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Cancellations in the Degree of the Colored Jones Polynomial

Christine Ruey Shan Lee¹ · Roland van der Veen²

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Abstract

We give an alternate expansion of the colored Jones polynomial of a pretzel link which recovers the degree formula in Garoufalidis et al. (*Internat. J. Math.* **31**(7):66, 2020). As an application, we determine the degrees of the colored Jones polynomials of a new family of 3-tangle pretzel knots.

Keywords Colored Jones polynomial · Pretzel link · Boundary slopes

Mathematics Subject Classification (2010) 57M25 · 57M27

1 Introduction

The discovery of the Jones polynomial and related quantum link and 3-manifold invariants has revolutionized knot theory. The colored Jones polynomial, an important example that is an invariant for a link in S^3 from the representation theory of the quantum group $U_q(\mathfrak{sl}_2)$, has been studied a great deal. For alternating and more generally adequate knots, it has been shown that the coefficients of their colored Jones polynomial stabilize [1, 3, 6] and give information on the hyperbolic volume and geometry of the knot complement [3, 4], and that their degrees relate to the boundary slopes of essential surfaces [5, 10].

For non-adequate knots, it is natural to ask the extent to which similar results hold. Results extending relationships observed for adequate knots exist [2, 8, 9, 13–15, 18]. However, it is difficult to study the colored Jones polynomial in complete generality, since the state sum which may be used to define the polynomial often has cancellations that are difficult to control. The purpose of this paper is to give an expansion of the colored Jones

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Fig. 1 Left: A diagram L for the pretzel link $P(w_0, w_1, \dots, w_m)$ with positive and negative twist regions. Middle: The n -cabled skein element L^n decorated by Jones-Wenzl projectors. Right: The skein element T resulting from applying the fusion and untwisting formulas to L^n . The center Jones-Wenzl projector to each twist region is shown as a black box

polynomial whose cancellations are more explicitly managed. The main ingredient is Khovanov's coefficient formulas for the expansion of the Jones-Wenzl projector in the canonical basis of the Temperley-Lieb algebra [12].

We will consider links in S^3 with diagrams from the projection to S^2 . Let $P(w_0, w_1, \dots, w_m)$ denote the pretzel link with standard diagram L consisting of $m+1$ vertical twist regions, each of $|w_i|$ crossings joined side by side, with the twist region of w_m joined to that of w_0 as shown on the left in Fig. 1. If $w_i > 0$ then the twist region is made up of positive crossings, and if $w_i < 0$, the twist region is made up of negative crossings.

Consider the diagram L as an element in the skein module $K(\mathbb{R}^2)$ of the plane $\mathbb{R}^2$. See Definition 2.1. For a fixed positive integer $n \geq 1$, cable a Jones-Wenzl projector as defined in Definition 2.2 and shown as a box $\bigcap_n$ on n strands in each component of L using the blackboard framing. Denote the resulting skein element by L^n . Using the idempotent property of the projector, double the projector(s) so that each twist region is framed by four projectors, then apply the fusion and untwisting formulas Fig. 7 to each of the twist regions of $P(w_0, w_1, \dots, w_m)$. Let $k = (k_0, k_1, \dots, k_m)$ be the set of fusion parameters. We get a skein element $T_k \in K(\mathbb{R}^2)$ that is the union of skein elements T_i in the Temperley-Lieb algebra TL_n for $0 \leq i \leq m$ decorated by Jones-Wenzl projectors joined side by side. For each T_i there is a center Jones-Wenzl projector from fusing and untwisting the twist region, and four framing projectors shared by adjacent skein elements T_{i-1}, T_{i+1} . See Fig. 1 for an illustration.

The Jones-Wenzl projector in TL_n has an expansion

$$\bigcap_n = \sum_{d \in B_n^{TL}} \mathbf{P}(d) d$$

in the canonical basis B_n^{TL} , where $\mathbf{P}(d)$ is the coefficient multiplying the skein element d of the basis in the expansion, see Definition 2.3. Let σ denote the choice of the skein element $\sigma = (d_1, \dots, d_m, \sigma_t^1, \sigma_b^1, \dots, \sigma_t^{m-1}, \sigma_b^{m-1})$ in the expansion of the Jones-Wenzl projectors decorating T_k . More precisely, for $1 \leq i \leq m$, $d_i \in B_n^{TL}$ is the choice of a skein element in the expansion of the center projector for T_i . For $1 \leq i \leq m-1$, $\sigma_t^i \in B_n^{TL}$ or $\sigma_b^i \in B_n^{TL}$ is the choice, in order, of a skein element of the top or bottom projector shared by T_i, T_{i+1} , where after choosing σ_t^i , we remove any circles attached to the bottom projector via Fig. 8, then we choose σ_b^i for the resulting bottom projector in TL_{n_i} . If the number of

circles removed is c_i , then $n_i = n - c_i$. For $n \geq 1$, we may write the $n + 1$ colored Jones polynomial of the pretzel link as follows. See Definition 2.5.

$$\begin{aligned} J_{P,n+1} &= \left((-1)^n q^{\frac{1}{2}} \right)^{w(L)(n^2+2n)} \langle L^n \rangle, \text{ where} \\ &= \sum_{0 \leq k_i \leq n, \sigma} \underbrace{\left(\prod_{i=0}^m \frac{[2k_i + 1]}{\theta(n, n, 2k_i)} U(w_i, k_i) \right) \left(\prod_{i=1}^m \mathbf{P}(d_i) \right) \left(\prod_{i=1}^{m-1} \mathbf{P}(\sigma_i^j) \mathbf{P}(\sigma_b^i) R(c_i) \right)}_{G_{k,\sigma}} \langle T_{k,\sigma}^n \rangle. \end{aligned} \quad (1)$$

Here $T_{k,\sigma}^n$ is the skein element that comes from applying σ to T_k , $\langle \cdot \rangle$ is the Kauffman bracket as in Definition 2.4, $w(L)$ is the writhe of the diagram L , $\frac{[2k_i+1]}{\theta(n,n,2k_i)} U(w_i, k_i)$ is a product of rational functions in q from the fusion and untwisting formulas Fig. 7, and $R(c_i)$ is the rational function resulting from removing c_i circles from a Jones-Wenzl projector in TL_n using Fig. 8.

Denote the rational function in q in (1) multiplying $\langle T_{k,\sigma}^n \rangle$ by $G_{k,\sigma}$. Our main result is the explicit formula of $G_{k,\sigma}$ in terms of quantum factorials for the leading terms of the state sum defining the colored Jones polynomial of a pretzel link.

Theorem 1.1 *Let $k = (k_0, k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ with $k_i \leq n$ and $\sigma = (d_1, \dots, d_m, \sigma_t^1, \dots, \sigma_b^{m-1})$ as described above. The $n + 1$ colored Jones polynomial of the pretzel link $P = P(w_0, w_1, \dots, w_m)$ with standard diagram L has the form*

$$\begin{aligned} J_{P,n+1} &= \left((-1)^n q^{\frac{1}{2}} \right)^{w(L)(n^2+2n)} \sum_{0 \leq k_i \leq n, \sigma} G_{k,\sigma} \langle T_{k,\sigma}^n \rangle \\ &= \left((-1)^n q^{\frac{1}{2}} \right)^{w(L)(n^2+2n)} \sum_{k_0 = \sum_{i=1}^m k_i} \mathcal{G}_k, \end{aligned}$$

where

$$\mathcal{G}_k = \left(\prod_{i=0}^m \frac{[2k_i+1]}{\theta(n,n,2k_i)} U(w_i, k_i) \right) \left(\prod_{i=1}^{m-1} \left(\frac{[n - \sum_{j=1}^i k_j]! [n - k_{i+1}]!}{[n - \sum_{j=1}^{i+1} k_j]! [n]!} \right)^2 \right) \langle T_{k_0} \rangle + l.o.t.$$

Here *l.o.t.* denotes lower order terms, and $\langle T_{k_0} \rangle$ is the Kauffman bracket of the following skein element shown in Fig. 2.

See (3) for the definition of the quantum factorial. As a direct corollary, we recover the degree formula¹ from [7].

Theorem 1.2 ([7, Theorem 3.2]) *Let $k = (k_0, k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ with $k_i \leq n$ and assume $|w_i| > 1$. The $n + 1$ colored Jones polynomial of a pretzel link $P = P(w_0, w_1, \dots, w_m)$ with diagram L has the form*

$$J_{P,n+1} = \left((-1)^n q^{\frac{1}{2}} \right)^{w(L)(n^2+2n)} \sum_{k_0 = \sum_{i=1}^m k_i} \mathcal{G}_k,$$

¹There is a slight difference in notation from the cited theorem where $\mathcal{G}_{c,k}$ there is our $2\mathcal{G}_k$.

Fig. 2 The skein element $\mathcal{T}_{k_0}$

where $\deg \mathcal{G}_k = (-1)^{w_0(n-k_0)+n+k_0+\sum_{i=1}^m(n-k_i)(w_i-1)} q^{\delta(n,k)} + l.o.t.^2$, and

$$\delta(n, k) = - \left((w_0 + 1)k_0^2 + \sum_{i=1}^m (w_i - 1)k_i^2 + \sum_{i=1}^m (-2 + w_0 + w_i)k_i - \frac{n(n+2)}{2} \sum_{i=0}^m w_i + (m-1)n \right).$$

The advantage of our approach is that each term of the state sum is explicit as described by (1) even if they are lower order. For 3-tangle pretzel knots $P = P(w_0, w_1, w_2)$ we show the following.

Theorem 1.3 Let $w = (w_0, w_1, w_2) \in \mathbb{Z}^3$ be such that $w_0 < -1 < 0 < 1 < w_1, w_2$. Define

$$s(w) = 1 + w_0 + \frac{1}{\sum_{i=1}^2 (w_i - 1)^{-1}} \text{ and } s_1(w) = \frac{\sum_{i=1}^2 (w_i + w_0 - 2)(w_i - 1)^{-1}}{\sum_{i=1}^2 (w_i - 1)^{-1}}.$$

Suppose w_1 is even and $-w_0 > \min\{w_1 - 1, w_2 - 1\}$. Let P denote the pretzel knot $P(w_0, w_1, w_2)$, and let $j_P(n)$ be the largest power of q in $J_{P,n}$. If $s(w) < 0$, we may write

$$j_P(n) = js_P n^2 + jx_P(n)n + c_P(n),$$

where jx_P, c_P are periodic functions in n . In particular we have

(a) For $n = \frac{-2+w_1+w_2}{\gcd(w_1-1, w_2-1)} j$, $j \geq 1$:

$$\begin{aligned} js_P &= -s(w) + w_0 + w_2, \\ jx_P(n) &= -s_1(w) + 2s(w) - (m-1) - 2 \frac{\min\{w_1-1, w_2-1\}}{-2+w_1+w_2}. \end{aligned} \quad (2)$$

(b) For $n \neq \frac{-2+w_1+w_2}{\gcd(w_1-1, w_2-1)} j$:

$$js_P = -s(w) + w_0 + w_2, \quad jx_P(n) = -s_1(w) + 2s(w) - (m-1).$$

²Denotes lower order terms.

Example 1.1 The pretzel knot $P(-5, 4, 3)$. Note $s(w) = -\frac{14}{5} < 0$ and $s_1(w) = -\frac{18}{5}$. Since

$$\frac{\min\{w_1 - 1, w_2 - 1\}}{\gcd(w_1 - 1, w_2 - 1)} = \frac{\min\{4 - 1, 3 - 1\}}{\gcd(4 - 1, 3 - 1)} = 2,$$

we have by Theorem 1.3, for $n \equiv 0 \pmod{5}$, $\text{js}_P = \frac{4}{5}$, and $\text{jx}_P(n) = -3 - \frac{4}{5}$. Otherwise for $n \not\equiv 0 \pmod{5}$, $\text{js}_P = \frac{4}{5}$, and $\text{jx}_P(n) = -3$.

The explicitly calculated *cancellation* of the terms in the expansion (1) of the n colored Jones polynomial is the decrease by $2 \frac{\min\{w_1-1, w_2-1\}}{-2+w_1+w_2}$ in $\text{jx}_P(n)$ in (2) for certain congruence of n . This is of interest as there is yet to be an explanation for the deviation of $\text{jx}_P(n)$ from the Euler characteristic of an essential surface in the complement of P from the viewpoint of the strong slope conjecture, see [5, 10]. We expect that the expansion would also apply to find explicit stable coefficients studied in [1, 6], and we will address this question in the future.

This paper is organized as follows: In Section 2 we summarize the background necessary to understand (1) by expanding the Jones-Wenzl projectors in the canonical basis. In Section 3, we prove lemmas useful for comparing degrees of $G_{k,\sigma}$. Then we prove Theorem 1.1 in Section 4, show how Theorem 1.2 follows, and conclude with the proof of Theorem 1.3 in Section 4.1.

2 Preliminaries

We follow [12]. We define for non-negative integers n, k , and indeterminate q ,

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! = [n][n-1] \cdots [1], \quad (3)$$

with the convention that $[0]! = 1$, and the binomial coefficient from the quantum factorial

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]}.$$

Definition 2.1 Let $\mathbb{C}(q)$ be the field of rational functions in q with complex coefficients. Given a connected, orientable surface F , the skein module $K(F)$ is the vector space over $\mathbb{C}(q)$ generated by the set of isotopy classes of framed unoriented and properly embedded link/tangle diagrams T modulo the Kauffman skein relations³ [11]

$$(1) \quad \bigcirc \sqcup T = (-q - q^{-1})T,$$

$$(2) \quad \times = q^{-1/2} \quad (+q^{1/2} \asymp).$$

Let D^2 be a disk in the plane whose boundary is viewed as a rectangle, with n marked points above and below. The *Temperley-Lieb algebra* TL_n is the skein module $K(D^2)$ of tangle/link diagrams in this disk, where we restrict to tangle diagrams whose endpoints are a

³Our convention differs from [12] in that q there is our q^{-1} .

Fig. 3 $T_1 \circ T_2$ in TL_n

subset of the $2n$ marked points on the boundary of the disk. There is a natural multiplication of two elements $T_1, T_2 \mapsto T_1 \circ T_2$ in TL_n by stacking the square containing T_1 above the square containing T_2 and identifying the n boundary points. See Fig. 3 below.

In this paper a rectangle D^2 with $2n$ marked boundary points defining TL_n will sometimes be depicted with a rotation from the horizontal position of Fig. 3. One can infer the direction of the multiplication based on the context of the figures. An algebra generator U_i of TL_n is shown below in Fig. 4.

The canonical dual basis B_n^{TL} of TL_n is the set of all crossingless matchings of the $2n$ boundary points of D^2 .

Definition 2.2 The Jones-Wenzl projector $\bigcap_n$ is an element in TL_n that is uniquely characterized by the following properties:

- $\bigcap_n \circ \bigcap_n = \bigcap_n, n \geq 1.$
- $\bigcap_n \circ U_i = 0 = U_i \circ \bigcap_n, 1 \leq i \leq n-1.$

Definition 2.3 Let d be an element in the canonical dual basis B_n^{TL} of TL_n . We define $P(d)$ to be the coefficient in the expansion of the Jones-Wenzl projector $\bigcap_n$ in TL_n .

$$[n]! \bigcap_n = \sum_{d \in B_n^{TL}} P(d) d.$$

$P(d)$ is a rational function in q with complex coefficients. We will denote by $\mathbf{P}(d)$ the coefficient divided by $[n]!$

$$\mathbf{P}(d) = \frac{P(d)}{[n]}.$$

Fig. 4 An algebra generator U_i of TL_n

It follows that

$$\bigcap_n = \sum_{d \in B_n^{TL}} \mathbf{P}(d)d.$$

From now on, whenever we say “expansion of the Jones-Wenzl projector”, we shall mean the expansion of the Jones-Wenzl projector in terms of the canonical basis as in Definition 2.3.

Definition 2.4 Consider the skein module $K(\mathbb{R}^2)$ of the plane $\mathbb{R}$. The *Kauffman bracket* of a skein element $\mathcal{S}$ in $K(\mathbb{R}^2)$, denoted by $\langle \mathcal{S} \rangle$, is the rational function in q multiplying the empty diagram after resolving $\mathcal{S}$ via the Kauffman skein relations.

Definition 2.5 Given a link diagram L and an integer $n \geq 1$, cable a Jones-Wenzl projector in each component using blackboard cabling and denote the resulting diagram by L^n . Let $\omega(L)$ be the writhe of the diagram L . Then the $n+1$ colored Jones polynomial $J_{L,n+1}$ may be defined as

$$J_{L,n+1} = \left((-1)^n q^{1/2} \right)^{\omega(L)(n^2+2n)} \langle L^n \rangle.$$

This material is well known and interested readers may consult [16, 19] for additional background.

Normalization With our conventions, the n (unreduced) colored Jones polynomial of the unknot is

$$J_{\bigcirc,n} = (-1)^{n-1} [n].$$

We will be using the following lemmas from [12]. All symbols $x, y, z, t, a, b, c, k, n$ will denote non-negative integers. A non-negative integer next to a strand indicates that number of parallel strands.

Lemma 2.1 [12, Proposition 4.8] *Let d_1 and d_2 be two diagrams in TL_n that differ as shown in Fig. 5.*

They are the same outside the part that is shown. Then

$$P(d_1) = \begin{bmatrix} x+y \\ x \end{bmatrix} P(d_2).$$

Lemma 2.2 [12, Proposition 4.9] *Let d_1, d_2 be two diagrams from TL_n that differ as depicted below in Fig. 6.*

They are the same outside the part that is shown. Then

$$P(d_1) = \begin{bmatrix} x+y \\ x \end{bmatrix} P(d_2).$$

Fig. 5 Elements d_1 and d_2 in TL_n

Fig. 6 Elements d_1 and d_2 in TL_n

Lemma 2.3 [12, Proposition 4.10] *Let d be a diagram as shown, where x, y, z, t are non-negative integers satisfying $x + y + z + t \geq 1$.*

Then

$$P(d) = \begin{bmatrix} x+y \\ y \end{bmatrix} \begin{bmatrix} t+y \\ y \end{bmatrix} [y]! [x+z+t+y]!$$

We will also take as given the following fusion and untwisting formulas in Figs. 7 and 8, and the formula for removing circles from a projector [17]. A triple of even integers (a, b, c) is called *admissible* if $a \leq b + c$, $b \leq a + c$, and $c \leq a + b$.

Define

$$U(w, k) = \left((-1)^{n-k} q^{n-k+\frac{n^2}{2}-k^2} \right)^w.$$

We have from Fig. 7,

Also define

$$R(c) := (-1)^c \frac{[n+2]}{[n+2-c]}.$$

Fig. 7 Fusion and untwisting formulas

Fig. 8 The formula for removing circles, innermost first, from a Jones-Wenzl projector

$$n - c \quad \Bigg| \quad \text{circle with } c \text{ strands} \quad = (-1)^c \frac{[n+2]}{[n+2-c]} \quad \Bigg| \quad n - c$$

This is the coefficient multiplying $n - c$ strands after removing c circles using Fig. 8.

For an admissible triple of integers (a, b, c) , the function $\theta(a, b, c)$ is $\langle \Theta(a, b, c) \rangle$ of the skein element $\Theta(a, b, c)$ as shown below in Fig. 9

$$\theta(a, b, c) = (-1)^{x+y+z} \frac{[x+y+z+1]![x]![y]![z]!}{[y+z]![z+x]![x+y]!},$$

where $x = \frac{a+b-c}{2}$, $y = \frac{b+c-a}{2}$, and $z = \frac{a+c-b}{2}$.

For a rational function $\mathcal{L}(q)$ in q with complex coefficients, the degree, $\deg(\mathcal{L})$ is the maximum power in q of the Laurent series expansion of $\mathcal{L}(q)$ whose degree is bounded from above. Note

$$\deg[c] = c - 1, \quad \text{and} \quad \deg \theta(a, b, c) = \frac{a + b + c}{2}.$$

3 Choosing Skein Elements in the Expansion of a Jones-Wenzl Projector

We collect useful lemmas classifying the possible choices of skein elements in the expansion of Jones-Wenzl projectors decorating certain skein elements. We will use these to justify our state sum of the colored Jones polynomial of pretzel links in Theorem 1.1. As assumed throughout the paper, the symbols labeling strands of tangle diagrams are non-negative integers, and they denote the number of parallel strands. We will use dashed bounding boxes to indicate the Jones-Wenzl projector we expand in each lemma.

Lemma 3.1 Suppose that we have a skein element as in Fig. 10.

Then a skein element d in the expansion of the center Jones-Wenzl projector (shown with a dashed bounding box) that does not result in a cap or a cup composed with one of the four framing Jones-Wenzl projectors is of the following form with $\tilde{k} \leq k$, see Fig. 11.

Fig. 9 The skein element $\Theta(a, b, c)$

Fig. 10 The center Jones-Wenzl projector is shown with a dashed border

The choice of the skein element d has coefficient

$$P(d) = \begin{bmatrix} k \\ k - \tilde{k} \end{bmatrix}^2 [k - \tilde{k}]! [k + \tilde{k}]!$$

in the expansion of the Jones-Wenzl projector.

Proof Due to the four projectors framing the center projector, none of the four endpoints on the dashed bounding box, each with k strands coming out, can be connected to itself through a choice of a skein element in the expansion. Otherwise, the resulting skein element would have a cap or a cup composed with one of the framing projectors. Nor can any of the four endpoints be connected to another across the diagonal, because it would result in a cap or a cup composed with the pair of projectors of the opposite diagonal. The coefficient follows directly from applying Lemma 2.3. $\square$

Definition 3.1 Given a crossingless diagram in TL_n without Jones-Wenzl projectors, a *through strand* is a strand of the diagram with one end in one of the top n points of the boundary of the disk D^2 defining TL_n , and the other end in one of the bottom n points.

Let $T \in TL_n$ be a skein element decorated by Jones-Wenzl projectors. We will denote by $\overline{T}$ the skein element obtained from T by replacing all its Jones-Wenzl projectors by the identity.

Lemma 3.2 Suppose we have a skein element T of the following form as in Fig. 12.

Denote by T_σ the skein element obtained from T by replacing the center Jones-Wenzl projector (shown with a dashed bounding box) by a choice of a skein element σ in its expansion. The choices of skein elements $\sigma = \sigma_i^1, \sigma_i^2$, or σ_i^3 which do not result in a cap or a cup composed with one of the four framing projectors have the following form as in Fig. 13.

Fig. 11 Possibilities for the skein element d in the expansion of the center Jones-Wenzl projector that does not result in a zero skein element

Fig. 12 The center Jones-Wenzl projector is indicated with a dashed rectangular box

The coefficients of the choices of skein elements σ_t^1 , σ_t^2 , and σ_t^3 are respectively

$$\begin{aligned} P(\sigma_t^1) &= \begin{bmatrix} b + \tilde{\ell}_1 \\ \tilde{\ell}_1 \end{bmatrix} \begin{bmatrix} t + \tilde{\ell}_1 \\ \tilde{\ell}_1 \end{bmatrix} [\tilde{\ell}_1]! [b + \tilde{\ell}_2 + t]!, \\ P(\sigma_t^2) &= \begin{bmatrix} b + \tilde{\ell}_2 \\ \tilde{\ell}_2 \end{bmatrix} \begin{bmatrix} t + \tilde{\ell}_2 \\ \tilde{\ell}_2 \end{bmatrix} [\tilde{\ell}_2]! [b + \tilde{\ell}_1 + t]!, \\ P(\sigma_t^3) &= \begin{bmatrix} b + \tilde{\ell}_1 \\ \tilde{\ell}_1 \end{bmatrix} \begin{bmatrix} t + \tilde{\ell}_1 \\ \tilde{\ell}_1 \end{bmatrix} [\tilde{\ell}_1]! [b + \tilde{\ell}_1 + t]!. \end{aligned}$$

Moreover, if $k_1 + k_2 \leq n$, then the maximum possible number of through strands of $\overline{T_\sigma}$ is $k_1 + k_2$, and there are three possible choices of skein elements in the expansions of the center Jones-Wenzl projector which achieve this: $\sigma = \overline{\sigma}_t^1$, $\overline{\sigma}_t^2$, or $\overline{\sigma}_t^3$ according to whether $k_1 < k_2$, $k_1 > k_2$, or $k_1 = k_2$, see Fig. 14.

The coefficients of the choices of skein elements are: $P(\overline{\sigma}_t^1) = P(\overline{\sigma}_t^2) = \frac{[n-k_1]![n-k_2]!}{[n-(k_1+k_2)]!}$, and $P(\overline{\sigma}_t^3) = \frac{([n-k_1]!)^2}{[n-2k_1]!}$.

Proof The first statement follows directly from examining the possibilities of connecting the endpoints on the dashed box of the center Jones-Wenzl projector. We enumerate the choices of a skein element in its expansion that would not result in a cap or a cup composed with a projector. The proof of the second statement in the case $k_1 + k_2 \leq n$ is similar. The formulas for the coefficients are obtained by definition and applying Lemma 2.3. $\square$

Fig. 13 Three choices of skein elements

Fig. 14 Choices of skein elements maximizing through strands when $k_1 + k_2 \leq n$

Lemma 3.3 Suppose we have a skein element in TL_n of the following form as shown in Fig. 15, where a skein element σ_t is previously chosen in the expansion of the Jones-Wenzl projector in the top dashed box. By Lemma 3.2, σ_t is one of the choices σ_t^1 , σ_t^2 , and σ_t^3 in Fig. 13.

Choosing σ_t results in a skein element which may have $c > 0$ circles attached to the bottom projector. After removing c circles from the bottom projector via Fig. 8, there are three choices, $\sigma = \sigma_b^1$, σ_b^2 , and σ_b^3 , up to mirror images via a reflection across the vertical axis, of a skein element in the expansion for the bottom Jones-Wenzl projector in TL_{n-c} (from the empty dashed box in Fig. 15) that does not result in a cap or a cup composed with the four framing Jones-Wenzl projectors in Fig. 16.

The coefficient of each choice of skein element σ_b^i for $1 \leq i \leq 3$ and their mirror images $\check{\sigma}_b^i$ are

$$P(\sigma_b^1) = P(\check{\sigma}_b^1) = \begin{bmatrix} r_1 + r_2 \\ r_1 \end{bmatrix} \begin{bmatrix} b + \ell \\ \ell \end{bmatrix} [\ell]! [b + s + \ell]!,$$

$$P(\sigma_b^2) = P(\check{\sigma}_b^2) = \begin{bmatrix} s + r_1 \\ r_1 \end{bmatrix}^{-1} \begin{bmatrix} r_1 + r_2 \\ r_1 \end{bmatrix} \begin{bmatrix} b + \ell \\ \ell \end{bmatrix} \begin{bmatrix} s + \ell \\ \ell \end{bmatrix} [\ell]! [b + s + \ell]!, \text{ and}$$

$$P(\sigma_b^3) = P(\check{\sigma}_b^3) = \begin{bmatrix} t + \ell \\ \ell \end{bmatrix} \begin{bmatrix} b + \ell \\ \ell \end{bmatrix} [\ell]! [b + t + \ell]!.$$

Proof Given a choice of skein element $\sigma_t = \sigma_t^1$ or σ_t^2 from Fig. 13 for the top center Jones-Wenzl projector, we have the following picture, shown in the middle of Fig. 17, of the strands coming out of the remaining bottom projector, up to reflection across the vertical axis.

After removing c circles connected to the bottom projector as the result of choosing σ_t , we have the picture on the right of Fig. 17. In order for a choice of a skein element in the

Fig. 15 The middle pair of projectors are shown with dashed boxes

Fig. 16 All the possibilities for a skein element in the expansion of the bottom projector after choosing a skein element in the expansion for the top projector. Note we must have $r_1 + r_2 = \ell$ for both σ_b^1 and σ_b^2

expansion of the bottom projector not to result in a cup or a cap composed with one of the four projectors, we need to connect the strands coming out of an endpoint on the box to another. Examination will show that the only possible cases are σ_b^1 or σ_b^2 as shown in Fig. 16, up to mirror images via a reflection across the vertical axis. For the choice of $\sigma_t = \sigma_t^3$ from Fig. 13, the choice of a skein element σ for the bottom projector has to be σ_b^3 as shown in Fig. 16 using similar arguments. The coefficients are computed for σ_b^1 , σ_b^2 and their mirror images by applying Lemmas 2.1 and 2.2, and the coefficient for σ_b^3 and its mirror image is computed by directly applying Lemma 2.3. $\square$

4 The Degree of the Colored Jones Polynomial of Pretzel Knots

Consider the skein element T comprised of two skein elements T_1, T_2 decorated with Jones-Wenzl projectors joined side by side as in the first picture of Fig. 18.

We define a sequence $\sigma = (d_1, d_2, \sigma^t, \sigma^b)$ of choices of skein elements for certain Jones-Wenzl projectors decorating T . First choose skein elements d_1, d_2 in the expansion of the center Jones-Wenzl projectors (marked with dashed boxes) as in Lemma 3.1. Then, using Lemma 3.2, pick the skein element σ_t in the expansion of the top projector shown with a dashed box. After σ_t is chosen, remove any circles attached to the bottom projector via Fig. 8, then use Lemma 3.3 to pick the skein element σ_b in the expansion of the bottom projector shown with a dashed box. Let $k = (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2$. Define $F_{k,\sigma}(q)$ to be the product of rational functions in q resulting from this sequence of moves, replacing projectors by

Fig. 17 We remove the circles shown in blue via Fig. 8

Fig. 18 The sequence of choices of expansions for an adjacent set of skein elements

skein elements in the expansion, and removing circles by Fig. 8.

$$F_{k,\sigma}(q) := \mathbf{P}(\sigma_t)\mathbf{P}(\sigma_b)R(c) \prod_{i=1,2} \mathbf{P}(d_i),$$

where c is the number of removed circles, and recall $R(c) = (-1)^c \frac{[n+2]}{[n+2-c]}$. Define $T_{k,\sigma}$ to be the skein element resulting from this sequence of moves applied to T .

$$T = \sum_{\substack{k,\sigma: 0 \leq k_i \leq n, \\ n, n, 2k_i \text{ admissible}}} F_{k,\sigma}(q) T_{k,\sigma}.$$

Before proving Theorem 1.1, we use Lemmas 3.1, 3.2, and 3.3 to establish a lemma comparing the degrees of $F_{k,\sigma}(q)$ coming from different choices of skein elements $\sigma = (d_1, d_2, \sigma_t, \sigma_b)$ in the expansions of the Jones-Wenzl projectors. Recall $\overline{T_{k,\sigma}}$ is the skein element obtained by replacing all the Jones-Wenzl projectors of $T_{k,\sigma}$ by the identity skein element id .

Lemma 4.1 Suppose we have parameters $0 \leq k_1 \leq n, 0 \leq k_2 \leq n$, and a choice of skein elements $\sigma = (d_1, d_2, \sigma_t, \sigma_b)$ is such that $\overline{T_{k,\sigma}}$ has $2(\ell_1 + \ell_2)$ through strands. Then there exists a set of parameters $\ell = (\ell_1, \ell_2)$ and a choice of skein elements $\overline{\sigma} = (\overline{d}_1 = id, \overline{d}_2 = id, \overline{\sigma}_t, \overline{\sigma}_b)$ in the expansions of the corresponding Jones-Wenzl projectors, such that $\overline{T_{\ell,\overline{\sigma}}}$ has the same number of through strands as $\overline{T_{k,\sigma}}$, $\ell_i \leq k_i$ for $i = 1, 2$, and $\ell_i \leq \ell_j$ if $k_i \leq k_j$. Moreover, letting $l_1 = k_1 - \ell_1$ and $l_2 = k_2 - \ell_2$ and $\tilde{\ell}_1, \tilde{\ell}_2$ be the intermediate through strands coming from choosing σ_t and removing circles attached to the bottom projector, see Fig. 18, we have

$$\deg F_{k,\sigma}(q) - \deg F_{\ell,\overline{\sigma}}(q) \leq \sum_{i=1}^2 l_i(2\ell_i + l_i). \quad (4)$$

Proof First note that the number of through strands $2(\ell_1 + \ell_2)$ of $\overline{T_{k,\sigma}}$ satisfies $2(\ell_1 + \ell_2) \leq 2n$. Let $\ell = (\ell_1, \ell_2)$, with $\ell_i \leq k_i$ and such that $\ell_1 + \ell_2 \leq n$. By Lemmas 3.2 and 3.3, we can always find a sequence of choices of skein elements $\overline{\sigma} = (\overline{d}_1, \overline{d}_2, \overline{\sigma}_t, \overline{\sigma}_b)$ with the number of through strands of $\overline{T_{\ell,\overline{\sigma}}}$ equal to that of $\overline{T_{k,\sigma}}$ by letting $\overline{d}_1, \overline{d}_2$ be the identity and

Fig. 19 A picture of the skein element $T_{\ell, \bar{\sigma}}$. The middle pair of projectors are shown with dashed boxes

$\bar{\sigma}_t = \sigma_t^1$ or σ_t^2 be as in Lemma 3.2, depending on whether $\ell_2 > \ell_1$ or $\ell_2 < \ell_1$, and $\bar{\sigma}_b$ its mirror image via a reflection across the horizontal axis, see Fig. 19.

Suppose for $\sigma = (d_1, d_2, \sigma_t, \sigma_b)$, the skein element d_i for the middle projector for each twist region w_i is not the identity, then choosing it results in $\tilde{k}_1, \tilde{k}_2$ parameters as in the first figure of the second row of Fig. 18. Note $\tilde{k}_1 + \tilde{k}_2 \geq \ell_1 + \ell_2$ and $\deg(P(d_1)P(d_2)) \leq \deg(P(\bar{d}_1)P(\bar{d}_2))$.

In the next step, a skein element σ_t is chosen for the top projector, which will result in intermediate $\tilde{\ell}_1 + \tilde{\ell}_2 + k_1 + k_2$ through strands for the top half of the skein element $\overline{T_{k, d_1, d_2, \sigma_t}}$, see Fig. 20. Note $\tilde{\ell}_1 + \tilde{\ell}_2 + k_1 + k_2 \geq 2(\ell_1 + \ell_2)$. If $\tilde{\ell}_1 + \tilde{\ell}_2 + k_1 + k_2 = 2(\ell_1 + \ell_2)$, then the choice of the skein element σ_b of the bottom projector is necessarily the mirror image of that of σ_t via a reflection across the horizontal axis, with top horizontal strands removed. Call this choice of expansion $\hat{\sigma} = (\hat{d}_1 = id, \hat{d}_2 = id, \hat{\sigma}_t, \hat{\sigma}_b)$. Denote the intermediate through strands of $\hat{\sigma}$ as $\hat{\ell}_1, \hat{\ell}_2$. See Fig. 21 for an example of $\hat{\sigma}_t$.

We first compare σ to $\hat{\sigma}$. Suppose the choice of skein elements (d_1, d_2, σ_t) results in intermediate through strands $\tilde{\ell}_1 + \tilde{\ell}_2 > \hat{\ell}_1 + \hat{\ell}_2$ for the top half of the skein element $\overline{T_{k, d_1, d_2, \sigma_t}}$ (the result of choosing d_1, d_2, σ_t for the Jones-Wenzl projectors of T then replacing all the remaining projectors by the identity). Without loss of generality assume $\tilde{\ell}_2 \geq \hat{\ell}_1$. Comparing the skein elements σ_t and $\hat{\sigma}_t$, this means $\tilde{\ell}_1 + \tilde{\ell}_2 > \hat{\ell}_1 + \hat{\ell}_2$ and $\tilde{t} + \tilde{b} < \hat{t} + \hat{b}$. We must have $\hat{\ell}_2 - \hat{\ell}_1 = \tilde{\ell}_2 - \tilde{\ell}_1$ and $\hat{\ell}_1 = \tilde{\ell}_1 - l$ for $l > 0$. This is because with fixed parameters k_1, k_2 and using Lemma 3.2, the choice of skein element for σ_t corresponding to a given number of through strands is determined by the number of turnbacks (which is ℓ_1 if $\tilde{\ell}_2 > \tilde{\ell}_1$ and $\tilde{\ell}_2$ otherwise). Then, since $\tilde{\ell}_1 + \tilde{\ell}_2 > \hat{\ell}_1 + \hat{\ell}_2$, the choice of the skein element σ_b must decrease the number of resulting through strands to $\hat{\ell}_1 + \hat{\ell}_2$ so that $\overline{T_{k, \sigma}}$ would still have $2(\ell_1 + \ell_2)$ through strands. The possibilities from Lemmas 3.2 and 3.3 are shown in Fig. 21.

We show $\deg F_{k, \sigma} \leq \deg F_{k, \hat{\sigma}}$ for all the cases of σ_t shown in Fig. 21 and note that the arguments are the same for their mirror images via a reflection across the vertical axis. We compute the difference between $\deg F_{k, \sigma}, \deg F_{k, \hat{\sigma}}$ using the explicit formulas of the quantum factorials. Since the algebraic computations based on the formulas of Lemmas 3.2 and 3.3 are straightforward, we will omit them and show the results of the computations. To simplify notations, we also let $p(x, y, z, t) = \deg P(d)$, where d is the skein element in

Fig. 20 The step generating intermediate through strands $\tilde{\ell}_1, \tilde{\ell}_2$ after first choosing a skein element σ_t on top

Fig. 21 Left: Comparing σ_t and $\hat{\sigma}_t$. Right: Possibilities for σ_b corresponding to σ_t . We changed ℓ to e from Lemma 3.3

Lemma 2.3. Note by the preceding discussion, we know $\tilde{\ell}_1 = \hat{\ell}_1 + l$, $\tilde{b} = \hat{b} - l$, $\tilde{t} = \hat{t} - l$, and $n = \hat{b} + \hat{\ell}_1 + \hat{\ell}_2 + \hat{t}$.

Case 1 $\tilde{\ell}_2 - \tilde{\ell}_1 = 0$. This would force $\sigma_b = \sigma_b^3$ as in Fig. 21, and $\hat{\ell}_2 - \hat{\ell}_1 = 0$. We have

$$\begin{aligned} & \deg(F_{k,\sigma}) - \deg(F_{k,\hat{\sigma}}) \\ &= \deg(\mathbf{P}(\sigma_t)\mathbf{P}(\sigma_b)R(\tilde{b})\mathbf{P}(d_1)\mathbf{P}(d_2)) - \deg(\mathbf{P}(\hat{\sigma}_t)\mathbf{P}(\hat{\sigma}_b)R(\hat{b})\mathbf{P}(id)\mathbf{P}(id)) \\ &= \underbrace{p(\tilde{b}, \tilde{\ell}_1, 0, \tilde{t})}_{\deg(\mathbf{P}(\sigma_t))} + \underbrace{\tilde{b}}_{\deg R(\tilde{b})} + \underbrace{p(b, e, 0, 0)}_{\deg(\mathbf{P}(\sigma_b))} - \deg[n - \tilde{b}]! \\ & \quad - \left(\underbrace{p(\hat{b}, \hat{\ell}_1, 0, \hat{t})}_{\deg(\mathbf{P}(\hat{\sigma}_t)\mathbf{P}(\hat{\sigma}_b))} + \underbrace{\hat{b}}_{\deg R(\hat{b})} - \underbrace{p(\hat{b}, \hat{\ell}_1, 0, 0)}_{\deg(\mathbf{P}(\hat{\sigma}_b))} - \deg[n - \hat{b}]! \right). \end{aligned}$$

We can use the fact that $T_{k,\sigma}$ and $T_{k,\hat{\sigma}}$ have the same number of through strands to get $b = \hat{t}$, $e = \hat{\ell}_1$, and $t = l_1$. Plugging in and simplifying, we get

$$\deg(F_{k,\sigma}) - \deg(F_{k,\hat{\sigma}}) \leq -l(1 + l + 2\hat{\ell}_1) \leq 0$$

since $l, \hat{\ell}_1 \geq 0$ and $\deg(\mathbf{P}(d_1)\mathbf{P}(d_2)) \leq \deg(\mathbf{P}(id)\mathbf{P}(id))$.

Case 2 $\tilde{\ell}_2 - \tilde{\ell}_1 \neq 0$. Either $\sigma_b = \sigma_b^2$ or σ_b^1 as in Fig. 21.

2 (a) $\sigma_b = \sigma_b^2$.

$$\begin{aligned} & \deg(F_{k,\sigma}) - \deg(F_{k,\hat{\sigma}}) \\ &= \deg(\mathbf{P}(\sigma_t)\mathbf{P}(\sigma_b)R(c)\mathbf{P}(d_1)\mathbf{P}(d_2)) - \deg(\mathbf{P}(\hat{\sigma}_t)\mathbf{P}(\hat{\sigma}_b)R(\hat{c})\mathbf{P}(id)\mathbf{P}(id)) \\ &= p(\tilde{b}, \tilde{\ell}_1, \tilde{\ell}_2 - \tilde{\ell}_1, \tilde{t}) + \tilde{b} + \deg \left[\begin{matrix} s + r_1 \\ s \end{matrix} \right]^{-1} + \deg \left[\begin{matrix} r_1 + r_2 \\ r_1 \end{matrix} \right] + p(b, e, 0, s) - \deg[n - \tilde{b}]! \\ & \quad - \left(p(\hat{b}, \hat{\ell}_1, 0, \hat{t}) + \hat{b} - p(\hat{b}, \hat{\ell}_1, 0, 0) - \deg[n - \hat{b}]! \right). \end{aligned}$$

Again using the fact that $T_{k,\sigma}$ and $T_{k,\hat{\sigma}}$ have the same number of through strands, we get $b = \hat{t}$, $e = \hat{\ell}_2$, $s = l - (\hat{\ell}_2 - \hat{\ell}_1)$, $r_1 = \hat{\ell}_2 - \hat{\ell}_1$, and $r_2 = \hat{\ell}_1$. Plug in and simplify

$$\deg(F_{k,\sigma}) - \deg(F_{k,\hat{\sigma}}) \leq -l(1 + l + 2\hat{\ell}_2) \leq 0.$$

2 (b) $\sigma_b = \sigma_b^1$. With similar arguments, we get

$$\deg(F_{k,\sigma}) - \deg(F_{k,\hat{\sigma}}) \leq -l(1 + l + 2\hat{\ell}_2) \leq 0.$$

Fig. 22 The case $\ell_2 \geq \ell_1$ and $k_2 \geq k_1$ is shown

Thus without loss of generality, to prove the last statement of the lemma, (4), we may assume $d_1 = d_2 = id$, and σ_b is the mirror image of σ_t via a reflection across the horizontal axis (top horizontal strands removed), with intermediate through strands satisfying $\tilde{\ell}_1 + \tilde{\ell}_2 + k_1 + k_2 = 2(\ell_1 + \ell_2)$. We compare σ_t to $\bar{\sigma}_t$ in Fig. 22.

Let $k_1 - \ell_1 = l_1$ and $k_2 - \ell_2 = l_2$, then $\tilde{\ell}_1 + l_1 = \ell_1$ and $\tilde{\ell}_2 + l_2 = \ell_2$. We have, plugging into the equations for the degrees and assuming $k_1 \leq k_2$ (the other case is similar),

$$\begin{aligned} \deg(F_{k,\sigma}) - \deg(F_{\ell,\bar{\sigma}}) &\leq k_1 - \tilde{\ell}_1 + 2l_2\tilde{\ell}_1 + 2l_1(l_2 + \tilde{\ell}_2) = 2l_1 + 2l_2\tilde{\ell}_1 + 2l_1l_2 + 2l_1\tilde{\ell}_2 \\ &\leq \sum_{i=1}^2 l_i(2\ell_i + l_i). \end{aligned}$$

The argument for when $k_1 \geq k_2$ is similar. $\square$

We prove a refined version of Theorem 1.1. We will focus on the degree of the Kauffman bracket and suppress the monomial from the writhe term in the n colored Jones polynomial.

Theorem 4.1 Let $k = (k_0, k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ with $k_i \leq n$ and $\sigma = (d_1, \dots, d_m, \sigma_t^1, \dots, \sigma_b^{m-1})$ as described in the introduction. For the pretzel link $P = P(w_0, w_1, \dots, w_m)$ with standard diagram L , we have

$$\langle L^n \rangle = \sum_{0 \leq k_i \leq n, \sigma} G_{k,\sigma} \langle T_{k,\sigma}^n \rangle = \sum_{k_0 = \sum_{i=1}^m k_i} \mathcal{G}_k,$$

where

$$\mathcal{G}_k = \left(\prod_{i=0}^m \frac{[2k_i+1]}{\theta(n,n,2k_i)} U(w_i, k_i) \right) \left(\prod_{i=1}^{m-1} \left(\frac{[n - \sum_{j=1}^i k_j]! [n - k_{i+1}]!}{[n - \sum_{j=1}^{i+1} k_j]! [n]!} \right)^2 \right) \langle T_{k_0} \rangle + l(k).$$

Fig. 23 The skein element T_{k_0}

Here $l(k)$ denotes lower order terms, and $\langle T_{k_0} \rangle$ is the Kauffman bracket of the following skein element in Fig. 23.

Moreover,

$$\deg \mathcal{G}_k - \deg l(k) \geq 2 \min_{0 \leq i \leq m} \{|w_i| - 1\} \min_{0 \leq i \leq m} \{k_i\}. \quad (5)$$

Proof Recall $T_{k,\sigma}^n$ is the skein element that comes from applying σ to T_k . This is a skein element of the form as shown below in Fig. 24.

Note that if $k_0 > c$ then $T_{k,\sigma}^n = 0$ as proved in [13]. Thus, we can assume $k_0 \leq c$. Let

$$d(\kappa, w) = \deg \left(\frac{[2\kappa + 1]}{\theta(n, n, 2\kappa)} U(w, \kappa) \right).$$

Note that $d(\kappa, w)$ monotonically increases in κ if $w < 0$. Therefore we can also assume $k_0 = c$. We organize the state sum (1) by the number of strands c of $T_{k,\sigma}^n$. A pair (k, σ) is *tight* if $c = k_0 = \sum_{i=1}^m k_i$.

For every pair of parameters and state (κ, τ) with $2c$ through strands for $T_{\kappa,\tau}^n$, there is a tight state $(k(\kappa, \tau), \sigma(\kappa, \tau))$ with the same number of through strands $2c$ and such that $k_i \leq \kappa_i$ and $k_i \leq k_j$ if $\kappa_i \leq \kappa_j$ for all $1 \leq i \leq m$ by Lemma 4.1. Note $\sigma(k, \tau)$ is determined by $k(\kappa, \tau) = (k_0, k_1, \dots, k_m)$ as follows: Using Lemma 3.2, choose $d_i = id$ for all $1 \leq i \leq m$ and choose σ_t^1, σ_b^1 to be the state that will result in $2(k_1 + k_2)$ through strands for $\overline{T_{k,d_1,\dots,d_m,\sigma_t^1,\sigma_b^1}}$, where $T_{k,d_1,\dots,d_m,\sigma_t^1,\sigma_b^1}$ is the skein element that comes from applying $d_1, d_2, \sigma_t^1, \sigma_b^1$ to the portion of T_k that is the union of T_1, T_2 . Then choose σ_t^2, σ_b^2 to be the state that will result in $2(k_1 + k_2 + k_3)$ through strands for $\overline{T_{k,d_1,\dots,d_m,\sigma_t^1,\sigma_b^1,\sigma_t^2,\sigma_b^2}}$, and so on for the rest of σ_t^i, σ_b^i , which are states that will result in $2(k_1 + k_2 + \dots + k_{i+1})$ through strands for $\overline{T_{k,d_1,\dots,d_m,\sigma_t^1,\sigma_b^1,\sigma_t^2,\sigma_b^2,\dots,\sigma_t^i,\sigma_b^i}}$, where $T_{k,d_1,\dots,d_m,\sigma_t^1,\sigma_b^1,\dots,\sigma_t^i,\sigma_b^i}$ is the skein element that comes from applying $d_1, d_2, \dots, d_m, \sigma_t^1, \sigma_b^1, \dots, \sigma_t^i, \sigma_b^i$ to the portion of T_k that is the union of $T_1, T_2, \dots, T_{i+1}$. It is possible to have more than one $(k(\kappa, \tau), \sigma(\kappa, \tau))$ for a pair (κ, τ) , since there could be more than one $k(\kappa, \tau)$ that satisfies the above conditions. The argument is not affected by the choice of $k(\kappa, \tau)$.

To prove that every term corresponding to the pair (κ, τ) that is not tight is a lower order term in the sum, we compare the degree of its coefficient function $G_{\kappa,\tau}$ to that of the tight

Fig. 24 $T_{k,\sigma}^n$

state corresponding to the pair $(k(\kappa, \tau), \sigma(\kappa, \tau))$. Recall the function $G_{\kappa, \tau}$ is the product of coefficients multiplying $\langle T_{\kappa, \tau}^n \rangle$ as in (1).

Write $k_i = k_i(\kappa, \tau)$ and let $\kappa_i = k_i + l_i$. We get

$$d(k_i + l_i, w_i) - d(k_i, w_i) = l_i - l_i(1 + 2k_i + l_i)w_i. \quad (6)$$

For an m -tangle pretzel link and the skein element T , choosing skein elements σ_i^1 and σ_b^1 between the pair of skein elements T_1 and T_2 creates a new skein element, say T_{12} . Similarly, we denote by T_{123} the skein element created by choosing σ_i^2 and σ_b^2 between T_{12} and T_3 , and so on, until $T_{123 \dots (m-1)}$. Let $F_1 = \deg F_{(\kappa_1, \kappa_2), (\tau_i^1, \tau_b^1)} = \deg(\mathbf{P}(\tau_i^1) \mathbf{P}(\tau_b^1) R(c_1))$, $\bar{F}_1 = \deg F_{(k_1, k_2), (\bar{\sigma}_i^1, \bar{\sigma}_b^1)} = \deg(\mathbf{P}(\bar{\sigma}_i^1) \mathbf{P}(\bar{\sigma}_b^1))$, where $2(k_1 + k_2) = \tilde{\ell}_1 + \tilde{\ell}_2 + \kappa_1 + \kappa_2$ from applying Lemma 4.1 to these skein elements successively. Similarly define

$$F_{12} = \deg F_{(\kappa_{12}=k_1+k_2, \kappa_3), (\tau_i^2, \tau_b^2)} = \deg(\mathbf{P}(\tau_i^2) \mathbf{P}(\tau_b^2) R(c_2))$$

$$\text{and } \bar{F}_{12} = \deg F_{(k_{12}=k_1+k_2, k_3), (\bar{\sigma}_i^2, \bar{\sigma}_b^2)} = \deg(\mathbf{P}(\bar{\sigma}_i^2) \mathbf{P}(\bar{\sigma}_b^2)),$$

$$F_{12 \dots (i)} = \deg F_{(\kappa_{1 \dots (i)}=k_1+\dots+k_i, \kappa_{i+1}), (\tau_i^i, \tau_b^i)},$$

$$\bar{F}_{12 \dots (i)} = \deg F_{(k_{1 \dots (i)}=k_1+\dots+k_i, k_{i+1}), (\bar{\sigma}_i^i, \bar{\sigma}_b^i)}.$$

Applying Lemma 4.1, we get

$$\begin{aligned} & \deg G_{\kappa, \tau} - \deg G_{k(\kappa, \sigma), \sigma(\kappa, \tau)} \\ &= \left(\sum_{i=0}^m d(k_i + l_i, w_i) - d(k_i, w_i) \right) + \left(\sum_{i=1}^{m-1} F_{12 \dots (i)} - \bar{F}_{12 \dots (i)} \right), \end{aligned}$$

where

$$\begin{aligned} \sum_{i=1}^{m-1} F_{12 \dots (i)} - \bar{F}_{12 \dots (i)} &\leq l_1(2k_1 + l_1) + l_2(2k_2 + l_2) + \dots + l_m(2k_m + l_m) \\ &= \sum_{i=1}^m l_i(2k_i + l_i). \end{aligned}$$

Thus

$$\deg G_{\kappa, \tau} - \deg G_{k(\kappa, \sigma), \sigma(\kappa, \tau)} \leq \left(\sum_{i=1}^m l_i - l_i(1 + 2k_i + l_i)w_i \right) + \left(\sum_{i=1}^m l_i(2k_i + l_i) \right).$$

Since $(\kappa, \tau) \neq (k(\kappa, \tau), \sigma(\kappa, \tau))$ by assumption, there is some $1 \leq i \leq m$ such that $l_i \neq 0$ and so we have

$$\deg G_{\kappa, \tau} - \deg G_{k(\kappa, \sigma), \sigma(\kappa, \tau)} \leq -2 \min_{0 \leq i \leq m} \{|w_i| - 1\} \min_{0 \leq i \leq m} \{k_i\}.$$

Therefore, every state corresponding to parameters (κ, τ) that are not tight can grouped into the lower order term $l(k)$ of some state with tight parameters (k, σ) in $\mathcal{G}_k$. $\square$

This gives the following corollary, Theorem 1.2, which we restate here for convenience.

Theorem 1.2 ([7, Theorem 3.2]) *Let $k = (k_0, k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ with $k_i \leq n$ and assume $|w_i| > 1$. The $n + 1$ colored Jones polynomial of a pretzel link $P = P(w_0, w_1, \dots, w_m)$*

with diagram L has the form

$$J_{P,n+1} = \left((-1)^n q^{\frac{1}{2}} \right)^{w(L)(n^2+2n)} \sum_{k_0=\sum_{i=1}^m k_i} \mathcal{G}_k,$$

where $\deg \mathcal{G}_k = (-1)^{w_0(n-k_0)+n+k_0+\sum_{i=1}^m (n-k_i)(w_i-1)} q^{\delta(n,k)} + l.o.t.$ ⁴, and

$$\begin{aligned} & \delta(n, k) \\ &= - \left((w_0 + 1)k_0^2 + \sum_{i=1}^m (w_i - 1)k_i^2 + \sum_{i=1}^m (-2 + w_0 + w_i)k_i - \frac{n(n+2)}{2} \sum_{i=0}^m w_i + (m-1)n \right). \end{aligned}$$

Proof From Theorem 4.1, we simply compute the degree of the leading term of $\mathcal{G}_k$ corresponding to the choice of skein elements k, σ with tight parameters $k_0 = \sum_{i=1}^m k_i$. Note $\deg(\mathcal{T}_{k_0})$ is n , half the number of circles in $\overline{\mathcal{T}_{k_0}}$, since it is an adequate skein element by [1]. Thus the degree is

$$\begin{aligned} & \underbrace{\sum_{i=0}^m w_i \left(n - k_i + \frac{n^2}{2} - k_i^2 \right) + \sum_{i=0}^m (k_i - n)}_{\text{fusion and untwisting}} \\ & + \sum_{i=1}^{m-1} \deg \left(\frac{\left[n - \sum_{j=1}^i k_j \right]! [n - k_{i+1}]!}{\left[n - \sum_{j=1}^{i+1} k_j \right]! [n]!} \right)^2 + \underbrace{n}_{\text{number of circles in } \mathcal{T}_{k_0}} \\ & = \frac{2n + n^2}{2} \sum_{i=0}^m w_i - \sum_{i=0}^m w_i k_i - \sum_{i=0}^m w_i k_i^2 + \sum_{i=0}^m k_i - nm - \left(\sum_{i=1}^m k_i \right)^2 + \sum_{i=1}^m k_i^2 + n. \end{aligned}$$

This proves the degree formula in the theorem after regrouping. The sign of the leading coefficient is obtained by multiplying the sign of each of the functions in the product. $\square$

4.1 3-Tangle Pretzel Knots $P(w_0, w_1, w_2)$.

We restate Theorem 1.3 here.

Theorem 1.3 Let $w = (w_0, w_1, w_2) \in \mathbb{Z}^3$ be such that $w_0 < -1 < 0 < 1 < w_1, w_2$. Define

$$s(w) = 1 + w_0 + \frac{1}{\sum_{i=1}^2 (w_i - 1)^{-1}} \text{ and } s_1(w) = \frac{\sum_{i=1}^2 (w_i + w_0 - 2)(w_i - 1)^{-1}}{\sum_{i=1}^2 (w_i - 1)^{-1}}.$$

Suppose w_1 is even and $-w_0 > \min\{w_1 - 1, w_2 - 1\}$. Let P denote the pretzel knot $P(w_0, w_1, w_2)$, and let $j_P(n)$ be the largest power of q in $J_{P,n}$. If $s(w) < 0$, we may write

$$j_P(n) = j s_P n^2 + j x_P(n)n + c_P(n),$$

⁴Denotes lower order terms.

where jx_P, c_P are periodic functions in n . In particular we have

(a) For $n = \frac{-2+w_1+w_2}{\gcd(w_1-1, w_2-1)}j, j \geq 1$:

$$js_P = -s(w) + w_0 + w_2,$$

$$jx_P(n) = -s_1(w) + 2s(w) - (m-1) - 2 \frac{\min\{w_1-1, w_2-1\}}{-2+w_1+w_2}.$$

(b) For $n \neq \frac{-2+w_1+w_2}{\gcd(w_1-1, w_2-1)}j$:

$$js_P = -s(w) + w_0 + w_2, jx_P(n) = -s_1(w) + 2s(w) - (m-1).$$

Proof Let L be the standard diagram of the pretzel knot $P(w_0, w_1, w_2)$. We apply Theorem 4.1 to write $\langle L^n \rangle$ as a sum $\sum_k \mathcal{G}_k$. We compute $\deg \langle L^n \rangle$ by first characterizing the terms $\mathcal{G}_k$ that maximize the degree $\delta(n, k)$ (see Theorem 1.2) using quadratic integer programming, and then determining the degree of the leading term that remains after possible cancellations of power series with the same degree but opposite-sign coefficients. Adding the writhe term finishes the proof.

For $x = (x_0, x_1, \dots, x_m) \in \mathbb{R}^{m+1}$, we compute the real maximum of the function $\delta(n, x)$, where $n = x_0 = \sum_{i=1}^m x_i$. Denote the critical points of the real maximum by $x^* = (x_0^*, x_1^*, \dots, x_m^*)$. In our case with the 3-pretzel, we get

$$x_1^* = \frac{-2n - w_1 + w_2 + 2nw_2}{2(-2 + w_1 + w_2)}, \quad x_2^* = n - x_1^*.$$

From here, we apply the quadratic integer programming method to the degree $\delta(n, k) = \deg \mathcal{G}_k, k = (k_0, k_1, \dots, k_m) \in \mathbb{Z}^{m+1}$ exactly as in [7]

$$\max_{\substack{k_i \geq 0 \\ \sum_{i=0}^{m-1} k_i = t = k_0 \leq n}} \delta(n, k) = \max_{0 \leq t \leq n} -s(w)t^2 - s_1(w)t + O(1). \quad (7)$$

From (7), we see that for $k_0 = k_1 + k_2 = t < n$, there are parameters $\tilde{k} = (\tilde{k}_0, \tilde{k}_1, \tilde{k}_2)$ with $\tilde{k}_0 = \tilde{k}_1 + \tilde{k}_2 = n$ such that the following inequality holds

$$\delta(n, \tilde{k}) - \delta(n, k) \geq 2 \left(-1 + \frac{-2 + w_1 + w_2}{\gcd(w_1 - 1, w_2 - 1)} j \right) \frac{j}{\gcd(w_1 - 1, w_2 - 1)} > 2 \min\{w_1 - 1, w_2 - 1\} \frac{j}{\gcd(w_1 - 1, w_2 - 1)}. \quad (8)$$

In other words, the degree of a term $\mathcal{G}_k$ with parameters k such that $k_0 = k_1 + k_2 = t < n$ is bounded away from that of a term $\mathcal{G}_{\tilde{k}}$ with parameters $\tilde{k}$ such that $\tilde{k}_0 = \tilde{k}_1 + \tilde{k}_2 = n$ by at least the amount indicated in the inequality. This means we may first search for maxima among terms $\mathcal{G}_k$ with $k_0 = k_1 + k_2 = n$. Since $-w_0 > \min\{w_1 - 1, w_2 - 1\}$ as part of the assumptions of Theorem 1.3, the quadratic on the right-hand side of (7) is negative definite, and the integer lattice maxima are achieved on $k_0 = k_1 + k_2 = n$ near the real maximum from critical points x^* .

Now we discuss when cancellations of these integer lattice maxima arise. Since the leading term of $\deg \mathcal{G}_k$ has coefficient $(-1)^{w_0(n-k_0)+n+k_0+\sum_{i=1}^m(n-k_i)(w_i-1)}$ and w_0, w_2 are odd while w_1 is even, we have that $(-1)^{n-k_1}$ determines the sign of the leading term of $\deg \mathcal{G}_k$ with degree $\delta(n, k)$. Therefore, if two parameters k_1 and k'_1 differ by 1, then $(-1)^{n-k_1} = -(-1)^{n-k'_1}$, and cancellation of lattice maxima of $\delta(n, k)$ occurs precisely when x_1^* in $x^* = (n, x_1^*, x_2^*)$ is a half integer. Let $k = (k_0 = n, k_1 = x_1^* + 1/2, k_2 = n - k_1)$ with respective choice of the skein element σ and $k' = (k_0 = n, k'_1 = x_1^* - 1/2 = k_1 - 1, k'_2 = n - k'_1)$

with respective choice of skein element σ' , then $\deg \mathcal{G}_k = \deg \mathcal{G}_{k'}$ and their leading terms have opposite signs. This results in cancellation.

For $x_1^* + \frac{1}{2}$ to be an integer, we must have

$$n = -1 + \frac{-2 + w_1 + w_2}{\gcd(w_1 - 1, w_2 - 1)} j$$

for a positive integer $j \geq 0$. When $n \neq -1 + \frac{-2 + w_1 + w_2}{\gcd(w_1 - 1, w_2 - 1)} j$, there is no cancellation and the degree of $J_{P,n}$ corresponds to the maximum of $\delta(n, k)$ found in (7).

Let $g = \gcd(w_1 - 1, w_2 - 1)$ and fix $n = -1 + \frac{-2 + w_1 + w_2}{g} j$ for $j \geq 1$. Let $v = (w_2 - 1) \frac{j}{g}$, and u be a non-negative integer such that $v + u \leq n$ and $v - u - 1 \geq 0$. Consider the pair of states $(\mathbf{k}_u, \sigma_u)$ and $(\mathbf{k}'_u, \sigma'_u)$ with parameters $\mathbf{k}_u = (n, v + u, n - v - u)$ and $\mathbf{k}'_u = (n, v - u - 1, n - v + u + 1)$, and the respective choices of skein elements σ_u, σ'_u in the expansions of the Jones-Wenzl projectors as specified in Theorem 4.1. These are the leading terms of $\mathcal{G}_{\mathbf{k}_u}$ and $\mathcal{G}_{\mathbf{k}'_u}$, respectively. Note the Kauffman brackets of the skein elements $T_{\mathbf{k}_u, \sigma_u}^n, T_{\mathbf{k}'_u, \sigma'_u}^n$ are the same independent of u , and we have, after getting rid of them and the common terms from applying the fusion and untwisting formula to the first twist region with $-\omega_0$ crossings,

$$\begin{aligned} & G_{\mathbf{k}_u, \sigma_u}(q) \langle T_{\mathbf{k}_u, \sigma_u}^n \rangle^{-1} \frac{\theta(n, n, 2n)}{[2n+1]} q^{-w_0 \frac{n^2}{2}} \\ &= \frac{[2(u+v)+1]}{\theta(n, n, 2(u+v))} q^{w_1(n - \frac{2(v+u)}{2} + \frac{n^2}{2} - \frac{(2(v+u))^2}{4})} \frac{[2n - 2(v+u) + 1]}{\theta(n, n, 2n - 2(v+u))} \\ &\quad \cdot q^{w_2(n - \frac{2n-2(v+u)}{2} + \frac{n^2}{2} - \frac{(2n-2(v+u))^2}{4})} \left(\frac{[n - (v+u)]![n - (n - (v+u))]}{[0]!} \right)^2 (-1)^{n-v-u}, \end{aligned}$$

where explicitly

$$\begin{aligned} \theta(n, n, 2(v+u)) &= \frac{[1+n+v+u]![n-v-u]![v+u]![v+u]!}{[2(v+u)]![n]!}, \text{ and} \\ \theta(n, n, 2n - 2(v+u)) &= \frac{[1+2n-(v+u)]![v+u]![n-(v+u)]![n-(v+u)]!}{[2n-2(v+u)]![n]![n]!}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} & G_{\mathbf{k}'_u, \sigma'_u}(q) \langle T_{\mathbf{k}'_u, \sigma'_u}^n \rangle^{-1} \frac{\theta(n, n, 2n)}{[2n+1]} q^{-w_0 \frac{n^2}{2}} \\ &= \frac{[2(v-u)-1]}{\theta(n, n, 2(v-u)-2)} q^{w_1(n - \frac{2(v-u)-2}{2} + \frac{n^2}{2} - \frac{(2(v-u)-2)^2}{4})} \\ &\quad \cdot \frac{[2n - 2(v-u) + 3]}{\theta(n, n, 2n - 2(v-u) + 2)} q^{w_2(n - \frac{2n-2(v-u)+2}{2} + \frac{n^2}{2} - \frac{(2n-2(v-u)+2)^2}{4})} \\ &\quad \cdot \left(\frac{[n - ((v-u)-1)]![n - (n - (v-u) + 1)]!}{[0]!} \right)^2 (-1)^{n-v+u+1}, \end{aligned}$$

where

$$\begin{aligned} \theta(n, n, 2(v-u)-2) &= \frac{[n+(v-u)]![1+n-(v-u)]![-1+(v-u)]![-1+(v-u)]!}{[-2+2(v-u)]![n]![n]!} \\ \theta(n, n, 2n - 2(v-u) + 2) &= \frac{[2+2n-(v-u)]![-1+(v-u)]![1+n-(v-u)]![1+n-(v-u)]!}{[2+2n-2(v-u)]![n]![n]!}. \end{aligned}$$

A direct computation shows $\deg G_{\mathbf{k}_u, \sigma_u} = \deg G_{\mathbf{k}'_u, \sigma'_u}$. Factoring out terms that are common to both

$$\begin{aligned} & \text{Common factors} \\ &= \langle T_{\mathbf{k}_u, \sigma_u}^n \rangle \frac{[2n+1]}{\theta(n, n, 2n)} q^{-w_0 \frac{n^2}{2} + \frac{1}{2} n^2 (s-t) + ns} ([n]!)^4 \\ & \quad \cdot \frac{[-2+2(v-u)]! [2n-2(v+u)]! [n-v-u]! [v-u+1]!}{[1+n-(v-u)]! [n-v-u]! ([-1+(v-u)]!)^2 [1+2n-v-u]! [-1+(v-u)]! ([n-v-u]!)^2} \end{aligned}$$

we get

$$\begin{aligned} & \frac{G_{\mathbf{k}_u, \sigma_u} - G_{\mathbf{k}'_u, \sigma'_u}}{\text{common factors}} \\ &= \frac{[2(u+v)+1][2n-2(v+u)+1]([2(v+u)] \cdots [2(v-u)-1])}{([1+n+v+u] \cdots [1+n-(v-u)+1])([v+u] \cdots [v-u])} \quad (9) \\ & \quad \cdot q^{n(2t(u+v)) - (u+v)t(-1+u+v) + s(1+u+v)} \end{aligned}$$

$$\begin{aligned} & - \frac{[2(v-u)-1][2n-2(v-u)+3]([2+2n-2(v-u)] \cdots [2n-2v-2u+1])}{([n+v-u] \cdots [n-v-u+1])([2+2n-v+u] \cdots [1+2n-v-u+1])} \quad (10) \\ & \quad \cdot q^{n(-2t(1+u-v)) - (s(u-v) + t(2+u-v))(1+u-v)}. \end{aligned}$$

To see the degree drop, we first compute the highest powers of q of $G_{\mathbf{k}_u, \sigma_u}$ and $G_{\mathbf{k}'_u, \sigma'_u}$. This involves adding up the highest powers of q for each term in the numerator, then subtracting from it the sum of the highest powers of q for each term in the denominator. This gives us, after substituting $n = -1 + \frac{-2+w_1+w_2}{g}j$ and $v = (w_2-1)\frac{j}{g}$ (the reader is invited to check the computations using software like *Mathematica*),

$$\begin{aligned} \deg \frac{G_{\mathbf{k}_u, \sigma_u}}{\text{common factors}} &= \deg \frac{G_{\mathbf{k}'_u, \sigma'_u}}{\text{common factors}} \\ &= -4 - \frac{8j}{g} - \frac{4j^2}{g^2} + \frac{3jw_1}{g} + \frac{j^2w_1}{g^2} + \frac{7jw_2}{g} + \frac{9j^2w_2}{g^2} - \frac{jw_1w_2}{g} \\ & \quad - \frac{2j^2w_1w_2}{g^2} - \frac{jw_2^2}{g} - \frac{6j^2w_2^2}{g^2} + \frac{j^2w_1w_2^2}{g^2} + \frac{j^2w_2^3}{g^2} - 4u \\ & \quad - \frac{4ju}{g} - w_1u + \frac{2jw_1u}{g} - w_2u + \frac{2jw_2u}{g} - w_1u^2 - w_2u^2. \quad (11) \end{aligned}$$

We compute the powers of the next highest-degree terms in $G_{\mathbf{k}_u, \sigma_u}$, $G_{\mathbf{k}'_u, \sigma'_u}$, respectively, and we show this determines the degree of the difference $\deg(G_{\mathbf{k}_u, \sigma_u} - G_{\mathbf{k}'_u, \sigma'_u})$. Note first that all terms of the form $(q - q^{-1})$ cancel out, since there is an equal number of those terms in the numerator and the denominator for both fractions. To obtain the Laurent series expansion of $G_{\mathbf{k}_u, \sigma_u}$ that is bounded from above, we can factor out q^m from each term in the denominator of the form $(q^m - q^{-m})$ then expand the resulting $1/(1 - q^{-2m}) = 1 + q^{-2m} + q^{-4m} + \cdots$. Thus for a Laurent series expansion of the following rational function

$$\frac{1}{(1 + q^{-2m_1})(1 + q^{-2m_2}) \cdots (1 + q^{-2m_\ell})},$$

the second highest degree is $-2 \min_{1 \leq i \leq \ell} m_i$. The expressions in (9) and (10) are then obtained by multiplying the Laurent series expansion above by a polynomial of the form

$$(1 + q^{-2n_1})(1 + q^{-2n_2}) \cdots (1 + q^{-2n_\ell})$$

after factoring out a monomial in q in the same way. The second highest degree term of the product

$$\frac{(1 + q^{-2n_1})(1 + q^{-2n_2}) \cdots (1 + q^{-2n_\ell})}{(1 + q^{-2m_1})(1 + q^{-2m_2}) \cdots (1 + q^{-2m_\ell})}$$

is then the term with degree $-2 \min\{\min_{1 \leq i \leq \ell} m_i, \min_{1 \leq j \leq \ell} n_j\}$. For $G_{\mathbf{k}_u, \sigma_u}$ /common factors, the minimum over all the quantum integers in the fraction is $\min\{v - u, n - v + u + 2\}$. For $G_{\mathbf{k}'_u, \sigma'_u}$ /common factors, it is $n - v - u + 1$. Getting the second highest degree term of the difference comes down to comparing $v - u$ and $n - v - u + 1$. Plugging in $n = -1 + \frac{-2+w_1+w_2}{g}j$ and $v = (w_2 - 1)\frac{j}{g}$. We see

$$v - u = (w_1 - 1)\frac{j}{g} - u \quad \text{and} \quad n - v - u + 1 = (w_2 - 1)\frac{j}{g} - u.$$

Note w_1 and w_2 have opposite parities, therefore these two terms $v - u$ and $n - v - u + 1$ cannot be the same. Either way, it is straightforward to see that the degree drop is $2 \min\{w_1 - 1, w_2 - 1\}\frac{j}{g} - 2u$ from $\deg G_{\mathbf{k}_u, \sigma_u}$.

Let $L(u) = \deg(G_{\mathbf{k}_u, \sigma_u} - G_{\mathbf{k}'_u, \sigma'_u})$. We established that $L(u) = \deg G_{\mathbf{k}_u, \sigma_u} - (2 \min\{w_1 - 1, w_2 - 1\}\frac{j}{g} - 2u)$. From (11), we also see that $\deg G_{\mathbf{k}_0, \sigma_0} - \deg G_{\mathbf{k}_u, \sigma_u} \geq 4u$. This shows

$$\begin{aligned} L(0) - L(u) &\geq \deg G_{\mathbf{k}_0, \sigma_0} - \left(2 \min\{w_1 - 1, w_2 - 1\}\frac{j}{g}\right) \\ &\quad - \left(\deg G_{\mathbf{k}_u, \sigma_u} - \left(2 \min\{w_1 - 1, w_2 - 1\}\frac{j}{g} - 2u\right)\right) \\ &\geq 2u. \end{aligned} \quad (12)$$

Thus $L(u) < L(0)$ when $u > 0$.

Finally, we show all other terms in the state sum have degrees that are more than $2 \min\{w_1 - 1, w_2 - 1\}\frac{j}{g}$ away from $L(0) + \deg\langle T_{\mathbf{k}_0, \sigma_0}^n \rangle$. Therefore, $L(0) + \deg\langle T_{\mathbf{k}_0, \sigma_0}^n \rangle = \deg G_{\mathbf{k}_0, \sigma_0} - 2 \min\{w_1 - 1, w_2 - 1\}\frac{j}{g} + \deg\langle T_{\mathbf{k}_0, \sigma_0}^n \rangle$ is the degree of $\langle L^n \rangle$, and this finishes the proof. Note first that for a term $\mathcal{G}_k$ of the state sum as in Theorem 4.1, if $k_0 < \sum_{i=1}^2 k_i$, then it has degree strictly smaller by at least $2 \min\{w_1 - 1, w_2 - 1\} \min\{k'_1, k'_2\}$ compared to $\deg \mathcal{G}_{k'}$, where $k'_0 = \sum_{i=1}^2 k'_i$ by (5) from Theorem 4.1. Thus it suffices to compare the degrees of states with tight parameters $k_0 = \sum_{i=1}^2 k_i$. For this case, as remarked following (8), if $k_0 < n$, then the degree of the corresponding term in the state sum is also more than $2 \min\{w_1 - 1, w_2 - 2\}j/g$ away from a term where $k_0 = n$.

Assume $k_0 = k_1 + k_2 = n$. Note that every set of parameters (k_1, k_2) such that $k_1 + k_2 = n$ satisfies $k_1 = v + u$ for some $\mathbf{k}_u$ or $k_1 = v - u - 1$ for some $\mathbf{k}'_u$. The leading terms corresponding to $(\mathbf{k}_u, \sigma_u)$ and $(\mathbf{k}'_u, \sigma'_u)$ form a canceling pair as we have seen. In fact, the lattice maxima are the pair of terms corresponding to $(\mathbf{k}_0, \sigma_0)$ and $(\mathbf{k}'_0, \sigma'_0)$, and the next highest degree terms are the terms corresponding to the pair $(\mathbf{k}_u, \sigma_u)$ and $(\mathbf{k}'_u, \sigma'_u)$ for $u > 0$. We have shown the degree $L(u) + \deg\langle T_{\mathbf{k}_0, \sigma_0}^n \rangle$ of such terms are strictly smaller than the proposed degree $L(0) + \deg\langle T_{\mathbf{k}_0, \sigma_0}^n \rangle$ in (12). $\square$

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References

1. Armond, C.: The head and tail conjecture for alternating knots. *Algebr. Geom. Topol.* **13**(5), 2809–2826 (2013)
2. Baker, K.L., Motegi, K., Takata, T.: The strong slope conjecture for graph knots. [arXiv:1809.01039](https://arxiv.org/abs/1809.01039) (2018)
3. Dasbach, O.T., Lin, X.S.: A volumish theorem for the Jones polynomial of alternating knots. *Pacific J. Math.* **231**(2), 279–291 (2007)
4. Futer, D., Kalfagianni, E., Purcell, J.: *Guts of Surfaces and the Colored Jones Polynomial*. Lecture Notes in Mathematics, vol. 2069. Springer, Heidelberg (2013)
5. Garoufalidis, S.: The Jones slopes of a knot. *Quantum Topol.* **2**(1), 43–69 (2011)
6. Garoufalidis, S., Lê, T.T.Q.: Nahm sums, stability and the colored Jones polynomial. *Res. Math. Sci.* **2**, Art. 1, 55 pp. (2015)
7. Garoufalidis, S., Lee, C.R.S., van der Veen, R.: The slope conjecture for Montesinos knots. *Internat. J. Math.* **31**(7), 66 (2020). Art. 2050,056
8. Garoufalidis, S., van der Veen, R.: Quadratic integer programming and the slope conjecture. *New York J. Math.* **22**, 907–932 (2016)
9. Howie, J.: Coiled surfaces and slope conjectures. In preparation
10. Kalfagianni, E., Tran, A.T.: Knot cabling and the degree of the colored Jones polynomial. *New York J. Math.* **21**, 905–941 (2015)
11. Kauffman, L.H.: State models and the Jones polynomial. *Topology* **26**(3), 395–407 (1987)
12. Khovanov, M.: *Graphical Calculus, Canonical Bases and Kazhdan-Lusztig Theory*. Thesis (Ph.D.)—Yale University, ProQuest LLC, Ann Arbor MI (1997)
13. Lee, C.R.S.: Jones slopes and coarse volume of near-alternating links. To appear in *Communications in Analysis and Geometry*
14. Lee, C.R.S., van der Veen, R.: Slopes for pretzel knots. *New York J. Math.* **22**, 1339–1364 (2016)
15. Leng, X., Yang, Z., Liu, X.: The slope conjectures for 3-string Montesinos knots. *New York J. Math.* **25**, 45–70 (2019)
16. Lickorish, W.B.R.: *An Introduction to Knot Theory* Graduate Texts in Mathematics, vol. 175. Springer-Verlag, New York (1997)
17. Masbaum, G., Vogel, P.: 3-valent graphs and the Kauffman bracket. *Pacific J. Math.* **164**(2), 361–381 (1994)
18. Motegi, K., Takata, T.: The slope conjecture for graph knots. *Math. Proc. Cambridge Philos. Soc.* **162**(3), 383–392 (2017)
19. Ohtsuki, T.: *Quantum invariants* world scientific (2001)

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