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Formation control using binary information[☆]

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ABSTRACT

In this paper, we study the problem of formation keeping of a network of strictly passive systems when very coarse information is exchanged. We assume that neighboring agents only know whether their relative position is larger or smaller than the prescribed one. This assumption results in very simple control laws that direct the agents closer or away from each other and take values in finite sets. We show that the task of formation keeping while tracking a desired velocity and rejecting matched disturbances is still achievable under the very coarse information scenario. In contrast with other results of practical convergence with coarse or quantized information, here the control task is achieved exactly.

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1. Introduction

Distributed motion coordination of mobile agents has attracted increasing attention in recent years owing to its wide range of applications from biology and social networks to sensor/robotic networks. Distributed formation keeping control is a specific case of motion coordination which aims at reaching a desired geometrical shape for the positions of the agents while tracking a desired velocity. This problem has been addressed with different approaches e.g. Arcak (2007), Bai, Arcak, and Wen (2011), Bullo, Cortés, and Martinez (2009), Mesbahi and Egerstedt (2010), Ren and Beard (2007) and Ren and Cao (2011). In problems of formation control, an important component, besides the dynamics of the agents and the graph topology, is the flow of information among the agents. In fact, the usual assumption in the literature on cooperative control is that a continuous flow of perfect information is exchanged among the agents. However, due to the coarseness of sensors and/or to communication constraints, the latter might be a restrictive requirement. To cope with the problem, the use of distributed

quantized feedback control has been proposed in the literature both for the discrete-time (Kashyap, Başar, & Srikant, 2007) and the continuous-time dynamic agents (Ceragioli, De Persis, & Frasca, 2011; Chen, Lewis, & Xie, 2011; Cortés, 2006; Dimarogonas & Johansson, 2010; Xargay, Choe, Hovakimyan, & Kaminer, 2012). In fact, in formation control by quantized feedback control, information is transmitted among the agents whenever measurements cross the thresholds of the quantizer. At these times, the corresponding quantized value taken from a discrete set is transmitted. This allows to deal naturally both with the continuous-time nature of the agents' dynamics and the discrete nature of the information transmission process without the need to rely on sampled-data models (Ceragioli et al., 2011; De Persis & Jayawardhana, 2012).

Main contribution. In this paper, we study the problem of distributed position-based formation keeping of a group of agents with strictly passive dynamics which exchange binary information. The binary information models a sensing scenario in which each agent detects whether or not the components of its current distance vector from a neighbor are above or below the prescribed distance and apply a force (in which each component takes a binary value) to reduce or respectively increase the actual distance. A similar coarse sensing scenario was considered in Yu, LaValle, and Liberzon (2012) in the context of the so-called “minimalist” robotics.

Remarkably, despite such a coarse information and control action, we show that the control law guarantees exact achievement of the desired formation. This is an interesting result, since statically quantized control inputs typically generate practical convergence, namely the achievement of an approximate formation in which the distance from the actual desired formation depends on

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the quantizer resolution (De Persis & Jayawardhana, 2012). Here the use of binary information allows us to conclude asymptotic convergence without the need to dynamically update the quantizer resolution.

Motivation. There are two main aspects that triggered our interest for the problem above.

- (i) The use of binary information in coordination problems (Chen et al., 2011; Cortés, 2006) has been proven useful to the design and real-time implementation of distributed controls for systems of first- or second-order agents in a cyber-physical environment (see e.g. De Persis & Frasca, 2013, De Persis, Frasca, & Hendrickx, 2013 and Nowzari & Cortés, 2012). We envision that a similar role will be played by the results in this paper for a larger class of coordination problems (see De Persis et al., 2013 for an early result in this respect).
- (ii) The resulting control laws are implemented by very simple directional commands (such as “move north”, “move north-east”, “stay still”, etc). We also show that the presence of coarse information does not affect the ability of the proposed controllers to achieve the formation in a leader-follower setting in which the desired reference velocity is only known to the leader.

This paper adopts a similar setting as in De Persis and Jayawardhana (2012) but controllers and analyses are different. Moreover, the paper investigates the formation control problem with unknown reference velocity tracking and matched disturbance rejection that was not considered in De Persis and Jayawardhana (2012). Compared with Yu et al. (2012), where also coarse information was used for rendezvous, the results in our contribution apply to a different class of systems and to a different cooperative control problem. Early results with the same sensing scenario but for a formation of double integrators have been presented in Jafarian and De Persis (2013).

The paper is organized as follows: Section 2 introduces the problem statement along with some motivations and notations. Analysis of the formation keeping problem with coarse data in the case of known/unknown reference velocity is studied in Section 3. Section 4 investigates the problem of formation keeping with coarse data in the presence of matched input disturbances. Related simulations are presented in Section 6. The paper is summarized in Section 7.

Notation. Given two sets S_1, S_2 , the symbol $S_1 \times S_2$ denotes the Cartesian product of two sets. This can be iterated. The symbol $\times_{k=1}^m S_k$ denotes $S_1 \times S_2 \times \dots \times S_m$. For a set S , $\text{card}(S)$ denotes the cardinality of the set S . Given a matrix M of real numbers, we denote by $\mathcal{R}(M)$ and $\mathcal{N}(M)$ the range and the null space, respectively. The symbols $\mathbf{1}, \mathbf{0}$ denote vectors or matrices of all 1 and 0 respectively. Sometimes the size of the matrix is explicitly given. Thus, $\mathbf{1}_N$ is the N -dimensional vector of all 1. I_p is the $p \times p$ identity matrix. Given two matrices A, B , the symbol $A \otimes B$ denotes the Kronecker product. $A = \text{block.diag}\{A_1, \dots, A_N\}$ denotes a diagonal matrix $A_{N \times N}$ such that A_i is its i th diagonal element.

2. Preliminaries

2.1. The multi-agent system

In this subsection, we review the passivity-based approach to coordination control (Arcak, 2007, see also Bai et al., 2011, Bürger, Zelazo, & Allgöwer, 2011, De Persis & Jayawardhana, 2012 and Munz, Papachristodoulou, & Allgöwer, 2011). A network of N agents in $\mathbb{R}^p$ are considered. For each agent i , $x_i \in \mathbb{R}^p$ represents its position. The communication topology of the network is assumed

Fig. 1. The dynamics of each agent is assumed to be composed of a strictly passive system and an integrator. The system $\mathcal{H}_i$ is strictly passive from an external input u_i to the velocity error y_i .

to be modeled by a connected and undirected graph $G = (V, E)$, where V is a set of N nodes and $E \subseteq V \times V$ is a set of M edges connecting the nodes. Label one end of each edge in E with a positive sign and the other end with a negative sign. We define the relative position z_k as follows

$$z_k = \begin{cases} x_i - x_j & \text{if node } i \text{ is the positive end of the edge } k \\ x_j - x_i & \text{if node } j \text{ is the positive end of the edge } k \end{cases}$$

where $x_i \in \mathbb{R}^p$ is the position of agent i expressed in an inertial frame. We define the $N \times M$ incidence matrix B associated with the graph G as follows

$$b_{ik} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of the edge } k \\ -1 & \text{if node } j \text{ is the positive end of the edge } k \\ 0 & \text{otherwise.} \end{cases}$$

By definition of B , we can represent the relative position variable z , with $z \triangleq [z_1^T \dots z_M^T]^T$, $z \in \mathbb{R}^{pM}$, as a function of the position variable x , namely

$$z = (B^T \otimes I_p)x, \quad (1)$$

which implies that z belongs to the range space $\mathcal{R}(B^T \otimes I_p)$.

We denote the network desired reference velocity by $v^*(t)$. This is the reference velocity to which the velocities of all the agents should converge. Moreover, each agent is expected to track its own desired (time-varying) reference velocity denoted by $v_i^r(t)$. If the desired network velocity is known to all of the agents, we have $v_i^r(t) = v^*(t)$. Otherwise, each agent will be equipped with an appropriate controller (as in Section 3.2) to recover $v^*(t)$, that is to guarantee that $v_i^r(t)$ converges to $v^*(t)$. This implies that by tracking $v_i^r(t)$, each agent eventually tracks $v^*(t)$.

We assume that the velocity of each agent is given by

$$\dot{x}_i = \mathcal{H}_i\{u_i\} + v_i^r, \quad (2)$$

where $\mathcal{H}_i\{u_i\}$ represents a system with dynamics (as in Fig. 1)

$$\mathcal{H}_i : \begin{cases} \dot{\xi}_i = f_i(\xi_i) + g_i(\xi_i)u_i \\ y_i = h_i(\xi_i) \end{cases} \quad (3)$$

where $\xi_i \in \mathbb{R}^{n_i}$ is the state variable, $u_i \in \mathbb{R}^p$ is the control input, $y_i \in \mathbb{R}^p$ is the velocity error, and the exogenous signal $v_i^r \in \mathbb{R}^p$ is the reference velocity for agent i . The maps f_i , g_i and h_i are assumed to be locally Lipschitz such that $f_i(\mathbf{0}) = \mathbf{0}$, $h_i(\mathbf{0}) = \mathbf{0}$, and $g_i(\mathbf{0})$ is full column-rank.

Notice that the output y_i of (3) is the velocity tracking error $\dot{x}_i - v_i^r$, namely

$$y_i = \dot{x}_i - v_i^r. \quad (4)$$

The system $\mathcal{H}_i$ is assumed to be strictly passive from the input u_i to the velocity error y_i . Since the system $\mathcal{H}_i$ is strictly passive, there is a continuously differentiable storage function $S_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ which is positive definite and radially unbounded and satisfies

$$\frac{\partial S_i}{\partial \xi_i} [f_i(\xi_i) + g_i(\xi_i)u_i] \leq -W_i(\xi_i) + y_i^T u_i \quad (5)$$

where W_i is a continuous positive function and $W_i(\mathbf{0}) = \mathbf{0}$.

For the sake of conciseness, Eqs. (3) and (4) are written in the compact form

$$\begin{aligned}\dot{x} &= \underbrace{\begin{pmatrix} h_1(\xi_1) \\ \vdots \\ h_N(\xi_N) \end{pmatrix}}_{h(\xi)} + \underbrace{\begin{pmatrix} v_1^r \\ \vdots \\ v_N^r \end{pmatrix}}_{v^r} \\ \dot{\xi} &= \underbrace{\begin{pmatrix} f_1(\xi_1) \\ \vdots \\ f_N(\xi_N) \end{pmatrix}}_{f(\xi)} + \underbrace{\begin{pmatrix} g_1(\xi_1) & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & g_N(\xi_N) \end{pmatrix}}_{g(\xi)} u \\ y &= h(\xi).\end{aligned}\quad (6)$$

We define the concatenated vectors $x \triangleq [x_1^T \dots x_N^T]^T$, $x \in \mathbb{R}^{pN}$, $x_i \in \mathbb{R}^p$, and $\xi \triangleq [\xi_1^T \dots \xi_N^T]^T$, $\xi \in \mathbb{R}^{pN}$, $\xi_i \in \mathbb{R}^p$. The desired position x^* , with $x^* \triangleq [x_1^{*T} \dots x_N^{*T}]^T$, $x^* \in \mathbb{R}^{pN}$, $x_i^* \in \mathbb{R}^p$ is given by $x^* = x_0^* + \mathbf{1}_N \otimes \int_0^t v^*(s)ds$, with x_0^* a constant vector. Analogously, $z^* \triangleq [z_1^{*T} \dots z_M^{*T}]^T$, $z^* \in \mathbb{R}^{pM}$, $z_i^* \in \mathbb{R}^p$, is the desired relative position vector. The two vectors are related by the identity $z^* = (B^T \otimes I_p)x^*$.

2.2. Problem statement

Our goal is to design coordination control laws to attain the following collective behaviors for the formation of dynamic agents (3), (4):

- (i) The velocity of each agent of the network asymptotically converges to the desired reference velocity $v^*(t) \in \mathbb{R}^p$

$$\lim_{t \rightarrow \infty} \|\dot{x}_i(t) - v^*(t)\| = \mathbf{0}, \quad i = 1, \dots, N. \quad (7)$$

- (ii) Each relative position vector z_k converges to the desired relative position vector z_k^*

$$\lim_{t \rightarrow \infty} \|z_k - z_k^*\| = \mathbf{0}, \quad z_k^* \in \mathbb{R}^p, \quad k = 1, \dots, M. \quad (8)$$

- (iii) In the presence of matched disturbances, i.e. in the case in which Eq. (3) is replaced by

$$\dot{\xi}_i = f_i(\xi_i) + g_i(\xi_i)(u_i + d_i)$$

where d_i is a disturbance signal generated by a suitable exosystem (see Section 4), the collective behaviors in (i) and (ii) are still guaranteed.

Differently from other work in formation control, we are interested in control laws that achieve complex coordination tasks using very coarse information about the relative positions of the agents. We assume that the agents of the network are equipped with sensors that are capable to detect whether the relative position of two agents is above or below a prescribed one. In line with our goals, for each agent i , the control law is designed as

$$u_i = - \sum_{k=1}^M b_{ik} \text{sign}(z_k - z_k^*), \quad i = 1 \dots N, \quad (9)$$

where $\text{sign}(\cdot)$ is the sign function and operates on each element of the p -dimensional vector $z_k - z_k^*$. The sign function $\text{sign}(\cdot) : \mathbb{R} \rightarrow \{-1, +1\}$ is defined as follows

$$\text{sign}(\zeta) = \begin{cases} +1 & \text{if } \zeta \geq 0 \\ -1 & \text{if } \zeta < 0. \end{cases}$$

Observe that, $b_{ik} \neq 0$ if and only if the edge k connects agent i to one of its neighbors. This implies the control law u_i uses only the information available to agent i . The above control law is inspired

by the passivity-based control design proposed in Arcak (2007).

Discussion about the control (9). The proposed control (9) has the following interpretation. Let the position of the agents be given with respect to an inertial frame in $\mathbb{R}^p$. Assign to each agent a local Cartesian coordinate system such that the current position of the agent is the origin of its local coordinate system. Therefore, there are p mutually perpendicular hyperplanes intersecting at the origin of the agent's local coordinate system and partitioning the state space into 2^p hyperoctants. For each neighbor of agent i , the corresponding p -dimensional vector $z_k - z_k^*$ lies in one of these hyperoctants (possibly at the boundary) and contributes the term $-b_{ik}\text{sign}(z_{ik} - z_{ik}^*)$ to the control u_i . The implementation of (9) only requires the agent to detect when $z_k - z_k^*$ is crossing the boundary between hyperoctants.¹ A few other advantages of this sensing scenario are discussed below:

- (i) The formation is achieved with large inaccuracies in the measurements of $z_k - z_k^*$. In fact, the calculated control vector corresponding to a given vector $-b_{ik}(z_k - z_k^*)$ is the same no matter where $z_k - z_k^*$ precisely lies in the hyperoctant. That is as far as the sign of $z_k - z_k^*$ is unchanged.
- (ii) If agent i detects that the vector $z_k - z_k^*$ is crossing the boundary of any two hyperoctants, then a new control vector is applied due to the term $-b_{ik}\text{sign}(z_k - z_k^*)$ in (9). In our analysis below, however, we show that the exact formation is still achieved if this control vector at the boundary takes any value in the convex hull of the two control vectors associated with the two hyperoctants. More precisely, if $z_k - z_k^* = \mathbf{0}$, then we allow for $-b_{ik}\text{sign}(z_k - z_k^*)$ to take on any value in $[-1, 1]^p$ (see (15) below). In this respect the control (9) is robust to possible uncertainties in the precise detection of the boundary crossing.
- (iii) The proposed control law guarantees the achievement of a desired formation by very simple directional commands, namely move closer if the actual position is larger than the desired one, move away if smaller.
- (iv) The input (9) provides a *finite*-valued control that uses *binary* measurements. However, this control action experiences a fast switching behavior while approaching the control target. This raises the issue on how to deal with chattering in practice. We discuss two possible methods.

The first one, studied in Ceragioli et al. (2011, Section 4), prescribes the introduction of hysteresis in the quantizers. The resulting system is hybrid and the existence of an inter-switching time can be proven. On the other hand, hysteric quantizers lead to results of practical coordination. From a mathematical point of view, one has to employ techniques from hybrid control systems theory rather than nonsmooth control tools.

The second method relies on event-based control techniques. In the case of networks of kinematic agents, the finite-valued control law of Cortés (2006) inspired the design of self-triggered control algorithms (see De Persis & Frasca, 2013). The underlying idea is the following. If it is possible to achieve coordination via binary information, then the controller only needs to keep track of the times when the measured information crosses zero. If one can predict when these time instants occur, and make sure that they do not accumulate in finite time, then the control will be updated only at these instants, thus preventing the occurrence of the fast switching behavior.

3. Analysis

In this section we investigate the stability properties of the closed-loop system. We define the error variable $\tilde{z} = z - z^*$ and

¹ In that regard, the control law is reminiscent of the controllers of Yu et al. (2012) that changes when a neighbor enters or leaves the field-of-view sector of the agent.

write the control laws u_i , $i = 1, 2, \dots, N$, in the following compact form

$$u = -(B \otimes I_p) \text{sign } \tilde{z}. \quad (10)$$

From Eq. (1), we can write $\tilde{z} = (B^T \otimes I_p)(x - x^*)$. We consider the stability of the origin of the error system, with state variable $(\tilde{z}^T, \xi^T)^T$. To derive the error system equations, we take the derivative of $\tilde{z}$, thus obtaining

$$\dot{\tilde{z}} = (B^T \otimes I_p)(\dot{x} - \dot{x}^*). \quad (11)$$

From Eq. (6), we further have

$$\dot{x} - \dot{x}^* = h(\xi) + v^r - \mathbf{1}_N \otimes v^* \quad (12)$$

where v^* is the desired reference velocity for the formation. By a property of the incidence matrix of a connected graph, $(B^T \otimes I_p)(\mathbf{1}_N \otimes v^*) = \mathbf{0}$. Thus, we represent the dynamics of the error system in the following general form

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)\dot{x} = (B^T \otimes I_p)(h(\xi) + v^r) \\ \dot{\xi} &= f(\xi) + g(\xi)(-B \otimes I_p) \text{sign } \tilde{z}. \end{aligned} \quad (13)$$

The system (13) has a discontinuous right-hand side due to the discontinuity of the sign function at zero. Before analyzing the system, we first define an appropriate notion of the solution. In this paper, the solutions to the system above are intended in the Krasowskii sense. As in Ceragioli et al. (2011) and De Persis and Jayawardhana (2012), the motivation to consider these solutions lies in the fact that there exist convenient Lyapunov methods to analyze their asymptotic behavior and that they include other notions of solutions such as Carathéodory solutions if the latter exists. Let $X = (\tilde{z}, \xi)$ and let $F(X)$ be the set-valued map

$$\begin{aligned} F(X) &= \left(\begin{array}{c} (B^T \otimes I_p)(h(\xi) + v^r) \\ f(\xi) \end{array} \right) \\ &\quad - \left(\begin{array}{c} \mathbf{0} \\ g(\xi)(B \otimes I_p) \end{array} \right) \mathcal{K} \text{sign } \tilde{z} \end{aligned} \quad (14)$$

where $\mathcal{K} \text{sign } \tilde{z} = \times_{k=1}^M \times_{\ell=1}^p \mathcal{K} \text{sign } \tilde{z}_{k\ell}$ and

$$\mathcal{K} \text{sign } \tilde{z}_{k\ell} = \begin{cases} \{\text{sign } \tilde{z}_{k\ell}\} & \text{if } \tilde{z}_{k\ell} \neq 0 \\ [-1, 1] & \text{if } \tilde{z}_{k\ell} = 0. \end{cases} \quad (15)$$

We define $X(t) = (\tilde{z}(t), \xi(t))$ a Krasowskii solution to (13) on the interval $[0, t_1]$ if it is an absolutely continuous function which satisfies the differential inclusion

$$\dot{X}(t) \in F(X(t)) \quad (16)$$

for almost every $t \in [0, t_1]$, with F defined as in (14). Notice that the relation between the differential inclusion (16) and the discontinuous ordinary differential equations (13) stems from the property

$$\mathcal{K}(f(X)) = \bigcap_{\delta > 0} \overline{\text{co}}(f(B(x, \delta))) \subseteq F(X), \quad (17)$$

with $f(X)$ the map defining the right-hand side of (13), $\overline{\text{co}}$ the closed convex hull of a set and $B(x, \delta)$ the ball centered at x and of radius δ . Local existence of Krasowskii solutions to the differential inclusion above is always guaranteed (see e.g. Ceragioli et al., 2011, Lemma 3 and Hájek, 1979).

3.1. Known reference velocity

In this section, we consider the case where the desired reference velocity, $v^*(t)$, is known to all the agents, namely $v^r = \mathbf{1}_N \otimes v^*$.

Proposition 1. Any Krasowskii solution to the closed-loop system

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)h(\xi) \\ \dot{\xi} &= f(\xi) + g(\xi)(-B \otimes I_p) \text{sign } \tilde{z} \end{aligned} \quad (18)$$

exists for all $t \geq 0$ and converges to the point $\tilde{z} = \mathbf{0}$, $\xi = \mathbf{0}$.

Proof. Since $v^r = \mathbf{1}_N \otimes v^*$, the closed-loop system in (13) simplifies to (18). Local existence of Krasowskii solutions is guaranteed by the fact that the right hand side of (18) is measurable and locally bounded (i.e., bounded on each bounded subset of its domain Hájek, 1979). Moreover, completeness of solutions can be deduced by their boundedness (Nemytskii & Stepanov, 1960). Boundedness of solutions to system (18) is proven below (see De Persis & Jayawardhana, 2012, Lemma 3.1, Theorem 3.2 for more details), thus showing completeness.

The convergence proof is based on the application of the nonsmooth La Salle's invariance principle (Bacciotti & Ceragioli, 1999). Consider the following locally Lipschitz Lyapunov function

$$V(\tilde{z}, \xi) = \|\tilde{z}\|_1 + S(\xi)$$

where $\|\tilde{z}\|_1$ denotes the 1-norm $\|\tilde{z}\|_1 = \sum_{k,\ell} |\tilde{z}_{k\ell}|$ and evaluate its set-valued derivative $\dot{V}(\tilde{z}, \xi)$ along (18). We have

$$\begin{aligned} \dot{V}(\tilde{z}, \xi) &= \{a \in \mathbb{R} : \exists w \in F(\tilde{z}, \xi) \text{ s.t.} \\ &\quad a = \langle w, p \rangle, \text{ for all } p \in \partial V(\tilde{z}, \xi)\}. \end{aligned}$$

By the definition of $F(\tilde{z}, \xi)$ in (14), for any $w \in F(\tilde{z}, \xi)$ there exists $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$ such that

$$w = \begin{pmatrix} (B^T \otimes I_p)h(\xi) \\ f(\xi) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \end{pmatrix} w^{\tilde{z}}.$$

Observe that the Clarke generalized gradient $\partial V(\tilde{z}, \xi)$ is given by

$$\begin{aligned} \partial V(\tilde{z}, \xi) &= \left\{ p : p = \begin{pmatrix} p^{\tilde{z}} \\ \nabla S(\xi) \end{pmatrix} \text{ s.t.} \right. \\ &\quad \left. p^{\tilde{z}}_{k\ell} \in \begin{cases} \{\text{sign } \tilde{z}_{k\ell}\} & \text{if } \tilde{z}_{k\ell} \neq 0 \\ [-1, +1] & \text{if } \tilde{z}_{k\ell} = 0 \end{cases} \right\}. \end{aligned}$$

Suppose that $\dot{V}(\tilde{z}, \xi) \neq \emptyset$ and take $a \in \dot{V}(\tilde{z}, \xi)$. Then by definition there exists $w \in F(\tilde{z}, \xi)$ such that $a = \langle w, p \rangle$ for all $p \in \partial V(\tilde{z}, \xi)$. Choose $p \in \partial V(\tilde{z}, \xi)$ such that $p^{\tilde{z}} = w^{\tilde{z}}$. Thus,

$$\begin{aligned} a &= \left\langle \begin{pmatrix} (B^T \otimes I_p)h(\xi) \\ f(\xi) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \end{pmatrix} w^{\tilde{z}}, \begin{pmatrix} w^{\tilde{z}} \\ \nabla S(\xi) \end{pmatrix} \right\rangle \\ &= \langle (B^T \otimes I_p)h(\xi), w^{\tilde{z}} \rangle + \langle f(\xi) - g(\xi)(B \otimes I_p)w^{\tilde{z}}, \nabla S(\xi) \rangle \\ &\leq -W(\xi). \end{aligned}$$

Hence, for any state $(\tilde{z}, \xi)$ such that $\dot{V}(\tilde{z}, \xi) \neq \emptyset$, $\dot{V}(\tilde{z}, \xi) = \{a \in \mathbb{R} : a \leq -W(\xi)\} \subseteq (-\infty, 0]$. Therefore, the solutions can be extended for all $t \geq 0$ and by La Salle's invariance principle they converge to the largest weakly invariant set where $\xi = \mathbf{0}$. Since $v^r = \mathbf{1}_N \otimes v^*$, from (14), any point $(\tilde{z}, \mathbf{0})$ on this invariant set must necessarily satisfy

$$\mathbf{0} = \begin{pmatrix} \mathbf{0} \\ g(\mathbf{0})(B \otimes I_p) \end{pmatrix} w^{\tilde{z}}$$

for some $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$. Since $g(\mathbf{0})$ is full-column rank, this implies $w^{\tilde{z}} \in \mathcal{N}(B \otimes I_p)$. Bearing in mind that $\tilde{z} \in \mathcal{R}(B^T \otimes I_p)$, then $\langle \tilde{z}, w^{\tilde{z}} \rangle = 0$. We claim that necessarily $\tilde{z} = \mathbf{0}$. In fact, by contradiction, $\tilde{z} \neq \mathbf{0}$ would imply the existence of at least a component $\tilde{z}_{k\ell} \neq 0$. As $\tilde{z}_{k\ell} \neq 0$ necessarily implies $w^{\tilde{z}}_{k\ell} \neq 0$ by definition of $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$, then it would imply $\tilde{z}_{k\ell} w^{\tilde{z}}_{k\ell} > 0$, which shows the contradiction. Hence, on the invariant set, $\tilde{z} = \mathbf{0}$ and all the system's solutions converge to the origin. $\square$

3.2. Unknown reference velocity

In the previous section, we assumed that the desired reference velocity is known to all of the agents. This assumption is very restrictive and one wonders whether it is possible to relax this assumption despite the very coarse information scenario we are confining ourselves. The positive answer to this question is given in the analysis below.

The problem of the reference velocity unknown to the agents is tackled here in the scenario in which at least one of the agents is aware of v^* . We refer to this agent as the leader (Mesbahi & Egerstedt, 2010; Ren & Beard, 2007). As in Bai et al. (2011), we further assume that the class of reference velocities that the formation can achieve are those generated by an autonomous system that we refer to as the exosystem. Given two matrices Φ , Γ^v , whose properties will be made precise later on, the exosystem obeys the following equations

$$\dot{w}^v = \Phi w^v, \quad v^* = \Gamma^v w^v. \quad (19)$$

Taking inspiration from the theory of output regulation (see e.g. Isidori, Marconi, & Serrani, 2003), an internal-model-based controller can be adopted for each agent $i = 2, 3, \dots, N$, namely

$$\begin{aligned} \dot{\eta}_i &= \Phi \eta_i + G \tilde{u}_i \\ v_i^r &= \Gamma^v \eta_i, \quad i = 2, 3, \dots, N. \end{aligned} \quad (20)$$

When $\tilde{u}_i = \mathbf{0}$ and the system is appropriately initialized, the latter system is able to generate any w^v solution to (19). Because v^* is known to agent 1, there is no need to implement system (20) at agent 1. The input $\tilde{u}_i$, as well as the input matrix G are to be designed later. Define $v^r = (v^{*T} \ v_2^{rT} \ \dots \ v_N^{rT})$ and the new error variables

$$\tilde{\eta}_i = \eta_i - w^v, \quad \tilde{v}_i = v_i^r - v^*, \quad i = 2, \dots, N,$$

along with the fictitious variables $\tilde{\eta}_1 = \mathbf{0}$, $\tilde{v}_1 = \mathbf{0}$. In compact form, we obtain

$$\begin{aligned} \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta} + \tilde{G} \tilde{u} \\ \dot{\tilde{v}} &= \tilde{\Gamma}^v \tilde{\eta}, \end{aligned} \quad (21)$$

where $\tilde{\Phi}$, $\tilde{G}$, and $\tilde{\Gamma}^v$ are $\tilde{\Phi} = \text{block.diag}\{\mathbf{0}, \Phi, \dots, \Phi\}$, $\tilde{G} = \text{block.diag}\{\mathbf{0}, G, \dots, G\}$, $\tilde{\Gamma}^v = \text{block.diag}\{\mathbf{0}, \Gamma^v, \dots, \Gamma^v\}$. From (13), the error position vector can be written as

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)(h(\xi) + v^r) \\ &= (B^T \otimes I_p)(h(\xi) + v^r - \mathbf{1}_N \otimes v^* + \mathbf{1}_N \otimes v^*) \\ &= (B^T \otimes I_p)(h(\xi) + \tilde{v}). \end{aligned}$$

The overall closed-loop system is

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ \dot{\xi} &= f(\xi) + g(\xi)u \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta} + \tilde{G} \tilde{u}. \end{aligned} \quad (22)$$

Proposition 2. Assume that Φ is skew-symmetric, namely

$$\Phi^T + \Phi = \mathbf{0}, \quad (23)$$

(Γ^v, Φ) is an observable pair and let

$$G = \Gamma^{vT}. \quad (24)$$

Then all the Krasowskii solutions to (22) in closed-loop with the control input

$$u = \tilde{u} = -(B \otimes I_p) \text{sign } \tilde{z} \quad (25)$$

converge to the point $\tilde{z} = \mathbf{0}$, $\xi = \mathbf{0}$, $\tilde{\eta} = \mathbf{0}$.

Proof. Any Krasowskii solution to (22) with the control law defined by (25) satisfies a differential inclusion of the form (14) where

$X = (\tilde{z}, \xi, \tilde{\eta})$ and $F(X)$ is equal to

$$\begin{pmatrix} (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) \\ \tilde{\Phi} \tilde{\eta} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \\ \tilde{G}(B \otimes I_p) \end{pmatrix} \mathcal{K} \text{sign } \tilde{z}.$$

Let

$$V(\tilde{z}, \tilde{v}, \tilde{\eta}) = \|\tilde{z}\|_1 + S(\xi) + \frac{1}{2} \tilde{\eta}^T \tilde{\eta}.$$

For any $w \in F(\tilde{z}, \xi, \tilde{\eta})$, there exists $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$ such that

$$w = \begin{pmatrix} (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) \\ \tilde{\Phi} \tilde{\eta} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \\ \tilde{G}(B \otimes I_p) \end{pmatrix} w^{\tilde{z}}.$$

Moreover for any $p \in \partial V(\tilde{z}, \xi, \tilde{\eta})$ there exists $p^{\tilde{z}} \in \partial \|\tilde{z}\|_1$, with

$$p_{k\ell}^{\tilde{z}} \in \begin{cases} \{\text{sign } \tilde{z}_{k\ell}\} & \text{if } \tilde{z}_{k\ell} \neq 0 \\ [-1, +1] & \text{if } \tilde{z}_{k\ell} = 0, \end{cases}$$

and $k = 1, 2, \dots, M$, $\ell = 1, 2, \dots, p$, such that

$$p = \begin{pmatrix} p^{\tilde{z}} \\ \nabla S(\xi) \\ \tilde{\eta} \end{pmatrix}.$$

For $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$, let $p \in \partial V(\tilde{z}, \xi, \tilde{\eta})$ such that $w^{\tilde{z}} = p^{\tilde{z}}$. Hence,

$$\begin{aligned} \langle w, p \rangle &= \langle (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}), w^{\tilde{z}} \rangle \\ &\quad + \langle f(\xi) - g(\xi)(B \otimes I_p)w^{\tilde{z}}, \nabla S(\xi) \rangle \\ &\quad + \langle \tilde{\Phi} \tilde{\eta} - \tilde{G}(B \otimes I_p)w^{\tilde{z}}, \tilde{\eta} \rangle. \end{aligned}$$

By (23) and (24), the equality above simplifies as

$$\begin{aligned} \langle w, p \rangle &= \langle (B^T \otimes I_p)h(\xi), w^{\tilde{z}} \rangle \\ &\quad + \langle f(\xi) - g(\xi)(B \otimes I_p)w^{\tilde{z}}, \nabla S(\xi) \rangle. \end{aligned}$$

Since the dynamic of each agent is strictly passive, we have

$$\langle f(\xi) - g(\xi)(B \otimes I_p)w^{\tilde{z}}, \nabla S(\xi) \rangle \leq -W(\xi) - \langle h(\xi), (B \otimes I_p)w^{\tilde{z}} \rangle$$

which implies $\langle w, p \rangle \leq -W(\xi)$.

Hence, similarly to the proof of Proposition 1, $\dot{\tilde{V}}(\tilde{z}, \xi, \tilde{\eta}) = \{a \in \mathbb{R} : a \leq -W(\xi)\} \subseteq (-\infty, 0]$, provided that $\tilde{V}(\tilde{z}, \xi, \tilde{\eta}) \neq \emptyset$. Because V is positive definite and proper, solutions are bounded and one can apply La Salle's invariance principle which shows that every Krasowskii solution converges to the largest weakly invariant set where $W(\xi) = \mathbf{0}$. By definition of $W(\xi)$, the latter is equivalent to have convergence to the largest weakly invariant set where $\xi = \mathbf{0}$. To characterize this set, we look for points $(\tilde{z}, \mathbf{0}, \tilde{\eta})$ such that $(\dot{\tilde{z}}, \mathbf{0}, \dot{\tilde{\eta}}) \in F(\tilde{z}, \mathbf{0}, \tilde{\eta})$. Having $f(\mathbf{0}) = h(\mathbf{0}) = \mathbf{0}$, this means that we seek points $(\tilde{z}, \mathbf{0}, \tilde{\eta})$ for which there exists $w^{\tilde{z}} \in \mathcal{K} \text{sign } \tilde{z}$ such that

$$\begin{pmatrix} \dot{\tilde{z}} \\ \mathbf{0} \\ \dot{\tilde{\eta}} \end{pmatrix} = \begin{pmatrix} (B^T \otimes I_p)(\tilde{\Gamma}^v \tilde{\eta}) \\ \mathbf{0} \\ \tilde{\Phi} \tilde{\eta} \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\mathbf{0})(B \otimes I_p) \\ (\tilde{\Gamma}^v)^T (B \otimes I_p) \end{pmatrix} w^{\tilde{z}}.$$

The equality $\mathbf{0} = -g(\mathbf{0})(B \otimes I_p)w^{\tilde{z}}$, implies that necessarily $\tilde{z} = \mathbf{0}$ (see the final part of the proof of Proposition 1). This also implies that on the invariant set the system evolves as

$$\begin{aligned} \mathbf{0} &= (B^T \otimes I_p)(\tilde{\Gamma}^v \tilde{\eta}) \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta}. \end{aligned}$$

From this point on, the argument is the same as in Bai et al. (2011). Indeed, as $\mathcal{N}(B^T) = \mathcal{R}(\mathbf{1}_N)$, by the block diagonal structure of $\tilde{\Gamma}^v$, we obtain that all the sub vectors of $\tilde{\Gamma}^v \tilde{\eta}$ must be the same and equal to zero, since the first p components of $\tilde{\Gamma}^v \tilde{\eta}$ are identically

zero. In other words, $\Gamma^v \tilde{\eta}_i = \mathbf{0}$ for all $i = 2, 3, \dots, N$. Hence we conclude that on the largest weakly invariant set where $\xi = \mathbf{0}$, we have

$$\mathbf{0} = \Gamma^v \tilde{\eta}_i$$

$$\dot{\tilde{\eta}}_i = \Phi \tilde{\eta}_i, \quad i = 2, 3, \dots, N.$$

By the observability of (Γ^v, Φ) , it holds that $\tilde{\eta}_i = \mathbf{0}$ for all $i = 2, 3, \dots, N$. This finally proves the claim. $\square$

4. Formation control with matched disturbance rejection

In this section, we study the problem of formation keeping with very coarse exchanged information in the presence of matched input disturbances. We consider a formation of strictly passive agents, where the dynamics of each agent of the network is

$$\dot{x}_i = y_i + v_i^r(t)$$

$$\mathcal{H}_i : \begin{cases} \dot{\xi}_i = f_i(\xi_i) + g_i(\xi_i)(u_i + d_i) \\ y_i = h_i(\xi_i), \end{cases} \quad (26)$$

where d_i is a disturbance. The network should converge to the prescribed formation and evolve with the given desired reference velocity v^* despite the action of the disturbance d . As in the previous section, we first consider the case in which the reference velocity is unknown to all the agents except one. As explained in Section 3.2, we adopt an internal-model-based controller to recover the desired reference velocity. Similarly we suppose that the disturbance signal d_i at agent i is generated by an exosystem of the form

$$\dot{w}_i^d = \Phi_i^d w_i^d, \quad d_i = \Gamma_i^d w_i^d, \quad i = 1, 2, \dots, N. \quad (27)$$

To counteract the effect of the disturbance, we introduce an additional internal-model-based controller given by

$$\dot{\theta}_i = \Phi_i^d \theta_i + G_i^d \tilde{u}_i$$

$$\hat{d}_i = \Gamma_i^d \theta_i, \quad i = 1, 2, \dots, N, \quad (28)$$

where $G_i^d, \tilde{u}_i$ will be designed later. We remark that the internal model for disturbance rejection is implemented also by agent 1 (the leader).

To both compensate the disturbance and achieve the desired formation, the proposed control law is composed of two parts: $\hat{u}$ guides the system to the desired formation, and $\hat{d}$ compensates the disturbance. Therefore, we can write the dynamics of the agents in the following compact form

$$u = \hat{u} - \hat{d}$$

$$\dot{\xi} = f(\xi) + g(\xi)(\hat{u} - \hat{d} + d). \quad (29)$$

Define the new variable $\tilde{d}_i = \hat{d}_i - d_i$ and $\tilde{\theta}_i = \theta_i - w_i^d$, then

$$\dot{\tilde{\theta}}_i = \Phi_i^d \tilde{\theta}_i + G_i^d \tilde{u}_i$$

$$\tilde{d}_i = \Gamma_i^d \tilde{\theta}_i. \quad (30)$$

In compact form we write

$$\dot{\tilde{\theta}} = \Phi^d \tilde{\theta} + G^d \tilde{u}$$

$$\tilde{d} = \Gamma^d \tilde{\theta}, \quad (31)$$

where the matrices Φ^d, G^d , and Γ^d are diagonal matrices whose diagonal elements are Φ_i^d, G_i^d , and Γ_i^d respectively.

From Eq. (22) together with Eqs. (29) and (31), the overall closed-loop system is

$$\dot{\tilde{z}} = (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta})$$

$$\dot{\xi} = f(\xi) + g(\xi)(\hat{u} - \Gamma^d \tilde{\theta})$$

$$\dot{\tilde{\eta}} = \tilde{\Phi} \tilde{\eta} + \tilde{G} \tilde{u}$$

$$\dot{\tilde{\theta}} = \Phi^d \tilde{\theta} + G^d \tilde{u}. \quad (32)$$

The following is proven:

Proposition 3. Assume that Φ and Φ_i^d , for $i = 1, 2, \dots, N$ are skew-symmetric matrices. If

$$G = (\Gamma^v)^T$$

$$G_i^d = (\Gamma_i^d)^T, \quad (33)$$

then all the Krasowskii solutions to (32) in closed-loop with the control input

$$\hat{u} = \tilde{u} = -(B \otimes I_p) \text{sign} \tilde{z}, \quad \tilde{u} = h(\xi) \quad (34)$$

are bounded and converge to the largest weakly invariant set for

$$\begin{pmatrix} \dot{\tilde{z}} \\ \mathbf{0} \\ \dot{\tilde{\eta}} \\ \dot{\tilde{\theta}} \end{pmatrix} \in \begin{pmatrix} (B^T \otimes I_p) \tilde{\Gamma}^v \tilde{\eta} \\ -(B \otimes I_p) \mathcal{K} \text{sign} \tilde{z} - \Gamma^d \tilde{\theta} \\ \tilde{\Phi} \tilde{\eta} - (\tilde{\Gamma}^v)^T (B \otimes I_p) \mathcal{K} \text{sign} \tilde{z} \\ \Phi^d \tilde{\theta} \end{pmatrix}, \quad (35)$$

such that $\xi = \mathbf{0}$.

Proof. Consider the Lyapunov function

$$V(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}) = \|\tilde{z}\|_1 + S(\xi) + \frac{1}{2} \tilde{\eta}^T \tilde{\eta} + \frac{1}{2} \tilde{\theta}^T \tilde{\theta}.$$

Any Krasowskii solution to (32) with the control law defined by (34) satisfies a differential inclusion of the form (16) where $X = (\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$ and $F(X)$ is

$$\begin{pmatrix} (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) - g(\xi) \Gamma^d \tilde{\theta} \\ \tilde{\Phi} \tilde{\eta} \\ \Phi^d \tilde{\theta} + G^d h(\xi) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \\ \tilde{G}(B \otimes I_p) \\ \mathbf{0} \end{pmatrix} \mathcal{K} \text{sign} \tilde{z}.$$

For any $w \in F(X)$, there exists $w^{\tilde{z}} \in \mathcal{K} \text{sign} \tilde{z}$ such that

$$w = \begin{pmatrix} (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) - g(\xi) \Gamma^d \tilde{\theta} \\ \tilde{\Phi} \tilde{\eta} \\ \Phi^d \tilde{\theta} + G^d h(\xi) \end{pmatrix} - \begin{pmatrix} \mathbf{0} \\ g(\xi)(B \otimes I_p) \\ \tilde{G}(B \otimes I_p) \\ \mathbf{0} \end{pmatrix} w^{\tilde{z}}.$$

Moreover for any $p \in \partial V(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$ there exists $p^{\tilde{z}} \in \partial \|\tilde{z}\|_1$, with

$$p_{k\ell}^{\tilde{z}} \in \begin{cases} \{\text{sign} \tilde{z}_{k\ell}\} & \text{if } \tilde{z}_{k\ell} \neq 0 \\ [-1, +1] & \text{if } \tilde{z}_{k\ell} = 0, \end{cases}$$

and $k = 1, 2, \dots, M, \ell = 1, 2, \dots, p$, such that

$$p = \begin{pmatrix} p^{\tilde{z}} \\ \nabla S(\xi) \\ \tilde{\eta} \\ \tilde{\theta} \end{pmatrix}.$$

For $w^{\tilde{z}} \in \mathcal{K} \text{sign} \tilde{z}$, let $p \in \partial V(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$ be such that $w^{\tilde{z}} = p^{\tilde{z}}$. Hence,

$$\langle w, p \rangle = \langle (B^T \otimes I_p)(h(\xi) + \tilde{\Gamma}^v \tilde{\eta}), w^{\tilde{z}} \rangle$$

$$+ \langle f(\xi) - g(\xi)((B \otimes I_p)w^{\tilde{z}} + \Gamma^d \tilde{\theta}), \nabla S(\xi) \rangle$$

$$+ \langle \tilde{\Phi} \tilde{\eta} - \tilde{G}(B \otimes I_p)w^{\tilde{z}}, \tilde{\eta} \rangle + \langle \Phi^d \tilde{\theta} + G^d h(\xi), \tilde{\theta} \rangle.$$

By condition (33), the above result simplifies as

$$\langle w, p \rangle = \langle (B^T \otimes I_p)h(\xi), w^{\tilde{z}} \rangle$$

$$+ \langle f(\xi) - g(\xi)((B \otimes I_p)w^{\tilde{z}} + \Gamma^d \tilde{\theta}), \nabla S(\xi) \rangle$$

$$+ \langle G^d h(\xi), \tilde{\theta} \rangle. \quad (36)$$

Since the dynamics of each agent is strictly passive, we have

$$\begin{aligned} & \langle f(\xi) - g(\xi)((B \otimes I_p)w^z + \Gamma^d \tilde{\theta}), \nabla S(\xi) \rangle \\ & \leq -W(\xi) - \langle h(\xi), (B \otimes I_p)w^z + \Gamma^d \tilde{\theta} \rangle. \end{aligned}$$

When it is replaced in (36), it gives $\langle w, p \rangle \leq -W(\xi)$, in view of $\Gamma^d = (G^d)^T$. As in the previous proofs, this shows $\tilde{V}(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}) = \{a \in \mathbb{R} : a \leq -W(\xi)\}$ and hence boundedness of the solutions and their convergence to the largest weakly invariant set for the closed-loop system such that $\xi = \mathbf{0}$. This proves the thesis. $\square$

Remark 1. The disturbance rejection controller for each agent i only uses the local information $y_i = h_i(\xi_i)$, despite the fact that the disturbance affects the input through which agent i coordinates with the other agents in the network. Note that the input of each agent is only influenced by a function of its own velocity $g_i(\xi_i)$ (see Eq. (6)).

Proposition 3 only proves boundedness of the solution and convergence of the velocity of each agent to its own reference velocity ($\xi = \mathbf{0}$). To prove the convergence of the network to the desired formation and evolution of the network with the desired velocity, one should additionally prove that $\tilde{z}$ converges to the origin. This result does not hold in general but there are a few cases of interest where this is true. In what follows we study two of these cases.

Case I: $\Phi = \mathbf{0}$, $\Phi_i^d = \mathbf{0}_{p \times p}$, $i = 1, 2, \dots, N$, and Γ^v , Γ_i^d , $i = 1, 2, \dots, N$, are nonsingular.

This case corresponds to the scenario in which the unknown reference velocity and the disturbances are constant signals. The arguments below show that the distributed control laws characterized in **Proposition 3** guarantee the achievement of the desired formation and the prescribed velocity while rejecting the disturbances.

From **Proposition 3** it is known that the system converges to the largest weakly invariant set for (35) such that $\xi = \mathbf{0}$. In the case $\Phi = \mathbf{0}$, $\Phi_i^d = \mathbf{0}_{p \times p}$, this implies that for any state $(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}) = (\tilde{z}, \mathbf{0}, \tilde{\eta}, \tilde{\theta})$ on this set, and for any $w \in F(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$, there exists $w^z \in \mathcal{K} \text{ sign } \tilde{z}$, such that

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)(\tilde{r}^v \tilde{\eta}) \\ \mathbf{0} &= -(B \otimes I_p)w^z - \Gamma^d \tilde{\theta} \\ \dot{\tilde{\eta}} &= -(\tilde{r}^v)^T (B \otimes I_p)w^z \\ \dot{\tilde{\theta}} &= \mathbf{0}. \end{aligned} \quad (37)$$

From $\dot{\tilde{\theta}} = \mathbf{0}$, we conclude that $\tilde{\theta}$ is a constant vector. Hence, considering the second equality in (37), we conclude that $(B \otimes I_p)w^z$ is a constant vector.

From the third equation, we deduce that each component of $\dot{\tilde{\eta}}(t)$ is identically zero. As a matter of fact, the first component is zero by construction. The vectors $\dot{\tilde{\eta}}_i(t)$, $i = 2, \dots, N$, are constant. If $\dot{\tilde{\eta}}_i(t)$ is a non-zero vector for some i , then at least one of its components, say $\dot{\tilde{\eta}}_{i\ell}(t)$, must be non-zero and then $\tilde{\eta}_{i\ell}(t)$ would grow with constant velocity and this would contradict the boundedness of the solutions. Hence, $\dot{\tilde{\eta}}(t) = \mathbf{0}$ and $\tilde{\eta}(t)$ is a constant vector. From the first equation, the latter implies that $\dot{\tilde{z}}(t)$ is constant as well. The same argument used for $\dot{\tilde{\eta}}(t)$ can be used again to show that $\dot{\tilde{z}}(t) = \mathbf{0}$. Now, consider the first equality in (37) and multiply both sides by $B \otimes I_p$. Since $\dot{\tilde{z}} = \mathbf{0}$, we obtain $\mathbf{0} = (B B^T \otimes I_p)(\tilde{r}^v \tilde{\eta})$. From the property of the Laplacian of a connected undirected graph, we conclude $\tilde{r}^v \tilde{\eta} = \mathcal{R}(\mathbf{1}_N \otimes I_p)$. From the definition of $\tilde{r}^v \tilde{\eta}$, we have $\tilde{r}^v \tilde{\eta} = \mathbf{0}$ and hence $\tilde{\eta} = \mathbf{0}$. The third identity in (37), $\dot{\tilde{\eta}} = \mathbf{0}$, and the structure of $\tilde{r}^v$ implies that

$$(B \otimes I_p)w^z = \begin{pmatrix} B_1^T \otimes I_p \\ B_2^T \otimes I_p \end{pmatrix} w^z = \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix}$$

for some constant $c \in \mathbb{R}^p$ (recall that it was proven previously that $(B \otimes I_p)w^z$ is a constant vector), and where $B_1^T \in \mathbb{R}^{1 \times M}$, $B_2^T \in \mathbb{R}^{(N-1) \times M}$ is a partition of the incidence matrix B . We now prove that $c = \mathbf{0}$. Multiplying on the left by $\mathbf{1}_N^T \otimes I_p$ the previous expression, one obtains

$$\mathbf{0} = (\mathbf{1}_N^T \otimes I_p)(B \otimes I_p)w^z = (\mathbf{1}_N^T \otimes I_p) \begin{pmatrix} c \\ \mathbf{0} \end{pmatrix} = c$$

as claimed. As a consequence, $(B \otimes I_p)w^z = \mathbf{0}$. With the same argument used in the proof of **Proposition 1**, this implies that $\tilde{z} = \mathbf{0}$. Moreover, from the second equation in (37), $\Gamma^d \tilde{\theta} = \mathbf{0}$, and this shows that $\tilde{\theta} = \mathbf{0}$.

The arguments above allow us to infer the following:

Corollary 1. Assume that $\Phi = \mathbf{0}$ and $\Phi_i^d = \mathbf{0}$, for $i = 1, 2, \dots, N$. Also assume that Γ^v , Γ_i^d , $i = 1, 2, \dots, N$, are non-singular and the graph G is a connected undirected graph. If

$$\begin{aligned} \tilde{G} &= (\tilde{r}^v)^T \\ G^d &= (\Gamma^d)^T, \end{aligned} \quad (38)$$

then all the Krasowskii solutions to the system (32) in closed-loop with the control input

$$\hat{u} = \tilde{u} = -(B \otimes I_p) \text{sign } \tilde{z}, \quad \check{u} = h(\xi) \quad (39)$$

are bounded and converge to the origin.

Case II: Harmonic disturbance rejection with known reference velocity.

We consider now the case in which the reference velocity is known and the controller only adopts an internal model to reject the disturbances. In this case, the system (32) becomes

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)h(\xi) \\ \dot{\xi} &= f(\xi) + g(\xi)(\hat{u} - \Gamma^d \tilde{\theta}) \\ \dot{\tilde{\theta}} &= \Phi^d \tilde{\theta} + G^d \check{u}. \end{aligned} \quad (40)$$

The choice $G^d = (\Gamma^d)^T$, $\hat{u} = -(B \otimes I_p) \text{sign } \tilde{z}$, $\check{u} = h(\xi)$ as designed in **Proposition 3** yields that all the Krasowskii solutions to the closed-loop system are bounded and converge to the largest weakly invariant set for (40). In the case the graph G has no loops the following holds:

Corollary 2. If the graph G has no loops and for $i = 1, 2, \dots, N$ the exosystems have matrices (Γ_i^d, Φ_i^d) of the form

$$\Phi_i^d = \text{block.diag} \left\{ \begin{pmatrix} 0 & \omega_{i1} \\ -\omega_{i1} & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \omega_{ip} \\ -\omega_{ip} & 0 \end{pmatrix} \right\},$$

with $\omega_{i\ell} \neq 0$ for all $\ell = 1, \dots, p$, and

$$\Gamma_i^d = \text{block.diag} \{ \Gamma_{i1}^d, \dots, \Gamma_{ip}^d \}$$

with $\Gamma_{i\ell}^d \neq 0$, for all $\ell = 1, \dots, p$, then all the Krasowskii solutions to (32) converge to the set of points where $\xi = \mathbf{0}$ and $\tilde{z} = \mathbf{0}$.

Proof. A solution on the weakly invariant set for (40), where $\xi = \mathbf{0}$, is such that $(\tilde{z}, \tilde{\theta})$ satisfies

$$\begin{aligned} \dot{\tilde{z}} &= \mathbf{0} \\ \dot{\tilde{\theta}} &= \Phi^d \tilde{\theta} \end{aligned} \quad (41)$$

and there exists $w^z \in \mathcal{K} \text{ sign } \tilde{z}$ such that $\mathbf{0} = g(\mathbf{0})(-(B \otimes I_p)w^z - \Gamma^d \tilde{\theta})$. Since by assumption $g(\mathbf{0})$ is full-column rank, the latter is equivalent to

$$\mathbf{0} = -(B \otimes I_p)w^z - \Gamma^d \tilde{\theta}. \quad (42)$$

If the graph G has no loops, the edge Laplacian matrix $B^T B$ is non-singular (Mesbahi & Egerstedt, 2010). Then, from (42) we obtain

$$w^z = -(B^T B \otimes I_p)^{-1} (B^T \otimes I_p) \Gamma^d \tilde{\theta}. \quad (43)$$

From (41), $\dot{\tilde{\theta}} = \Phi^d \tilde{\theta}$ implies that each $\Gamma_{i\ell}^d \tilde{\theta}_{i\ell}$ is a harmonic signal. One can write

$$\Gamma_{i\ell}^d \tilde{\theta}_{i\ell} = \alpha_{i\ell} \cos(\omega_{i\ell} t + \varphi_{i\ell})$$

where $\alpha_{i\ell}$, $\varphi_{i\ell}$ are constants. Therefore, (43) implies that each $w_{k\ell}^z$ is a linear combination of harmonic signals. One obtains

$$w_{k\ell}^z = - \sum_{i,\ell} \beta_{i\ell} \cos(\omega_{i\ell} t + \varphi_{i\ell}) \quad (44)$$

for some constants $\beta_{i\ell}$ (where the dependence on k was neglected). We claim that (44) implies that $\tilde{z}_{k\ell}$ is equal to zero. By contradiction, assume that $\tilde{z}_{k\ell}$ is not zero. Since $\tilde{z}_{k\ell} = 0$ (from (41)), then $\tilde{z}_{k\ell} = c$, where $c \in \mathbb{R}$ is a non-zero constant. As a result, $w_{k\ell}^z$ is either $+1$ or -1 . Without loss of generality, let us assume $w_{k\ell}^z = 1$. Now, integrate both sides of (44) from $t = 0$ to an arbitrary value λ . The right-hand side of (44) yields

$$\begin{aligned} \int_0^\lambda - \sum_{i,\ell} \beta_{i\ell} \cos(\omega_{i\ell} t + \varphi_{i\ell}) dt \\ = \sum_{i,\ell} \beta_{i\ell} \left(\frac{\sin(\varphi_{i\ell}) - \sin(\lambda \omega_{i\ell} + \varphi_{i\ell})}{\omega_{i\ell}} \right). \end{aligned} \quad (45)$$

Since $\omega_{i\ell} \neq 0$, N, p are finite numbers, the above sum has a bounded value that is independent of the choice of λ . On the other hand, the integration of the left-hand side of (44) is equal to λ . Therefore, if we choose λ big enough, it can be concluded that the equality (44) under the assumption $\tilde{z}_{k\ell} = c$ cannot hold. Therefore, $\tilde{z}_{k\ell} = 0$. Since the same holds true for any $k\ell$, then $\tilde{z} = \mathbf{0}$. $\square$

Remark 2. The condition on the graph to have no cycles is not necessary and one can find alternative statements that do not require the graph to be a tree, for instance, introducing conditions (again, not necessary) on the frequencies $\omega_{i\ell}$.

5. A different controller design for velocity tracking and disturbance rejection

In the previous section, the distributed controllers that allow to keep the formation while tracking an unknown reference trajectory and rejecting disturbances apply only to some special cases. To overcome this limitation, we propose in this section a slightly different controller design. The alternative design is carried out for a more restricted class of systems. As a matter of fact, we assume that the dynamics (3) of the agents satisfies

$$g_i(\xi_i) = g_i, \quad y_i = h_i(\xi_i) = \xi_i,$$

that is the input vector field is constant (and has full-column rank) and the passive output y_i coincides with the state ξ_i . Finally, we ask the function $W_i(\xi_i)$ in (5) to be lower bounded by a quadratic term, namely $W_i(\xi_i) \geq \|\xi_i\|^2$.

The main difference in the design lies in the controller (28) introduced to counteract the effect of the disturbances. Although we still adopt the same structure, namely

$$\begin{aligned} \dot{\theta}_i &= \Phi_i^d \theta_i + G_i^d \tilde{u}_i \\ \dot{\tilde{d}}_i &= \Gamma_i^d \tilde{\theta}_i, \quad i = 1, 2, \dots, N, \end{aligned} \quad (46)$$

we give $G_i^d \tilde{u}_i$ a different interpretation, namely we let

$$\tilde{u}_i = y_i - \hat{\xi}_i,$$

where $\hat{\xi}_i$ is an additional state of the controller that obeys the equation

$$\dot{\hat{\xi}}_i = f_i(y_i) + g_i(u_i + \Gamma_i^d \theta_i) + H_i(\xi_i - \hat{\xi}_i),$$

and G_i^d, H_i are two gain matrices. The variables

$$\tilde{\xi}_i = \xi_i - \hat{\xi}_i, \quad \tilde{\theta}_i = \theta_i - w_i^d$$

obey the equations

$$\begin{aligned} \dot{\tilde{\xi}}_i &= -g_i \Gamma_i^d \tilde{\theta}_i - H_i \tilde{\xi}_i \\ \dot{\tilde{\theta}}_i &= \Phi_i^d \tilde{\theta}_i + G_i^d \tilde{\xi}_i. \end{aligned} \quad (47)$$

The pair

$$\begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \Phi_i^d \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} & -g_i \Gamma_i^d \\ \Phi_i^d & \Phi_i^d \end{pmatrix}$$

is observable provided that g_i is full-column rank and the pair (Γ_i^d, Φ_i^d) is observable. Under these conditions, there exist matrix gains G_i^d, H_i such that the estimation error system (47) converges exponentially to zero. Let $P_i = P_i^T > 0$ be a matrix such that

$$\begin{pmatrix} -H_i & -g_i \Gamma_i^d \\ G_i^d & \Phi_i^d \end{pmatrix}^T P_i + P_i \begin{pmatrix} -H_i & -g_i \Gamma_i^d \\ G_i^d & \Phi_i^d \end{pmatrix} = -2 \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \gamma I \end{pmatrix}$$

where γ is a constant satisfying $\gamma \geq \max\{\|\Gamma_i^d\|_2^2, 1 \leq i \leq N\}$. Consider the overall system

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)(\xi + \tilde{r}^v \tilde{\eta}) \\ \dot{\xi} &= f(\xi) + g(\hat{u} - \Gamma^d \tilde{\theta}) \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta} + \tilde{G} \tilde{u} \\ \dot{\tilde{\theta}} &= \Phi^d \tilde{\theta} + G^d \tilde{\xi} \\ \dot{\tilde{\xi}} &= -g \Gamma^d \tilde{\theta} - H \tilde{\xi} \end{aligned} \quad (48)$$

where $u + \Gamma^d w^d = \hat{u} - \Gamma^d \theta + \Gamma^d w^d = \hat{u} - \Gamma^d \tilde{\theta}$, $H = \text{block.diag}\{H_1, H_2, \dots, H_N\}$, and design the control inputs as

$$\hat{u} = \tilde{u} = -(B \otimes I_p) \text{sign} \tilde{z},$$

so that the closed-loop system becomes

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p)(\xi + \tilde{r}^v \tilde{\eta}) \\ \dot{\xi} &= f(\xi) + g(-(B \otimes I_p) \text{sign} \tilde{z} - \Gamma^d \tilde{\theta}) \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta} - \tilde{G}(B \otimes I_p) \text{sign} \tilde{z} \\ \dot{\tilde{\theta}} &= \Phi^d \tilde{\theta} + G^d \tilde{\xi} \\ \dot{\tilde{\xi}} &= -g \Gamma^d \tilde{\theta} - H \tilde{\xi}. \end{aligned} \quad (49)$$

Proposition 4. Assume that the pairs (Γ^v, Φ) and (Γ_i^d, Φ_i^d) , for $i = 1, 2, \dots, N$, are observable. Let $G = (\Gamma^v)^T$. Then, all the Krasovskii solutions to the closed-loop system (49) converge to the origin.

Proof. The analysis follows a similar trail as for the other proofs, with a few significant variations. We focus on the set-valued derivative

$$\begin{aligned} \dot{V}(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}, \tilde{\xi}) &= \{a \in \mathbb{R} : \exists w \in F(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}, \tilde{\xi}) \text{ s.t.} \\ &a = \langle w, p \rangle, \text{ for all } p \in \partial V(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}, \tilde{\xi})\}. \end{aligned}$$

In this case the Lyapunov function is

$$\begin{aligned} V(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta}, \tilde{\xi}) &= \|\tilde{z}\|_1 + S(\xi) + \frac{1}{2} \tilde{\eta}^T \tilde{\eta} \\ &+ \frac{1}{2} \sum_{i=1}^N (\tilde{\xi}_i^T \tilde{\theta}_i^T) P_i (\tilde{\xi}_i^T \tilde{\theta}_i^T)^T, \end{aligned}$$

where $P = \text{block.diag}\{P_1, \dots, P_N\}$ and F is obtained from the right-hand side of (49), namely

$$\begin{pmatrix} (B^T \otimes I_p)(\xi + \tilde{\Gamma}^v \tilde{\eta}) \\ f(\xi) - g\Gamma^d \tilde{\theta} \\ \tilde{\Phi} \tilde{\eta} \\ \Phi^d \tilde{\theta} + G^d \tilde{\xi} \\ -g\Gamma^d \tilde{\theta} - H \tilde{\xi} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -g(B \otimes I_p) \\ -\tilde{G}(B \otimes I_p) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathcal{K} \text{sign} \tilde{z}.$$

Following the line of arguments already used in this paper, for each state such that $\tilde{V}$ is a non-empty set, any element of $\tilde{V}$ will satisfy

$$\langle p, v \rangle \leq -W(\xi) - \xi^T \Gamma^d \tilde{\theta} - \|\tilde{\xi}\|^2 - \gamma \|\tilde{\theta}\|^2.$$

By the condition on $W(\xi)$ and the definition of γ , a completion of the squares argument implies that $\langle p, v \rangle \leq -\frac{\|\xi\|^2}{2} - \gamma \frac{\|\tilde{\theta}\|^2}{2} - \|\tilde{\xi}\|^2$.

As a result, for all the states for which $\tilde{V} \neq \emptyset$, we have $\dot{\tilde{V}} = \{a \in \mathbb{R} : a \leq -\frac{\|\xi\|^2}{2} - \gamma \frac{\|\tilde{\theta}\|^2}{2} - \|\tilde{\xi}\|^2\} \subseteq (-\infty, 0]$. This implies convergence to the largest weakly invariant set where $\xi = \mathbf{0}$, $\tilde{\theta} = \mathbf{0}$, $\tilde{\xi} = \mathbf{0}$. On this invariant set, system (49) becomes

$$\begin{aligned} \dot{\tilde{z}} &= (B^T \otimes I_p) \tilde{\Gamma}^v \tilde{\eta} \\ \mathbf{0} &= -(B \otimes I_p) \text{sign} \tilde{z} \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta} - \tilde{G}(B \otimes I_p) \text{sign} \tilde{z}. \end{aligned}$$

The identity $\mathbf{0} = -(B \otimes I_p) \text{sign} \tilde{z}$ implies $\tilde{z} = \mathbf{0}$ and the equations above can be further simplified as

$$\begin{aligned} \mathbf{0} &= (B^T \otimes I_p) \tilde{\Gamma}^v \tilde{\eta} \\ \dot{\tilde{\eta}} &= \tilde{\Phi} \tilde{\eta}. \end{aligned}$$

Hence, $\tilde{\Gamma}^v \tilde{\eta} = \mathbf{0}$ and by observability of (Γ^v, Φ) , it follows that $\tilde{\eta} = \mathbf{0}$. $\square$

The result can be rephrased as follows. For the formation of agents

$$\begin{aligned} \dot{x}_i &= y_i + v_i^r \\ \dot{\xi}_i &= f_i(\xi_i) + g_i(u_i + d_i) \\ y_i &= \xi_i \end{aligned}$$

with prescribed reference velocity v^* generated by (19) and with disturbances d_i generated by (27), the distributed controller

$$\begin{aligned} \dot{\eta} &= \tilde{\Phi} \eta - \tilde{G}(B \otimes I_p) \text{sign} \tilde{z} \\ \dot{\theta} &= \Phi^d \theta + G^d(y - \hat{\xi}) \\ \dot{\hat{\xi}} &= f(y) + g(-(B \otimes I_p) \text{sign} \tilde{z} + \Gamma^d \theta) + H(y - \hat{\xi}) \\ u &= -(B \otimes I_p) \text{sign} \tilde{z} - \Gamma^d \theta \end{aligned}$$

(where the first component of v^r is v^*) guarantees boundedness of all the states, and convergence to the desired formation ($\tilde{z} = \mathbf{0}$) with asymptotic tracking of the reference velocity ($\xi = \mathbf{0}$, $\tilde{\eta} = \mathbf{0}$) as well as disturbance rejection.

6. Simulations

In this section we present the simulation results for a group of five strictly passive systems in $\mathbb{R}^2$. The dynamics of each agent is given by

$$\mathcal{H}_i : \begin{cases} \dot{\xi}_i = -\xi_i + u_i \\ y_i = \xi_i \end{cases} \quad (50)$$

where, comparing with (3), $f_i(\xi_i) = -\xi_i$, $g_i(\xi_i) = I_2$, and $h_i(\xi_i) = \xi_i$. The agents exchange information over a connected graph. The associated incidence matrix is equal to $B = (B_1^T \dots B_5^T)^T$ with $B_1 =$

Fig. 2. The estimated reference velocity v_i^r for each agent i . The leader generates the desired velocity $v^* = (-\sin(t), \cos(t))^T$.

$(-1 \ 0 \ 0 \ 0 \ 0 \ 0)$, $B_2 = (+1 \ -1 \ -1 \ 0 \ 0 \ 0)$, $B_3 = (0 \ +1 \ 0 \ -1 \ -1 \ 0)$, $B_4 = (0 \ 0 \ +1 \ +1 \ 0 \ -1)$, $B_5 = (0 \ 0 \ 0 \ 0 \ +1 \ +1)$. The desired formation has a pentagonal shape with edge length equal to 2 and is defined by the following inter-agent distance vectors: $z_1^* = (0, 2)^T$, $z_2^* = (1, \sqrt{3})^T$, $z_3^* = (2, 0)^T$, $z_4^* = (1, \sqrt{3})^T$, $z_5^* = (1, -2-\sqrt{3})^T$, $z_6^* = (0, -2)^T$. The number of edges of the graph is six. The initial position of the agent is set to $x(0) = (0.5, -0.5, 0.5, 1, 1, 0.5, 0.8, 0, 1.1, 0)^T$. Figs. 2 to 3 refer to the case of a formation evolving with a time-varying reference velocity known only to the formation leader (agent 1). The internal model parameters for the agents are: $\Phi = I_2 \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Gamma^v = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0.5 \end{pmatrix}$, $G = \Gamma^{vT}$, $w^v(0) = (1, 1, 1, 1)^T$, and $\eta_i(0) = (0, 0, 0, 0)^T$, for $i = 2, \dots, 5$. Fig. 2 shows the horizontal and vertical components of the reference velocity $v^* = (-\sin(t), \cos(t))^T$ and the estimated velocities v_i^r . The leader generates a desired sinusoidal velocity and the follower agents estimate the same reference velocity after some time. Fig. 3 shows the time behavior of the horizontal component of $\tilde{z}_1$, $\text{sign} \tilde{z}_1$ and the corresponding control $\tilde{u}_1$. As time elapses, $\tilde{z}_1$ converges to the origin implying convergence to the desired relative position. While $\tilde{z}_1$ converges to the origin, $\text{sign} \tilde{z}_1$ and $\tilde{u}_1$ converge to the discontinuity surface and oscillate between $+1$ and -1 . The state variables associated with the other agents exhibit a similar behavior and are not shown. In practice the observed fast oscillations can be overcome by the hysteric quantizers studied in Ceragioli et al. (2011) or the self-triggered controllers of De Persis and Frasca (2013).

Simulation results in the presence of matched disturbances are presented next. Fig. 4 shows the state $(\tilde{z}, \xi, \tilde{\eta}, \tilde{\theta})$ of the system with constant disturbances and a constant reference velocity (Case I). The reference velocity is only known to agent one. The following are the parameters chosen for the simulation: $\Phi_i^d = \Phi = \mathbf{0}_{2 \times 2}$, $\Gamma^v = \Gamma_i^d = I_2$, $G = G_i^d = I_2$, $w_i^d(0) = (i, i+3)^T$ for $i = 1 \dots 5$ and $w^v(0) = (1, 1)^T$, $\eta_i(0) = (0, 0)^T$ for $i = 2 \dots 5$. As expected, the formation is achieved, the desired velocity is reached by all the agents and the disturbances are rejected. Fig. 5 shows the state of the system with harmonic disturbances and a known reference velocity (Case II). The graph is taken to be a tree with an incidence matrix B' which is obtained by removing the two last columns of the proposed matrix B . The desired reference velocity is $v^* = (1, 1)^T$ and it is known to all of the agents. In this example, the following are set as the internal model parameters: Φ_1^d and Γ_1^d follow Φ and Γ^v in the example related to Fig. 2, $G_1^d = \Gamma_1^{dT}$, $w_1^d(0) = (0.1, 0.1, 0.1, 0.1)^T$, $\Phi_2^d = 2\Phi_1^d$, $\Phi_1^d = \Phi_3^d = \Phi_5^d$, $\Phi_4^d = 0.5\Phi_1^d$. The result

Fig. 3. The plot of the (horizontal) x -component of $\tilde{z}_1$, $\text{sign } \tilde{z}_1$ and the corresponding control $\tilde{u}_1$.

Fig. 4. The x -component of w^d together with the x -components of the states of the system (Case I).

Fig. 5. The x -component of w^d together with the x -components of the states of the system (Case II).

Fig. 6. The x - y components of w^d and the x -component of the states of the system (Section 5).

confirms that both the desired formation and the desired velocity are achieved. Simulation result related to Section 5 is shown in Fig. 6. The disturbance is assumed to be a linear combination of a constant and a harmonic signal for all five agents. The desired constant reference velocity, $v^* = (1, 1)^T$, is only known to the formation leader (agent 1). In this example, the following are set as the parameters of the disturbance and the controller:

$$\Phi_i^d = I_2 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix},$$

$$\Gamma_i^d = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & -0.5 & 0.5 \end{pmatrix},$$

$G_i^d = \Gamma_i^{dT}$, $w_i^d(0) = (0.1, 0.1, 0.1, 0.1, 0.1, 0.1)^T$, for $i = 1, \dots, 5$. Also, $\Phi = \mathbf{0}_{2 \times 2}$, $\Gamma^v = I_2$, $G = I_2$, $w^v(0) = (1, 1)^T$, and $\eta_i(0) = (0, 0)^T$, for $i = 2, \dots, 5$. The observer gain H_i is set to $50 I_2$, and the velocity error dynamics is $f_i(\xi_i) = -30 I_2 \xi_i$ and $g_i = 10 I_2$. The result confirms that the desired formation and the desired velocity are attained. In addition, the disturbance is rejected.

7. Conclusion

In this paper we considered a formation control problem with very coarse information for a network of strictly passive systems. We showed that despite the very coarse information, exact formation is achieved. Moreover, the formation tracks a desired reference velocity even when the reference velocity is only available to one of the agents (the so-called leader). We also studied the problem of rejecting input disturbances within the coarse information scenario. Possible future avenues of research include the extension of the results to deal with time-varying topologies and (asymmetric) measurement noise. The oscillations observed in the simulations could be overcome by the hysteric quantizer of Ceragioli et al. (2011) or the self-triggered approach of De Persis and Frasca (2013). This is another interesting topic that deserves attention.

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