Properties of Pattern Matrices With Applications to Structured Systems
B. M. Shali, H. J. van Waarde, M. K. Camlibel, Member, IEEE, and H. L. Trentelman, Fellow, IEEE

Abstract—In cases where we do not have the exact parameter values of a mathematical model, we often have at least some structural information, e.g., that some parameters are nonzero. Such information can be captured by so-called pattern matrices, whose symbolic entries are used to represent the available information about the corresponding parameters. In this letter, we focus on pattern matrices with three types of symbolic entries: those that represent zero, nonzero, and arbitrary parameters. We formally define and study addition and multiplication of such pattern matrices. The results are then used in the algebraic characterization of three strong structural properties. In particular, we provide sufficient conditions for controllability of linear descriptor systems, necessary and sufficient conditions for input-state observability, and sufficient conditions for output controllability of linear systems.

Index Terms—Control of networks, differential-algebraic systems, linear systems.

I. INTRODUCTION

The concept of structure was introduced more than 40 years ago by Lin [1] in order to obtain more realistic models of physical systems. Lin considers a linear time-invariant system with a single input, where the numerical entries of the system matrices are not known precisely, but are known to be fixed zeros or arbitrary real numbers. This pattern of fixed zero entries gives the system its structure, and is what makes it a structured system. Based on this concept of structure, a structured system is said to be (weakly) structurally controllable if there exists at least one controllable system with the given structure.

Following this, there have been a number of papers that deal with structured systems and their properties. The results on (weak) structural controllability have been extended to multi-input systems [2] and shown to be generic [3]. Requiring all systems with a given structure to be controllable has led to the introduction and characterization of strong structural controllability [4], [5]. More recently, strong structural controllability has been characterized in [6] for a general class of structured systems whose parameters are allowed to be fixed zeros, nonzero or arbitrary real numbers. This has been done with the help of a particular type of pattern matrix with three types of symbolic entries that represents the structure of the structured system.

Some of the most noteworthy applications of the concept of structure are in the context of networked systems. The general idea there is that a weighted graph represents the network, where the topology of the graph is known but the precise values of the edge weights are not. Assuming that the edge weights are nonzero real numbers, the networked system can be interpreted as a structured system with a zero-nonzero structure, in the sense that a nonzero entry corresponds to an edge and a zero entry corresponds to the absence of an edge. Then one can study structural properties of the network, i.e., properties that depend solely on its topology. This has already been done for a variety of relevant system theoretic properties, among which are controllability [7]–[9], target controllability [10]–[13] and input-state observability [14]–[17].

In this letter, we will follow the pattern matrix framework as introduced in [6]. In contrast to [6], we will be interested in strong structural controllability of linear descriptor systems, as well as input-state observability and output controllability of linear systems. We will see that these properties can be characterized elegantly once we have established suitable notions of addition and multiplication for pattern matrices. As such, the main contributions of this letter are as follows. First, we formally define addition and multiplication for pattern matrices and analyse some of their relevant properties. Second, we use these properties to algebraically characterize strong structural controllability of linear descriptor systems, and strong structural input-state observability and output controllability of linear systems.

All of these properties have been studied in some form already. Strong structural controllability of linear descriptor systems has been studied in [18] under the name of selective strong structural controllability. In comparison, the approach that we take here has the advantages of being conceptually
simple and broadly applicable in the study of strong structural properties. In fact, a major advantage of our approach is that it also enables the simple characterization of strong structural output controllability and input-state observability.

The latter have been studied in the context of networked systems [11], [12], [15]–[17], where output controllability is typically referred to as target controllability. However, the class of networked systems considered in these papers imposes additional assumptions on the pattern matrices that represent it. For example, diagonal entries are assumed to be arbitrary in [11], [12], and all entries are either zero or nonzero in [15]–[17]. We make no such assumptions, hence our results can be used to assess strong structural target controllability and input-state observability of a more general class of networked systems. We even show (see Example 2) that our results on strong structural target controllability can be conclusive while existing results are not.

The outline of this letter is as follows. In Section II, we review the concept of pattern matrix and define addition and multiplication for pattern matrices. In Section III, we apply our results on pattern matrices to the algebraic characterization of strong structural controllability of linear descriptor systems, and strong structural input-state observability and output controllability of linear systems. We finish with concluding remarks in Section IV.

II. PATTERN MATRICES

In this section, we will review the concept of pattern matrix and then further develop it by defining addition and multiplication for pattern matrices. To begin with, a particular type of pattern matrix was introduced in [6] in order to formalize the idea of matrices whose entries are not known precisely, but are known to be fixed zeros, nonzero or arbitrary real numbers. More precisely, a pattern matrix is a matrix with entries from the set of symbols \{0, *, ?\}, where * represents nonzero real numbers and ? represents arbitrary real numbers. This is captured in the following definition.

Definition 1: The pattern class of the pattern matrix \( A \in \{0, *, ?\}^{m \times n} \) is defined as

\[
\mathcal{P}(A) = \left\{ A \in \mathbb{R}^{m \times n} \mid A_{ij} = 0 \text{ if } A_{ij} = 0 \right\}
\]

We can define properties of pattern matrices in terms of the properties of the real matrices in their pattern classes. For example, we say that a pattern matrix \( A \) has full rank if \( A \) has full rank for all \( A \in \mathcal{P}(A) \). Rank properties will be crucial in the applications to structured systems since most system-theoretic properties are characterized in terms of full rank conditions. Fortunately, conditions under which a pattern matrix has full row rank exist and can be verified using a simple algorithm (see [6, Th. 11, Lemma 21]). Naturally, one can check if a matrix has full column rank by checking if its transpose has full row rank.

In practice, we will be working with several “unknown” matrices that belong to the pattern classes of some known pattern matrices. This will naturally lead to expressions involving sums and products. To understand the results of such expressions, we will define a sensible way of adding and multiplying pattern matrices. Here, sensible means that the result of adding and multiplying pattern matrices gives us some useful information on the result of adding and multiplying matrices belonging to their pattern classes.

To this end, given a pair of pattern matrices, we want the sum of any pair of real matrices from their pattern classes to be contained in the pattern class of the sum of the pattern matrices. We know that the sum of zero and any real number is just the number itself, while the sum of two nonzero real numbers can be any real number. Motivated by this, we define addition for the set \{0, *, ?\} as shown in Table I. Then addition for pattern matrices is defined element-wise.

Definition 2: Let \( A, B \in \{0, *, ?\}^{m \times n} \). Their sum \( A + B \in \{0, *, ?\}^{m \times n} \) is defined as

\[
(A + B)_{ij} = A_{ij} + B_{ij}
\]

for all \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, n\} \).

By definition, if \( A \) and \( B \) are pattern matrices of the same dimensions, then \( \mathcal{P}(A + B) \supset \mathcal{P}(A) + \mathcal{P}(B) \), where

\[
\mathcal{P}(A) + \mathcal{P}(B) = \{ A + B \mid A \in \mathcal{P}(A), \ B \in \mathcal{P}(B) \}
\]

is the Minkowski sum of sets. It turns out that the converse is true as well.

Theorem 1: If \( A \) and \( B \) are pattern matrices of the same dimensions, then \( \mathcal{P}(A + B) = \mathcal{P}(A) + \mathcal{P}(B) \).

Proof: The inclusion \( \mathcal{P}(A + B) \supset \mathcal{P}(A) + \mathcal{P}(B) \) follows from the definition of addition. For the converse inclusion, let \( C \in \mathcal{P}(A + B) \) and consider an entry \( C_{ij} \). The goal is to show that there exist entries \( A_{ij} \in \mathcal{P}(A_{ij}) \) and \( B_{ij} \in \mathcal{P}(B_{ij}) \) such that

\[
C_{ij} = A_{ij} + B_{ij}
\]

We will consider the cases \( C_{ij} = 0 \) and \( C_{ij} \neq 0 \) separately.

Suppose that \( C_{ij} = 0 \). Then either \( A_{ij} + B_{ij} = 0 \) or \( A_{ij} + B_{ij} = \). In the former, we must have that \( A_{ij} = 0 \) and \( B_{ij} = 0 \), hence \( A_{ij} = 0 \) and \( B_{ij} = 0 \) would work. In the latter, there are three cases:

1) If \( A_{ij}, B_{ij} \in \{ *, ? \} \), then \( A_{ij} = B_{ij} = 1 \).
2) If \( A_{ij} = 0 \) and \( B_{ij} = \), then \( A_{ij} = B_{ij} = 0 \).
3) If \( A_{ij} = \) and \( B_{ij} = 0 \), then \( A_{ij} = B_{ij} = 0 \).

Suppose that \( C_{ij} \neq 0 \). Then either \( A_{ij} + B_{ij} = * \) or \( A_{ij} + B_{ij} = \). In the former, exactly one of \( A_{ij} \) and \( B_{ij} \) is * and the other one is 0, hence we can pick either \( A_{ij} = C_{ij} \) and \( B_{ij} = 0 \), or \( A_{ij} = 0 \) and \( B_{ij} = C_{ij} \). In the latter, there are three cases again:

1) If \( A_{ij}, B_{ij} \in \{ *, ? \} \), then \( A_{ij} = B_{ij} = \frac{1}{2} C_{ij} \).
2) If \( A_{ij} = 0 \) and \( B_{ij} = \), then \( A_{ij} = 0 \) and \( B_{ij} = C_{ij} \).
3) If \( A_{ij} = \) and \( B_{ij} = 0 \), then \( A_{ij} = C_{ij} \) and \( B_{ij} = 0 \).

The element \( C_{ij} \) was chosen arbitrarily, hence we can always find matrices \( A \in \mathcal{P}(A) \) and \( B \in \mathcal{P}(B) \) such that \( A + B = C \) and thus \( \mathcal{P}(A + B) \subset \mathcal{P}(A) + \mathcal{P}(B) \).

In the same vein, we now turn to the definition of multiplication for pattern matrices. Note that the product of zero and any real number is just zero, while the product of two nonzero real numbers is always a nonzero real number. This motivates the definition of multiplication for the set \{0, *, ?\} shown in

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TABLE II

MULTIPLICATION FOR THE SET \{0, *, ?\}

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Table II. Then we can define pattern matrix multiplication in the usual way.

**Definition 3:** Let \( A \in \{0, *, ?\}^{m \times p} \) and \( B \in \{0, *, ?\}^{p \times n} \). Their product \( AB \in \{0, *, ?\}^{m \times n} \) is defined as

\[
(AB)_{ij} = \sum_{k=1}^{p} A_{ik}B_{kj}
\]

for all \( i \in \{1, \ldots, m\} \) and \( j \in \{1, \ldots, n\} \).

By definition, if \( A \) and \( B \) are of appropriate dimensions, then \( \mathcal{P}(AB) \supseteq \mathcal{P}(A)\mathcal{P}(B) \), where

\[
\mathcal{P}(A)\mathcal{P}(B) = \{AB | A \in \mathcal{A}, B \in \mathcal{B}\}.
\]

Unfortunately, the converse is generally not true. When multiplying matrices with at least two rows or columns, we typically create dependencies between the entries of the product. These dependencies cannot be inferred from the product of pattern matrices.

**Example 1:** Consider the pattern vectors \( A = [\ast \ast]^\top \) and \( B = [\ast \ast] \). It is easy to see that

\[
AB = [\ast \ast] \quad \text{and} \quad [1 1 2] \in \mathcal{P}(AB).
\]

Note that the latter is a matrix of rank 2 and thus it cannot be written as the outer product of two vectors. In other words, the fact that the columns (or rows) of \( AB \), where \( A \in \mathcal{P}(A) \) and \( B \in \mathcal{P}(B) \), are linearly dependent cannot be inferred from the product \( AB \).

Although the equality \( \mathcal{P}(AB) = \mathcal{P}(A)\mathcal{P}(B) \) does not hold in general, there are special cases of \( A \) and \( B \) for which equality does hold. A notable special case is the one where either \( A \) or \( B \) is the “identity” pattern matrix \( I \) of appropriate dimensions, defined as a diagonal matrix with *’s on the diagonal. Indeed, suppose that \( A = I \). It is not difficult to see that \( IB = B \) and \( \mathcal{P}(I)\mathcal{P}(B) = \mathcal{P}(B) \), and thus \( \mathcal{P}(IB) = \mathcal{P}(B) = \mathcal{P}(I)\mathcal{P}(B) \). In the case where \( B = I \), we can prove the equality in an analogous way.

Finally, we provide the following lemma, which will be relevant in the applications to structured systems in the next section.

**Lemma 1:** Let \( A, B \in \{0, *, ?\}^{m \times n} \). Then \( A - \lambda B \) has full rank for all \( A \in \mathcal{P}(A), B \in \mathcal{P}(B) \) and nonzero \( \lambda \in \mathbb{C} \) if and only if \( A + B \) has full rank.

**Proof:** Suppose that \( A - \lambda B \) has full rank for all \( A \in \mathcal{P}(A), B \in \mathcal{P}(B) \) and nonzero \( \lambda \in \mathbb{C} \) if and only if \( A + B \) has full rank.

**III. Strong Structural Properties**

In this section, we will show how \( \{0, *, ?\} \) pattern matrices can be used to characterize strong structural properties of structured systems. We will extend the work on strong structural controllability in [6] by studying strong structural controllability of linear descriptor systems. In addition, we will also study strong structural input-state observability and output controllability of linear systems. As we will see, addition and multiplication of pattern matrices will play an important role in the study of each of these three properties.

**A. Controllability of Linear Descriptor Systems**

In this subsection, we will extend the results on strong structural controllability from [6] to linear descriptor systems. Let \((E, A, B)\) denote the system

\[
E\dot{x}(t) = Ax(t) + Bu(t),
\]

where \( t \geq 0 \), \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the input, \( E \in \mathbb{R}^{m \times n} \), \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \). The system \((E, A, B)\) is called regular if \( A - \lambda E \) is invertible for some \( \lambda \in \mathbb{C} \). Regularity of \((E, A, B)\) guarantees the existence and uniqueness of solutions to (4), given an initial state and a sufficiently differentiable input (see [19]), hence regularity of \((E, A, B)\) is a desirable property. Typically, the matrix \( E \) is singular and thus (4) puts algebraic constraints on the state. This leads to \((E, A, B)\) having special features that are not found in systems in which the state is not constrained algebraically. Consequently, there are different kinds of controllability notions defined for \((E, A, B)\), some of which make sense only in the presence of algebraic constraints.

We will not go into the analysis of descriptor systems, and will instead focus on a particular definition of controllability...
and its characterization, as presented in [19]. To this end, let \( x(t; x_0, u) \) denote the state trajectory at time \( t \geq 0 \) for the initial condition \( x(0) = x_0 \in \mathbb{R}^n \) and input \( u \). In the following definition, \( h \) denotes the index of the descriptor system (see [19, Ch. 1]), and \( C_{p-1}^p \) denotes the class of \((h-1)\)-times piecewise continuously differentiable functions.

**Definition 4:** The regular system \((E, A, B)\) is controllable if for any \( T > 0 \), \( x_0 \in \mathbb{R}^n \) and \( x_1 \in \mathbb{R}^n \), there exists an input function \( u \in C_{p-1}^p \) such that \( x(T; x_0, u) = x_1 \).

Then we have the following characterization of controllability for regular \((E, A, B)\).

**Proposition 1** [19, Th. 2-2.1]: The regular system \((E, A, B)\) is controllable if and only if
\[
\text{rank}[E \quad B] = \text{rank}[A - \lambda E \quad B] = n \quad (5)
\]
for all \( \lambda \in \mathbb{C} \).

Now, suppose that \( E, A \) and \( B \) are not known precisely but are known to belong to the pattern classes of some known pattern matrices. In other words, we know that \( E \in \mathcal{P}(E) \), \( A \in \mathcal{P}(A) \) and \( B \in \mathcal{P}(B) \) for given pattern matrices \( E \in \{0, *, ? \}^{n \times n} \), \( A \in \{0, *, ? \}^{n \times n} \) and \( B \in \{0, *, ? \}^{n \times m} \). This naturally leads to a family of systems as \( E, A \) and \( B \) range over the respective pattern classes. This family is completely characterized by \( E, A \) and \( B \), hence we denote it by \((E, A, B)\) and refer to it as a structured system.

We are interested in conditions under which all regular \((E, A, B) \in \mathcal{P}(E) \times \mathcal{P}(A) \times \mathcal{P}(B)\) are controllable. This motivates the following definition.

**Definition 5:** The structured system \((E, A, B)\) is regularly strongly structurally controllable if all regular systems \((E, A, B) \in \mathcal{P}(E) \times \mathcal{P}(A) \times \mathcal{P}(B)\) are controllable.

Making use of the results from Section II, we now have the following theorem that provides a sufficient condition for regular strong structural controllability.

**Theorem 2:** The rank conditions (5) hold for all \((E, A, B) \in \mathcal{P}(E) \times \mathcal{P}(A) \times \mathcal{P}(B)\) if and only if \([E \quad B], \ [A \quad B] \) and \([A + E \quad B] \)

have full row rank. Moreover, if these pattern matrices have full row rank then \((E, A, B)\) is regularly strongly structurally controllable.

**Proof:** Note that the rank conditions (5) hold for all \((E, A, B) \in \mathcal{P}(E) \times \mathcal{P}(A) \times \mathcal{P}(B)\) if and only if
\[
\text{rank}[E \quad B] = \text{rank}[A] = \text{rank}[A - \lambda E \quad B] = n
\]
for all nonzero \( \lambda \in \mathbb{C} \), \( E \in \mathcal{P}(E) \), \( A \in \mathcal{P}(A) \) and \( B \in \mathcal{P}(B) \).

The result then follows from Lemma 1.

The lack of necessity in the algebraic characterization of regular strong structural controllability in Theorem 2 stems from the fact that generally not all \((E, A, B) \in \mathcal{P}(E) \times \mathcal{P}(A) \times \mathcal{P}(B)\) are regular. For example, one can show that the structured system \((E, A, B)\), with
\[
E = \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}, \quad A = \begin{bmatrix} ? & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ * \end{bmatrix},
\]
is regularly strongly structurally controllable, although the pattern matrix \([A \ B] \) does not have full row rank.

**Remark 1:** In the special case where \( E = \mathcal{I} \), all systems \((E, A, B) \in \mathcal{P}(\mathcal{I}) \times \mathcal{P}(A) \times \mathcal{P}(B)\) are regular. In this case, \( E^{-1} \) always exists and we can also write (4) as
\[
\dot{x} = E^{-1}Ax + E^{-1}Bu.
\]
Clearly, \( E^{-1} \in \mathcal{P}(\mathcal{I}) \) for all \( E \in \mathcal{P}(\mathcal{I}) \). Since \( \mathcal{P}(I)\mathcal{P}(A) = \mathcal{P}(A) \) and \( \mathcal{P}(I)\mathcal{P}(B) = \mathcal{P}(B) \), we see that regular strong structural controllability of \((I, A, B)\) is equivalent to strong structural controllability of \((A, B)\), as defined in [6]. In fact, in the special case \( E = \mathcal{I} \), the conditions of Theorem 2 coincide with the conditions for strong structural controllability given in [6, Th. 7]. To see this, note that \([I \quad B]\) has full row rank for all \( B \). In addition, the matrix \( \tilde{A} := A + \mathcal{I} \) is the pattern matrix obtained from \( A \) by changing the diagonal entries of \( A \) to
\[
\tilde{A}_{kk} = \begin{cases} * & \text{if } A_{kk} = 0, \\ ? & \text{otherwise.} \end{cases}
\]

As such, Theorem 2 requires \([A \ B] \) and \([\tilde{A} \ B] \) to have full row rank, which are exactly the two conditions of [6, Th. 7]. These conditions are, in fact, necessary and sufficient for controllability of \((A, B)\).

**B. Input-State Observability**

In this section, we will use the techniques developed for the analysis of strong structural controllability to characterize another property, namely, input-state observability. Let \((A, B, C, D)\) denote the system
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*}
\]
where \( t \geq 0 \) represents time, \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the input, \( y(t) \in \mathbb{R}^p \) is the output, \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \) and \( D \in \mathbb{R}^{p \times m} \). For a given initial condition \( x(0) = x_0 \in \mathbb{R}^n \) and input function \( u \), we denote the corresponding output trajectory at time \( t \geq 0 \) by \( y(t; x_0, u) \). Then we consider the following definition.

**Definition 6:** The system \((A, B, C, D)\) is input-state observable if \( y(t; x_1, u_1) = y(t; x_2, u_2) \) for all \( t \geq 0 \) implies that \( x_1 = x_2 \) and \( u_1(t) = u_2(t) \) for all \( t \geq 0 \).

In other words, a system \((A, B, C, D)\) is input-state observable if different initial conditions and inputs can be distinguished on the basis of the output of the system. Input-state observability has been called left-invertibility with unknown initial state, and it is characterized as follows.

**Proposition 2** [20, Th. 3]: The system \((A, B, C, D)\) is input-state observable if and only if
\[
\text{rank}[A - \lambda I \quad C \quad D] = n + m
\]
for all \( \lambda \in \mathbb{C} \).

As before, instead of considering a single system \((A, B, C, D)\), we consider the family of systems where \( A \in \mathcal{P}(A) \), \( B \in \mathcal{P}(B) \), \( C \in \mathcal{P}(C) \) and \( D \in \mathcal{P}(D) \) for given pattern matrices \( A, B, C \) and \( D \) of appropriate dimensions. We denote this family by \((A, B, C, D)\) and refer to it as a structured system. We are interested in finding necessary and sufficient conditions under which \((A, B, C, D)\) is guaranteed to be input-state observable for all \( A \in \mathcal{P}(A) \), \( B \in \mathcal{P}(B) \), \( C \in \mathcal{P}(C) \) and \( D \in \mathcal{P}(D) \).
Definition 7: The structured system \((A, B, C, D)\) is strongly structurally input-state observable if \((A, B, C, D)\) is input-state observable for all \(A \in \mathcal{P}(A), B \in \mathcal{P}(B), C \in \mathcal{P}(C)\) and \(D \in \mathcal{P}(D)\).

In view of Proposition 2 and the results presented so far, the following algebraic characterization of strong structural input-state observability follows naturally.

Theorem 3: The structured system \((A, B, C, D)\) is strongly structurally input-state observable if and only if
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
A + I & B \\
C & D
\end{bmatrix}
\]
have full column rank.

Proof: We claim that \((A, B, C, D)\) is strongly structurally input-state observable if and only if
\[
\text{rank}\left[\begin{bmatrix} A - \lambda I - B \\ C & D \end{bmatrix}\right] = n + m
\]
(7)

for all \(\lambda \in \mathbb{C}, \Delta \in \mathcal{P}(\mathbb{I}), A \in \mathcal{P}(A), B \in \mathcal{P}(B), C \in \mathcal{P}(C)\) and \(D \in \mathcal{P}(D)\). Indeed, (7) holds if and only if
\[
\text{rank}\left[\begin{bmatrix} \Delta^{-1}A - \lambda I & \Delta^{-1}B \\ C & D \end{bmatrix}\right] = n + m,
\]
where we have \(\Delta^{-1}A \in \mathcal{P}(A)\) and \(\Delta^{-1}B \in \mathcal{P}(B)\) since \(\Delta^{-1} \in \mathcal{P}(\mathbb{I}), \mathcal{P}(\mathbb{I}) \mathcal{P}(A) = \mathcal{P}(A)\) and \(\mathcal{P}(\mathbb{I}) \mathcal{P}(B) = \mathcal{P}(B)\). Therefore, \((A, B, C, D)\) is strongly structurally input-state observable if and only if
\[
\text{rank}\left[\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right] = \text{rank}\left[\begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right] = n + m
\]
for all nonzero \(\lambda \in \mathbb{C}, \Delta \in \mathcal{P}(\mathbb{I}), A \in \mathcal{P}(A), B \in \mathcal{P}(B), C \in \mathcal{P}(C)\) and \(D \in \mathcal{P}(D)\). In view of Lemma 1, the latter holds if and only if the pattern matrices in (6) have full column rank.

C. Output Controllability

In this section, we will show how pattern matrix multiplication and its properties can be used to characterize strong structural output controllability. To this end, consider the system \((A, B, C, D)\) as defined in Section III-B.

Definition 8: The system \((A, B, C, D)\) is output controllable if for any \(x_0 \in \mathbb{R}^n\) and \(y_1 \in \mathbb{R}^p\), there exist a time \(T > 0\) and an input \(u\) such that \(y(T; x_0, u) = y_1\).

The following is a well-known characterization of output controllability of \((A, B, C, D)\), see [21, Exercise 3.22].

Proposition 3: The system \((A, B, C, D)\) is output controllable if and only if
\[
\text{rank}\left[\begin{bmatrix} D & CB & CAB & \cdots & CA^{n-1}B \end{bmatrix}\right] = p.
\]

Now, consider the structured system \((A, B, C, D)\) as defined in Section III-B.

Definition 9: The structured system \((A, B, C, D)\) is strongly structurally output controllable if \((A, B, C, D)\) is output controllable for all \(A \in \mathcal{P}(A), B \in \mathcal{P}(B), C \in \mathcal{P}(C)\) and \(D \in \mathcal{P}(D)\).

We are interested in conditions under which \((A, B, C, D)\) is strongly structurally output controllable. Note that the condition for output controllability of \((A, B, C, D)\) involves products of system matrices, unlike the conditions for controllability of \((E, A, B)\) or input-state observability of \((A, B, C, D)\). This suggests that we need to consider products of pattern matrices when investigating strong structural output controllability of \((A, B, C, D)\). Unfortunately, since products of pattern matrices do not share the same favourable property as sums, i.e., \(\mathcal{P}(AB) \neq \mathcal{P}(A)\mathcal{P}(B)\), we cannot easily derive necessary and sufficient conditions. Nevertheless, we state and prove the following sufficient condition.

Theorem 4: The structured system \((A, B, C, D)\) is strongly structurally output controllable if
\[
\begin{bmatrix} D & CB & CAB & \cdots & CA^{n-1}B \end{bmatrix}
\]
has full row rank.

Proof: Let \(A \in \mathcal{P}(A), B \in \mathcal{P}(B), C \in \mathcal{P}(C)\) and \(D \in \mathcal{P}(D)\). Recall that \(\mathcal{P}(C)\mathcal{P}(B) \subset \mathcal{P}(CB)\), that is, \(CB \in \mathcal{P}(CB)\) for all \(C \in \mathcal{P}(C)\) and \(B \in \mathcal{P}(B)\). Similarly, we can show that \(\mathcal{P}(A)\mathcal{P}(B) \subset \mathcal{P}(AB)\) by induction, hence
\[
\mathcal{P}(C)\mathcal{P}(A)^k \mathcal{P}(B) \subset \mathcal{P}(C)\mathcal{P}(A)\mathcal{P}(B) \subset \mathcal{P}(C)\mathcal{P}(A)\mathcal{P}(B),
\]
for all positive integers \(k\). In other words, we have that
\[
\begin{bmatrix} D & CB & CAB & \cdots & CA^{n-1}B \end{bmatrix} \subset \mathcal{P}\left[\begin{bmatrix} D & CB & CAB & \cdots & CA^{n-1}B \end{bmatrix}\right],
\]
hence \((A, B, C, D)\) is output controllable due to Proposition 3. As \((A, B, C, D)\) is output controllable, it follows that \((A, B, C, D)\) is strongly structurally output controllable.

As already mentioned in the introduction, strong structural output controllability is closely related to strong structural target controllability of networked systems. To show this, we will follow the exposition in [11]. Consider the graph \(G = (V, E)\) with vertex set \(V = \{1, \ldots, n\}\) and edge set \(E \subset V \times V\). The qualitative class \(Q(G)\) of \(G\) is defined as
\[
Q(G) = \{A \in \mathbb{R}^{n \times n} \mid \text{for } i \neq j, \text{ } A_{ij} \neq 0 \iff (i, j) \in E\}.
\]

For subsets \(V_r, V_c \subset V\), let \(P(V_r; V_c)\) denote the submatrix of the \(n \times n\) identity matrix whose rows and columns are indexed by \(V_r\) and \(V_c\), respectively. Now, consider a leader set \(V_L \subset V\) and a target set \(V_T \subset V\). The triple \((G; V_L; V_T)\) defines the family of systems \((A, B, C, 0)\), where \(A \in Q(G), B = P(V; V_L)\) and \(C = P(V_T; V)\). The triple \((G; V_L; V_T)\) is said to be strongly structurally target controllable if \((A, B, C, 0)\) is output controllable for all \(A \in Q(G)\). This already suggests a connection between strong structural target controllability and strong structural output controllability.

To make this explicit, let \(A \in [0, *, \ldots, n]^{n \times n}\) be such that
\[
A_{ij} = \begin{cases} \ast & \text{if } i \neq j, \\ 0 & \text{otherwise}, \\ ? & \text{if } i = j, \end{cases}
\]
and note that \(\mathcal{P}(A) = Q(G)\). Moreover, let \(B\) and \(C\) be the pattern matrices obtained from \(B\) and \(C\) by replacing all 1’s with \(*\)’s. Given the special structure of \(B\) and \(C\), any matrix \(B \in \mathcal{P}(B)\) can be obtained from \(B\) by an appropriate nonzero scaling of its columns, and any matrix \(C \in \mathcal{P}(C)\) can be obtained from \(C\) by an appropriate nonzero scaling of its rows. Since the rank of a matrix is invariant under nonzero scaling of its rows and columns, it follows that
\[
\text{rank}\left[\begin{bmatrix} \bar{C}B & \bar{C}AB & \cdots & \bar{C}A^{n-1}B \end{bmatrix}\right] = p
\]
Fig. 1. The graph $G = (V, E)$.

for all $A \in Q(G)$ if and only if

$$\text{rank}[CB \ CAC \ \cdots \ CA^{n-1}B] = p$$

for all $A \in \mathcal{P}(A)$, $B \in \mathcal{P}(B)$ and $C \in \mathcal{P}(C)$. This implies that $(G; V_L; V_T)$ is strongly structurally target controllable if and only if $(A, B, C, 0)$ is strongly structurally output controllable, hence we can use Theorem 4 to check for strong structural target controllability. In fact, Theorem 4 can reveal that $(G; V_L; V_T)$ is strongly structurally target controllable in cases where even the sharpest theorem in [11], Theorem VI.6, is inconclusive. We demonstrate this in the following example, which is borrowed from [11, Th. 6.6].

**Example 2:** Consider the graph $G(V, E)$ depicted in Figure 1. Let $V_L = \{1, 2\}$ and $V_T = \{1, \ldots, 7\}$ be the leader and targets sets, respectively. Let $A$ be the pattern matrix for which $\mathcal{P}(A) = Q(G)$ and let $B$ and $C$ be the pattern matrices obtained from $P(V; V_L)$ and $P(V_T; V)$ by replacing all 1’s with *’s. With $D = 0$, the matrix in Theorem 4 is given by

$$
\begin{bmatrix}
0 & 0 & * & ? & ? & ? & \ldots \\
0 & 0 & * & ? & ? & ? & \ldots \\
0 & 0 & 0 & * & ? & ? & \ldots \\
0 & 0 & 0 & 0 & * & ? & \ldots \\
0 & 0 & 0 & 0 & 0 & ? & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & * \\
\end{bmatrix}
$$

which has full row rank due to the upper triangular structure when the first two columns are neglected. In view of Theorem 4 and the discussion above, we conclude that $(G; V_L; V_T)$ is strongly structurally target controllable. Note that the authors of [11] could not make this conclusion, as explained in the last paragraph of [11, Sec. VI].

IV. Conclusion

In this letter, we adopted and expanded the pattern matrix framework introduced in [6] in order to study structured systems and their strong structural properties. In particular, we defined addition and multiplication for the set of symbols $\{0, *, ?\}$, which allowed us to define addition and multiplication for pattern matrices with entries in the set $\{0, *, ?\}$. These definitions are such that the pattern class of a sum (product) of pattern matrices is contained in the sum (product) of their pattern classes. We showed that the converse is true for sums, but generally not true for products.

Using these operations and their properties, we characterized strong structural controllability of linear descriptor systems, as well as strong structural input-state observability and output controllability of linear systems. All characterizations are algebraic and take the form of full rank conditions on suitably chosen pattern matrices. While the conditions are necessary and sufficient for input-state observability, they are only sufficient for the other two properties.

Finally, it would be worthwhile to investigate extensions of our work to more general classes of structured systems, e.g., structured systems that allow given nonzero or arbitrary entries to be constrained to take identical values (see [9], [22]). The latter has been motivated by the fact that nonzero parameters in physical systems are seldom independent, e.g., due to symmetry or due to physical constraints.

**References**


