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On the Controllability and Stabilizability of Linear Complementarity Systems

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Abstract: This paper studies controllability and stabilizability of linear complementarity systems that can be cast as continuous piecewise affine dynamical systems. Under a certain right-invertibility assumption, we present a la Hautus necessary and sufficient conditions for both controllability and stabilizability.

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1. INTRODUCTION

Controllability is not only a fundamental system-theoretic property but also a prerequisite virtually for all control design problems. Kalman’s and Hautus’ tests for controllability are among the classical results of linear system theory. Apart from finite-dimensional linear time-invariant systems, there are only a few cases for which easily verifiable controllability conditions exist. For instance, necessary and sufficient conditions that are linear in nature exist for linear switched systems with state-independent switching Sun et al. (2002). This result relies on the fact that the switching signal can be considered as an external input. Yet another instance for which controllability admits a simple characterization is the case of continuous piecewise affine dynamical systems Thuan and Camlibel (2014). Contrary to linear switched systems, piecewise affine dynamical systems exhibit state-dependent switching phenomenon that makes the analysis much harder. Nevertheless, Hautus-like tests can be devised by exploiting the structure imposed by continuity of the vector field. Moreover, this Hautus-like controllability test also extends to stabilizability.

In this paper, we study a particular class of continuous piecewise affine dynamical systems, namely linear complementarity systems. Complementarity systems are encountered in a multitude of applications in various disciplines of science and engineering including electrical/mechanical engineering and operations research van der Schaft and Schumacher (1998, 2000); Brogliato (2003); Heemels and Brogliato (2003); Camlibel et al. (2004); Schumacher (2004). We refer to van der Schaft and Schumacher (1996); Heemels et al. (2000); Camlibel (2001); Camlibel and Schumacher (2002); Camlibel et al. (2003); Heemels et al. (2002); Camlibel et al. (2002); Camlibel (2007); Camlibel et al. (2008b, 2014) for the work on the analysis of general LCSs.

Under certain conditions, a linear complementarity system can be cast as a continuous piecewise affine dynamical system. We will exploit the structure of such piecewise affine systems and the results in Thuan and Camlibel (2014) in order to present tailor-made necessary and sufficient conditions for controllability and stabilizability of linear complementarity systems.

The organization of the paper is as follows. In Section 2, we will introduce the class of linear complementarity systems. Section 3 is devoted to continuous piecewise affine dynamical systems and Hautus-like controllability/stabilizability tests for these systems. Subsequently, Section 4 will discuss the relationships between linear complementarity systems and continuous piecewise affine systems. This will be followed by the main results and their proofs in Section 5. Finally, the paper closes with the conclusions in Section 6.

2. LINEAR COMPLEMENTARITY SYSTEMS

Consider the linear system

\[ \dot{x}(t) = Ax(t) + Bu(t) + Ez(t) \]  \hspace{1cm} (1a)
\[ w(t) = Cx(t) + Du(t) + Fz(t) \]  \hspace{1cm} (1b)

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, \) and \( (z, w) \in \mathbb{R}^{p+p} \). When the external variables \( (z, w) \) satisfy the so-called complementarity relations

\[ 0 \leq z(t) \perp w(t) \geq 0 \]  \hspace{1cm} (1c)

where the inequalities are componentwise and \( \perp \) denotes orthogonality, that is \( z \perp w \) if and only if \( z^T w = 0 \). We call the overall system (1) a linear complementarity system (LCS).

3. CONTINUOUS PIECEWISE AFFINE SYSTEMS

In order to introduce continuous piecewise affine dynamical systems, we first begin with piecewise affine functions and their properties.

A function \( \psi : \mathbb{R}^r \to \mathbb{R}^\ell \) is said to be affine if there exist a matrix \( H \in \mathbb{R}^{\ell \times r} \) and a vector \( g \in \mathbb{R}^\ell \) such that \( \psi(x) = Hx + g \) for all \( x \in \mathbb{R}^r \). A function \( \phi \) is called piecewise affine if there exists a finite family of affine
functions $\psi_i : \mathbb{R}^r \rightarrow \mathbb{R}^\ell$ with $i = 1, 2, \ldots, r$ such that $\phi(x) \in \{\psi_1(x), \psi_2(x), \ldots, \psi_r(x)\}$ for all $x \in \mathbb{R}^r$.

Continuous piecewise affine functions admit useful geometrical representations that will be used later on this paper. To elaborate, we first need to introduce some terminology.

A set of $\mathbb{R}^r$ is called a polyhedron if it is the intersection of a finite family of closed half-spaces or hyperplanes in $\mathbb{R}^r$. If $P \subseteq \mathbb{R}^r$ is a polyhedron, then there exist a matrix $P \in \mathbb{R}^{k \times r}$ and a vector $q \in \mathbb{R}^k$ such that $P = \{x \in \mathbb{R}^r \mid Px \geq q\}$. A finite collection $\Xi$ of polyhedra of $\mathbb{R}^r$ is said to be a polyhedral subdivision of $\mathbb{R}^r$ if (see Facchinei and Pang (2002) for more details)

i. the union of all polyhedra in $\Xi$ is equal to $\mathbb{R}^r$,
ii. each polyhedron in $\Xi$ is of dimension $n$, and
iii. the intersection of any two polyhedra in $\Xi$ is either empty or a common proper face of both polyhedra

As stated next, polyhedral subdivisions appear in the description of continuous piecewise affine functions.

**Proposition 1.** (Facchinei and Pang (2002), Prop. 4.2.1.)
A continuous function $\psi : \mathbb{R}^r \rightarrow \mathbb{R}^\ell$ is piecewise affine if and only if there exist a polyhedral subdivision $\Xi = \{\Xi_1, \Xi_2, \ldots, \Xi_r\}$ of $\mathbb{R}^r$ and a finite family of affine functions $\psi_i : \mathbb{R}^r \rightarrow \mathbb{R}^\ell$ with $i = 1, 2, \ldots, r$ such that each $\psi_i$ coincides on $\Xi_i$ for every $i = 1, 2, \ldots, r$.

By a continuous piecewise affine system (CPAS), we mean a dynamical system of the form

$$
\begin{align*}
x'(t) &= Ax(t) + Bu(t) + \Phi(y(t)) \quad (2a) \\
y(t) &= Cx(t) + Du(t) \quad (2b)
\end{align*}
$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, all involved matrices are of appropriate sizes, and $\Phi : \mathbb{R}^p \rightarrow \mathbb{R}^\ell$ is a continuous piecewise affine function.

Since the piecewise affine function $\Phi$ is continuous, the right-hand side of $(2a)$ is globally Lipschitz (see e.g. (Facchinei and Pang, 2002, Proposition 4.2.2)). Thus, it follows from the theory of ordinary differential equations that the system $(2)$ must admit a unique solution for each initial state $x^0$ and locally integrable input $u$. We denote this solution by $x^u(t; x^0)$, and denote the corresponding output by $y^u(t; x^0)$.

Next, we define controllability and stabilizability for CPASs.

**Definition 2.** We say that the system $(2)$ is

- **controllable** if for any two states $x^0, x^1$ there exist $T > 0$ and a locally integrable input $u$ such that $x^u(T; x^0) = x^1$,
- **stabilizable** if for any initial state $x^0$ there exists a locally integrable input $u$ such that $\lim_{t \to \infty} x^u(t; x^0) = 0$.

In this paper, the following blanket assumption will be in force.

**Assumption 3.** The transfer matrix $D + C(sI - A)^{-1}B$ is right invertible as a rational matrix.

Similar invertibility assumptions are common in the analysis of linear systems with output constraints (see e.g. Saberi et al. (2002); Shi et al. (2003); Saberi et al. (2003); Heemels and Camlibel (2008)). The main restriction of the assumption is that the number of inputs should be at least as the number of outputs. Whenever this is the case, right invertibility is generically satisfied.

A complete characterization of controllability and stabilizability of CPASs are established in Thuan and Camlibel (2014). In order to quote the results of Thuan and Camlibel (2014), we need to introduce the following terminology. For a nonempty convex set $S \subseteq \mathbb{R}^n$, $S^-$ denotes the dual cone

$$
S^- := \{x \in \mathbb{R}^n \mid \xi^T x \leq 0 \text{ for all } x \in S\},
$$

$S^\perp$ denotes orthogonal complement in $\mathbb{C}^n$, and $C_r(S)$ denotes the recession cone

$$
C_r(S) := \{x \in \mathbb{R}^n \mid z + \lambda x \in \Lambda \text{ for all } \lambda \geq 0 \text{ and } z \in S\}.
$$

Necessary and sufficient conditions for controllability of CPASs are stated next.

**Proposition 4.** (Thuan and Camlibel (2014), Thm. 3.4).
Consider a CPAS $(2)$ satisfying Assumption 3. Suppose that the origin is contained in the closure of the convex hull of the image of the function $\Phi$. Let $\Xi = \{\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_h\}$ be a polyhedral subdivision of $\mathbb{R}^p$ induced by $\Phi$. Also let

$$
\Phi(y) = H^i y - g^i \text{ whenever } y \in \mathcal{Y}_i.
$$

Then, the system $(2)$ is controllable if and only if the following four implications hold:

i. $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$, $z^* g^i = 0$, and $\left[ A + H^i C - \lambda I \ B + H^i D \right] \geq 0 \text{ and } \Rightarrow \ z = 0$.

ii. $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$, $(z, w^i) \in \{g^i \times \mathcal{Y}_i\}^-$, and

$$
\left[ \begin{array}{c}
z^i \\
w^i \\
\end{array} \right] \left[ \begin{array}{c}
A + H^i C - \lambda I \\
B + H^i D \\
\end{array} \right] \geq 0 \text{ and } \Rightarrow \ z = 0.
$$

iii. $\lambda \in \mathbb{C}$, Re($\lambda$) $\neq 0$, $z \in \mathbb{C}^n$, $w^i \in C_r(\mathcal{Y}_i)^-$, and

$$
\left[ \begin{array}{c}
z^i \\
w^i \\
\end{array} \right] \left[ \begin{array}{c}
A + H^i C - \lambda I \\
C \\
\end{array} \right] \geq 0 \text{ and } \Rightarrow \ z = 0.
$$

iv. $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $z \in \mathbb{R}^n$, $(z, w^i) \in \{g^i \times \mathcal{Y}_i\}^-$, and

$$
\left[ \begin{array}{c}
z^i \\
w^i \\
\end{array} \right] \left[ \begin{array}{c}
A + H^i C - \lambda I \\
D \\
\end{array} \right] \geq 0 \text{ and } \Rightarrow \ z = 0.
$$

Stabilizability of CPASs can also be characterized by similar spectral conditions.

**Proposition 5.** (Thuan and Camlibel, 2014, Thm. 3.6).
Consider a CPAS $(2)$ satisfying Assumption 3. Suppose that $\Phi(0) = 0$. Let $\Xi = \{\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_h\}$ be a polyhedral subdivision of $\mathbb{R}^p$ induced by $\Phi$. Also let

$$
\Phi(y) = H^i y - g^i \text{ whenever } y \in \mathcal{Y}_i.
$$

Then, the system $(2)$ is stabilizable if and only if the following four implications hold:

i. $\lambda \in \mathbb{C}$, Re($\lambda$) $\geq 0$, $z \in \mathbb{C}^n$, $z^* g^i = 0$ and $\left[ A + H^i C - \lambda I \ B + H^i D \right] \geq 0 \text{ and } \Rightarrow \ z = 0$.

ii. $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $z \in \mathbb{R}^n$, $(z, w^i) \in \{g^i \times \mathcal{Y}_i\}^-,$ and
where $\Psi$ is a Lipschitz continuous function given by

$$\Psi(q) = H^\alpha y_\alpha,$$

and

$$\bar{\mathcal{Y}}^\alpha = \{ y \in \mathbb{R}^p \mid -F_{\alpha\alpha}^{-1} 0_{|\alpha| \times |\alpha|} F_{\alpha\alpha}^{-1} 0_{|\alpha| \times |\alpha|} I_{|\alpha|} \geq 0 \}.$$  

5. MAIN RESULTS

In this section, our aim is first to streamline the conditions in Proposition 4 and Proposition 5.

**Proposition 6.** Consider an LCS of the form (1) satisfying Assumption 3. Suppose that $F$ is a $P$-matrix. Then, the following statements hold:

(a) The system (2) is controllable if and only if the following two implications hold:

i. $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$, and

$$z^* \left[ A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I - B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\bullet\bullet} \right] = 0$$

for all $\alpha \subseteq \{ 1, 2, \ldots, p \}$. This means that $z^*$ is stabilizable if and only if the follow-

ii. $\lambda \in \mathbb{R}$, $\lambda > 0$, $z \in \mathbb{R}^n$, and

$$z^* \left[ A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I - B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\bullet\bullet} \right] = 0$$

for all $\alpha \subseteq \{ 1, 2, \ldots, p \}$. This means that $z^*$ is stabilizable if and only if the fol-

(b) The system (2) is stabilizable if and only if the following two implications hold:

i. $\lambda \in \mathbb{C}$, $\text{Re}(\lambda) \geq 0$, $z \in \mathbb{C}^n$, and

$$z^* \left[ A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I - B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\bullet\bullet} \right] = 0$$

for all $\alpha \subseteq \{ 1, 2, \ldots, p \}$. This means that $z^*$ is stabilizable if and only if the fol-

ii. $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $z \in \mathbb{R}^n$, and

$$z^* \left[ A - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} C_{\alpha\bullet} - \lambda I - B - E_{\bullet\alpha} F_{\alpha\alpha}^{-1} D_{\bullet\bullet} \right] = 0$$

for all $\alpha \subseteq \{ 1, 2, \ldots, p \}$. This means that $z^*$ is stabilizable if and only if the fol-

To prove this proposition, we need the following auxiliary result.

**Lemma 7.** Let $C \subseteq \mathbb{R}^p$ be a polyhedral cone given by $C = \{ \xi \in \mathbb{R}^p \mid M\xi \geq 0 \}$ for some matrix $M \in \mathbb{R}^{n \times p}$. Then, $C^- = \{ \eta \in \mathbb{R}^p \mid M^T\eta \leq 0 \}$ for some $\eta \in \mathbb{R}^p$. Moreover, if $\ker M^T = \{ 0 \}$, then $C^- = \{ 0 \}$.

**Proof.** The first statement readily follows from Farkas’ Lemma (see e.g. (Bertsekas, 2009, Prop. 2.3.1)). To prove the second statement, note that $C^- = -C \cap C^-$. Therefore, $\eta \in C^-$ if and only if there exist $\zeta_1 \geq 0$ and $\zeta_2 \leq 0$ such that $\eta = M^T\zeta_1 = M^T\zeta_2$. This means that $\zeta_1 - \zeta_2 \in \ker M^T$ and hence $\zeta_1 = \zeta_2 = 0$. Consequently, $\eta = 0$.}

Now, we are in a position to prove Proposition 6.

**Proof of Proposition 6:** The idea of the proof is to apply Proposition 4 for controllability and Proposition 5
for stabilizability. First note that, we have $\Psi(0) = 0$ due to (7)-(9). Therefore, hypotheses of both propositions are satisfied.

It follows from (9) that $\mathcal{Y}^\alpha$ is a polyhedral cone for all $\alpha \subseteq \{1, 2, \ldots, p\}$. As such, we have $C_t(\mathcal{Y}^\alpha) = \mathcal{Y}^\alpha$. Moreover, it follows from Lemma 7 that $\mathcal{Y}^{\emptyset} = \{0\}$ for all $\alpha \subseteq \{1, 2, \ldots, p\}$ since the matrix

$\begin{bmatrix} -(F_{\alpha\alpha})^{-1} & 0_{|\alpha| \times |\alpha|} \\ -F_{\alpha'\alpha}(F_{\alpha\alpha})^{-1} & I_{|\alpha'|} \end{bmatrix}$

is nonsingular.

As (8) does not contain translation term, that is $g^\alpha = 0$, we see that the condition (iv) of Proposition 4 is implied by the condition (ii) of Proposition 4 whereas the condition (iii) of Proposition 4 is implied by the condition (i) of Proposition 4. Moreover, the same reasoning holds if we replace Proposition 4 by Proposition 5 above.

Note that

$H^\alpha C = -E_{\alpha\alpha} F_{\alpha\alpha}^{-1} C_{\alpha} \quad \text{and} \quad H^\alpha D = -E_{\alpha\alpha} F_{\alpha\alpha}^{-1} D_{\alpha}.$

Therefore, we can conclude that (a) is implied by Proposition 4 and (b) by Proposition 5.

Proposition 6 requires checking the involved conditions for all index sets $\alpha \subseteq \{1, 2, \ldots, p\}$. It turns out that the structure of LCSs can be exploited to obtain more streamlined conditions that can be checked much more easily as stated next.

**Theorem 8.** Consider an LCS of the form (1) satisfying Assumption 3. Suppose that $F$ is a $P$-matrix. Then, the following statements hold:

(a) The system (2) is controllable if and only if the following two implications hold:

i. $(A, [B \ E])$ is controllable.

ii. $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$, $w \leq 0$, and

$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} E \\ F \end{bmatrix} \geq 0$

implies $z = 0$ and $w = 0$.

(b) The system (2) is stabilizable if and only if the following two implications hold:

i. $(A, [B \ E])$ is stabilizable.

ii. $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $z \in \mathbb{R}^n$, $w \leq 0$, and

$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} E \\ F \end{bmatrix} \geq 0$

implies $z = 0$ and $w = 0$.

The proof of this theorem will rely on the following intermediate result.

**Lemma 9.** Suppose that $(A, B, C, D)$ satisfies Assumption 3 and $F$ is a $P$-matrix. Then, the following statements hold:

(i) Let $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Then, for all $\alpha \subseteq \{1, 2, \ldots, p\}$

$z^* \left[ A - E_{\alpha\alpha} F_{\alpha\alpha}^{-1} C_{\alpha} - \lambda I \ B - E_{\alpha\alpha} F_{\alpha\alpha}^{-1} D_{\alpha} \right] = 0$

if and only if

$z^* \left[ A - \lambda I \ B \ E \right] = 0.$

(ii) Let $\lambda \in \mathbb{R}$ and $z \in \mathbb{R}^n$. Then, for all $\alpha \subseteq \{1, 2, \ldots, p\}$ there exists $w^\alpha \in (\mathcal{Y}^\alpha)^-$ such that

$\begin{bmatrix} z \\ w^\alpha \end{bmatrix}^T \left[ A - E_{\alpha\alpha} F_{\alpha\alpha}^{-1} C_{\alpha} - \lambda I \ B - E_{\alpha\alpha} F_{\alpha\alpha}^{-1} D_{\alpha} \right] = 0$

if and only if there exists $w \leq 0$ such that

$\begin{bmatrix} z \\ w \end{bmatrix}^T \left[ A - \lambda I \ B \right] = 0 \quad \text{and} \quad \begin{bmatrix} z \\ w \end{bmatrix}^T \left[ E \ F \right] \geq 0.$

**Proof.** For the ‘only if’ part of (i), note that the choice $\alpha = \emptyset$ results in

$z^* \left[ A - \lambda I \ B \right] = 0.$

Therefore, it remains to show that $z^* E = 0$. Let $\alpha = \{1, 2, \ldots, p\}$. Then, we have

$z^* \left[ A - EF^{-1} C - \lambda I \ B - EF^{-1} D \right] = 0.$

By using (10), we further obtain

$0 = z^* \left[ EF^{-1} C \ EF^{-1} D \right] = z^* EF^{-1} \left[ C \ D \right].$  \hspace{1cm} (11)

Note that $[C \ D]$ is full row rank due to Assumption 3. Therefore, it follows from (11) that $z^* E = 0$.

For the ‘if’ part of (i), note that $z^* E = 0$ implies that $z^* E_{\alpha\alpha} = 0$ for all $\alpha \subseteq \{1, 2, \ldots, p\}$. Therefore,

$z^* \left[ A - \lambda I \ B \ E \right] = 0$

implies

$z^* \left[ A - E_{\alpha\alpha} F_{\alpha\alpha}^{-1} C_{\alpha} - \lambda I \ B - E_{\alpha\alpha} F_{\alpha\alpha}^{-1} D_{\alpha} \right] = 0$

for all $\alpha \subseteq \{1, 2, \ldots, p\}$.

For the ‘only if’ part of (ii), we claim that $w^\emptyset \leq 0$,

$\begin{bmatrix} z \\ w^\emptyset \end{bmatrix}^T \left[ A - \lambda I \ B \right] = 0,$  \hspace{1cm} (12)

and

$\begin{bmatrix} z \\ w^\emptyset \end{bmatrix}^T \left[ E \ F \right] \geq 0.$  \hspace{1cm} (13)

Since (12) readily follows from the choice of $\alpha = \emptyset$, it remains to show that $w^\emptyset \leq 0$ and (13) holds. To do so, we first note that $\mathcal{Y}^\emptyset = \mathbb{R}^n_+$ due to (9). Since $w^\emptyset \in (\mathcal{Y}^\emptyset)^-$, we see that $w^\emptyset \leq 0$. For the rest, let $\bar{\alpha} = \{1, 2, \ldots, p\}$ and $\bar{w} = w^\emptyset$. Then, we have

$\begin{bmatrix} z \\ w - w^\emptyset \end{bmatrix}^T \left[ A - EF^{-1} C - \lambda I \ B - EF^{-1} D \right] = 0.$

By using (12), we obtain

$\begin{bmatrix} z \\ w - w^\emptyset \end{bmatrix}^T \left[ -EF^{-1} C \ -EF^{-1} D \right] = 0.$
Hence, we see that
\[(\bar{w}^T - (w^o)^T - z^T EF^{-1}) [C \; D] = 0.\]
Since \([C \; D]\) is full row rank due to Assumption 3, we obtain
\[
\begin{bmatrix}
z \\
w^o
\end{bmatrix}^T [E \; F] = \bar{w}^T F.
\]
As such, it is enough to show that \(\bar{w}^T F \geq 0\). Note that \(\bar{w} \in (\mathcal{Y}^o)^-\). Since \(CD\), we obtain
\[
\mathcal{Y}^\bar{a} = \{y \mid -F^{-1} y \geq 0\}
\]
due to (9), it follows from Lemma 7 that
\[
(\mathcal{Y}^\bar{a})^- = \{\eta \mid \eta = (F^T)^{-1} \zeta \text{ for some } 0 \leq \zeta\}.
\]
Therefore, \(\bar{w} = (F^T)^{-1} \bar{c} \zeta \geq 0\). Note that
\[
\bar{w}^T F = \bar{c}^T \zeta \geq 0.
\]
For the ‘if’ part of (ii), \(w\) be such that \(w \leq 0\),
\[
\begin{bmatrix}
z \\
w
\end{bmatrix}^T [A - \lambda I \; B] = 0,
\]
and
\[
\begin{bmatrix}
z \\
w
\end{bmatrix}^T [E \; F] \geq 0.
\]
For all \(\alpha \subseteq \{1, 2, \ldots, p\}\), define \(w^\alpha\) by
\[
w^\alpha = w_{-c} \quad \text{and} \quad (w^\alpha)^T = (w^\alpha)^T + z^T E_{\alpha} F^{-1}_{\alpha}
\]
With this definition, we have
\[
\begin{bmatrix}
z \\
w^\alpha
\end{bmatrix}^T [A - E_{\alpha} F^{-1}_{\alpha} C_{\alpha} - \lambda I \; B - E_{\alpha} F^{-1}_{\alpha} D_{\alpha}] = 0.
\]
It remains to show that \(w^\alpha \in (\mathcal{Y}^\alpha)^-\). From (15), we see that
\[
z^T E_{\alpha} + w_{-c}^T F_{\alpha} + w_{-c}^T F_{-\alpha} \geq 0.
\]
By using (16), we get
\[
z^T E_{\alpha} + ((w^\alpha)^T - z^T E_{\alpha} F^{-1}_{\alpha}) F_{\alpha} + (w_{-c}^\alpha)^T F_{-\alpha} \geq 0.
\]
This leads to
\[
(w^\alpha)^T F_{\alpha} + (w_{-c}^\alpha)^T F_{-\alpha} \geq 0.
\]
In addition, we have
\[
w^\alpha_{-c} \leq 0
\]
since \(w \leq 0\). By combining (17) and (18), we can write
\[
\begin{bmatrix}
(F_{\alpha})^T & (F_{-\alpha})^T \\
0 & -I
\end{bmatrix}
\begin{bmatrix}
w^\alpha \\
w_{-c}
\end{bmatrix} \geq 0.
\]
It follows from (9) and Lemma 7 that \(\eta \in (\mathcal{Y}^\alpha)^- \iff \eta = (F_{\alpha})^T (F_{\alpha})^{-T} ((F_{-\alpha})^T)^T \zeta\), for some \(\zeta \geq 0\). Note that
\[
\begin{bmatrix}
(F_{\alpha})^T & (F_{-\alpha})^T \\
0 & -I
\end{bmatrix}
\begin{bmatrix}
(F_{\alpha})^T & (F_{-\alpha})^T \\
0 & -I
\end{bmatrix} = I.
\]
Therefore, (19) implies that \(w^\alpha \in (\mathcal{Y}^\alpha)^-\).

Now, we are in a position to prove Theorem 8.

**Proof of Theorem 8:** In view of Lemma 9, the conditions in the statement (a) of Proposition 6 are equivalent with the conditions in the statement of Theorem 8. This proves the controllability claims of Theorem 8. The claims about stabilizability also follows from Lemma 9 in a similar fashion.

### 6. CONCLUSIONS

In this paper, we studied controllability and stabilizability of linear complementarity systems. The main results are Hautus-like necessary and sufficient conditions for both controllability and stabilizability. By working under the assumption that the transfer matrix of the underlying linear system from inputs to one of the complementarity variable is right invertible, we first showed that the existing tests for controllability and stabilizability of continuous piecewise affine systems can be streamlined. Based on these streamlined tests, we presented tailor-made tests for linear complementarity systems.

Future research directions include relaxing the right-invertibility assumption on the one hand and studying linear complementarity systems that cannot be cast as continuous piecewise affine systems on the other hand.

### REFERENCES


