v = -K_d \bar{y}$ which yields

$$\begin{aligned} \dot{x} &= [J_d(x) - R_d(x) - g(x) K_d g^\top(x)] \frac{\partial H_d}{\partial x}(x) + g(x) d \\ \bar{y} &= g(x)^\top \frac{\partial H_d}{\partial x}(x). \end{aligned} \quad (36)$$

In a similar way as in the proof of Proposition 1, the HJI (21) for the closed-loop system without notch (36) with $\mathcal{W} = H(x)$ yields $\gamma^2 I \geq (2K_d - I)^{-1}$ for $K_d > (1/2)I$. We can tune K_d such to achieve an L_2 -gain bound γ as in (28), which again corresponds to increasing the damping in the system. However, damping slows down the transient response of a system. Application of the inverse notch filters also increases the damping, but only for the disturbance frequency. All other frequency components in the feedback signal remain relatively underdamped. The closed-loop system (36) realizes the same steady-state attenuation as (26), however, in (36) all frequency components have the same relatively high damping. Intuitively we can expect the transient response for (36) to be slower. Inverse notch filters can then be used to attenuate the effects of specific disturbance frequencies, without overdamping the system. Notice furthermore how the bound on γ_2 given in (28) relates to the amplitude of the (base) disturbance frequency ω_d . When the amplitude of the higher harmonics are negligible, then $\alpha_{i,1} \approx 1$, which then gives $\hat{K} \approx \lambda K^2$. On the other hand, if the amplitude of the base harmonic is negligible compared to the higher harmonics, then $\alpha_{i,1} \approx 0$ and $\hat{K} \approx K^2$, which (as expected) coincides with the case when there is a significant difference between ω_0 and ω_d .

To conclude this section, in Proposition 1 we assumed only one disturbance frequency ω_d . The results presented in this section can be applied for a combination of different frequencies, as this coincides with repeating the procedure of Proposition 1 on the resulting closed-loop system for each disturbance frequency. For simplicity we

¹For simplicity of notation, we leave out the arguments of $J(x)$, $R(x)$, $g(x)$

assumed each notch filter to have the same parameters β_1 and β_2 . Obviously, it is possible to let each inverse notch to have different parameters.

B. Robustness Analysis

Based on how the inverse notch filter is tuned, the gain for frequencies close to ω_0 are also amplified (but obviously less than a factor λ). Since (28) is given in terms of the base harmonic ω_d , we briefly look at (34) when there is a small discrepancy between ω_0 and ω_d .

Consider (30) and assume $\omega_0 = (1 + \epsilon)\omega_d$, with ϵ a small constant. We then have for (30)

$$\begin{aligned} -\frac{\beta_2\omega_0^2}{\beta_1 - \beta_2}\tau_{i,2}^2 &= -\frac{4\beta_2(\beta_1 - \beta_2)(1 + \epsilon)^2}{\epsilon^4 + 4\epsilon^3 + 4\epsilon^2 + 4\beta_2^2\epsilon^2 + 8\beta_2^2\epsilon + 4\beta_2^2} \\ &\quad \times k_i^2 A_i^2 \cos^2(\omega_d t + \varphi_i + \Phi) \\ &= -\frac{(\beta_1 - \beta_2)}{\beta_2} \cdot \Psi k_i^2 A_i^2 \cos^2(\omega_d t + \varphi_i + \Phi) \\ &= -(\lambda - 1)\Psi k_i^2 A_i^2 \cos^2(\omega_d t + \varphi_i + \Phi) \end{aligned} \quad (37)$$

where

$$\Psi = \frac{\epsilon^2 + 2\epsilon + 1}{\epsilon^2 + 2\epsilon + 1 + \frac{1}{4\beta_2^2}(\epsilon^4 + 4\epsilon^3 + 4\epsilon^2)}. \quad (38)$$

For a small ϵ , first notice that Ψ is positive and $\Psi < 1$. We then have that

$$-\frac{\beta_2\omega_0^2}{\beta_1 - \beta_2}\tau_{i,2}^2 \leq -(\lambda - 1)\Psi k_i^2 \alpha_{i,1} \bar{A}_i. \quad (39)$$

Since $\Psi < 1$ and positive, the bound is larger than in (34), which then gives a higher bound on γ_2 in (28).

V. SIMULATION EXAMPLE

In this section, we show an example of PBC combined with a PH inverse notch filter for control of a nonlinear mechanical system. The system is a robot manipulator with rigid joints under the influence of sinusoidal input disturbances. The manipulator system belongs to the class of standard mechanical systems, which are described in the PH form (1) by

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & -D \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ u + d \end{bmatrix}, \quad y = \frac{\partial H}{\partial p} = \dot{q} \quad (40)$$

with $q = (q_1, \dots, q_k)^\top$ the vector of generalized configuration coordinates, $p = (p_1, \dots, p_k)^\top$ the vector of generalized momenta, D the positive semidefinite damping matrix, I the identity matrix, u the input vector, d the input disturbance vector and y the output vector. The Hamiltonian of the system is equal to the sum of kinetic and potential energy

$$H(q, p) = \frac{1}{2}p^\top M^{-1}(q)p + V(q) \quad (41)$$

where $M(q) = M^\top(q) > 0$ is the system mass-inertia matrix and $V(q)$ the potential energy.

Consider a rigid joint robot with 2 links and denote the length of link i by l_i , the angle of link i by θ_i , the distance from the joint to the center of gravity of the link i by r_i , the mass of link i by m_i and the inertia of link i by I_i . The control input is given by $u = (u_1, u_2)^\top$, with u_i the input torque of joint i and the input disturbance vector $d = (d_1, d_2)^\top$. Describe the mass-inertia matrix of the robot by

$$M(q) = \begin{bmatrix} a_1 + a_2 + 2b \cos \theta_2 & a_2 + b \cos \theta_2 \\ a_2 + b \cos \theta_2 & a_2 \end{bmatrix} \quad (42)$$

with constants

$$a_1 = m_1 r_1^2 + m_2 l_1^2 + I_1, \quad a_2 = m_2 r_2^2 + I_2, \quad b = m_2 l_1 r_2.$$

TABLE I
2R MANIPULATOR PARAMETER VALUES

Parameter	Value	Parameter	Value	Parameter	Value
m_1	0.5	l_1	0.2	D_1	0.1
m_2	1	l_2	0.25	D_2	0.1
I_1	0.01	r_1	0.1	q_{1d}	0.5π
I_2	0.01	r_2	0.1	q_{2d}	0.25π

Fig. 2. Error trajectories for the 2R manipulator. Initial conditions $[q(0)p(0)] = [0 \ 0 \ 0 \ 0]$.

The damping matrix is assumed to be $D = \text{diag}\{D_1, D_2\}$, with positive constants D_1, D_2 . For simplicity, we assume a constant damping, however non-constant damping can also be included in a PH model, see [8], [13]. The robot in this example is described by coordinates $q = (\theta_1, \theta_2)^\top$, $p = M(q)\dot{q}$ and potential energy

$$V(q) = m_1 g_0 r_1 \cos(\theta_1) + m_2 g_0 (l_1 \cos(\theta_1) + r_2 \cos(\theta_1 + \theta_2)) \quad (43)$$

with $g_0 = 9.81 \text{ m/s}^2$ the gravity constant. Assume the control input u is described by

$$u = a(q, p) + v \quad (44)$$

with $a(q, p)$ the energy-shaping control input vector for the system (40) when $d = 0$ and $v = (v_1, v_2)^\top$ the new control input. In the PBC literature [14], [20], the control input

$$a(q, p) = \frac{\partial V}{\partial q}(q) - K_p(q - q_d) \quad (45)$$

with K_p positive definite, shapes the potential energy from $V(q)$ into a function $V_d(q)$ with desired equilibrium point $q_d = (q_{1d}, q_{2d})^\top$, i.e.,

$$V_d(q) = \frac{1}{2}(q - q_d)^\top K_p(q - q_d). \quad (46)$$

Global asymptotic stability is then realized by damping injection, i.e., $v = -K_d y$ with K_d a positive definite matrix.

When there is input disturbance, the manipulator can be interconnected in a passive way with the necessary numbers of PH inverse notch filters by (25). In this simulation example, we assume the disturbance $d_i = c_i \sin(\omega_d t)$. For simplicity, we take $c_1 = c_2 = 2$ and $\omega_d = 3 \text{ rad/s}$. The manipulator parameters are shown in Table I. The parameters, except for the friction, describe an available experimental manipulator from Quanser. The friction values have not yet been verified on the experimental setup. Fig. 2 shows the results for $K_p = \text{diag}\{10, 10\}$ and $K_d = \text{diag}\{4, 4\}$. Fig. 3 compares the results when we add an inverse notch filter with the results when we only use high K_d gains. In both cases, the influence of the disturbance is significantly reduced, however, the transient response is faster when the inverse notch is applied. As explained in the previous section and shown in this example, only applying high K_d gains increases (overall) damping and slows down the transient response.

We also did simulations for $\omega_0 = 2.5 \text{ rad/s}$ and $\omega_0 = 3.5 \text{ rad/s}$, while keeping the real disturbance frequency at $\omega_d = 3 \text{ rad/s}$. The results showed very small differences compared to Fig. 3, showing the

Fig. 3. Error trajectories for the 2R manipulator. Initial conditions $[q(0) \ p(0)] = [0 \ 0 \ 0 \ 0]$. Blue lines: trajectories with inverse notch filters with $\omega_0 = 3$ rad/s, $\beta_1 = 1$, $\beta_2 = 0.01$. Dashed red lines: trajectories without inverse notch and $K_d = \text{diag}\{50, 50\}$.

robustness of the inverse notch filter. Tuning of β_1 and β_2 can make the inverse notch wide enough to deal with uncertainties in the disturbance frequency.

VI. FINAL REMARKS

The notch filter is a well-known control tool and popular in many control applications. The contribution of this technical note is to describe the standard notch filter in the PH framework. By having a PH description of the notch filter it can be interconnected to any other PH system and preserve the PH structure. The advantages of the PH structure for analysis and control design are then combined with the notch filter, which is tuned based on frequency requirements.

In a similar way, we also give a PH description for the inverse notch filter. Contrary to a normal notch filter, the inverse notch filter only passes signals of a specific frequency. We also analyze the disturbance attenuation properties of the inverse notch filter for (nonlinear) PH systems.

A simulation example is given for the application of the inverse notch filter for disturbance attenuation of a robot manipulator. The simulation results show, as expected, disturbance attenuation without overdamping the system. Although only illustrated for an example of a standard mechanical system, the port-Hamiltonian notch/inverse notch filter can be applied in a similar way in other physical domains.

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