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Dynamic coupling design for nonlinear output agreement and time-varying flow control[☆]

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ABSTRACT

This paper studies the problem of output agreement in networks of nonlinear dynamical systems under time-varying disturbances, using dynamic diffusive couplings. Necessary conditions are derived for general networks of nonlinear systems, and these conditions are explicitly interpreted as conditions relating the node dynamics and the network topology. For the class of incrementally passive systems, necessary and sufficient conditions for output agreement are derived. The approach proposed in the paper lends itself to solve flow control problems in distribution networks. As a first case study, the internal model approach is used for designing a controller that achieves an optimal routing and inventory balancing in a dynamic transportation network with storage and time-varying supply and demand. It is in particular shown that the time-varying optimal routing problem can be solved by applying an internal model controller to the dual variables of a certain convex network optimization problem. As a second case study, we show that droop-controllers in microgrids have also an interpretation as internal model controllers.

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1. Introduction

Output agreement has evolved as one of the most important control objectives in cooperative control. It appears in various contexts, ranging from distributed optimization (Tsitsiklis, Bertsekas, & Athans, 1986), formation control (Olfati-Saber, Fax, & Murray, 2007) up to oscillator synchronization (Stan & Sepulchre, 2007). Over the past years, it has become evident that the *internal model principle* takes a central role in output agreement problems, see e.g. Bai, Arcak, and Wen (2011), De Persis (2013), De Persis and Jayawardhana (2014), Wieland, Sepulchre, and Allgöwer (2011). Independently and in parallel, it was shown that *passivity* and related systems theoretic concepts have outstanding relevance in the analysis and synthesis of synchronizing networks and output agreement problems, see e.g. Arcak (2007), Bai et al. (2011), Bürger, Zelazo, and Allgöwer (2014), De Persis and Jayawardhana (2012), Scardovi, Arcak, and Sontag (2010), Stan and Sepulchre (2007), van der Schaft and Maschke (2013).

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The present paper studies output agreement in networks of heterogeneous nonlinear dynamical systems affected by external disturbances and presents an approach that combines elements from internal model control with those known in passivity-based cooperative control. We follow here the trail opened in Pavlov and Marconi (2008) for centralized output regulation and provide necessary and sufficient conditions for the solution of the output agreement problem for the class of *incrementally passive* systems. Our results provide a bridge connecting the two complementary approaches for output agreement problems, namely the internal model approach on the one hand, and the passivity-based approach on the other hand.

The proposed approach is inherently different from other internal model approaches such as Isidori, Marconi, and Casadei (2013), Wieland et al. (2011), and Wieland, Wu, and Allgöwer (2013). In Wieland et al. (2011) each node is augmented with a local controller that contains a reference system, identical for all nodes, and the local controllers track the reference system. The local (“virtual”) copies of the reference system are then synchronized with static diffusive couplings. The approach considered in the present paper differs in central points. Most obviously, dynamic couplings, rather than local controllers, are investigated. Furthermore, external signals are assumed to affect the node dynamics, a case that is not covered in Wieland et al. (2011). Incrementally passive systems and disturbance rejection are also dealt with in De Persis and Jayawardhana (2014). However, the framework we propose here,

inspired by Pavlov and Marconi (2008), is completely different and leads to a family of new distinct results that have not been considered in De Persis and Jayawardhana (2014).

The main contribution of this paper is the development of a control framework for network systems that integrates the ideas of internal model and passivity-based cooperative control. The proposed framework leads to both, constructive methods for the design of distributed controllers and a novel understanding of existing control approaches. In particular, we consider networks of heterogeneous nonlinear systems, interacting according to an undirected network topology. The objective is to design dynamic controllers placed on the edges of the network such that output agreement is achieved. Necessary conditions for the feasibility of the problem are presented for the general case of heterogeneous nonlinear systems with external disturbances. Sufficient conditions for convergence to output agreement are derived for *incrementally passive* systems. Following this, we present a relevant class of heterogeneous nonlinear systems for which all assumptions are met and the coupling controllers can be found following a constructive design procedure. The constructiveness of the result is further demonstrated via the design of *optimal routing controllers* for distribution systems with time-varying demand. To explain the relation to existing approaches the special situations where output agreement can be reached with static diffusive couplings or where the disturbances are constant are discussed. Based on these results, it is shown that *droop-controllers* in microgrids, as, e.g., studied in Simpson-Porco, Dörfler, and Bullo (2013), are designed exactly in accordance with the internal model control approach. Early results on the internal model approach to output agreement have been presented in Bürger and De Persis (2013).

The remainder of the paper is organized as follows. The problem formulation and necessary conditions for output agreement are presented in Section 2. Sufficient conditions for output agreement in networks of incrementally passive systems are discussed in Section 3. A constructive procedure for the design of such controllers for a class of nonlinear systems is presented in Section 4. The design procedure is applied to a time-varying optimal distribution problem in Section 5. In Section 6, the relation to known methods in the literature is formally discussed, and an interpretation of droop-controllers as internal model controllers is provided in Section 7.

Notation: The set of (positive) real numbers is denoted by $\mathbb{R}_{\geq}$. Given two matrices A and B , the *Kronecker product* is denoted by $A \otimes B$. The Moore–Penrose inverse (or pseudo-inverse) of a non-invertible matrix A is denoted by $A^\dagger$. The *range-space* and *null-space* of a matrix B are denoted by $\mathcal{R}(B)$ and $\mathcal{N}(B)$, respectively. A graph $\mathcal{G} = (V, E)$ is an object consisting of a finite set of nodes, $|V| = n$, and edges, $|E| = m$. The incidence matrix $B \in \mathbb{R}^{n \times m}$ of the graph $\mathcal{G}$ with arbitrary orientation, is a $\{0, \pm 1\}$ matrix with $[B]_{ik}$ having value ‘+1’ if node i is the initial node of edge k , ‘−1’ if it is the terminal node, and ‘0’ otherwise.

2. Problem formulation and necessary conditions

We consider a network of dynamical systems defined on a connected, undirected graph $\mathcal{G} = (V, E)$. Each node represents a non-linear system

$$\begin{aligned} \dot{x}_i &= f_i(x_i, u_i, w_i) \\ y_i &= h_i(x_i, w_i), \quad i = 1, 2, \dots, n, \end{aligned} \quad (1)$$

where $x_i \in \mathbb{R}^{r_i}$ is the state, and $u_i, y_i \in \mathbb{R}^p$ are the input and output, respectively. Each system (1) is driven by the time-varying signal $w_i \in \mathbb{R}^{q_i}$, representing, e.g., a disturbance or reference. We assume that the exogenous signals w_i are generated by systems of the form

$$\dot{w}_i = s_i(w_i), \quad w_i(0) \in \mathcal{W}_i, \quad (2)$$

where $\mathcal{W}_i$ is a set whose properties are specified below. A common assumption in nonlinear output regulation theory is neutral stability¹ of the exo-systems (Isidori & Byrnes, 2008). For the purpose of this paper, it is advantageous to restrict the discussion to a slightly smaller class of exo-systems.

Assumption 1. The vector field $s_i(w_i)$ satisfies for all w_i, w'_i the inequality

$$(w_i - w'_i)^T (s_i(w_i) - s_i(w'_i)) \leq 0. \quad (3)$$

Remarkably, Assumption 1 includes in particular neutrally stable linear exo-systems, i.e., a linear function $s_i(w_i) = S_i w_i$ with skew-symmetric matrix S_i , i.e., $S_i^T + S_i = \mathbf{0}$ satisfies the requirements. These linear exosystems can generate signals that are combinations of constant and periodic modes.

In addition, our assumption includes various nonlinear dynamical systems. For example, it has been shown in DeLellis, di Bernardo, and Garofalo (2009, Sec. 4.3) that nonlinear Chua’s oscillators satisfy Assumption 1. We stack together the signals w_i , for $i = 1, 2, \dots, n$, and obtain the vector $w \in \mathbb{R}^q$, which satisfies the equation $\dot{w} = s(w)$. In what follows, whenever we refer to the solutions of $\dot{w} = s(w)$, we assume that the initial condition is chosen in a compact set $\mathcal{W} = \mathcal{W}_1 \times \dots \times \mathcal{W}_n$. The set $\mathcal{W}$ is assumed to be forward invariant for the system $\dot{w} = s(w)$. Similarly, let x, u , and y be the stacked vectors of x_i, u_i , and y_i , respectively. Using this notation, the totality of all systems is

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(x, u, w) \\ y &= h(x, w) \end{aligned} \quad (4)$$

with state space $\mathcal{W} \times \mathcal{X}$ and $\mathcal{X}$ a compact subset of $\mathbb{R}^{r_1} \times \dots \times \mathbb{R}^{r_n}$.

The control objective is to reach output agreement of all nodes in the network, independent of the exact representation of the time-varying external signals. We aim to achieve this control objective by a suitable design of dynamic couplings between the systems. This means, between any pair of neighboring nodes, i.e., on any edge of $\mathcal{G}$, a dynamical system (in the following called “controller”) is placed, taking the form

$$\begin{aligned} \dot{\xi}_k &= F_k(\xi_k, v_k) \\ \lambda_k &= H_k(\xi_k, v_k), \quad k = 1, 2, \dots, m, \end{aligned} \quad (5)$$

with state $\xi_k \in \mathbb{R}^{l_k}$, input $v_k \in \mathbb{R}^p$ and output $\lambda_k \in \mathbb{R}^p$. Using the same notational convention as before, we define ξ and λ as the stacked state and output vector. Together, the controllers (5) give rise to the overall controller

$$\begin{aligned} \dot{\xi} &= F(\xi, v) \\ \lambda &= H(\xi, v), \end{aligned} \quad (6)$$

where $\xi \in \mathcal{E}$, a compact subset of $\mathbb{R}^{l_1} \times \dots \times \mathbb{R}^{l_m}$. The collection (6) of dynamical systems (5) generates the overall output λ that determines the control input u applied to the network system (4) via the interconnection (9) below. This motivates the choice of referring to the dynamical systems (5) as controllers.

Throughout the paper the following interconnection structure between the plants, placed on the nodes of $\mathcal{G}$, and the controllers, placed on the edges of $\mathcal{G}$, is considered. A controller (5), associated with edge k connecting nodes i, j , has access to the relative outputs $y_i - y_j$. In vector notation, the relative outputs of the systems are

$$z = (B^T \otimes I_p) y, \quad (7)$$

¹ The dynamics (2) is said to be neutrally stable if it admits a Lyapunov stable equilibrium point for $w = 0$ and there exists a neighborhood of Poisson stable points around $w = 0$.

Fig. 1. Structure of the internal model control scheme.

where B is the $(n \times m)$ signed incidence matrix of the graph $\mathcal{G}$. The controllers are then driven by the systems via the interconnection condition

$$v = -z, \quad (8)$$

where v are the stacked inputs of the controllers. Additionally, the output of the controllers influences the incident systems via the interconnection²

$$u = (B \otimes I_p)\lambda. \quad (9)$$

Due to this interconnection structure the dynamics on the network can be represented as a closed-loop dynamics as illustrated in Fig. 1. We are now ready to formally introduce the output agreement problem.

Definition 1 (Output Agreement Problem). The output agreement problem is solvable for the process (4) under the interconnection relations (7)–(9), if there exist controllers (6), such that every solution $(w(t), x(t), \xi(t))$ originating from $\mathcal{W} \times \mathcal{X} \times \mathcal{E}$ is bounded and satisfies $\lim_{t \rightarrow \infty} (B^T \otimes I_p)y(t) = \mathbf{0}$.

2.1. Necessary conditions

We first investigate the *necessary conditions* for the output agreement problem to be solvable. To this purpose, we strengthen the requirement on the convergence of the regulation error to the origin, requiring that $\lim_{t \rightarrow \infty} (B^T \otimes I_p)y(t) = \mathbf{0}$ uniformly in the initial condition (Isidori & Byrnes, 2008). The closed-loop system (4) and (6)–(9) can be written as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(x, (B \otimes I_p)H(\xi, -(B^T \otimes I_p)h(x, w)), w) \\ \dot{\xi} &= F(\xi, -(B^T \otimes I_p)h(x, w)). \end{aligned} \quad (10)$$

Definition 2 (ω -limit set). The ω -limit set $\Omega(\mathcal{W} \times \mathcal{X} \times \mathcal{E})$ is the set of points (w, x, ξ) for which there exists a sequence of pairs $(t_k, (w_k, x_k, \xi_k))$ with $t_k \rightarrow +\infty$ and $(w_k, x_k, \xi_k) \in \mathcal{W} \times \mathcal{X} \times \mathcal{E}$ such that $\varphi(t_k, (w_k, x_k, \xi_k)) \rightarrow (w, x, \xi)$ as $k \rightarrow +\infty$, where $\varphi(\cdot, \cdot)$ is the flow of (10).

If the output agreement problem is solvable, then the ω -limit set $\Omega(\mathcal{W} \times \mathcal{X} \times \mathcal{E})$ is nonempty, compact, invariant and uniformly attracts $\mathcal{W} \times \mathcal{X} \times \mathcal{E}$ under the flow of (10). Furthermore, the

ω -limit set must satisfy

$$\begin{aligned} \Omega(\mathcal{W} \times \mathcal{X} \times \mathcal{E}) \\ \subseteq \{(w, x, \xi) \in \mathcal{W} \times \mathcal{X} \times \mathcal{E} : (B^T \otimes I_p)h(x, w) = \mathbf{0}\}. \end{aligned}$$

This set is the graph of a map defined on the whole $\mathcal{W}$ and is invariant for the closed-loop system. By the invariance, for any solution w of the exosystem originating from $\mathcal{W}$, there exists (x^w, u^w, ξ^w) such that

$$\begin{aligned} \dot{x}^w &= f(x^w, u^w, w) \\ \mathbf{0} &= (B^T \otimes I_p)h(x^w, w) \end{aligned} \quad (11)$$

and

$$\begin{aligned} \dot{\xi}^w &= F(\xi^w, \mathbf{0}) \\ u^w &= (B \otimes I_p)H(\xi^w, \mathbf{0}). \end{aligned} \quad (12)$$

Proposition 1. If the output agreement problem is solvable, then, for every w solution to $\dot{w} = s(w)$ originating in $\mathcal{W}$, there must exist solutions (x^w, u^w, ξ^w) such that Eqs. (11) and (12) are satisfied.

In a controller-independent form, the constraints (11) and (12) require that there exists (x^w, u^w) satisfying

$$\begin{aligned} \dot{x}^w &= f(x^w, u^w, w), \quad y^w = h(x^w, w) \\ u^w &\in \mathcal{R}(B \otimes I_p), \quad y^w \in \mathcal{N}(B^T \otimes I_p), \end{aligned} \quad (13)$$

where $u^w \in \mathcal{R}(B \otimes I_p)$ denotes that at every time t the vector $u^w(t)$ is contained in the respective vector space. Let in the following u^w be a solution to (13), and λ_p^w be a trajectory satisfying $u^w = (B \otimes I_p)\lambda_p^w$. The trajectory λ_p^w is uniquely defined if and only if the graph $\mathcal{G}$ has no cycles. Otherwise, the matrix B has a nontrivial nullspace, see Godsil and Royle (2001). In the most general form, the existence of a feedforward controller is equivalent to the constraint that there exist an integer d and maps $\tau : \mathcal{W} \rightarrow \mathbb{R}^d$, $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^{mp}$ satisfying

$$\begin{aligned} \frac{\partial \tau}{\partial w} s(w) &= \phi(\tau(w)) \\ \lambda_p^w + \lambda_0^w &= \psi(\tau(w)), \quad \lambda_0^w \in \mathcal{N}(B \otimes I_p). \end{aligned} \quad (14)$$

Note that there might be an infinite number of possible controllers that can generate the desired steady state input u^w . If the constraint (14) holds, the system

$$\begin{aligned} \dot{\eta} &= \phi(\eta), \quad \eta \in \mathbb{R}^d \\ \lambda &= \psi(\eta) \end{aligned} \quad (15)$$

has the property that if $\eta_0 = \tau(w(0))$, then the solution $\eta(t)$ to (15) starting from η_0 is such that $(B \otimes I_p)\lambda(t) = u^w(t)$ for all $t \geq 0$. We denote by η^w such a solution to (15) such that $(B \otimes I_p)\psi(\eta^w(t)) = u^w(t)$ for all $t \geq 0$. We then let $\lambda^w(t) := \psi(\eta^w(t))$. Here $\lambda^w(t)$ is one of the infinite many realizations of the map $\lambda_p^w(t) + \lambda_0^w(t)$, with $\lambda_0^w \in \mathcal{N}(B \otimes I_p)$.

To design a controller that decomposes into controllers on the edges of $\mathcal{G}$, we introduce a vector $\eta_k \in \mathbb{R}^d$ for each edge $k = 1, \dots, m$, and denote with ψ_k the entries of the vector valued function ψ corresponding to the edge k . Each edge is now assigned a controller of the form

$$\dot{\eta}_k = \phi(\eta_k), \quad \lambda_k = \psi_k(\eta_k), \quad k = 1, 2, \dots, m. \quad (16)$$

With the stacked vector $\eta = [\eta_1^T, \dots, \eta_m^T]^T$, and vector valued functions $\tilde{\phi}(\eta) = [\phi(\eta_1)^T, \dots, \phi(\eta_m)^T]^T$, $\tilde{\psi}(\eta) = [\psi_1(\eta_1)^T, \dots, \psi_m(\eta_m)^T]^T$, the overall controller (16) is

$$\begin{aligned} \dot{\eta} &= \tilde{\phi}(\eta) \\ \lambda &= \tilde{\psi}(\eta). \end{aligned} \quad (17)$$

If the initial condition is chosen as $\eta_0 = \mathbf{1}_m \otimes \tau(w(0))$ then the solution $\eta(t)$ to (15) starting from η_0 is such that $\lambda(t) = \lambda^w(t)$ for all $t \geq 0$.

² The interconnection structure (7), (9) naturally represents a canonical structure for distributed control laws. This structure is often considered in the context of passivity-based cooperative control, see e.g., Arcak (2007), Bai et al. (2011), Bürger et al. (2014), De Persis and Jayawardhana (2012), van der Schaft and Maschke (2013). In fact, it can also model a physical coupling of the dynamical systems in the network, as for the case studies in Sections 5 and 7.

Remark 1. Necessary conditions for output agreement of nonlinear systems without disturbances and node controllers were presented in [Wieland et al. \(2013\)](#). We formalize here the necessary conditions in the presence of external disturbances and dynamic couplings. The main difference between the two formulations, is that in the present formulation the synchronous solution depends on the disturbance representation and we allow for non-zero inputs from the network in agreement, i.e., in general we will have $u^w \neq 0$ in (13). As it will become evident later on, this formulation allows us to solve problems where the approach of [Wieland et al. \(2013\)](#) is not applicable.

2.2. Discussion: The regulator equations

The necessary conditions (11) and (12) are a weaker form of the regulator equations of [Isidori and Byrnes \(1990\)](#). We discuss next that in the considered networked systems the classical regulator equations might fail to admit a solution while the output agreement problem is still feasible. This motivates the presentation of the more general condition (11). However, analyzing the regulator equations still leads to valuable insights, as it reveals the outstanding role of passivity in the context of networked systems.

If the systems (1) are such that for each given exogenous input $w(t)$ there exists a *unique* steady state response, and the ω -limit set can be expressed as $\Omega(\mathcal{W} \times \mathcal{X} \times \mathcal{E}) = \{(w, x, \xi) : x = \pi(w), \xi = \pi_c(w)\}$, then $x^w = \pi(w)$ and the regulator equation (11) express the existence of an invariant manifold where the “regulation error” $(B^T \otimes I_p)y$ is identically zero provided that the control input u^w is applied. Furthermore, (12) express the existence of a controller able to provide u^w . In this case, (11) and (12) take the familiar expressions, see e.g. [Isidori and Byrnes \(1990\)](#):

$$\begin{aligned} \frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), (B \otimes I_p)H(\pi_c(w)), w) \\ 0 &= (B^T \otimes I_p)h(\pi(w), w) \end{aligned} \quad (18)$$

and

$$\frac{\partial \pi_c}{\partial w} s(w) = F(\pi_c(w), 0). \quad (19)$$

However, there is a substantial structural difference between the output agreement problem considered here and output regulation problems, that can be best seen for linear dynamical systems. Suppose each system (1) is of the form

$$\begin{aligned} \dot{x}_i &= A_i x_i + G_i u_i + P_i w_i \\ y_i &= C_i x_i, \end{aligned} \quad (20)$$

with a linear exosystem $\dot{w}_i = S_i w_i$. Let in the following $\bar{A} = \text{block.diag}(A_1, \dots, A_n)$, $\bar{G} = \text{block.diag}(G_1, \dots, G_n)$, $\bar{P} = \text{block.diag}(P_1, \dots, P_n)$ and $\bar{C} = \text{block.diag}(C_1, \dots, C_n)$. The exosystems are stacked into the dynamics $w = \bar{S}w$, with $\bar{S} = \text{block.diag}(S_1, \dots, S_n)$. The classical result of [Francis \(1976\)](#) states that one can take $x^w = \Pi w$ and $u^w = \Gamma w$ such that the regulator equations (18) take the form of Sylvester equations

$$\begin{aligned} \Pi \bar{S} &= \bar{A} \Pi + \bar{G} (B \otimes I_p) \Gamma + \bar{P} \\ (B^T \otimes I_p) \bar{C} \Pi &= 0. \end{aligned} \quad (21)$$

Under controllability and observability assumptions, feasibility of (21) is necessary and sufficient for output regulation of linear systems. We will see next, that due to the networked structure of the considered problems the assumptions might fail to hold, although the output agreement problem is solvable (as we show in the next sections). First note that the regulator equations (21) have a solution if and only if

$$\text{rank} \left(\begin{bmatrix} \bar{A} - sI_r & \bar{G} (B \otimes I_p) \\ (B^T \otimes I_p) \bar{C} & 0_{mp \times mp} \end{bmatrix} \right) = \# \text{rows}, \quad (22)$$

for all $s \in \sigma(\bar{S})$, where $r = \sum_{i=1}^n r_i$ and $\sigma(\bar{S})$ is the spectrum of $\bar{S}$. The condition states that no pole of the stacked exosystem is a transmission zero of the system from input λ to output $z = (B \otimes I_p)^T y$. To focus the discussion on the impact of the constraints resulting from the network, we impose the following assumption:

Assumption 2. For each system $i \in \{1, \dots, n\}$

$$\text{rank} \left(\begin{bmatrix} A_i - sI_{r_i} & G_i \\ C_i & 0_{p \times p} \end{bmatrix} \right) = r_i + p, \quad \forall s \in \sigma(\bar{S}).$$

The main observation is that the rank condition can be violated due to the networked structure of the problem. We summarize this in the result below.

Proposition 2. Suppose [Assumption 2](#) holds. The rank condition (22) is violated if either of the following holds:

1. $\mathcal{G}$ contains a cycle;
2. $\mathcal{R}(\bar{H}(s)(B \otimes I_p)) \cap \mathcal{N}((B^T \otimes I_p)) \neq \{0\}$ for some $s \in \sigma(\bar{S})$, where $\bar{H}(s) = \bar{C}(sI - \bar{A})^{-1} \bar{G}$.

Moreover, the conditions are necessary provided that for all $s \in \sigma(\bar{S})$, $s \notin \sigma(\bar{A})$.

The proof is presented in the [Appendix](#). The first condition shows that the regulator equations (21) have no solution if graph contains cycles or if the transfer functions of the dynamical systems “rotate” $\mathcal{R}(B \otimes I_p)$ in such a way that it intersects nontrivially its orthogonal space $\mathcal{N}(B^T \otimes I_p)$. The previous result gives an intuition about a class of systems for which the output agreement problem is feasible.

Corollary 1. Assume [Assumption 2](#) holds and $\mathcal{G}$ contains no cycles. Suppose furthermore that all eigenvalues of $\bar{S}$ have zero real part. Then [Eqs. \(21\)](#) are feasible if $\bar{H}(s)$ is strictly positive real.³

The proof is presented in the [Appendix](#). The result suggests that passivity takes an outstanding role in output agreement problems. In fact, we show next, that a certain type of passivity is sufficient for the feasibility of the output agreement problem.

3. Incremental passivity as sufficient condition for output agreement

In this section we highlight that the output agreement problem is solvable for a special class of nonlinear systems, namely *incrementally passive* systems. Our approach follows the line of [Pavlov and Marconi \(2008\)](#), where the notion of a regular storage function was introduced.

Definition 3 ([Pavlov & Marconi, 2008](#)). A storage function $V(t, x, x')$ is called regular if for any sequence (t_k, x_k, x'_k) , $k = 1, 2, \dots$, such that x'_k is bounded, t_k tends to infinity, and $|x_k| \rightarrow \infty$, it holds that $V(t_k, x_k, x'_k) \rightarrow \infty$, as $k \rightarrow \infty$.

The dissipativity characterization of incremental passivity provided in [Pavlov and Marconi \(2008\)](#) is as follows.

Definition 4. The system (1) is said to be *incrementally passive* if there exists a C^1 regular storage function $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^{r_i} \times \mathbb{R}^{r_i} \rightarrow \mathbb{R}_{\geq 0}$ such that for any two inputs u_i, u'_i and any two solutions x_i, x'_i , corresponding to these inputs, the respective outputs y_i, y'_i satisfy

$$\begin{aligned} \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(x_i, u_i, w_i) + \frac{\partial V_i}{\partial x'_i} f_i(x'_i, u'_i, w_i) \\ \leq (y_i - y'_i)^T (u_i - u'_i). \end{aligned} \quad (23)$$

³ We refer to [Khalil \(2002, Def. 6.4\)](#) for the definition of a strictly positive real transfer function.

Incremental passivity is an input–output property that compares two arbitrary trajectories of the dynamical system. While it is more restrictive than classical passivity it has been a relevant concept in modern control theory, see e.g., Desoer and Vidyasagar (1974), Pavlov and Marconi (2008), Sastry (1999). Related concepts have proven to be very useful in the study of network systems ((Scardovi et al., 2010; Stan & Sepulchre, 2007)—more details on the comparison between our approach and the one in those papers are found in the first paragraph of Section 6.1). In Pavlov and Marconi (2008, Theorem 2), a procedure is proposed to render a class of nonlinear systems incrementally passive by the design of a suitable local feedback controller. In the context of this paper, the following two classes of incrementally passive systems take an outstanding role.

Example 1. Linear systems of the form (20) that are passive from the input u_i to the output y_i are also incrementally passive, with $V_i = \frac{1}{2}(x_i - x'_i)^T Q_i (x_i - x'_i)$ and $Q_i = Q_i^T > 0$ the matrix such that $A_i^T Q_i + Q_i A_i \leq 0$ and $Q_i G_i = C_i^T$. Remarkably, system (20) is incrementally passive if its transfer function is positive real. Thus, in view of Corollary 1, incremental passivity takes an outstanding role for the feasibility of output agreement problems.

Example 2. Nonlinear systems of the form

$$\begin{aligned}\dot{x}_i &= f_i(x_i) + G_i u_i + P_i w_i \\ y_i &= C_i x_i\end{aligned}\quad (24)$$

with $f_i(x_i) = \nabla F_i(x_i)$, $F_i(x_i)$ twice continuously differentiable and concave, and $G_i = C_i^T$ are incrementally passive. In fact, by concavity of $F_i(x_i)$, $(x_i - x'_i)^T (f_i(x_i) - f_i(x'_i)) \leq 0$, and $V_i = \frac{1}{2}(x_i - x'_i)^T (x_i - x'_i)$ is the incremental storage function.

For the sake of brevity, we will in the following sometimes write $\dot{V}_i$ for the directional derivative $\frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} f_i(x_i, u_i, w_i) + \frac{\partial V_i}{\partial x'_i} f_i(x'_i, u'_i, w'_i)$.

In the previous section, it was shown that the controllers at the edges have to take the form (16). Now, they must be completed by considering additional control inputs that guarantee the achievement of the steady state. While we require the internal model to be identical for all edges, i.e., $\phi(\eta_k)$, the augmented systems might be different. Then, the controllers (16) modify as

$$\begin{aligned}\dot{\eta}_k &= \phi_k(\eta_k, v_k) \\ \lambda_k &= \psi_k(\eta_k), \quad k = 1, 2, \dots, m,\end{aligned}\quad (25)$$

where all controllers reduce to the common internal model if no external forcing is applied, i.e., $\phi_k(\eta_k, 0) = \phi(\eta_k)$. The controller is then said to have the *internal model property*. The following is the main standing assumption that the controllers must satisfy.

Assumption 3. For each $k = 1, 2, \dots, m$, there exist regular storage functions $W_k(\eta_k, \eta'_k)$, with $W_k : \mathbb{R}^{q_k} \times \mathbb{R}^{q_k} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\frac{\partial W_k}{\partial \eta_k} \phi_k(\eta_k, v_k) + \frac{\partial W_k}{\partial \eta'_k} \phi_k(\eta'_k, v'_k) \leq (\lambda_k - \lambda'_k)^T (v_k - v'_k). \quad (26)$$

The existence of an incrementally passive controller is in general hard to verify. In fact, it depends on the structure of the exosystem (2) as well as on the properties of the node dynamics, i.e., in particular on u^w as defined by (13). However, there is an important class of problems, where an incrementally passive controller can be designed in a constructive way.

Example 3. The important example when the design is possible is when the feedforward control input is linear, that is (14) is satisfied with $\tau = \text{Id}$, $\phi = s$ and ψ is a linear function of its argument. In

this case, we let

$$\phi_k(\eta_k, 0) = s(\eta_k), \quad \psi_k(\eta_k) = M_k \eta_k$$

and define

$$\phi_k(\eta_k, v_k) = s(\eta_k) + M_k^T v_k. \quad (27)$$

Then, by definition of s as the gradient of a concave function, the storage function $W_k(\eta_k, \eta'_k) = \frac{1}{2}(\eta_k - \eta'_k)^T (\eta_k - \eta'_k)$ satisfies

$$\begin{aligned}\frac{\partial W_k}{\partial \eta_k} \phi_k(\eta_k, v_k) + \frac{\partial W_k}{\partial \eta'_k} \phi_k(\eta'_k, v'_k) \\ = (\eta_k - \eta'_k)^T (s(\eta_k) - s(\eta'_k)) + (\eta_k - \eta'_k)^T M_k^T (v_k - v'_k) \\ \leq (\psi_k(\eta_k) - \psi_k(\eta'_k))^T (v_k - v'_k),\end{aligned}\quad (28)$$

that is (26).

We state below the main result of the section that, while extending to networked systems the results of Pavlov and Marconi (2008), provide a solution to the output agreement problem in the presence of time-varying disturbances.

Theorem 1. Consider the network $\mathcal{G}$ with dynamics on the nodes (4). Suppose all exosystems satisfy (3), the regulator equations (11) hold, and all node dynamics are incrementally passive. Consider the controllers

$$\begin{aligned}\dot{\eta} &= \bar{\phi}(\eta, v) \\ \lambda &= \bar{\psi}(\eta) + v\end{aligned}\quad (29)$$

where $\bar{\phi}$ and $\bar{\psi}$ are the stacked functions of $\phi_k(\eta_k, v_k)$ and $\psi_k(\eta_k)$, and v is an additional input to be designed. Suppose the controllers have the internal model property and satisfy Assumption 3. Then, the controller (29) with the interconnection structure

$$u = (B \otimes I_p) \lambda, \quad v = -(B^T \otimes I_p) y, \quad (30)$$

and $v := v = -(B^T \otimes I_p) y$ solves the output agreement problem, that is every solution starting from $\mathcal{W} \times \mathcal{X} \times \mathcal{E}$ is bounded and

$$\lim_{t \rightarrow +\infty} (B^T \otimes I_p) y(t) = 0.$$

Proof. By the incremental passivity property of the x subsystem in (4) and (11), it is true that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, u, w) + \frac{\partial V}{\partial x^w} f(x^w, u^w, w) \leq (y - y^w)^T (u - u^w),$$

where $V = \sum_i V_i$. Similarly by Assumption 3, the system (29) satisfies

$$\frac{\partial W}{\partial \eta} \bar{\phi}(\eta, v) + \frac{\partial W}{\partial \eta^w} \bar{\phi}(\eta^w) \leq (\lambda - \lambda^w)^T v - (v - v^w)^T v,$$

with $W = \sum_k W_k$ and $\bar{\phi}(\eta^w) = \mathbf{1}_m \otimes \phi(\eta^w)$. Bearing in mind the interconnection constraints $u = (B \otimes I_p) \lambda$, $u^w = (B \otimes I_p) \lambda^w$, and $v = -(B^T \otimes I_p) y$, and letting $U((x, x^w), (\eta, \eta^w)) = V(x, x^w) + W(\eta, \eta^w)$ we obtain

$$\begin{aligned}\dot{U}((x, x^w), (\eta, \eta^w)) &:= \dot{V}(x, x^w) + \dot{W}(\eta, \eta^w) \\ &\leq (y - y^w)^T (u - u^w) + (\lambda - \lambda^w)^T v - (v - v^w)^T v \\ &= (y - y^w)^T (B \otimes I_p) (\lambda - \lambda^w) \\ &\quad - (\lambda - \lambda^w)^T (B^T \otimes I_p) y + (v - v^w)^T (B^T \otimes I_p) y.\end{aligned}$$

By definition of output agreement, $(B^T \otimes I_p) y^w = 0$ and the previous equality becomes

$$\begin{aligned}\dot{U}((x, x^w), (\eta, \eta^w)) &\leq v^T (B^T \otimes I_p) y \\ &= -\|(B^T \otimes I_p) y\|^2 = -z^T z,\end{aligned}\quad (31)$$

by definition of $v = -z$ and $v^w = 0$. Since U is non-negative and non-increasing, then $U(t)$ is bounded. As x^w, η^w are bounded⁴ and U is regular, then x, η are bounded as well. Hence the solutions exist for all t . Integrating the latter inequality we obtain

$$\int_0^{+\infty} z^T(s)z(s)ds \leq U(0).$$

By Barbalat's lemma, if one proves that $\frac{d}{dt}z^T(t)z(t)$ is bounded then one can conclude that $z^T(t)z(t) \rightarrow 0$. Now, $z(t) = (B^T \otimes I_p)y = (B^T \otimes I_p)h(x, w)$ is bounded because x, w are bounded. If h is continuously differentiable and $\dot{x}, \dot{w}$ are bounded, then $\dot{z}$ is bounded and one can infer that $\frac{d}{dt}z^T(t)z(t)$ is bounded. By assumption, w is the solution of $\dot{w} = s(w)$ starting from a forward invariant compact set. Hence, both w and $\dot{w}$ are bounded. On the other hand, $\dot{x}$ satisfies

$$\dot{x} = f(x, (B \otimes I_p)\bar{\psi}(\eta) - z, w)$$

which proves that it is bounded because x, η, z were proven to be bounded, while w is bounded by assumption. Therefore, $\dot{x}, \dot{w}$ are bounded and this implies that $\frac{d}{dt}z^T(t)z(t)$ is bounded. Then by Barbalat's lemma we have $\lim_{t \rightarrow +\infty} z(t) = \mathbf{0}$ as claimed.

Remarkably, the assumptions on the interconnections can be weakened, if stronger assumptions on the node dynamics are imposed.

Corollary 2. Let all assumptions of Theorem 1 hold, but assume furthermore that all node dynamics are output strictly incrementally passive, that is, there exist a C^1 regular storage function V_i , and a positive definite function $\rho_i : \mathbb{R}^p \rightarrow \mathbb{R}$, such that for any two inputs u_i, u'_i and corresponding outputs y_i, y'_i

$$\dot{V} \leq -\rho_i(y_i - y'_i) + (y_i - y'_i)^T(u_i - u'_i). \quad (32)$$

Then the output agreement problem is feasible with the interconnection (30) and $v = \mathbf{0}$.

Proof. Consider the storage function used in the proof of Theorem 1, i.e., $U((x, x^w), (\eta, \eta^w)) = V(x, x^w) + W(\eta, \eta^w)$. After repeating the steps of the proof of Theorem 1, but using the output strict passivity property and setting $v = \mathbf{0}$, (31) is now replaced by $\dot{U}((x, x^w), (\eta, \eta^w)) \leq -\sum_{i=1}^n \rho_i(y_i - y'_i)$, where $y'_i = y_j^w$ for all $i \neq j$. Now, we can proceed as in the proof of Theorem 1.

4. Constructive controller design for a class of nonlinear systems

We present now a fairly large class of heterogeneous nonlinear systems for which all assumptions of Theorem 1 are satisfied and the controller design follows a constructive procedure. Consider the systems introduced in Example 2, namely

$$\begin{aligned} \dot{w}_i &= s_i(w_i) \\ \dot{x}_i &= f_0(x_i) + Gu_i + P_i w_i \\ y_i &= Cx_i \quad i = 1, 2, \dots, n \end{aligned} \quad (33)$$

where, compared with (24), we have chosen the systems to have the same dynamics, i.e. $f_i(x_i) = f_0(x_i)$ for all $i = 1, 2, \dots, n$, and we have set $G_i = C_i^T = G = C^T$. Assuming that the dynamics of the systems are the same facilitate the design of incrementally passive distributed controllers, as we see in the proof below.

Proposition 3. Consider systems (33), where $f_0 = \nabla F$ and F is a twice continuously differentiable and concave map, $G = C^T$ and full column rank matrix, and the maps s_i satisfy (3). Moreover, assume that $\mathcal{R}(P_i) \subseteq \mathcal{R}(G)$ for $i = 1, 2, \dots, n$. Given the vector of disturbances $w = (w_1, \dots, w_n)$, assume there exists a bounded solution x_* to

the system

$$\dot{x} = f_0(x) + \frac{\sum_{i=1}^n P_i w_i}{n}. \quad (34)$$

Then:

1. there exists a bounded solution x^w, u^w to the regulator equations (11);
2. there exist controllers at the edges of the form

$$\begin{aligned} \dot{\eta}_k &= s(\eta_k) + H_k v_k \\ \lambda_k &= H_k^T \eta_k + v_k, \quad k = 1, 2, \dots, m \end{aligned} \quad (35)$$

such that the output agreement problem is solved for the systems (33), interconnected with the controllers (35) via the conditions (30).

Proof. Take any solution x_*^w to (34). By definition

$$\dot{x}_*^w = f_0(x_*^w) + \frac{\sum_{i=1}^n P_i w_i}{n}. \quad (36)$$

Observe that such a solution x_*^w is necessarily bounded. As a matter of fact, in view of the assumptions on f_0 , the incremental dissipation inequality (23) holds and the incremental storage function $V(x_*^w, x_*) = \frac{1}{2}(x_*^w - x_*)^T(x_*^w - x_*)$ satisfies $\dot{V} \leq 0$ (in system (34) inputs are absent). Hence $V(x_*^w, x_*)$ is bounded and by regularity of V and boundedness of x_*, x_*^w is bounded. Define now

$$Gu_i^w = - \left(P_i w_i - \frac{\sum_{i=1}^n P_i w_i}{n} \right). \quad (37)$$

Observe that since $\sum_{i=1}^n Gu_i^w = \mathbf{0}$ by construction, and G is full-column rank, then $u^w \in \mathcal{R}(B \otimes I_p)$, i.e., the requirement imposed by the interconnection condition (30) is fulfilled. An explicit expression for u^w can be given. Let

$$(I_n \otimes G)u^w = \left(\left(\frac{\mathbf{1}_n \mathbf{1}_n^T}{n} - I_n \right) \otimes I_r \right) Pw =: (Y \otimes I_r)Pw$$

where r is the dimension of the state space of each system. Hence (37) can be rewritten as

$$Gu_i^w = \sum_{j=1}^n Y_{ij} P_j w_j, \quad \text{with } Y_{ij} = [Y]_{ij}.$$

There exists a solution u_i^w to the latter equation if and only if $GG^\dagger b_i = b_i$ with $b_i = \sum_{j=1}^n Y_{ij} P_j w_j$, where $G^\dagger$ is the Moore–Penrose pseudo inverse. Recalling that $\mathcal{R}(P_j) \subseteq \mathcal{R}(G)$, we can assume the existence of matrices Γ_j such that

$$P_j w_j = G \Gamma_j w_j.$$

As a result

$$b_i = \sum_{j=1}^n Y_{ij} P_j w_j = \sum_{j=1}^n Y_{ij} G \Gamma_j w_j = G \sum_{j=1}^n Y_{ij} \Gamma_j w_j$$

that is $b_i \in \mathcal{R}(G)$, and $GG^\dagger b_i = GG^\dagger G \sum_{j=1}^n Y_{ij} \Gamma_j w_j = G \sum_{j=1}^n Y_{ij} \Gamma_j w_j = \sum_{j=1}^n Y_{ij} G \Gamma_j w_j = \sum_{j=1}^n Y_{ij} P_j w_j = b_i$. Then the unique solution to (37) is $u_i^w = G^\dagger \sum_{j=1}^n Y_{ij} P_j w_j$. Replacing (37) into (36), the latter becomes

$$\dot{x}_*^w = f_0(x_*^w) + Gu_i^w + P_i w_i.$$

The latter holds true for all $i = 1, 2, \dots, n$ thus showing that $(x^w, u^w) = ((\mathbf{1}_n \otimes I_r)x_*^w, ((u^w)^T, \dots, (u_n^w)^T)^T)$ solves the regulator equations.

⁴ By definition, (w, x^w, η^w) belongs to the ω -limit set, which is compact. Hence, x^w, η^w are bounded.

Bearing in mind that $(B \otimes I_p)\lambda^w = u^w$, we have

$$\begin{aligned}\lambda^w &= -(B^\dagger \otimes I_p)(I_n \otimes G^\dagger)(Y \otimes I_r)Pw \\ &= (I_n \otimes G^\dagger)(B^\dagger Y \otimes I_r)Pw \\ &= (B^\dagger Y \otimes G^\dagger)Pw,\end{aligned}\quad (38)$$

that is $\lambda^w = Hw$. Using the embedding (14) with $\tau = Id$, $\phi = s$ and $\psi(\eta) = H\eta$, and an analogous decomposition as in (16), the internal model controller takes the form

$$\begin{aligned}\dot{\eta}_k &= s(\eta_k) \\ \lambda_k &= H_k^T \eta_k, \quad k = 1, 2, \dots, m.\end{aligned}$$

The addition of the control term $H_k v_k$

$$\begin{aligned}\dot{\eta}_k &= s(\eta_k) + H_k v_k \\ \lambda_k &= H_k^T \eta_k, \quad k = 1, 2, \dots, m,\end{aligned}$$

renders the system incrementally passive, in view of the incrementally passive nature of the map $s(\cdot)$. Recall (see Example 2) that the condition on F that defines the dynamics of the systems according to the identity $f_0 = \nabla F$ guarantees incremental passivity of systems (33). Hence, we are under the conditions of Theorem 1 and one concludes that the controllers

$$\begin{aligned}\dot{\eta}_k &= s(\eta_k) - H_k z_k \\ \lambda_k &= H_k^T \eta_k - z_k, \quad k = 1, 2, \dots, m,\end{aligned}\quad (39)$$

with $z = (B^T \otimes I_p)y$ guarantee that the output agreement problem is solved.

The proof is constructive. The controllers, designed as in (39), provide an explicit solution to the output agreement problem for systems (33). In (39), the vector field s is given by the exosystems generating the disturbances affecting (33), while the input matrix H_k is obtained from the matrix H defined in (38).

The controllers (39) can be stacked together to the dynamics

$$\begin{aligned}\dot{\eta} &= \bar{s}(\eta) - \bar{H}v \\ \lambda &= \bar{H}^T \eta - v,\end{aligned}\quad (40)$$

where $\eta = [\eta_1^T, \dots, \eta_m^T]^T \in \mathbb{R}^{mr}$, $\bar{s}(\eta) = [s(\eta_1)^T, \dots, s(\eta_m)^T]^T$, and $\bar{H} = \text{block.diag}\{H_1, \dots, H_m\}$. Bearing in mind the controllers (40), one conclusion that follows immediately from the proof of the result is that steady state solution of the controllers η^w can be taken as $\eta^w = \mathbf{1}_n \otimes w$. That is, one possible steady state solution of the output agreement problem is that each controller dynamics reproduces exactly the disturbance signal. This observation can be used to redesign the controllers. In particular, additional communication between the different (distributed) controllers can be used to improve the convergence of the controllers.

4.1. Adding communication between controllers

Consider an additional communication network $\mathcal{G}_{\text{comm}}$, having one node for each controller, and one edge if the controllers can exchange data.⁵ For simplicity, we assume that $\mathcal{G}_{\text{comm}}$ is an undirected connected graph with Laplacian matrix $L_{\text{comm}} \in \mathbb{R}^{m \times m}$. As we shall see below, the additional communication term allows us to add a diffusive coupling between the various controllers that explicitly enforces the convergence of all the controllers states η_k to the same signal. This in turn guarantees that the stacked vector $\bar{H}\bar{\eta} = \text{block.diag}\{H_1^T, \dots, H_m^T\}(\eta_1^T \dots \eta_m^T)^T$ converges to $H\eta^*$, for some η^* . We recall that under the conditions that the convergence to the solution of the output agreement problem is uniform in the

initial conditions, such a signal η^* must satisfy $(B \otimes I_p)H\eta^* = u^w$. If in addition the graph is acyclic and the system $\dot{\eta} = s(\eta)$, $\lambda = H\eta$ is incrementally observable,⁶ then necessarily, $\eta^* = w$, i.e. the internal model controllers asymptotically synchronize to the disturbance w .

By revisiting now the proof of Theorem 1, one can directly see that the assumption of incremental passivity of the controllers, i.e., Assumption 3, is stricter than necessary. In particular, one can require the incremental passivity property (26) not to hold with respect to any two trajectories, but only with respect to the real and the steady state trajectory, i.e., with $\eta'_k = \eta_k^w$, $v'_k = 0$, $\lambda'_k = \lambda_k^w$. Thus, one can replace Assumption 3 with the following weaker assumption.

Assumption 3a. Let $\eta^w = \tau(w)$ and $\lambda^w = \psi(\tau(w))$ be a solution to (14), and let $v^w = 0$. For each $k = 1, 2, \dots, m$, there exist regular storage functions $W_k(\eta_k, \eta_k^w)$, with $W_k : \mathbb{R}^{q_k} \times \mathbb{R}^{q_k} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\frac{\partial W_k}{\partial \eta_k} \phi_k(\eta_k, v_k) + \frac{\partial W_k}{\partial \eta_k^w} \phi_k(\eta_k^w, 0) \leq (\lambda_k - \lambda_k^w)^T v_k.$$

It can be readily seen that the proof of Theorem 1 remains valid if Assumption 3 is replaced by Assumption 3a. In particular, the following result holds.

Proposition 4. Let all assumptions of Proposition 3 hold and let $L_{\text{comm}} \in \mathbb{R}^{m \times m}$ be the Laplacian matrix of communication graph. Then the distributed controller with communication of the form

$$\begin{aligned}\dot{\eta} &= \bar{s}(\eta) - (L_{\text{comm}} \otimes I_r)\eta - \bar{H}v \\ \lambda &= \bar{H}^T \eta\end{aligned}\quad (41)$$

interconnected with the node dynamics (33) according to (30), solves the output agreement problem. Furthermore, $\lim_{t \rightarrow \infty} \|\eta_k(t) - \eta_j(t)\| = 0$ for all $k \neq j$.

Proof. Under the assumptions of Proposition 3 it holds that $\eta^w = w$. Consequently $(L_{\text{comm}} \otimes I_r)\bar{\eta}^w = 0$. The controller (41) satisfies Assumption 3a, since the directional derivative of $W = \frac{1}{2}(\bar{\eta} - \bar{\eta}^w)^T(\bar{\eta} - \bar{\eta}^w)$ is

$$\dot{W} \leq -(\eta - \eta^w)^T (L_{\text{comm}} \otimes I_r)(\eta - \eta^w) + (\lambda - \lambda^w)^T v.$$

Mirroring the proof of Theorem 1, the derivative of the storage function $U((x, x^w), (\eta, \eta^w))$ satisfies

$$\dot{U} \leq -\|(B^T \otimes I_p)y\|^2 - \eta^T (L_{\text{comm}} \otimes I_r)\eta.$$

Thus, with the same arguments as in the proof of Theorem 1, convergence can be concluded. Additionally, this proves that $\lim_{t \rightarrow \infty} \|\eta_k(t) - \eta_j(t)\| = 0$ for all $k \neq j$.

5. Time-varying optimal distribution control

We illustrate next the applicability of the proposed controller design method to an (optimal) distribution control problem in networks with storage. This is a simple yet meaningful class of systems for which the theory developed so far provides distributed flow controllers that can be proven to be optimal at steady state.

Consider an inventory system with n inventories and m transportation lines, and let B be the incidence matrix of the transportation network. The dynamics of the inventory system is given as

$$\dot{x} = B\lambda + Pw, \quad (42)$$

⁵ For instance, two controllers can exchange data if their corresponding edges are incident to the same node in the original graph $\mathcal{G}$.

⁶ The system $\dot{\eta} = s(\eta)$, $\lambda = H\eta$ is incrementally observable if any two solutions η, η' to $\dot{\eta} = s(\eta)$ which yield the same output necessarily coincide, i.e. $\eta = \eta'$.

where $x \in \mathbb{R}^n$ represents the storage level, $\lambda \in \mathbb{R}^m$ the flow along one line, and Pw an external in-/outflow of the inventories, i.e., the supply or demand. This basic model is studied, e.g., in Bauso, Blanchini, and Pesenti (2006) and van der Schaft and Wei (2012) or in a discrete-time form in Baric and Borrelli (2012) and Danielson, Borelli, Oliver, Anderson, and Phillips (2013).

We assume here that the exact realization of the supply/demand w is unknown, while it is known that it is generated by a dynamics of the form (2). The *distribution and balancing problem* is to design controllers on the edges of the network, using only measurements of the storage levels of the incident inventories and regulating the flows λ_k such that instantaneously all possible supply/demand is satisfied and all inventory levels evolve synchronously. By choosing $u = B\lambda$, the problem can be readily formulated as an output agreement problem with time varying disturbance.

The regulator equations (13) for the distribution problem are

$$\begin{aligned} \dot{x}^w(t) &= u^w(t) + Pw(t), \\ u^w(t) &\in \mathcal{R}(B), \quad x^w(t) \in \mathcal{N}(B^T). \end{aligned} \quad (43)$$

The solution to the regulator equation (13) is

$$x^w(t) = \mathbf{1}_n \left(x_0^w + \int_0^t \frac{\mathbf{1}_n^T Pw(s)}{n} ds \right) \quad (44)$$

where x_0^w belongs to the projection of $\omega(W \times X \times \mathcal{E})$ onto $\mathbb{R}^{r_1 + \dots + r_n}$. To see (44), note that $u^w(t) \in \mathcal{R}(B) \Leftrightarrow \mathbf{1}_n^T u^w(t) = 0$. Let now $x^w(t) = \mathbf{1}_n x_*^w(t)$, for some $x_*^w(t) \in \mathbb{R}$. Then, multiplying (43) from the left with the all ones vector gives $n\dot{x}^w = \mathbf{1}^T Pw(t)$, leading to the desired expression. The following observation is now a direct consequence.

Proposition 5. *The output agreement problem is feasible only if the accumulated imbalance $\bar{w}(t) = \int_0^t \frac{\mathbf{1}_n^T Pw(s)}{n} ds$ is bounded for all $t \geq 0$.*

Otherwise the inventory levels (i.e., x^w) will grow unbounded. The corresponding input is naturally given as $u^w(t) = -\Delta_n Pw(t)$, with $\Delta_n = (I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T)$, namely the projection of the supply/demand vector to the space orthogonal to $\text{span}\{\mathbf{1}_n\}$. Next, we verify that the necessary conditions for the output agreement problem are satisfied by showing feasibility of (14). Note that the controller output must satisfy $B\lambda^w = -(I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T)Pw(t)$.

Proposition 6. *If the network contains a spanning tree then the condition (14) is feasible.*

Proof. Let $\mathcal{T} \subseteq \mathcal{G}$ be a spanning tree. Assume without loss of generality that the edges are labeled in such a way that the flow vector can be written as $\lambda^w = [\lambda_{\mathcal{T}}^w, \lambda_{\bar{\mathcal{T}}}^w]^T$, where $\lambda_{\mathcal{T}}^w$ are the flows on the edges in $\mathcal{T}$ and $\lambda_{\bar{\mathcal{T}}}^w$ are the flows in all other edges. Similarly, the incidence matrix can be represented as $B = [B_{\mathcal{T}}, B_{\bar{\mathcal{T}}}]$. A feasible flow solution λ^w can now be chosen as $\lambda_{\bar{\mathcal{T}}}^w = 0$ and $\lambda_{\mathcal{T}}^w = -(B_{\mathcal{T}}^T B_{\mathcal{T}})^{-1} B_{\mathcal{T}}^T Pw(t)$. Note that $\lambda_{\mathcal{T}}^w$ routes exactly the balanced component of the supply/demand through the network, since $\lambda_{\mathcal{T}}^w = -(B_{\mathcal{T}}^T B_{\mathcal{T}})^{-1} B_{\mathcal{T}}^T (I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T) Pw(t)$ since $B_{\mathcal{T}}^T \mathbf{1}_n = 0_n$. Define now

$$H = \begin{bmatrix} -(B_{\mathcal{T}}^T B_{\mathcal{T}})^{-1} B_{\mathcal{T}}^T P \\ 0 \end{bmatrix},$$

and note that $\lambda_p^w = Hw$. Thus, $\tau = Id$, $\phi(\cdot) = s(\cdot)$ and ψ being the linear function defined by H , solve (14).

After augmenting the controller with external outputs, a possible routing controller is

$$\begin{aligned} \dot{\eta} &= s(\eta) + H^T v \\ \lambda &= H\eta + v. \end{aligned} \quad (45)$$

Note that if $s(\cdot)$ satisfies the standing assumption (3), then this controller is incrementally passive.

5.1. Optimal distribution control

We enlarge our control objective and aim to design a feedback controller that achieves an *optimal* routing. That is, we want to regulate the flows such that they minimize the quadratic cost function

$$\mathcal{P}(\lambda) = \frac{1}{2} \lambda^T Q \lambda, \quad (46)$$

with $Q = \text{diag}(q_1, \dots, q_m)$ and $q_k > 0$.

We exploit therefore that the internal model controller achieving balancing is not unique. In particular, we redesign the controller (45) in such a way that it routes the balanced component of the flow through the network in such a way that at each time instant the cost (46) is minimized. That is, asymptotically the routing should be such that at each time instant t the following static optimization problem is solved

$$\min_{\lambda} \mathcal{P}(\lambda), \quad \text{s.t. } \mathbf{0} = B\lambda + \Delta_n Pw, \quad (47)$$

where $w = w(t)$ is the supply at the respective time. Let now $\zeta \in \mathbb{R}^n$ be the multiplier for the equality constraint. The Lagrangian function of (47) is

$$\mathcal{L}(\lambda, \zeta) = \frac{1}{2} \lambda^T Q \lambda + \zeta^T (B\lambda + \Delta_n Pw).$$

One can express the optimality conditions in terms of the dual solution as

$$Q\lambda + B^T \zeta = \mathbf{0}, \quad B\lambda + \Delta_n Pw = \mathbf{0},$$

from which $B(Q^{-1} B^T \zeta) + \Delta_n Pw = \mathbf{0}$, with the optimal routing being $\lambda = -Q^{-1} B^T \zeta$. Thus the optimal routing/supply pairs are defined as the set

$$\Gamma = \{(\lambda, w) : Q\lambda \in \mathcal{R}(B^T), B\lambda + \Delta_n Pw = \mathbf{0}\}.$$

We formalize the optimal distribution problem as follows:

Definition 5. The time-varying optimal distribution problem is solvable for the system (42), if there exists a controller (6) such that any solution originating from $W \times X \times \mathcal{E}$ satisfies (i) $\lim_{t \rightarrow \infty} B^T x(t) = \mathbf{0}$ and (ii) $\lim_{t \rightarrow \infty} \text{dist}_{\Gamma}(\lambda(t), w(t)) = 0$.

To solve the problem, we proceed in this way. Instead of designing the controller directly for the flows, we design the controllers for the *multipliers*. We take $\tau = Id$ and $\phi(\cdot) = s(\cdot)$ and design a controller of the form

$$\begin{aligned} \dot{\eta} &= s(\eta) + H_v v \\ \zeta &= H_{\zeta} \eta, \end{aligned}$$

where H_{ζ} and H_v are suitable input and output matrices to be designed next. The routing will then be defined as $\lambda(t) = Q^{-1} B^T \zeta(t)$. For designing H_{ζ} , note that, provided that $v = \mathbf{0}$ and the initial condition $\eta(0)$ is properly chosen, the system above generates the solution $\eta^w(t) = w(t)$. Then H_{ζ} must be designed in such a way that $\zeta^w(t) = H_{\zeta} \eta^w(t)$ satisfies the optimality condition

$$BQ^{-1} B^T H_{\zeta} \eta^w(t) + \Delta_n Pw(t) = \mathbf{0}.$$

The matrix $L_Q = BQ^{-1} B^T$ is a weighted Laplacian matrix. As L_Q has one eigenvalue at zero, with the corresponding eigenvector $\mathbf{1}$, it is not invertible. However, since $\eta^w(t) = w(t)$, one possible solution is

$$H_{\zeta} = -L_Q^{\dagger} P, \quad (48)$$

where $L_Q^{\dagger}$ is the Moore–Penrose-inverse of L_Q , see e.g., Gutman and Xiao (2004). From the properties of $L_Q^{\dagger}$ follows that $BQ^{-1} B^T H_{\zeta} \eta^w + \Delta_n Pw = -BQ^{-1} B^T L_Q^{\dagger} P \eta^w + \Delta_n Pw = -\Delta_n P \eta^w + \Delta_n Pw = \mathbf{0}$ as desired. Now, as the controller should be incrementally passive

with input v and output λ , we can design it in the form (45) taking

$$H = Q^{-1}B^T H_\zeta. \quad (49)$$

Then, to have incremental passivity, we simply choose $H_v = H^T$. This choice of the input and output matrices for the controller (45) ensures that the optimal distribution problem is solved.

Proposition 7. Consider the inventory system (42) with the supply generated by the linear dynamics $\dot{w} = s(w)$, satisfying (3). Consider the controller

$$\dot{\eta} = s(\eta) - H^T z$$

$$\lambda = H\eta - z,$$

with the interconnection condition $z = B^T x$. Then, every solution of the closed-loop system is bounded and (i) $\lim_{t \rightarrow +\infty} B^T x = 0$, and (ii) $\lim_{t \rightarrow +\infty} \text{dist}_r(\lambda(t), w(t)) = 0$, that is the time-varying optimal distribution problem is solvable.

Proof. First note that the optimal routing $\lambda^w(t) = Q^{-1}B^T \zeta^w(t)$ satisfies the identity

$$\begin{aligned} B\lambda^w(t) + Pw(t) &= -BQ^{-1}B^T L_Q^\dagger P\eta^w(t) + Pw(t) \\ &= -\Delta_n P\eta^w(t) + Pw(t) = \mathbf{1}^T Pw(t) \frac{1}{n}. \end{aligned}$$

Since $\mathbf{1}^T Pw(t) = \dot{x}^w$ (see (44)), the optimal routing is such that $\dot{x}^w(t) = B\lambda^w(t) + Pw(t)$.

Now, consider the storage function $U(x - x^w, \eta, \eta^w) = \frac{1}{2}\|x - x^w\|^2 + \frac{1}{2}\|\eta - \eta^w\|^2$ along the solutions of the autonomous system $\frac{d}{dt}(x - x^w) = B(\lambda - \lambda^w) = -BB^T(x - x^w) + BH\eta - BH\eta^w$, $\dot{\eta} = s(\eta) - H^T B^T(x - x^w)$, $\dot{\eta}^w = s(\eta^w)$. It satisfies

$$\begin{aligned} \dot{U} &= -(x - x^w)^T BB^T(x - x^w) + (x - x^w)^T BH(\eta - \eta^w) \\ &\quad + (\eta - \eta^w)^T (s(\eta) - s(\eta^w)) - (\eta - \eta^w)^T H^T B^T x \\ &\leq -\|B^T x\|^2, \end{aligned}$$

due to the incremental passivity of the exosystem, i.e., $(\eta - \eta^w)^T (s(\eta) - s(\eta^w)) \leq 0$. Since U is positive semidefinite and η^w is bounded (again by the incremental passivity property of the exosystem), we have that $x - x^w$, η , η^w are all bounded. Then, by LaSalle's invariance principle, the trajectories converge to the largest invariant set such that $B^T(x - x^w) = B^T x = 0$. Thus, there exists x_* such that on this set $x - x^w = x_* \mathbf{1}$ and the dynamics evolves as

$$\dot{x}_* \mathbf{1} = BH\eta - BH\eta^w, \quad \dot{\eta} = s(\eta), \quad \dot{\eta}^w = s(\eta^w). \quad (50)$$

After multiplying by $\frac{1}{n}\mathbf{1}^T$ from the left, it follows $\dot{x}_* = 0$, proving that x must approach x^w modulo a constant. This proves the claim (i) of the statement.

To prove claim (ii), note that inserting $\dot{x}_* = 0$ into (50) and bearing in mind that $\eta^w = w$ gives the necessary condition that in the set where $B^T(x - x^w) = 0$ it must hold that

$$\begin{aligned} 0 &= BH\eta - BH\eta^w = BH\eta - BHw \\ &= -BQ^{-1}B^T L_Q^\dagger P(\eta - w) = -\Delta_n P(\eta - w). \end{aligned}$$

Hence, $\Delta_n P\eta = \Delta_n Pw$. The flow on the invariant set is $\lambda = Q^{-1}B^T L_Q^\dagger P\eta$, while the optimal flow is $\lambda^w = Q^{-1}B^T L_Q^\dagger Pw$. Together with the previous condition, this implies that there is a vector $v \in \mathcal{N}(B)$ such that $\lambda = \lambda^w + v$. We will show next that v must be identical to zero. Note that $v = \lambda - \lambda^w$, and must therefore satisfy

$$v = Q^{-1}B^T L_Q^\dagger P(\eta - w).$$

Multiplying the previous equation from the left by $v^T Q$ leads to

$$v^T Q v = 0$$

Fig. 2. Inventory network with four inventories and five transportation lines.

Fig. 3. Inventory levels $x_i(t)$, $i \in \{1, 2, 3, 4\}$, under time-varying supply/demand.

since $v^T B^T = 0$. As Q is by assumption positive definite, the only solution is $v = 0$. This proves that in the set where $B^T x = 0$ it must hold that $\lambda = \lambda^w$, completing the proof.

If additionally communication between the controllers is allowed, the controller can be augmented with a consensus term of the form (41).

5.2. Simulation example

We illustrate the performance of the controller on a design example. Consider a network with four inventories and five transportation lines as illustrated in Fig. 2. The supply/demand at each inventory is generated by the linear dynamics

$$\dot{w}_i = \begin{bmatrix} 0 & -s_i \\ s_i & 0 \end{bmatrix} w_i$$

with $s_1 = 0.1, s_2 = 0.7, s_3 = -0.4$ and $s_4 = -0.2$ and initial conditions $w_i(0) = [1, 1]^T$. The flow cost function is of the form (46) with $Q = \text{diag}\{1, 2, 3, 4, 5\}$. The controller is implemented in the distributed form with communication (41), where H is chosen to satisfy (49), and $\mathcal{G}_{\text{comm}}$ is chosen such that two controllers communicate if they are incident to the same inventory.

The simulation results for the inventory levels are shown in Fig. 3. Note that the supply/demand is not balanced, but the accumulated imbalance is bounded. The controller achieves a balancing of the inventory levels. As an example, the flow $\lambda_1(t)$ is shown in Fig. 4. The flow approaches fairly quickly the time-varying optimal flow. The simulations illustrate that the controller achieves both objectives, the balancing of the inventory levels and the optimal routing of the flow through the network.

To illustrate the relevance of the dynamic coupling design, we compare in Fig. 5 the long-time behavior of our controller to an integral routing controller that is designed assuming an unknown but static supply and demand. It was shown in Wei and van der Schaft (2013) that integral controllers can solve distribution problems with static inflows, and in Bürger, Zelazo, and Allgöwer (2013)

Fig. 4. Flow $\lambda_1(t)$ (solid) and optimal flow on the corresponding edge (dashed).

Fig. 5. Long-term inventory levels with the internal model controller (solid, black) and the integral controller (dashed, red).

that they can in this case also solve an optimal distribution problem. Following the design of Wei and van der Schaft (2013) (and combining it with the optimal design proposed in Bürger et al. (2013)) the integral routing controller takes the form (see Bürger et al. (2013) or Wei and van der Schaft (2013))

$$\dot{\eta}_k = v_k, \quad \lambda_k = -q_k^{-1} \eta_k - v_k. \quad (51)$$

Fig. 5 compares the long-term behavior of the integral controller (51) (dashed, red) to the internal model controller (black, solid). One can readily observe that the integral controllers do not achieve a balancing of the inventory levels, while the internal model controller achieves an exact balancing. Note that the integral controller (51) would solve the optimal distribution problem (47) asymptotically exactly if the supply was constant and balanced. However, for the time-varying supply the resulting steady state flow violates the equality constraint of (47). Thus, the integral controller cannot achieve asymptotically an optimal routing.

6. Relation to known results

The presented controller design methodology can be seen as an extension of several existing control approaches for networked systems. We clarify next the relation of the proposed approach to existing ones.

6.1. Static couplings

Significant work has been dedicated to the study of synchronizing oscillatory systems without disturbances using *static diffusive couplings*. In the early work Hale (1997), Hale studied the synchronization of nonlinear oscillatory systems with infinitely strong couplings. Subsequent works studied synchronization of identical nonlinear systems that have a property known as *relaxed cocoerciveness* (DeLellis, Bernardo, & Russo, 2011; Scardovi et al., 2010; Stan & Sepulchre, 2007) or *semi-passivity* (Pogromsky & Nijmeijer, 2001). We show next that, while synchronization of heterogeneous systems with external disturbances requires in general dynamic couplings, also in our framework static diffusive couplings are sufficient if all systems already share a common *internal model*.

Proposition 8 (Static Coupling). *Consider the system (4) and suppose all node dynamics are incrementally passive. If there exists a solution to the regulator equations (11) with $u^w(t) = 0$, then, the static controller $\lambda = v$ with the interconnection $u = (B \otimes I_p)\lambda$, and $v = -(B^T \otimes I_p)y$ solves the output agreement problem.*

Proof. By the incremental passivity property of the subsystems it is true that $\dot{V} \leq (y - y^w)^T(u - u^w)$, where $V = \sum_{i=1}^n V_i$. Now, since $u^w = 0$ and $y^w \in \mathcal{N}((B^T \otimes I_p))$ the coupling $u = -Ly = -(B \otimes I_p)(B^T \otimes I_p)y = -(B \otimes I_p)(B^T \otimes I_p)(y - y^w)$ gives $\dot{V} \leq -(y - y^w)^T L(y - y^w)$. Convergence and boundedness can now be shown as in the proof of Theorem 1.

Please note that the input to the systems computes in this case as $u = -(L \otimes I_p)y$, where $L = BB^T$ is the *Laplacian matrix* of the (undirected) graph. Thus, for homogeneous systems, our controller design method reduces to the well-known Laplacian coupling, as studied, e.g., in DeLellis et al. (2011), Scardovi et al. (2010) and Stan and Sepulchre (2007).

6.2. Static disturbances

For the special case of output agreement problems with *constant disturbances*, our approach provides an interpretation of integral controllers that have been intensively studied in various different contexts (Bürger et al., 2013, 2014; Simpson-Porco et al., 2013; van der Schaft & Wei, 2012; Wei & van der Schaft, 2013). Control of passive system with constant disturbances is studied, e.g., in Hines, Arcak, and Packard (2011) or Jayawardhana, Ortega, Garcia-Canseco, and Castanos (2007). The stability of passive networks with static disturbance signals has been discussed in van der Schaft and Wei (2012). We derive here slightly more general controllers (or dynamic couplings) as Bürger et al. (2014) and Bürger et al. (2013), using the internal model control approach.

Proposition 9. *Consider the network $\mathcal{G}$ with dynamics on the nodes (4). Suppose w_i is some constant signal, i.e., $s_i(w_i) = 0$, the regulator equations (11) hold and (4) are incrementally passive. Then, any controller of the form*

$$\begin{aligned} \dot{\eta}_k &= v_k \\ \lambda_k &= \psi_k(\eta_k) + v_k, \quad k \in \{1, \dots, m\} \end{aligned} \quad (52)$$

with $\psi_k(\cdot)$ satisfying the strong monotonicity condition

$$(\psi_k(\eta) - \psi_k(\eta'))^T(\eta - \eta') \geq c \|\eta - \eta'\|^2, \quad \forall \eta, \eta' \quad (53)$$

for some positive constant c , and interconnection constraints (30) solves the output agreement problem.

Proof. Let x^w and u^w be solutions to the regulator equations (11). By the structure of (11) follows immediately that $v^w = -(B^T \otimes I_p)h(x^w, w) = 0$. Since the disturbance is static, i.e., $\dot{w} = 0$, the conditions (14) are solved with $\phi(\cdot) = 0$ and $\tau(w)$ such that

$$\lambda_p^w + \lambda_0^w = \psi(\tau(w)) \quad (54)$$

for some λ_p^w satisfying $u^w = (B \otimes I_p) \lambda_p^w$ and some $\lambda_0^w \in \mathcal{N}(B \otimes I_p)$ and constant. Thus, there is not a unique solution to (14), but rather for any $\lambda_0^w \in \mathcal{N}(B \otimes I_p)$ there exists exactly one $\tau(w)$ solving (14) (the existence of more than one solution $\tau(w)$ would contradict the strong monotonicity condition (53)). Select now $\eta^w = \tau(w)$ as a solution to (54) an arbitrary $\lambda_0^w \in \mathcal{N}(B \otimes I_p)$, and let $\lambda^w = \lambda_p^w + \lambda_0^w$.

One can construct now for each controller a storage function W_k that satisfies Assumption 3a, i.e., that shows passivity with respect to the constant signals λ_k^w and $v_k^w = 0$. Let in the following $\Psi_k : \mathbb{R}^q \rightarrow \mathbb{R}$ be a twice continuously differentiable function such that $\nabla \Psi_k(\eta_k) = \psi_k(\eta_k)$. Since, by assumption, ψ_k satisfy the monotonicity condition (53) all Ψ_k are strongly convex. Consider now the storage function (Bürger et al., 2014; Jayawardhana et al., 2007):

$$W_k(\eta_k, \eta_k^w) = \Psi_k(\eta_k) - \Psi_k(\eta_k^w) - \nabla \Psi_k^T(\eta_k^w)(\eta_k - \eta_k^w). \quad (55)$$

Since Ψ_k is convex and, by the global under-estimator property of the gradient, we have $\Psi_k(\eta_k) \geq \Psi_k(\eta_k^w) + \nabla \Psi_k^T(\eta_k^w)(\eta_k - \eta_k^w)$ for each η_k, η_k^w . Since Ψ_k is strongly convex, then it is in particular strictly convex and the previous inequality holds if and only if $\eta_k = \eta_k^w$. Then W_k is regular (Jayawardhana et al., 2007). Hence, W_k is a positive regular storage function. Furthermore,

$$\begin{aligned} \frac{\partial W_k}{\partial \eta_k} \phi_k(\eta_k, v_k) &= (\psi_k(\eta_k) - \psi_k(\eta_k^w))^T v_k \\ &= (\lambda_k - \lambda_k^w)^T v_k. \end{aligned}$$

In the case of constant disturbances Assumption 3a is always fulfilled by controllers of the form (52). Mimicking now the proof of Theorem 1, using the storage function $U((x, x^w), (\eta, \eta^w)) = V(x, x^w) + W(\eta, \eta^w)$, with $W(\eta, \eta^w) = \sum_{k=1}^m W_k(\eta_k, \eta_k^w)$, one obtains $\dot{U} \leq -\|(B^T \otimes I_p)y\|^2$. With the same arguments as used in the proof of Theorem 1, it follows that the controller (52) solves the output agreement problem in the case of static disturbances, that is $\lim_{t \rightarrow \infty} (B^T \otimes I_p)y(t) = 0$.

7. Power systems droop-control as internal model control

The previous discussion suggests, that the incremental passivity approach can provide novel insights into existing results. In the same direction, we show next that a dynamic oscillatory model of microgrids with frequency-droop controllers studied in Simpson-Porco et al. (2013) can be re-interpreted in the internal model framework. This re-interpretation has several merits. It unveils the framework underlying some recent synchronization results in oscillatory networks, thus opening a new perspective on the analysis (and control) of these networks. Moreover, it provides an interesting case study that has attracted a great deal of attention and to which our approach conveniently applies.

The model of Simpson-Porco et al. (2013) for the frequency-droop controller is

$$D_i \dot{\theta}_i = P_i^* - P_{e,i}, \quad i \in \{1, \dots, n\}, \quad (56)$$

where D_i is the inverse of the controller gain, P_i^* is the inverters nominal power, and $P_{e,i}$ is the active electric power. The active electric power is given by

$$P_{e,i} = \sum_{j=1}^n \alpha_{ij} \sin(\theta_i - \theta_j), \quad (57)$$

where α_{ij} are constants depending on the node voltages and the line admittance. The coefficients are symmetric $\alpha_{ij} = \alpha_{ji}$ and only non-zero if the two nodes i and j are connected by a line. We refer to Simpson-Porco et al. (2013) for a detailed discussion of the model. As in Simpson-Porco et al. (2013), we restrict the discussion in the following to *acyclic* networks.

In the model, the dynamics in the nodes (56) represents the controllers, while the couplings between the nodes, i.e., (57), are physical laws. Although the situation is reversed to the basic setup of this paper, we can still interpret the droop-controller as an internal model controller. Consider the node dynamics (1) as (56), with node state $x_i = \theta_i$, constant external signal $w_i = P_i^*$, satisfying $\dot{w}_i = 0$, input $u_i = -P_{e,i}$ and output $y_i = \dot{\theta}_i$, i.e.,

$$D_i \dot{x}_i = P_i^* + u_i, \quad y_i = \dot{x}_i. \quad (58)$$

The node dynamics is *output strictly incrementally passive* since for any two inputs u_i, u_i' and the two corresponding outputs y_i, y_i' , it holds that

$$\begin{aligned} (y_i - y_i')(u_i - u_i') &= (y_i - y_i')(D_i y_i - P_i^* - D_i y_i' + P_i^*) \\ &= D_i \|y_i - y_i'\|^2. \end{aligned}$$

From the interpretation of the node dynamics (1) as (56), one notices that $u = -P_e = -BA \sin(B^T x)$, where $\sin(z) = [\sin(z_1), \dots, \sin(z_m)]^T$, $A = \text{diag}\{a_1, \dots, a_m\}$, $a_k = \alpha_{ij} = \alpha_{ji}$, and k is the label of the edge connecting nodes i, j . One can then interpret the latter equation as the first one of the interconnection conditions (30), provided that $\lambda = A \sin(B^T x)$. Set now $\eta = B^T x$. Then

$$\begin{aligned} \dot{\eta} &= -B^T y \\ \lambda &= A \sin(\eta). \end{aligned} \quad (59)$$

This can be understood as the stacked controllers (25), where, for all k , $\phi_k(\eta_k, v_k) = v_k$, $v = B^T y$, $\psi_k(\eta_k) = a_k \sin(\eta_k)$. Hence, rewriting the model (56)–(57) in this way leads directly to an interpretation as an internal model control loop of the form (52), where the feed-through term can be omitted, i.e., $v = 0$, since the node dynamics is output strictly incrementally passive (see Corollary 2). We can now restate the result of Simpson-Porco et al. (2013) in the context of internal model control.

Proposition 10. Consider the droop-controller dynamics in the form (58) and (59) and let the underlying network $\mathcal{G}$ be acyclic. Then

1. the regulator equations (11) are solved by $\dot{x}_i^w = \frac{\sum_{i=1}^n P_i^*}{\sum_{i=1}^n D_i} =: y^w$ and $u_i^w = D_i \frac{\sum_{i=1}^n P_i^*}{\sum_{i=1}^n D_i} - P_i^*$ for all $i = 1, 2, \dots, n$;
2. the embedding condition (14) is feasible if and only if $\|A^{-1}(B^T B)^{-1} B^T u^w\|_\infty < 1$;
3. if the necessary conditions (11) and (14) hold, the solutions to the closed loop dynamics (58) and (59) with interconnection $u = B\lambda$ and that originate sufficiently close to x^w and $\eta^w := \sin^{-1}(A^{-1}(B^T B)^{-1} B^T u^w)$ satisfy $\lim_{t \rightarrow \infty} \|y_i - y^w\| \rightarrow 0$.

The proof follows completely along the lines of the internal model control approach (except the local nature of the stability result) and exploits in particular the results for static disturbances of Section 6.2. For completeness, we provide the proof in the Appendix.

8. Conclusions

The paper has investigated output agreement problems in the presence of time-varying disturbances, with a particular focus on the role of dynamic internal-model-based controllers. The proposed methodology is applicable to a variety of problems, including time-varying distribution control and frequency droop control, and lends itself to several possible extensions. In many other distribution networks, similar to the inventory system, the constraints imposed by the network induce a non-unique solution such that it becomes meaningful to design controllers with optimal features. Our approach naturally lends itself to providing such solutions. Moreover, the potentials of our approach

in the context of the two case studies have not been fully explored yet. Phenomena to be studied are for instance the presence of constraints on the input and state variables. For the case of power systems, other classes of controllers could be considered, dealing for instance with the presence of time-varying exogenous inputs.

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Appendix

A.1. Proof of Proposition 2

Proof (Necessity). If the rank condition in (22) is violated then there exists a nonzero vector (x_0, λ_0) such that

$$\begin{bmatrix} \bar{A} - sI_r & \bar{G}(B \otimes I_p) \\ (B \otimes I_p)^T \bar{C} & \mathbf{0}_{np \times np} \end{bmatrix} \begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix} = \mathbf{0}. \quad (\text{A.1})$$

This equality can be made explicit as

$$\begin{aligned} (\bar{A} - sI_r)x_0 + \bar{G}(B \otimes I_p)\lambda_0 &= \mathbf{0} \\ (B \otimes I_p)^T \bar{C}x_0 &= \mathbf{0}. \end{aligned} \quad (\text{A.2})$$

As $s \notin \sigma(\bar{A})$, then necessarily $\lambda_0 \neq \mathbf{0}$.

The graph $\mathcal{G}$ may or may not contain a cycle. If it does, then statement (1) of the thesis holds. If not, then $(B \otimes I_p)\lambda_0 \neq \mathbf{0}$ (recall that $\lambda_0 \neq \mathbf{0}$). Since s is not an eigenvalue of any A_i , then from (A.2)

$$\begin{aligned} x_0 &= (sI_r - \bar{A})^{-1} \bar{G}(B \otimes I_p)\lambda_0 \\ (B \otimes I_p)^T \bar{C}(sI_r - \bar{A})^{-1} \bar{G}(B \otimes I_p)\lambda_0 &= \mathbf{0}. \end{aligned}$$

The latter shows that $\mathcal{R}(\bar{H}(s)(B \otimes I_p)) \cap \mathcal{N}((B \otimes I_p)^T) \neq \{\mathbf{0}\}$, that is statement (2) of the thesis.

(Sufficiency) If $\mathcal{G}$ contains cycles, then there exists $\lambda_0 \neq \mathbf{0}$, such that $(B \otimes I_p)\lambda_0 = \mathbf{0}$. Consider now the nonzero vector $(x_0, \lambda_0) = (\mathbf{0}, \lambda_0)$. The product

$$\begin{bmatrix} \bar{A} - sI_r & \bar{G}(B \otimes I_p) \\ (B \otimes I_p)^T \bar{C} & \mathbf{0}_{mp \times mp} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \lambda_0 \end{bmatrix}$$

returns a zero vector. This shows that the null space of the matrix on the left-hand side is non-trivial, that is condition (22) is violated.

If $\mathcal{R}(\bar{H}(s)(B \otimes I_p)) \cap \mathcal{N}((B^T \otimes I_p)) \neq \{\mathbf{0}\}$, then there exists a vector $\lambda_0 \neq \mathbf{0}$ such that

$$(B^T \otimes I_p)\bar{H}(s)(B \otimes I_p)\lambda_0 = \mathbf{0}.$$

By definition of $\bar{H}(s) = \bar{C}(sI_r - \bar{A})^{-1}\bar{G}$, the latter equality becomes $(B^T \otimes I_p)\bar{C}(sI_r - \bar{A})^{-1}\bar{G}(B \otimes I_p)\lambda_0 = \mathbf{0}$.

Define $x_0 = (sI_r - \bar{A})^{-1}\bar{G}(B \otimes I_p)\lambda_0$. Then

$$\begin{aligned} (B \otimes I_p)^T \bar{C}x_0 &= \mathbf{0} \\ (\bar{A} - sI_r)x_0 + \bar{G}(B \otimes I_p)\lambda_0 &= \mathbf{0}, \end{aligned}$$

which is the same as (A.1). But this shows that condition (22) is violated.

A.2. Proof of Corollary 1

Proof. Under the given assumptions, Eqs. (21) are feasible if and only if Condition 2 in Proposition 2 does not hold. Suppose, by contradiction, that $H(s)$ is strictly positive real and Condition 2 in Proposition 2 holds. The condition can equivalently be expressed as follows: there exist vectors $v \in \mathbb{R}^{np}$ and $\mathbf{0} \neq \beta \in \mathbb{R}^p$ such that

$$\bar{H}(s)(B \otimes I_p)v = \mathbf{1}_n \otimes \beta, \quad \forall s \in \sigma(\bar{S}).$$

Multiplying the previous condition from the left by $v^T(B^T \otimes I_p)$ leads to

$$v^T(B^T \otimes I_p)\bar{H}(s)(B \otimes I_p)v = 0, \quad \forall s \in \sigma(\bar{S}).$$

Since all $s \in \sigma(\bar{S})$ have zero real part, is equivalent to $\tilde{v}^T(\bar{H}(j\omega) + \bar{H}(-j\omega))\tilde{v} = 0$ for all $\tilde{v} = (B \otimes I_p)v$ and for some $\omega \in \mathbb{R}$.

This is a contradiction since $H(s)$ being strictly positive real implies that $(\bar{H}(j\omega) + \bar{H}(-j\omega))$ is positive definite for all $\omega \in \mathbb{R}$. This proves the statement.

A.3. Proof of Proposition 10

Proof. The first statement follows directly after summing all Eqs. (58) and noting that there must be a scalar valued function $y^* \in \mathbb{R}$ such that $y_i = \dot{x}_i^w = y^*$ for all $i \in \{1, \dots, n\}$.

To prove the second statement, we choose $\phi(\tau(w)) = 0$ since $\dot{w} = 0$. Now, note that $u^w = B\lambda^w$, and since the network is acyclic, λ^w is uniquely defined as $\lambda^w = (B^T B)^{-1}B^T u^w$. Since for an acyclic network $\mathcal{N}(B) = \{\mathbf{0}\}$, the second condition in (14) becomes $(B^T B)^{-1}B^T u^w = \psi(\tau(w))$, where we can take $\psi(\tau(w)) = A \sin(\tau(w))$. Thus, we have

$$\tau(w) = \sin^{-1}(A^{-1}(B^T B)^{-1}B^T u^w),$$

which exists if and only if $\|A^{-1}(B^T B)^{-1}B^T u^w\|_\infty < 1$.

To prove local stability we use the standard storage function (55). Note that it can be defined with $\Psi_k(\eta_k) = a_k(1 - \cos(\eta_k))$. If the conditions (14) hold, choose $\eta^w = \tau(w)$. Stability follows now with the storage function $U(x, x^w, \eta, \eta^w) = \sum_{i=1}^n V_i(x_i, x_i^w) + \sum_{k=1}^m W_k(\eta_k, \eta_k^w)$, with $V_i(x_i, x_i^w) = 0$ and W_k defined as (55). Note that U is positive semidefinite in a neighborhood around x^w and η^w and such that $\eta, \eta^w \in (-\frac{\pi}{2}, \frac{\pi}{2})^m$, and satisfies

$$\begin{aligned} \dot{U} &\leq - \sum_{i=1}^n D_i \|y_i(t) - y_i^w(t)\|^2 \\ &= - \sum_{i=1}^n D_i^{-1} \left\| \sum_{k=1}^m b_{ik} a_k (\sin(\eta_k(t)) - \sin(\eta_k^w(t))) \right\|^2, \end{aligned} \quad (\text{A.3})$$

due to output strict incremental passivity of (58). Note that the latter inequality involves only the variables η, η^w . Hence, it shows that the trajectories of the closed-loop system $\dot{\eta} = -B^T y = -B^T D^{-1}(P^* + BA \sin(\eta))$ are bounded and converge to the set of points where $BA \sin(\eta) = BA \sin(\eta^w)$ (i.e., to the set of points where $\sin(\eta) = \sin(\eta^w)$, since the graph has no cycles and A is a diagonal matrix) or, equivalently, to the set of points where $y_i = y^w = \dot{x}_i^w$ for all i . Thus, any trajectory originating sufficiently close to x^w and η^w satisfies $\lim_{t \rightarrow \infty} \|y_i - y^w\| \rightarrow 0$.

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