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Dynamic Pricing Control for Constrained Distribution Networks With Storage

Mathias Bürger, Claudio De Persis, and Frank Allgöwer

Abstract—This paper studies the problem of optimal flow control in dynamic inventory systems. A dynamic optimal distribution problem, including time-varying supply and demand, capacity constraints on the transportation lines, and convex flow cost functions of Legendre-type, is formalized and solved. The time-varying optimal flow is characterized in terms of the time-varying dual variables of a corresponding network optimization problem. A dynamic feedback controller is proposed that regulates the flows asymptotically to the optimal flows and achieves in addition a balancing of all storage levels.

Index Terms—Graphs, inventory control, network theory, nonlinear control systems.

I. INTRODUCTION

DISTRIBUTION networks are systems composed of several inventories, where goods/materials/products are stored, and transportation lines along which the goods are transported. The control problem in distribution networks is to assign the flows to the transportation lines so that an external supply of goods, arriving at some of the inventories, is distributed to inventories in such a way that a requested demand can always be satisfied. The static problem of distributing a constant flow to transportation lines according to some cost functions has been at the heart of the mathematical theory called *network optimization* [2]–[4]. However, real networks must react dynamically on changes in the external conditions, such as time-varying supply and demand. If the networks operate on a fast time scale, continuous feedback controllers are required to adapt the flow instantaneously to external changes. Beyond that, feedback controllers also promise to provide desirable robustness properties to the network.

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The control objective of the feedback controllers is twofold. First, in order to minimize the variation in the storage levels of the single inventories, the controllers should distribute the goods that have to be stored in the network (i.e., those goods that are supplied but not requested) equally to all inventories in the system. Second, the controllers should adjust the flows on the transportation lines in such a way that at each time instant the cost of transporting the goods is minimized.

Feedback control of inventory systems has recently found significant attention. The control of distribution networks with unknown supply and demand has been investigated in [5], using a combination of robust optimization and control theory. A control strategy is derived that ensures robust feasibility as well as a min-max-type optimality. Robust control strategies for distribution networks have been considered, for example, in [6] and [7]. The aspect of balancing in distribution networks has been studied in [8] using a one-step model predictive control scheme. In [9] and [10], distribution networks with constant supply and demand were considered and it was shown that the balancing problem can be formulated in a Hamiltonian framework.

Network flow and balancing problems are intimately related to consensus or output agreement problems. In [11] and [12], we showed that networks of passive systems with constant external disturbances solve asymptotically certain network flow problems. Furthermore, the phenomena of output agreement and clustering are mathematically related to optimal network flow problems [13]. The connection between output agreement and network flow control has been further studied from an internal model control perspective in [14]–[16]. Considering demands and supplies that are generated by an external dynamical system (exosystem), distributed dynamic controllers were proposed that ensure exact routing as well as a balancing of the inventory levels. In [16], controllers were also proposed that route the flow asymptotically optimal through the network, provided that the transportation capacity is unlimited and the underlying cost functions are quadratic.

We address here the problem of balancing and optimal routing in time-varying distribution networks with convex routing costs and capacity constraints on the transportation lines. In this regard, we address several aspects of the dynamic optimal distribution problem that have not been considered in the mentioned references. The time-varying optimal flow problem is formulated, assuming that the supply/demand is generated by an exosystem. It is shown that for a certain class of convex cost functions, namely *Legendre-type* functions, the time-varying optimal flow can be characterized in terms of the time-varying dual solutions. Legendre-type cost functions are smooth,

strictly convex functions, that grow unbounded on the boundary of the constrained set. We show that the required Legendre-type cost functions can be constructed, for example, by augmenting a strictly convex cost function with logarithmic barrier functions to incorporate the capacity constraints.

Assuming that the cost functions are of Legendre-type, we derive an explicit differential equation for the dual solutions. The construction of this differential equation requires finding the analytic solution to a *generalized Dirichlet problem* [17]. Based on this characterization of the time-varying dual variables, a feedback controller is proposed that achieves a balancing of the storage levels and routes the flow asymptotically optimal through the network. We first present the controller for the special case that the transportation network as a whole has infinite transportation capacities (while single lines can still have finite capacity). Then, we show how the controller can be modified for networks with finite transportation capacity.

The remainder of this paper is organized as follows. The inventory system model and the relevant assumptions on the supply are introduced in Section II. The time-varying optimal flow problem is presented in Section III, where also a differential equation for the optimal dual variables is derived. A feedback controller, ensuring balancing and asymptotic optimal routing, is proposed in Section IV. The design procedure is illustrated in Section V.

Preliminaries: The notation we employ is standard. The set of (non negative) real numbers is denoted by $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$). The distance of a point q from a set $\mathcal{A}$ is defined as $\text{dist}_{\mathcal{A}} q = \inf_{p \in \mathcal{A}} |p - q|$. Given a matrix B (vector x), the notation $[B]_{ik}$ ($[x]_i$) denotes its ik th (i th) entry. The range space and null space of a matrix B are denoted by $\mathcal{R}(B)$ and $\mathcal{N}(B)$, respectively. A graph $\mathcal{G} = (V, E)$ is an object consisting of a finite set of nodes $|V| = n$ and edges $|E| = m$. Throughout this paper, we assume that a graph $\mathcal{G}$ is assigned an (arbitrary) orientation. The *incidence matrix* $B \in \mathbb{R}^{n \times m}$ of the graph $\mathcal{G}$ is a $\{0, \pm 1\}$ matrix with $[B]_{ik}$ having a value of “+1” if node i is the initial node of edge k , “−1” if it is the terminal node, and “0” otherwise.

For a convex function $\mathcal{P} : \mathbb{R}^p \rightarrow \mathbb{R}$, the *effective domain* is $\text{dom} \mathcal{P} = \{s : \mathcal{P}(s) < \infty\}$. The convex conjugate of $\mathcal{P}(s)$ is $\mathcal{P}^*(r) = \sup_s \{r^T s - \mathcal{P}(s) : s \in \text{dom} \mathcal{P}\}$.

II. TIME-VARYING DISTRIBUTION PROBLEMS

Consider an inventory network with n inventories and m transportation lines, described by a graph $\mathcal{G} = (V, E)$, and let B be the incidence matrix (with arbitrary orientation). The dynamics of the inventory system are given as

$$\dot{x}(t) = B\lambda(t) + Pw(t) \quad (1)$$

where $x = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ represents the storage level, $\lambda = [\lambda_1, \dots, \lambda_m]^T \in \mathbb{R}^m$ is the flow along the lines, and Pw is an external in-/outflow of the inventories, that is, the supply or demand. This basic model (or its discrete time version) is studied in [5], [7], [8], [9], and [14], and is a relevant model for supply chains [18] or data networks [19].

The objective of this paper is to design distribution control laws for the flows λ , that achieve two objectives: 1) balancing

of the storage levels, that is, $\lim_{t \rightarrow \infty} |x_i(t) - x_j(t)| = 0$ for all $i, j \in \{1, \dots, n\}$ and 2) an optimal routing of the flows through the network, that is, $\lim_{t \rightarrow \infty} \lambda(t) = \lambda^w(t)$, where $\lambda^w(t)$ is at each time instant the optimizer of a certain network flow problem.

We assume throughout this paper that the supply and demand vary over time and, in particular, that they as well as their derivatives are known, that is, all information about the evolution of the flow is known. In compliance with the control-theoretic approach of this paper, we assume that the supply/demand is generated by a dynamical system

$$\dot{w}(t) = s(w(t)), \quad w \in \mathbb{R}^r \quad (2)$$

with initial condition $w(0) = w_0$.

Assumption 1: There exists a compact set $\mathcal{W}$ that is forward invariant under (2) and the initial conditions $w(0)$ are contained in this set $\mathcal{W}$.

Note that this assumption ensures the boundedness of the supply and demand vector.

At each time instant, the imbalance between supply and demand, that is, the amount of goods that must be either stored or supplied from the inventories, is computed as

$$\Delta(t) = \mathbf{1}_n^T Pw(t).$$

Assumption 2: The accumulated imbalance of the supply/demand of the network is bounded, that is, there exists $C < \infty$ such that

$$\left| \int_0^t \Delta(\tau) d\tau \right| \leq C \quad \text{for all } t \geq 0.$$

If we now assume that the stored goods are distributed equally to all inventories, then the goods that have to be transported to/from the inventories become

$$Pw(t) - \frac{1}{n} \mathbf{1}_n \Delta(t) = \left(I - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) Pw(t).$$

Let the projection matrix for the space orthogonal to the all ones vector be

$$Q = \left(I - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right). \quad (3)$$

In what follows, we will call the vector:

$$\bar{w}(t) = QPw(t) \quad (4)$$

the *balanced component* of the supply/demand vector, since it holds that $\mathbf{1}_n^T \bar{w}(t) = 0$.

We assume that the flow on each line can flow in both directions, but is restricted to satisfy certain capacity bounds. That is, we assume that each line $k \in \{1, \dots, m\}$, there exists $\bar{\lambda}_k \in \mathbb{R}_{>0}$ and the flow must satisfy

$$-\bar{\lambda}_k \leq \lambda_k(t) \leq \bar{\lambda}_k \quad \text{for all } t \geq 0. \quad (5)$$

If the flow on one line is unconstrained, we let $\bar{\lambda}_k$ be infinity. In the following, we will also write $-\bar{\lambda} \leq \lambda \leq \bar{\lambda}$, to indicate that the inequalities hold for all components of the vector.

A *cut* of the network is defined to be a signed edge set¹ $Q \subset E$ (with positive and negative parts denoted by Q^+ and Q^-) such that for some node set $S \subset V$

$$\begin{aligned} Q^+ &= \{k \in E : k = (i, j) \text{ with } i \in S, j \in V \setminus S\} \\ Q^- &= \{k \in E : k = (j, i) \text{ with } i \in S, j \in V \setminus S\}. \end{aligned} \quad (6)$$

We will denote the cut induced by a set of nodes $S \subset V$ with $Q(S)$. The *capacity* of a cut is

$$\bar{\lambda}_{Q(S)} = \sum_{k \in Q(S)} \bar{\lambda}_k.$$

To ensure feasibility of the problem, we impose the following condition.

Assumption 3: There exists $\epsilon > 0$ such that for any set of nodes $S \subseteq V$ it holds that

$$-\bar{\lambda}_{Q(S)} + \epsilon \leq \sum_{i \in S} \bar{w}_i \leq \bar{\lambda}_{Q(S)} - \epsilon.$$

The assumption ensures that the “in/out-flow” to any set of nodes is strictly smaller than the capacity of the corresponding induced cut. The following result summarizes the conditions that ensure feasibility of the distribution problem.

Proposition 1 (Feasibility): Consider the inventory system (1) with a time-varying supply/demand generated by the exosystem (2) and capacity constraints (5). Suppose Assumptions 1, 2, and 3 hold. Then, 1) for any flow vector $\lambda \in \mathbb{R}^m$ the average inventory level $x^{\text{avg}}(t) = (1/n)\mathbf{1}_n^T x(t)$ remains bounded and 2) there exists a continuous flow trajectory $\lambda : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ such that for each edge k the bounds (5) hold with strict inequality and $B\lambda(t) + \bar{w}(t) = 0$.

Proof: To see the first claim, note that the average $x^{\text{avg}}(t)$ evolves as $\dot{x}^{\text{avg}} = (1/n)\mathbf{1}_n^T B\lambda + (1/n)\mathbf{1}_n^T Pw(t)$. Thus, $x^{\text{avg}}(t) = x^{\text{avg}}(0) + (1/n)\int_0^t \Delta(\tau)d\tau$. The boundedness of $x^{\text{avg}}(t)$ follows now immediately from Assumption 2.

To prove the second claim, we show first that at each fixed time t , there exists a feasible flow vector. To indicate that we consider a single time instant, we write in the following equation $\bar{w} = \bar{w}(t)$. Assume without loss of generality that all transportation lines have a finite capacity, that is, $\bar{\lambda}_k < \infty$. Let $\Lambda = \text{diag}\{\bar{\lambda}_1, \dots, \bar{\lambda}_m\}$. Consider the linear optimization problem

$$\min_{\lambda} \|\Lambda^{-1}\lambda\|_{\infty} \quad \text{s.t. } B\lambda = -\bar{w}. \quad (7)$$

The linear programming dual can be derived as

$$\max_u -\bar{w}^T u \quad \text{s.t. } \|\Lambda B^T u\|_1 \leq 1. \quad (8)$$

Note now that the extreme points of the polytope defined by $\|\Lambda B^T u\|_1 \leq 1$ are multiples of the vectors whose elements are zeros and ones (see, for example, [20]). Thus, each extreme point must be of the form $u = c_Q e_Q$, where Q is a cut induced by some node set $S \subseteq V$ according to (6), $c_Q \in \mathbb{R}$ is a constant such that $|c_Q| = \bar{\lambda}_Q^{-1}$, and $e_Q \in \mathbb{R}^n$ is a vector with the entries being $[e_Q]_i = 1$ if node $i \in S$ and $[e_Q]_i = 0$ if $i \in V \setminus S$. It

follows now from Assumption 3 that (8) must always have an optimal value strictly smaller than one. From the strong duality of linear programs follows that (7) must also have an optimal value strictly smaller than one. This shows that at each fixed time instant t , there exists a feasible flow vector $\lambda(t)$ that satisfies all constraints, that is, $-\bar{\lambda} < \lambda(t) < \bar{\lambda}$.

It remains to prove that continuous function $\lambda : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ exists that represents a feasible flow trajectory. Consider now the set of feasible flows for a given supply/demand vector w , i.e.,

$$\Lambda(w) = \{\lambda \in \mathbb{R}^m : B\lambda = -\bar{w}, -\bar{\lambda} \leq \lambda \leq \bar{\lambda}\}.$$

The set $\Lambda(w)$ is compact, convex, and has a nonempty relative interior. One can understand Λ as a *set-valued map* from $\mathbb{R}^r$ to $\mathbb{R}^m$. As a set-valued map, Λ is *lower semicontinuous*.² The existence of a continuous selection $\lambda(w)$ of Λ follows now from Michael's Theorem ([21, Theor. 9.1.2]). Since, in addition, $w(t)$ is a continuous function of time (i.e., it respects the dynamics (2)), there exists a continuous function $\lambda(t)$ such that $\lambda(t) \in \Lambda(w(t))$, that is, there is a continuous feasible flow trajectory.

Remark 1: In the proposition from before, we investigate the existence of a flow λ for which the equality $B\lambda(t) + \bar{w}(t) = 0$. The reason why we consider this equality rather than $B\lambda(t) + w(t) = 0$ is that the existence of a solution to the balancing of the inventory levels of system (1) requires necessarily the fulfillment of the equality $B\lambda(t) + \bar{w}(t) = 0$ (see the proof of Theorem 1 below).

III. OPTIMAL TIME-VARYING ROUTING

In the following text, we assume that the routing of flow along transportation lines of the network incorporates a cost. We seek a control strategy to route the time-varying flow through the network in such a way that the routing cost is minimal at each time instant. We next characterize a class of feed-forward controllers that achieve such an optimal routing.

The flow cost on a transportation line is described by a convex C^∞ cost function

$$\mathcal{P}_k(\lambda_k), \quad k = 1, 2, \dots, m$$

for the flow λ_k . We aim to characterize the steady-state flows that are such that at each time instant the following *static optimal distribution problem* is solved

$$\lambda^w = \arg \min_{\lambda} \sum_{k=1}^m \mathcal{P}_k(\lambda_k) \quad 0 = B\lambda + \bar{w} \quad (9)$$

where $\bar{w} = \bar{w}(t)$ is the supply/demand at time t . We will use in the following the short-hand notation $\mathcal{P}(\lambda) = \sum_{k=1}^m \mathcal{P}_k(\lambda_k)$.

The convex cost functions are such that they incorporate the capacity constraints of the transportation lines, in the sense that $\mathcal{P}_k(\lambda_k) = \infty$ for $|\lambda_k| > \bar{\lambda}_k$. More precisely, we assume that the cost functions are of *Legendre type* [22, p. 258].

¹We assume that the direction of the edges is chosen in accordance with the definition of the incidence matrix B .

²The set-valued function Λ is lower semicontinuous at a point $w \in \text{dom}\Lambda$ if and only if for any $\lambda \in \Lambda(w)$ and any sequence w_k converging to w , there exists a sequence $\lambda_k \in \Lambda(w_k)$ converging to λ . It is said to be lower semicontinuous if it is lower semicontinuous at every point $w \in \text{dom}\Lambda$.

Assumption 4: The functions $\mathcal{P}_k$, $k \in \{1, \dots, m\}$ are strictly convex and differentiable on the open set $\{\lambda_k : -\bar{\lambda}_k < \lambda_k(t) < \bar{\lambda}_k\}$, and $\lim_{\ell \rightarrow \infty} |\nabla \mathcal{P}(\lambda_k^{[\ell]})| = +\infty$ for any sequence $\lambda_k^{[1]}, \lambda_k^{[2]}, \dots$ such that $|\lambda_k^{[\ell]}| < \bar{\lambda}_k$ and $\lim_{\ell \rightarrow \infty} |\lambda_k^{[\ell]}| = \bar{\lambda}_k$.

With cost functions of this form, any finite solution to (9) represents a feasible flow. Note that if the assumptions of Proposition 1 hold, then the optimization problem (9) has always a finite optimal solution.

We can now characterize the optimality conditions of (9). The Lagrangian function associated with (9) with the Lagrange multiplier ζ is

$$\mathcal{L}(\lambda, \zeta) = \mathcal{P}(\lambda) + \zeta^T (-B\lambda - \bar{w}).$$

Now, (λ^w, ζ^w) is an optimal primal/dual solution pair to (9) if the following nonlinear equations hold:

$$\nabla \mathcal{P}(\lambda^w) - B^T \zeta^w = 0 \quad B\lambda^w + \bar{w} = 0. \quad (10)$$

Note that the first condition simply implies $\nabla \mathcal{P}(\lambda^w) \in \mathcal{R}(B^T)$ since ζ^w has no further constraints. We define the set of optimal routing/supply pairs as

$$\Gamma = \{(\lambda, w) \in \mathbb{R}^m \times \mathcal{W} : \nabla \mathcal{P}(\lambda) \in \mathcal{R}(B^T), \\ B\lambda + QPw = 0\} \quad (11)$$

where we used that $\bar{w} = QPw$. In particular, $(\lambda^w, w) \in \Gamma$ if and only if λ^w is an optimal solution to the static optimal distribution problem (9) with the supply vector $\bar{w}$.

Proposition 2 (Feedforward Controller): Suppose Assumptions 1, 2, 3, and 4 hold. Then there exist functions $\Phi : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$, such that for any w_0 , there exists $\zeta^w(0)$ such that the solution $\zeta^w(t)$ to the dynamical system

$$\dot{\zeta} = \Phi(\zeta, w), \quad \lambda = \Psi(\zeta) \quad (12)$$

originating at $\zeta^w(0)$ is such that $\lambda^w(t) = \Psi(\zeta^w(t))$ satisfies $(\lambda^w(t), w(t)) \in \Gamma$ for all $t \geq 0$, where $w(t)$ is the solution to (2) originating at $w(0) = w_0$.

The proof of this result provides a method to construct the dynamical system (12). However, before we present the proof, some additional discussion is required.

The main difficulty associated with the proposed characterization of the set Γ is related to the constraint $\nabla \mathcal{P}(\lambda) \in \mathcal{R}(B^T)$. To avoid this constraint, we express the optimality conditions in terms of the *dual solutions* ζ . Under the given assumption that the cost functions $\mathcal{P}_k$ are of Legendre type, the gradients $\nabla \mathcal{P}_k$ are invertible on the open sets $\{\lambda_k : -\bar{\lambda}_k < \lambda_k < \bar{\lambda}_k\}$ [4]. Since the optimal solution must always be strictly feasible, the two optimality conditions (10) can be combined to

$$B\nabla \mathcal{P}^{-1}(B^T \zeta^w) + \bar{w} = 0. \quad (13)$$

Bearing this in mind, we define the set of optimal dual solutions as

$$\Gamma_D = \{(\zeta, w) \in \mathbb{R}^n \times \mathcal{W} : B\nabla \mathcal{P}^{-1}(B^T \zeta) + QPw = 0\}$$

where we recalled that $\bar{w} = QPw$. We want to emphasize two properties of Γ_D . First, if $(\zeta^w, w) \in \Gamma_D$, then $(\zeta^w + c\mathbf{1}, w) \in \Gamma_D$

for any $c \in \mathbb{R}$. Second, $(\zeta^w, w) \in \Gamma_D$ if and only if the routing strategy $\lambda^w = \nabla \mathcal{P}^{-1}(B^T \zeta^w)$ satisfies $(\lambda^w, w) \in \Gamma$.

Remark 2: In the considered framework, the inverse of the gradient of $\mathcal{P}_k$ is the gradient of the convex conjugate $\mathcal{P}_k^*$, that is, $\nabla \mathcal{P}^{-1}(\cdot) = \nabla \mathcal{P}^*(\cdot)$ [22]. For completeness, we want to point out that the dual problem to (9) is

$$\min_{z, \zeta} \sum_{k=1}^m \mathcal{P}_k^*(z_k) + \bar{w}^T \zeta \quad z = B^T \zeta \quad (14)$$

where $\mathcal{P}_k^*$ are the convex conjugates of $\mathcal{P}_k$.

The main idea pursued here is to design the feedforward controller for the *dual variables*. We need to design a dynamical system that generates a solution $\zeta(t)$ such that the constraint (13) is satisfied for all times. Suppose now that (13) is feasible at the initial time, that is

$$B\nabla \mathcal{P}^{-1}(B^T \zeta^w(0)) = -\bar{w}(0). \quad (15)$$

Then, $\dot{\zeta}^w(t)$ must be such that

$$(BD\mathcal{P}^{-1}(B^T \zeta^w(t))B^T) \dot{\zeta}^w(t) = -\dot{\bar{w}}(t) \quad (16)$$

where $D\mathcal{P}^{-1}(\cdot) = \text{diag}[\nabla^2 \mathcal{P}_1^{-1}(\cdot), \dots, \nabla^2 \mathcal{P}_m^{-1}(\cdot)]$. Note that $D\mathcal{P}^{-1}(\cdot)$ is the Hessian matrix of $\mathcal{P}^*$.

Equation (16) defines an implicit differential equation for ζ^w . For the design of the feedforward controller (12), we need to transform it into an explicit differential equation.

Under the stated assumptions, $D\mathcal{P}^{-1}(\cdot)$ is a diagonal matrix, with all entries on the diagonal being positive. Thus, the matrix

$$L(\zeta^w) := BD\mathcal{P}^{-1}(B^T \zeta^w)B^T \quad (17)$$

is a weighted, state-dependent *Laplacian matrix* of the network, with the weights on the edges being the second derivatives of the cost function's convex conjugates. The Laplacian matrix has always one eigenvalue at zero and is therefore not invertible. To find an explicit differential equation for ζ^w , we make the following observation. For the balanced component of the supply/demand, there always exists $\omega(t) \in \mathbb{R}^m$ such that

$$QPw(t) = B\omega(t).$$

One can always define $\omega(t) = B^T(BB^T)^\dagger QPw(t) = B^T \times (BB^T)^\dagger Pw(t)$, where $(BB^T)^\dagger$ is the *generalized inverse* of the matrix BB^T .³ Now, the conditions in (16) can be written as

$$L(\zeta^w(t)) \dot{\zeta}^w(t) = -B\dot{\omega}(t). \quad (18)$$

Following [17, Ch. 16] we call the problem of finding $\dot{\zeta}^w(t)$ a *generalized Dirichlet problem*.

It is described in [17, p. 659] how an analytic solution for $\dot{\zeta}^w$ in (18) can be constructed. In particular, one can find analytically a matrix $X(\zeta^w) \in \mathbb{R}^{m \times n}$ such that

$$\dot{\zeta}^w(t) = -X(\zeta^w(t))^T D\mathcal{P}(B^T \zeta^w(t)) \dot{\omega}(t) \quad (19)$$

³By the properties of the generalized inverse, it is readily verified that $B\omega(t) = BB^T(BB^T)^\dagger Pw(t) = QPw(t) = \bar{w}(t)$. It can be easily verified that the matrix $B^T(BB^T)^\dagger$ is, in fact, the Moore–Penrose inverse of the incidence matrix B .

solves (16).⁴ The algorithmic procedure for the construction of $X(\zeta^w)$ is provided in [17, p. 659] and is reviewed in the Appendix. The construction of $X(\zeta^w)$ has an explicit graph-theoretic interpretation and the matrix has a characteristic structure. The entries of $X(\zeta^w)$ are fractions of polynomials, with each polynomial being the sum of products of different functions $\nabla^2 \mathcal{P}_k^{-1}(\cdot)$. Since, by assumption, all $\nabla^2 \mathcal{P}_k^{-1}(\cdot)$ are strictly positive and continuously differentiable, $X(\zeta^w)$ is continuously differentiable. Note that if the assumptions of Proposition 1 hold, the optimal flow problem (9) has, at each time instant, a feasible solution and, thus, admits a finite dual solution. We are now ready to prove Proposition 2.

Proof (of Proposition 2): We prove the proposition by constructing the corresponding feedforward controller. Consider the dynamical system

$$\begin{aligned}\dot{\zeta} &= -X(\zeta)^T \mathcal{D} \mathcal{P}(B^T \zeta) B^T (BB^T)^\dagger P s(w) \\ \lambda &= \nabla \mathcal{P}^{-1}(B^T \zeta)\end{aligned}\quad (20)$$

where $X(\zeta)$ is as constructed in (19) and $(BB^T)^\dagger$ is the generalized inverse of BB^T . If the initial condition $\zeta(0)$ is chosen as an optimal dual solution at time $t = 0$, that is, $(\zeta(0), w(0)) \in \Gamma_D$, then the solution $\zeta(t)$ to (20) originating at $\zeta(0)$ will be a solution to (16) and will therefore be an optimal dual solution for all times, that is, $(\zeta(t), w(t)) \in \Gamma_D$ for all $t \geq 0$. Furthermore, if $\zeta(t)$ is an optimal dual solution at time t , then $\lambda(t) = \nabla \mathcal{P}^{-1}(B^T \zeta(t))$ is an optimal primal solution, that is, $(\lambda(t), w(t)) \in \Gamma$. Thus, Proposition 2 follows with $\Phi(\zeta, w) = -X(\zeta)^T \mathcal{D} \mathcal{P}(B^T \zeta) B^T (BB^T)^\dagger P s(w)$, $\Psi(\zeta) = \nabla \mathcal{P}^{-1}(B^T \zeta)$, and $\zeta(0)$ such that $(\zeta(0), w(0)) \in \Gamma_D$.

Remark 3: The feedforward controller (20) (initialized with the correct initial condition) generates the exact optimal routing of the time-varying flows. Since the optimal solution follows directly from the problem itself, also the number of states of (20) is defined by the problem. Please note that the number of states is exactly the number of inventories in the network.

IV. FEEDBACK CONTROLLER DESIGN

We can use the proposed feedforward controller now to design a dynamic feedback controller for the inventory system (1) that ensures convergence to the optimal routing without being initialized at an optimal solution. For the feedback controller design, we extend now the control objective and require in addition that the feedback controller achieve a *balancing* of the inventory levels of (1). Network balancing is widely accepted as the control objective in storage networks, since it promises to make the best use of the available storage [8], [14].

Definition 1: The *time-varying optimal distribution problem* is solvable for the system (1), if there exists a controller of the form

$$\dot{\zeta} = \Phi(\zeta, w, x) \quad \lambda = \Psi(\zeta, x) \quad (21)$$

⁴Please note that the solution to (18) could equivalently be expressed as $\dot{\zeta}^w(t) = -L^\dagger(\zeta^w(t)) B \dot{w}(t)$, where $L^\dagger(\zeta^w(t))$ is the generalized inverse of the weighted Laplacian matrix (17). However, to do this, one needs an analytic expression for $L^\dagger(\zeta^w(t))$, that is, for the generalized inverse of a state-dependent matrix. This analytic solution is, in fact, what is defined by (19).

such that any solution $(x(t), w(t), \zeta(t))$ of the closed-loop system (1), (2), (21) is bounded and satisfies 1) $\lim_{t \rightarrow \infty} B^T x(t) = 0$ and 2) $\lim_{t \rightarrow \infty} \text{dist}_\Gamma(\lambda(t), w(t)) = 0$.

Note that the flow constraints are treated here as soft constraints that must be satisfied only asymptotically. Furthermore, the proposed problem formulation allows using the supply/demand vector w directly in the controller. We next present a constructive way for designing a controller (21) that solves the time-varying optimal distribution problem on the basis of the feedforward controller designed in the previous section.

A. Controller Design for Infinite Cut Capacity

To start the discussion, we use a stronger assumption on the feasibility of the flows (i.e., Assumption 3), that is, we assume that each cut has an infinite capacity. The new assumption is introduced for technical reasons and will be relaxed later on.

Assumption 5: For any set of nodes $S \subset V$ it holds that $\bar{\lambda}_{Q(S)} = \infty$.

This assumption implies that there are transportation lines forming the spanning tree in the network that have infinite capacity. Note that it does not require that all lines are unconstrained. Assumption 5 will allow us to present the controller design in a straightforward manner. In order to present the controller design in a compact way, we define the matrix

$$\Xi(\zeta) := X(\zeta)^T \mathcal{D} \mathcal{P}(B^T \zeta). \quad (22)$$

The main result of this paper is a controller that routes the flow asymptotically optimal through the network.

Theorem 1 (Feedback Controller): Consider the inventory dynamics (1) with the supply generated by the dynamical system (2). Suppose the Assumptions 1, 2, 4, and 5 hold. Then, the dynamic controller

$$\begin{aligned}\dot{\zeta} &= -Q \Xi(\zeta) (B^T (BB^T)^\dagger (P s(w) + x + k P w) \\ &\quad + k \nabla \mathcal{P}^{-1}(B^T \zeta)) \\ \lambda &= \nabla \mathcal{P}^{-1}(B^T \zeta) - B^T x\end{aligned}\quad (23)$$

with $k > 0$ being a scalar controller gain, solves the time-varying optimal distribution problem for any initial inventory levels $x(0)$ and any controller initialization $\zeta(0)$, that is, all trajectories remain bounded, and 1) $\lim_{t \rightarrow \infty} B^T x = 0$ and 2) $\lim_{t \rightarrow \infty} \text{dist}_\Gamma(\lambda(t), w(t)) = 0$.

Proof: To achieve a balancing of all inventory levels, there must exist $x^w(t) = x_0^w(t) \mathbf{1}_n$ and $\lambda^w(t)$ such that

$$\dot{x}_0^w(t) \mathbf{1}_n = B \lambda^w(t) + P w(t).$$

Choosing $x_0^w(0) = x^{\text{avg}}(0)$ and multiplying the previous expression from the left with $(1/n) \mathbf{1}_n$ gives $x^w(t) = x^{\text{avg}}(t) \mathbf{1}_n$. Note that

$$\begin{aligned}\dot{x}^{\text{avg}}(t) \mathbf{1}_n &= B \lambda^w(t) + P w(t) \\ \Leftrightarrow 0 &= B \lambda^w(t) + \left(I - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) P w = B \lambda^w(t) + \bar{w}(t).\end{aligned}$$

Thus, a flow $\lambda^w(t)$ that can keep all inventory levels balanced must necessarily be such that it routes the balanced component of the supply/demand through the network. Note

for the choice $x^w(t) = x^{\text{avg}}(t)\mathbf{1}_n$, it holds that $\mathbf{1}_n^T(x(0) - x^w(0)) = 0$ and $\mathbf{1}_n^T(\dot{x}(t) - \dot{x}^w(t)) = \mathbf{1}_n^T(B\lambda(t) + Pw(t) - B\lambda^w(t) - Pw(t)) = 0$ and therefore

$$(x(t) - x^w(t)) \in \mathcal{R}(B).$$

Furthermore, we have that the average of the controller variables $(1/n)\mathbf{1}_n^T\zeta(t)$ is constant since $\mathbf{1}_n^T\dot{\zeta} = 0$. The optimal controller (i.e., dual) variables satisfy

$$B\nabla\mathcal{P}^{-1}(B^T\zeta^w(t)) + QPw(t) = 0. \quad (24)$$

In view of the considerations presented before, we can now set $\lambda^w(t) = \nabla\mathcal{P}^{-1}(B^T\zeta^w(t))$, which gives

$$\dot{x}^w(t) = B\nabla\mathcal{P}^{-1}(B^T\zeta^w(t)) + Pw(t). \quad (25)$$

To prove the desired convergence result, consider a composed storage function of the form

$$U(x - x^w, \zeta, w) = V(x - x^w) + W(\zeta, w). \quad (26)$$

As the first component of the storage function, we consider

$$V(x - x^w) = \frac{1}{2}\|x - x^w\|^2. \quad (27)$$

This function satisfies

$$\begin{aligned} \dot{V} &= (x - x^w)^T(B\lambda + Pw - B\lambda^w - Pw) \\ &= -(x - x^w)^T BB^T(x - x^w) \\ &\quad + (x - x^w)^T(B\nabla\mathcal{P}^{-1}(B^T\zeta) - B\nabla\mathcal{P}^{-1}(B^T\zeta^w)) \\ &= -(x - x^w)^T BB^T(x - x^w) \\ &\quad + (x - x^w)^T(B\nabla\mathcal{P}^{-1}(B^T\zeta) + QPw) \end{aligned}$$

where we exploited the two relations (24) and (25). Now, as the second component of U , we consider the positive semidefinite function

$$W(\zeta, w) = \frac{1}{2}\|B\nabla\mathcal{P}^{-1}(B^T\zeta) + QPw\|^2. \quad (28)$$

The function satisfies $W(\zeta, w) = 0$ if and only if $(\zeta, w) \in \Gamma_D$, that is, if $\zeta = \zeta^w$.

In order to keep the notation compact, we introduce the vector valued function

$$Z(\zeta, w) := B\nabla\mathcal{P}^{-1}(B^T\zeta) + QPw. \quad (29)$$

The directional derivative of W satisfies

$$\begin{aligned} \dot{W} &= Z(\zeta, w)^T \left(B D\mathcal{P}^{-1}(B^T\zeta) B^T \dot{\zeta} + Q P \dot{w} \right) \\ &= Z(\zeta, w)^T \left(-L(\zeta) \Xi(\zeta) B^T (BB^T)^\dagger (Ps(w) + x + kPw) \right. \\ &\quad \left. - kL(\zeta) \Xi(\zeta) \nabla\mathcal{P}^{-1}(B^T\zeta) + QPs(w) \right) \end{aligned}$$

where we defined analogous to (17)

$$L(\zeta) := (B D\mathcal{P}^{-1}(B^T\zeta) B^T)$$

and used the fact that $L(\zeta)Q = L(\zeta)$.

Recalling the definition of $\Xi(\zeta)$ in (22) and the definition of $X(\zeta)$ provided in the Appendix, it follows that:

$$\begin{aligned} L(\zeta) \Xi(\zeta) B^T (BB^T)^\dagger Ps(w) \\ = L(\zeta) X(\zeta)^T D\mathcal{P}(B^T\zeta) B^T (BB^T)^\dagger Ps(w) \end{aligned}$$

$$\begin{aligned} &= BB^T (BB^T)^\dagger Ps(w) = \left(I - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) Ps(w) \\ &= QPs(w). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} L(\zeta) \Xi(\zeta) B^T (BB^T)^\dagger x \\ = L(\zeta) X(\zeta) D\mathcal{P}(B^T\zeta) B^T (BB^T)^\dagger x = BB^T (BB^T)^\dagger x \\ = \left(I - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T \right) x = (x - x^{\text{avg}} \mathbf{1}_n) = (x - x^w). \end{aligned}$$

Along the same line of argumentation, we note that

$$\begin{aligned} L(\zeta) \Xi(\zeta) (B^T (BB^T)^\dagger Pw + L(\zeta) \Xi(\zeta) \\ \nabla\mathcal{P}^{-1}(B^T\zeta)) = BB^T (BB^T)^\dagger Pw + B\nabla\mathcal{P}^{-1}(B^T\zeta) \\ = QPw + B\nabla\mathcal{P}^{-1}(B^T\zeta) = Z(\zeta, w). \end{aligned}$$

From these considerations, it follows that:

$$\dot{W} = -Z(\zeta, w)^T (x - x^w) - kZ(\zeta, w)^T Z(\zeta, w).$$

To complete the proof of convergence, we consider now the composed (positive semidefinite) storage function

$$\begin{aligned} U(x - x^w, \zeta, w) &= V(x - x^w) + W(\zeta, w) \\ &= \frac{1}{2}\|x - x^w\|^2 + \frac{1}{2}\|Z(\zeta, w)\|^2. \end{aligned}$$

The derivative of this function satisfies

$$\begin{aligned} \dot{U} &= -(x - x^w)^T BB^T (x - x^w) + (x - x^w)^T Z(\zeta, w) \\ &\quad - Z(\zeta, w)^T (x - x^w) - kZ(\zeta, w)^T Z(\zeta, w) \\ &= -(x - x^w)^T BB^T (x - x^w) - kZ(\zeta, w)^T Z(\zeta, w). \quad (30) \end{aligned}$$

Thus, the function U is nonincreasing under the closed-loop dynamics. Note that $\mathbf{1}^T(x - x^w) = 0$ and, therefore, it holds that

$$(x - x^w)^T BB^T (x - x^w) \geq s_2 \|x - x^w\|^2 \quad (31)$$

where $s_2 > 0$ is the smallest nonzero eigenvalue of the Laplacian matrix BB^T .⁵

The derivative of $U(x - x^w, \zeta, w)$ satisfies therefore

$$\dot{U} \leq -\sigma U \quad (32)$$

where $\sigma = \min\{s_2, k\}$. It follows now from the Comparison Lemma [23, Lemma 3.4] that

$$U(x(t) - x^w(t), \zeta(t), w(t)) \leq e^{-\sigma t} U(x(0) - x^w(0), \zeta(0), w(0))$$

and, consequently, $\lim_{t \rightarrow \infty} U(x(t) - x^w(t), \zeta(t), w(t)) = 0$. This proves that $\lim_{t \rightarrow \infty} \|x(t) - x^w(t)\| = 0$, which implies $\lim_{t \rightarrow \infty} B^T x(t) = 0$ and $\lim_{t \rightarrow \infty} Z(\zeta(t), w(t)) = 0$. These conditions imply $\lim_{t \rightarrow \infty} \text{dist}_\Gamma(\lambda(t), w(t)) = 0$.

It remains to show the boundedness of all trajectories. The uniform boundedness of $U(x(t) - x^w(t), \zeta(t), w(t))$ over time implies directly that $V = \|x(t) - x^w\|^2$ must remain bounded and, consequently, $x(t)$ must remain bounded. Now, the boundedness of U implies the boundedness of $W = \|B\nabla\mathcal{P}^{-1} \times (B^T\zeta) - QPw(t)\|^2$. By Assumption 1, $w(t)$ is bounded and, consequently, also $\|B\nabla\mathcal{P}^{-1}(B^T\zeta(t))\|$ must remain bounded.

⁵The scalar s_2 is known as the *algebraic connectivity* of the network.

Thus, it holds that at each time instant, there exists a vector $c(t)$ with finite norm such that

$$B\nabla\mathcal{P}^{-1}(B^T\zeta(t)) = c(t).$$

It follows with the same argumentation as used previously to derive (13) that at each time instant, the vector $\zeta(t)$ is an optimal dual solution to a network flow problem of the form

$$\min \sum_{k=1}^m P_k(\tilde{\lambda}_k(t)) \quad \text{s.t. } B\tilde{\lambda}(t) = c(t). \quad (33)$$

Due to Assumption 5, the problem has a feasible solution and due to strict convexity of the cost functions (Assumption 4), the optimal solution is unique. From the optimality conditions, it follows that there exists at each time instant a unique $z(t)$ such that $\nabla\mathcal{P}(\tilde{\lambda}(t)) = z(t)$ and $z(t) = B^T\zeta(t)$. Since, in addition, at each time instant $\mathbf{1}^T\zeta(t) = \mathbf{1}^T\zeta(0)$, the vector $\zeta(t)$ is uniquely defined and must be bounded. This proves that all trajectories remain bounded.

Remark 4: The controller (23) requires the explicit computation of the inverse gradient $\nabla\mathcal{P}_k^{-1}(\cdot)$ and the inverse Hessian $\nabla^2\mathcal{P}_k^{-1}(\cdot)$. In various relevant problems, these quantities can be determined analytically (an important example has been presented in [1]). However, for general cost functions, analytic expressions of these cost functions might be hard to find. In this case, one can employ a simple trick. Let k be the edge with the positive end i and negative end j and let ζ_i and ζ_j be two variables associated with the nodes i and j . Now, on the edge k , the solution to the unconstrained optimization problem

$$\hat{\lambda}_k := \arg \min \mathcal{P}_k(\lambda_k) - (\zeta_i - \zeta_j)\lambda_k \quad (34)$$

satisfies $\hat{\lambda}_k = \nabla\mathcal{P}_k^{-1}(\zeta_i - \zeta_j)$, which is in stacked form

$$\hat{\lambda} = \nabla\mathcal{P}^{-1}(B^T\zeta).$$

Furthermore, to compute the Hessian $\nabla^2\mathcal{P}_k^{-1}(\zeta_i - \zeta_j)$, one can simply compute the inverse of the scalar $\nabla^2\mathcal{P}_k(\hat{\lambda})$. Thus, instead of computing the inverse gradient analytically, one can at each time instant, take the current controller state $\zeta(t)$, and solve the unconstrained optimization problem (34). Note that (34) can be solved individually for each line, once the (dual) ζ variables are fixed. This leads to a *hierarchical* scheme, where a central unit, that is, the ζ -dynamics in (23), generates dual solutions, which are used to define a set of decoupled optimization problems on the lower level. This hierarchical scheme is well known in static optimization theory, where the cost of (34) is often called the “reduced cost.” The decomposition scheme suggests the interpretation of the dual variables as *prices*. Remarkably, the proposed feedback controller leads to a similar hierarchical structure as classical dual decomposition methods.

The storage function (26) at the initialization, that is, $U(0) := V(x(0) - x^w(0)) + W(\zeta(0), w(0))$ measures the initial imbalance of the inventories and the initial dual error of the optimal flow problem. In fact, this initialization error bounds the performance of the feedback controller.

Corollary 1 (Constraint Violation): Suppose there exists $C_2 > 0$ such that $(1/n) \int_0^t \Delta(\tau) d\tau \geq C_2 - x^{\text{avg}}(0)$. If the controller (23) initialized at $\zeta(0)$ and the initial inventory

levels $x(0)$ are such that $U(0) := V(x(0) - x^w(0)) + W(\zeta(0), w(0))$ satisfies $\sqrt{2U(0)} \leq C_2$ then all inventory levels remain positive, that is, $x_i(t) > 0$ for all $t \geq 0$. Furthermore, the flows on the lines do not exceed the capacity bounds by more than $\sqrt{8U(0)}$.

The proof is straightforward, and is omitted for brevity of the presentation.

B. Controller Design for Finite Cut Capacity

To derive Theorem 1, we assumed that the flow capacity of each cut is infinite (i.e., Assumption 5). This assumption is required to ensure in the proof of Theorem 1 the boundedness of the controller states (i.e., the existence of a bounded dual solution to (33)). It will be shown next how the controller (23) can be modified, such that Assumption 5 can be overcome, and the controller can handle flows that satisfy only the original Assumption 3. We propose here to modify the cost functions (and, in particular, their inverse gradients) to ensure the boundedness of the dual variables in the transient phase, as well as convergence to the optimal constrained flow.

In particular, we suggest modifying the inverse gradients of the cost functions as follows. Let $\delta > 0$ be a sufficiently large constant and define the function

$$\begin{aligned} \nabla\tilde{\mathcal{P}}_{k,\delta}^{-1}(z_k) &= \begin{cases} \nabla^2\mathcal{P}_k^{-1}(\delta)(z_k - \delta) + \nabla\mathcal{P}_k^{-1}(\delta) & \text{if } z_k > \delta \\ \nabla\mathcal{P}_k^{-1}(z_k) & \text{if } -\delta \leq z_k \leq \delta \\ \nabla^2\mathcal{P}_k^{-1}(-\delta)(z_k + \delta) + \nabla\mathcal{P}_k^{-1}(-\delta) & \text{if } z_k < -\delta. \end{cases} \end{aligned} \quad (35)$$

Note that (35) is a linear continuation of $\nabla\mathcal{P}_k^{-1}(z_k)$ outside the set $[-\delta, \delta]$. By construction, $\nabla\tilde{\mathcal{P}}_{k,\delta}^{-1}(z_k)$ is a continuously differentiable, strongly monotone function. Since (35) is strongly monotone, it is invertible, and the inverse is given by

$$\begin{aligned} \nabla\tilde{\mathcal{P}}_{k,\delta}(\lambda_k) &= \begin{cases} \nabla^2\mathcal{P}_k(\nu_k^+)(\lambda_k - \nu_k^+) + \nabla\mathcal{P}_k(\nu_k^+) & \text{if } \lambda_k > \nu_k^+ \\ \nabla\mathcal{P}_k(\lambda_k) & \text{if } \nu_k^- \leq \lambda_k \leq \nu_k^+ \\ \nabla^2\mathcal{P}_k(\nu_k^-)(\lambda_k - \nu_k^-) + \nabla\mathcal{P}_k(\nu_k^-) & \text{if } \lambda_k < \nu_k^- \end{cases} \end{aligned} \quad (36)$$

where $\nu_k^+ = \nabla\mathcal{P}_k^{-1}(\delta)$ and $\nu_k^- = \nabla\mathcal{P}_k^{-1}(-\delta)$. The integral function $\tilde{\mathcal{P}}_{k,\delta}(\lambda_k)$ of (36) can be understood as a *modified flow cost function* associated with (35). Note that while the original cost function has an effective domain corresponding to the interior of the line's capacity, the effective domain of the modified cost functions $\tilde{\mathcal{P}}_{k,\delta}(\lambda_k)$ is now $\mathbb{R}$. Furthermore, note that $\tilde{\mathcal{P}}_{k,\delta}(\lambda_k)$ are strongly convex and continuously differentiable functions. The modified optimal flow problem, parameterized by the constant δ , then becomes

$$\lambda_\delta^w = \arg \min \sum_{k=1}^m \tilde{\mathcal{P}}_{k,\delta}(\lambda_k) \quad 0 = B\lambda + QPw. \quad (37)$$

Using our standard notation, we will write in the following $\tilde{\mathcal{P}}_\delta(\lambda) = \sum_{k=1}^m \tilde{\mathcal{P}}_{k,\delta}(\lambda_k)$.

Proposition 3: Suppose Assumption 3 holds. Then there exists $\delta^* > 0$ such that for all $\delta \geq \delta^*$, $\lambda_\delta^w = \lambda^w$, where λ^w is defined in (9).

Proof: Due to Assumption 3, the optimal flow problem (9) has a finite cost and there exists a dual solution ζ^w with a finite norm such that

$$B\nabla\mathcal{P}^{-1}(B^T\zeta^w) = -QPw, \text{ and } \lambda^w = \nabla\mathcal{P}^{-1}(B^T\zeta^w).$$

On the other hand, the optimal solution to (37) satisfies

$$B\nabla\tilde{\mathcal{P}}_\delta^{-1}(B^T\zeta_\delta^w) = -QPw, \text{ and } \lambda_\delta^w = \nabla\tilde{\mathcal{P}}_\delta^{-1}(B^T\zeta_\delta^w).$$

Thus, we have

$$B\nabla\mathcal{P}^{-1}(B^T\zeta^w) = B\nabla\tilde{\mathcal{P}}_\delta^{-1}(B^T\zeta_\delta^w). \quad (38)$$

Choose now δ^* such that $-\delta^*\mathbf{1} < B^T\zeta^w < \delta^*\mathbf{1}$, then for any $\delta > \delta^*$, (38) implies immediately that $\nabla\mathcal{P}^{-1}(B^T\zeta^w) = \nabla\mathcal{P}^{-1}(B^T\zeta_\delta^w) = \nabla\tilde{\mathcal{P}}_\delta^{-1}(B^T\zeta_\delta^w)$. Thus, $\lambda^w = \lambda_\delta^w$. The immediate implication of Proposition 3 is as follows. One can replace the Legendre-type cost functions $\mathcal{P}_k(\cdot)$ used in (9) with strongly convex cost functions $\tilde{\mathcal{P}}_{k,\delta}(\cdot)$, while obtaining the same optimal solution. That is, for an appropriate choice of δ , the solution to (37) is such that all capacity constraints of the transportation lines are satisfied. This observation now enables the design of a feedback controller solving the time-varying optimal distribution problem, even if *all* lines have transportation constraints. Let in the following Ξ_δ be as in (22), but with $\nabla\mathcal{P}_k$ being replaced by $\nabla\tilde{\mathcal{P}}_{k,\delta}$.

Corollary 2 (Relaxed Feedback Controller): Consider the inventory dynamics (1) with the supply generated by the dynamical system (2). Suppose that Assumptions 1, 2, and 4 hold. Then, there exists $\delta^* > 0$ such that for all $\delta \geq \delta^*$, the dynamic controller

$$\begin{aligned} \dot{\zeta} &= -Q\Xi_\delta(\zeta)(B^T(BB^T)^\dagger(PS(w) + x + kPw) \\ &\quad + k\nabla\tilde{\mathcal{P}}_\delta^{-1}(B^T\zeta)) \\ \lambda &= \nabla\tilde{\mathcal{P}}_\delta^{-1}(B^T\zeta) - B^Tx \end{aligned} \quad (39)$$

with $k > 0$ a scalar controller gain, solves the time-varying optimal distribution problem for any initial inventory levels $x(0)$ and any controller initialization $\zeta(0)$, that is, all trajectories remain bounded, and 1) $\lim_{t \rightarrow \infty} B^Tx = 0$ and 2) $\lim_{t \rightarrow \infty} \text{dist}_\Gamma(\lambda(t), w(t)) = 0$.

Proof: The proof follows exactly along the lines of the proof of Theorem 1. However, in (33), the cost functions $\mathcal{P}_k(\cdot)$ are replaced by $\tilde{\mathcal{P}}_k(\cdot)$. This ensures that (33) has a bounded primal and a dual solution for all $c(t)$. These arguments prove convergence of the trajectories to the solution of (37). Now, in view of Proposition 1, there exists $\delta^* > 0$ such that for all $\delta \geq \delta^*$, the solution to (37) is the set Γ , as defined in (11).

V. DESIGN EXAMPLE

We illustrate in this section the proposed controller design on a small example network. In particular, we consider an inventory system with three inventories and three transportation lines as illustrated in Fig. 1.

After assigning an arbitrary orientation to the three connecting lines, one obtains the incidence matrix

$$B = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Fig. 1. Illustration of the inventory system.

Fig. 2. Legendre-type cost function $\mathcal{P}_k(\lambda_k)$ (solid) and the relaxed cost function $\tilde{\mathcal{P}}_{k,\delta}(\lambda_k)$ (dashed) with $\delta = 1 \times 10^4$.

We assume that the first line (connecting inventories 1 and 2) has an infinite capacity, while the second and third lines (connecting inventories 2 and 3 or 1 and 3, respectively) are constrained such that $|\lambda_k| < 3, k \in \{2, 3\}$. This is $\bar{\lambda}_k = 3$ for $k \in \{2, 3\}$. We assume that each line is assigned a quadratic cost function that is augmented with a *logarithmic barrier function* (see [24, Sec. 11.2.1]), i.e.,

$$\mathcal{P}_k(\lambda_k) = \frac{q_k}{2} \lambda_k^2 - \epsilon_k \log(-\lambda_k + \bar{\lambda}_k) - \epsilon_k \log(\lambda_k + \bar{\lambda}_k) \quad (40)$$

for some positive constants q_k and ϵ_k . If we further define $\mathcal{P}_k(\lambda_k) = \infty$ for $|\lambda_k| > \bar{\lambda}_k$, the cost function is of Legendre-type and satisfies Assumption 4. For the simulations we have chosen, $\epsilon_1 = 0$ (since the first line is unconstrained) and $\epsilon_2 = \epsilon_3 = 0.2$, and $q_1 = 1000, q_2 = 1$ and $q_3 = 1$. The cost functions are chosen such that it is most expensive to use the unconstrained line. For the implementation, we construct the relaxed cost functions $\tilde{\mathcal{P}}_{k,\delta}(\lambda_k)$ as constructed for (37), with $\delta = 2 \times 10^4$. The cost function is illustrated in Fig. 2. We do not compute the inverse of the gradient or the inverse of the Hessian analytically, but use the trick described in Remark 4, that is, we solve numerically the unconstrained optimization problem with the reduced cost (34) at each time instant (assuming this is done instantaneously) and use the optimal solution as the inverse gradient.

To construct the controller, the matrix $X(\zeta)$, as defined in (19), is required. For convenience, we will use in the following notation: $h_k = \nabla^2 \mathcal{P}_k^{-1}(z_k), k \in \{1, 2, 3\}$, where $z = B^T\zeta$. Following the procedure described in the Appendix, one can determine X as

$$X = \frac{1}{\bar{h}} \begin{bmatrix} 0 & h_1 h_3 + h_1 h_2 & h_1 h_2 \\ h_2 h_3 & 0 & h_2 h_3 + h_1 h_2 \\ h_2 h_3 + h_1 h_3 & h_1 h_3 & 0 \end{bmatrix}$$

where $\bar{h} = h_1 h_2 + h_2 h_3 + h_1 h_3$.

For the simulations, we assume that the supply and demand is generated by linear dynamical systems and is consequently

Fig. 3. Inventory levels (top left, x_1 solid, x_2 dashed, x_3 dotted) and flows on the three different lines, that is, λ_1 top right, λ_2 bottom left, and λ_3 bottom right. Note that asymptotically the inventory levels evolve synchronously and the capacity constraints (black dashed lines) are satisfied.

a harmonic signal. In particular, we consider the supply and demand at one node to be generated as

$$w_i(t) = \kappa_i \sin(\omega_i t + \phi_i).$$

For the simulations, we have chosen $\kappa_1 = 4$, $\kappa_2 = 5$, $\kappa_3 = 6$, $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 4$, and $\phi_1 = 0$, $\phi_2 = 3.1$, $\phi_3 = 0.3$.

The simulation results are shown in Fig. 3. It can be readily seen that the controller asymptotically ensures a synchronous behavior of all inventory levels (top left). The flows are asymptotically routed in such a way that they approach the optimal flows and satisfy, in particular, the capacity constraints (dashed lines in the two bottom plots). Note that the capacity constraints are slightly violated in the initial phase, but are all asymptotically satisfied exactly. It seems remarkable that the controller routed flow over the very expensive line 1 only if one of the two other lines is at its capacity limit.

VI. CONCLUSIONS

We proposed in this paper a feedback controller for optimal routing control in distribution networks with storage and time-varying supply. The controller is composed of a feedforward component, generating the time-varying optimal dual variables, and a feedback component, regulating the flows to the optimal trajectory. The construction of the feedforward component of the controller utilizes the analytic solution to a generalized Dirichlet problem. With certain choices of the cost functions, namely with Legendre-type cost functions, the proposed controller will also respect capacity constraints on the transportation lines. One can expect that the presented results can be generalized in various directions. A relevant extension will be a distributed implementation of the controller, achieving eventually only an approximate optimality of the steady-state

flows. Furthermore, future research will focus on a controller design that does not require a measurement of the external supply and demand, and the use of other flow-cost functions, such a quasiconvex functions. Beyond this, a relevant open question is a precise characterization of the relation between the choice of the relaxed cost functions and the possible violation of the capacity constraints by the controller.

APPENDIX

A. Analytic Solution to the Generalized Dirichlet Problem

We review here the analytic solution for generalized Dirichlet problems provided in [17]. Consider the problem of finding a vector z that solves an equation of the form

$$(BHB^T)z = Bv \quad (41)$$

where $B \in \mathbb{R}^{n \times m}$ is the incidence matrix of a graph $\mathcal{G} = (V, E)$ (with arbitrary orientation), H is a diagonal matrix with positive elements on the diagonal, and v is a given vector. It is known [17] that a solution to (41) is

$$z = X^T H^{-1} v. \quad (42)$$

The analytic construction of the matrix X is reported and justified in [17, p. 659] and recalled below. In what follows, we say that the node $i \in V$ is the positive end of edge $k \in E$ if $[B]_{i,k} = 1$ and the negative end if $[B]_{i,k} = -1$.

We call a *spanning tree* of the graph $\mathcal{G}$ an acyclic, connected subgraph $\mathcal{T} \subset \mathcal{G}$, defined on the same node set $\mathcal{T} = (V, T)$ having an edge set T being exactly of size $|T| = n - 1$.

Consider now any edge $l \in T$. After removing l from T , the spanning tree will partition into two connected components. The connected component that contains the positive end of l

will be denoted with $\mathcal{T}_l^+$. For the set of nodes forming the connected component $\mathcal{T}_l^+$, we will write $U(\mathcal{T}, l)$.

Each edge $k \in E$ has a weight h_k , which is the k th diagonal element of the matrix H . The weight of a spanning tree is now defined as $h(\mathcal{T}) = \prod_{k \in \mathcal{T}} h_k$. Furthermore, we define the sum of the weights of all spanning trees of $\mathcal{G}$ by $\bar{h} = \sum_{\mathcal{T}} h(\mathcal{T})$. We can now construct the matrix $X \in \mathbb{R}^{m \times n}$ in (42) according to the following procedure. For each edge $k \in E$ and each node $i \in V$, define the corresponding entry of X as

$$(X)_{ki} = \frac{1}{\bar{h}} \sum_{\{\mathcal{T}: i \in U(\mathcal{T}, k)\}} h(\mathcal{T})$$

that is, the sum of the weights of all spanning trees for which the node i is contained in the positive connected component.

The proposed method can naturally be used to solve (16). There, the weighting matrix is $H = D\mathcal{P}^{-1}(B^T \zeta^w(t))$. The time-varying nature of the weights has no influence on the solution. In fact, since the solution is analytically constructed, we will obtain expressions for X and H that depend explicitly on the state.

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