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Gradient Flows in QCD-like Theories

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Gradient Flows in QCD-like Theories

Marco Boers

Van Swinderen Institute PhD series 2021

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Gradient Flows in QCD-like Theories

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Chapter 1

Introduction

The importance of non-Abelian gauge theories in modern particle physics is hard to overstate. First and foremost, two of the three fundamental Standard Model (SM) interactions — the weak and strong — are described using this type of quantum field theory. The theory that describes the latter of these two interactions is based on the gauge group $SU(3)_c$, with the c -subscript standing for *color*, and is known as *quantum chromodynamics* (QCD). The associated gauge bosons, referred to as *gluons*, interact both with the *quarks* — the fundamental fermionic constituents of hadrons, such as protons and neutrons — and among themselves. Generalizations of QCD, based on the gauge group $SU(N)$ for N colors and any number of fermions in any of its representations, are the focus of this thesis — though we assume the fermions to be in the fundamental representation of the gauge group unless explicitly stated otherwise. We refer to these theories as *QCD-like* theories, distinguishing them from non-Abelian gauge theories containing elementary scalars.

Wildly successful as it is in describing nearly all experimentally observed elementary particle physics phenomena [1], there are multiple indications that the Standard Model is incomplete and should be thought of as the low-energy effective field theory realization of a more fundamental theory. Hence, a lot of effort has been put in formulating beyond-the-Standard-Model (BSM) theoretical descriptions. An appealing concept is the high-energy unification of the fundamental forces, and also here non-Abelian gauge theories play a crucial

role: the assumption is that the three Standard Model gauge groups¹ are resulting from the spontaneous breaking of a larger gauge group, for example $SU(5)$ or $SO(10)$. Often combined with supersymmetry, these types of theories aim to explain the phenomena not described by the Standard Model, such as neutrino oscillations and dark matter. This thesis will not treat the phenomenology further, but rather continues exploring the theoretical foundations of non-Abelian gauge theories and their modifications.

In the absence of matter fields, non-Abelian gauge theories are referred to as *pure Yang-Mills* theories. Their two most important features are *asymptotic freedom* and *color confinement*. The former concerns the asymptotic vanishing of the renormalized gauge coupling at asymptotically high energies, which results from the anti-screening effect due to the self-interactions of the gluons, and has the consequence that perturbation theory is reliable at high energies. In QCD-like theories, the screening effect of the fermionic fields counteracts this anti-screening due to the gluonic self-interactions, though the matter content of real-world² QCD is such that asymptotic freedom persists. Color confinement refers to the fact that physical states are exclusively color-charge neutral. Even though all evidence, both experimental and from lattice simulations, verifies this property, the theoretical derivation from first principles for generic non-Abelian gauge theories — and pure Yang-Mills theories in particular — remains elusive³.

Another essential aspect is the breaking of chiral symmetry in the presence of fermionic matter. At least perturbatively, i.e. barring the $U(1)$ axial anomaly, the QCD-like interactions respect all continuous global symmetries among the different flavors of quarks, some of which are only approximate when quark masses are taken into account — we will work with massless fermions throughout this thesis. However, when the number of massless fermionic fields is sufficiently small, the exact global axial flavor symmetries, acting differently on left- and right-handed quarks, are spontaneously broken. Nonvanishing quark masses induce an additional explicit breaking of these spontaneously broken symmetries, and in real-world QCD the associated pseudo-Goldstone bosons

¹These are the $SU(3)$ color, the $SU(2)$ weak isospin and $U(1)$ weak hypercharge groups [1].

²Real-world QCD has $N = 3$ colors and $N_f = 6$ flavors of massive quarks, known as (from light to heavy): up, down, strange, charm, bottom/beauty and top, whose masses cover several orders of magnitude: $m_{u,d} \sim 1 \text{ MeV}$, $m_s \sim 10^2 \text{ MeV}$, $m_{c,b} \sim 10^3 \text{ MeV}$, $m_t \sim 10^5 \text{ MeV}$ [1]. The number of active flavors hence depends on the energy at which the theory is probed.

³Recently, progress has been made in this direction in the large- N limit of $SU(N)$ Yang-Mills theory [2].

Figure 1.1: Qualitative partly-conjectured phase diagram of real-world QCD in the temperature (T) and baryon chemical potential (μ_B) plane. First-order phase transitions are represented by solid black lines, which end in critical points (black dots). At the crossover ($T \sim 150$ MeV) only the three lightest quark flavors are active — similarly at the gray area, where the precise structure of the phase diagram is still uncertain. Color superconductivity is theoretically conjectured and studied for low T and asymptotically high μ_B . For an in-depth review of the QCD phase diagram see [4].

are identified as the pions. Whenever the masses of all quarks are equal — be it zero or nonzero — the vector flavor symmetries, acting the same on left- and right-handed quarks, remain unbroken. This implies conservation and non-renormalization of the Noether currents associated with these flavor symmetries — we will come back to the highly non-trivial connection between conservation and nonrenormalization of currents later on.

When also taking into account nonzero temperature and density, QCD-like theories reveal a rich phase diagram. In particular, for vanishing baryon chemical potential μ_B and temperatures around 150 MeV [3] (the sun’s core temperature is roughly 1 keV), real-world QCD undergoes a crossover, see figure 1.1. The resulting high-temperature phase, describing the state of quarks and gluons in the post-inflationary universe up until around 10^{-6} seconds after the Big Bang,

Figure 1.2: Qualitative phase diagram in the plane of the temperature (T) and number of massless fermions in the fundamental representation (N_f) for QCD-like theories, given any number of colors N . The upper edge of the conformal window is located at $N_f^{AF} = \frac{11}{2}N$. The curve separating the hadronic and the quark-gluon plasma phases is the chiral phase boundary, whose endpoint can be used to determine the location of the lower edge of the conformal window [7].

is known as the quark-gluon plasma, and is studied in the laboratory through heavy-ion collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). Another interesting phase of QCD is expected to be found in the cores of neutron stars, where the temperature is relatively low ($\sim 10^2$ eV), and the density high — the gray area in figure 1.1. Models describing this phase remain of a speculative nature due to the lack of experimental data, though astrophysical observations do provide some information, with the exciting prospect of a large amount of gravitational-wave data in the near future [5, 6].

By varying the number of massless fermionic matter fields N_f at zero temperature and vanishing baryon chemical potential, more phases emerge, as illustrated in figure 1.2. For relatively few massless fundamental fermions, QCD-like theories with N colors still exhibit asymptotic freedom, chiral symmetry is spontaneously broken, and color is confined. However, when increasing the matter content keeping the number of colors fixed, at some point the *conformal window* opens up, see figure 1.2. The conformal window is the range of values for N_f for

which there exists a nontrivial infra-red (IR) fixed point for the renormalized gauge coupling, which in that case converges to a finite nonzero value for asymptotically low energies. The upper edge of the conformal window is the value of N_f for which asymptotic freedom is lost — i.e. the theory becomes IR free — and is located at the value $N_f^{AF} = \frac{11}{2}N$ for fundamental fermions.⁴ In the Veneziano limit, consisting of taking both $N_f \rightarrow \infty$ and $N \rightarrow \infty$ while keeping the ratio N_f/N fixed, one can consider the ratio N_f/N such that the theory is inside the conformal window and infinitesimally close to the upper edge. There, the existence of the nontrivial IR fixed point can be established exactly within perturbation theory [8,9]. At any other location inside the window, and beyond the Veneziano limit, nonperturbative methods are required — for example, for establishing the existence of the nontrivial IR fixed point for $SU(3)$ with $N_f = 12$, numerical studies were conducted where its existence was inferred from the direct observation of the flattening of the running coupling [10,11], signatures of phase transitions were examined in a thermodynamic study [12], thereby providing rather complete evidence for its existence, and the spectrum of the theory was analyzed [13], solidifying the evidence even further. In particular, finding the location of the lower edge is a nonperturbative problem. In supersymmetric non-Abelian gauge theories it can be analytically determined, making use of the additional R -symmetry — in $\mathcal{N} = 1$ SQCD this is a global $U(1)_R$ group, which also plays a crucial role in establishing the electromagnetic duality in the supersymmetric conformal window [14]. Similar analytical methods are not available in generic nonsupersymmetric QCD-like theories⁵, though information about the lower edge's location can still be obtained through nonperturbative techniques, such as the formulation of the theory on a Euclidean spacetime lattice, and subsequently performing simulations [7].

These lattice simulations can be viewed as consisting of the following steps. First, the theory is formulated in a finite Euclidean spacetime box, and spacetime is discretized, i.e. a nonzero minimal distance between sites of the lattice exists — the lattice spacing — which acts as an ultra-violet (UV) cutoff of the

⁴ The beta function captures the energy-scale dependence of the running gauge coupling. In perturbation theory it is a power series in the coupling, and the sign (+/−) of its first nonzero coefficient determines whether the theory is IR (+) or UV (−) free. In QCD-like theories, the leading order coefficient is negative for $N_f < N_f^{AF}$, and vanishes at $N_f = N_f^{AF}$, where the next-to-leading order coefficient is positive, and the theory is IR free. See also section 3.4.

⁵However, see [15] for a nonsupersymmetric nonperturbative analytical study working in the Veneziano limit, where the lower edge is determined to be located at $N_f/N = \frac{5}{2}$.

Figure 1.3: A local (composite) operator $O(x)$ consists of elementary fields — and possibly derivatives thereof — at the spacetime point x . When evolved by the gradient flow, the smeared operator $O(t, x)$ now depends on the elementary fields in a spacetime region of mean-square radius $\sqrt{8t}$ around the point x . The sensitivity to the elementary fields away from x decreases exponentially.

theory. Next, Monte-Carlo methods and related algorithms are used to simulate the theory on this finite lattice. Finally, the obtained results have to be extrapolated to continuum spacetime — i.e. removing the UV cutoff — and infinite volume. With the rapid rise of computational power, lattice simulations have become an increasingly important nonperturbative calculational method in QCD and related theories [1, 16]. A particular tool, known as the *gradient flow*, has become progressively present throughout these studies during the last decade, though many aspects of its application to numerical studies need further investigation.

The gradient flow

The gradient flow consists of a diffusion equation which *smears* fields over a region in spacetime, whose mean-square radius is proportional to the flow time t , which has dimensions of a length squared, see figure 1.3. Its smoothing feature makes it a useful tool in quantum field theories, where by locally modifying the short-distance properties of the theory, the flow is able to provide specific insights.

Focusing on non-Abelian gauge theories, the simplest gradient flow of the gauge

field is called the *Yang-Mills gradient flow*⁶. It is defined by taking the gradient in gauge-field space of the Yang-Mills action, thereby resulting in a flow that drives the gauge field towards the stationary points of this action along the direction of steepest descent. It was first formulated in the classical — as opposed to quantum — Yang-Mills theory in a purely mathematical context [17, 18], where it found its use in the study of four-manifolds⁷. Two decades later in 2006, the first quantum field theory application followed, when a lattice formulation of the flow was used in the large- N study of Yang-Mills theory [26]. The quantum properties of the flow in continuum spacetime were explored later in 2010 in [27], where the one-loop renormalization of the gradient-flow evolved Yang-Mills Lagrangian density in the Wilsonian normalization was determined. This latter quantity is the simplest purely-gluonic gauge-invariant observable, and it is the continuum limit of the *plaquette*, the fundamental gauge building block in lattice studies. A systematic analysis of the perturbative all-order renormalization properties of gradient-flow evolved local operators — both elementary and composite — followed [28, 29], where it was established that condensates and correlation functions of gradient-flow evolved purely-gluonic operators are finite, i.e. they do not require any UV-divergent renormalization.

There exists a close relation between the gradient flow and stochastic quantization of Euclidean quantum field theories [30], where fields are coupled to a Gaussian-noise term, causing them to wander randomly on their respective manifolds, in accordance with their corresponding Langevin equation. In the equilibrium limit — after an infinite amount of wandering-time — stochastic averages are equivalent to ordinary Euclidean vacuum expectation values. Notably, the Yang-Mills gradient-flow equation for gauge fields coincides with the Langevin equation for the stochastic quantization of a Yang-Mills theory with the Gaussian-noise term removed, supplemented with the appropriate boundary conditions.

Up to now we have left out the fermionic degrees of freedom from our discussion of the gradient flow — in fact, we can study QCD-like theories while solely evolving the gauge fields. We can additionally evolve the fermion fields according to their own gradient-flow equations [31]. In the latter case, the renormalization

⁶In the context of lattice applications it is also referred to as the *Wilson flow* or the *Symanzik flow*, depending on which lattice action is used to define the flow equation.

⁷A similar diffusion equation, the *Ricci flow* [19], was extensively used to prove the Poincaré conjecture [20–24], dealing with the topology of three-manifolds. For applications of the Ricci flow in physics see [25].

properties of correlation functions containing these evolved fermions are altered, though not in the same way as with the gauge fields — it was found that divergences still appear in contrast to the evolved purely-gluonic case. Hence, the evolved elementary fermions still require field-renormalization factors, though these differ from their nonevolved counterparts. The remarkable difference, however, is in the renormalization of composite operators. In the nonevolved case, all the different fermion-bilinear operators require their own unique divergent subtractions, see e.g. [32], with the important exception of conserved flavor currents, which are finite. When evolving the two constituent fermion fields of the bilinears, the resulting operators renormalize with a single universal renormalization factor for all evolved fermion-bilinear operators. More generally, any evolved composite operator — possibly also containing evolved gauge fields — straightforwardly renormalizes with the product of the renormalization factors of its evolved constituent fermion fields, no additional renormalization due to the compositeness is required.

Lattice applications

Though the work presented in this thesis is solely concerned with continuum quantum field theories, due to the importance and multitude of applications of the gradient flow in lattice studies, we want to mention some of these applications here.

First of all, the Yang-Mills gradient flow is widely used for scale setting, in which the value of the lattice spacing is determined as a function of the bare gauge coupling. This is indispensable for making continuum extrapolations and for determining dimensional quantities in physical units. In [33] different scale setting methods — including and emphasizing the Yang-Mills gradient flow — are reviewed and critically compared.

Besides scale setting, the Yang-Mills gradient flow is used to determine topological properties of lattice simulations [27, 34], though we should caution the reader by noting that there are recent studies showing that the gradient flow can alter the topological charge of a system by promoting small fluctuations to instanton-like objects [35]. For a comparison with cheaper ‘cooling’ methods, see [36].

Lattice simulations are used to nonperturbatively determine the running of the gauge coupling, employing step-scaling and rather established and well-tested

methods such as the Schrödinger functional [37]. More recently, the gradient flow is also combined with step-scaling [38–40]. This approach has resulted in confusion, ranging from comparable studies yielding contradictory results — e.g. see [41] and [42] — to publications failing to reproduce the universal two-loop running of the perturbatively determined coupling, the minimum requirement for the QCD running coupling. Related to this last point, see for example [43], where the there-determined gradient-flow running coupling does not have any overlapping region with the perturbatively determined coupling, and hence there is no way to directly establish that it reproduces the running implied by the universal two-loop beta function. The authors attempt to compensate for this problem by using the (non gradient-flow) Schrödinger-functional coupling in an energy region that overlaps with both the perturbatively determined coupling at high energies, and the gradient-flow coupling at low/intermediate energies. However, we note that the Schrödinger-functional and gradient-flow coupling curves do not actually match, a fact that is left uncommented, see figure 2 in [43]. In relation to the arisen confusion see also [44], where the gradient-flow coupling studies are critically analyzed. We would additionally like to refer to [45], where it seems to be indicated that the short-distance running is completely altered once the gradient flow is applied. Taking all these considerations into account, we think that the community should currently refrain from drawing conclusions from studies that use gradient-flow running coupling schemes, and, in this context, give the highest priority to more carefully conducted and critically evaluated tests.

Finally, the Yang-Mills gradient flow is used to study thermodynamics of QCD-like theories on the lattice. This is done by utilizing the energy-momentum tensor and the *small flow-time expansion* [46–48], which is closely related to the operator product expansion, though in the former the expansion is made in the flow time t instead of a small spacetime separation, and there is not necessarily a product of operators involved⁸. Additionally, the deconfining phase transition can be studied using the Polyakov loop, and the gradient flow has been used to renormalize this quantity [50, 51].

We would like to stress that the dire situation regarding the gradient-flow running coupling schemes, where there appears to be a lack of control over some nonperturbative effects, lattice artifacts, finite volume and/or topological effects, also casts a veil of doubt over the validity and reliability of the other

⁸There is also a gradient-flow evolved version of the operator product expansion [49].

mentioned applications of the gradient flow in lattice studies. Though these issues are partly what motivated the research presented in this thesis, they will not feature in the body of this work — we will briefly return to them in the conclusion, chapter 7. Instead we focus on gaining a more complete understanding of the gradient flow in continuum quantum field theories.

Continuum studies

Although the gradient flow’s main application is in lattice studies, a lot of theoretical studies using continuous spacetime have been carried out in order to improve our understanding of the flow, a selection of which we proceed to discuss next.

The perturbative expansion of the expectation value of the Yang-Mills Lagrangian density is currently known up to next-to-next-to-leading order in the gauge coupling [52, 53]. The gradient-flow evolved energy-momentum tensor [46, 54–56] and its small flow-time expansion coefficients have also been determined to this order [57], similarly to several other quantities [58, 59]. These expansion coefficients are particularly important, since they serve as input for lattice studies on the thermodynamic properties of QCD-like theories, for example including the transport coefficients of the quark-gluon plasma [60].

Generally, having lattice applications in mind, it is natural to consider gradient-flow evolved quantities in finite volume, while retaining continuous spacetime. Calculations of the expectation value of the evolved Yang-Mills Lagrangian density in finite volume crucially led to the definition of the gradient-flow running coupling, depending on both the flow-time t and the linear size of the finite-volume box L [38]. Note that different choices for temporal and spatial boundary conditions lead to different definitions of the gradient-flow running coupling [39, 61, 62].

The fermionic flow has also found several applications, for example in the study of the partial-conserved axial current (PCAC) relation [31]. These, and other chiral Ward identities, are studied from a slightly different perspective in [63], see also [64]. The former three studies solely deal with flavor non-singlet currents; for the application of the gradient flow to the flavor singlet axial current and the associated $U(1)$ axial anomaly see [65]. Closely related to the spontaneous breaking of the (approximate) chiral symmetry is the effective field theory approach of chiral perturbation theory, and there are studies applying the tools

provided by the Yang-Mills gradient flow to this approach [66, 67].

Finally, when searching for BSM physics, one typically uses higher-dimensional operators in an effective field theory setting. The Yang-Mills gradient flow can be used to renormalize these operators, and can thus be employed in BSM searches [68–71]. The gradient flow greatly simplifies the renormalization properties — any composite operator renormalizes with the product of the universal renormalization factors of its constituent fermion fields (if present) — which, combined with the small flow-time expansion, makes it a powerful tool in such studies. By smoothing the short-distance fluctuations of operators, it parametrizes and disentangles the, possibly divergent, short-distance behavior from the long-distance behavior in correlation functions, thereby providing additional conceptual insight.

The application of gradient-flow techniques is not limited to QCD-like theories; they are used in a wide variety of settings, for example in scalar field theory with cubic interaction [72, 73], the Gross-Neveu model [74], the two-dimensional $O(N)$ nonlinear sigma model [75], its large- N limit [76, 77] and $\mathcal{N} = 1$ supersymmetric version [78], as well as $\mathcal{N} = 1, 2$ supersymmetric Yang-Mills theories in four dimensions [79–83]. Another application, different in spirit to what we have discussed up to now, is to the subject of holography. There, the gradient flow is involved in the appearance of nontrivial geometries [84, 85]. For a relation to the anti-de Sitter/conformal field theory (AdS/CFT) correspondence, see [86]. We also like to touch on the close relation between the Wilsonian/exact renormalization group and the gradient flow, which is made explicit in [87–90]. As a final example of the many applications of gradient flows we like to mention sphaleron studies [91, 92], which might contribute to our understanding of certain baryon-number violating processes in the early universe, with the aim of explaining the observed baryon asymmetry in the present observable universe [93].

Research presented in this thesis

Although there are gradient-flow studies focusing on the spontaneously broken chiral symmetry, as well as on the anomalous axial $U(1)$, no specific studies have been conducted where the focus is on the exact flavor symmetry, its conserved currents, and the impact of the gradient flow on this conservation. Since there is a direct connection — with some to-be-discussed caveats — between the exactness of a global symmetry and the nonrenormalization property of its associated conserved Noether current, we consider it interesting to establish how

the gradient flow affects both these properties, and their relation.

These questions were addressed in [94] and will be discussed in chapter 5, where we focus on the conserved flavor-singlet vector current associated with the exact global $U(1)$ symmetry related to baryon-number conservation. In addition to the analysis of the interrelation between conservation and nonrenormalization, and especially the gradient flow's impact on it, we also present the first explicit calculation of a continuum gauge-invariant gradient-flow evolved 2-point correlation function, up to next-to-leading order in the gauge coupling. We mainly use coordinate-space methods instead of the more conventional momentum-space methods, thereby providing further conceptual insight into the workings of the gradient flow on the computational level. Renormalization group methods are employed to analyze the asymptotic behavior of the evolved vector current 2-point correlation function, which now depends on two scales (flow time and spacetime separation; the fermions are massless). Furthermore, the small flow-time expansion method is used in order to connect to the nonevolved result, thereby elucidating the role of the gradient flow simultaneously in the appearance of soft symmetry-breaking terms *and* the appearance of a t -dependent anomalous dimension.

In all gradient-flow studies in QCD-like theories mentioned so far, the flow equation for the gauge fields is always defined by the gradient of the Yang-Mills action, regardless of whether there are fermions present in the theory or not. Since in pure Yang-Mills theory certain properties of gradient-flow evolved quantities have been attributed to the fact that the flow drives the gauge field towards the stationary points of the Yang-Mills action, we asked ourselves what would happen if we take the gradient of the *full* QCD-like action — including the fermions — when defining the flow equation for the gauge field when fermions are present.

The resulting novel generalization of the Yang-Mills gradient flow and fermion flows were presented in [95], and are referred to as *nonminimal gradient flows*. This approach has an even closer relation to stochastic quantization than the conventional Yang-Mills gradient flow. If we relax the requirement that the gauge field ought to be driven towards the stationary points of the action, we can further generalize the flow equations, by changing the relative normalization (and thereby the relative importance) of the different terms in the flow equations. The nonminimal flows will be defined and analyzed in chapter 6. We will furthermore anticipate possible applications to lattice studies in the conclusions

presented in chapter 7.

Outline of the thesis

In chapter 2 we discuss the properties of QCD-like theories that will be relevant throughout the rest of the thesis. We start with a basic review of the continuous global internal symmetries of QCD-like theories, then briefly discuss the path-integral quantization procedure we are employing throughout, and finally discuss some more advanced renormalization topics such as operator mixing and Ward identities, culminating in the demonstration of how the conserved flavor currents do not require any renormalization.

Chapter 3 focuses on the renormalization group methods that are utilized in later chapters, starting with a detailed historical overview. The Callan-Symanzik equation is presented, and the running of the gauge coupling is analyzed. After reviewing the renormalization-scheme dependence of the beta function and anomalous dimensions, we conclude with the asymptotic solution of the Callan-Symanzik equation.

In chapter 4 we provide a technical introduction to the Yang-Mills gradient flow, first discussing the flow equation, its solution, and the perturbative expansion thereof. This is followed by a similar analysis for the fermion flow, after which we provide additional tools for performing perturbative calculations using the evolved fields in coordinate space, such as the evolved propagators. Subsequently, we review the renormalization properties of the evolved fields and composite operators, and end the chapter discussing the small flow-time expansion.

Chapter 5 is based on our work from [94] on the gradient-flow evolved conserved vector current, and contains the calculation of the 2-point correlation function of the evolved flavor-singlet conserved vector current up to next-to-leading order in the gauge coupling, employing coordinate-space techniques. We discuss its renormalization properties and small flow-time expansion, focusing on the intimate interplay between current conservation, (non)renormalization, and the gradient-flow evolution.

In chapter 6 we present our work from [95], introducing the nonminimal gradient flows. We examine whether there are any qualitative changes in the all-order renormalization features of nonminimally-evolved quantities, calculate the first

nontrivial order in the gauge coupling of the fermion renormalization factors and the Yang-Mills Lagrangian density using the different flows, and analyze the latter's behavior when changing the number of massless fermions.

We present our conclusions in chapter 7. Additional material is included in appendices A–D.

Chapter 2

QCD-like theories

In the previous chapter we motivated the usefulness of studying generalizations of QCD, where both the number of colors N and the massless fermionic matter content can be varied. In this chapter we provide a more technical review of these theories, focusing on aspects that are relevant in later parts of this thesis. Specifically, after introducing the basics (details are provided in appendix A) we discuss continuous global symmetries and the associated conserved currents. Next we discuss quantization of these theories, using the path integral formalism, thereby introducing the Faddeev-Popov ghosts, and we review the BRST symmetry that replaces gauge invariance. In the final section we briefly review renormalization in QCD-like theories, and focus specifically on composite operators, Ward identities, and the nonrenormalization of exactly conserved currents.

2.1 Introducing the basics

Since this thesis is focusing on QCD-like theories we start by summarizing some basic properties. Certain specific details are provided in appendix A, and more can be found in introductory quantum field theory textbooks; we suggest [96–98].

2.1.1 Gauge group and matter content

We are studying non-Abelian gauge theories with special unitary gauge group $SU(N)$ in Euclidean spacetime, where N is the number of colors, and the bare¹ gauge coupling associated with the interaction is denoted by g_0 , see appendix A.1 for the $SU(N)$ and spacetime conventions used throughout this thesis. The bare gauge potential (i.e. gauge field) is denoted $A_{0,\mu}(x) = A_{0,\mu}^a(x)T^a$ where the index $a = 1, \dots, N^2 - 1$ is implicitly summed over, with T^a the anti-Hermitian generators of the Lie algebra $su(N)$ satisfying $[T^a, T^b] = f^{abc}T^c$ with f^{abc} the fully anti-symmetric real structure constants, and the 0-subscript indicating it is a bare field will be omitted when no ambiguity can arise. Under a gauge transformation $\Lambda(x) = e^{\omega^a(x)T^a}$ with $\omega^a(x)$ real and $\Lambda^\dagger(x) = \Lambda^{-1}(x)$, the bare gauge field transforms as²:

$$A'_\mu(x) = \Lambda A_\mu \Lambda^{-1} + \frac{1}{g_0} \Lambda \partial_\mu \Lambda^{-1} \quad (2.1)$$

where g_0 is the bare gauge coupling. The field strength tensor is given by:

$$F_{\mu\nu}(x) = F_{\mu\nu}^a(x)T^a = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + g_0[A_\mu(x), A_\nu(x)] \quad (2.2)$$

and it transforms as:

$$F'_{\mu\nu}(x) = \Lambda(x)F_{\mu\nu}(x)\Lambda^{-1}(x) \quad (2.3)$$

The theories we study contain $N_f(R)$ number of *flavors* of bare massless spin- $\frac{1}{2}$ Dirac fermion fields $\psi_0(x)$ in the representation R of $SU(N)$, and an identical number of bare massless spin- $\frac{1}{2}$ Dirac fermion fields $\bar{\psi}_0(x)$ in the dual representation of R , i.e. $\bar{R}$. Again, the bare-indicating 0-subscript will be omitted when no ambiguity can arise, and we also will often omit the argument R from N_f . When in the (anti)fundamental representation $F(\bar{F})$, the (anti)fermion fields transform as:

$$\psi'(x) = \Lambda(x)\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)\Lambda^{-1}(x) \quad (2.4)$$

Covariant derivatives transform as:

$$D'_\mu = \Lambda D_\mu \Lambda^{-1} \quad (2.5)$$

¹We will discuss the difference and relation between bare and renormalized quantities in section 2.4.1.

²The gauge transformation properties of the gauge and matter fields are more elaborately discussed in appendix in A.2.1, and are repeated here for the reader's convenience.

When acting on (anti-)fundamental fermion fields, the covariant derivatives take on the form:

$$\begin{aligned} D_\mu \psi(x) &= (\partial_\mu + g_0 A_\mu(x)) \psi(x) \\ \bar{\psi}(x) \overleftarrow{D}_\mu &= \partial_\mu \bar{\psi}(x) - g_0 \bar{\psi}(x) A_\mu(x) \end{aligned} \quad (2.6)$$

and they act on fields in any other representation R of $SU(N)$ as:

$$D_\mu = \partial_\mu + g_0 A_\mu^a T_R^a \quad (2.7)$$

with the R -subscript on the generators T_R^a indicating the representation they are in. For the adjoint representation the generators are related to the structure constants, $(T_{\text{adj}}^a)^{bc} = -f^{abc}$ in our conventions (see appendix A), hence in that case we can write the covariant derivative as:

$$D_\mu = \partial_\mu + g_0 [A_\mu, \cdot] \quad (2.8)$$

2.1.2 Most general Lagrangian

The most general renormalizable Lagrangian of QCD-like theories in terms of the bare fields, consistent with gauge invariance, with canonically normalized kinetic terms is given by:

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_F = -\frac{1}{2} \text{tr}(F_{\mu\nu} F_{\mu\nu}) + \sum_{N_f} \bar{\psi} \not{D} \psi \quad (2.9)$$

where $-\frac{1}{2} \text{tr}(F_{\mu\nu} F_{\mu\nu}) = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$ (again, for conventions see appendix A.1).

There is one other term consistent with gauge invariance, which is the CP -violating term proportional to $\theta \text{tr}(F_{\mu\nu} \tilde{F}_{\mu\nu})$, where $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$ is the dual of $F_{\mu\nu}$, with $\epsilon_{\mu\nu\rho\sigma}$ the totally anti-symmetric tensor with $\epsilon_{1234} = 1$, and θ is known as the *vacuum angle*, which parametrizes the topologically nonequivalent vacua of the theory [99, 100]. Using the Bianchi identity $\epsilon_{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma} = 0$, it can be rewritten as a total derivate:

$$\text{tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}) = \partial_\mu K_\mu, \quad K_\mu = \epsilon_{\mu\nu\rho\sigma} \text{tr} \left(A_\nu F_{\rho\sigma} - \frac{2}{3} g_0 A_\nu A_\rho A_\sigma \right) \quad (2.10)$$

and consequently it does not contribute at any order in perturbation theory, since this implies that its momentum space interaction vertex is proportional

to the sum of the ingoing momenta minus the sum of the outgoing momenta, which vanishes due to momentum conservation. It nevertheless has important nonperturbative physical consequences; e.g. it contributes to the electric dipole moment of the neutron when θ is nonvanishing³. We will not consider the contribution of the θ -term to the Lagrangian in (2.9) in the rest of this thesis (except briefly when we discuss the anomalous breaking of the axial $U(1)$ symmetry in section 2.2). Hence, the theory given by the Lagrangian (2.9) has only one free parameter, i.e. the bare gauge coupling g_0 .

We like to point out that although gauge invariance is dictating our description of the fundamental interactions, it also introduces a certain amount of redundancy into this description. These redundancies have to be removed via a process known as *gauge fixing*, which breaks the gauge invariance. We will come back to this point in section 2.3.2.

2.2 Global symmetries

While the interactions are dictated by gauge invariance — which is not an actual symmetry of the theory — the Lagrangian in (2.9) does possess a number of *global symmetries*. Specifically, we will focus on the continuous global internal symmetries⁴.

In appendix A.4 we briefly review Noether’s theorem, establishing the link between conservation laws and continuous global symmetries.

2.2.1 Continuous global symmetries of the Lagrangian

The Lagrangian (2.9) was constructed as the most general renormalizable Lagrangian invariant under the $SU(N)$ gauge transformations from section 2.1.1, see also appendix A.2.1. The gauge transformation parameters $\omega^a(x)$ are x -dependent; gauge transformations are *local*. Clearly, *global* transformations are

³In the Standard Model, the unexplained smallness of the CP -violating effective angle $\bar{\theta}$ — which includes the effects of diagonalizing the (complex) quark mass matrix — is known as the *strong CP problem*: theoretically, θ is expected to be $\mathcal{O}(1)$, while experimental searches for the neutron electric dipole moment indicate that $|\bar{\theta}| \lesssim 10^{-10}$ [1].

⁴*Internal* as opposed to *spacetime* symmetries; the continuous global spacetime symmetries of the classical Lagrangian from (2.9) are the conformal transformations, consisting of Poincaré transformations (i.e. Lorentz transformations and translations, see e.g. [101]), dilatations and special conformal transformations [102, 103]. For discrete global symmetries like charge conjugation (C), parity transformation (P) and time reversal (T), and combinations thereof, see e.g. [96].

a subclass of these, taking $\omega^a(x) = \omega^a$ to be constant throughout spacetime. Therefore, the theory has a global $SU(N)$ color symmetry. This is the only continuous global internal symmetry of the Lagrangian from (2.9) acting on both the gauge and fermion fields.

For the matter sector in (2.9) we can identify additional global symmetries, usually referred to as the *flavor symmetries*. Decomposing the massless fermions into left-handed and right-handed components, see appendix A.3, the total flavor symmetry group is given by:

$$U(N_f)_L \times U(N_f)_R \quad (2.11)$$

where $U(N_f)_L$ and $U(N_f)_R$ act only on the left- and right-handed fermions, respectively. We however will use the basis where one $U(N_f)$ has a trivial Dirac matrix structure, while the other carries the nontrivial γ_5 structure. These will be referred to as the *vector* (V) and *axial* (A) groups, respectively. See appendix A.3 for the explicit group action in different bases.

It will be useful to decompose both the $U(N_f)_V$ and $U(N_f)_A$ groups into $SU(N_f)$ and $U(1)$ groups. The precise relation between these groups is that $U(N_f)$ is the inner semidirect product of $U(1)$ with $SU(N_f)$. Given an element of $U(1)$, there are N_f distinct ways we can combine this with an $SU(N)$ element by placing it in one of the N_f entries of the $N_f \times N_f$ unit matrix. Practically speaking, $U(N_f)$ contains all the individual $U(1)$ phase factor transformations of all the distinct N_f flavors, as well as the $SU(N_f)$ transformations in flavor space.

With these subtleties in mind, we write for the total flavor group:

$$SU(N_f)_V \times SU(N_f)_A \times U(1)_B \times U(1)_A \quad (2.12)$$

where the B subscript on the vector $U(1)$ refers to *baryon number* — the conserved charge associated with this symmetry. The $SU(N_f)_V$ factor is associated with the transformations:

$$\psi'(x) = e^{\theta^a T^a} \psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x) e^{-\theta^a T^a} \quad (2.13)$$

where the T^a are now the anti-Hermitian generators of $SU(N_f)$, and θ^a are real numbers, with $a = 1, \dots, N_f^2 - 1$. Since the gauge field $A_\mu(x)$ does not transform under $SU(N_f)$, the Lagrangian (2.9) is clearly invariant under the transformations (2.13).

As mentioned above, the transformations of the axial $SU(N_f)_A$ factor contain a γ_5 matrix. They act on the fermions as:

$$\psi'(x) = e^{\theta^a T^a \gamma_5} \psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x) e^{\theta^a T^a \gamma_5} \quad (2.14)$$

and we readily verify that these transformations also leave the Lagrangian (2.9) invariant.

Next is $U(1)_B$, under which the fermions transform as:

$$\psi'(x) = e^{i\theta}\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{-i\theta} \quad (2.15)$$

with θ some real number. The γ_5 -containing counterpart of this is $U(1)_A$:

$$\psi'(x) = e^{i\theta\gamma_5}\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)e^{i\theta\gamma_5} \quad (2.16)$$

again with θ real; note the sign difference in the exponent of the transformation for $\bar{\psi}(x)$ in (2.16) with respect to (2.15).

The conserved currents derived using Noether's theorem (see appendix A.4, specifically equation (A.39)) associated with the flavor symmetries of the massless fermions from (2.9) are given by (note that the T^a 's are the generators of $SU(N_f)$):

$$SU(N_f)_V : \quad j_\mu^a(x) = \bar{\psi}(x)\gamma_\mu T^a \psi(x) \quad (2.17a)$$

$$SU(N_f)_A : \quad j_{\mu,5}^a(x) = \bar{\psi}(x)\gamma_\mu\gamma_5 T^a \psi(x) \quad (2.17b)$$

$$U(1)_B : \quad j_\mu(x) = \bar{\psi}(x)\gamma_\mu \psi(x) \quad (2.17c)$$

$$U(1)_A : \quad j_{\mu,5}(x) = \bar{\psi}(x)\gamma_\mu\gamma_5 \psi(x) \quad (2.17d)$$

Let us for a moment shift our attention back to the global color group $SU(N)$. The associated matter current — $\bar{\psi}(x)\gamma_\mu T^a \psi(x)$ with T^a now the generators of $SU(N)$ — is *not* conserved: applying (A.39) it is clear that the *conserved* current also has a gauge field part, and is given by:

$$SU(N) : \quad \mathcal{J}_\mu^a(x) = \bar{\psi}(x)\gamma_\mu T^a \psi(x) + f^{abc} A_\nu^b(x) F_{\mu\nu}^c(x) \quad (2.18)$$

These color non-singlet currents, however, are not of physical interest, since they are not gauge invariant, i.e. they are not observable, in contrast to the flavor currents given in (2.17) which are color-neutral.

When the number of massless fermions N_f is such⁵ that there is a dynamical formation of a chiral condensate $\langle\bar{\psi}\psi\rangle \neq 0$ (see e.g. [104]), then this leads to the spontaneous breaking of the axial⁶ $SU(N_f)_A$ flavor symmetry — known

⁵As we mentioned in chapter 1, when the matter content is increased, at some point the conformal window opens up and the theory becomes deconfined and chiral symmetry is restored.

⁶When the vacuum θ -angle discussed around (2.10) is zero, it is not possible to spontaneously break the vector symmetry $SU(N_f)_V$ [105].

as *chiral symmetry breaking* — which gives rise to $N_f^2 - 1$ massless Goldstone bosons. The associated flavor non-singlet axial current $j_{\mu,5}^a(x)$ from (2.17b) is nonanomalous and conserved [106]:

$$\partial_\mu j_{\mu,5}^a(x) = 0 \quad (2.19)$$

The $U(1)_A$ symmetry would also appear to be spontaneously broken in this scenario, however it is actually anomalous; it is broken by quantum effects, and the associated flavor singlet axial current $j_{\mu,5}(x)$ from (2.17d) is no longer conserved when the theory is quantized:

$$\partial_\mu j_{\mu,5}(x) \sim \text{tr}(F_{\mu\nu}(x)\tilde{F}_{\mu\nu}(x)) \neq 0 \quad (2.20)$$

where $\text{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu})$ is given in (2.10). This $U(1)_A$ anomaly is one-loop exact and can be elegantly derived using a method due to Fujikawa [107, 108].

In real-world QCD there are six flavors of *massive* quarks, and the Lagrangian from (2.9) is supplemented by a mass term:

$$\mathcal{L}_F \xrightarrow{QCD} \sum_{i,j=1}^6 \bar{\psi}_i(\delta_{ij}\not{D} + M_{0,ij})\psi_j \quad (2.21)$$

where we explicitly display the flavor indices. The bare mass matrix M_0 is diagonal, with as entries the distinct bare masses of the quarks. These bare masses explicitly break both the flavor singlet and non-singlet axial symmetries (and also the flavor non-singlet vector symmetry since the masses are nondegenerate, see also footnote 7):

$$\begin{aligned} \partial_\mu j_{\mu,5}(x) &\sim \bar{\psi}(x)\gamma_5 M_0 \psi(x) + \text{anomaly} \\ \partial_\mu j_{\mu,5}^a(x) &\sim \bar{\psi}(x)\gamma_5 \{T^a, M_0\} \psi(x) \\ \partial_\mu j_\mu^a(x) &\sim \bar{\psi}(x)[T^a, M_0] \psi(x) \end{aligned} \quad (2.22)$$

At low (i.e. hadronic) energy scales, only the two lightest quarks (up and down) contribute, which explicitly break these symmetries only softly due to their small (with respect to the hadronic scales) and almost identical masses. The additional *spontaneous* breaking of the softly broken axial $SU(2)_A$ symmetry is accompanied by $2^2 - 1 = 3$ massive *pseudo*-Goldstone bosons, i.e. the pions.

The same analysis can also be carried out with the strange quark included, even though its mass is not small compared to hadronic energy scales [98]. This results in $3^2 - 1 = 8$ pseudo-Goldstone bosons; the three pions are accompanied

by the four kaons and the eta meson. Historically, the eta-prime meson — i.e. the would-be pseudo-Goldstone boson associated with the spontaneous breaking of $U(1)_A$ — turned out to be heavier than expected from group-theoretical considerations [109]; this became known as the *$U(1)$ problem*. It was resolved by the realization that the $U(1)_A$ symmetry is anomalously broken by nonperturbative effects [110], see (2.20).

The only current that is conserved both in massless and in massive QCD-like theories with generic⁷ values for the quark masses is the flavor singlet vector current $j_\mu(x)$ from (2.17c), associated with baryon number conservation:

$$\partial_\mu j_\mu(x) = 0 \quad (2.23)$$

The fact that it is conserved will have important consequences for its renormalization properties; in fact, it does not require any renormalization at all, as we will show in section 2.4.4. The effect of the Yang-Mills gradient flow on these renormalization properties of $j_\mu(x)$ is the subject of [94] and will be discussed in chapter 5.

2.3 Quantization

2.3.1 Path integral quantization

Throughout this work, correlation functions are defined using the path integral quantization procedure (for a review of other quantization procedures for non-Abelian gauge theories see e.g. [111] and references therein). Crudely put, the *bare* (as opposed to *renormalized*, which we discuss in section 2.4.1) Euclidean n -point correlation function of some operators⁸ $O_i(x)$ is defined as:

$$\langle O_{i_1}(x_1) \dots O_{i_n}(x_n) \rangle \equiv \frac{\int \mathcal{D}\phi_i O_{i_1}(x_1) \dots O_{i_n}(x_n) e^{-S(\phi_i)}}{\int \mathcal{D}\phi_i e^{-S(\phi_i)}} \quad (2.24)$$

where $S(\phi_i) = \int d^4x \mathcal{L}(\phi_i)$ is the Euclidean action of the theory, and the measure $\mathcal{D}\phi_i \equiv \prod_i \mathcal{D}\phi_i$ implies the integration over the values of the bare fields $\phi_i(x)$

⁷In the case of $n \leq N_f$ degenerate masses, there is the additional conserved flavor non-singlet vector current $j_\mu^a(x)$ with $a = 1, \dots, n$.

⁸This nomenclature is inherited from the canonical quantization procedure, where the fields are operator-valued. In the path integral paradigm, all fields are classical-valued as opposed to operator-valued.

at every spacetime point x . In other words, $\langle \dots \rangle$ is the weighted average, where the denominator in (2.24) represents the vacuum of the theory.

Correlation functions of the bare elementary fields ϕ_i are conveniently derived from the so-called *generating functional*:

$$Z[J_i] = \int \mathcal{D}\phi_i e^{-S(\phi_i) + \sum_i \int_x J_i(x) \phi_i(x)} \quad (2.25)$$

by taking functional derivatives with respect to the bare sources $J_i(x)$, and at the end setting these sources to zero:

$$\begin{aligned} \langle \phi_{i_1}(x_1) \dots \phi_{i_n}(x_n) \rangle &\equiv Z[0]^{-1} \frac{\delta}{\delta J_{i_1}(x_1)} \dots \frac{\delta}{\delta J_{i_n}(x_n)} Z[J_i] \Big|_{J_i=0} \\ &= \frac{\int \mathcal{D}\phi_i \phi_{i_1}(x_1) \dots \phi_{i_n}(x_n) e^{-S(\phi_i)}}{\int \mathcal{D}\phi_i e^{-S(\phi_i)}} \end{aligned} \quad (2.26)$$

We will use the generating functional in section 2.4.3 when deriving the Ward identities. For an extensive review of the path integral quantization procedure we recommend [98].

2.3.2 Ghosts and BRST symmetry

When quantizing non-Abelian gauge theories such as (2.9) using the path integral procedure, we have to take into account that certain field configurations can be related by gauge transformations — these gauge equivalent states constitute a *gauge orbit*. Care has to be taken to integrate only once over such gauge equivalent states. This can be implemented by using the famous method developed by Faddeev and Popov [112], where the singling out of a representative from every⁹ gauge orbit, i.e. fixing the gauge, is translated to the path integral by introducing the so-called *ghost fields*. These are unphysical Grassmann-valued (anti-commuting) scalar fields — they do not obey the *spin-statistics theorem* [114–116] — and will never show up in physical external states.

The way in which the ghosts show up in the theory depends on the choice of gauge. For example, one can eliminate the ghosts completely by using axial gauges $n_\mu A_\mu = 0$ with n_μ some constant vector, see [117] for a critical and

⁹In non-Abelian gauge theories it is in general not possible to *globally* define a covariant gauge condition such that every gauge orbit contributes exactly once; this is known as *Gribov ambiguity* [113]. This is however not a problem in perturbation theory, where we only need to ensure that such a condition is obeyed *locally*.

diverse review of this subject. We will be using the so-called R_ξ gauges in this thesis, see (2.28) below.

After following the Faddeev-Popov procedure briefly outlined above (see e.g. [98] for a detailed treatment), the Lagrangian is given by:

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_F + \mathcal{L}_{gf} + \mathcal{L}_{c\bar{c}} \quad (2.27)$$

with $\mathcal{L}_{YM} + \mathcal{L}_F$ given in (2.9) and:

$$\mathcal{L}_{gf} = -\frac{1}{\xi} \text{tr} \{ (\partial_\mu A_\mu)^2 \}, \quad \mathcal{L}_{c\bar{c}} = -2 \text{tr} (\partial_\mu \bar{c} D_\mu c) \quad (2.28)$$

where $c(x) = c^a(x)T^a$, $\bar{c}(x) = \bar{c}^a(x)T^a$ are the bare ghost and antighost fields, respectively, with $a = 1, \dots, N^2 - 1$, D_μ is defined in (2.7), and let us for clarity present the full (bare) Lagrangian:

$$\mathcal{L} = -\frac{1}{2} \text{tr} (F_{\mu\nu} F_{\mu\nu}) + \sum_{N_f} \bar{\psi} \not{D} \psi - \frac{1}{\xi} \text{tr} \{ (\partial_\mu A_\mu)^2 \} - 2 \text{tr} (\partial_\mu \bar{c} D_\mu c) \quad (2.29)$$

In the process of singling out gauge orbit representatives, we have traded gauge invariance for Becchi-Rouet-Stora-Tyutin (BRST) invariance [118, 119]. A BRST transformation can be viewed as a gauge transformation $\Lambda(x) = e^{\omega^a(x)T^a}$, but now making the identification $\omega^a(x) = -g_0 c^a(x)$ with $c^a(x)$ the ghost field, which is Grassmann-valued. The BRST variations of the fields are thus given by (which can be compared to the infinitesimal gauge transformations provided in appendix A.2.1):

$$\begin{aligned} \delta A_\mu(x) &= D_\mu c(x) & \delta c(x) &= -g_0 c^2(x) \\ \delta \psi(x) &= -g_0 c(x) \psi(x) & \delta \bar{c}(x) &= \frac{1}{\xi} \partial_\mu A_\mu(x) \\ \delta \bar{\psi}(x) &= -g_0 \bar{\psi}(x) c(x) \end{aligned} \quad (2.30)$$

where δ can be viewed as a nilpotent operator, which anticommutes with Grassmann variables, i.e. the (anti)ghosts and (anti)fermions.

BRST symmetry is extremely powerful when we are dealing with renormalization, especially with all-order statements in perturbation theory. It will be employed in chapter 6 in the demonstration of such all-order statements.

We conclude this section by presenting the final gauge fixed description of the bare Euclidean n -point correlation functions of QCD-like theories using the path integral approach. For n operators $O_i(x)$ consisting of the bare fields A_μ , ψ , $\bar{\psi}$, c

and/or $\bar{c}$, or combinations thereof¹⁰ (possibly containing spacetime derivatives), the correlator is defined as:

$$\langle O_{i_1}(x_1) \dots O_{i_n}(x_n) \rangle \equiv \frac{\int \mathcal{D}\Phi_i O_{i_1}(x_1) \dots O_{i_n}(x_n) e^{-S(\Phi_i)}}{\int \mathcal{D}\Phi_i e^{-S(\Phi_i)}} \quad (2.31)$$

where Φ_i collectively stands for the bare fields A_μ , ψ , $\bar{\psi}$, c and $\bar{c}$, $\mathcal{D}\Phi_i \equiv \prod_i \mathcal{D}\Phi_i$, and the Euclidean action $S(\Phi_i) = \int d^4x \mathcal{L}(\Phi_i)$ with the Lagrangian $\mathcal{L}(\Phi_i)$ given in (2.29).

We like to point out that we are mostly interested in correlation functions of observable, i.e. gauge invariant, operators, since they contain the physics. Gauge invariant operators are clearly composite, and in this thesis we will focus on two distinct types: the color-neutral (flavor-singlet) fermion bilinears:

$$\bar{\psi}(x)\Gamma\psi(x), \quad \Gamma = \mathbb{1}, \gamma_\mu, \gamma_\mu\gamma_5, [\gamma_\mu, \gamma_\nu] \quad (2.32)$$

and the Yang-Mills Lagrangian density $\sim \text{tr}(F_{\mu\nu}(x)F_{\mu\nu}(x))$. Other gauge invariant operators that are of interest in perturbative studies are e.g. higher spin operators [98], though they will not be discussed in this thesis.

2.4 Renormalization properties

Now that we have introduced the bare Euclidean n -point correlation functions of QCD-like theories using the path integral we are ready to cover their renormalization properties. We will keep the following aspects rather general for now: divergences will not be *explicitly* regularized, and the renormalization factors will remain unspecified (beyond the fact that they cancel these encountered divergences); these subjects will be covered (among other things) in chapter 3.

2.4.1 Renormalizing QCD-like theories

When the bare correlation functions defined in (2.31) are evaluated in perturbation theory by expanding in powers of the gauge coupling g_0 — which corresponds to an expansion in loops — one generally encounters divergent contributions. The systematic removal of these divergences is called *renormalization*, and we call a theory *renormalizable* if all divergences that are encountered in bare correlation functions of elementary fields can be removed by reexpressing

¹⁰When the operator $O_i(x)$ consists of more than one elementary field it is referred to as a *composite operator*. We will discuss these types of operators in more detail in section 2.4.2.

the fields and parameters of the original Lagrangian¹¹ in terms of renormalized quantities. The QCD-like theories we are studying are of this type.

Up to now, all fields and parameters we have discussed were bare, and in the following we will explicitly indicate this with a 0-subscript. The relation between the bare and renormalized elementary fields is established using multiplicative¹² field renormalization factors:

$$\Phi_{0,i}(x) = Z_i \Phi_{R,i}(x) \quad (2.33)$$

where $\Phi_{0,i}$ collectively represents the bare fields $A_{0,\mu}$, ψ_0 , $\bar{\psi}_0$, c_0 , $\bar{c}_0$, and $\Phi_{R,i}$ the renormalized fields $A_{R,\mu}$, ψ_R , $\bar{\psi}_R$, c_R , $\bar{c}_R$, and no summation is implied on the right-hand side. The two bare parameters in the Lagrangian — the gauge coupling g_0 and the gauge-fixing parameter ξ_0 — are also reexpressed in terms of renormalized parameters and multiplicative renormalization factors:

$$g_0 = Z_g g_R, \quad \xi_0 = Z_\xi \xi_R \quad (2.34)$$

We postpone a detailed analysis of the explicit structure and parameter-dependence of the renormalization factors Z_i , Z_g , Z_ξ until section 3.2.

Correlation functions of the renormalized fields $\Phi_{R,i}$, expressed in terms of the renormalized quantities g_R , ξ_R , are finite. They are systematically constructed from their bare counterparts using the R -operation¹³ [122–124]:

$$\begin{aligned} R(\langle \Phi_{0,i_1}(x_1) \dots \Phi_{0,i_n}(x_n) \rangle) &= Z_{i_1}^{-1} \dots Z_{i_n}^{-1} \langle \Phi_{0,i_1}(x_1) \dots \Phi_{0,i_n}(x_n) \rangle \\ &= \langle \Phi_{R,i_1}(x_1) \dots \Phi_{R,i_n}(x_n) \rangle \end{aligned} \quad (2.35)$$

where the inverse Z -factors in the first equality are subtracting the divergences from the bare correlation function. The R -operation is not restricted to correlation functions of elementary fields, and the renormalization of correlation functions containing composite operators will be the subject of section 2.4.2.

The subtraction of divergences from bare correlation functions of elementary fields can alternatively be accomplished by including counterterms in the orig-

¹¹When new terms have to be added to the Lagrangian in order to remove the divergences encountered in correlation functions we speak of a *non-renormalizable* theory. These new terms consist of higher dimensional operators with corresponding dimensionful couplings.

¹²Additional *additive* renormalization can be required in momentum space when renormalizing correlation functions of gauge invariant composite operators [32].

¹³The R -operation is specifically renormalizing the short-distance (i.e. ultraviolet) divergences that occur in Feynman loop diagrams. Possible long-distance (i.e. infrared) divergences can similarly be renormalized using the so-called R^* -operation [120, 121]. For our purposes it suffices to consider only the R -operation; see [106] for a detailed review.

inal Lagrangian¹⁴, see e.g. [106]. Let us explicitly illustrate these two distinct approaches — subtracting at the correlation-function level versus subtracting via counterterms in the Lagrangian — and their equivalence when renormalizing correlation functions of bare elementary fields, since this difference will be important in derivation of the nonrenormalization of conserved currents in section 2.4.4. First, consider the approach where we subtract divergences at the level of the correlation function, i.e. explicitly using the path integral formulation from (2.31) in (2.35):

$$\begin{aligned}
& R(\langle \Phi_{0,i_1}(x_1) \dots \Phi_{0,i_n}(x_n) \rangle) \\
&= R\left(\mathcal{N} \int \mathcal{D}\Phi_{0,i} \Phi_{0,i_1}(x_1) \dots \Phi_{0,i_n}(x_n) e^{-S(\Phi_{0,i})}\right) \\
&= Z_{i_1}^{-1} \dots Z_{i_n}^{-1} \mathcal{N} \int \mathcal{D}\Phi_{0,i} \Phi_{0,i_1}(x_1) \dots \Phi_{0,i_n}(x_n) e^{-S(\Phi_{0,i})} \\
&= \mathcal{N} \int \mathcal{D}\Phi_{0,i} Z_{i_1}^{-1} \Phi_{0,i_1}(x_1) \dots Z_{i_n}^{-1} \Phi_{0,i_n}(x_n) e^{-S(\Phi_{0,i})} \\
&= \mathcal{N} \int \mathcal{D}\Phi_{R,i} \Phi_{R,i_1}(x_1) \dots \Phi_{R,i_n}(x_n) e^{-S(Z_i \Phi_{R,i})} \\
&= \langle \Phi_{R,i_1}(x_1) \dots \Phi_{R,i_n}(x_n) \rangle
\end{aligned} \tag{2.36}$$

with $\mathcal{N}^{-1} = \int \mathcal{D}\Phi_{0,i} e^{-S(\Phi_{0,i})}$, where in the second equality the inverse Z -factors are introduced in order to subtract the divergences from the bare correlator, as in (2.35), and where in the fourth equality we reexpressed the fields according to (2.33), though note that for the path integral measure we have taken $\mathcal{D}\Phi_{0,i} = \mathcal{D}\Phi_{R,i}$; this is only correct in perturbation theory when using dimensional regularization (which we explain in section 3.2 and appendix B), otherwise a rescaling of the fields is accompanied by a Jacobian factor, see e.g. [125]. Next

¹⁴The divergences appearing in correlation functions containing composite operators cannot be removed in this way in general. This will also be discussed in section 2.4.2.

consider the approach using counterterms in the Lagrangian:

$$\begin{aligned}
& R(\langle \Phi_{0,i_1}(x_1) \dots \Phi_{0,i_n}(x_n) \rangle) \\
&= R\left(\mathcal{N} \int \mathcal{D}\Phi_{0,i} \Phi_{0,i_1}(x_1) \dots \Phi_{0,i_n}(x_n) e^{-S(\Phi_{0,i})}\right) \\
&= R\left(\mathcal{N} \int \mathcal{D}\Phi_{R,i} \Phi_{R,i_1}(x_1) \dots \Phi_{R,i_n}(x_n) e^{-S(\Phi_{R,i})}\right) \\
&= \mathcal{N} \int \mathcal{D}\Phi_{R,i} \Phi_{R,i_1}(x_1) \dots \Phi_{R,i_n}(x_n) e^{-\int d^4x (\mathcal{L}(\Phi_{R,i}) + \mathcal{L}_{ct}(\Phi_{R,i}))} \\
&= \mathcal{N} \int \mathcal{D}\Phi_{R,i} \Phi_{R,i_1}(x_1) \dots \Phi_{R,i_n}(x_n) e^{-S(Z_i \Phi_{R,i})} \\
&= \langle \Phi_{R,i_1}(x_1) \dots \Phi_{R,i_n}(x_n) \rangle
\end{aligned} \tag{2.37}$$

with again $\mathcal{N}^{-1} = \int \mathcal{D}\Phi_{0,i} e^{-S(\Phi_{0,i})}$, where in the second equality we relabeled the fields $\Phi_{0,i} \rightarrow \Phi_{R,i}$, and in the third equality $\mathcal{L}(\Phi_{R,i})$ is the original Lagrangian from (2.29) with the relabeling $\Phi_i \rightarrow \Phi_{R,i}$, and $\mathcal{L}_{ct}(\Phi_{R,i})$ consists of the divergent counterterms. In the fourth equality we make the identification:

$$\mathcal{L}(\Phi_{R,i}) + \mathcal{L}_{ct}(\Phi_{R,i}) = \mathcal{L}(Z_i \Phi_{R,i}) \tag{2.38}$$

where we explicitly make use of the fact that the theories we consider are renormalizable.

We like to stress that the statements in this section have been highly non-trivial, and detailed treatments of them are beyond the scope and aim of this thesis. For a very complete exposition of renormalization and all the concepts involved, we recommend [106].

For later use, let us conclude this section with the generating functional of the Euclidean n -point functions of the renormalized elementary fields in QCD-like theories. It is identical to the generating functional of the bare correlation functions from (2.25) — where $\phi_i \rightarrow \Phi_{0,i}$ includes the bare ghost fields, see also (2.31) — though now reexpressed in terms of the renormalized quantities $\Phi_{R,i}, g_R, \xi_R$, and it is a functional of the *renormalized sources* $J_{R,i}(x) = Z_i J_{0,i}(x)$:

$$Z[J_{R,i}] = \int \mathcal{D}\Phi_{R,i} e^{-S(Z_i \Phi_{R,i}) + \sum_i \int_x J_{R,i}(x) \Phi_{R,i}(x)} \tag{2.39}$$

Renormalized correlation functions are constructed from the generating functional by functionally differentiating with respect to the renormalized sources

Figure 2.1: The bare operators $O_{0,1}(x)$ (left) and $O_{0,2}(x)$ (right) from (2.41).

$J_{R,i}(x)$, and at the end setting these sources to zero:

$$\langle \Phi_{R,i_1}(x_1) \dots \Phi_{R,i_n}(x_n) \rangle = Z[0]^{-1} \frac{\delta}{\delta J_{R,i_1}(x_1)} \dots \frac{\delta}{\delta J_{R,i_n}(x_n)} Z[J_{R,i}] \Big|_{J_{R,i}=0} \quad (2.40)$$

2.4.2 Composite operators

In general, additional renormalization beyond that of the bare constituent fields — as in (2.35) — of a bare composite operator will be required in order to render renormalized correlators finite when such a bare composite operator is inserted. Also, whenever there are operators with the same (or lower) mass dimension and the same symmetry properties as the bare composite operator we are renormalizing, then these former operators will mix with the latter under renormalization.

In order to clarify these properties, we will now consider QCD-like theories with *massive* fermions. For simplicity we take the masses to be identical, and denote the bare mass m_0 . Specifically, we consider the renormalization and mixing of the CP -even¹⁵ Lorentz-scalar mass-dimension-four operators — i.e. all the terms in the Lagrangian (2.29), supplemented with the fermion mass term $m_0 \bar{\psi}(x)\psi(x)$ — closely following [126], see also [127]. We are interested in correlators of gauge-invariant operators. In general, these operators mix under renormalization with operators that vanish under the equations of motion (EoM) and Q-exact operators¹⁶. However, the correlators of Q-exact operators with gauge-invariant operators vanish, while the correlators of EoM operators with gauge-invariant operators lead to contact terms at most. Therefore, in the correlators of gauge invariant operators the above mixing can be ignored.

The two bare gauge-invariant operators that do not vanish upon applying

¹⁵In other words, we will not consider the topological term $\sim F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a$ discussed below (2.9).

¹⁶These operators are the BRST variation of some operator, with Q the nilpotent BRST charge.

Figure 2.2: Divergences of the bare operator $O_{0,1}(x)$ when inserted in a finite correlation function.

Figure 2.3: Counterterms subtracting the divergences from figure 2.2.

the EoM that mix upon renormalization are:

$$O_{0,1}(x) = \frac{1}{4} F_{0,\mu\nu}^a(x) F_{0,\mu\nu}^a(x), \quad O_{0,2}(x) = m_0 \bar{\psi}_0(x) \psi_0(x) \quad (2.41)$$

and are diagrammatically represented in figure 2.1. Let us first consider the renormalization of $O_{0,1}(x)$. When it is inserted in any finite correlation function $\langle X_R \rangle$, where X_R is some collection of renormalized elementary fields $\Phi_{R,i}$:

$$\langle O_{0,1}(x) X_R \rangle \quad (2.42)$$

it will in general generate divergences. The divergent quantum corrections to the operator $O_{0,1}(x)$ are schematically depicted in figure 2.2; the gray blobs contain divergences. Now, in order to subtract the divergences represented in figure 2.2 we need operators with the same overall structure as those in figure 2.2, with divergent coefficients that exactly cancel the divergences from the blobs. These are represented in figure 2.3. In other words, the divergences present in the diagrams from figure 2.2 dictate the necessity of mixing of operators under renormalization, and we write:

$$O_{0,1}(x) = Z_{11}[O_{R,1}(x)] + Z_{12}[O_{R,2}(x)] \quad (2.43)$$

where the square brackets signal that when these operators are inserted into a finite correlation function, it yields a finite result. We call such operators *renormalized composite operators*, and Z_{11}, Z_{12} are the renormalization factors

Figure 2.4: Mixing between fermionic and gluonic operators.

containing the divergent structure. Equation (2.43) is the analogue of (2.33) in the case of elementary fields.

Using the R -operation introduced in the previous section, the renormalization of the bare composite operator $O_{0,1}(x)$ when inserted into any finite correlation function $\langle X_R \rangle$ is given by:

$$R(\langle O_{0,1}(x)X_R \rangle) = \sum_{j=1,2} Z_{1j}^{-1} \langle O_{0,j}(x)X_R \rangle = \langle [O_{R,1}(x)]X_R \rangle \quad (2.44)$$

where Z_{ij}^{-1} with $i, j = 1, 2$ is the inverse of the renormalization matrix Z_{ij} , i.e. $Z_{ik}Z_{kj}^{-1} = \delta_{ij}$, implicitly summing over $k = 1, 2$. Renormalization and mixing of the fermionic $O_{0,2}(x)$ operator is trivial [126]: $Z_{21} = Z_{21}^{-1} = 0$ and $Z_{22} = Z_{22}^{-1} = 1$, i.e.:

$$R(\langle O_{0,2}(x)X_R \rangle) = \langle O_{0,2}(x)X_R \rangle = \langle [O_{R,2}(x)]X_R \rangle \quad (2.45)$$

The nontrivial renormalization factor Z_{11}^{-1} from (2.44) can be linked in straightforward manner to Z_g from (2.34), and Z_{12}^{-1} is related to the mass renormalization factor Z_m , defined by $m_0 = Z_m m_R$ [126].

In massless theories, $O_{0,1}(x)$ does not mix with any operator. One might be tempted to think that it can mix with the lower-mass-dimensional $\bar{\psi}(x)\psi(x)$ operator, however the Dirac gamma matrix structure of the mixing diagram, shown in figure 2.4, prevents this; such a diagrammatic structure always contains the trace of an odd number of gamma matrices. The only possible mixing between fermionic and gluonic operators in theories with massless fermions is between the derivative of the flavor singlet pseudo-vector current from (2.17), $\partial_\mu j_{\mu 5}(x)$, and the topological gluonic term $\sim F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a$; the fact that these operators mix is a manifestation of the chiral anomaly, also briefly discussed towards the end of section 2.2.

In general, we can denote the mixing of composite operators under renormalization as:

$$O_{0,i}(x) = Z_{ij}[O_{R,j}(x)] \quad (2.46)$$

where the $O_{0,i}$'s form a complete (closed) set of operators under renormalization. The renormalization of a bare composite operator $O_{0,i}(x)$ when inserted into a finite correlator $\langle X_R \rangle$ is given by:

$$R(\langle O_{0,i}(x)X_R \rangle) = Z_{ij}^{-1} \langle O_{0,j}(x)X_R \rangle = \langle [O_{R,i}(x)]X_R \rangle \quad (2.47)$$

Note that the R -operation can also be applied to correlators containing solely bare fields (both composite and elementary):

$$R(\langle O_{0,i}(x)X_0 \rangle) = \langle [O_{R,i}(x)]X_R \rangle \quad (2.48)$$

yielding the same result as (2.47).

2.4.3 Ward identities

When discussing Noether's theorem in appendix A.4 we deal with classical field theory. Translating these classical conservation laws to the quantized theory leads us to the *Ward identities*¹⁷. We will present the derivation of the Ward identities, since some intermediate steps will be used in the demonstration of the nonrenormalization of conserved currents in the following section.

We consider an infinitesimal transformation of the fields under a continuous internal symmetry (though we will keep the derivation rather general, it can be useful to have the flavor symmetries in mind that we discussed in section 2.2):

$$\Phi_i(x) \rightarrow \Phi_i(x) + \alpha^a(x)(\delta\Phi_i)^a(x) \quad (2.49)$$

where $\alpha^a(x)$ are the parameters of the transformation¹⁸. We do not specify whether the fields are the bare or renormalized ones; they transform in the same way since $\Phi_{0,i}(x) = Z_i\Phi_{R,i}(x)$, where Z_i only contains the divergent structure and does not transform. Applying this infinitesimal transformation to the fields in the generating functional of the bare correlators (2.25), and taking the measure to be invariant¹⁹, we have:

$$Z[J_{0,i}] \rightarrow \int \mathcal{D}\Phi_{0,i} e^{-S(\Phi_{0,i} + \alpha^a(\delta\Phi_{0,i})^a) + \int_x J_{0,i}(x)(\Phi_{0,i}(x) + \alpha^a(x)(\delta\Phi_{0,i})^a(x))} \quad (2.50)$$

¹⁷These identities are sometimes referred to as (generalized) Ward-Takahashi identities [128, 129] in the context of QED, see e.g. [98], and Slavnov-Taylor identities in the context of gauge invariance in QCD [130, 131]. We will follow [106] and in the overarching generalized case simply refer to them as Ward identities.

¹⁸Note that the transformation parameters $\alpha^a(x)$ explicitly depend on x ; the results derived in this way nevertheless also hold for continuous global symmetries [106].

¹⁹Here we again implicitly assume that we work with dimensional regularization; see also the comment below (2.36).

where the summation over i in the source term in the exponent is implicit. Next we expand in $\alpha^a(x)$:

$$Z[J_{0,i}] \rightarrow Z[J_{0,i}] + \alpha^a(x)(\delta Z[J_{0,i}])^a(x) \quad (2.51)$$

where:

$$\begin{aligned} (\delta Z[J_{0,i}])^a(x) = \int \mathcal{D}\Phi_{0,i} \left(- \frac{\delta S(\Phi_{0,i} + \alpha^b(\delta\Phi_{0,i})^b)}{\delta\alpha^a(x)} \right. \\ \left. + J_{0,i}(x)(\delta\Phi_{0,i})^a(x) \right) e^{-S(\Phi_{0,i}) + \int J_{0,i}\Phi_{0,i}} \end{aligned} \quad (2.52)$$

The generating functional is invariant under field redefinitions, i.e. $(\delta Z[J_{0,i}])^a = 0$, and we obtain:

$$\begin{aligned} \int \mathcal{D}\Phi_{0,i} \frac{\delta S(\Phi_{0,i} + \alpha^b(\delta\Phi_{0,i})^b)}{\delta\alpha^a(x)} e^{-S(\Phi_{0,i}) + \int J_{0,i}\Phi_{0,i}} \\ = \int \mathcal{D}\Phi_{0,i} J_{0,i}(x)(\delta\Phi_{0,i})^a(x) e^{-S(\Phi_{0,i}) + \int J_{0,i}\Phi_{0,i}} \end{aligned} \quad (2.53)$$

The assumption that the variation in (2.49) constitutes a symmetry of the theory implies via Noether's theorem that the variation of the action is proportional to the derivative of the associated conserved current, which vanishes in the classical theory, see appendix A.4. This is however not true off-shell, i.e. without employing the Euler-Lagrange equations, and we write:

$$\frac{\delta S(\Phi_{0,i} + \alpha^b(\delta\Phi_{0,i})^b)}{\delta\alpha^a(x)} = \partial_\mu j_{0,\mu}^a(x) \quad (\text{off-shell}) \quad (2.54)$$

with $j_{0,\mu}^a(x)$ given in (A.39), now in terms of the bare fields. Therefore, (2.53) can be written as:

$$\begin{aligned} \int \mathcal{D}\Phi_{0,i} \partial_\mu j_{0,\mu}^a(x) e^{-S(\Phi_{0,i}) + \int J_{0,i}\Phi_{0,i}} \\ = \int \mathcal{D}\Phi_{0,i} J_{0,i}(x)(\delta\Phi_{0,i})^a(x) e^{-S(\Phi_{0,i}) + \int J_{0,i}\Phi_{0,i}} \end{aligned} \quad (2.55)$$

Taking functional derivatives with respect to the sources on both sides of (2.55), and afterwards setting them to zero, yields the Ward identities. For example, taking one functional derivative we obtain (the derivative with respect to x can be taken outside the averaging [106]):

$$\partial_\mu^x \langle j_{0,\mu}^a(x) \Phi_{0,i}(y) \rangle = \delta^{(4)}(x-y) \langle (\delta\Phi_{0,i})^a(y) \rangle \quad (2.56)$$

The general form of the Ward identities is then obtained by taking n functional derivatives with respect to $J_{0,i}$ both on the left- and right-hand side of (2.55), and afterwards setting $J_{0,i} = 0$:

$$\begin{aligned} & \partial_\mu^x \langle j_{0,\mu}^a(x) \Phi_{0,i_1}(y_1) \dots \Phi_{0,i_n}(y_n) \rangle \\ &= \delta^{(4)}(x - y_1) \langle (\delta \Phi_{0,i_1})^a(y_1) \Phi_{0,i_2}(y_2) \dots \Phi_{0,i_n}(y_n) \rangle \\ & \quad + \delta^{(4)}(x - y_2) \langle \Phi_{0,i_1}(y_1) (\delta \Phi_{0,i_2})^a(y_2) \Phi_{0,i_3}(y_3) \dots \Phi_{0,i_n}(y_n) \rangle \\ & \quad + \dots + \delta^{(4)}(x - y_n) \langle \Phi_{0,i_1}(y_1) \dots (\delta \Phi_{0,i_n})^a(y_n) \rangle \end{aligned} \quad (2.57)$$

which completes the derivation.

It is convenient to introduce a more compact notation and write (2.57) as:

$$\partial_\mu^x \langle j_{0,\mu}^a(x) X_0 \rangle = \delta_x^a \langle X_0 \rangle \quad (2.58)$$

where X_0 is representing the collection of bare fields $\Phi_{0,i_1}(y_1) \dots \Phi_{0,i_n}(y_n)$, and δ_x^a acts on any correlator as:

$$\begin{aligned} \delta_x^a \langle \Phi_{i_1}(y_1) \dots \Phi_{i_n}(y_n) \rangle &= \sum_{k=1}^n \langle \Phi_{i_1}(y_1) \dots \Phi_{i_k}(y_k) \dots \Phi_{i_n}(y_n) \rangle \\ & \quad \text{with} \quad \Phi_{i_k}(y_k) \rightarrow \delta^{(4)}(x - y_k) (\delta \Phi_{i_k})^a(y_k) \end{aligned} \quad (2.59)$$

and indeed, δ_x^a acting on $\langle X_0 \rangle$ produces the right-hand side of (2.57).

When we repeat the same steps as above, but now using the generating functional of the renormalized correlation functions from (2.39), we obtain the Ward identities:

$$\partial_\mu^x \langle j_{0,\mu}^a(x) X_R \rangle = \delta_x^a \langle X_R \rangle \quad (2.60)$$

where the collection of fields X_R now represents $\Phi_{R,i_1}(y_1) \dots \Phi_{R,i_n}(y_n)$, keeping in mind that the fields Φ_i represent the elementary fields. We kept the bare subscript on the current, since:

$$\frac{\delta S(Z_i \Phi_{R,i} + \alpha^b (\delta Z_i \Phi_{R,i})^b)}{\delta \alpha^a(x)} = \frac{\delta S(\Phi_{0,i} + \alpha^b (\delta \Phi_{0,i})^b)}{\delta \alpha^a(x)} = \partial_\mu j_{0,\mu}^a(x) \quad (2.61)$$

Another way of seeing that (2.60) is correct is to recall the relation between the bare and renormalized fields from (2.33), resulting in:

$$\begin{aligned} X_0 &= \Phi_{0,i_1}(y_1) \dots \Phi_{0,i_n}(y_n) \\ &= \left(\prod_{k=1}^n Z_{i_k} \right) \Phi_{R,i_1}(y_1) \dots \Phi_{R,i_n}(y_n) \equiv Z^{(n)} X_R \end{aligned} \quad (2.62)$$

Combining this with the fact that $(\delta\Phi_{0,i})^a = Z_i(\delta\Phi_{R,i})^a$, implying that $\delta_x^a \langle X_0 \rangle = Z^{(n)} \delta_x^a \langle X_R \rangle$, we immediately see that the bare Ward identity (2.58) implies (2.60).

2.4.4 Nonrenormalization of conserved currents

We will now combine the results from the previous sections to show that an exactly conserved current $j_\mu^a(x)$ does not require any renormalization²⁰. Let us be more specific about what we mean by this.

Let $j_{0,\mu}^a(x)$ be the bare conserved current, which is in general a composite operator consisting of bare elementary fields. Let $\langle X_R \rangle$ be any finite n -point correlation function consisting of renormalized elementary fields. Saying that $j_{0,\mu}^a(x)$ does not require any renormalization²¹ means:

$$R(\langle j_{0,\mu}^a(x) X_R \rangle) \equiv \langle [j_{R,\mu}^a(x)] X_R \rangle = \langle j_{0,\mu}^a(x) X_R \rangle \quad (2.63)$$

We thus set out to show that $j_{0,\mu}^a(x) = [j_{R,\mu}^a(x)]$.

In order to derive this, we need one more path-integral ingredient, which we will review now.

Consider the generating functional of the (finite) correlation functions of the renormalized elementary fields, given in (2.39). Now, set all Z -factors equal to 1, i.e. $S(Z_i \Phi_{R,i}) \rightarrow S(\Phi_{R,i})$:

$$Z[J_{R,i}; Z_i = 1] = \int \mathcal{D}\Phi_{R,i} e^{-S(\Phi_{R,i}) + \sum_i \int_x J_{R,i}(x) \Phi_{R,i}(x)} \quad (2.64)$$

Clearly, this is simply the generating functional of the bare correlation functions from (2.25) with the fields and sources relabeled: $\Phi_{0,i} \rightarrow \Phi_{R,i}$, $J_{0,i} \rightarrow J_{R,i}$. Therefore, the correlation functions constructed from (2.64) will have the same divergences as the bare correlation functions constructed from the ‘bare’ generating functional (2.25). Let us denote the correlation functions of elementary fields generated with (2.64) as:

$$\langle X_R \rangle_{Z=1} = Z[0; Z_i = 1]^{-1} \frac{\delta}{\delta J_{R,i_1}(x_1)} \cdots \frac{\delta}{\delta J_{R,i_n}(x_n)} Z[J_{R,i}; Z = 1] \Big|_{J_{R,i}=0} \quad (2.65)$$

²⁰There are some caveats to this statement, which will be discussed towards the end of this section. See also [106].

²¹We should point out that there are cases where the right-hand side of (2.63) is proportional to a *finite* renormalization [106]. In that case, a renormalized conserved current obeying the correct Ward identity can still be defined, taking this finite renormalization into account. An example is the flavor non-singlet axial current, which we briefly discuss toward the end of this section.

Using the R -operation introduced at the end of section 2.4.1, we have:

$$R(\langle X_R \rangle_{Z=1}) = \langle X_R \rangle = \text{finite} \quad (2.66)$$

and equivalently acting on the path integral representation (see also (2.37)):

$$R\left(\int \mathcal{D}\Phi_{R,i} X_R e^{-S(\Phi_{R,i})}\right) = \int \mathcal{D}\Phi_{R,i} X_R e^{-S(Z_i \Phi_{R,i})} = \text{finite} \quad (2.67)$$

We now explicitly note that $\langle X_R \rangle_{Z=1} = \langle X_0 \rangle$; it merely constitutes a relabeling of the fields.

Now, back to showing that $j_{0,\mu}^a(x) = [j_{R,\mu}^a(x)]$. We first repeat the steps involved in deriving the Ward identities in section 2.4.3, now using the generating functional with $Z_i = 1$ from (2.64). We obtain for the intermediate step (2.55) (where now $j_{R,\mu}^a(x)$ is *not* the renormalized composite operator, but is the bare current with the bare constituent fields replaced by the renormalized fields $\Phi_{0,i} \rightarrow \Phi_{R,i}$):

$$\begin{aligned} & \int \mathcal{D}\Phi_{R,i} \partial_\mu j_{R,\mu}^a(x) e^{-S(\Phi_{R,i}) + \int J_{R,i} \Phi_{R,i}} \\ &= \int \mathcal{D}\Phi_{R,i} J_{R,i}(x) (\delta\Phi_{R,i})^a(x) e^{-S(\Phi_{R,i}) + \int J_{R,i} \Phi_{R,i}} \end{aligned} \quad (2.68)$$

where $j_{R,\mu}^a(x)$ is defined via:

$$\frac{\delta S(\Phi_{R,i} + \alpha^b (\delta\Phi_{R,i})^b)}{\delta \alpha^a(x)} = \partial_\mu j_{R,\mu}^a(x) \quad (2.69)$$

Note the difference between (2.69) and (2.61): the former is equal to the left-hand side of the latter with $Z_i = 1$. Again, it is important to note that the R -subscript on $j_{R,\mu}^a(x)$ does not imply that it is a renormalized operator; renormalized composite operators are indicated with square brackets. The R -subscript is merely there to remind us that the current consists of elementary renormalized fields²².

The Ward identities derived from (2.68) are summarized as:

$$\partial_\mu^x \langle j_{R,\mu}^a(x) X_R \rangle_{Z=1} = \delta_x^a \langle X_R \rangle_{Z=1} \quad (2.70)$$

²²This difference is subtle and important. If the reader should experience confusion about this point we recommend another close look at section 2.4.2.

Since this is basically (2.58) with all 0-subscripts replaced by R -subscripts, we know that in general both the left- and right-hand sides of (2.70) contain divergences. Applying the renormalization operation R to both sides we obtain:

$$R \left(\partial_\mu^x \langle j_{R,\mu}^a(x) X_R \rangle_{Z=1} \right) = \partial_\mu^x \langle [j_{R,\mu}^a(x)] X_R \rangle = \delta_x^a \langle X_R \rangle \quad (2.71)$$

and note the use of the square brackets to indicate that the composite operator is renormalized.

Now compare (2.71) with (2.60): the right-hand sides are the same, implying that $j_{0,\mu}^a$ and $[j_{R,\mu}^a]$ obey the same Ward identities. Let us define their difference as:

$$\varepsilon_\mu^a(x) = [j_{R,\mu}^a(x)] - j_{0,\mu}^a(x) \quad (2.72)$$

It now follows from (2.60) and (2.71) that this quantity $\varepsilon_\mu^a(x)$ obeys:

$$\partial_\mu^x \langle \varepsilon_\mu^a(x) X_R \rangle = 0 \quad (2.73)$$

without employing any equations of motion. If there exists no such $\varepsilon_\mu^a(x)$ that satisfies (2.73), then this implies that $\varepsilon_\mu^a(x) = 0$, i.e. that $[j_{R,\mu}^a(x)] = j_{0,\mu}^a(x)$. This means that the conserved current does not acquire any renormalization.

Let us analyze the implications for the flavor currents discussed in section 2.2. The exact symmetries are the vector $SU(N_f)_V$ with associated conserved currents $j_\mu^a(x)$ ($a = 1, \dots, N_f^2 - 1$), and the baryon number $U(1)_B$ with $j_\mu(x)$ its associated conserved current, given in equation (2.17a) and (2.17c), respectively. In both cases it is not possible to find a quantity $\varepsilon_\mu^a(x)$ ($\varepsilon_\mu(x)$ in the case of $U(1)_B$) that satisfies (2.73); all available operators with the correct mass dimension have nonvanishing color charge, like e.g. $\partial_\nu F_{0,\mu\nu}^a(x)$, with $a = 1, \dots, N^2 - 1$. Therefore, (2.73) implies that $\varepsilon_\mu^a(x) = 0$, and these currents will not receive any renormalization. In chapter 5 we will discuss what happens to the nonrenormalization of the flavor-singlet vector current when we apply the gradient flow evolution to the constituent fields of the current.

The case of the conserved flavor non-singlet axial current $j_{0,\mu,5}^a(x)$ is more involved, since the presence of γ_5 allows for anomalous contributions that require renormalization [106]. After a long analysis it turns out that the minimally subtracted composite operator $[j_{R,\mu,5}^a(x)]$ does renormalize, but that it is related to the *conserved* renormalized current via a *finite* renormalization, such that the latter does not have an anomalous dimension²³, nor an anomalous Ward

²³We will cover the topic of anomalous dimensions in chapter 3.

identity [106].

There are cases for which the symmetry is exact, and yet one can find a non-vanishing $\varepsilon_\mu^a(x)$ that has a vanishing derivative, also off-shell, and nonrenormalization is no longer implied. An often-overlooked example is the bare electromagnetic vector current from quantum electrodynamics (QED), $j_{0,\mu}(x) = \bar{\psi}_0(x)\gamma_\mu\psi_0(x)$, associated with lepton number conservation. Many textbooks (also [97]) claim that since this current is conserved, it indeed does not renormalize. However, in [132] it is shown that the bare current mixes with the derivative of the bare electromagnetic field strength tensor $\partial_\nu F_{0,\mu\nu}(x)$ upon renormalization; both operators have mass-dimension 3, are Lorentz vectors, and are charge-neutral. Most importantly, $\varepsilon_\mu(x) = \partial_\nu F_{0,\mu\nu}(x)$ obeys (2.73); its derivative vanishes identically, since the field strength tensor is antisymmetric in its Lorentz indices. Therefore, the nonrenormalization argument does not apply, and the conserved Noether current indeed acquires renormalization. It is nevertheless still possible to define a conserved current that does not have an anomalous dimension, and does give rise to the correct lepton number operator, by adding a term proportional to $\partial_\nu F_{0,\mu\nu}(x)$ to the Noether current $j_{0,\mu}(x)$ [132].

Chapter 3

Renormalization group methods

In this chapter we introduce the renormalization group (RG) methods that are being used throughout this thesis. We start with a historical overview in section 3.1, covering the 20-year period from the first formulation of the renormalization group in 1953 up to the discovery of asymptotic freedom in 1973. In section 3.2 we consider the explicit form of the renormalization factors introduced in section 2.4.1 using dimensional regularization, after which we discuss the Callan-Symanzik equation in section 3.3, containing the information on the quantum-induced scale dependence of correlation functions. The scale dependence of the renormalized gauge coupling is analyzed in section 3.4, and renormalization scheme dependence of this running coupling and the anomalous dimensions entering the correlation functions are discussed in section 3.5. We close the chapter by studying asymptotic solutions to the Callan-Symanzik equation in section 3.6.

3.1 Historical overview

The renormalization group was introduced in the early 1950's by Stueckelberg and Petermann¹. They noticed that there was a certain arbitrariness in the choice of constants involved in removing divergences from S-matrix elements,

¹For an account of the events leading to the formulation of renormalization theory we recommend [133].

and that a change in these constants could be translated into a *renormalization* of the couplings of the theory. The operators generating the infinitesimal changes in these constants in the S-matrix elements form a Lie group [134, 135], which was coined the *normalization group*².

Soon after, in 1954 Gell-Mann and Low [136] explicitly derived the renormalization group equations for the photon propagator in QED. They realized that since the renormalized coupling depends on the external momentum, a change in momentum scale can be compensated by a change in the renormalized coupling. Their work was complemented and generalized by Bogoliubov and Shirkov in 1956 [137], who coined the term *renormalization group*, while extending the analysis to other QED amplitudes, and making the connection to the earlier work of Stueckelberg and Petermann.

There were hardly any applications for the RG until the 1970's, when the discovery of non-Abelian gauge theories as suitable candidates for the interactions among elementary particles revived the interest in perturbative quantum field theory [138]. These theories were introduced as early as 1954 by Yang and Mills [139] as a generalization of QED, but were rejected at the time partly because they seemed to introduce new massless vector bosons, which were not observed. It was through a series of works that the validity of non-Abelian gauge theory as a bona fide description of particle interactions was established; in 1961 Glashow introduced a model [140] based on the Yang-Mills Lagrangian in order to describe the electroweak interactions, though with some modifications in order to include a mass for the vector bosons. In 1964 the Brout-Englert-Higgs mechanism was discovered [141–144], making it possible to assign a mass to the non-Abelian vector bosons without explicitly adding a mass term to the Lagrangian. This was subsequently applied to the electroweak interactions by Weinberg and Salam [145, 146]. Throughout these years the renormalizability of Yang-Mills theories was studied by Veltman [147–151], and following the 1971 works of 't Hooft [152, 153] — focusing on Yang-Mills theories both with and without the Brout-Englert-Higgs mechanism — this culminated in their joint work proving the renormalizability of non-Abelian gauge theories using dimensional regularization [154].

Around the same time an observation was made that would lead to a very powerful RG tool. In 1970 Coleman and Jackiw [155] were studying the anoma-

²The renormalization group is usually considered not to be an actual group. However, in the approach we take in this chapter it definitely is. The modern interpretation and overarching principle of the *Wilsonian renormalization group* indeed is not a group, but rather a *semigroup*. We will comment on the differences and connection between these approaches later in this section.

lous breaking of scale invariance: even a theory that is scale invariant at the classical level will in general not be so when quantum effects are included, since the renormalization procedure always introduces an arbitrary scale μ . Therefore, the Ward identities for n -point Green's functions associated with scale invariance must be modified at the quantum level. This modification was provided by Callan [156] and Symanzik [157, 158], and the resulting differential equations are known as the Callan-Symanzik equations³. See [103] for a nice review.

Meanwhile, experimentalists had observed Björken scaling [161] in deep inelastic scattering of electrons on nucleons [162, 163], leading Feynman to hypothesize the existence of *partons*: the nucleons' constituents that move around approximately freely at the high momenta where Björken scaling is observed [164]. At that point in time there was no theory describing the partons or, indeed, the observed Björken scaling.

This changed in 1973, when the perturbative asymptotic behavior of non-Abelian gauge theories was studied. It was found by Politzer [165], Gross and Wilczek [166] and 't Hooft⁴ that non-Abelian gauge theories exhibit the opposite behavior as Abelian gauge theories such as QED: instead of becoming more strongly coupled towards the UV, non-Abelian gauge theories become more weakly coupled. This feature became known as *asymptotic freedom*, and it is generally considered to be the pinnacle of RG methods' applications. Together with the hypothesization of quarks by Gell-Mann and Zweig [167, 168] and the SU(3) color description of the strong interactions pioneered in the mid 1960's by Greenberg, Han and Nambu [169, 170], the discovery of asymptotic freedom provided the explanation for the observed Björken scaling, and a solid high-energy basis for the theory of strong (nuclear) interactions: quantum chromodynamics (QCD).

Note on Wilson's renormalization group

As anticipated in footnote 2, we will now elucidate the connection and differences between the RG methods we are discussing in this chapter, and the modern approach to the subject due to Wilson [171–173]. The latter is inspired by the idea of *spin blocking* due to Kadanoff [174] in the context of spin systems, of which the quantum field theory analogue is the *integrating out of high energy*

³It appears that the Callan-Symanzik equations were already derived as early as 1956 by Ovsiannikov [159, 160].

⁴It is clear from papers and talks around that time that 't Hooft was aware of this fact. See also his own recollection of events from this period [138].

modes. This block-spin operation is irreversible; the coarse-graining involved implies a loss of information about the original UV degrees of freedom. This means that Wilson’s renormalization group (WRG) is actually a semigroup⁵.

Since we will not be using the WRG in this thesis, we will not go into detail here. For good expositions on the subject we refer the reader to [97, 98]. Very crudely summarized: the WRG describes flow trajectories in *theory space*. The RG methods we are considering in this chapter always act within one specific theory at a time; in the context of the WRG this corresponds to one particular trajectory in theory space.

The methods developed by Wilson (and others later on, e.g. [175, 176]) have proven to be extremely useful and have taught us a huge amount about quantum field theories. An example is the concept of universality, which means that theories with non-identical degrees of freedom in the UV can be shown to describe the same physics in the IR. Related to this is the usefulness of the WRG approach when the UV theory is not completely known. Indeed, the modern *effective field theory* approach to particle physics — and also gravitational physics, cosmology, condensed matter physics, et cetera — is highly influenced by Wilson’s work.

However, when studying asymptotically free theories like QCD in isolation, there is no ambiguity about what the UV theory ‘might be’; these theories are well-defined up to arbitrarily high energy scales, and are referred to as *UV-complete* theories. In WRG language: the asymptotic freedom property singles out one particular trajectory in theory space, namely the one originating from the free UV fixed point. Hence, we know which trajectory we are on, and therefore we can resort to the RG methods described in this chapter when studying asymptotically free QCD-like theories.

3.2 Renormalization with dimensional regularization

When we introduced the renormalization factors Z_i, Z_g, Z_ξ in equations (2.33) and (2.34), we limited ourselves to a qualitative description of them, namely that they subtract the divergences encountered in the correlation functions of the bare elementary fields. In this section we will explicate their structure in perturbation theory.

⁵Semigroups are a generalization of groups; the requirements of the existence of the identity element and inverses are being dropped.

The loop integrals appearing in the perturbative expansion of correlation functions in QCD-like theories in 4 spacetime dimensions generically have a divergent short-distance — i.e. UV — part⁶, and these UV divergences are made tractable by *regularizing* them⁷. We will use the method of *dimensional regularization* [154, 177–180] throughout this thesis, which is based on the observation that loop integrals which are UV divergent in 4 spacetime dimensions become finite when the number of dimensions is lowered. Explicitly, these loop integrals are evaluated for generic $d \neq 4$ spacetime dimensions, after which the result is analytically continued to all d and subsequently is expressed as a Laurent series around $4 - d$, thereby isolating the divergences which are parametrized by setting $d = 4 - 2\varepsilon$, with ε the infinitesimal positive regulator (see appendix B for details).

Dimensional regularization is the most common regularization method used when performing perturbative calculations in non-Abelian gauge theories, due to its symmetry-preserving qualities; it respects both Poincaré invariance — the spacetime symmetry consisting of Lorentz transformations and translations — as well as gauge invariance. Nonchiral global internal symmetries, such as the vector-like flavor symmetry $U(N_f)_V$ discussed in section 2.2, are also preserved under dimensional regularization. Care has to be taken with the axial flavor group $U(N_f)_A$ since γ_5 has no straightforward d -dimensional generalization⁸, and the associated renormalized axial currents require an additional *finite* renormalization [106, 181].

An important consequence of generalizing QCD-like theories to d spacetime dimensions is that the bare gauge coupling acquires a mass-dimension: $[g_0] = \frac{4-d}{2} = \varepsilon$, which is straightforwardly derived from the fact that the action — $S = \int d^d x \mathcal{L}$, with $\mathcal{L}$ given in (2.29) — must remain dimensionless in any number of spacetime dimensions. By requiring that the renormalized coupling introduced in (2.34) remains dimensionless in $4 - 2\varepsilon$ spacetime dimensions, we necessarily have to introduce an arbitrary mass scale μ , and consequently in dimensional regularization the parametrization of the bare coupling in terms of the renormalized coupling from (2.34) takes on the form $g_0 = \mu^\varepsilon Z_g g_R$.

⁶We will not consider IR divergences in this thesis, though we should point out that when dimensional regularization is used, automatically also these divergences are regularized. This requires extra care to be taken when we are solely interested in the UV divergences, making sure we distinguish between the UV and IR nature of the divergences.

⁷It is also possible to renormalize without the use of a regulator, by directly subtracting at the level of the integrand of the divergent integrals, prior to performing the integration [106, 124].

⁸The issue lies in the fact that the Euclidean $\gamma_5 \sim \epsilon_{\mu_1 \dots \mu_d} \gamma_{\mu_1} \dots \gamma_{\mu_d}$, and the Levi-Civita symbol $\epsilon_{\mu_1 \dots \mu_d}$ is only defined for integer d . See (B.15) for a d -dimensional definition of γ_5 .

Now, since the renormalization factors Z_i, Z_g, Z_ξ from (2.33) and (2.34) subtract the divergences — which manifest themselves as ε -poles — encountered order by order in the perturbative expansion of the correlation functions, they will have the following generic form⁹:

$$Z(g_R, \varepsilon) = 1 + \sum_{n=0}^{\infty} \frac{z_n(g_R)}{\varepsilon^n}, \quad z_n(g_R) = \sum_{i=n}^{\infty} z_{ni} g_R^{2i} \quad (3.1)$$

with z_{ni} real numbers, and the $z_0(g_R)$ term contains finite contributions. These z_{ni} coefficients are not unique; they in general depend on the *renormalization scheme* (with some notable exceptions, to be discussed in section 3.5), which is a manifestation of the fact that the parametrization in (2.33) and (2.34) of bare quantities in terms of renormalized ones is not unique. When we *only* subtract the ε -poles — i.e. $z_{0i} = 0$ for all i and all renormalization factors, hence no $z_0(g_R)$ term in (3.1) — then this uniquely fixes the scheme, which is known as *minimal subtraction* (MS) [182]. In the remainder of this chapter we will assume the use of the MS-scheme, unless stated otherwise.

Since dimensional analysis forces the parametrization of the bare coupling to assume the form $g_0 = \mu^\varepsilon Z_g g_R$ where Z_g has the general form from (3.1), the fact that the bare coupling does not depend on the arbitrary scale μ implies (dropping the R -subscript on the renormalized coupling):

$$g_0 = \mu^\varepsilon Z_g(g(\mu), \varepsilon) g(\mu) \quad (3.2)$$

i.e. the renormalized coupling must depend on the arbitrary scale μ , inducing a *running* of the renormalized coupling $g(\mu)$, which will be discussed in depth in section 3.4. The renormalization factor for the coupling in the MS-scheme explicitly reads:

$$Z_g(g(\mu), \varepsilon) = 1 + \sum_{n=1}^{\infty} \frac{a_n(g(\mu))}{\varepsilon^n}, \quad a_n(g(\mu)) = \sum_{i=n}^{\infty} a_{ni} g^{2i}(\mu) \quad (3.3)$$

with a_{ni} real.

Though we will use the MS-scheme throughout most of this chapter, it is important to note that the value of the renormalization scale μ in the factor

⁹ In addition, the renormalization factors of the elementary fields from (2.33) depend on the gauge fixing parameter ξ . In the MS-like schemes we introduce below, this is not the case for the renormalization factors of the gauge coupling, masses in massive theories, and gauge invariant composite operators [106].

μ^ε in (3.2) is totally arbitrary, and by changing it we obtain a one-parameter family of renormalization schemes, which are collectively referred to as *MS-like schemes*. Taking $\tilde{\mu} = a\mu$ with a any strictly positive real number, (3.2) becomes:

$$g_0 = \tilde{\mu}^\varepsilon Z_g(g_a(\mu), \varepsilon) g_a(\mu) = a^\varepsilon \mu^\varepsilon Z_g(g_a(\mu), \varepsilon) g_a(\mu) \quad (3.4)$$

with Z_g given in (3.3), and a parameterizes the scheme. The a -subscript on the renormalized coupling indicates that it depends on the scheme; this has to be the case, since the bare coupling cannot depend on particularities of its own parametrization. In section 3.5 we will discuss renormalization scheme dependence — and, most importantly, independence — in more detail.

When performing calculations, it is often most practical to use a particular MS-like scheme known as *modified minimal subtraction* ($\overline{\text{MS}}$) [183], which corresponds to taking $a = (e^{\gamma_E}/4\pi)^{1/2}$ in (3.4), with γ_E the Euler-Mascheroni constant. Using this scheme results in the cleanest answers.

3.3 The Callan-Symanzik equation

Analogously to the case of the gauge coupling, the μ -independence of bare correlation functions implicitly conveys information on the anomalous — i.e. quantum-induced — scale dependence of their renormalized counterparts. This scaling behavior of renormalized correlation functions is captured by the *Callan-Symanzik equation*, which we now proceed to derive.

Consider the n -point correlator of the bare¹⁰ operator $O_1 = \frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a$, similarly to the example discussed in section 2.4.2, though now with massless fermions, thereby eliminating operator mixing¹¹. The operators renormalize as:

$$O_1(x) = Z_1(g(\mu), \varepsilon)[O_1(x)] \quad (3.5)$$

where the renormalization factor reads:

$$Z_1(g(\mu), \varepsilon) = 1 + \sum_{n=1}^{\infty} \frac{b_n(g(\mu))}{\varepsilon^n}, \quad b_n(g(\mu)) = \sum_{i=n}^{\infty} b_{ni} g^{2i}(\mu) \quad (3.6)$$

¹⁰We drop the 0-subscript on all bare operators, and we drop the R -subscript on renormalized composite operators; the latter will be indicated by solely using the hard brackets [...]. Renormalized elementary fields will still be denoted using the R -subscript.

¹¹The case with massive fermions, which includes operator mixing is a straightforward generalization of the massless case; most notably, renormalization factors become matrices. Since the required extra notation somewhat obscures the discussion, together with the fact that we only consider massless theories in the rest of this thesis, we choose to restrict ourselves to the simpler massless case. For the massive case see e.g. [106, 126].

and hence the bare n -point correlation function is related to the renormalized one via:

$$\begin{aligned}\Pi_0(x_1, \dots, x_n; g_0, \varepsilon) &\equiv \langle O_1(x_1) \dots O_1(x_n) \rangle \\ &= Z_1^n(g(\mu), \varepsilon) \langle [O_1(x_1)] \dots [O_1(x_n)] \rangle \\ &\equiv Z_1^n(g(\mu), \varepsilon) \Pi_R(x_1, \dots, x_n; g(\mu), \mu, \varepsilon)\end{aligned}\quad (3.7)$$

Note that although the renormalized correlation function Π_R has an explicit dependence on the regulator ε , by definition this dependence vanishes smoothly as $\varepsilon \rightarrow 0$.

Now, the Callan-Symanzik equation is derived using the μ -independence of the bare correlation function, i.e. $\mu \frac{d}{d\mu} \Pi_0 = 0$, which through (3.7) implies (dropping the μ argument from the renormalized coupling where no confusion can arise):

$$\left\{ \frac{\partial}{\partial \log \mu} + \beta(g, \varepsilon) \frac{\partial}{\partial g} + n\gamma_1(g) \right\} \Pi_R(x_1, \dots, x_n; g, \mu, \varepsilon) = 0 \quad (3.8)$$

with:

$$\beta(g, \varepsilon) \equiv \frac{dg}{d \log \mu}, \quad \gamma_1(g) \equiv \frac{d \log Z_1(g, \varepsilon)}{d \log \mu} \quad (3.9)$$

where $\beta(g, \varepsilon)$ is known as the *beta function*, and $\gamma_1(g)$ as the *anomalous dimension* for the operator O_1 . They can be perturbatively expressed in terms of the coefficients in the renormalization factors. Starting with the beta function, utilizing the expression for the renormalized coupling implied by (3.2) and (3.3):

$$\begin{aligned}\beta(g, \varepsilon) &= -\varepsilon g \left(1 + \frac{d \log Z_g}{d \log g} \right)^{-1} \\ &= -\varepsilon g \left(1 - \frac{\frac{dZ_g}{d \log g}}{Z_g + \frac{dZ_g}{d \log g}} \right) \\ &= -\varepsilon g + g \frac{da_1(g)}{d \log g} - \varepsilon g \left(\frac{1}{\varepsilon} \frac{da_1(g)}{d \log g} - \frac{\frac{dZ_g}{d \log g}}{Z_g + \frac{dZ_g}{d \log g}} \right)\end{aligned}\quad (3.10)$$

and we demand that this last term in brackets in the last line vanishes, since the beta function should be finite as $\varepsilon \rightarrow 0$. This allows us to iteratively express the higher order coefficient functions $a_n(g), n > 1$ in terms of $a_1(g)$:

$$\frac{da_{n+1}(g)}{d \log g} = \frac{da_1(g)}{d \log g} \left(\frac{da_n(g)}{d \log g} + a_n(g) \right) \quad (3.11)$$

and these differential equations can be solved order by order in the coupling, using the explicit expressions for the coefficient functions $a_n(g)$ given in (3.3). The perturbative expression for $\beta(g, \varepsilon)$ is now obtained by inserting the explicit expression for $a_1(g)$ from (3.3):

$$\beta(g, \varepsilon) = -\varepsilon g + g \frac{da_1(g)}{d \log g} = -\varepsilon g + 2a_{11}g^3 + 4a_{12}g^5 + \mathcal{O}(g^7) \quad (3.12)$$

The anomalous dimension $\gamma_1(g)$ defined in (3.9), with $Z_1(g, \varepsilon)$ given in (3.6), is similarly expressed as:

$$\begin{aligned} \gamma_1(g) &= \frac{\beta(g, \varepsilon)}{g} \frac{d \log Z_1(g, \varepsilon)}{d \log g} \\ &= \left(-\varepsilon + \frac{da_1(g)}{d \log g} \right) Z_1^{-1}(g, \varepsilon) \frac{dZ_1(g, \varepsilon)}{d \log g} \\ &= -\frac{db_1(g)}{d \log g} + \varepsilon \left\{ \frac{1}{\varepsilon} \frac{db_1(g)}{d \log g} - \left(1 - \frac{1}{\varepsilon} \frac{da_1(g)}{d \log g} \right) Z_1^{-1}(g, \varepsilon) \frac{dZ_1(g, \varepsilon)}{d \log g} \right\} \end{aligned} \quad (3.13)$$

where the last term in the curly brackets in the last line must vanish, since also $\gamma_1(g)$ should remain finite as the regulator is removed. This leads to the following iterative expression:

$$\frac{db_{n+1}(g)}{d \log g} = \frac{da_1(g)}{d \log g} \frac{db_n(g)}{d \log g} + \frac{db_1(g)}{d \log g} b_n(g) \quad (3.14)$$

allowing us to determine the coefficients b_{nj} for $n > 1$ from (3.6) in terms of the coefficients a_{1j} and b_{1j} from (3.3) and (3.6), respectively. The perturbative expression for $\gamma_1(g)$ is obtained by inserting $b_1(g)$ from (3.6):

$$\gamma_1(g) = -\frac{db_1(g)}{d \log g} = -2b_{11}g^2 - 4b_{12}g^4 + \mathcal{O}(g^6) \quad (3.15)$$

The Callan-Symanzik equation allows one to study the impact of quantum corrections on the scaling properties of correlation functions. In section 3.6 we will focus on the asymptotic solutions to the Callan-Symanzik equation, explicating these properties. However, before confronting this task, it is relevant to discuss the asymptotic behavior of the running coupling $g(\mu)$.

3.4 The running coupling

The beta function $\beta(g, \varepsilon)$ defined in (3.9) contains the information on the renormalization scale dependence of the running coupling. Taking $\varepsilon \rightarrow 0$, hence in

$d = 4$ spacetime dimensions, its perturbative expansion reads:

$$\beta(g) = g \frac{da_1(g)}{d \log g} = \sum_{n=1}^{\infty} 2n a_{1n} g^{2n+1} = 2a_{11}g^3 + 4a_{12}g^5 + \mathcal{O}(g^7) \quad (3.16)$$

It is conventional in QCD-like theories to express the perturbative series of the beta function in terms of β_n coefficients, and they are related to the a_{1n} coefficients via:

$$a_{1n} = -\frac{\beta_{n-1}}{2n(4\pi)^{2n}} \quad (3.17)$$

where the coefficients of the first two orders are given by [8, 184]:

$$\beta_0 = \frac{11}{3}N - \frac{4}{3} \sum_R N_f(R)T(R) \quad (3.18a)$$

$$\beta_1 = \frac{34}{3}N^2 - \left(\frac{26}{3}N - \frac{2}{N} \right) \sum_R N_f(R)T(R) \quad (3.18b)$$

where N is the number of colors, $N_f(R)$ the number of fermions in representation R , and $T(R)$ the index of this representation. The coefficients are known up to and including β_4 [185], though only the first two coefficients are *universal*, i.e. renormalization scheme independent — we will explicitly show this in section 3.5. Throughout the rest of this chapter we will stick to the a_{1n} notation.

The sign of the first nonzero¹² coefficient a_{1n} , say a_{1st} , determines whether the coupling decreases towards the UV or IR. The two scenarios are:

(a) $a_{1st} > 0$: the running coupling decreases towards the infrared (IR), i.e. long distances. This scenario is presented in figure 3.1(a), and the theory is referred to as *IR free*. An example of a theory possessing this property is quantum electrodynamics. For QCD-like theories this scenario occurs when $\sum_R N_f(R)T(R) \geq \frac{11}{4}N$.

(b) $a_{1st} < 0$: the running coupling decreases towards the ultraviolet (UV), i.e. short distances. This scenario is presented in figure 3.1(b). An example of a theory exhibiting this behavior is pure Yang-Mills theory. Theories of this kind

¹²In the hypothetical case where all a_{1n} from (3.16) vanish, the beta function is identically zero to all orders in perturbation theory, and the coupling does not run; this cannot happen in QCD-like theories as (3.18) shows. An example of a theory where this is the case is $\mathcal{N} = 4$ super Yang-Mills in 4 spacetime dimensions [186–189].

Figure 3.1: Running coupling $g(\mu)$ in the vicinity of the free fixed point. The left graph (a) is showing the scenario where $a_{1\text{st}} > 0$; the coupling decreases towards the IR ($\mu \rightarrow 0$). The right graph (b) represents asymptotic freedom, $a_{1\text{st}} < 0$; the coupling decreases towards the UV ($\mu \rightarrow \infty$).

are called asymptotically free, and this is the case for QCD-like theories with $\sum_R N_f(R)T(R) < \frac{11}{4}N$.

3.4.1 Fixed points

When the beta function vanishes, this implies the existence of a *fixed point* of the theory, with coupling g^* , whose value is renormalization scheme dependent:

$$\beta(g^*) = 0 \quad (3.19)$$

At a fixed point the theory becomes scale invariant¹³. It is clear from (3.16) that there is always a fixed point at $g^* = 0$, which is referred to as the *trivial* or *free* fixed point¹⁴. Whether it is located in the UV or IR is again determined by the sign of $a_{1\text{st}}$: $a_{1\text{st}} > 0$ leads to a trivial IR fixed point (IRFP), and $a_{1\text{st}} < 0$ to a trivial UV fixed point (UVFP).

Interacting infrared fixed points

When $g^* \neq 0$, the fixed point is referred to as *non-trivial*, or *interacting*. In QCD-like theories it is possible to have a perturbatively accessible interacting

¹³In most cases this implies conformal invariance; see [190] for a review of the conditions involved.

¹⁴It is also often referred to as the *Gaussian* fixed point, stemming from the fact that the Euclidean path integration for free scalar fields is Gaussian. However, since this is not the case for free Grassmann-valued fermion fields, we will use the adjectives *trivial* or *free*.

Figure 3.2: Beta function of the 't Hooft coupling in the Banks-Zaks limit. The point where the line crosses the horizontal axis corresponds to equation (3.20).

fixed point in the IR, known as the Banks-Zaks¹⁵ fixed point [9]. In order to find it, one has to take the Veneziano limit [192], which consists of taking the large- N 't Hooft limit of $SU(N)$ gauge theory [193] — where the perturbative expansion rearranges itself as a series in the 't Hooft coupling $\lambda = \frac{Ng^2}{(2\pi)^2}$ — while simultaneously taking the large- N_f limit, keeping the ratio N_f/N fixed. This ratio can be tuned such that we are infinitesimally close to the point of losing asymptotic freedom, known as the upper edge of the conformal window, which is located at $(N_f/N)_{\text{edge}} = 11/2$ for fermions in the fundamental representation. Expanding in $(N_f/N)_{\text{edge}} - N_f/N \equiv \varepsilon \ll 1$, the Banks-Zaks fixed point is located at $(\beta(\lambda) = \frac{d \log \lambda}{d \log \mu})$:

$$\beta(\lambda^*) = -\frac{2\beta_0}{N}\lambda^* - \frac{2\beta_1}{N^2}\lambda^{*2} + \mathcal{O}(\lambda^{*3}) = 0 \quad (3.20)$$

with β_0, β_1 given in (3.18). Therefore, the fixed point 't Hooft coupling λ^* is given by:

$$\lambda^* = -\frac{N\beta_0}{\beta_1} + \mathcal{O}(\lambda^{*2}) = \frac{4}{75}\varepsilon + \mathcal{O}(\varepsilon^2) \quad (3.21)$$

and we indeed find that λ^* can be kept infinitesimally close to zero¹⁶. The beta function $\beta(\lambda)$, including the Banks-Zaks fixed point, is depicted in figure 3.2.

¹⁵The existence of this fixed point had already been established by Caswell [8], and independently by Belavin and Migdal [191], though the latter paper contained an error in the numerical coefficient proportional to N^2 in the next-to-leading order term of $\beta(g)$.

¹⁶One often encounters perturbative studies in the literature where claims are made regarding the existence of interacting fixed points at finite values of the coupling that are not infinitesimally close to zero, while the existence of such fixed points can only be established using nonperturbative methods. We can still make a perturbative expansion around $(g^* - g)$, and by doing so analyze properties of the fixed point, such as its critical exponents [97].

Another example of an interacting IR fixed point can be found in $\mathcal{N} = 1$ supersymmetric QCD (SQCD). There, the beta function can nonperturbatively be expressed in a closed form [194, 195]:

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \frac{3N - \sum_i T(R_i)(1 - \gamma_i(g))}{1 - 2N \frac{g^2}{(4\pi)^2}} \quad (3.22)$$

where the sum is over all matter supermultiplets in representation R_i of the gauge group $SU(N)$, and $\gamma_i(g)$ are the associated anomalous dimensions. The fixed point is identified by setting the numerator in (3.22) to zero, provided that the denominator does not vanish¹⁷ for values of the coupling between 0 and g^* . For example, taking N_f matter supermultiplets in the fundamental and antifundamental representation, it follows that:

$$\beta(g^*) = 0 \implies \gamma^* = \gamma(g^*) = 1 - \frac{3N}{N_f} \quad (3.23)$$

We like to stress that this holds nonperturbatively. Note that we can also perturbatively access this fixed point; close to the upper edge of the conformal window, now located at $(N_f/N)_{\text{edge}} = 3$, we can guarantee that $\beta(g^*) = 0$, with $Ng^{*2} \sim \varepsilon$, with $(N_f/N)_{\text{edge}} - N_f/N \equiv \varepsilon \ll 1$, by perturbatively expanding $\gamma(g)$ in terms of the gauge coupling $g(\mu)$ as in (3.15).

There also exists a nonperturbative closed form for the beta function in non-supersymmetric large- N QCD-like theories in the massless Veneziano limit [15], similar to (3.22). These nonperturbative closed form beta functions in both the supersymmetric and non-supersymmetric theories have proven to be very useful; for example they were used to show that the onset of conformality at the lower edge of the conformal window — which is solely nonperturbatively accessible — does not come about through a merging of UV and IR fixed points [197].

For completeness we should also mention the possibility of the combination of a free IRFP and interacting UVFP. This scenario is known as *asymptotic safety*. There are studies of non-Abelian gauge theories including fermions where asymptotic safety can be achieved in perturbation theory, though the presence of elementary scalar fields is essential in that case [198]. In SQCD the possibility of asymptotic safety can be ruled out via nonperturbative methods like unitarity bounds and the a -theorem [197, 199]. However, there are other

¹⁷Consequences of a pole in the beta function — resulting in a so-called *cusp singularity* in the coupling — are discussed in [196].

supersymmetric theories which can achieve asymptotic safety, for example by considering the product gauge group $SU(N_1) \times SU(N_2)$ [200].

Asymptotic safety is also one of the avenues pursued in the search for a viable predictive UV-complete description of quantum gravity, outside of string theory. In this paradigm the non-predictivity due to the non-renormalizability of the theory is cured by entertaining the option of the existence of an interacting UVFP [201], see also e.g. [202].

3.4.2 Explicit expressions for the running coupling

We can find an expression for the running coupling at a scale μ' in terms of the running coupling at another scale μ by integrating the beta function:

$$\int_{g(\mu)}^{g(\mu')} \frac{dg}{\beta(g)} = \log \frac{\mu'}{\mu} \quad (3.24)$$

When $g(\mu), g(\mu') \ll 1$ we can use the perturbative form of the beta function (3.16), and at one-loop the value of g at the nearby scale μ' is given by:

$$g^2(\mu') = \frac{g^2(\mu)}{1 - 2a_{11}g^2(\mu) \log\left(\frac{\mu'^2}{\mu^2}\right)} = g^2(\mu) + 2a_{11}g^4(\mu) \log\left(\frac{\mu'^2}{\mu^2}\right) + \dots \quad (3.25)$$

From (3.25) we indeed verify that when $a_{11} < 0$, $g(\mu') < g(\mu)$ for $\mu' > \mu$, i.e. the running coupling decreases when the scale is increased.

When there are no interacting fixed points in the theory, we can define a renormalization group invariant scale by utilizing the fact that the coupling diverges; consider (3.24) and take M to be the scale where the coupling becomes infinite:

$$\int_{g(\mu)}^{\infty} \frac{dg}{\beta(g)} = \log\left(\frac{M}{\mu}\right) + c \implies M = \mu \exp\left(-c + \int_{g(\mu)}^{\infty} \frac{dg}{\beta(g)}\right) \quad (3.26)$$

where $g(\mu) \ll 1$ — i.e $\mu \ll M$ when the theory is IR free and $\mu \gg M$ when it is UV free — and c is a constant of integration which can be absorbed in M . This phenomenon where the running coupling defines a scale is known as *dimensional transmutation*, and it shows — at least perturbatively — how a fundamental scale can emerge at the quantum level in a classically scale-free theory. In general, $\beta(g)$ will depend on the renormalization scheme used beyond the next-to-leading order (NLO), which we shall discuss in more depth in section 3.5.

This implies that the scale M will also depend on the renormalization scheme used, and the order up to which $\beta(g)$ is known.

Taking only the leading order of β into account we have for M (neglecting the integration constant at this order):

$$M_{LO} = \mu \exp \left(\frac{1}{4a_{11}g^2(\mu)} \right) \quad (3.27)$$

and inverting this equation leads to the one-loop approximation of the running coupling:

$$g^2(\mu) = \frac{1}{2a_{11} \log \frac{M_{LO}^2}{\mu^2}} \quad (3.28)$$

Taking also the next-to-leading order (NLO) into account when integrating (3.16), and iteratively solving for $g(\mu)$, we obtain the renormalization group improved two-loop approximation for the running coupling:

$$g^2(\mu) = -\frac{1}{2a_{11}l} - \frac{a_{12}}{2a_{11}^3l^2} \log l + \mathcal{O} \left(\frac{\log(l)^2}{l^3} \right) \quad (3.29)$$

where $l = \log \frac{\mu^2}{M^2}$. In deriving (3.29) the integration constant c from (3.26) is chosen such that the $\mathcal{O}(l^{-2})$ term is canceled [203], which is the case when:

$$c = \frac{a_{12}}{2a_{11}^2} \log \left(-\frac{a_{12}}{a_{11}^2} \right) \quad (3.30)$$

See also [204] for an insightful discussion regarding renormalization group improvement. In [1] the approximation for $g^2(\mu)$ in terms of l is given up to $\mathcal{O}(\log(l)^4 l^{-5})$.

Quantum chromodynamics, the theory describing the strong nuclear interactions, has $N = 3$ colors and $N_f = 6$ flavors of massive quarks in the fundamental representation of $SU(3)$. Some of the quark masses significantly differ in magnitude, which causes the number of active quark flavors to be dependent on the energy at which the theory is probed: the heaviest quark (top) has a mass ≈ 170 GeV, while the second heaviest (bottom) has a mass ≈ 4 GeV, and the lightest quark (up) ≈ 2 MeV [1]. This meant that e.g. at LEP (Large Electron-Positron collider, LHC's predecessor) which reached maximum energies ~ 200 GeV, the top quark effectively did not play any role, and the number of active quarks was 5 at the highest. For a nice exposition of the effect of these quark thresholds on the running of the coupling see e.g. [205].

As discussed above, a theory is asymptotically free when the first term of the beta function is negative. We deduce from (3.18) that for QCD this means that the theory is asymptotically free when $N_f \leq 16$, which is the case at any energy scale.

3.5 Renormalization scheme dependence

We briefly touched upon changing renormalization schemes when defining the MS-like schemes in (3.4). Since the bare coupling is independent of its parametrization, the expressions given in (3.2) and (3.4) must be equal:

$$\begin{aligned} g_0 &= \mu^\varepsilon Z_g(g(\mu), \varepsilon) g(\mu) \\ &= \tilde{\mu}^\varepsilon Z_g(g(\tilde{\mu}), \varepsilon) g(\tilde{\mu}) \\ &= \mu^\varepsilon a^\varepsilon Z_g(g_a(\mu), \varepsilon) g_a(\mu) \end{aligned} \quad (3.31)$$

where we made the identification $g(\tilde{\mu}) = g_a(\mu)$, and we can perturbatively express the MS-like coupling $g_a(\mu)$ in terms of the MS coupling $g(\mu)$, using (3.25):

$$g_a(\mu) = g(\mu) + 2a_{11} \log(a) g^3(\mu) + \dots \quad (3.32)$$

where the ellipsis represents the higher order terms starting at $\mathcal{O}(g^5)$. In order to take scheme changes outside of MS-like schemes into account, we must consider more general redefinitions than those of (3.32). Define a new coupling $g'(\mu)$ associated with the new scheme in terms of a series of powers of the MS coupling $g(\mu)$:

$$g'(g(\mu)) = g(\mu) + \sum_{i=1}^{\infty} A_i g^{2i+1}(\mu) = g(\mu) + A_1 g^3(\mu) + A_2 g^5(\mu) + \dots \quad (3.33)$$

We demand that the bare coupling remains the same, so we need to change the coefficients a_{ni} in $Z_g(g, \varepsilon)$ from (3.3):

$$g_0 = \mu^\varepsilon \left(1 + \sum_{n=1}^{\infty} \frac{a'_n(g')}{\varepsilon^n} \right) g', \quad a'_n(g') = \sum_{i=n}^{\infty} a'_{ni} g'^{2i} \quad (3.34)$$

where we again keep the μ dependence of g' implicit. The beta function of the new coupling g' is derived following the same steps as in (3.10), and in $d = 4$

reads:

$$\begin{aligned}
 \beta'(g') &= g' \frac{da'_1(g')}{d \log g'} \\
 &= 2a'_{11}g'^3 + 4a'_{12}g'^5 + \dots \\
 &= 2a'_{11}g^3(1 + A_1g^2 + A_2g^4)^3 + 4a'_{12}g^5(1 + A_1g^2 + \dots)^5 + \dots \\
 &= 2a'_{11}g^3 + (4a'_{12} + 6a'_{11}A_1)g^5 + \dots
 \end{aligned} \tag{3.35}$$

where in the third line we used (3.33). On the other hand, the beta function of the new coupling is also obtained by applying the chain rule:

$$\begin{aligned}
 \beta'(g') &= \beta(g) \frac{d}{dg} g'(g) = (2a_{11}g^3 + 4a_{12}g^5 + \dots)(1 + 3A_1g^2 + \dots) \\
 &= 2a_{11}g^3 + (4a_{12} + 6a_{11}A_1)g^5 + \dots
 \end{aligned} \tag{3.36}$$

Comparing the two expressions for $\beta'(g')$ from (3.35) and (3.36) we find that $a'_{11} = a_{11}$ and $a'_{12} = a_{12}$. In other words, the coefficients of the first two orders of the expansion in the coupling of the beta function are *renormalization scheme independent*, also referred to as *universal*. All higher order coefficients a_{1n} with $n > 2$ are dependent on the scheme being used. Whenever $a_{11} = 0$, then only the first nonzero coefficient a_{1n} with $n > 1$ is scheme independent [106].

We can follow a similar procedure for the anomalous dimensions. Let us consider the renormalization of the bare operator $O_1(x)$, given in (3.5). In the new scheme it renormalizes as:

$$O_1(x) = Z'_1(g', \varepsilon)[O'_1(x)] \tag{3.37}$$

where:

$$Z'_1(g', \varepsilon) = 1 + \sum_{n=1}^{\infty} \frac{b'_n(g')}{\varepsilon^n}, \quad b'_n(g') = \sum_{i=n}^{\infty} b'_{ni}g'^{2i} \tag{3.38}$$

Following the derivation in (3.13), the anomalous dimension associated with the operator $[O'_1(x)]$ is given by:

$$\gamma'_1(g') = -\frac{db'_1(g')}{d \log g'} = -2b'_{11}g'^2 - 2b'_{12}g'^4 + \dots \tag{3.39}$$

Simultaneously, the scheme change can be absorbed by a change in the normalization — i.e. a *finite* renormalization — of the operator $[O_1(x)]$ [106]; the

relation between the operators in both schemes is given by:

$$[O'_1(x)] = \zeta(g)[O_1(x)], \quad \zeta(g) = 1 + \sum_{i=1}^{\infty} B_i g^{2i} = 1 + B_1 g^2 + B_2 g^4 + \dots \quad (3.40)$$

Comparing (3.37) to (3.5) using (3.40), we make the identification:

$$Z'_1(g', \varepsilon) = \zeta^{-1}(g) Z_1(g, \varepsilon) \quad (3.41)$$

from which it follows that:

$$\begin{aligned} \gamma'_1(g') &= \gamma_1(g) - \beta(g) \frac{d\zeta(g)}{dg} \\ &= -2b_{11}g^2 - (4b_{12} + 4a_{11}B_1)g^4 + \dots \end{aligned} \quad (3.42)$$

Comparing the expressions for the anomalous dimension $\gamma'_1(g')$ in (3.39) and (3.42), we find that now only the first coefficient from the expansion in powers of the coupling is universal: $b'_{11} = b_{11}$.

It is important to note that when the first loop coefficient b_{11} of γ_1 vanishes, then there exists a scheme in which γ_1 vanishes to all orders in perturbation theory; we can always find B_i 's in (3.40) such that all higher orders in (3.42) vanish. This fact will be used in section 6.6.

3.6 Asymptotic solution of the Callan-Symanzik equation

We now have all the required theoretical tools to study the asymptotic solution of the Callan-Symanzik equations, i.e. the solution in the vicinity of a fixed point. For simplicity and later use we focus on the Callan-Symanzik equation of a 2-point correlation function of some bare Lorentz-scalar operator $O(x)$ of canonical mass dimension Δ_O , which can be thought of as e.g. $\frac{1}{4}F_{\mu\nu}^a F_{\mu\nu}^a$, or $\bar{\psi}\psi$; for our following discussion¹⁸ this does not matter. We assume that all matter is massless, hence no mixing is present. The operator renormalizes as:

$$O(x) = Z_O(g, \varepsilon)[O(x)] \quad (3.43)$$

¹⁸The majority of the material presented in this section is based on the relevant part from [94], which in turn was partly inspired on [206].

and the relation between its bare and renormalized 2-point correlation functions reads:

$$\begin{aligned}\Pi_0(x; g_0, \varepsilon) &= \langle O(x)O(0) \rangle \\ &= Z_O^2(g, \varepsilon) \langle [O(x)][O(0)] \rangle \\ &= Z_O^2(g, \varepsilon) \Pi_R(x; g, \mu, \varepsilon)\end{aligned}\tag{3.44}$$

where we left the μ -dependence of the renormalized gauge coupling g implicit. Repeating the steps from section 3.3, the Callan-Symanzik equation is now derived using $\mu \frac{d}{d\mu} \Pi_0 = 0$, and in $d = 4$ reads:

$$\left\{ \frac{\partial}{\partial \log \mu} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_O(g) \right\} \Pi_R(x; g, \mu) = 0\tag{3.45}$$

with $\beta(g)$ given in (3.16), and for $\gamma_O(g)$ we use the same notation as for $\gamma_1(g)$ from (3.15):

$$\begin{aligned}\gamma_O(g) &\equiv \frac{d \log Z_O}{d \log \mu} = -\frac{db_1(g)}{d \log g} \\ &= -2b_{11}g^2 - 4b_{12}g^4 + \dots\end{aligned}\quad b_1(g) = \sum_{n=1}^{\infty} b_{1n}g^{2n}\tag{3.46}$$

Solution to the Callan-Symanzik equation

Next we analyze the solution to (3.45). First we note that in the free limit, i.e. $g(\mu) = 0$, conformal invariance dictates the scaling behavior of the free correlation function, and it allows us to write:

$$\Pi_R(x; g(\mu), \mu) = \frac{1}{x^{2\Delta_O}} \bar{\Pi}_R(x\mu, g(\mu))\tag{3.47}$$

with $\bar{\Pi}_R$ a dimensionless function, $x = \sqrt{x^2}$, and Δ_O the canonical mass dimension of the operator O . Since $\bar{\Pi}_R$ only depends on the product $x\mu$, we can write the Callan-Symanzik equation for $\bar{\Pi}_R$ as:

$$\left\{ \frac{\partial}{\partial \log x} + \beta(g(\mu)) \frac{\partial}{\partial g(\mu)} + 2\gamma_O(g(\mu)) \right\} \bar{\Pi}_R(x\mu, g(\mu)) = 0\tag{3.48}$$

The general solution to this equation can be parametrized by:

$$\bar{\Pi}_R(x\mu, g(\mu)) = F(g(x)) \mathcal{Z}(g(x), g(\mu))\tag{3.49}$$

where $F(g(x))$ is an RG invariant function — therefore not determined by (3.48) — of the RG invariant coupling $g(x)$, which is short for $g(x^{-1}) = g(x\mu, g(\mu))$ with boundary condition $g(1, g(\mu)) = g(\mu)$, and the latter is implicitly given by integrating the beta function:

$$\int_{g(x)}^{g(\mu)} \frac{dg}{\beta(g)} = \int_{-\log x}^{\log \mu} d \log \mu' = \log(x\mu) \quad (3.50)$$

The renormalization function $\mathcal{Z}$ from (3.49), solving (3.48), reads:

$$\mathcal{Z}(g(x), g(\mu)) = \exp \left(-2 \int_{g(x)}^{g(\mu)} dg \frac{\gamma_O(g)}{\beta(g)} \right) \quad (3.51)$$

We can explicitly verify that (3.49) indeed solves (3.48) by noting that:

$$\frac{\partial \mathcal{Z}(g(x), g(\mu))}{\partial \log x} = -2\gamma_O(g(x)) \mathcal{Z}(g(x), g(\mu)) \quad (3.52)$$

and:

$$\frac{\partial \mathcal{Z}(g(x), g(\mu))}{\partial g(\mu)} = \left\{ -2 \frac{\gamma(g(\mu))}{\beta(g(\mu))} + 2 \frac{\partial g(x)}{\partial g(\mu)} \frac{\gamma(g(x))}{\beta(g(x))} \right\} \mathcal{Z}(g(x), g(\mu)) \quad (3.53)$$

both of which follow directly from the definition of $\mathcal{Z}(g(x), g(\mu))$ given in (3.51), and implemented in the left-hand side of (3.48) lead to:

$$\begin{aligned} & \left\{ \frac{\partial}{\partial \log x} + \beta(g(\mu)) \frac{\partial}{\partial g(\mu)} + 2\gamma_O(g(\mu)) \right\} F(g(x)) \mathcal{Z}(g(x), g(\mu)) \\ &= \left\{ \frac{\partial g(x)}{\partial \log x} + \beta(g(\mu)) \frac{\partial g(x)}{\partial g(\mu)} \right\} \frac{\partial F(g(x))}{\partial g(x)} \mathcal{Z}(g(x), g(\mu)) \\ & \quad + \left\{ -2\gamma_O(g(x)) + 2\beta(g(\mu)) \frac{\partial g(x)}{\partial g(\mu)} \frac{\gamma_O(g(x))}{\beta(g(x))} \right\} F(g(x)) \mathcal{Z}(g(x), g(\mu)) \quad (3.54) \\ &= \left\{ \beta(g(x)) + \beta(g(\mu)) \frac{\partial g(x)}{\partial g(\mu)} \right\} \left\{ \frac{\partial \log F(g(x))}{\partial g(x)} - 2 \frac{\gamma(g(x))}{\beta(g(x))} \right\} \\ & \quad \times F(g(x)) \mathcal{Z}(g(x), g(\mu)) \end{aligned}$$

which vanishes due to the renormalization group invariance of $g(x) = g(x\mu, g(\mu))$:

$$0 = \frac{dg(x)}{d \log \mu} = \frac{\partial g(x)}{\partial \log \mu} + \beta(g(\mu)) \frac{\partial g(x)}{\partial g(\mu)} = \beta(g(x)) + \beta(g(\mu)) \frac{\partial g(x)}{\partial g(\mu)} \quad (3.55)$$

The full renormalized 2-point correlation function is thus given by:

$$\Pi_R(x; g(\mu), \mu) = \frac{1}{x^{2\Delta_O}} F(g(x)) \mathcal{Z}(g(x), g(\mu)) \quad (3.56)$$

with $\mathcal{Z}(g(x), g(\mu))$ in (3.51), and $F(g(x))$ a renormalization-group invariant function which is not determined by the Callan-Symanzik equation (3.45)¹⁹. We subsequently use the form in (3.56) to analyze the correlation function's asymptotic behavior.

Asymptotics

We can study the solution (3.56) in some more detail in the vicinity of fixed points. Let us first assume that the theory has an interacting fixed point in the IR, and we analyze (3.56) in the domain of attraction of the IR fixed point (see section 3.4), i.e. $\mu, x^{-1} \ll M$, with $x\mu \sim 1$, where M is the dimensional transmutation scale of the theory we are considering. We can rewrite $\mathcal{Z}$ from (3.51) as:

$$\begin{aligned} \mathcal{Z}(g(x), g(\mu)) &= \exp \left(-2 \int_{-\log x}^{\log \mu} d \log \mu' \{ \gamma_O(g(\mu')) - \gamma_O^* + \gamma_O^* \} \right) \\ &= (x\mu)^{-2\gamma_O^*} \exp \left(-2 \int_{-\log x}^{\log \mu} d \log \mu' \{ \gamma_O(g(\mu')) - \gamma_O^* \} \right) \end{aligned} \quad (3.57)$$

with $\gamma_O^* = \gamma_O(g^*)$, and we see that in this case the asymptotic scaling of the 2-point correlation function in the deep IR as $x \rightarrow \infty$ goes like:

$$\lim_{x \rightarrow \infty} \Pi_R(x; g(\mu), \mu) \sim \frac{1}{(x^2)^{\Delta_O + \gamma_O^*}} \quad (3.58)$$

This is the reason for the nomenclature ‘anomalous dimension’; it is the quantum correction to the classical scaling dimension of the correlation function.

Next we study the correlation function from (3.56) in the vicinity of the trivial UV fixed point. We can directly plug in the perturbative expansions for

¹⁹Note that in the free limit $g(\mu) = g(x) = 0$, the correlation function is simply the product of its conformal scaling factor, and a constant color and/or flavor etc. factor; the latter are collected in $F(0)$. Asymptotically close to the free fixed point, $F(g(x))$ can be perturbatively expressed as a renormalization-group invariant series in $g(x)$: $F(g(x)) = F(0) + \mathcal{O}(g^2(x))$.

the beta function (3.16) and the anomalous dimension (3.46) in the expression for $\mathcal{Z}$ from (3.51):

$$\begin{aligned}
& \mathcal{Z}(g(x), g(\mu)) \\
&= \exp \left(2 \int_{g(x)}^{g(\mu)} dg \frac{2b_{11}g^2 + 4b_{12}g^4 + \dots}{2a_{11}g^3 + 4a_{12}g^5 + \dots} \right) \\
&= \exp \left(2 \int_{g(x)}^{g(\mu)} dg \left\{ \frac{b_{11}}{a_{11}} \frac{1}{g} + \frac{a_{11}b_{12} - a_{12}b_{11}}{a_{11}^2} g + \mathcal{O}(g^3) \right\} \right) \\
&= \exp \left(2 \frac{b_{11}}{a_{11}} \log \left(\frac{g(\mu)}{g(x)} \right) \right. \\
&\quad \left. - \frac{a_{11}b_{12} - a_{12}b_{11}}{a_{11}^2} (g^2(x) - g^2(\mu)) + \mathcal{O}(g^4) \right) \\
&= \left(\frac{g^2(x)}{g^2(\mu)} \right)^{-\frac{b_{11}}{a_{11}}} e^{\mathcal{O}(g^2(x) - g^2(\mu))}
\end{aligned} \tag{3.59}$$

In the last line, the leading logarithmic behavior associated with asymptotic freedom can be revealed by perturbatively expressing $g(x)$ in terms of $g(\mu)$ using the last equality from (3.25), with μ'^{-1} replaced by x , and for small logarithms.

The techniques that have been presented in this section will be used extensively throughout chapter 5, where we investigate the properties of 2-point functions of fermion bilinear currents, and their gradient flow evolved counterparts.

Chapter 4

The gradient flow in QCD-like theories

In this chapter we provide a technical introduction to the gradient flow in QCD-like theories. In particular, it covers the Yang-Mills gradient flow for gauge fields as formulated in [27], and a flow equation based on the operator D^2 for fermions [31]. We present the necessary tools for performing perturbative calculations using these flows, which will subsequently be employed in chapters 5 and 6. The renormalization properties of the gradient-flow evolved fields will also be discussed, and we explain the small (flow-)time expansion (STE) which is used to make the connection to the nonevolved fields and correlation functions.

4.1 The Yang-Mills gradient flow

The Yang-Mills gradient-flow equation is given by:

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x), \quad B_\mu(0, x) = A_\mu(x) \quad (4.1)$$

where $D_\mu = \partial_\mu + g_0[B_\mu, \cdot]$, $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + g_0[B_\mu, B_\nu]$, and $A_\mu(x)$ in the initial condition is the nonevolved bare gauge field introduced in section 2.1.1. It respects all the symmetries of the nonevolved theory. For example, it is invariant under gauge transformations $\Lambda(x) \in SU(N)$, see appendix A.2.1.

The flow time t has mass-dimension -2 , i.e. the dimension of a length-squared, and it is a measure for the amount of *smearing* that the field undergoes;

effectively, the flow equation (4.1) acts as a UV filter on the x -dependence of the gauge field $A_\mu(x)$, by an amount t [207].

Note that the right-hand side of the flow equation (4.1) is proportional to the gradient of the Yang-Mills action in field space:

$$\partial_t B_\mu(t, x) = - \frac{\delta S_{YM}}{\delta A_\mu(x)} \Big|_{A_\mu(x)=B_\mu(t, x)}, \quad B_\mu(0, x) = A_\mu(x) \quad (4.2)$$

with the Euclidean action:

$$S_{YM} = -\frac{1}{2} \int_x \text{tr}(F_{\mu\nu} F_{\mu\nu}) \quad (4.3)$$

Therefore, the Yang-Mills gradient flow is driving the evolved gauge field $B_\mu(t, x)$ towards stationary points of the classical Yang-Mills action S_{YM} . In chapter 6 we will investigate the consequences of using the full QCD-like action — i.e. including the fermionic contributions to the Lagrangian — in the definition of the flow equation (4.2).

When performing perturbative calculations, the flow equation (4.1) is usually conveniently modified by adding a term that in the context of stochastic quantization¹ is known as the *Zwanziger term* [30]:

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) + \alpha_0 D_\mu \partial_\nu B_\nu(t, x), \quad B_\mu(0, x) = A_\mu(x) \quad (4.4)$$

The solution to this modified flow equation is related to the solution of (4.1) via a t -dependent gauge transformation, as we will now show.

Under a gauge transformation $\Lambda(t, x) \in SU(N)$, the gauge field B_μ transforms as:

$$B'_\mu(t, x) = \Lambda(t, x) B_\mu(t, x) \Lambda^{-1}(t, x) + \frac{1}{g_0} \Lambda(t, x) \partial_\mu \Lambda^{-1}(t, x) \quad (4.5)$$

Let the primed field be the solution to the modified flow equation, and the nonprimed field the solution to the nonmodified flow equation. The latter is obtained from the former by performing the transformation (4.5) with:

$$\begin{aligned} \partial_t \Lambda(t, x) &= -\alpha_0 g_0 (\partial_\nu B'_\nu(t, x)) \Lambda(t, x) \\ \partial_t \Lambda^{-1}(t, x) &= \alpha_0 g_0 \Lambda^{-1}(t, x) (\partial_\nu B'_\nu(t, x)) \end{aligned} \quad (4.6)$$

¹In section 6.1.1 we will elucidate the connection between the Yang-Mills gradient flow and the Langevin equation used in stochastic quantization.

and it is then straightforward to verify that (4.4) is equivalent to (4.1) by means of (4.6). The gauge invariance of (4.4) is verified by imposing:

$$\partial_t \Lambda(t, x) = \alpha_0 D_\nu \partial_\nu \Lambda(t, x), \quad \partial_t \Lambda^{-1}(t, x) = \alpha_0 D_\nu \partial_\nu \Lambda^{-1}(t, x) \quad (4.7)$$

instead of (4.6) on $\Lambda(t, x)$. The equivalence of (4.4) and (4.1) implies that observables will be independent of α_0 , and we will soon see that a convenient choice is $\alpha_0 = 1$.

The solution to (4.4) for generic α_0 reads:

$$B_\mu(t, x) = \int_y K_{t,\mu\nu}(x - y) A_\nu(y) + \int_y \int_0^t ds K_{t-s,\mu\nu}(x - y) R_\nu(s, y), \quad (4.8)$$

where the kernel can be written in terms of its Fourier transform:

$$K_{t,\mu\nu}(x) = \int_p \frac{e^{ipx}}{p^2} \left\{ (\delta_{\mu\nu} p^2 - p_\mu p_\nu) e^{-tp^2} + p_\mu p_\nu e^{-\alpha_0 t p^2} \right\} \quad (4.9)$$

and:

$$\begin{aligned} R_\mu &= 2g_0[B_\nu, \partial_\nu B_\mu] - g_0[B_\nu, \partial_\mu B_\nu] + g_0(\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] \\ &\quad + g_0^2[B_\nu, [B_\nu, B_\mu]] \end{aligned} \quad (4.10)$$

The convenience of choosing $\alpha_0 = 1$ is now apparent. With this choice the kernel (4.9) becomes $K_{t,\mu\nu}(x) = \delta_{\mu\nu} K_t(x)$, with the scalar kernel:

$$K_t(x) = \int_p e^{ipx} e^{-tp^2} = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{x^2}{4t}} \quad (4.11)$$

In [94] we introduced the ‘exponential-of-Laplacian’ notation, in which the scalar kernel is written as a nonlocal Gaussian regulator — a smearing — of the Dirac delta distribution:

$$K_t(x - y) = e^{t\Delta_x} \delta^{(d)}(x - y) \quad (4.12)$$

with $\Delta_x = \partial_x^2$. We thus conveniently rewrite the solution to the modified flow equation (4.8) with $\alpha_0 = 1$ as:

$$B_\mu(t, x) = e^{t\Delta_x} A_\mu(x) + \int_0^t ds e^{(t-s)\Delta_x} R_\mu(s, x) \quad (4.13)$$

For perturbative calculations it is useful to expand the evolved bare gauge field (4.13) as a series in powers of the bare gauge coupling g_0 :

$$B_\mu(t, x) = \sum_{n=0}^{\infty} g_0^n B_{\mu,n}(t, x) \quad (4.14)$$

with the first few orders iteratively given by²:

$$\begin{aligned}
B_{\mu,0}(t, x) &= e^{t\Delta_x} A_\mu(x) \\
B_{\mu,1}(t, x) &= \int_0^t ds e^{(t-s)\Delta_x} \{[B_{\nu,0}, 2\partial_\nu B_{\mu,0} - \partial_\mu B_{\nu,0}]\} \\
B_{\mu,2}(t, x) &= \int_0^t ds e^{(t-s)\Delta_x} \{[B_{\nu,1}, 2\partial_\nu B_{\mu,0} - \partial_\mu B_{\nu,0}] \\
&\quad + [B_{\nu,0}, 2\partial_\nu B_{\mu,1} - \partial_\mu B_{\nu,1}] + [B_{\nu,0}, [B_{\nu,0}, B_{\mu,0}]]\}
\end{aligned} \tag{4.15}$$

All fields in the flow integral in (4.15) are functions of (s, x) , the derivatives are always with respect to x , and the ‘exponential-of-Laplacian’ acts on the expressions inside brackets $\{\dots\}$. The expressions in (4.15) are diagrammatically represented in figure 4.1.

Note that at the leading order in the coupling, $B_\mu(t, x)$ is simply a Gaussian smearing of the nonevolved field $A_\mu(x)$; writing the latter in terms of its Fourier transform $\tilde{A}_\mu(p)$, we obtain:

$$B_\mu(t, x) = \int_y K_t(x - y) A_\mu(y) + \mathcal{O}(g_0) = \int_p e^{ipx} e^{-tp^2} \tilde{A}_\mu(p) + \mathcal{O}(g_0) \tag{4.16}$$

Combining (4.16) with the explicit expression for the scalar kernel $K_t(x - y)$ from (4.11), we deduce that the gradient flow is smearing the gauge field over a sphere in spacetime with a mean-square radius of $\sqrt{8t}$ in $d = 4$ [27].

4.2 The fermion flow

The conventional flow equations for the fermions [31] read:

$$\begin{aligned}
\partial_t \chi(t, x) &= \Delta \chi(t, x), & \chi(0, x) &= \psi(x) \\
\partial_t \bar{\chi}(t, x) &= \bar{\chi}(t, x) \overleftarrow{\Delta}, & \bar{\chi}(0, x) &= \bar{\psi}(x)
\end{aligned} \tag{4.17}$$

where $\psi(x), \bar{\psi}(x)$ are the bare fermion fields introduced in section 2.1.1, and:

$$\begin{aligned}
\Delta &= D^2, & D_\mu &= \partial_\mu + g_0 B_\mu \\
\overleftarrow{\Delta} &= \overleftarrow{D}^2, & \overleftarrow{D}_\mu &= \overleftarrow{\partial}_\mu - g_0 B_\mu
\end{aligned} \tag{4.18}$$

²We should point out that in most of the literature the Yang-Mills gradient-flow equation is defined using the Wilsonian normalization for the gauge field, to be discussed in section 4.4.1. In those studies, the nonevolved gauge field is rescaled to canonical normalization *after* the iterative solution (4.8) is written, which results in a different labeling of the $B_{\mu,i}$ ’s in (4.14): there the lowest order is denoted $B_{\mu,1}$, to be identified with our $B_{\mu,0}$.

Figure 4.1: Diagrammatic representation of the first three orders of the evolved gauge field's expansion in the coupling g_0 from (4.14). Shown are, from (4.15): $B_{\mu,0}$ in (a), $B_{\mu,1}$ in (b), the last term from $B_{\mu,2}$ in (c), and the first two terms from $B_{\mu,2}$ collectively in (d). Double lines stand for kernels ('exponentials-of-Laplacian') together with a flow-time integration, white blobs represent gradient-flow vertices, and the crossed blobs are the external points. Note that we make no distinction between evolved and nonevolved external fields; this will always be clear from the context.

where g_0 is the bare gauge coupling, and B_μ is the solution to (4.1). In section 6.4 we investigate the consequences of using $\overleftarrow{D}^2$ and $\overleftarrow{\overline{D}}^2$ instead of D^2 and $\overleftarrow{D}^2$ in (4.18).

The flow equations (4.17) are invariant under gauge transformations, and the evolved fields $\chi(t, x)$ and $\bar{\chi}(t, x)$ inherit all symmetry properties from the nonevolved fields $\psi(x)$, $\bar{\psi}(x)$. In particular, under a t -dependent gauge transformation $\Lambda(t, x) \in SU(N)$, the evolved fields transform as:

$$\begin{aligned}\chi'(t, x) &= \Lambda(t, x)\chi(t, x) \\ \bar{\chi}'(t, x) &= \bar{\chi}(t, x)\Lambda^{-1}(t, x)\end{aligned}\tag{4.19}$$

The fermion flow equations can be conveniently modified by a term analogous

to the Zwanziger term in (4.4):

$$\begin{aligned}\partial_t \chi(t, x) &= \{\Delta - \alpha_0 g_0 \partial_\mu B_\mu(t, x)\} \chi(t, x), & \chi(0, x) &= \psi(x) \\ \partial_t \bar{\chi}(t, x) &= \bar{\chi}(t, x) \{\overleftarrow{\Delta} + \alpha_0 g_0 \partial_\mu B_\mu(t, x)\}, & \bar{\chi}(0, x) &= \bar{\psi}(x)\end{aligned}\quad (4.20)$$

where now the B_μ fields are solutions to (4.4), and the solutions to (4.20) are related to those of (4.17) by the t -dependent gauge transformation $\Lambda(t, x)$ from (4.19) satisfying (4.6). We also note that the modified flow equations (4.20) are gauge invariant when we demand that $\Lambda(t, x)$ satisfies (4.7).

The solutions to (4.20) are given by:

$$\begin{aligned}\chi(t, x) &= \int_y K_t(x - y) \psi(y) + \int_y \int_0^t ds K_{t-s}(x - y) \Delta' \chi(s, y) \\ \bar{\chi}(t, x) &= \int_y K_t(x - y) \bar{\psi}(y) + \int_y \int_0^t ds \bar{\chi}(s, y) \overleftarrow{\Delta}' K_{t-s}(x - y)\end{aligned}\quad (4.21)$$

where:

$$\begin{aligned}\Delta' &= g_0(1 - \alpha_0) \partial_\mu B_\mu + 2g_0 B_\mu \partial_\mu + g_0^2 B_\mu B_\mu \\ \overleftarrow{\Delta}' &= -g_0(1 - \alpha_0) \partial_\mu B_\mu - 2g_0 \overleftarrow{\partial}_\mu B_\mu + g_0^2 B_\mu B_\mu\end{aligned}\quad (4.22)$$

and with the scalar kernel $K_t(x)$ in (4.11). Hence, the flow equations are effectively smearing the fermion fields over a sphere of mean-square radius $\sqrt{8t}$ in $d = 4$, analogously to the evolved gauge field $B_\mu(t, x)$.

By utilizing the ‘exponential-of-Laplacian’ notation from [94], and setting $\alpha_0 = 1$, the solutions to the modified flow equations from (4.20) read:

$$\begin{aligned}\chi(t, x) &= e^{t\Delta_x} \psi(x) + \int_0^t ds e^{(t-s)\Delta_x} \{\Delta' \chi(s, x)\} \\ \bar{\chi}(t, x) &= e^{t\Delta_x} \bar{\psi}(x) + \int_0^t ds e^{(t-s)\Delta_x} \{\bar{\chi}(s, x) \overleftarrow{\Delta}'\}\end{aligned}\quad (4.23)$$

where now:

$$\Delta' = 2g_0 B_\mu \partial_\mu + g_0^2 B_\mu B_\mu, \quad \overleftarrow{\Delta}' = -2g_0 \overleftarrow{\partial}_\mu B_\mu + g_0^2 B_\mu B_\mu \quad (4.24)$$

Analogously to (4.14), we write the evolved fermion fields as a series in powers of the bare coupling g_0 :

$$\chi(t, x) = \sum_{n=0}^{\infty} g_0^n \chi_n(t, x), \quad \bar{\chi}(t, x) = \sum_{n=0}^{\infty} g_0^n \bar{\chi}_n(t, x) \quad (4.25)$$

and the first few orders read:

$$\begin{aligned}
\chi_0(t, x) &= e^{t\Delta_x} \psi(x), & \bar{\chi}_0(t, x) &= e^{t\Delta_x} \bar{\psi}(x) \\
\chi_1(t, x) &= 2 \int_0^t ds e^{(t-s)\Delta_x} \{B_{\rho,0}(s, x) \partial_\rho \chi_0(s, x)\} \\
\bar{\chi}_1(t, x) &= -2 \int_0^t ds e^{(t-s)\Delta_x} \{\partial_\rho \bar{\chi}_0(s, x) B_{\rho,0}(s, x)\} \\
\chi_2(t, x) &= \int_0^t ds e^{(t-s)\Delta_x} \{B_{\mu,0} B_{\mu,0} \chi_0 + 2B_{\mu,1} \partial_\mu \chi_0 + 2B_{\mu,0} \partial_\mu \chi_1\} \\
\bar{\chi}_2(t, x) &= \int_0^t ds e^{(t-s)\Delta_x} \{\bar{\chi}_0 B_{\mu,0} B_{\mu,0} - 2\partial_\mu \bar{\chi}_0 B_{\mu,1} - 2\partial_\mu \bar{\chi}_1 B_{\mu,0}\}
\end{aligned} \tag{4.26}$$

with the $B_{\mu,n}$'s in (4.15). In the expressions for $\chi_2(t, x)$ and $\bar{\chi}_2(t, x)$ all fields in the flow integral are functions of (s, x) . Furthermore, derivatives are always with respect to x , and the 'exponential-of-Laplacian' acts on the expressions inside brackets $\{\dots\}$. The expressions from (4.26) are diagrammatically represented in figure 4.2.

4.3 Gradient-flow evolved propagators

The free propagator in Feynman gauge, i.e. $\xi = 1$ in (2.29), for the nonevolved gauge fields is:

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle \big|_{\mathcal{O}(g_0^0)} = D_{\mu\nu}^{ab}(x - y) = \delta^{ab} \delta_{\mu\nu} D(x - y) \tag{4.27}$$

where $D(x - y)$ is the scalar propagator in d -dimensional Euclidean spacetime:

$$\begin{aligned}
D(x - y) &= \int_p e^{ip(x-y)} \frac{1}{p^2} \\
&= \frac{\Gamma(\frac{d}{2} - 1)}{4\pi^{d/2}} \frac{1}{((x - y)^2)^{d/2-1}}
\end{aligned} \tag{4.28}$$

which satisfies:

$$\Delta_x D(x - y) = \Delta_y D(x - y) = -\delta^{(d)}(x - y) \tag{4.29}$$

The nonevolved free fermion propagator is denoted by:

$$\langle \psi_i(x) \bar{\psi}_j(y) \rangle \big|_{\mathcal{O}(g_0^0)} = \delta_{ij} S(x - y) \tag{4.30}$$

Figure 4.2: Diagrammatic representation of the first three orders of the evolved fermion field's expansion in the coupling g_0 from (4.25). Shown are, from (4.26): χ_0 in (a), χ_1 in (b), and the first, second and last term from χ_2 in (c), (d) and (e), respectively. Also here double lines stand for kernels ('exponentials-of-Laplacian') together with a flow-time integration, white blobs represent gradient-flow vertices, and the crossed blobs are the external points. The diagrams for the $\bar{\chi}_n$ expressions are identical, but with opposite arrows.

with $SU(N)$ indices (which we will later keep implicit) $i, j = 1, \dots, d(R)$, where $d(R)$ is the dimension of the representation R , and:

$$\begin{aligned}
 S(x-y) &= \int_p e^{ip(x-y)} \frac{-i\not{p}}{p^2} \\
 &= \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}} \frac{\not{x} - \not{y}}{((x-y)^2)^{d/2}}
 \end{aligned} \tag{4.31}$$

which satisfies:

$$\not{\partial}_x S(x-y) = -\not{\partial}_y S(x-y) = \delta^{(d)}(x-y) \tag{4.32}$$

We also recall the relation between the fermion and scalar propagators:

$$\not{\partial}_x D(x-y) = S(y-x) = -S(x-y) \tag{4.33}$$

The above formulas are readily generalized to the gradient flow evolved case,

where the free propagators do not receive contributions from the second flow-time integral terms in (4.8) and (4.21). Hence:

$$\begin{aligned} \langle B_\mu^a(t, x) B_\nu^b(s, y) \rangle \big|_{\mathcal{O}(g_0^0)} &= \langle e^{t\Delta_x} A_\mu^a(x) e^{s\Delta_y} A_\nu^b(y) \rangle \big|_{\mathcal{O}(g_0^0)} \\ &= \delta^{ab} \delta_{\mu\nu} e^{t\Delta_x + s\Delta_y} D(x - y) \\ &\equiv \delta^{ab} \delta_{\mu\nu} D(\bar{x}_t - \bar{y}_s) \end{aligned} \quad (4.34)$$

and:

$$\begin{aligned} \langle \chi(t, x) \bar{\chi}(s, y) \rangle \big|_{\mathcal{O}(g_0^0)} &= \langle e^{t\Delta_x} \psi(x) e^{s\Delta_y} \bar{\psi}(y) \rangle \big|_{\mathcal{O}(g_0^0)} \\ &= e^{t\Delta_x + s\Delta_y} S(x - y) \\ &\equiv S(\bar{x}_t - \bar{y}_s) \end{aligned} \quad (4.35)$$

where we have introduced a convenient notation: the bar over the coordinate in the last line of (4.34) and (4.35) represents the exponential-of-Laplacian, and the subscript the associated flow time. With this notation the scalar kernel in (4.11) now reads: $K_t(x - y) = \delta^{(d)}(\bar{x}_t - \bar{y}_t) = \delta^{(d)}(x - \bar{y}_t)$.

The evolved gauge and fermion propagators now satisfy, respectively:

$$\Delta_x D(\bar{x}_t - \bar{y}_s) = \Delta_y D(\bar{x}_t - \bar{y}_s) = -K_{t+s}(x - y) \quad (4.36)$$

and:

$$\not{\partial}_x S(\bar{x}_t - \bar{y}_s) = -\not{\partial}_y S(\bar{x}_t - \bar{y}_s) = K_{t+s}(x - y) \quad (4.37)$$

where the evolved scalar propagator is:

$$\begin{aligned} D(\bar{x}_t - \bar{y}_s) &= \int_p e^{ip(x-y)} e^{-(t+s)p^2} \frac{1}{p^2} \\ &= \frac{\gamma(\frac{d}{2} - 1, \frac{(x-y)^2}{4(t+s)})}{4\pi^{d/2}} \frac{1}{((x-y)^2)^{d/2-1}} \end{aligned} \quad (4.38)$$

and the evolved fermion propagator is:

$$\begin{aligned} S(\bar{x}_t - \bar{y}_s) &= \int_p e^{ip(x-y)} e^{-(t+s)p^2} \frac{-i\not{p}}{p^2} \\ &= \frac{\gamma(\frac{d}{2}, \frac{(x-y)^2}{4(t+s)})}{2\pi^{d/2}} \frac{\not{x} - \not{y}}{((x-y)^2)^{d/2}} \end{aligned} \quad (4.39)$$

with $\gamma(a, z)$ the lower incomplete gamma function [94]. The gradient-flow evolution thus amounts to replacing the gamma functions in (4.28) and (4.31) with their lower incomplete counterparts, whose integral representation is:

$$\gamma(a, z) = \int_0^z dt t^{a-1} e^{-t} \quad (4.40)$$

for $\text{Re}(a) > 0^3$. For $z \ll 1$, the expansion:

$$\gamma(a, z) = \frac{z^a}{a} + \mathcal{O}(z^{a+1}) \quad (4.41)$$

applied to (4.38) and (4.39) for vanishing separation at fixed positive flow time (hence, $(x - y)^2/t \rightarrow 0$) illustrates how the gradient flow regulates the short-distance singularities in x -space of the free propagators at fixed positive flow time. Indeed, in this limit one has [94]:

$$D(\bar{x}_t - \bar{y}_t) = \frac{4}{(d-2)(8\pi)^{d/2}} \frac{1}{t^{d/2-1}} \left(1 + \mathcal{O}\left(\frac{(x-y)^2}{t}\right) \right) \quad (4.42)$$

and:

$$S(\bar{x}_t - \bar{y}_t) = \frac{1}{d(8\pi)^{d/2}} \frac{\not{x} - \not{y}}{t^{d/2}} \left(1 + \mathcal{O}\left(\frac{(x-y)^2}{t}\right) \right) \quad (4.43)$$

Note that the evolved fermion propagator in (4.43) actually vanishes in this limit due to its Lorentz structure⁴, differently from the scalar propagator in (4.42).

Vice versa, in the limit $t \rightarrow 0$ at fixed nonzero separation, the exact forms in (4.38) and (4.39) recover the nonevolved result, since $\gamma(a, (x-y)^2/(8t)) \rightarrow \Gamma(a)$ in this limit.

The result of the combined limits, $t \rightarrow 0$ and $(x - y) \rightarrow 0$, thus depends on the order in which the two limits are taken in the following sense. For the scalar propagator one always produces a singularity, which is in x -space — the one of the nonevolved case — or in t -space, i.e., in the flow coordinate, when taking first $t \rightarrow 0$, or $(x - y) \rightarrow 0$, respectively. The fermion propagator, instead, vanishes when taking first $(x - y) \rightarrow 0$, whereas it recovers the original x -space singularity when taking first $t \rightarrow 0$. This makes clear that we need to consider the latter limit, i.e., $t \rightarrow 0$ at nonzero separations in order to make contact with the nonevolved correlators of the original quantum field theory.

³When analytically continuing to $\text{Re}(a) \leq 0$, $\gamma(a, z)$ has the same poles and residues at $a = 0, -1, \dots$ as $\Gamma(a)$ has at $a = 0, -1, \dots$, provided $z \neq 0$ [208].

⁴For Euclidean distances $(x - y)^2 = 0$ implies $x_\mu = y_\mu$.

4.4 Renormalization properties

One of the main reasons for the interest in the Yang-Mills gradient flow is the way in which correlation functions of evolved operators renormalize. We will discuss the renormalization properties of the evolved gauge fields, fermion fields, and composite operators in the next subsections.

4.4.1 Finiteness of evolved gauge fields

First of all, correlation functions of bare evolved gauge fields multiplied by the bare gauge coupling, i.e. $g_0 B_\mu$, do not require any renormalization. This was first proven in [28], and later an alternative proof was provided in [29]. These proofs use Wilsonian normalization for the gauge fields — both nonevolved and evolved — which is obtained from the canonical normalization used in this thesis by taking $A_{0,\mu}(x) = \mathcal{A}_{0,\mu}(x)/g_0$, where the curly notation indicates the use of Wilsonian normalization. All expressions in the previous sections of this chapter can be straightforwardly rewritten in terms of $\mathcal{B}_\mu(t, x)$; e.g. the Yang-Mills gradient-flow equation from (4.1) becomes:

$$\partial_t \mathcal{B}_\mu(t, x) = D_\nu \mathcal{G}_{\nu\mu}(t, x), \quad \mathcal{B}_\mu(0, x) = \mathcal{A}_\mu(x) \quad (4.44)$$

where now $D_\mu = \partial_\mu + [\mathcal{B}_\mu, \cdot]$ and $\mathcal{G}_{\mu\nu} = \partial_\mu \mathcal{B}_\nu - \partial_\nu \mathcal{B}_\mu + [\mathcal{B}_\mu, \mathcal{B}_\nu]$. In [28] it was proven that:

$$R(\langle \mathcal{B}_{\mu_1}(t_1, x_1) \dots \mathcal{B}_{\mu_n}(t_n, x_n) \rangle) = \langle \mathcal{B}_{\mu_1}(t_1, x_1) \dots \mathcal{B}_{\mu_n}(t_n, x_n) \rangle \quad (4.45)$$

where $R(\dots)$ is the R -operation described in section 2.4.1, which is subtracting the divergences from the correlation function, order-by-order in the perturbative expansion. We provide a concise review of the proof of this statement in section 6.2.2, and we will subsequently generalize the proof to incorporate the nonminimal flows which are the subject of chapter 6.

Now, returning to canonical normalization, equation (4.45) implies⁵:

$$\begin{aligned} R(g_0^n \langle B_{\mu_1}(t_1, x_1) \dots B_{\mu_n}(t_n, x_n) \rangle) \\ = g_0^n \langle B_{\mu_1}(t_1, x_1) \dots B_{\mu_n}(t_n, x_n) \rangle \\ = g^n(\mu) Z_g^n(g(\mu), \varepsilon) \mu^{n\varepsilon} \langle B_{\mu_1}(t_1, x_1) \dots B_{\mu_n}(t_n, x_n) \rangle \end{aligned} \quad (4.46)$$

⁵There is no rescaling anomaly in the path integral when dimensional regularization is used; this statement is true at the nonperturbative level [125].

where in the last line we used the definition of the MS renormalized coupling $g(\mu)$ from (3.2). Since $g(\mu)$ is finite by construction, we can deduce from (4.46) that:

$$R(\langle B_{\mu_1}(t_1, x_1) \dots B_{\mu_n}(t_n, x_n) \rangle) = Z_g^n(g(\mu), \varepsilon) \langle B_{\mu_1}(t_1, x_1) \dots B_{\mu_n}(t_n, x_n) \rangle \quad (4.47)$$

and therefore we could make the identification:

$$B_\mu(t, x) = Z_g^{-1}(g(\mu), \varepsilon) B_{R,\mu}(t, x) \quad (4.48)$$

i.e. identify Z_g^{-1} as the field renormalization factor of the evolved gauge field, similar to nonevolved case in (2.33) where we have $A_\mu(x) = Z_A A_{R,\mu}(x)$. We will however refrain from doing this, since it is of no added value and obscures the fact that the Yang-Mills gradient flow profoundly alters the renormalization properties of the gauge field, namely that the operator consisting of the combination $g_0 B_\mu$ does not require any renormalization.

4.4.2 Renormalizing evolved fermion fields

In [31] the first serious effort was made to incorporate fermions in the gradient flow framework, by evolving them according to the flow equations in (4.17). Contrary to expectations [28] it turned out that the evolved fermions do require renormalization. The renormalized fields are given by⁶:

$$\chi_R(t, x) = Z_\chi^{1/2} \chi(t, x), \quad \bar{\chi}_R(t, x) = Z_\chi^{1/2} \bar{\chi}(t, x) \quad (4.49)$$

where $\chi(t, x)$ and $\bar{\chi}(t, x)$ are the bare fields, and Z_χ is the field renormalization factor⁷ which will be subtracting the UV divergences that show up when calculating correlation functions containing the bare evolved fermions. The simplest of such correlators is the fermion 2-point function $\langle \chi(t, x) \bar{\chi}(t, y) \rangle$, and we can use it to determine Z_χ order by order in perturbation theory by means of the relation:

$$R(\langle \chi(t, x) \bar{\chi}(t, y) \rangle) \equiv \langle \chi_R(t, x) \bar{\chi}_R(t, y) \rangle = Z_\chi \langle \chi(t, x) \bar{\chi}(t, y) \rangle \quad (4.50)$$

⁶Note that the Z_χ factor has the opposite sign in the exponent with respect to our definition for the nonevolved fields from (2.33). This is standard practice in the literature, and it makes more sense when we take the discussion around (6.31) into account.

⁷We want to stress that this field renormalization factor Z_χ is fundamentally different from Z_ψ in (2.33); it is genuinely associated with the flow and cannot be derived without evolving the fields. This will become even more clear in chapter 6, where we explicitly derive how the $\mathcal{O}(g_0^2)$ contribution to Z_χ depends on which particular flow equations are being used.

Figure 4.3: Next-to-leading order diagrams for the 2-point function of fermions evolved with the flow equation in (4.20). White blobs represent gradient-flow vertices, black blobs the usual QCD vertices, and double lines indicate a kernel including integration over the associated flow time.

The diagrams contributing at $\mathcal{O}(g_0^2)$ are shown in figure 4.3. We collect the dimensionally regularized results in table 4.1, where $S(\bar{x}_t - \bar{y}_t)$ is the evolved fermion propagator from (4.39), i.e. $\langle \chi(t, x) \bar{\chi}(t, y) \rangle$ at $\mathcal{O}(g_0^0)$. The sum of the contributions in table 4.1 at $\mathcal{O}(g_0^2)$ yields:

$$\langle \chi(t, x) \bar{\chi}(t, y) \rangle \big|_{\mathcal{O}(g_0^2)} = -C_2(R) \frac{3}{\varepsilon} \frac{g_0^2}{(4\pi)^2} S(\bar{x}_t - \bar{y}_t) + \mathcal{O}(\varepsilon^0) \quad (4.51)$$

so that the right-hand side of equation (4.50) becomes:

$$\begin{aligned} Z_\chi \langle \chi(t, x) \bar{\chi}(t, y) \rangle \\ = Z_\chi S(\bar{x}_t - \bar{y}_t) \left(1 - C_2(R) \frac{3}{\varepsilon} \frac{g_0^2}{(4\pi)^2} + \mathcal{O}(\varepsilon^0 g_0^2) + \mathcal{O}(g_0^4) \right) \end{aligned} \quad (4.52)$$

and hence the renormalization factor for fermions evolved with the flow equation (4.20), in terms of the MS renormalized coupling $g(\mu)$ defined in (3.2), reads:

$$Z_\chi(g(\mu), \varepsilon) = 1 + C_2(R) \frac{3}{\varepsilon} \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4(\mu)) \quad (4.53)$$

This result was first obtained in [31], and the calculation was repeated in [95], where we subsequently generalized to the nonminimal gradient flow cases. The

diagram	value
YM1	$-\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)$
YM2 + YM3	$-\frac{2}{\varepsilon} + \mathcal{O}(\varepsilon^0)$
YM4 + YM5	$\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)$
YM6 + YM7	$-\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)$
YM8	$\mathcal{O}(\varepsilon^0)$

Table 4.1: UV divergences of the diagrams from figure 4.3 in units of $C_2(R) \frac{g_0^2}{(4\pi)^2} S(\bar{x}_t - \bar{y}_t)$.

latter will be the subject of chapter 6.

To summarize, any correlator consisting of m operators $g_0 B_\mu$, n $\bar{\chi}$ and n χ fields, is rendered finite by multiplication with Z_χ^n :

$$\begin{aligned}
& R(\langle g_0 B_{\mu_1}(t_1, x_1) \dots g_0 B_{\mu_m}(t_m, x_m) \bar{\chi}(s_1, y_1) \dots \\
& \quad \dots \bar{\chi}(s_n, y_n) \chi(u_1, z_1) \dots \chi(u_n, z_n) \rangle) \\
& = g^m(\mu) Z_g^m(g(\mu), \varepsilon) Z_\chi^n(g(\mu), \varepsilon) \mu^{m\varepsilon} \langle B_{\mu_1}(t_1, x_1) \dots \\
& \quad \dots B_{\mu_m}(t_m, x_m) \bar{\chi}(s_1, y_1) \dots \bar{\chi}(s_n, y_n) \chi(u_1, z_1) \dots \chi(u_n, z_n) \rangle
\end{aligned} \tag{4.54}$$

which follows from (4.46) and (4.50).

4.4.3 Gradient-flow evolved composite operators

In section 2.4.2 we discussed the renormalization properties of nonevolved composite operators. The gradient flow significantly alters these properties, and the result can be summarized as follows: there are no renormalization factors associated with gradient-flow evolved composite operators, other than those of the evolved elementary fermion fields contained in the composite operators. This is due to the fact that the divergent contributions associated with the short-

Figure 4.4: General mechanism generating UV divergences associated with composite operators, illustrated using the t -channel. The s and u -channels generate similar divergences.

distance behavior — when the spacetime positions of two (or more) operators are getting arbitrarily close — are rendered finite by the gradient flow.

Intuitively this last statement makes sense: short-distance singularities occur due to the quantum fluctuations of the fields at these short-distances; however, the gradient flow smoothens these violent fluctuations on length-scales smaller than $\sim \sqrt{8t}$, and hence no singular short-distance behavior is to be expected.

We can illustrate what happens in a diagrammatic way. In figure 4.4 we show the mechanism responsible for generating divergences specifically associated with the compositeness of nonevolved composite operators: when the two distinct spacetime locations coincide, a closed loop is formed, which generically leads to UV divergences not present in the original correlation function, and these must be subtracted⁸.

Now, we take as the external fields the evolved gauge fields $B_{\mu,0}$ with coordinates (t_1, x_1) and (t_2, x_2) , and consider the limit $x_1, x_2 \rightarrow x$. The propagators attached to the external point x both carry a Gaussian damping factor $\sim e^{-tp^2}$, with $t = t_1, t_2$ and p_μ the loop momentum, which will guarantee the finiteness of the loop integral.

Next consider the case where we have two flow-time integrations, as shown in figure 4.5. The flow-time integrations are over s_1, s_2 and go from 0 to t , with damping factors $\sim e^{-(t-s)p^2}$, with $s = s_1, s_2$, see the expression for $B_{\mu,1}$ from (4.15). Thus, the only propagator left in the loop will be damped whenever $s_1, s_2 < t$, or when only one s equals t , and will result in a finite loop momentum integration. The remaining question is what happens when both $s_1 = s_2 = t$. In this case, the propagator carries the damping factor $\sim e^{-2tp^2}$, as can be seen

⁸ Alternatively it may lead to a situation where the newly-formed divergent diagram cancels an already existing divergence; this is what happens when the newly-formed composite operator is a (nonanomalous) conserved current.

Figure 4.5: Construction of composite operator, illustrated using the t -channel, with two flow-time integrations.

from the flow-time integration's integrand:

$$\begin{aligned}
 & \int_y e^{(2t-s_1-s_2)\Delta_x} \left\{ \delta^{(d)}(x-y) D(\bar{x}_{s_1} - \bar{y}_{s_2}) \right\} \Big|_{s_1=s_2=t} \\
 &= \int_y \delta^{(d)}(x-y) D(\bar{x}_t - \bar{y}_t) \\
 &= \lim_{y \rightarrow x} D(\bar{x}_t - \bar{y}_t)
 \end{aligned} \tag{4.55}$$

which is finite according to (4.42). Therefore, we can conclude that in the gradient-flow evolved case there are no divergences associated with the coinciding-spacetime-points limit inherent in the construction of composite operators from elementary fields, at this order in the coupling. This reasoning generalizes to all orders in the coupling, and any number of coinciding operator positions, by noticing that any closed loop containing only flow-time integrations, i.e. double lines, vanishes identically, and that having more nested flow-time integrations does not change any of the conclusions drawn above [29].

The gradient-flow evolved composite operators that we will use in the rest of this thesis are the Yang-Mills Lagrangian density operator [27]:

$$E(t, x) = g_0^2 \mathcal{L}_{YM} \Big|_{A_\mu(x)=B_\mu(t, x)} = \frac{g_0^2}{4} G_{\mu\nu}^a(t, x) G_{\mu\nu}^a(t, x) \tag{4.56}$$

where all the fields are bare, and the evolved fermion bilinear operators. Since we will work with massless fermions there is no mixing between any of these operators upon renormalization. Due to the renormalization properties of the evolved elementary fields summarized in (4.54), the evolved composite operator $E(t, x)$ from (4.56) simply renormalizes as:

$$E(t, x) = [E(t, x)] \tag{4.57}$$

where operators in square brackets denote the renormalized operators introduced in section 2.4.2. In the case of fermion bilinears, (4.54) implies:

$$J_\Gamma(t, x) \equiv \bar{\chi}_0(t, x) \Gamma \chi_0(t, x) = Z_\chi^{-1} \bar{\chi}_R(t, x) \Gamma \chi_R(t, x) = Z_\chi^{-1} [J_\Gamma(t, x)] \quad (4.58)$$

with $\Gamma = \mathbb{1}, \gamma_\mu, [\gamma_\mu, \gamma_\nu], \gamma_5, \gamma_\mu \gamma_5$, possibly combined with color and/or flavor matrices.

4.4.4 Small flow-time expansion

All the differences in renormalization properties between evolved and nonevolved operators — both elementary and composite — have now been discussed. The next step is to establish the connection between the evolved and the nonevolved operators. This is done by studying the small flow-time t limit, first introduced in [28].

The operator expansion that we can make in the small- t limit is similar in spirit and form to the conventional *operator product expansion* (OPE) [209], which deals with the small spacetime separation limit of operators: consider two renormalized operators $[O_1], [O_2]$ separated by a small distance x , then the OPE reads:

$$[O_1(x)][O_2(0)] \sim \sum_n c_{12}^n(x) [O_n(0)] \quad (4.59)$$

with the coefficients $c_{12}^n(x)$ containing the possibly divergent behavior as $x \rightarrow 0$, the operators $[O_n]$ have the same symmetry properties as the product $[O_1][O_2]$, and the canonical mass-dimensions of the $[O_n]$ operators are equal or smaller than that of the product $[O_1][O_2]$; when the dimensions are equal the $c_{12}^n(x)$ coefficients may be logarithmically divergent as $x \rightarrow 0$, and when the canonical mass-dimensions of the $[O_n]$ are smaller than that of the product $[O_1][O_2]$ the coefficients have a power-like divergence dictated by dimensional analysis, possibly accompanied by logarithmic corrections. Corrections to the right-hand side of (4.59) are of the order $\mathcal{O}(x^2)$. For an in-depth review of the OPE see e.g. [97].

The small flow-time expansion, though distinctly different from the OPE, follows the same recurring philosophy: a limit of an argument of an operator is taken which causes divergences to arise, and these divergences have the most general structure consistent with the symmetries involved; this was the essence of operator mixing in section 2.4.2, and it is the essence of the OPE in (4.59). So, for a generic renormalized gradient-flow evolved operator $[O(t, x)]$, we have

as $t \rightarrow 0$:

$$[O(t, x)] \sim \sum_n c_O^n(t) [O_n(x)] \quad (4.60)$$

where the coefficients $c_O^n(t)$ contain the possibly divergent behavior as $t \rightarrow 0$, the operators $[O_n(x)]$ have the same symmetry properties as $[O(t, x)]$, and the canonical mass-dimensions of $[O_n(x)]$ are smaller or equal to that of $[O(t, x)]$, fully analogous to the OPE in (4.59). When the mass-dimensions are the same, $c_O^n(t)$ may diverge logarithmically as $t \rightarrow 0$, and when the mass-dimensions of $[O_n(x)]$ are smaller than that of $[O(t, x)]$, the $c_O^n(t)$ contain a power-like divergence possibly with logarithmic corrections.

For illustration, we consider the small flow-time expansion of $E(t, x)$ in (4.56), recalling that $[E(t, x)] = E(t, x)$. Since it is a gauge-invariant Lorentz scalar it can mix with the identity⁹, and the small flow-time expansion reads:

$$E(t, x) = c_{-2}(t) + c_E(t)[E(x)] + \mathcal{O}(t) \quad (4.61)$$

where the c -coefficients are singular functions of t : $c_{-2}(t) \sim t^{-2}$ and $c_E(t) \sim \log t$, and furthermore depend on the renormalized coupling $g(\mu)$. Taking the expectation value on both sides of (4.61), we arrive at [28]:

$$E(t, x) = \langle E(t) \rangle + c_E(t) ([E(x)] - \langle [E] \rangle) + \mathcal{O}(t) \quad (4.62)$$

where the gluonic condensate $\langle [E] \rangle$ is often omitted, since it vanishes to all orders in perturbation theory; it is a purely nonperturbative object, see e.g. [210].

Next we consider the small flow-time expansion of the renormalized scalar fermion-bilinear $[J(t, x)] = \bar{\chi}_R(t, x)\chi_R(t, x)$. Since it is a gauge-invariant Lorentz scalar, it too mixes with the identity:

$$[J(t, x)] = c_{-3/2}(t) + c_{\bar{\psi}\psi}(t)[j(x)] + \mathcal{O}(t) \quad (4.63)$$

with $[j(x)] = [\bar{\psi}_R(x)\psi_R(x)]$, and the singular coefficients now go like $c_{-3/2} \sim t^{-3/2}$ and $c_{\bar{\psi}\psi}(t) \sim \log t$. Taking the expectation value at both sides of (4.63), we obtain:

$$[J(t, x)] = \langle [J(t)] \rangle + c_{\bar{\psi}\psi}(t) ([j(x)] - \langle [j] \rangle) + \mathcal{O}(t) \quad (4.64)$$

where both condensates on the right-hand side are nonvanishing only at the nonperturbative level¹⁰.

⁹Since we are considering massless fermions, there is no additional mixing; see section 2.4.2.

¹⁰When we do include massive fermions, $\langle J(t) \rangle$ is nonvanishing also at the perturbative level, see [211].

Renormalization group analysis

We can use the renormalization group methods introduced in chapter 3 to obtain information on the singular coefficients $c_E(t)$ and $c_{\bar{\psi}\psi}(t)$ [28, 211]. Due to the finiteness of $E(t, x)$ we can derive via (4.62):

$$\frac{d}{d \log \mu} \left(c_E(t) ([E(x)] - \langle [E] \rangle) + \mathcal{O}(t) \right) = 0 \quad (4.65)$$

Since there is no mixing we have:

$$[E(x)] = Z_E^{-1}(g(\mu), \varepsilon) E(x) \quad (4.66)$$

and note that since $c_E(t)$ is a dimensionless function, it will depend only on the dimensionless product $\sqrt{t}\mu$ and the renormalized coupling $g(\mu)$. Combining (4.65) with (4.66), and recalling that bare quantities do not depend on μ , we obtain:

$$\left(\frac{\partial}{\partial \log \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_E(g) \right) c_E(\sqrt{t}\mu, g(\mu)) = \mathcal{O}(t) \quad (4.67)$$

with $\beta(g) = \frac{dg}{d \log \mu} \big|_{\varepsilon=0}$ and $\gamma_E(g) = \frac{d \log Z_E}{d \log \mu}$.

Following the methods described in section 3.6 — where now we have the scale $\sqrt{t}$ instead of x — we parametrize the solution to (4.67) as:

$$c_E(\sqrt{t}\mu, g(\mu)) = F(g(\sqrt{t})) \mathcal{C}_E(g(\sqrt{t}), g(\mu)) \quad (4.68)$$

where $F(g(\sqrt{t}))$ is a renormalization-group invariant dimensionless function of $g(\sqrt{t})$ which is left unspecified by (4.67), in contrast to $\mathcal{C}_E$, which solves (4.67):

$$\mathcal{C}_E(g(\sqrt{t}), g(\mu)) = \exp \left(\int_{g(\sqrt{t})}^{g(\mu)} dg \frac{\gamma_E(g)}{\beta(g)} \right) \quad (4.69)$$

which can be compared with (3.51); the sign difference in the exponents in (3.51) and (4.69) can be traced back to the sign difference in front of the anomalous dimensions in (3.48) versus (4.67), which in turn stems from the fact that the former renormalization group equation deals with the anomalous scaling behavior of a correlation function, while the latter deals with the anomalous small- t scaling behavior of the coefficient $c_E(t)$.

We can perform the same analysis for the small- t expansion of the fermion bilinear given in (4.63). The main difference is that $\bar{\chi}\chi$ is not renormalization-group invariant, due to the fermionic renormalization factor Z_χ which we introduced in section 4.4.2, i.e.:

$$[J(t, x)] = \bar{\chi}_R(t, x) \chi_R(t, x) = Z_\chi(g(\mu), \varepsilon) \bar{\chi}(t, x) \chi(t, x) \quad (4.70)$$

while its nonevolved counterpart renormalizes via:

$$[j(x)] = Z_{\bar{\psi}\psi}^{-1}(g(\mu), \varepsilon) j(x) \quad (4.71)$$

and the explicit form of $Z_{\bar{\psi}\psi}$ in the $\overline{\text{MS}}$ -scheme, up to and including $\mathcal{O}(g^8(\mu))$, can be reconstructed from the results presented in [32]. The renormalization-group equation for $c_{\bar{\psi}\psi}(\sqrt{t}\mu, g(\mu))$ now follows from:

$$\frac{d}{d \log \mu} \left([J(t, x)] - \langle [J(t)] \rangle - c_{\bar{\psi}\psi}(\sqrt{t}\mu, g) ([j(x)] - \langle [j] \rangle) \right) = \mathcal{O}(t) \quad (4.72)$$

which, combined with (4.70) and (4.71), and noting that bare quantities do not depend on μ , reads:

$$\left(\frac{\partial}{\partial \log \mu} + \beta(g) \frac{\partial}{\partial g} + \gamma_{\bar{\chi}\chi}(g) - \gamma_{\bar{\psi}\psi}(g) \right) c_{\bar{\psi}\psi}(\sqrt{t}\mu, g(\mu)) = \mathcal{O}(t) \quad (4.73)$$

with $\gamma_{\bar{\chi}\chi} = -\frac{d \log Z_{\bar{\chi}\chi}}{d \log \mu}$ and $\gamma_{\bar{\psi}\psi} = \frac{d \log Z_{\bar{\psi}\psi}}{d \log \mu}$. The minus sign in the definition of $\gamma_{\bar{\chi}\chi}$ stems from the fact that $Z_{\bar{\chi}\chi}$ is defined with the opposite sign in the exponent with respect to our convention in (2.33); see (4.49) and also footnote 6. The sign-difference between the anomalous dimensions in (4.73) is however crucial, and is independent of the conventions chosen.

We again parametrize the solution for $c_{\bar{\psi}\psi}$ as in (4.68):

$$c_{\bar{\psi}\psi}(\sqrt{t}\mu, g(\mu)) = F'(g(\sqrt{t})) \mathcal{C}_{\bar{\psi}\psi}(g(\sqrt{t}, g(\mu))) \quad (4.74)$$

where $F'(g(\sqrt{t}))$ is an RG invariant function unspecified by (4.73), and $\mathcal{C}_{\bar{\psi}\psi}$ reads [94]:

$$\mathcal{C}_{\bar{\psi}\psi}(g(\sqrt{t}), g(\mu)) = \exp \left(\int_{g(\sqrt{t})}^{g(\mu)} dg \frac{\gamma_{\bar{\psi}\psi}(g) - \gamma_{\bar{\chi}\chi}(g)}{\beta(g)} \right) \quad (4.75)$$

and we see that the anomalous dimension of $c_{\bar{\psi}\psi}(t)$ is proportional to the difference of the nonevolved and evolved anomalous dimension of the fermion bilinear.

The coefficients $c_E(t)$ and $c_{\bar{\psi}\psi}(t)$ can be further analyzed in the asymptotic UV regime, following the methods described in section 3.6.

In section 5.4.4 we will determine the logarithmic small- t coefficient of the vector current $\bar{\chi}\gamma_\mu\chi$ explicitly from its 2-point correlation function.

Chapter 5

Conserved vector current and the gradient flow

This chapter is based on the paper [94].

5.1 Introduction

In this chapter we further explore the properties of gradient-flow evolved composite operators in the fermion sector of QCD-like theories with massless fermions. In particular, we will compute to $\mathcal{O}(g_0^2)$ the Euclidean 2-point correlation function of the flavor-singlet vector current evolved¹ to a gradient-flow time t :

$$\Pi_{\mu\nu}^V(t, x - y) = \langle J_\mu(t, x) J_\nu(t, y) \rangle \quad (5.1)$$

where:

$$J_\mu(t, x) = \bar{\chi}(t, x) \gamma_\mu \chi(t, x) \quad (5.2)$$

which is the gradient-flow evolved version of the conserved current $j_\mu(x) = \bar{\psi}(x) \gamma_\mu \psi(x)$ associated with $U(1)_B$ in (2.17), and $\chi(t, x), \bar{\chi}(t, x)$ given in (4.23). We will show that the evolved 2-point correlation function requires a multiplicative renormalization consistent with the universal Z_χ factor introduced in section 4.4.2. This suggests that the only renormalization of evolved fermionic composite operators is the one induced by the renormalization of the evolved

¹We use the upper-case notation J to indicate that the current is gradient-flow evolved.

elementary fermion fields and the gauge coupling, independently of their tensor structure, as anticipated in section 4.4.3. Therefore, our result yields the leading $\mathcal{O}(g^2)$ contribution to the anomalous dimension of all evolved fermion-bilinear currents. This also means that the evolved vector current acquires a t -dependent anomalous dimension, in contrast to the nonevolved case, where it has no anomalous dimension, as discussed in section 2.4.4.

Moreover, we will make contact with the nonevolved case by deriving the leading contribution to the small flow-time expansion (STE) — see section 4.4.4 — of the evolved fermion-bilinear currents from the corresponding connected 2-point correlators in the limit of small flow-time t , at fixed separation $|x - y|$.

We will also clarify how, in the evolved case, current conservation up to terms that vanish for small t no longer entails nonrenormalization.

At various intermediate steps we will be employing coordinate space techniques, instead of the more conventional momentum space approach. They are especially useful to illustrate the behavior of two limits: vanishing flow time at fixed separation and vanishing separation at fixed flow time.

The chapter is organized as follows. Section 5.2 discusses the vanishing of the 1-point correlator of the divergence of the evolved vector current $\langle \partial_\mu J_\mu(t, x) \rangle$, and its main purpose is to introduce some coordinate space techniques. In section 5.3 we review the nonevolved flavor singlet 2-point vector correlator to $\mathcal{O}(g_0^2)$, generalizing the original calculation [32] to $SU(N)$. Section 5.4 is devoted to the calculation of the evolved 2-point vector correlator, where sections 5.4.1 and 5.4.2 contain the analytic calculation to $\mathcal{O}(g_0^2)$. Renormalization at $\mathcal{O}(g_0^2)$ is treated in section 5.4.3. In section 5.4.4 we determine the leading contribution to the STE of evolved fermion-bilinear currents from the corresponding connected 2-point correlators. Section 5.5 discusses the conservation of the evolved vector current up to terms of $\mathcal{O}(t)$.

Perturbative expansion of the evolved vector current

Before commencing we introduce the expansion in powers of the bare coupling g_0 of the evolved vector current from (5.2), analogously to the expansions for $B_\mu(t, x)$ in (4.14) and $\chi(t, x), \bar{\chi}(t, x)$ in (4.25):

$$J_\mu(t, x) = \sum_{n=0}^{\infty} g_0^n J_{\mu,n}(t, x) \quad (5.3)$$

where $J_{\mu,n}(t, x)$ for $n = 0, 1, 2$ read:

$$\begin{aligned} J_{\mu,0}(t, x) &= \bar{\chi}_0(t, x) \gamma_\mu \chi_0(t, x) \\ J_{\mu,1}(t, x) &= \bar{\chi}_1(t, x) \gamma_\mu \chi_0(t, x) + \bar{\chi}_0(t, x) \gamma_\mu \chi_1(t, x) \\ J_{\mu,2}(t, x) &= \bar{\chi}_2(t, x) \gamma_\mu \chi_0(t, x) + \bar{\chi}_0(t, x) \gamma_\mu \chi_2(t, x) + \bar{\chi}_1(t, x) \gamma_\mu \chi_1(t, x) \end{aligned} \quad (5.4)$$

with χ_n and $\bar{\chi}_n$, for $n = 0, 1, 2$ given in (4.26).

5.2 The 1-point correlator $\langle \partial_\mu J_\mu(t, x) \rangle$ evolved by the gradient flow

Given the operators $O_\pm(t, x)$:

$$O_\pm(t, x) = \bar{\chi}(t, x) \not{D}_\pm \chi(t, x) \quad (5.5)$$

with:

$$\not{D}_\pm = \overrightarrow{\not{D}} \pm \overleftarrow{\not{D}} \quad (5.6)$$

and ($D_\mu = \partial_\mu + g_0 B_\mu$):

$$\begin{aligned} \not{D}_+ &= \overrightarrow{\not{D}} + \overleftarrow{\not{D}} \\ \not{D}_- &= \overrightarrow{\not{D}} - \overleftarrow{\not{D}} + 2g_0 \not{B} \end{aligned} \quad (5.7)$$

we recognize that O_+ is the divergence of the vector current $O_+ = \partial_\mu J_\mu$, whereas O_- is the operator that enters the fermion equation of motion. The 1-point correlator $\langle O_-(t, x) \rangle$ has been studied to $\mathcal{O}(g_0^2)$ in [55], where, differently from its nonevolved counterpart, it was found to be nonzero and to renormalize with a new counterterm induced by the gradient flow.

In this section we consider the 1-point correlator $\langle O_+(t, x) \rangle = \langle \partial_\mu J_\mu(t, x) \rangle$. Its nonevolved version vanishes (before and after subtraction of divergences) independently of whether j_μ is conserved or not. In fact, $\langle [\partial_\mu j_\mu(x)] \rangle = \partial_\mu \langle [j_\mu(x)] \rangle = 0$ holds for the simple reason that the momentum at the vertex for j_μ vanishes, but, in the vector case, also because $\langle [j_\mu(x)] \rangle$ itself vanishes due to the Euclidean equivalent of Lorentz invariance, i.e. 4 dimensional space(time) rotations, and the fact that j_μ is odd under charge conjugation².

For the evolved correlator, the same chain of identities:

$$\langle [O_+(t, x)] \rangle = \langle [\partial_\mu J_\mu(t, x)] \rangle = \partial_\mu \langle [J_\mu(t, x)] \rangle = 0 \quad (5.8)$$

² $\langle j_\mu(x) \rangle = \langle C^\dagger C j_\mu(x) C^\dagger C \rangle = -\langle j_\mu(x) \rangle$.

Figure 5.1: Diagrams that contribute to the 1-point correlator $\langle \partial_\mu J_\mu(t, x) \rangle$ at leading order (D01) and next-to-leading order (D02-D06). Notation is explained in figures 4.1 and 4.2, with the addition that black blobs represent the usual QCD vertices.

holds provided one can write the evolved current in terms of its Fourier transform as $J_\mu(t, x) = \int_p e^{ipx} \tilde{J}_\mu(t, p)$. This is true for the bare current, with χ and $\bar{\chi}$ in (4.21), and it cannot be spoiled by renormalization, analogously to the nonevolved case. Again, the last equality in (5.8) is also implied by Euclidean rotational invariance and the fact that $J_\mu(t, x)$ is odd under charge conjugation.

We verify (5.8) diagrammatically up to $\mathcal{O}(g_0^2)$ to illustrate the coordinate space approach in the context of evolved correlators. The diagrams that contribute to $\langle \partial_\mu J_\mu(t, x) \rangle$ to order g_0^2 are shown in figure 5.1³

³The naming of the diagrams corresponds to the one in [55].

5.2.1 Leading order, $\mathcal{O}(g_0^0)$

The leading order contribution is given by diagram D01 in figure 5.1 and reads:

$$\begin{aligned}
 \langle \partial_\mu J_{\mu,0}(t, x) \rangle_0 &= \int_y \delta^{(d)}(x - y) \langle \bar{\chi}_0(t, x) (\overrightarrow{\not{\partial}}_y + \overleftarrow{\not{\partial}}_x) \chi_0(t, y) \rangle_0 \\
 &= - \int_y \delta^{(d)}(x - y) \text{tr}[(\not{\partial}_y + \not{\partial}_x) S(\bar{y}_t - \bar{x}_t)] \\
 &= -4 d(R) \int_y \delta^{(d)}(x - y) (\delta^{(d)}(\bar{x}_t - \bar{y}_t) - \delta^{(d)}(\bar{x}_t - \bar{y}_t)) \\
 &= 0
 \end{aligned} \tag{5.9}$$

where $d(R)$ is the dimension of the fermion representation R (see appendix A.1), and $\langle \dots \rangle_0$ stands for the connected contribution to the path-integral average over the Euclidean free-theory measure. In the third line we used (4.32), which implies that the correlator vanishes at leading order.

5.2.2 Next-to-leading order, $\mathcal{O}(g_0^2)$

The next-to-leading order contribution is given by diagrams D02 through D06 in figure 5.1. Diagram D02 contains the insertion of two vertices from the QCD action — QCD vertices in short —, and is the only diagram present at $\mathcal{O}(g_0^2)$ in the nonevolved case, where it vanishes. Diagram D03 contains the insertion of one QCD vertex, while D04-D06 do not contain QCD vertices. We show that each one of these five contributions vanishes separately. The contribution from D02 reads:

$$\begin{aligned}
 \text{D02} &= \frac{1}{2} \langle \partial_\mu J_{\mu,0}(t, x) \int_{z_1} (-g_0 \bar{\psi} \not{A} \psi) \int_{z_2} (-g_0 \bar{\psi} \not{A} \psi) \rangle_0 \\
 &= -\text{tr}(T^a T^a) g_0^2 \int_{z_1, z_2} D(z_1 - z_2) \text{tr} \left[S(z_2 - \bar{x}_t) (\overrightarrow{\not{\partial}} + \overleftarrow{\not{\partial}}) S(\bar{x}_t - z_1) \right. \\
 &\quad \left. \times \gamma_\rho S(z_1 - z_2) \gamma_\rho \right] \\
 &= \text{tr}(T^a T^a) (2 - d) g_0^2 \int_{z_1, z_2} D(z_1 - z_2) \delta^{(d)}(\bar{x}_t - z_1) \text{tr} \left[S(\bar{x}_t - z_2) \right. \\
 &\quad \left. \times (S(z_1 - z_2) + S(z_2 - z_1)) \right] \\
 &= 0
 \end{aligned} \tag{5.10}$$

where we employed $\gamma_\rho \gamma_\alpha \gamma_\rho = (2-d)\gamma_\alpha$, the relation:

$$S(z_2-x)(\vec{\partial} + \overleftarrow{\partial})S(x-z_1) = -\delta^{(d)}(x-z_1)S(x-z_2) - \delta^{(d)}(x-z_2)S(x-z_1) \quad (5.11)$$

and the symmetry relations $D(z_1-z_2) = D(z_2-z_1)$ and $S(z_1-z_2) = -S(z_2-z_1)$. The contribution from D03 reads:

$$\begin{aligned} D03 &= g_0 \langle \partial_\mu J_{\mu,1}(t, x) \int_z (-g_0 \bar{\psi} A \psi) \rangle_0 \\ &= 2g_0^2 \text{tr}(T^a T^a) \int_z \int_0^t ds \text{tr} \left[\vec{\partial}_x \left\{ e^{(t-s)\square_x} \left\{ D(\bar{x}_s - z) \right. \right. \right. \\ &\quad \times S(\bar{x}_s - z) \overleftarrow{\partial}_x \left. \right\} S(z - \bar{x}_t) \left. \right\} \Big] \\ &\quad - 2g_0^2 \text{tr}(T^a T^a) \int_z \int_0^t ds \text{tr} \left[\vec{\partial}_x \left\{ e^{(t-s)\square_x} \left\{ D(\bar{x}_s - z) \right. \right. \right. \\ &\quad \times \overleftarrow{\partial}_x S(z - \bar{x}_s) \left. \right\} S(\bar{x}_t - z) \left. \right\} \Big] \\ &= 2g_0^2 \text{tr}(T^a T^a) \int_z \int_0^t ds \text{tr} \left[\vec{\partial}_x \left\{ e^{(t-s)\square_x} \left\{ D(\bar{x}_s - z) \left(S(\bar{x}_s - z) \overleftarrow{\partial}_x \right. \right. \right. \right. \\ &\quad \left. \left. \left. + \overleftarrow{\partial}_x S(z - \bar{x}_s) \right) \right\} S(z - \bar{x}_t) \right\} \right] = 0 \end{aligned} \quad (5.12)$$

where we employed (4.32). The final contribution is given by:

$$D04 + D05 + D06 = g_0^2 \langle \partial_\mu J_{\mu,2}(t, x) \rangle_0 \quad (5.13)$$

By (5.4), one obtains:

$$\begin{aligned} D05 &= g_0^2 \langle \bar{\chi}_1(t, x) (\vec{\partial} + \overleftarrow{\partial}) \chi_1(t, x) \rangle_0 \\ &= g_0^2 \int_y \delta^{(d)}(x-y) \langle \bar{\chi}_1(t, x) (\vec{\partial}_y + \overleftarrow{\partial}_x) \chi_1(t, y) \rangle_0 \\ &= 4g_0^2 \int_y \delta^{(d)}(x-y) \int_0^t ds_1 \int_0^t ds_2 e^{(t-s_1)\Delta_x} e^{(t-s_2)\Delta_y} \left\{ \text{tr} \left[(\vec{\partial}_x + \vec{\partial}_y) \right. \right. \\ &\quad \times \left. \left. \left\{ \partial_\mu^x \partial_\mu^y S(\bar{y}_{s_2} - \bar{x}_{s_1}) D(\bar{x}_{s_1} - \bar{y}_{s_2}) \right\} \right] \right\} \\ &= 4g_0^2 \int_y \delta^{(d)}(x-y) \int_0^t ds_1 \int_0^t ds_2 e^{(t-s_1)\Delta_x} e^{(t-s_2)\Delta_y} \left\{ \text{tr} \left[\partial_\mu^x \partial_\mu^y \right. \right. \\ &\quad \times \left. \left. (\vec{\partial}_x + \vec{\partial}_y) S(\bar{y}_{s_2} - \bar{x}_{s_1}) \right] D(\bar{x}_{s_1} - \bar{y}_{s_2}) \right. \\ &\quad \left. \left. + \text{tr} \left[(\vec{\partial}_x + \vec{\partial}_y) D(\bar{x}_{s_1} - \bar{y}_{s_2}) \partial_\mu^x \partial_\mu^y S(\bar{y}_{s_2} - \bar{x}_{s_1}) \right] \right\} = 0 \end{aligned} \quad (5.14)$$

where in the last line we used $(\not{\partial}_x + \not{\partial}_y)S(\bar{y}_{s_2} - \bar{x}_{s_1}) = 0$ and $(\not{\partial}_x + \not{\partial}_y)D(\bar{x}_{s_1} - \bar{y}_{s_2}) = -S(\bar{x}_{s_1} - \bar{y}_{s_2}) + S(\bar{x}_{s_1} - \bar{y}_{s_2}) = 0$. Finally⁴:

$$\begin{aligned}
D04 + D06 &= g_0^2 \langle \bar{\chi}_2(t, x) (\vec{\not{\partial}} + \overleftarrow{\not{\partial}}) \chi_0(t, x) + \bar{\chi}_0(t, x) (\vec{\not{\partial}} + \overleftarrow{\not{\partial}}) \chi_2(t, x) \rangle_0 \\
&= g_0^2 \int_y \delta^{(d)}(x - y) \left\{ \langle \bar{\chi}_2(t, y) (\vec{\not{\partial}}_x + \overleftarrow{\not{\partial}}_y) \chi_0(t, x) \rangle_0 \right. \\
&\quad \left. + \langle \bar{\chi}_0(t, x) (\vec{\not{\partial}}_y + \overleftarrow{\not{\partial}}_x) \chi_2(t, y) \rangle_0 \right\} \\
&= -4 \text{tr}(T^a T^a) g_0^2 \int_y \delta^{(d)}(x - y) \int_0^t ds e^{(t-s)\Delta_y} (\partial_\lambda^x + \partial_\lambda^y) \\
&\quad \times \left\{ \int_z \delta^{(d)}(y - z) D(\bar{y}_s - \bar{z}_s) (S_\lambda(\bar{y}_s - \bar{x}_t) + S_\lambda(\bar{x}_t - \bar{y}_s)) \right\} \\
&\quad - 16 \text{tr}(T^a T^a) g_0^2 \int_y \delta^{(d)}(x - y) \int_0^t ds e^{(t-s)\Delta_y} (\partial_\lambda^x + \partial_\lambda^y) \\
&\quad \times \left\{ \int_z \delta^{(d)}(y - z) \int_0^s du e^{(s-u)\Delta_y} \partial_\mu^y \left\{ D(\bar{y}_u - \bar{z}_s) \partial_\mu^y (S_\lambda(\bar{y}_u - \bar{x}_t) \right. \right. \\
&\quad \left. \left. + S_\lambda(\bar{x}_t - \bar{y}_u)) \right\} \right\} \\
&\quad - 16(2 - d) g_0^2 \int_y \delta^{(d)}(x - y) \int_0^t ds e^{(t-s)\Delta_y} (\partial_\lambda^x + \partial_\lambda^y) \left\{ \partial_\mu^y S_\lambda(y - x) \right. \\
&\quad \left. \times \int_0^s du e^{(s-u)\Delta_y} \left\{ \int_z \delta^{(d)}(y - z) \partial_\mu^z D(\bar{y}_u - \bar{z}_u) \right\} \right\} \delta^{ab} \delta_{ij} [T^a, T^b]_{ij} \\
&= 0
\end{aligned} \tag{5.15}$$

where we employed the evolved fields in (4.15) and (4.26). In the third equality of (5.15), the first term comes from $D06$ and it vanishes because $S_\lambda(y - x) + S_\lambda(x - y) = 0$, the second term is from $D04$ and vanishes for the same reason, the last term is also from $D04$ and vanishes since $\delta^{ab} \delta_{ij} [T^a, T^b]_{ij} = \text{tr}\{[T^a, T^a]\} = 0$.

Hence, we have shown diagrammatically how the 1-point correlator of the divergence of the evolved vector current vanishes to $\mathcal{O}(g_0^2)$, as anticipated in (5.8) and in full analogy with the nonevolved case.

⁴We use the notation $S(x - y) = \gamma_\mu S_\mu(x - y)$.

5.3 The nonevolved 2-point vector correlator

Now we shift our attention to the 2-point correlation function of the nonevolved bare vector currents in Euclidean coordinate space:

$$\Pi_{\mu\nu}^V(x-y) = \langle j_\mu(x) j_\nu(y) \rangle \quad (5.16)$$

where $\langle \dots \rangle$ defines the path-integral average in the Euclidean theory and $j_\mu = \bar{\psi} \gamma_\mu \psi$ is the bare flavor singlet vector current whose conservation is associated with the $U(1)_B$ symmetry, as is described around (2.17). For nonzero separations, contact terms do not occur in (5.16) and the correlator is multiplicatively renormalizable. Results up to $\mathcal{O}(g^8)$ for (5.16) both in momentum and coordinate space are provided in [32] with $N = 3$.

5.3.1 Solution of the Callan-Symanzik equation

For later comparison with the gradient-flow evolved case, we recall the renormalization-group equation and its solution for the connected 2-point correlator of a general gauge invariant and multiplicatively renormalizable current of canonical energy-dimension D , as discussed in section 3.6. For simplicity, we consider the scalar case $\Pi = \langle j(x) j(y) \rangle_{conn}$ with $j = \bar{\psi} \psi$. For correlators of pure-gluon operators and further details of this construction see [206].

Multiplicative renormalization in coordinate space at nonzero separation implies:

$$\Pi_R(x-y; g(\mu), \mu) = Z_j^{-2}(g(\mu), \varepsilon) \Pi(x-y; g_0, \varepsilon) \quad (5.17)$$

where Π_R and Π are the renormalized and bare correlators, respectively, Z_j is the multiplicative renormalization factor, ε is the regulator from dimensional regularization ($d = 4 - 2\varepsilon$), μ is the renormalization scale and $g(\mu)$ is the running gauge coupling. Following the same steps as in section 3.6, we conclude that the dimensionful correlator in (5.17) factorizes according to (3.56):

$$\Pi_R(x-y; g(\mu), \mu) = \frac{1}{|x-y|^{2D}} F(g(|x-y|)) \mathcal{Z}_j^2(g(|x-y|), g(\mu)) \quad (5.18)$$

where F is a dimensionless renormalization-group invariant (RGI) function of the RGI running coupling $g(|x-y|)$ — therefore not determined by the Callan-Symanzik equation — and $\mathcal{Z}_j^2$ is determined by integrating the Callan-Symanzik

equation between a reference scale μ^{-1} and $|x - y|$:

$$\begin{aligned}
\mathcal{Z}_j^2(g(|x - y|), g(\mu)) &= \exp \left(-2 \int_{g(|x - y|)}^{g(\mu)} dg \frac{\gamma_j(g)}{\beta(g)} \right) \\
&= \left(\frac{g^2(|x - y|)}{g^2(\mu)} \right)^{\frac{\gamma_j^{(0)}}{\beta_0}} e^{\frac{\beta_0 \gamma_j^{(1)} - \beta_1 \gamma_j^{(0)}}{\beta_0^2} (g^2(|x - y|) - g^2(\mu)) + \dots} \quad (5.19) \\
&= (1 + g^2(\mu) 2\beta_0 \log(|x - y|\mu) + \dots)^{\frac{\gamma_j^{(0)}}{\beta_0}} e^{\mathcal{O}(g^2(|x - y|) - g^2(\mu))} \\
&= (1 + g^2(\mu) 2\gamma_j^{(0)} \log(|x - y|\mu) + \dots) e^{\mathcal{O}(g^2(|x - y|) - g^2(\mu))}
\end{aligned}$$

where we the anomalous dimension $\gamma_j(g)$ and the beta function $\beta(g)$ are defined through:

$$\gamma_j(g) = \frac{d \log Z_j(g, \varepsilon)}{d \log \mu}, \quad \beta(g) = \frac{dg}{d \log \mu} \Big|_{\varepsilon=0} \quad (5.20)$$

and we introduced the perturbative expansions:

$$\begin{aligned}
\gamma_j(g) &= -\gamma_j^{(0)} g^2 - \gamma_j^{(1)} g^4 + \mathcal{O}(g^6) \\
\beta(g) &= -\beta_0 g^3 - \beta_1 g^5 + \mathcal{O}(g^7)
\end{aligned} \quad (5.21)$$

with β_0 and β_1 the universal, i.e., renormalization-scheme independent one- and two-loop coefficients of the QCD beta function, respectively, see (3.18).

Importantly, the second line of (5.19) determines the universal UV asymptotics of $\mathcal{Z}_j^2$ as $g(|x - y|) \rightarrow 0$:

$$\mathcal{Z}_j^2(g(|x - y|), g(\mu)) \sim \left(\frac{g^2(|x - y|)}{g^2(\mu)} \right)^{\frac{\gamma_j^{(0)}}{\beta_0}} e^{\mathcal{O}(g^2(\mu))} \quad (5.22)$$

Furthermore, the third and fourth line of (5.19) contain the perturbative expansion of $\mathcal{Z}_j^2$ by means of the perturbative expansion of $g^2(z)$ [206]:

$$\begin{aligned}
g^2(z) &= g^2(\mu) + g^4(\mu) 2\beta_0 \log(z\mu) \\
&\quad + g^6(\mu) (2\beta_1 \log(z\mu) + 4\beta_0^2 \log^2(z\mu)) + \mathcal{O}(g^8)
\end{aligned} \quad (5.23)$$

with $z = |x - y|$, in terms of $g^2(\mu)$ to order $g^6(\mu)$, and valid for scales μ^{-1} and z close to zero and $\mathcal{O}(1)$ logarithms $\log(z\mu)$.

For massless fermions, the validity of the above analysis extends with appropriate account of the Lorentz structure to the connected 2-point correlators of all flavor singlet and non-singlet fermion bilinear currents — scalar, pseudoscalar, vector, axial and tensor⁵.

In the case of the 2-point vector correlator in (5.16), the well-known all-order result $Z_j = 1$, hence $\gamma_j = 0$ through (5.20), is a consequence of the conservation of the flavor singlet vector current, as explained in section 2.4.4. It entails the renormalization-group invariance of the correlator $\Pi_{R,\mu\nu}^V = \Pi_{\mu\nu}^V$ and that of the vector current $[j_\mu] = j_\mu$.

An important observation is that, in order to fulfill these properties and at difference with the scalar case, the Lorentz structure of the vector correlator starts to change at $\mathcal{O}(g^4)$ in perturbation theory. This is explicitly shown in [32]. Therefore, in this case we replace the all-order solution in (5.18) with:

$$\begin{aligned} \Pi_{R,\mu\nu}^V(x-y; g(\mu), \mu) \\ = \frac{1}{(x-y)^{2D}} \sum_{n=0}^{\infty} g^{2n}(|x-y|) \left(A_n \frac{\delta_{\mu\nu}}{2} - B_n \frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2} \right) \end{aligned} \quad (5.24)$$

in terms of dimensionless coefficients A_n and B_n . The leading ($n = 0$) and next-to-leading ($n = 1$) orders have $A_n = B_n$. See also the discussion below (5.27).

5.3.2 Leading order, $\mathcal{O}(g_0^0)$ and next-to-leading order, $\mathcal{O}(g_0^2)$

We now proceed to review the perturbative expression for the correlator $\Pi_{R,\mu\nu}^V$ up to next-to-leading order $\mathcal{O}(g_0^2)$. The calculation up to $\mathcal{O}(g^8)$ for $N = 3$ and fermions in the fundamental representation can be found in [32].

As also for the evolved case in section 5.4, we present our results in the most general case, i.e., for N colors and N_f Dirac flavors in a representation R .

The leading order result, from the left diagram in figure 5.2, is:

$$\begin{aligned} \Pi_{\mu\nu}^V(x-y) &= -d(R) \operatorname{tr} [\gamma_\mu S(x-y) \gamma_\nu S(y-x)] + \mathcal{O}(g_0^2) \\ &= -\frac{2d(R)}{\pi^4} \frac{1}{((x-y)^2)^3} \left(\frac{\delta_{\mu\nu}}{2} - \frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2} \right) + \mathcal{O}(g_0^2) \end{aligned} \quad (5.25)$$

The next-to-leading order contributions are associated to the last two diagrams

⁵A detailed analysis of the multiplicative renormalization properties of flavor singlet and

Figure 5.2: Contributions to $\Pi_{\mu\nu}^V(x-y)$ at $\mathcal{O}(g_0^0)$ (left) and $\mathcal{O}(g_0^2)$ (center, right).

in figure 5.2. The UV divergences of both diagrams exactly cancel each other⁶, so that the next-to-leading order result is renormalization-group invariant. Indeed, we have verified that the results reported in [32] for the vector correlator in coordinate space with $N = 3$ and fermions in the fundamental, up to and including $\mathcal{O}(g^6)$, can be rewritten in terms of $g(|x-y|)$ in (5.23) only.

For comparison with the evolved case, our generalized result to order $g^2(\mu)$ reads:

$$\begin{aligned} \Pi_{\mu\nu}^V(x-y) = & -\frac{2d(R)}{\pi^4} \frac{1}{((x-y)^2)^3} \left\{ \left(\frac{\delta_{\mu\nu}}{2} - \frac{(x-y)_\mu(x-y)_\nu}{(x-y)^2} \right) \right. \\ & \left. \times \left(1 + 3C_2(R) \frac{g^2(\mu)}{(4\pi)^2} \right) + \mathcal{O}(g^4(\mu)) \right\} \end{aligned} \quad (5.26)$$

It manifestly satisfies the transversality condition:

$$\partial_\mu^x \Pi_{\mu\nu}^V(x-y) = \partial_\mu^y \Pi_{\mu\nu}^V(x-y) = 0 \quad (5.27)$$

to order $g^2(\mu)$. Equation (5.27) is equivalent to the statement that the corresponding current is conserved and not renormalized.

At order $g^4(\mu)$ and higher, transversality forces the Lorentz structure of the vector correlator to change. We can understand it as follows. The 2-point correlator of the conserved vector current does not renormalize multiplicatively, thus only the RGI coupling $g(|x-y|)$ induces a coordinate dependence beyond the one of the leading order correlator. This dependence starts at order $g^4(\mu)$, as implied by (5.23). Hence, the Lorentz structure of the vector correlator must change at order $g^4(\mu)$ in order to still guarantee transversality. This applies iteratively to higher orders.

non-singlet axial and pseudoscalar currents and the role of the axial anomaly can be found in [181] (see also references therein).

⁶One can verify the exact cancellation of the short-distance divergences in coordinate space following a known method nicely explained in [106], chapter 11.

For the 2-point correlators of possibly higher-spin nonconserved currents, where operator mixing may or may not occur, a change of the Lorentz structure at higher orders is in general allowed. Interestingly, the 2-point correlator of the multiplicatively renormalizable nonconserved tensor current $\bar{\psi}\sigma_{\mu\nu}\psi$ does not change its Lorentz structure to the orders computed in [32], i.e., to $\mathcal{O}(g^6)$.

5.4 The 2-point vector correlator evolved by the gradient flow

We now consider the Euclidean 2-point correlator of flavor singlet vector currents evolved to the same flow time t . Analogously to (5.3), we expand the bare correlator as follows:

$$\Pi_{\mu\nu}^V(t, x - y) = \langle J_\mu(t, x) J_\nu(t, y) \rangle = \sum_{n=0}^{\infty} g_0^{2n} \Pi_{\mu\nu, 2n}^V(t, x - y) \quad (5.28)$$

The leading order contribution reads:

$$\Pi_{\mu\nu, 0}^V(t, x - y) = \langle J_{\mu, 0}(t, x) J_{\nu, 0}(t, y) \rangle_0 \quad (5.29)$$

with $J_{\mu, 0}$ in (5.4). $\Pi_{\mu\nu, 0}^V$ is derived in section 5.4.1.

The next-to-leading order contribution is derived in section 5.4.2 and can be conveniently divided into four terms:

$$\Pi_{\mu\nu, 2}^V(t, x - y) = \text{I} + \text{II} + \text{III} + \text{IV} \quad (5.30)$$

where:

$$\begin{aligned} \text{I} &= \frac{1}{2} \langle J_{\mu, 0}(t, x) J_{\nu, 0}(t, y) \int_{z_1} (-\bar{\psi} A \psi) \int_{z_2} (-\bar{\psi} A \psi) \rangle_0 \\ \text{II} &= \langle J_{\mu, 1}(t, x) J_{\nu, 0}(t, y) \int_z (-\bar{\psi} A \psi) \rangle_0 + (x \leftrightarrow y; \mu \leftrightarrow \nu) \\ \text{III} &= \langle J_{\mu, 2}(t, x) J_{\nu, 0}(t, y) \rangle_0 + (x \leftrightarrow y; \mu \leftrightarrow \nu) \\ \text{IV} &= \langle J_{\mu, 1}(t, x) J_{\nu, 1}(t, y) \rangle_0 \end{aligned} \quad (5.31)$$

with the currents given in (5.4). The first two classes of diagrams, I and II, contain respectively the insertion of two QCD vertices and one QCD vertex.

Our aim is now to establish the renormalization properties of the evolved $\Pi_{\mu\nu}^V$, and how it differs from the nonevolved case where it does not renormalize.

Figure 5.3: Leading order diagram for the gradient-flow evolved 2-point vector correlator. Lines are gradient-flow evolved propagators — see notation in figures 4.1 and 4.2.

Therefore, when deriving the next-to-leading order contribution in section 5.4.2, we will concentrate on the divergent parts only. The diagrammatic notation for sections 5.4.1 and 5.4.2 is explained in figures 4.1 and 4.2, with the addition that the black blobs are the usual QCD vertices.

5.4.1 Leading order, $\mathcal{O}(g_0^0)$

The leading order contribution is given by:

$$\begin{aligned} \Pi_{\mu\nu,0}^V(t, x-y) &= -d(R) \operatorname{tr} [\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu S(\bar{y}_t - \bar{x}_t)] \\ &= -\frac{2d(R)}{\pi^4} \frac{\gamma\left(2, \frac{(x-y)^2}{8t}\right)^2}{((x-y)^2)^3} \left(\frac{\delta_{\mu\nu}}{2} - \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right) \end{aligned} \quad (5.32)$$

from the diagram in figure 5.3, and $S(\bar{x}_t - \bar{y}_t)$ is the evolved fermion propagator in (4.39). The comparison of (5.32) with the nonevolved result in (5.25) shows that the gradient flow evolution replaces $\Gamma(\frac{d}{2})$ in d spacetime dimensions with the lower incomplete gamma function $\gamma(\frac{d}{2}, \frac{(x-y)^2}{8t})$ introduced in section 4.3. This has the following implications.

As $t \rightarrow 0$ at fixed nonzero separation, one recovers the nonevolved result in (5.25) up to exponentially suppressed contributions:

$$\Pi_{\mu\nu,0}^V(t, x-y) \Big|_{t \sim 0} = \Pi_{\mu\nu,0}^V(x-y) \left(1 + \mathcal{O}\left(\frac{(x-y)^2}{t} e^{-\frac{(x-y)^2}{t}} \right) \right) \quad (5.33)$$

where $\Pi_{\mu\nu,0}^V(x-y)$ is the nonevolved leading order contribution in (5.25), and we employed $\gamma(2, z) = 1 - e^{-z}(1+z) \sim 1 - ze^{-z}$ for z large⁷.

⁷ $\gamma(n+1, z) = n!(1 - e^{-z}e_n(z))$, with $e_n(z) = \sum_{k=0}^n z^k/k!$, for $n = 0, 1, 2, \dots$

Thus, the spacetime derivative of the small- t expansion in (5.33) yields:

$$\partial_\mu^x \left(\Pi_{\mu\nu,0}^V(t, x-y) \Big|_{t \sim 0} \right) = \Pi_{\mu\nu,0}^V(x-y) \frac{x_\mu - y_\mu}{t} \mathcal{O} \left(\frac{(x-y)^2}{t} e^{-\frac{(x-y)^2}{t}} \right) \quad (5.34)$$

showing that the small- t evolution induces an exponentially soft violation of transversality.

We recall that in the nonevolved case the transversality condition is respected up to contact terms. In fact, in the evolved case, the spacetime derivative of (5.32) reads:

$$\begin{aligned} \partial_\mu^x \Pi_{\mu\nu,0}^V(t, x-y) &= -d(R) \partial_\mu^x \left\{ \text{tr} [\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu S(\bar{y}_t - \bar{x}_t)] \right\} \\ &= 2d(R) K_{2t}(x-y) \text{tr} [\gamma_\nu S(\bar{x}_t - \bar{y}_t)] \\ &= \frac{4d(R)}{\pi^2} K_{2t}(x-y) \frac{(x_\nu - y_\nu)}{((x-y)^2)^2} \gamma \left(2, \frac{(x-y)^2}{8t} \right) \\ &= -\frac{16d(R)}{\pi^2} \partial_\nu^x K_{2t}(x-y) \frac{t}{((x-y)^2)^2} \gamma \left(2, \frac{(x-y)^2}{8t} \right) \\ &= -\frac{16d(R)}{\pi^2} \partial_\nu^x \left(e^{2t\Delta_x} \delta^{(4)}(x-y) \right) \frac{t}{((x-y)^2)^2} \gamma \left(2, \frac{(x-y)^2}{8t} \right) \end{aligned} \quad (5.35)$$

showing that the violation of transversality at a generic fixed t is proportional to the derivative of a smeared Dirac delta distribution, i.e., a contact term smeared by the gradient flow. For each equality in (5.35) we employed in order (5.32), (4.37), (4.39) for the evolved fermion propagator, and finally (4.11) and (4.12) for the scalar kernel K_{2t} .

We may also consider the alternative limit of vanishing separation $|x-y| \rightarrow 0$ at fixed flow time. In this case, the expansion in (4.41) inserted in (5.32) yields for the 2-point correlator:

$$\begin{aligned} \Pi_{\mu\nu,0}^V(t, x-y) \Big|_{x \sim y} &= -\frac{d(R)}{2\pi^4} \frac{(x-y)^2}{(8t)^4} \left(\frac{\delta_{\mu\nu}}{2} - \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right) \left(1 + \mathcal{O} \left(\frac{(x-y)^2}{t} \right) \right) \end{aligned} \quad (5.36)$$

showing that it vanishes at zero spacetime separation, i.e., the short-distance singularity has been removed by the gradient flow.

This exercise makes manifest the noncommutativity of the two limits $t \rightarrow 0$ and $|x-y| \rightarrow 0$ at the level of 2-point correlators.

Figure 5.4: Type I diagrams for the evolved 2-point vector correlator at $\mathcal{O}(g_0^2)$.

5.4.2 Next-to-leading order, $\mathcal{O}(g_0^2)$

We treat separately the four contributions in (5.31).

Type I contribution

The first contribution in (5.31) is associated to the diagrams in figure 5.4 and it is given by:

$$\begin{aligned}
 (\text{I.1}) = & -2\text{tr}(T^a T^a) \int_{z_1, z_2} \text{tr} \left[\gamma_\mu S(\bar{x}_t - z_1) \gamma_\rho S(z_1 - z_2) \gamma_\rho S(z_2 - \bar{y}_t) \right. \\
 & \left. \times \gamma_\nu S(\bar{y}_t - \bar{x}_t) \right] D(z_1 - z_2)
 \end{aligned} \tag{5.37}$$

and:

$$\begin{aligned}
 (\text{I.2}) = & -\text{tr}(T^a T^a) \int_{z_1, z_2} \text{tr} \left[\gamma_\mu S(\bar{x}_t - z_1) \gamma_\rho S(z_1 - \bar{y}_t) \gamma_\nu S(\bar{y}_t - z_2) \right. \\
 & \left. \times \gamma_\rho S(z_2 - \bar{x}_t) \right] D(z_1 - z_2)
 \end{aligned} \tag{5.38}$$

Type I diagrams are the direct generalization of the two diagrams that contribute at $\mathcal{O}(g_0^2)$ to the nonevolved correlator, see section 5.3. In the latter case, the UV divergence of I.1 cancels the one of I.2 rendering the correlator finite. In particular, the divergence, i.e., the non-integrable short-distance singularity of the self-energy contribution I.1 arises at $z_1 = z_2$ — the coordinates of the internal vertices in figure 5.4 —, while the divergence of I.2 is at $z_1 = z_2 = x$ and $z_1 = z_2 = y$.

In the evolved case, we note that the propagators that contribute to the divergence of I.1 are not modified by the flow. Hence, I.1 is UV divergent and it generates the same divergence as in the nonevolved case. In I.2, the

Figure 5.5: Type II diagrams for the evolved 2-point vector correlator at $\mathcal{O}(g_0^2)$.

two fermion propagators that potentially contribute to the divergence are now modified by the flow, thus their short-distance behavior is altered as explained in section 4.3. This is enough to render I.2 finite, in contrast to the nonevolved case. In appendix C.2 we rewrite all contributions in terms of the momentum space representations of the (evolved) propagators and (evolved) Dirac deltas. Then, the fact that I.2 is finite is again manifest due to the exponential flow factors e^{-tk^2} , with t the external flow time and k the internal loop momentum associated to the propagators that generate the UV divergence in the nonevolved case.

Type II contribution

The second contribution in (5.31) is associated to the diagrams in figure 5.5 and it is given by:

$$\begin{aligned}
 (\text{II.1}) = & 4\text{tr}(T^a T^a) \int_0^t ds \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu S(\bar{y}_t - z) \right. \\
 & \left. \times e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(\bar{x}_s - z) D(\bar{x}_s - z) \right\} \right] + (x \leftrightarrow y; \mu \leftrightarrow \nu)
 \end{aligned} \tag{5.39}$$

and:

$$\begin{aligned}
 (\text{II.2}) = & -4\text{tr}(T^a T^a) \int_0^t ds \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - z) \gamma_\rho S(z - \bar{y}_t) \gamma_\nu \right. \\
 & \left. \times e^{(t-s)\Delta_x} \left\{ \partial_\rho^x S(\bar{y}_t - \bar{x}_s) D(\bar{x}_s - z) \right\} \right] + (x \leftrightarrow y; \mu \leftrightarrow \nu)
 \end{aligned} \tag{5.40}$$

Diagram II.1 is divergent, where the short-distance singularity arises when the black and white blobs in figure 5.5 coalesce. Diagram II.2 is instead finite, due to the presence of evolved propagators. Again, this is also manifest in the

Figure 5.6: Type III diagrams for the evolved 2-point vector correlator at $\mathcal{O}(g_0^2)$.

momentum integrals in appendix C.2, for the same reasons as I.2 in type I contribution.

Type III contribution

The third contribution in (5.31) is associated to the diagrams in figure 5.6, and we employed:

$$\begin{aligned} \langle (\bar{\chi}_2 \gamma_\mu \chi_0)_x (\bar{\chi}_0 \gamma_\nu \chi_0)_y \rangle &= \langle (\bar{\chi}_2 \gamma_\mu \chi_0)_x (\bar{\chi}_0 \gamma_\nu \chi_0)_y \rangle^\dagger \\ &= \langle (\bar{\chi}_0 \gamma_\mu \chi_2)_x (\bar{\chi}_0 \gamma_\nu \chi_0)_y \rangle \end{aligned} \quad (5.41)$$

when inserting $J_{\mu,2}$ of (5.4) in the correlator. Diagrams III.1 and III.2 come from the term $\bar{\chi}_2 \gamma_\mu \chi_0 (\bar{\chi}_0 \gamma_\mu \chi_2)$ in $J_{\mu,2}$. In fact, $\bar{\chi}_2 (\chi_2)$ in (4.26) has three terms, yet, the contribution coming from $B_{\mu,2}$ vanishes. Diagram III.3 comes from the term $\bar{\chi}_1 \gamma_\mu \chi_1$ in $J_{\mu,2}$.

It is convenient to rewrite the total contribution from III.1 and III.2 as follows, see appendix C.1 for the derivation:

$$(III.1) + (III.2) = (III.1') + (III.2') - (II.1) \quad (5.42)$$

where (II.1) is the type II contribution in (5.39) and:

$$\begin{aligned} (III.1') &= \frac{d-2}{d} (III.1) \\ &= -2(d-2) \text{tr}(T^a T^a) \int_0^t ds \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu \right. \\ &\quad \times e^{(t-s)\Delta_x} \left\{ S(\bar{y}_t - \bar{x}_s) \delta^{(d)}(x-z) D(\bar{x}_s - \bar{z}_s) \right\} \left. \right] + (x \leftrightarrow y; \mu \leftrightarrow \nu) \end{aligned} \quad (5.43)$$

(IV.1)

(IV.2)

Figure 5.7: Type IV diagrams for the evolved 2-point vector correlator at $\mathcal{O}(g_0^2)$.

$$\begin{aligned}
 (\text{III.2}') &= 8\text{tr}(T^a T^a) \int_0^t ds \int_0^s du \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu \right. \\
 &\quad \times e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(x-z) e^{(s-u)\Delta_x} \left\{ D(\bar{x}_u - \bar{z}_s) \not{\partial}^x \delta^{(d)}(\bar{y}_t - \bar{x}_u) \right\} \right\} \Big] \quad (5.44) \\
 &\quad + (x \leftrightarrow y; \mu \leftrightarrow \nu)
 \end{aligned}$$

Finally, III.3 reads:

$$\begin{aligned}
 (\text{III.3}) &= 4\text{tr}(T^a T^a) \\
 &\quad \times \int_0^t ds_1 \int_0^t ds_2 \int_z \text{tr} \left[\gamma_\mu e^{(t-s_2)\Delta_x} \left\{ \delta^{(d)}(x-z) \partial_\rho^x S(\bar{x}_{s_2} - \bar{y}_t) \right\} \right. \quad (5.45) \\
 &\quad \times \gamma_\nu e^{(t-s_1)\Delta_x} \left\{ D(\bar{x}_{s_1} - \bar{z}_{s_2}) \partial_\rho^x S(\bar{y}_t - \bar{x}_{s_1}) \right\} \Big] + (x \leftrightarrow y; \mu \leftrightarrow \nu)
 \end{aligned}$$

The contribution III.1' is UV divergent and we further derive it in section 5.4.3. The remaining two contributions III.2' and III.3 are both finite, though this is less straightforward to see in coordinate-space expressions due to nested flow-time integrals and exponential-of-Laplacian actions. The finiteness of III.2' is further established in appendix C.3 using integration by parts in the momentum expression of appendix C.2. Finally, the finiteness of III.3 is manifest in its momentum expression in appendix C.2, analogously to I.2 and II.2.

Type IV contribution

The last contribution in (5.31) is associated to the diagrams in figure 5.7 and it is given by:

$$\begin{aligned}
 (\text{IV.1}) &= -8\text{tr}(T^a T^a) \int_0^t ds_1 \int_0^t ds_2 \int_z \text{tr} \left[\gamma_\mu e^{(t-s_2)\Delta_y} \left\{ \delta^{(d)}(y-z) \right. \right. \\
 &\quad \times \partial_\rho^y S(\bar{x}_t - \bar{y}_{s_2}) \Big\} \gamma_\nu e^{(t-s_1)\Delta_x} \left\{ D(\bar{x}_{s_1} - \bar{z}_{s_2}) \partial_\rho^x S(\bar{y}_t - \bar{x}_{s_1}) \right\} \Big] \quad (5.46)
 \end{aligned}$$

and:

$$\begin{aligned}
 (\text{IV.2}) = & 8\text{tr}(T^a T^a) \int_0^t ds_1 \int_0^t ds_2 \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu e^{(t-s_1)\Delta_x + (t-s_2)\Delta_y} \right. \\
 & \left. \times \left\{ D(\bar{x}_{s_1} - \bar{y}_{s_2}) \partial_\rho^x \partial_\rho^y S(\bar{y}_{s_2} - \bar{x}_{s_1}) \right\} \right]
 \end{aligned} \tag{5.47}$$

Both IV.1 and IV.2 are finite. This is established by considering the momentum space expressions in appendix C.2, analogously to I.2, II.2 and III.3; the k integrals for both contributions contain the factor e^{-tk^2} , thereby excluding the possibility of developing a divergence.

5.4.3 Total UV divergence at $\mathcal{O}(g_0^2)$

We have established in section 5.4.2 that the contributions I.2, II.2, III.2', III.3 and all type IV contributions to the evolved 2-point vector correlator are finite. We have also found that the contribution II.1, which is UV divergent, is canceled by the type III contributions in (5.42). Hence, we proceed to derive the UV divergent part of the remaining contributions⁸ I.1 and III.1' in dimensional regularization, with $d = 4 - 2\varepsilon$.

We have noticed in section 5.4.2 that the contribution I.1 in (5.37) is UV divergent due to the one-loop fermion self-energy insertion, which is not modified by the flow. Appendix B.2 shows the final result for I.1 in d dimensions — where the gradient-flow modifications occur outside the self-energy insertion — and its ε -expansion. The latter yields:

$$(\text{I.1}) = -\frac{2}{\varepsilon} t^\varepsilon C_2(R) \frac{1}{(4\pi)^2} \Pi_{\mu\nu,0}^V(t, x-y) + \dots \tag{5.48}$$

where we employed $\text{tr}(T^a T^a) = -C_2(R)d(R)$, $\Pi_{\mu\nu,0}^V(t, x-y)$ is the leading order contribution in (5.32) and the ellipsis represents the finite contributions that do not enter the renormalization of the correlator.

The last divergent contribution is III.1' in (5.43). Its calculation is straight-

⁸All contributions are IR finite. This is important, otherwise spurious UV divergences would be produced in dimensional regularization.

forward upon noticing that the integral in z in (5.43) yields:

$$\begin{aligned}
& \int_z \delta^{(d)}(x-z) D(\bar{x}_s - \bar{z}_s) \\
&= \frac{1}{4\pi^{d/2}} \lim_{z \rightarrow x} \left\{ ((x-z)^2)^{1-d/2} \gamma\left(\frac{d}{2} - 1, \frac{(x-z)^2}{8s}\right) \right\} \\
&= \frac{2^{2-d/2}}{(d-2)(4\pi)^{d/2}} s^{1-d/2}
\end{aligned} \tag{5.49}$$

where we employed the propagator in (4.38) and the expansion of the γ function in (4.41), see appendix B.2 for the ε -expansion. Thus we obtain:

$$\begin{aligned}
& \text{(III.1')} \\
&= -\text{tr}(T^a T^a) \frac{2^{4-d/2}}{(4\pi)^{d/2}} \int_0^t ds s^{1-d/2} \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu S(\bar{y}_t - \bar{x}_t) \right] \\
&= -\frac{2}{(4-d)} \text{tr}(T^a T^a) \frac{2^{4-d/2}}{(4\pi)^{d/2}} t^{2-d/2} \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu S(\bar{y}_t - \bar{x}_t) \right] \\
&= -\frac{4}{\varepsilon} t^\varepsilon \text{tr}(T^a T^a) \frac{1}{(4\pi)^2} \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu S(\bar{y}_t - \bar{x}_t) \right] + \dots \\
&= -\frac{4}{\varepsilon} t^\varepsilon C_2(R) \frac{1}{(4\pi)^2} \Pi_{\mu\nu,0}^V(t, x-y) + \dots
\end{aligned} \tag{5.50}$$

where $\Pi_{\mu\nu,0}^V(t, x-y)$ is the leading order contribution in (5.32) and the ellipsis represents the finite contributions that do not enter the renormalization of the correlator.

We conclude that the bare 2-point vector correlator evolved by the gradient flow is no longer UV finite at the next-to-leading order in perturbation theory, i.e., $\mathcal{O}(g_0^2)$. The total UV divergence at $\mathcal{O}(g_0^2)$, from the sum of (5.48) and (5.50) and after multiplying $\Pi_{\mu\nu,2}^V(t, x-y)$ by the bare coupling g_0^2 reads:

$$g_0^2 \Pi_{\mu\nu,2}^V(t, x-y) = -\frac{6}{\varepsilon} C_2(R) \frac{g_0^2}{(4\pi)^2} \Pi_{\mu\nu,0}^V(t, x-y) + \mathcal{O}(\varepsilon^0) \tag{5.51}$$

Thus the evolved correlator acquires a renormalization not present in the non-evolved case.

Including the gradient-flow renormalization factor, Z_χ

The presence of a UV divergence and thus of a renormalization in the 2-point correlator of the evolved vector current may initially come as a surprise. How-

ever, this calculation with the result in (5.51) provides an explicit verification of the fact that the renormalization factor Z_χ introduced in [31] (and discussed in section 4.4.2) has indeed a universal nature, arising as a new renormalization of the fermion fields $\chi(t, x)$ and $\bar{\chi}(t, x)$ evolved by the gradient flow. In fact, by introducing the renormalized evolved fermion fields in the representation R :

$$\chi_R(t, x) = Z_\chi^{1/2} \chi(t, x), \quad \bar{\chi}_R(t, x) = Z_\chi^{1/2} \bar{\chi}(t, x) \quad (5.52)$$

with renormalization factor [31]:

$$Z_\chi^{1/2}(g(\mu), \varepsilon) = 1 + \frac{g^2(\mu)}{(4\pi)^2} C_2(R) \frac{3}{2\varepsilon} + \mathcal{O}(g^4(\mu)) \quad (5.53)$$

where $g(\mu)$ is the renormalized coupling, and employing the renormalization factor $Z_J \equiv Z_\chi^{-1}$ for the evolved current, we obtain the renormalized 2-point vector correlator at $\mathcal{O}(g^2(\mu))$:

$$\begin{aligned} \Pi_{R, \mu\nu}^V(t, x-y; g(\mu), \mu) &= Z_J^{-2}(g(\mu), \varepsilon) \Pi_{\mu\nu}^V(t, x-y; g_0, \varepsilon) \\ &= \left(1 + \frac{g^2(\mu)}{(4\pi)^2} \frac{6}{\varepsilon} C_2(R)\right) \Pi_{\mu\nu, 0}^{V(\varepsilon)}(t, x-y) \\ &\quad + g^2(\mu) \mu^{2\varepsilon} \Pi_{\mu\nu, 2}^{V(\varepsilon)}(t, x-y) + \mathcal{O}(g^4(\mu)) \\ &= \Pi_{\mu\nu, 0}^V(t, x-y) + \frac{g^2(\mu)}{(4\pi)^2} C_2(R) \left(\frac{6}{\varepsilon} - \frac{6}{\varepsilon} - 6 \log(t\mu^2)\right. \\ &\quad \left.+ \text{finite terms}\right) \Pi_{\mu\nu, 0}^V(t, x-y) + \dots \\ &= \left(1 - \frac{g^2(\mu)}{(4\pi)^2} 6C_2(R) \log(t\mu^2) + \text{finite terms}\right) \Pi_{\mu\nu, 0}^V(t, x-y) + \dots \end{aligned} \quad (5.54)$$

where $\Pi_{\mu\nu}^V$ in the first equality is the bare correlator in dimensional regularization with bare coupling g_0 , $\Pi_{\mu\nu, 0}^{V(\varepsilon)}$ and $\Pi_{\mu\nu, 2}^{V(\varepsilon)}$ in the second equality are derived in $d = 4 - 2\varepsilon$ dimensions in appendix B.2, (B.31) and (B.30), respectively, and $\Pi_{\mu\nu, 0}^V$ is the leading order contribution in (5.32). In the last two equalities, finite terms stand for finite non-logarithmic contributions at $\mathcal{O}(g^2(\mu))$ that multiply $\Pi_{\mu\nu, 0}^V$, whereas the ellipses stand for possible finite non-logarithmic $\mathcal{O}(g^2(\mu))$ contributions with Lorentz structure different from $\Pi_{\mu\nu, 0}^V$ ⁹ and $\mathcal{O}(g^4(\mu))$ contributions.

⁹We refer to a Lorentz structure of the type $\delta_{\mu\nu}/2 - a(x-y)_\mu(x-y)_\nu/(x-y)^2$ with $a \neq 1$.

Equation (5.54) is written in terms of the renormalized vector current $[J_\mu] = Z_J^{-1} J_\mu = \bar{\chi}_R \gamma_\mu \chi_R$ and the QCD renormalized coupling $g(\mu)$, which is related to the bare coupling g_0 as follows:

$$g_0^2 = Z_g^2 \mu^{2\varepsilon} g^2(\mu) \quad (5.55)$$

with:

$$Z_g^2(g(\mu), \varepsilon) = 1 - \frac{\beta_0 g^2(\mu)}{\varepsilon} - \frac{\beta_1 g^4(\mu)}{2\varepsilon} + \frac{\beta_0^2 g^4(\mu)}{\varepsilon^2} + \mathcal{O}(g^6(\mu)) \quad (5.56)$$

see sections 3.2 and 3.4. Since the $1/\varepsilon$ poles of the dimensionally regularized expression in (5.54) exactly cancel, we conclude that the renormalization of the evolved elementary fermion fields and that of the coupling is the only one required for the evolved 2-point vector correlator at $\mathcal{O}(g_0^2)$ in perturbation theory.

The arguments presented in [28, 29, 31] further suggest that this property extends to all orders in perturbation theory, see also our concise review of these arguments in section 6.2.2. Moreover, the results in [28, 29, 31] imply that all evolved fermion-bilinear currents acquire the same renormalization factor Z_J , and thus the same anomalous dimension. The latter can be obtained from its definition in (5.20), with Z_j replaced by Z_J . In MS-like schemes (see section 3.2) it reads:

$$\begin{aligned} \gamma_J(g) &= \frac{d \log Z_J}{d \log \mu} = \beta(g, \varepsilon) \frac{\partial \log Z_J}{\partial g} \\ &= (-\varepsilon g + \beta(g)) \frac{\partial \log Z_J}{\partial g} \\ &= 6C_2(R) \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4(\mu)) \end{aligned} \quad (5.57)$$

with $\beta(g)$ in (5.20) and $Z_J = Z_\chi^{-1}$ in (5.53)¹⁰. The $\mathcal{O}(g^4(\mu))$ contribution to γ_J in (5.57) is renormalization-scheme dependent, as is shown explicitly in section 3.5.

5.4.4 STE of the evolved currents from their 2-point correlators

In full analogy with the analysis in section 5.3 for the connected 2-point correlator of a generic nonevolved gauge-invariant and multiplicatively renormalizable

¹⁰Note that in (5.57) one should use the beta function in $d = 4 - 2\varepsilon$ dimensions.

current, the Callan-Symanzik equation for the connected 2-point correlator of an evolved gauge-invariant current of canonical mass-dimension D , and that renormalizes as $[J(t, x)] = Z_J^{-1} J(t, x)$ reads:

$$\left(\frac{\partial}{\partial \log \mu} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_J(g) \right) \Pi_R(t, x - y; g(\mu), \mu) = 0 \quad (5.58)$$

with γ_J in (5.57). Equation (5.58) is implied by the renormalization-group invariance (μ independence) of the evolved bare correlator Π , related to Π_R as:

$$\Pi_R(t, x - y; g(\mu), \mu) = Z_J^{-2}(g(\mu), \varepsilon) \Pi(t, x - y; g_0, \varepsilon) \quad (5.59)$$

analogously to the nonevolved case in (5.17). However, differently from the nonevolved case, the Callan-Symanzik equation (5.58) poses a two-scale problem, with scales $\sqrt{t}$ and $|x - y|$. We can solve the equation in the limit in which one of the two scales dominates in the UV. Specifically, we are interested in the limit $t \rightarrow 0$ at fixed $|x - y|$, hence $\sqrt{t} \ll |x - y|$, to establish an asymptotic relation between the evolved and nonevolved correlators.

The validity of the STE for composite operators evolved by the gradient flow has been verified in [28, 31], and we reviewed it in section 4.4.4. Thus, for a generic multiplicatively renormalizable evolved fermion-bilinear current in the small- t limit we write, generalizing (4.64):

$$[J(t, x)] = \langle J(t) \rangle + c(t)[j(x)] + \mathcal{O}(t) \quad (5.60)$$

where $\langle \dots \rangle$ is its vacuum expectation value, $[j(x)]$ is the renormalized nonevolved current and $c(t)$ is the leading coefficient of the STE, see also section 4.4.4.

We can determine $c(t)$ from the small- t expansion of the corresponding evolved 2-point correlator Π_R . Such small- t expansion reads:

$$\begin{aligned} \Pi_R(t, x - y; g(\mu), \mu) &= C(\sqrt{t}\mu, g(\mu)) \Pi_R(x - y; g(\mu), \mu) \\ &\quad + \Delta \Pi_R(t, x - y; g(\mu), \mu) \end{aligned} \quad (5.61)$$

where in the right-hand side the dimensionless coefficient $C = c^2$ is the square of the STE coefficient $c(t)$ in (5.60), and Π_R is the renormalized nonevolved correlator. The latter is given by (5.18) for a multiplicatively renormalizable nonevolved current $[j(x)] = Z_J^{-1} j(x)$ in (5.60).

The factorization of the dependence on $\sqrt{t}$ and $|x - y|$ occurs in the first term of the right-hand side in (5.61), which is of $\mathcal{O}(t^0)$ times logarithms of $\sqrt{t}\mu$.¹¹

¹¹Note that in this expansion μ must be of order $1/\sqrt{t}$ to avoid large logarithms.

The term $\Delta\Pi_R$ refers to contributions of $\mathcal{O}(t)$ times logarithms of $\sqrt{t}\mu$ and $|x - y|\mu$. These higher order contributions have in general a nonfactorizable dependence on the two scales $\sqrt{t}$ and $|x - y|$.

Therefore, for multiplicatively renormalizable $[J(t, x)]$ and $[j(x)]$ the renormalization-group equation for C is now implied by (5.59) for the evolved correlator and (5.17) for the nonevolved one, which yield:

$$\left(\frac{\partial}{\partial \log \mu} + \beta(g) \frac{\partial}{\partial g} + 2\gamma_J(g) - 2\gamma_j(g)\right) C(\sqrt{t}\mu, g(\mu)) = 0 \quad (5.62)$$

Hence, C has an anomalous dimension given by the difference of the anomalous dimensions of the evolved and the nonevolved Π_R . The solution can be written as:

$$C(\sqrt{t}\mu, g(\mu)) = F(g(\sqrt{t})) \mathcal{C}^2(g(\sqrt{t}), g(\mu)) \quad (5.63)$$

where F is a dimensionless RGI function of the RGI running coupling $g(\sqrt{t})$, with $F \rightarrow 1$ as $g(\sqrt{t}) \rightarrow 0$ implied by perturbation theory, and:

$$\mathcal{C}^2(g(\sqrt{t}), g(\mu)) = \exp \left(2 \int_{g(\sqrt{t})}^{g(\mu)} dg \frac{\gamma_j(g) - \gamma_J(g)}{\beta(g)} \right) \quad (5.64)$$

Equation (5.64) coincides with (5.19) for the nonevolved case once γ_j has been replaced with $\gamma_J - \gamma_j$ and $g(|x - y|)$ with $g(\sqrt{t})$. Then, by means of the latter substitutions, the UV asymptotic expression for $\mathcal{C}^2$ in (5.64) is given by the second line of (5.19).

With C given by (5.63), the small- t expansion of the evolved correlator thus reads:

$$\begin{aligned} \Pi_R(t, x - y; g(\mu), \mu) &= \left(\frac{g^2(\sqrt{t})}{g^2(\mu)} \right)^{\frac{\gamma_J^{(0)} - \gamma_j^{(0)}}{\beta_0}} e^{\mathcal{O}(g^2(\sqrt{t}) - g^2(\mu))} F(g(\sqrt{t})) \\ &\times \Pi_R(x - y; g(\mu), \mu) + \mathcal{O}(t) \end{aligned} \quad (5.65)$$

showing that the evolved correlator has the same Lorentz structure of the nonevolved one at leading order in the small- t expansion, i.e., up to $\mathcal{O}(t)$ contributions.

The leading universal UV asymptotics of (5.65) as $g(\sqrt{t}) \rightarrow 0$ then reads:

$$\Pi_R(t, x - y; g(\mu), \mu) \sim \left(\frac{g^2(\sqrt{t})}{g^2(\mu)} \right)^{\frac{\gamma_J^{(0)} - \gamma_j^{(0)}}{\beta_0}} e^{\mathcal{O}(g^2(\mu))} \Pi_R(x - y; g(\mu), \mu) \quad (5.66)$$

and the leading universal UV asymptotics of the first STE coefficient:

$$c(\sqrt{t}\mu, g(\mu)) = F^{1/2}(g(\sqrt{t}))\mathcal{C}(g(\sqrt{t}), g(\mu)) \quad (5.67)$$

for $[J(t, x)]$ in (5.60) thus follows¹²:

$$c(\sqrt{t}\mu, g(\mu)) \sim \left(\frac{g(\sqrt{t})}{g(\mu)} \right)^{\frac{\gamma_J^{(0)} - \gamma_j^{(0)}}{\beta_0}} e^{\mathcal{O}(g^2(\mu))} \quad (5.68)$$

which is indeed consistent with (4.75). In the case of the evolved vector current one has $\gamma_j^{(0)} = 0$. We have also noticed that in the nonevolved vector case, (5.18) is replaced with (5.24), which takes into account a change of the Lorentz structure at higher orders in perturbation theory.

Accordingly, by employing (5.24) the small- t expansion for the evolved flavor-singlet vector correlator has the explicit form:

$$\begin{aligned} \Pi_{R,\mu\nu}^V(t, x-y; g(\mu), \mu) &= \left(\frac{g^2(\sqrt{t})}{g^2(\mu)} \right)^{\frac{\gamma_J^{(0)}}{\beta_0}} e^{\mathcal{O}(g^2(\sqrt{t}) - g^2(\mu))} F(g(\sqrt{t})) \\ &\times \frac{1}{(x-y)^{2D}} \sum_{n=0}^{\infty} g^{2n}(|x-y|) \left(A_n \frac{\delta_{\mu\nu}}{2} - B_n \frac{(x-y)_\mu (x-y)_\nu}{(x-y)^2} \right) + \mathcal{O}(t) \end{aligned} \quad (5.69)$$

with $A_n = B_n$ for $n = 0, 1$, and we employed the coefficient C from (5.63) and (5.64) with $\gamma_j = 0$. Equation (5.69) shows explicitly that the leading contribution to the small- t expansion of the evolved vector correlator in (5.69) inherits its Lorentz structure from the nonevolved correlator order-by-order in $g^2(|x-y|)$.

Finally, we note that the leading $n = 0$ term in (5.69) reproduces the explicit $\mathcal{O}(g^2(\mu))$ result in (5.54) by means of the perturbative expansion:

$$\left(\frac{g^2(\sqrt{t})}{g^2(\mu)} \right)^{\frac{\gamma_J^{(0)}}{\beta_0}} = 1 + g^2(\mu) \gamma_J^{(0)} \log(t\mu^2) + \dots \quad (5.70)$$

with $\gamma_J^{(0)} = -6C_2(R)/(4\pi)^2$, and $F(g(\sqrt{t})) = 1 + \mathcal{O}(g^2(\sqrt{t}))$.

We conclude this section with a brief analysis of the finite contributions to the evolved $\Pi_{R,\mu\nu}^V$ and their asymptotic behavior.

¹²We note that the one-loop calculations in [211] are all described by the one-loop expansion of the single universal (5.68), where the anomalous dimension of all evolved currents is given by (5.57) and the anomalous dimensions of the nonevolved currents are given in [32, 181].

At $\mathcal{O}(g_0^2)$ the finite contributions are given by I.2, II.2, III.2', III.3, IV.1 and IV.2, see section 5.4.2. Only I.2 is already present in the nonevolved case, where it is UV divergent at $d = 4$ and cancels the UV divergence of I.1. Therefore, I.2 does not vanish as $t \rightarrow 0$ at fixed nonzero separation, and it contributes to the leading STE term in (5.69). All the other finite contributions manifestly vanish in the same limit and only contribute at higher orders to the STE.

In the opposite limit of vanishing separation at fixed $t > 0$, all of the above finite contributions are expected to vanish as a consequence of the regulating effect of the gradient-flow smearing.

Finally, we mention that “evolved contact terms” resulting from the smearing of nonevolved contact terms, may be expected to contribute to evolved 2-point correlators. In the vector case, dimensional analysis tells us that nonevolved contact terms are of the type $\delta_{\mu\nu}\Delta\delta^{(4)}(x-y)$ and $\partial_\mu\partial_\nu\delta^{(4)}(x-y)$. Hence, they may lead to evolved contact terms of the type $\delta_{\mu\nu}\Delta K_t(x-y)$ and $\partial_\mu\partial_\nu K_t(x-y)$, respectively. These terms no longer vanish at nonzero separation for small t , and they can contribute starting at $\mathcal{O}(t)$ to the STE in (5.69).

5.5 Current conservation and renormalization

5.5.1 Nonevolved case: conservation implies nonrenormalization

In section 2.4.4 we showed that conserved currents do not require renormalization, barring some exceptions when the current mixes with operators whose divergence vanishes identically. The prime example of the latter case is the vector current in QED, which mixes with the divergence of the field strength [132]. However, the conserved current we are considering here, associated with the exact — i.e. nonanomalous — global $U(1)_B$ symmetry discussed in section 2.2, does not mix with any operator: all available operators with the correct mass-dimension carry color charge. Therefore, the arguments presented in section 2.4.4 hold, and the nonevolved current does not renormalize: $j_\mu(x) = [j_\mu(x)]$.

The nonrenormalization and conservation of $j_\mu(x)$ in turn imply the transversality of the corresponding 2-point correlator.

5.5.2 Evolved case: conservation does not imply nonrenormalization

The situation is fundamentally different for the evolved vector current $J_\mu(t, x)$. We have seen that the latter acquires an anomalous dimension γ_J , which enters the leading term of the small- t expansion of the evolved 2-point vector correlator as shown in (5.69). Yet, the same leading term fulfils transversality and current conservation, despite the presence of an anomalous dimension. Indeed, specifically:

$$\begin{aligned} \partial_\mu^x \Pi_{R,\mu\nu}^V(t, x-y; g(\mu), \mu) &= C(\sqrt{t}\mu, g(\mu)) \partial_\mu^x \Pi_{R,\mu\nu}^V(x-y; g(\mu), \mu) \\ &\quad + \partial_\mu^x \Delta \Pi_{R,\mu\nu}(t, x-y; g(\mu), \mu) \\ &= \partial_\mu^x \Delta \Pi_{R,\mu\nu}(t, x-y; g(\mu), \mu) \end{aligned} \quad (5.71)$$

with $\Delta \Pi_{R,\mu\nu}$ the $\mathcal{O}(t)$ contribution to the small- t expansion of $\Pi_{R,\mu\nu}^V$ introduced in (5.61). The first term in the right-hand side of the first equality in (5.71) has an anomalous dimension, but vanishes because the nonevolved correlator is transversal, i.e., $\partial_\mu^x \Pi_{R,\mu\nu}^V(x-y; g(\mu), \mu) = 0$. It is then clear that a nonzero anomalous dimension is allowed because the evolved current depends on the additional (unphysical) gradient-flow scale $\sqrt{t}$, which parametrizes its nonlocality.

On the other hand, the violation of transversality and the nonconservation of the evolved vector current do occur through the second term $\partial_\mu^x \Delta \Pi_R$ in the right-hand side of (5.71). Hence, they are a soft-breaking effect of $\mathcal{O}(t)$ induced by the smearing action of the gradient flow that vanishes as $t \rightarrow 0$. This agrees with the explicit results at $\mathcal{O}(g_0^0)$ in (5.34) and (5.35) and the classical leading-order consideration in appendix C.4.

As a side note, one could also relate the nonconservation of the evolved vector current to the lack of a corresponding exact symmetry in a $d+1$ -dimensional theory that includes the flow direction, along the lines of [28, 31, 63].

The conclusions of this chapter are presented in section 7.1.

Chapter 6

Nonminimal gradient flows in QCD-like theories

This chapter is based on the paper [95].

6.1 Introduction

Up to now we have been using a flow equation for the gauge fields based on the gradient of the Yang-Mills action:

$$\partial_t B_\mu(t, x) = - \frac{\delta S_{YM}}{\delta A_\mu(x)} \Big|_{A_\mu(x)=B_\mu(t, x)}, \quad B_\mu(0, x) = A_\mu(x) \quad (6.1)$$

also when (nonevolved) fermions are present. We considered this equation in depth in chapter 4. A much-studied observable using (6.1) is:

$$E(t, x) = g_0^2 \mathcal{L}_{YM} \Big|_{A_\mu(x)=B_\mu(t, x)} \quad (6.2)$$

which we briefly discussed in section 4.4.3. Since $E(t, x)$ solely consists of evolved gauge fields, it acquires no multiplicative renormalization factor. Its expectation value $\langle E(t) \rangle$ is used in lattice studies for e.g. scale setting [33] and defining finite volume running coupling schemes¹ [212].

¹There is some degree of disagreement in the lattice community on the validity of certain schemes defined in this way, especially in relatively large N_f studies, see e.g. [41, 42].

In chapter 4 we pointed out that the gradient flow (6.1) drives the gauge field $B_\mu(t, x)$ towards the stationary points of the Yang-Mills action for increasing flow time t . Since some properties of the expectation value of $E(t, x)$ in pure Yang-Mills theory are attributed to this fact [27], one might wonder what happens if one adds the fermionic term to the action in (6.1), i.e. $S_{YM} \rightarrow S_{YM} + S_F$, with the actions in (2.9). The consequences of this change are the subject of this chapter.

There exists a close relation between the gradient flow and the Langevin equation used in stochastic quantization [213]. Since it is customary to include the fermionic matter term in the Langevin equation when stochastically quantizing QCD-like theories, we will make this relation more explicit.

6.1.1 Motivation from stochastic quantization

In stochastic quantization the basic idea is to introduce an extra scale t — we will shortly see how it is related to the flow time t of (6.1) — and a Gaussian noise field $\eta(t, x)$, and describe the t evolution of some field $\phi(t, x)$ by the generalized Langevin equation:

$$\partial_t \phi(t, x) = - \int_y M(x, y) \frac{\delta S}{\delta \phi(t, y)} + \eta(t, x) \quad (6.3)$$

where $M(x, y)$ is some kernel², S is the Euclidean action for the theory under study, and $\frac{\delta S}{\delta \phi(t, y)}$ is short for $\frac{\delta S}{\delta \phi(y)} \big|_{\phi(y)=\phi(t, y)}$. The Gaussian noise satisfies:

$$\langle \eta(t, x) \eta(s, y) \rangle_\eta = 2M(x, y) \delta(t - s), \quad \langle \eta(t, x) \rangle_\eta = 0 \quad (6.4)$$

where the η subscript on the angle brackets indicates that the average over η is being taken, see e.g. [30] for details. Note that the presence of the noise in (6.3) makes the value of ϕ at large t completely independent of the boundary condition at $t = 0$, and a common choice is $\phi(0, x) = 0$. We now imagine that the fields are coupled to some heat reservoir. It can be shown that in the large t limit we reach equilibrium [30], and all stochastic correlators converge to their Euclidean quantum field theory counterparts:

$$\langle \phi(t, x) \phi(t, y) \dots \rangle_\eta \xrightarrow{t \rightarrow \infty} \langle \phi(x) \phi(y) \dots \rangle \quad (6.5)$$

Now, the gradient flow evolution can be understood as follows³: at some large time $t = t_0$ we abruptly switch off the heat reservoir and the noise field. At

²The standard Langevin equation is the special case $M(x, y) = \delta^{(d)}(x - y)$.

³This reasoning is based on [207].

some later time $t_1 > t_0$, the fields will have evolved via the flow equation:

$$\partial_t \phi(t, x) = - \int_y M(x, y) \frac{\delta S}{\delta \phi(t, y)}, \quad \phi(t_0, x) = \phi(x) \quad (6.6)$$

which is exactly the gradient flow equation.

When stochastically quantizing pure Yang-Mills theory, the following Langevin equation is used [30] — here we neglect the possible presence of a ‘gauge fixing term’, usually called a *Zwanziger term* in this context:

$$\partial_t A_\mu(t, x) = D_\nu F_{\nu\mu}(t, x) + \eta(t, x) \quad (6.7)$$

with $D_\mu = \partial_\mu + g_0[A_\mu, \cdot]$ the covariant derivative, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g_0[A_\mu, A_\nu]$ the field strength. However, when considering QCD-like theories, a different Langevin equation is being used for the gauge field in the presence of fermionic matter:

$$\partial_t A_\mu(t, x) = D_\nu F_{\nu\mu}(t, x) - g_0 \bar{\psi}(t, x) \gamma_\mu T^a \psi(t, x) T^a + \eta(t, x) \quad (6.8)$$

with T^a the $SU(N)$ generators, where for the Langevin evolution of the fermions one can take [214]:

$$\begin{aligned} \partial_t \psi(t, x) &= \not{D}^2 \psi(t, x) - \not{D} \theta(t, x) \\ \partial_t \bar{\psi}(t, x) &= \bar{\psi}(t, x) \overleftarrow{\not{D}}^2 + \bar{\theta}(t, x) \end{aligned} \quad (6.9)$$

where $\theta, \bar{\theta}$ are Gaussian noise fields.

6.1.2 Nonminimal gradient flows

In gradient flow studies it is customary instead to use the Yang-Mills gradient flow of (6.1), i.e. (6.7) with⁴ $\eta = 0$, also when fermions are present. We set out to investigate what will change if we instead use (6.8) with $\eta = 0$ as the basis for the gradient flow, i.e. the case where we include the fermion-bilinear color non-singlet vector current in the right-hand side of the flow equation. We will refer to this as the *nonminimal gradient flow* equation.

For the evolution of the fermions we are considering two different flow equations. Firstly we use the one that is standard in gradient flow studies [31] using

⁴And with $A_\mu(t, x) \rightarrow B_\mu(t, x)$ with boundary condition $B_\mu(0, x) = A_\mu(x)$.

the operator D^2 which we discussed in section 4.2, and secondly we use the operator $\not{D}^2$ instead, i.e. (6.9) with $\theta = \bar{\theta} = 0$.

All flows must respect the symmetries of the nonevolved theory, and the Yang-Mills gradient flow is indeed the simplest possibility. However, the non-minimal cases we will investigate have the conceptual advantage that they are actually driving the gauge field towards the stationary points of the full QCD-like action, not just of the Yang-Mills action.

That said, we also carry out the exercise of writing the most general flow equations for the gauge field and the fermions, respecting the symmetries of the nonevolved theory, for completeness.

We use the nonminimal flows to calculate the expectation value of the operator $E(t, x)$ from (6.2) to next-to-leading order in the coupling, generalizing the result first presented in [27]. We also generalize the demonstration of the renormalization properties of the evolved fields of [28, 31] (see also [29]), and calculate the evolved fermions' field renormalization factor Z_χ for the different nonminimal flows. We find that there exists a one-parameter family of flow equation systems for which there is a renormalization scheme in which the evolved fermion anomalous dimension vanishes to all orders in perturbation theory.

We again solely consider massless fermions. The quantity $\langle E(t) \rangle$ will have a mass dependence at next-to-leading order in the gauge coupling, see e.g. [52] for the Yang-Mills gradient flow case. However, the $\overline{\text{MS}}$ expressions for Z_χ and the other renormalization properties discussed in this chapter are independent of the presence of fermion masses.

6.1.3 Organization of the chapter

In section 6.2 we review the perturbative result for $\langle E \rangle$ with E in (6.2) up to next-to-leading order in the coupling, and the renormalization properties of the evolved fields, using the Yang-Mills gradient flow (YMGF). In section 6.3 we introduce the nonminimal gradient flow (NMGF), calculate $\langle E \rangle$, present the generalization of the arguments for the renormalization properties for the nonminimally evolved fields, and calculate the field renormalization factor of the evolved fermions. Subsequently, in section 6.4 we include the effects of replacing the operator D^2 in the fermion flow equation with $\not{D}^2$, coining this case the 'slashed' nonminimal gradient flow (sNMGF). In section 6.5 we investigate the N_f dependence of $\langle E \rangle$ with $N = 3$, calculated with the three different flows. In section 6.6 we generalize the flow equations by putting them in the most general form compliant with all the symmetries of the nonevolved theory.

6.2 Yang-Mills gradient flow

We will now first review some results obtained using the YMGF described in chapter 4, which we will generalize in the following sections.

6.2.1 Perturbative calculation of $\langle E \rangle$

For later comparison, we briefly review the perturbative calculation of $\langle E \rangle$ up to $\mathcal{O}(g_0^4)$ of [27]. At this order in the coupling it only involves the evolved gauge fields given in (4.13); the fermionic contribution will solely be coming from the nonevolved path integral averaging.

The observable $E(t, x)$ from (6.2) written in terms of the B_μ field reads:

$$\begin{aligned} E &= \frac{g_0^2}{4} G_{\mu\nu}^a G_{\mu\nu}^a \\ &= \frac{1}{2} g_0^2 \partial_\mu B_\nu^a (\partial_\mu B_\nu^a - \partial_\nu B_\mu^a) + g_0^3 f^{abc} \partial_\mu B_\nu^a B_\mu^b B_\nu^c \\ &\quad + \frac{1}{4} g_0^4 f^{abe} f^{cde} B_\mu^a B_\nu^b B_\mu^c B_\nu^d \end{aligned} \quad (6.10)$$

and the expectation value $\langle E \rangle$ can be computed order-by-order in the coupling [27, 52], using (4.15) combined with the contributions stemming from the nonevolved path integral averaging. The result up to next-to-leading order in the coupling is given by (using dimensional regularization with $d = 4 - 2\varepsilon$, see appendix B):

$$\langle E \rangle = \frac{1}{2} g_0^2 \frac{(N^2 - 1)}{(8\pi t)^{d/2}} (d - 1) \{1 + c_1 g_0^2 + \mathcal{O}(g_0^4)\} \quad (6.11)$$

$$\begin{aligned} c_1 &= \frac{1}{(4\pi)^2} (4\pi)^\varepsilon (8t)^\varepsilon \\ &\quad \times \left\{ N \left(\frac{11}{3\varepsilon} + \frac{52}{9} - 3 \log 3 \right) - N_f \left(\frac{2}{3\varepsilon} + \frac{4}{9} - \frac{4}{3} \log 2 \right) + \mathcal{O}(\varepsilon) \right\} \end{aligned} \quad (6.12)$$

This observable is rendered finite by the coupling renormalization of the non-evolved theory. In the $\overline{\text{MS}}$ scheme one has (see section 3.4):

$$g_0^2 = \mu^{2\varepsilon} (4\pi e^{-\gamma_E})^{-\varepsilon} g^2(\mu) Z_g^2(g(\mu), \varepsilon) \quad (6.13)$$

with $g(\mu)$ the renormalized coupling, and:

$$Z_g^2(g(\mu), \varepsilon) = 1 - \frac{\beta_0}{\varepsilon} \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4(\mu)), \quad \beta_0 = \frac{11}{3} N - \frac{4}{3} T(R) N_f \quad (6.14)$$

After renormalization⁵ we express the renormalized coupling $\alpha(\mu) = \frac{g^2(\mu)}{4\pi}$ in terms of the renormalization group invariant coupling $\alpha(q)$ by inverting (D.6), and subsequently setting $q = (8t)^{-1/2}$ and $N = 3$ we arrive at:

$$\langle E \rangle = \frac{3}{4\pi t^2} \alpha(q) \{1 + k_1 \alpha(q) + \mathcal{O}(\alpha^2)\}, \quad k_1 = 1.0978 + 0.0075 \times N_f \quad (6.15)$$

For the renormalized result for general N we refer the reader to appendix D.

6.2.2 Renormalization of evolved fields

It has been shown in [28] (see also [29]) that in Wilsonian normalization — see section 4.4.1 — all correlators consisting of only B_μ fields are finite, i.e. they do not require renormalization beyond the usual renormalization of the parameters of the four-dimensional theory over which the path integral average is being taken. When switching to canonical normalization, i.e. taking $B_\mu = \mathcal{B}_\mu/g_0$, any correlator consisting of m number of B_μ fields now automatically contains a factor of g_0^m , which will renormalize as given in (6.13) in the case of $\overline{\text{MS}}$, see the discussion around (4.46).

The fermion fields χ and $\bar{\chi}$ do require a renormalization factor [31], namely:

$$\chi_R(t, x) = Z_\chi^{1/2} \chi(t, x), \quad \bar{\chi}_R(t, x) = Z_\chi^{1/2} \bar{\chi}(t, x) \quad (6.16)$$

with:

$$Z_\chi(g(\mu), \varepsilon) = 1 + C_2(R) \frac{3}{\varepsilon} \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4) \quad (6.17)$$

and the contributing self-energy diagrams together with their values are presented in section 4.4.2. Any bare evolved correlator consisting of an arbitrary number of $g_0 B_\mu$'s, and n number of χ 's and $\bar{\chi}$'s is now rendered finite by multiplicative renormalization with Z_χ^n , supplemented with the usual renormalization of the parameters⁶ of the nonevolved theory. An important feature is that this also holds true when some of the evolved fields have coinciding spacetime positions at strictly positive flow time; i.e. composite operators of evolved fields do not acquire any additional renormalization factor, in contrast to the nonevolved

⁵We would like to emphasize that the fact that $\langle E \rangle$ is rendered finite by the coupling renormalization at this order does not constitute an example of the ‘nonrenormalization’ of the evolved gauge field B_μ , to be discussed in section 6.2.2. In order to achieve this, one has to use the 2-loop universality of Z_g from (6.14) and calculate $\langle E \rangle$ to next-to-next-to-leading order. This has been done in [52], where the ‘nonrenormalization’ is indeed confirmed.

⁶With ‘parameters’ we mean the gauge coupling and the gauge fixing parameter. See section 2.3 for the Euclidean action of the nonevolved theory.

case, see section 4.4.3.

The all order demonstration of these statements is found in [28] and [29]. In the remainder of this section we present the general reasoning for this demonstration, without any intention or pretension of being complete. We nevertheless choose to include it here, since these will be the arguments that we generalize later on in sections 6.3.3 and 6.4.2.

The starting point of the demonstration from [28] is to describe the theory in $d + 1$ dimensions, the extra dimension being the flow time direction $t \in [0, \infty)$. The evolved fields are being considered as independent from the nonevolved elementary fields, apart from their implicit dependence through the boundary conditions. The $d + 1$ dimensional ‘bulk’ action is given by:

$$S_{bulk} = S_{G,fl} + S_{d\bar{d}} + S_{F,fl} \quad (6.18)$$

with:

$$S_{G,fl} = -2 \int_0^\infty dt \int_x \text{tr} \{ L_\mu (\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu) \} \quad (6.19)$$

$$S_{F,fl} = \int_0^\infty dt \int_x \{ \bar{\lambda} (\partial_t - \Delta + \alpha_0 g_0 \partial_\nu B_\nu) \chi + \bar{\chi} (\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0 g_0 \partial_\nu B_\nu) \lambda \} \quad (6.20)$$

$$S_{d\bar{d}} = -2 \int_0^\infty dt \int_x \text{tr} \{ \bar{d} (\partial_t d - \alpha_0 D_\mu \partial_\mu d) \} \quad (6.21)$$

where we introduced the lie algebra valued lagrange multiplier field $L_\mu(t, x) = L_\mu^a(t, x) T^a$ with purely imaginary components, the two fermionic lagrange multipliers $\lambda(t, x)$ and $\bar{\lambda}(t, x)$ which carry the same indices as the quark fields, and the bulk ghost fields $d(t, x)$ and $\bar{d}(t, x)$. The ghost d has boundary condition:

$$d(0, x) = c(x) \quad (6.22)$$

while the other new fields do not obey any. Note that the lagrange multipliers L_μ , λ and $\bar{\lambda}$ are there to enforce the flow equations upon variation. The $\bar{d}$ field acts as a lagrange multiplier enforcing the correct diffusion equation on the $d + 1$ dimension ghost, such that the flow equations are invariant under an infinitesimal gauge transformation $\Lambda(t, x) = e^{\varepsilon \omega(t, x)}$, with $\omega(t, x) = -g_0 d(t, x)$ (see sections 2.3.2, 4.1 and 4.2). This generalizes the BRST symmetry to the $d + 1$ dimensional bulk action:

$$\delta S_{bulk} = 0 \quad (6.23)$$

with $\delta = \delta_{BRST}$, which can be checked using the following BRST variations of the bulk fields:

$$\begin{aligned}
\delta B_\mu &= D_\mu d, & \delta L_\mu &= g_0[L_\mu, d] \\
\delta \chi &= -g_0 d \chi, & \delta \lambda &= -g_0 d \lambda \\
\delta \bar{\chi} &= -g_0 \bar{\chi} d, & \delta \bar{\lambda} &= -g_0 \bar{\lambda} d, \\
\delta d &= -g_0 d^2, & \delta \bar{d} &= D_\mu L_\mu - g_0 \{d, \bar{d}\}
\end{aligned} \tag{6.24}$$

The nontrivial variation of $\bar{d}$ is chosen such that $\delta S_{bulk} = 0$, and $\delta^2 \bar{d} = 0$. Useful identities in checking $\delta S_{bulk} = 0$ are obtained by defining:

$$\begin{aligned}
E_\mu &= \partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu, & f &= (\partial_t - \Delta + \alpha_0 g_0 \partial_\nu B_\nu) \chi \\
e &= \partial_t d - \alpha_0 D_\mu \partial_\mu d, & \bar{f} &= \bar{\chi} (\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0 g_0 \partial_\nu B_\nu)
\end{aligned} \tag{6.25}$$

and noting:

$$\begin{aligned}
\delta E_\mu &= g_0[E_\mu, d] + D_\mu e, & \delta f &= -g_0 df - g_0 e \chi \\
\delta e &= -g_0 \{e, d\}, & \delta \bar{f} &= -g_0 \bar{f} d - g_0 \bar{\chi} e
\end{aligned} \tag{6.26}$$

The same identities hold when Δ and $\overleftarrow{\Delta}$ are replaced by $\mathbb{A}$ and $\overleftarrow{\mathbb{A}}$, as in section 6.4.

Together with the BRST invariance of the d dimensional QCD action discussed in section 2.3.2, and the fact that the path integral measure is invariant, we have:

$$\langle \delta O \rangle = 0 \tag{6.27}$$

with O any combination of evolved and nonevolved elementary fields. Equation (6.27) has some important consequences. For our purposes the most important relation that follows, for reasons becoming obvious below, is:

$$\frac{1}{\xi_0} \langle B_\mu^a(t, x) \partial_\nu A_\nu^b(y) \partial_\rho A_\rho^c(z) \rangle = - \langle (D_\mu d)^a(t, x) \bar{c}^b(y) \partial_\rho A_\rho^c(z) \rangle \tag{6.28}$$

where ξ_0 is the bare gauge-fixing parameter introduced in (2.28). The next step in the demonstration is to exclude the possibility of counterterms. These are either localized in the $d+1$ dimensional $\mathbb{R}^d \times (0, \infty)$ bulk or at the d dimensional $\mathbb{R}^d$ boundary [215]. We first focus on the former.

The impossibility of bulk divergences is shown by first eliminating the elementary nonevolved fields from the quadratic part of the bulk action, by plug-

ging in:

$$B_\mu(t, x) = e^{t\Delta_x} A_\mu(x) + b_\mu(t, x) \quad (6.29a)$$

$$\chi(t, x) = e^{t\Delta_x} \psi(x) + \Psi(t, x) \quad (6.29b)$$

$$\bar{\chi}(t, x) = e^{t\Delta_x} \bar{\psi}(x) + \bar{\Psi}(t, x) \quad (6.29c)$$

where $b_\mu, \Psi, \bar{\Psi}$ obey homogeneous boundary conditions. The terms involving the nonevolved elementary fields all drop out, since they solve the linearized flow equations. Therefore, the only nonzero 2-point bulk correlators are $\langle bL \rangle$, $\langle \Psi \bar{\lambda} \rangle$, $\langle \lambda \bar{\Psi} \rangle$ and $\langle d\bar{d} \rangle$. By subsequently analyzing the interaction part of the bulk action it becomes clear that the only nonzero bulk correlators of these fields are trees, i.e. they can never form loops, and thus will be finite. Therefore, there will be no bulk counterterms.

The remaining possibility is the presence of boundary counterterms. Dimensional analysis together with symmetry requirements and ghost number conservation tells us that the only allowed new counterterms are of the form:

$$\begin{aligned} \text{Boundary CTs} \sim & \int_x \text{tr} \{ z_1 L_\mu(0, x) A_{R,\mu}(x) + z_2 \bar{d}(0, x) c_R(x) \} \\ & + z_3 \int_x \{ \bar{\lambda}(0, x) \psi_R(x) + \bar{\psi}_R(x) \lambda(0, x) \} \end{aligned} \quad (6.30)$$

At this point, the importance of (6.28) becomes apparent; the tree level computation of its renormalized version⁷ forces $z_1 = z_2 = 0$, i.e. there is no field renormalization for L_μ and $\bar{d}$, and thus no field renormalization for B_μ and d (since there can be no bulk counterterms). However, there is no restriction on z_3 , and consequently $\bar{\lambda}$ and λ do get a field renormalization factor:

$$\bar{\lambda}(t, x) = Z_\chi^{1/2} \bar{\lambda}_R(t, x), \quad \lambda(t, x) = Z_\chi^{1/2} \lambda_R(t, x) \quad (6.31)$$

which implies the reciprocal field renormalization factors for χ and $\bar{\chi}$ in order not to have a bulk counterterm.

This completes the reasoning for the case of correlators consisting of the evolved fields at distinct points in spacetime. In order to include composite operators it is sufficient to notice the following: the small — or indeed vanishing — distance limit of any number of evolved fields inside a correlator will always remain finite for $t > 0$ due to the exponential suppression of the high frequency modes. This argument is more elaborately presented in section 4.4.3.

Missing details of the demonstration can be found in [28] and [29].

⁷Only the fields and parameters of the four-dimensional theory are renormalized here, i.e. $\lambda_0 \rightarrow \lambda$, $A_\mu \rightarrow A_{R,\mu}$, $c \rightarrow c_R$. See [28] for the explicit computation.

6.3 Nonminimal gradient flow

6.3.1 Defining the nonminimal gradient flow

We now include fermionic matter in the flow equation for the gauge field, giving rise to the nonminimal gradient flow (NMGF) equation:

$$\partial_t B_\mu = - \frac{\delta(S_{YM} + S_F)}{\delta B_\mu} \Big|_{A_\mu=B_\mu, \psi=\chi, \bar{\psi}=\bar{\chi}} = D_\nu G_{\nu\mu} - g_0 j_\mu^a T^a \quad (6.32)$$

with boundary condition $B_\mu|_{t=0} = A_\mu$, where:

$$j_\mu^a(t, x) = \bar{\chi}(t, x) \gamma_\mu T^a \chi(t, x) \quad (6.33)$$

is the gauge covariant matter current. The quark fields χ and $\bar{\chi}$ are the same as in section 4.2, apart from the fact that the B_μ field in their flow equations is now the solution of the NMGF equation (6.32). Importantly, the NMGF equation is gauge invariant, as is demonstrated by combining the transformation property of the current:

$$j_\mu^{a'} T^a = \Lambda j_\mu^a T^a \Lambda^{-1} \quad (6.34)$$

with the transformation of the YM equation, provided in section 4.1.

We can modify the flow equation (6.32) by the same Zwanziger term as in (4.4) by utilizing a particular t -dependent gauge transformation, see again section 4.1. This gives rise to the modified NMGF equation:

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) - g_0 j_\mu^a(t, x) T^a + \alpha_0 D_\mu \partial_\nu B_\nu(t, x) \quad (6.35)$$

with boundary condition $B_\mu(0, x) = A_\mu(x)$. For $\alpha_0 = 1$ the solution to the modified nonminimal gradient flow equation reads:

$$\begin{aligned} B_\mu(t, x) &= e^{t\Delta_x} A_\mu(x) + \int_0^t ds e^{(t-s)\Delta_x} \{R_\mu(s, x) - g_0 j_\mu^a(s, x) T^a\} \\ &\equiv B_\mu^{YM}(t, x) + \mathcal{F}_\mu(t, x) \end{aligned} \quad (6.36)$$

where R_μ is still the same as in (4.10), though now the B_μ 's are the solution to (6.35) instead of (4.4), and we have explicitly put the superscript 'YM' on the solution (4.8) of the Yang-Mills gradient-flow equation (4.4). Again, we write

the fields as a series in powers of the bare coupling:

$$\begin{aligned}\chi(t, x) &= \sum_{n=0}^{\infty} g_0^n \chi_n(t, x), & \bar{\chi}(t, x) &= \sum_{n=0}^{\infty} g_0^n \bar{\chi}_n(t, x), \\ B_\mu(t, x) &= \sum_{n=0}^{\infty} g_0^n B_{\mu,n}(t, x), & B_{\mu,n}(t, x) &= B_{\mu,n}^{YM}(t, x) + \mathcal{F}_{\mu,n}(t, x)\end{aligned}\tag{6.37}$$

and thus we have for the first three orders of B_μ :

$$B_{\mu,0}^a(t, x) = B_{\mu,0}^{YM,a}(t, x) = e^{t\Delta_x} A_\mu^a(x) \tag{6.38}$$

$$\begin{aligned}B_{\mu,1}^a(t, x) &= \int_0^t ds e^{(t-s)\Delta_x} \{f^{abc} B_{\nu,0}^b (2\partial_\nu B_{\mu,0}^c - \partial_\mu B_{\nu,0}^c) - \bar{\chi}_0 \gamma_\mu T^a \chi_0\} \\ &= B_{\mu,1}^{YM,a}(t, x) + \mathcal{F}_{\mu,1}^a(t, x)\end{aligned}\tag{6.39}$$

$$B_{\mu,2}^a(t, x) = B_{\mu,2}^{YM,a}(t, x) + \mathcal{F}_{\mu,2}^a(t, x) \tag{6.40}$$

where:

$$\mathcal{F}_{\mu,1}^a(t, x) = - \int_0^t ds e^{(t-s)\Delta_x} \{ \bar{\chi}_0(s, x) \gamma_\mu T^a \chi_0(s, x) \} \tag{6.41}$$

$$\begin{aligned}\mathcal{F}_{\mu,2}^a(t, x) &= - \int_0^t ds e^{(t-s)\Delta_x} j_{\mu,1}^a(s, x) \\ &\quad + f^{abc} \int_0^t ds e^{(t-s)\Delta_x} \left\{ (\partial_\mu \mathcal{F}_{\nu,1}^b - 2\partial_\nu \mathcal{F}_{\mu,1}^b) B_{\nu,0}^c \right. \\ &\quad \left. + (\partial_\mu B_{\nu,0}^b - 2\partial_\nu B_{\mu,0}^b) \mathcal{F}_{\nu,1}^c \right\}\end{aligned}\tag{6.42}$$

with:

$$j_{\mu,1}^a(s, x) = \bar{\chi}_1(s, x) \gamma_\mu T^a \chi_0(s, x) + \bar{\chi}_0(s, x) \gamma_\mu T^a \chi_1(s, x) \tag{6.43}$$

and:

$$\bar{\chi}_1(s, x) = -2 \int_0^s du e^{(s-u)\Delta_x} \{ \partial_\rho \bar{\chi}_0(u, x) B_{\rho,0}(u, x) \} \tag{6.44}$$

$$\chi_1(s, x) = 2 \int_0^s du e^{(s-u)\Delta_x} \{ B_{\rho,0}(u, x) \partial_\rho \chi_0(u, x) \} \tag{6.45}$$

These quantities are diagrammatically represented in figure 6.1.

Figure 6.1: Diagrammatic representation of the expansion of the fields in (6.37). The top row is $\mathcal{O}(g_0)$: (a) represents χ_1 from (6.45), (b) $\bar{\chi}_1$ from (6.44), and (c) is a genuine new contribution due to the NMGF, namely $\mathcal{F}_{\mu,1}$ from (6.41). The bottom row is $\mathcal{O}(g_0^2)$, and are all terms of $\mathcal{F}_{\mu,2}$ from (6.42): (d) and (e) represent the first line while (f) collectively represents the second and third line in (6.42).

6.3.2 Perturbative calculation of $\langle E \rangle$

We present the calculation of the new contributions — collectively called $\mathcal{E}_\chi$ — to $\langle E \rangle$ stemming from the fermionic term $\mathcal{F}_\mu$ in B_μ , see (6.36). The full observable has the same form as in (6.10), though now the B_μ 's represent solutions to the NMGF, so we have:

$$\langle E \rangle = \frac{g_0^2}{4} \langle G_{\mu\nu}^a G_{\mu\nu}^a \rangle = \langle E \rangle \big|_{B_\mu = B_\mu^{YM}} + \mathcal{E}_\chi \quad (6.46)$$

The new contributions start at $\mathcal{O}(g_0^4)$, and are given by:

$$\begin{aligned} \mathcal{E}_{\chi,1} &= g_0^3 \langle \partial_\mu \mathcal{F}_{\nu,1}^a (\partial_\mu B_{\nu,0}^a - \partial_\nu B_{\mu,0}^a) \rangle \\ &= 4g_0^4 N_f \text{tr}(T^a T^a) \left\{ (d-3)I_1 + 2(I_2 + 2I_3 + I_4) \right\} \end{aligned} \quad (6.47)$$

$$\begin{aligned} \mathcal{E}_{\chi,2} &= \frac{1}{2} g_0^4 \langle \partial_\mu \mathcal{F}_{\nu,1}^a (\partial_\mu \mathcal{F}_{\nu,1}^a - \partial_\nu \mathcal{F}_{\mu,1}^a) \rangle \\ &= 4g_0^4 N_f \text{tr}(T^a T^a) \left\{ I_5 + (d-1)I_6 + (d-2)I_7 \right\} \end{aligned} \quad (6.48)$$

Figure 6.2: Diagrammatic representation of the contributions to $\langle E \rangle$ due to the NMGF, whose total value is given in (6.53). (a) represents $\mathcal{E}_{\chi,1}$ from (6.47), (b) represents $\mathcal{E}_{\chi,2}$ from (6.48), and the combination of (c) and (d) represents $\mathcal{E}_{\chi,3}$ from (6.49).

$$\begin{aligned}\mathcal{E}_{\chi,3} &= g_0^4 \langle \partial_\mu \mathcal{F}_{\nu,2}^a (\partial_\mu B_{\nu,0}^a - \partial_\nu B_{\mu,0}^a) \rangle \\ &= 16g_0^4 N_f \text{tr}(T^a T^a) \{I_8 - I_9\}\end{aligned}\quad (6.49)$$

and the integrals I_1 – I_9 are listed in appendix B.3. These contributions are diagrammatically represented in figure 6.2. The total fermionic contribution is subsequently given by:

$$\begin{aligned}\mathcal{E}_\chi &= \sum_{i=1}^3 \mathcal{E}_{\chi,i} \\ &= 4g_0^4 N_f \text{tr}(T^a T^a) ((d-2)I_1 + I_5 + (d-1)I_6 + (d-2)I_7 + 4I_8)\end{aligned}\quad (6.50)$$

The integrals I_1 , I_5 – I_8 , are solved using Schwinger parametrization:

$$\frac{1}{p^2} = \int_0^\infty du e^{-up^2} \quad (6.51)$$

and Gaussian integration:

$$\int_{p,q} e^{-\vec{p}^T A \vec{p}} = (4\pi)^{-d} \det(A)^{-d/2} \quad (6.52)$$

The expressions obtained in this way contain gamma functions, incomplete beta functions and hypergeometric functions, which depend on d only; the t dependence factors out. The ε -expansions with $d = 4 - 2\varepsilon$ of I_1 , I_5 – I_8 are provided in appendix B.3. Adding all contributions, we find for (6.50):

$$\mathcal{E}_\chi = -\frac{1}{2} \frac{(N^2 - 1)}{(8\pi t)^{d/2}} (d-1) N_f \frac{g_0^4}{(4\pi)^2} \left(2 + \frac{4}{3} \log 2 - \log 3 + \mathcal{O}(\varepsilon) \right) \quad (6.53)$$

which is manifestly finite, as is to be expected anticipating the results of section 6.3.3. We now have for the full observable (6.46):

$$\langle E \rangle = \frac{1}{2} g_0^2 \frac{(N^2 - 1)}{(8\pi t)^{d/2}} (d - 1) \{1 + c'_1 g_0^2 + \mathcal{O}(g_0^4)\} \quad (6.54)$$

$$c'_1 = \frac{1}{(4\pi)^2} (4\pi)^\varepsilon (8t)^\varepsilon \times \left\{ N \left(\frac{11}{3\varepsilon} + \frac{52}{9} - 3 \log 3 \right) - N_f \left(\frac{2}{3\varepsilon} + \frac{22}{9} - \log 3 \right) + \mathcal{O}(\varepsilon) \right\} \quad (6.55)$$

Following the same steps as in section 6.2.1, i.e. renormalizing the coupling as in (6.13), expressing the renormalized coupling $\alpha(\mu)$ in terms of the RG invariant coupling $\alpha(q)$ by inverting (D.6) and setting $q = (8t)^{-1/2}$ and $N = 3$, we now have:

$$\langle E \rangle = \frac{3}{4\pi t^2} \alpha(q) \{1 + k'_1 \alpha(q) + \mathcal{O}(\alpha^2)\}, \quad k'_1 = 1.0978 - 0.1377 \times N_f \quad (6.56)$$

Note that the N_f contribution in k'_1 is negative, in contrast to the result from (6.15) that is obtained using the Yang-Mills gradient flow. See appendix D for the renormalized result for general N . We further comment on our result (6.56) in section 6.5, where we will make a more elaborate comparison with the YMGF result from (6.15).

6.3.3 Renormalization of evolved fields

The result for $\langle E \rangle$ from the previous section seems to indicate that the nonminimally evolved gauge field remains devoid of any renormalization. However, one might suspect that at the next order in g_0 , i.e. $\mathcal{O}(g_0^6)$, divergences arise from the fermionic contribution; at that order the self-energy contribution to the evolved fermionic propagator might have to be canceled with Z_χ factors. Our aim here is to show that this is not the case, and any correlator consisting only of evolved gauge fields $B_\mu - g_0 B_\mu$ when canonically normalized — is still finite, without any renormalization beyond that of the nonevolved d dimensional theory that is being used in the path integral averaging.

The demonstration is a straightforward generalization of the one from [28, 31], see also [29], that we have reviewed in section 6.2.2. We will now show where the changes are with respect to the YMGF case, and why the same reasoning still holds.

First of all, the gauge bulk action $S_{G,fl}$ from (6.19) changes in order to comply with the NMGF equation (6.35):

$$S_{G,fl} = -2 \int_0^\infty dt \int_x \text{tr} \{ L_\mu (\partial_t B_\mu - D_\nu G_{\nu\mu} + g_0 j_\mu^a T^a - \alpha_0 D_\mu \partial_\nu B_\nu) \} \quad (6.57)$$

The BRST variations from (6.24) also change slightly: we now have:

$$\delta \bar{d} = D_\mu L_\mu - g_0 \{d, \bar{d}\} + g_0 \bar{\lambda} T^a \chi T^a - g_0 \bar{\chi} T^a \lambda T^a \quad (6.58)$$

which results in a change in E_μ from (6.25):

$$E_\mu = \partial_t B_\mu - D_\nu G_{\nu\mu} + g_0 j_\mu^a T^a - \alpha_0 D_\mu \partial_\nu B_\nu \quad (6.59)$$

while δE_μ from (6.26) remains the same. Using these BRST variations we establish that the complete bulk action is again invariant:

$$\delta S_{bulk} = 0 \quad (6.60)$$

Also, the path integral measure has not been changed and is still BRST invariant, therefore $\langle \delta O \rangle = 0$ still holds, where O is any combination of evolved and nonevolved elementary fields. Therefore, also (6.28) still holds.

The argument for the absence of bulk counterterms translates one-to-one from the YMGF to the NMGF case, with one addition. There now is an additional interaction term proportional to $\bar{\chi} L_\mu \chi$, which upon the substitution of (6.29) yields the bulk interaction $\bar{\Psi} L_\mu \Psi$. However, recalling that the only nonzero propagators are $\langle bL \rangle$, $\langle \Psi \bar{\lambda} \rangle$, $\langle \lambda \bar{\Psi} \rangle$ and $\langle d\bar{d} \rangle$, in order to form a loop in the bulk the presence of the interaction term $\bar{\lambda} B_\mu \lambda$ is essential. It is clear from the fermionic bulk action (6.20) that this term is absent, and thus no loops can be formed. Therefore, all bulk correlators still solely consist of trees and no divergences can arise, i.e. there will be no bulk counterterms.

The allowed boundary counterterms remain to be of the form given in (6.30). Also, since we still have $\langle \delta O \rangle = 0$, and thus in particular (6.28) still holds, we have $z_1 = z_2 = 0$, i.e. there is again no field renormalization factor for L_μ , and consequently also no field renormalization factor for B_μ .

Now, returning to the issue raised at the beginning of this section: might it be that the χ and $\bar{\chi}$ field in j_μ^a in the solution (6.36) of the NMGF equation (6.35) have to be renormalized in order to render correlators of B_μ finite? It is now clear that the answer is no. A Z_χ^{-1} factor in the flow equation (6.35) would imply a Z_χ^{-1} factor multiplying j_μ^a in the gauge bulk action (6.57). This in turn would imply a bulk counterterm, unless $L_\mu = Z_\chi L_{R,\mu}$ i.e. that L_μ renormalizes.

Figure 6.3: New next-to-leading order diagrams for the NMGF evolved fermion 2-point function. Combined with the diagrams from figure 4.3 they represent the full NMGF evolved fermion 2-point function, which is rendered finite by multiplying with the renormalization factor Z_χ from (6.61).

diagram	value
NM1 + NM2	$\frac{1}{\varepsilon} + \mathcal{O}(\varepsilon^0)$

Table 6.1: UV divergence of the diagrams from figure 6.3 in units of $C_2(R) \frac{g_0^2}{(4\pi)^2} S(\bar{x}_t - \bar{y}_t)$.

The bulk counterterm is excluded due to the absence of loops in the bulk, and the renormalization of L_μ is excluded due to BRST symmetry.

The discussion presented above does not make any statement about the value of Z_χ , and there is no reason why it should remain the same as when using the YMGF, (6.17). The new vertex at $\mathcal{O}(g_0)$ due to the NMGF — see (6.41) and corresponding figure 6.1.c — gives rise to a new contribution to Z_χ . In figure 6.3 we show the diagrams that contribute to the evolved fermion 2-point function, and their values are presented in table 6.1 in units of $C_2(R) \frac{g_0^2}{(4\pi)^2} S(\bar{x}_t - \bar{y}_t)$, where $S(\bar{x}_t - \bar{y}_t)$ is the evolved fermion propagator from (4.39). Combining these new contributions with the self-energy diagrams presented in section 4.4.2 we find:

$$\text{NMGF :} \quad Z_\chi(g(\mu), \varepsilon) = 1 + C_2(R) \frac{2}{\varepsilon} \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4) \quad (6.61)$$

6.4 Using $\overleftarrow{D}^2$ in the fermion flow equations

Next we present the fermion flow equation using $\overleftarrow{D}^2$ instead of D^2 , and calculate its contribution to $\langle E \rangle$. The well-known relation between these operators is:

$$\overleftarrow{D}^2 = D^2 + \frac{g_0}{2} \sigma_{\mu\nu} G_{\mu\nu} \quad (6.62)$$

with $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. The modified fermion flow equations from (4.17) now read:

$$\begin{aligned} \partial_t \chi(t, x) &= \{\overleftarrow{D} - \alpha_0 g_0 \partial_\mu B_\mu(t, x)\} \chi(t, x), & \chi(0, x) &= \psi(x) \\ \partial_t \bar{\chi}(t, x) &= \bar{\chi}(t, x) \{\overleftarrow{D} + \alpha_0 g_0 \partial_\mu B_\mu(t, x)\}, & \bar{\chi}(0, x) &= \bar{\psi}(x) \end{aligned} \quad (6.63)$$

with $\overleftarrow{D} = \overleftarrow{D}^2$, $\overleftarrow{D} = \overleftarrow{D}^2$. The solutions to these flow equations are:

$$\begin{aligned} \chi(t, x) &= e^{t\Delta_x} \psi(x) + \int_0^t ds e^{(t-s)\Delta_x} \{\overleftarrow{D}' \chi(s, x)\} \\ \bar{\chi}(t, x) &= e^{t\Delta_x} \bar{\psi}(x) + \int_0^t ds e^{(t-s)\Delta_x} \{\bar{\chi}(s, x) \overleftarrow{D}'\} \end{aligned} \quad (6.64)$$

with:

$$\overleftarrow{D}' = \Delta' + \frac{g_0}{2} \sigma_{\mu\nu} G_{\mu\nu}, \quad \overleftarrow{D}' = \overleftarrow{\Delta}' + \frac{g_0}{2} \sigma_{\mu\nu} G_{\mu\nu} \quad (6.65)$$

where Δ' and $\overleftarrow{\Delta}'$ are given in (4.24). The combination of these fermion flow equations and the nonminimal gauge field flow from section 6.3 will be referred to as the ‘slashed’ nonminimal gradient flow (sNMGF).

It is convenient to define the extra contribution in $\chi(t, x)$ stemming from the $\sigma_{\mu\nu}$ term in (6.62):

$$\begin{aligned} \chi_{\overleftarrow{D}}(t, x) &= \chi_{\Delta}(t, x) + \chi^*(t, x) \\ \bar{\chi}_{\overleftarrow{D}}(t, x) &= \bar{\chi}_{\Delta}(t, x) + \bar{\chi}^*(t, x) \end{aligned} \quad (6.66)$$

where $\chi_{\overleftarrow{D}}$ is representing the solution from (6.64), and χ_{Δ} the solution from (4.23). The extra contributions $\chi^*, \bar{\chi}^*$ are given by:

$$\begin{aligned} \chi^*(t, x) &= \int_0^t ds e^{(t-s)\Delta_x} \{\Delta' \chi^*(s, x) + \frac{g_0}{2} \sigma_{\mu\nu} G_{\mu\nu}(s, x) \chi_{\overleftarrow{D}}(s, x)\} \\ \bar{\chi}^*(t, x) &= \int_0^t ds e^{(t-s)\Delta_x} \{\bar{\chi}^*(s, x) \overleftarrow{\Delta}' + \frac{g_0}{2} \bar{\chi}_{\overleftarrow{D}}(s, x) \sigma_{\mu\nu} G_{\mu\nu}(s, x)\} \end{aligned} \quad (6.67)$$

Figure 6.4: Diagrammatic representation of the new terms due to the sNMGF in the expansion of the fields in (6.68). The new interaction vertices due to the sNMGF are represented by white blocks instead of blobs. (a) and (b) are $\mathcal{O}(g_0)$ and represent χ_1^* and $\bar{\chi}_1^*$ from (6.69), while (c) and (d) are $\mathcal{O}(g_0^2)$ and collectively represent $\mathcal{F}_{\mu,2}^*$ from (6.70).

Writing these contributions as a series in powers of g_0 :

$$\chi^*(t, x) = \sum_{n=0}^{\infty} g_0^n \chi_n^*(t, x), \quad \bar{\chi}^*(t, x) = \sum_{n=0}^{\infty} g_0^n \bar{\chi}_n^*(t, x) \quad (6.68)$$

we have for the first two orders:

$$\begin{aligned} \chi_0^*(t, x) &= \bar{\chi}_0^*(t, x) = 0 \\ \chi_1^*(t, x) &= \frac{1}{2} \sigma_{\mu\nu} \int_0^t ds e^{(t-s)\Delta_x} \{(\partial_\mu B_{\nu,0}(s, x) - \partial_\nu B_{\mu,0}(s, x)) \chi_0(s, x)\} \\ \bar{\chi}_1^*(t, x) &= \frac{1}{2} \int_0^t ds e^{(t-s)\Delta_x} \{\bar{\chi}_0(s, x) (\partial_\mu B_{\nu,0}(s, x) - \partial_\nu B_{\mu,0}(s, x))\} \sigma_{\mu\nu} \end{aligned} \quad (6.69)$$

The first — and at this order in g_0 only — nonzero new contribution to B_μ from (6.36) is then given by:

$$\mathcal{F}_{\mu,2}^{*a}(t, x) = - \int_0^t ds e^{(t-s)\Delta_x} j_{\mu,1}^{*a}(s, x) \quad (6.70)$$

where:

$$j_{\mu,1}^{*a}(t, x) = \bar{\chi}_1^*(t, x) \gamma_\mu T^a \chi_0(t, x) + \bar{\chi}_0(t, x) \gamma_\mu T^a \chi_1^*(t, x) \quad (6.71)$$

Figure 6.5: Diagrammatic representation of the contribution to $\langle E \rangle$ due to the sNMGF, whose total value is given in (6.75). (a) and (b) collectively represent $\mathcal{E}_\chi^*$ from (6.73).

These new contributions are diagrammatically represented in figure 6.4. Note that the new interaction vertices due to the sNMGF are represented by white blocks instead of blobs.

6.4.1 Perturbative calculation of $\langle E \rangle$

The calculation of $\langle E \rangle$ now consists of calculating one additional contribution with respect to the calculation presented in section 6.3.2. Comparing to (6.46), we now have:

$$\langle E \rangle = \langle E \rangle \big|_{B_\mu = B_\mu^{YM}} + \mathcal{E}_\chi + \mathcal{E}_\chi^* \quad (6.72)$$

with $\mathcal{E}_\chi$ as presented in section 6.3.2, and:

$$\mathcal{E}_\chi^* = g_0^4 \langle \partial_\mu \mathcal{F}_{\nu,2}^a (\partial_\mu B_{\nu,0}^a - \partial_\nu B_{\mu,0}^a) \rangle \quad (6.73)$$

which is diagrammatically represented in figure 6.5. Employing the expressions presented in the previous section, we evaluate this term as:

$$\begin{aligned} \mathcal{E}_\chi^* &= -8(d-1)g_0^4 N_f \text{tr}(T^a T^a) \\ &\quad \times \int_0^t ds \int_0^s du \int_{p,q} e^{-2sp^2 - 2tq^2 - 2(s-u)p \cdot q} \frac{p \cdot q}{p^2} \\ &\equiv -4(d-1)g_0^4 N_f \text{tr}(T^a T^a) \{I_{10} - I_{11}\} \end{aligned} \quad (6.74)$$

The integrals I_{10}, I_{11} and their ε -expansions are listed in appendix B.3. The new contribution is then given by:

$$\mathcal{E}_\chi^* = -\frac{1}{2} \frac{(N^2 - 1)}{(8\pi t)^{d/2}} (d-1) N_f \frac{g_0^4}{(4\pi)^2} (4 \log 2 - 2 \log 3 + \mathcal{O}(\varepsilon)) \quad (6.75)$$

and, similarly to $\mathcal{E}_\chi$ from (6.53), this is manifestly finite. Again, this is to be expected, anticipating the arguments provided in section 6.4.2.

We obtain for the total fermionic contribution:

$$\mathcal{E}_\chi + \mathcal{E}_\chi^* = -\frac{1}{2} \frac{(N^2 - 1)}{(8\pi t)^{d/2}} (d-1) N_f \frac{g_0^4}{(4\pi)^2} \left(2 + \frac{16}{3} \log 2 - 3 \log 3 + \mathcal{O}(\varepsilon) \right) \quad (6.76)$$

We therefore now have for the full observable from (6.72):

$$\langle E \rangle = \frac{1}{2} g_0^2 \frac{(N^2 - 1)}{(8\pi t)^{d/2}} (d-1) \{ 1 + \tilde{c}_1 g_0^2 + \mathcal{O}(g_0^4) \} \quad (6.77)$$

$$\begin{aligned} \tilde{c}_1 = \frac{1}{(4\pi)^2} (4\pi)^\varepsilon (8t)^\varepsilon \left\{ N \left(\frac{11}{3\varepsilon} + \frac{52}{9} - 3 \log 3 \right) \right. \\ \left. - N_f \left(\frac{2}{3\varepsilon} + \frac{22}{9} + 4 \log 2 - 3 \log 3 \right) + \mathcal{O}(\varepsilon) \right\} \end{aligned} \quad (6.78)$$

Repeating the procedure from sections 6.2.1 and 6.3.2, i.e. renormalizing the coupling as in (6.13), expressing the renormalized coupling $\alpha(\mu)$ in terms of the RG invariant coupling $\alpha(q)$ by inverting (D.6) and setting $q = (8t)^{-1/2}$ and $N = 3$, we now obtain:

$$\langle E \rangle = \frac{3}{4\pi t^2} \alpha(q) \left\{ 1 + \tilde{k}_1 \alpha(q) + \mathcal{O}(\alpha^2) \right\}, \quad \tilde{k}_1 = 1.0978 - 0.1835 \times N_f \quad (6.79)$$

Again, for the renormalized result for general N we refer the reader to appendix D. We will compare the sNMGF result (6.79) with the NMGF and YMGF results (6.56) and (6.15), respectively, in section 6.5.

6.4.2 Renormalization of evolved fields

The arguments presented in section 6.3.3 are fully applicable to the case presented here. The added gamma matrix structure in (6.63) does not change any of the BRST transformation properties presented in sections 6.2.2 and 6.3.3, and we still have $\delta S_{bulk} = 0$. The appearance of the new terms in the fermionic bulk action (6.20) — now with $\bar{\Delta}$ instead of Δ — proportional to $\bar{\lambda} \sigma_{\mu\nu} G_{\mu\nu} \chi$ and $\bar{\chi} \sigma_{\mu\nu} G_{\mu\nu} \lambda$ also does not change any of the reasoning presented in sections 6.2.2 and 6.3.3. Thus, all conclusions of those sections carry over.

With regard to the renormalization factor Z_χ , there is no reason for it not to change, and indeed it does. The new $\mathcal{O}(g_0)$ sNMGF vertex represented by

Figure 6.6: New next-to-leading order diagrams for the sNMGF evolved fermion 2-point function. Combined with the diagrams from figure 6.3 and figure 4.3 they represent the full sNMGF evolved fermion 2-point function, which is rendered finite by multiplying with the renormalization factor Z_χ from (6.80).

the white block in figure 6.4.a and figure 6.4.b, combined with the QCD vertex and the $\mathcal{O}(g_0)$ vertices from figure 6.1, yields 10 new contributions to the evolved fermion 2-point function. These are represented in figure 6.6, and their values are given in table 6.2. The full sNMGF evolved fermion 2-point function is given by the combined contributions from figures 4.3, 6.3 and 6.6, and it is rendered finite by multiplying with the sNMGF evolved fermion renormalization factor:

$$\text{sNMGF :} \quad Z_\chi(g(\mu), \varepsilon) = 1 - C_2(R) \frac{4}{\varepsilon} \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4) \quad (6.80)$$

6.5 N_f dependence of $\langle E \rangle$ for the different flow equations

Next we compare the N_f dependence of the different expressions for $\langle E \rangle$ with $N = 3$, namely (6.15), (6.56) and (6.79), which are respectively the Yang-Mills (YM), the nonminimal (NM), and the ‘slashed’ nonminimal (sNM) gradient flow cases. For general N see appendix D. Collecting the results, again setting

diagram	value
sNM1 + sNM2	$\frac{3}{\varepsilon} + \mathcal{O}(\varepsilon^0)$
sNM3 + sNM4	$\mathcal{O}(\varepsilon^0)$
sNM5 + sNM6	$\mathcal{O}(\varepsilon^0)$
sNM7 + sNM8	$\frac{3}{2\varepsilon} + \mathcal{O}(\varepsilon^0)$
sNM9 + sNM10	$\frac{3}{2\varepsilon} + \mathcal{O}(\varepsilon^0)$

Table 6.2: UV divergences of the diagrams from figure 6.6 in units of $C_2(R) \frac{g_0^2}{(4\pi)^2} S(\bar{x}_t - \bar{y}_t)$.

$q = (8t)^{-1/2}$, we have:

$$\langle E \rangle_i = \frac{3}{4\pi t^2} \alpha(q) \{1 + k_i \alpha(q) + \mathcal{O}(\alpha^2)\} \quad (6.81)$$

with $i = \{YM, NM, sNM\}$, $\alpha(q)$ the renormalization group invariant $\overline{\text{MS}}$ coupling, and:

$$k_{YM} = 1.0978 + 0.0075 \times N_f \quad (6.82a)$$

$$k_{NM} = 1.0978 - 0.1377 \times N_f \quad (6.82b)$$

$$k_{sNM} = 1.0978 - 0.1835 \times N_f \quad (6.82c)$$

It is convenient to express $\alpha(q)$ in terms of the fundamental scale Λ_{QCD} ⁸ using the universal 2-loop renormalization-group improved UV asymptotic expression [8], see (D.7). We expand the coupling in powers of $1/l$, with $l = -\log(8t\Lambda_{\text{QCD}}^2)$, and we obtain for (6.81):

$$\langle E \rangle_i = \frac{3}{t^2 \beta_0 l} \left\{ 1 + \frac{1}{\beta_0 l} \left(4\pi k_i - \frac{\beta_1}{\beta_0} \log(l) \right) \right\} + \mathcal{O}(l^{-3}) \quad (6.83)$$

⁸In general the renormalization-group invariant scale Λ_{QCD} depends on N , N_f , the value of the coupling at some given scale, and the renormalization scheme used. We will solely use it as a measure for the flow time t and will not worry about its value or how it changes with N_f ; when comparing theories with different N_f we will keep the dimensionless product $t\Lambda_{\text{QCD}}^2$ fixed.

Figure 6.7: Behavior of the different $t^2 \langle E \rangle_i$'s for $N_f = 4, 8$, as a function of t in units of $\Lambda_{\text{QCD}}^{-2}$, see (6.83). From top to bottom: the YM (solid purple), the NM (dashed red), and the sNM (blue dot-dashed) gradient flow cases. The horizontal gray dotted line is there to better visualize the relative changes.

The values for $t^2 \langle E(t) \rangle_i$ are plotted for $N_f = 4, 8$ in figure 6.7.

In [27] it was found, using lattice simulations, that for $N = 3$, $N_f = 0$, beyond the perturbative regime, $t^2 \langle E(t) \rangle$ grows roughly linearly with t . There, as a possible explanation for this behavior the author alluded to the fact that the gradient flow drives the gauge fields towards stationary points of the YM action. For these configurations the right-hand side of the flow equation (4.1) is small, implying that E will change relatively little with t .

When considering cases with $N_f \neq 0$, this same reasoning should result in a more pronounced slowing down of $t^2 \langle E(t) \rangle$ in the (s)NM cases compared to the YMGF, since the latter now no longer drives the gauge field towards the stationary points of the full action of the theory.

The graphs in figure 6.7 seem to confirm this reasoning. Moreover, as N_f is increased, the difference between the use of the different flow equations becomes more clear. It would be interesting to see this behavior being reproduced in lattice simulations when the different flows are implemented.

6.5.1 Sensitivity to change in N_f

We can study the sensitivity to changes in N_f of the different $\langle E \rangle_i$'s from (6.81), (6.82) by considering the relative change:

$$\frac{\Delta \langle E \rangle_{N_f}}{\langle E \rangle_{N_f}} = \frac{\langle E \rangle_{N_f+1} - \langle E \rangle_{N_f}}{\langle E \rangle_{N_f}} \quad (6.84)$$

Figure 6.8: Relative change for the different $\langle E \rangle_i$'s from (6.81), (6.82) at the scale $t = 0.008/\Lambda_{\text{QCD}}^2$ when N_f is increased by 1, see (6.84). Differences decrease for smaller values of $t\Lambda_{\text{QCD}}^2$, but the qualitative structure remains the same.

The results are plotted in figure 6.8. It shows that the sNMGF provides a much more stable value for $\langle E \rangle$ under changes of N_f than the YMGF. For example, for $N_f = 4 \rightarrow 5$, at $t = 0.008/\Lambda_{\text{QCD}}^2$, the sNMGF result changes by 2%, while for the YMGF we have a change of around 13%.

We would again like to stress that although Λ_{QCD} will also change with N_f , the dimensionless product $t\Lambda_{\text{QCD}}^2$ of the flow time, or ‘smearing radius’ $\sqrt{t}$, and the fundamental physical scale Λ_{QCD} is kept fixed.

6.6 Generalized flows

The extensions of the Yang-Mills gradient flow presented in sections 6.3 and 6.4 are motivated by the similarities between the Langevin equations used in stochastic quantization and the gradient flow, and by the observation that the YMGF does not drive the gauge field towards the stationary points of the full action of the QCD-like theory when fermionic matter is present.

However, we also noted that in principle any gradient-flow equation respecting all the symmetries of the nonevolved theory is equally valid from a purely mathematical point of view. Therefore, we also present the most general flow equations:

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) - a_1 g_0 j_\mu^a(t, x) T^a + \alpha_0 D_\mu \partial_\nu B_\nu(t, x) \quad (6.85a)$$

$$\partial_t \chi(t, x) = \left\{ \Delta + a_2 \frac{g_0}{2} \sigma_{\mu\nu} G_{\mu\nu}(t, x) - \alpha_0 g_0 \partial_\mu B_\mu(t, x) \right\} \chi(t, x) \quad (6.85b)$$

$$\partial_t \bar{\chi}(t, x) = \bar{\chi}(t, x) \left\{ \overleftarrow{\Delta} + a_2 \frac{g_0}{2} \sigma_{\mu\nu} G_{\mu\nu}(t, x) + \alpha_0 g_0 \partial_\mu B_\mu(t, x) \right\} \quad (6.85c)$$

with boundary conditions $B_\mu(0, x) = A_\mu(x)$, $\chi(0, x) = \psi(x)$ and $\bar{\chi}(0, x) = \bar{\psi}(x)$, where we include the modification terms proportional to α_0 , and a_1, a_2 are real constants parametrizing the amount of ‘nonminimality’. The YMGF is the minimal case: $(a_1, a_2) = (0, 0)$, the NMGF presented in section 6.3 is obtained setting $(a_1, a_2) = (1, 0)$, and the sNMGF from section 6.4 corresponds to $(a_1, a_2) = (1, 1)$. It should be clear from the discussions in sections 6.2.2, 6.3.3 and 6.4.2 that the qualitative renormalization properties remain the same in this generalized case.

The solutions to the generalized flow equations from (6.85) are straightforward generalizations of those given in sections 6.3 and 6.4, and read (setting $\alpha_0 = 1$):

$$B_\mu(t, x) = e^{t\Delta_x} A_\mu(x) + \int_0^t ds e^{(t-s)\Delta_x} \{R_\mu(s, x) - a_1 g_0 j_\mu^a(s, x) T^a\} \quad (6.86a)$$

$$\chi(t, x) = e^{t\Delta_x} \psi(x) + \int_0^t ds e^{(t-s)\Delta_x} \left\{ \left(\Delta' + a_2 \frac{g_0}{2} \sigma_{\mu\nu} G_{\mu\nu} \right) \chi(s, x) \right\} \quad (6.86b)$$

$$\bar{\chi}(t, x) = e^{t\Delta_x} \bar{\psi}(x) + \int_0^t ds e^{(t-s)\Delta_x} \left\{ \bar{\chi}(s, x) \left(\overleftarrow{\Delta}' + a_2 \frac{g_0}{2} \sigma_{\mu\nu} G_{\mu\nu} \right) \right\} \quad (6.86c)$$

Using these solutions we calculate the expectation value of the observable $E(t, x)$ from (6.2). For general N we obtain:

$$\langle E(a_1, a_2; t) \rangle = \frac{3(N^2 - 1)}{32\pi t^2} \alpha(\mu) \{1 + k(a_1, a_2; \mu^2 t) \alpha(\mu) + \mathcal{O}(\alpha^2)\} \quad (6.87)$$

where:

$$k(a_1, a_2; \mu^2 t) = k_{YM}(\mu^2 t) - \frac{N_f a_1}{12\pi} \left\{ 4 + 40 \log 2 - 24 \log 3 \right. \\ \left. + a_1 (2 - 36 \log 2 + 21 \log 3) + 6a_2 (2 \log 2 - \log 3) \right\} \quad (6.88)$$

and $k_{YM}(\mu^2 t)$ is given in (D.2). The k_i ’s from (6.82) are special cases of $k(a_1, a_2)$: $k_{YM} = k(0, 0) = k(0, 1)$, $k_{NM} = k(1, 0)$ and $k_{sNM} = k(1, 1)$.

The evolved fermion renormalization factor is also calculated using the generalized flow equations, and reads:

$$Z_\chi(a_1, a_2; g(\mu), \varepsilon) = 1 + \frac{3 - a_1(1 + 6a_2)}{\varepsilon} C_2(R) \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4) \quad (6.89)$$

Thus, the anomalous dimension associated with the evolved fermion fields is given by:

$$\gamma_\chi(a_1, a_2; g) \equiv -\frac{1}{2} \frac{d \log Z_\chi}{d \log \mu} = (3 - a_1 (1 + 6a_2)) C_2(R) \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4) \quad (6.90)$$

Note that when $a_1(1 + 6a_2) = 3$ the first coefficient of the anomalous dimension vanishes. This implies that there exists a renormalization scheme such that γ_χ vanishes to all orders in the gauge coupling, as we explained in section 3.5.

The conclusions of this chapter are presented in section 7.2.

Chapter 7

Conclusion

In this thesis we focused on the Yang-Mills gradient flow and its generalizations, applied to QCD-like theories in Euclidean spacetime with massless fermions. Before discussing the specific results obtained, we recall our motivation for studying these flow equations.

The most notable feature of gradient-flow evolved quantities is the way in which they renormalize, as reviewed in section 4.4; evolved gauge fields do not require any renormalization, though evolved fermions do still require a renormalization factor which in general differs from its nonevolved-fermions' counterpart. Evolved composite operators straightforwardly renormalize with the renormalization factors of their constituent fermion fields, in contrast to the nonevolved case. These relatively simple renormalization properties stem from the smearing effect of the gradient flow, which effectively smooths short-distance quantum fluctuations.

After applying the gradient flow, the expectation values of evolved gauge-invariant operators are probing the theory at length-scales on the order of the flow-time $\sqrt{t}$. The simplest among the purely-gluonic operators is the Yang-Mills Lagrangian density, which has been used extensively on the lattice for scale setting, and for devising finite-volume running coupling schemes — though many open questions and problems remain there, to which we return in section 7.3. Flowed fermionic operators, though acquiring a renormalization factor, have also found their use in lattice studies, e.g. in studying the chiral condensate, and in determining the pseudo-scalar decay constant in the chiral limit [31].

While already finding many applications on the lattice, we observed that some subtle questions were not yet answered to a satisfactory degree in our view. Especially when it came to the question of universality of the evolved fermion renormalization factor there remained some doubt. For instance, in [28] it was argued that evolved fermions would renormalize the same as gauge fields, i.e. not renormalize at all. This turned out to be false, and arguments were provided why this is the case in [29,31]. However, we found these arguments somewhat elementary when concerning composite operators, and moreover we wanted to see an evolved 2-point correlation function example, providing supporting evidence for the expositions in [28,29,31]. These considerations, together with the question of what would happen to conserved currents when evolved, motivated the research presented in [94], which was discussed in chapter 5 of this thesis.

7.1 Conserved vector current and the Yang-Mills gradient flow

In this section we review the results obtained in [94] as presented in chapter 5. We studied the renormalization properties of the gradient-flow evolved flavor-singlet 2-point vector correlator in QCD-like theories with massless fermions in Euclidean spacetime. We mainly employed coordinate-space techniques instead of the more conventional momentum-space methods, thereby providing further conceptual insight into the perturbative approach to the gradient flow at the computational level. We showed that, in contrast to the nonevolved case, the 2-point correlator is renormalized and the evolved vector current acquires an anomalous dimension (see section 5.4):

$$\gamma_J = 6C_2(R) \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4(\mu)) \quad (7.1)$$

Our result confirms that this anomalous dimension is induced by the renormalization of the evolved elementary fermion field first derived in [31], and thus applies to all evolved fermion-bilinear currents. Our result is also in agreement with results in the literature for 1-point correlators of evolved fermion bilinears [31,55,58].

The Callan-Symanzik equation for the connected 2-point correlators of generic multiplicatively renormalizable evolved fermion-bilinear currents now involves two scales: the flow-time $\sqrt{t}$ and the spacetime separation $|x - y|$. In

section 5.4.4, we made connection with the nonevolved case by deriving the solution of the Callan-Symanzik equation in the limit of small flow-time $\sqrt{t}$, at fixed separation $|x - y|$. Incidentally, the leading-order contribution to this expansion also determines the leading small flow-time expansion coefficient for the corresponding evolved current.

We also discussed how, interestingly, conservation of the evolved vector current and transversality of the corresponding 2-point correlator no longer imply nonrenormalization, at difference with the nonevolved case, see section 5.5. In particular, the leading contribution to the small flow-time expansion of the evolved 2-point vector correlator — which is $\mathcal{O}(t^0)$ up to powers of logarithms — fulfils transversality and the current is conserved despite the presence of the anomalous dimension γ_J , as is shown in (5.71). This is due to the presence of the additional gradient-flow scale $\sqrt{t}$, so that, in the aforementioned terms, the renormalization logarithms that do not arise from the coupling renormalization only depend on the product $\sqrt{t}\mu$. Violation of transversality and nonconservation do occur, instead, at $\mathcal{O}(t)$ in the small flow time expansion as a soft breaking effect induced by the nonlocality of the evolved current.

After working extensively with evolved fermions during the research presented in chapter 5, the author of this thesis became interested in what happens when a fermionic term is included in the gradient flow equation for the gauge field. Work in this direction culminated in [95], which was presented in chapter 6.

7.2 Nonminimal gradient flows

Here we review and discuss the results obtained in [95] as presented in chapter 6. We investigated some of the properties and consequences of nonminimal gradient flows in non-Abelian gauge theories containing fermions, i.e. flow equations which are defined using the full QCD-like action, instead of only the pure Yang-Mills action. The main new feature is that the flow equation for the gauge field acquires a purely-fermionic term proportional to the gauge-covariant vector current, see (6.32). In addition to the conventional fermion flow equations based on the operator D^2 , we also studied those based on $\not{D}^2$ in section 6.4.

We found that the same qualitative renormalization properties hold for the nonminimally-evolved fields as in the Yang-Mills gradient flow case. That is, any correlator — possibly containing evolved composite operators — consisting of n (s)NMGF evolved fermion fields χ and n (s)NMGF evolved fields $\bar{\chi}$ and any number of (s)NMGF evolved gauge fields B_μ accompanied with a factor of

the bare gauge coupling g_0 , is rendered finite by multiplicative renormalization with Z_χ^n , supplemented with the usual gauge coupling renormalization, where the different Z_χ factors for the different flows read:

$$Z_\chi(g(\mu), \varepsilon) = 1 + \left\{ \begin{array}{ll} \text{YM:} & +3 \\ \text{NM:} & +2 \\ \text{sNM:} & -4 \end{array} \right\} \times \frac{1}{\varepsilon} C_2(R) \frac{g^2(\mu)}{(4\pi)^2} + \mathcal{O}(g^4(\mu)) \quad (7.2)$$

Furthermore, we extended the nonminimal flows to the most general form compliant with the symmetries of the nonevolved theory, resulting in (6.85). We showed that the most general Z_χ in (6.89) depends on two constants a_1, a_2 , which are fully independent of any other parameters of the theory such as N and N_f , and that there exists a particular combination of these constants for which the $\mathcal{O}(g^2)$ term vanishes. This implies that there exists a one-parameter (a) family of flow systems for which there is a renormalization scheme in which the anomalous dimension in (6.90) of the evolved fermion field vanishes to all orders in the gauge coupling. This parameter a , for which $(a_1, a_2) = (a, (3-a)/2a)$, can take on any nonzero real value. It is unlike a gauge-fixing parameter, since observables such as $\langle E(t) \rangle$ do depend on it, see (6.87) and (6.88).

Additionally, we calculated the expectation value of the Yang-Mills Lagrangian density multiplied by the square of the bare coupling, i.e. $\langle E(t) \rangle$ where $E(t, x) = \frac{1}{4} g_0^2 G_{\mu\nu}^a(t, x) G_{\mu\nu}^a(t, x)$, up to next-to-leading order in the coupling using the generalized nonminimal gradient flow, of which the YMGF, NMGF and sNMGF are special cases, see (6.87). We found that out of these three different cases $\langle E(t) \rangle$ calculated with the sNMGF has the smallest dependence on N_f , as presented in section 6.5. Also, for fixed $N_f \neq 0$, $t^2 \langle E(t) \rangle_{\text{sNM}}$ grows slower with t than $t^2 \langle E(t) \rangle_{\text{NM}}$, which in turn grows slower with t than the conventionally employed $t^2 \langle E(t) \rangle_{\text{YM}}$. This behavior might be attributed to the fact that the gauge fields are now driven towards the stationary points of the full action, and not those of the pure-gauge Yang-Mills action.

Possible lattice applications

It may be interesting to see if the above-mentioned properties of $\langle E(t) \rangle$ will persist beyond the perturbative regime. The nonminimal gradient flows might have useful applications in lattice studies involving fermions.

First of all, the gradient-flow derived scales t_0 and w_0 (see e.g. [33]) might have a smaller N_f dependence when determined using the nonminimal flows. A

much-used definition for t_0 is:

$$t_0^2 \langle E(t_0) \rangle = 0.3 \quad (7.3)$$

and figure 6.7 seems to imply that when using the YMGF at $N_f = 8$ this definition leads to a relatively small value for the dimensionless quantity $t_0 \Lambda_{\text{QCD}}^2$. The nonminimal flows will lead to a significantly larger value for $t_0 \Lambda_{\text{QCD}}^2$ when using the same definition (7.3) — more comparable to the pure Yang-Mills case [27] — possibly making them in practice more useful in lattice studies including fermions.

Secondly, the application of the generalized nonminimal gradient flow on the lattice will give the scales t_0, w_0 a continuous dependence on the parameters a_1, a_2 which are defining the flows in (6.85). This might provide useful additional tests of lattice implementations of the gradient flow, which even at the level of scale-setting might be affected by the same problems that trouble the gradient-flow running coupling studies.

Finally, it may be interesting to investigate the effect of the nonminimal flows on the gradient-flow evolved topological charge¹. In the chiral limit at finite volume these charges should be absent, yet a recent study [35] shows that on the lattice the gradient flow sometimes promotes gauge-field vacuum fluctuations to instanton-like objects, resulting in a nonzero net topological charge. The nonminimal flows might be helpful in investigating this effect and eventually minimizing it.

7.3 Concluding remarks

In its early stages, our research presented in this thesis was motivated by the — in our eyes — insufficiently-convincing evidence warranting some of the uses of the gradient flow. Specifically, there were doubts regarding the claimed universality of the evolved fermions' renormalization factor in fermion-bilinear composite operators, and the use of the gradient-flow running coupling on the lattice — reservations towards the latter remain.

Concerning the renormalization factor of the evolved fermions, we provided supporting evidence for its universal nature, and in the process performed the first calculation of a gradient-flow evolved gauge-invariant 2-point correlation function, using coordinate-space methods. Moreover, we cleared up the status

¹We thank the reviewer of [95] for pointing this out.

of the conservation — and associated (non)renormalization — of the evolved flavor-singlet vector current. Combined with the obtained results utilizing the nonminimal flows, both for the all-order renormalization properties of the condensates and correlation functions of nonminimally evolved operators, as for the fermionic field renormalization factor up to next-to-leading order in the coupling, we are led to conclude that the universality claim holds true.

Along the way, we generalized the framework of the Yang-Mills gradient flow, by constructing the most general flow equations respecting the symmetries of the theory, and while doing so, we discovered a one-parameter family of flow systems for which there exists a renormalization scheme which renders all evolved quantities finite, to all orders in perturbation theory. Though we did not find a direct use for this result yet, we expect there to be practical applications in the future; be it on the lattice, or in more theoretically-oriented lines of research.

With regard to the gradient-flow running coupling, we remain unable to fully convince ourselves of its viability, at least at the level of lattice implementations, where nonperturbative effects, lattice artifacts, finite volume, and topological effects can potentially severely impact the conclusions. The presence of conflicting claims from comparable lattice studies — especially at relatively large number of fermions [41, 42] — combined with the lacking possibility to directly compare this coupling to the perturbatively-determined one [43], causes us to have reservations. Since the aforementioned problems seem to increase as the number of fermions increases, it might be interesting here to experiment with implementing the nonminimal gradient flows, as they seem to be less sensitive to changes in the number of fermions. Still, the expectation that this behavior will persist in the nonperturbative regime remains speculative, and more preliminary studies in this direction are desirable.

Aside from the more obvious applications of the results and methods described in this thesis, there might be uses in different fields of research. For example, we could speculate that some of the perturbative coordinate-space methods might be useful in studies on the quantum effects of nonlocal theories of gravity, see e.g. [216] and references therein. There, differently from our setup, the nonlocality is inherently built into the definition of the theory, while in our case we solely evolve the fields in correlation functions — not the fields in the actual path integral used in the averaging. Nevertheless, the nonlocal signatures are strikingly similar, and some of the tools we developed might be useful in the above-mentioned context.

List of publications

[94] Marco Boers and Elisabetta Pallante. Conserved vector current in QCD-like theories and the gradient flow. *JHEP*, 10:034, 2020

[95] Marco Boers. Nonminimal gradient flows in QCD-like theories. *JHEP*, 01:204, 2021

Nederlandse samenvatting

Niet-Abelse ijktheorieën staan aan de basis van het Standaard Model van elementaire deeltjesinteracties — zowel de zwakke als de sterke kernkracht laten zich vatten in dit raamwerk. Dit proefschrift houdt zich bezig met bepaalde aspecten van gradiënt-stromingen in de context van niet-Abelse ijktheorieën. De Yang-Mills gradiënt-stroming vindt zijn oorsprong in de zuivere wiskunde, waar het werd gebruikt bij de studie naar vier-variëteiten. Later realiseerde men zich dat de afvlakkingseigenschappen van deze diffusie-achtige stromingsvergelijking kunnen worden gebruikt in kwantumveldentheorieën bij het omgaan met korte-afstandsfluctuaties. Wanneer we ons focussen op kwantumchromodynamica-achtige theorieën — generalisaties van de theorie van de sterke interactie tussen quarks en gluonen — evolueert de Yang-Mills gradiënt-stroming het ijkveld, terwijl fermionische velden kunnen worden geëvolueerd met hun eigen stromingsvergelijkingen. Wanneer de respectievelijke stromingen op deze velden worden toegepast, worden deze in de ruimtetijd over een gebied uitgesmeerd, waardoor hun renormalisatie-eigenschappen op korte afstand veranderen. Dit is vooral opvallend in het geval van lokale samengestelde operatoren, waar in het niet-geëvolueerde geval deze korte-afstandseffecten gecompliceerde ultraviolette renormalisaties vereisen, terwijl wanneer ze worden geëvolueerd door de gradiënt-stroming, ze eenvoudigweg renormaliseren met het product de veld-renormalisatiefactoren van hun samenstellende velden — wat in het geval van het geëvolueerde elementaire ijkveld triviaal is. Derhalve renormaliseren alle bilineaire operatoren van fermionen op dezelfde universele manier, en zijn samengestelde puur-gluonische operatoren ultraviolet eindig.

Deze renormalisatie-eigenschappen hebben geresulteerd in significante toepassingen van de Yang-Mills gradiënt-stroming in roosterstudies, waarin de niet-Abelse ijktheorieën worden gesimuleerd op een gediscretiseerde Euclidische ruimtetijd met eindig volume, wat een manier biedt om resultaten te verkrijgen buiten het gebruik van storingsrekening om. De Yang-Mills gradiënt-stroming

wordt hierbij gebruikt als één van de methoden voor schaalzetting, waarbij de roosterafstand wordt bepaald als een functie van de kale ijk koppeling. Tevens wordt het gebruikt bij het bepalen van de schaalafhankelijkheid van de gerenormaliseerde ijk koppeling — deze hangt af van de energieschaal waarop de theorie wordt onderzocht, en door dit op een bepaalde manier te relateren aan de schaal die is verbonden aan de hoeveelheid stroming, gecombineerd met de schaal van het eindige roostervolume, beweren studies dat ze met succes de schaalafhankelijkheid van de ijk koppeling kunnen bepalen in energiegebieden die gewoonlijk niet toegankelijk zijn met andere methoden. Er blijven echter problemen en open vragen. Het is bijvoorbeeld doorgaans niet mogelijk om de gradiënt-stroming-ijk koppeling direct te vergelijken met de storingsrekening-ijk koppeling, en waar dit wel mogelijk is, slaagt het er vaak niet in om de schaalafhankelijkheid te reproduceren die wordt geïmpliceerd door de universele twee-lus bètafunctie. Dit soort problemen, gecombineerd met spanningen tussen vergelijkbare resultaten van verschillende onderzoeksgroepen, waren een belangrijke motivatie voor het werk dat in dit proefschrift wordt gepresenteerd.

Om meer inzicht te krijgen de precieze werking van de gradiënt-stromingsmethoden, hebben we ons beperkt tot de toepassing ervan op niet-Abelse ijktheorieën in oneindig volume en continuümruimtetijd, en ons gefocust op het verhelderen van open vragen en problemen in dit scenario. Hier was reeds veel onderzoek naar gedaan, maar een aantal vragen bleef open. Een daarvan, die we in dit proefschrift behandelen, betreft het effect van de gradiënt-stromings-evolutie op het behoud en de nauw daarmee samenhangende niet-renormalisatie van geconserveerde smaakstromen. In het niet-geëvolueerde geval verhindert de exacte smaaksymmetrie dat de fermion-bilineaire geconserveerde stromen divergenties ontwikkelen, terwijl de universele aard van de geëvolueerde-fermionveld-renormalisatiefactor impliceert dat divergenties in het stroming-geëvolueerde geval wél voorkomen. We verduidelijken dit probleem door eraan te herinneren dat in het niet-geëvolueerde geval een ingewikkelde opheffing tussen verschillende divergerende bijdragen optreedt, terwijl de gradiënt-stroming alleen een specifiek soort korte-afstandsdivergentie afvlakt en eindig maakt — de met vacuümpolarisatie geassocieerde divergenties blijven bestaan — en de onderlinge opheffing vindt niet plaats. In de loop van het afleiden van dit resultaat hebben we bepaalde coördinaatruimtetekniken ontwikkeld, waardoor extra conceptueel inzicht wordt verkregen in de gradiënt-stroming op het storingsrekeningniveau, en welke wellicht toepassing kunnen vinden in andere onderzoeksgebieden, zoals kwantum-aspecten van specifieke niet-lokale zwaartekracht theorieën. Verder hebben we de eerste expliciete gradiënt-stroming-geëvolueerde tweepuntsrelatiefunctieberekening uitgevoerd, tot en met de tweede-

ordebenadering in de ijk koppeling.

Ten slotte hebben we de meest algemene generalisatie van de Yang-Mills gradiënt-stroming gepresenteerd, waarbij een fermionvelden-bevattende term is opgenomen in de stromingsvergelijking voor de ijkvelden. Gecombineerd met de generalisatie van de fermion-stromingsvergelijkingen, resulteert dit in een twee-parameter afhankelijkheid van de universele geëvolueerde-fermionveld-renormalisatiefactor, wat zich vertaalt in een twee-parameter afhankelijkheid van de universele fermion-bilineaire anomale dimensie. De renormalisatie van het geëvolueerde ijkveld is niet veranderd en blijft triviaal. We hebben een familie van stromingssystemen met één parameter gevonden waarvoor de leidende term in de storingsrekening van de geëvolueerde-fermion anomale dimensie verdwijnt, wat impliceert dat er een renormalisatieschema bestaat waarin deze anomale dimensie verdwijnt voor alle orden in de storingsrekening, met het gevolg dat alle stromings-geëvolueerde objecten ultraviolet eindig zijn. We verwachten toepassingen van deze niet-minimale stromingsvergelijkingen voor roosterstudies, waarbij de toegevoegde vrijheid van de twee parameters extra controles zou kunnen bieden. Tevens wordt, voor bepaalde natuurlijke keuzes van de parameters, de grootheid die wordt gebruikt bij het schaalzetten en het construeren van gradiënt-stromingsschaal-afhankelijke koppelingen minder gevoelig voor het *aantal* fermionische velden, en zou daarom wellicht toepassingen kunnen vinden in studies met veel smaken.

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Appendix A

Basics of QCD-like theories

A.1 Conventions

Throughout this thesis we work in natural units: $\hbar = c = 1$.

A.1.1 $SU(N)$

We are considering theories with special unitary gauge group $SU(N)$, which consists of all $N \times N$ unitary matrices $U(x)$ with determinant 1:

$$U^\dagger(x) = U^{-1}(x), \quad \det U(x) = 1 \quad (\text{A.1})$$

The group elements are represented by:

$$\Lambda(x) = e^{\omega^a(x)T^a} \quad (\text{A.2})$$

where T^a are the traceless anti-Hermitian generators spanning the Lie algebra $su(N)$ of the group $SU(N)$, and $\omega^a(x)$ are referred to as the transformation parameters, with $a = 1, \dots, d(G)$, and $d(G) = N^2 - 1$ is the dimension of the group.

The traceless anti-Hermitian generators T^a satisfy:

$$[T^a, T^b] = f^{abc}T^c \quad (\text{A.3})$$

with fully anti-symmetric real structure constants f^{abc} . Traces of the generators are normalized as:

$$\text{tr}_R(T^a T^b) = -T(R)\delta^{ab}, \quad \text{tr}_R(1) = d(R) \quad (\text{A.4})$$

where $T(R)$ and $d(R)$ are the index and dimension of the representation R , respectively. We usually omit the subscript R on the trace, since it will mostly be taken over the fundamental indices (unless stated otherwise):

$$\mathrm{tr}(T^a T^b) = \mathrm{tr}_F(T^a T^b) = -\frac{1}{2}\delta^{ab}, \quad \mathrm{tr}(1) = N \quad (\text{A.5})$$

with F denoting the fundamental representation. The quadratic Casimir of the representation R is related to the generators via:

$$T^a T^a = -C_2(R)1 \quad (\text{A.6})$$

and thus:

$$C_2(R) = -\frac{\mathrm{tr}_R(T^a T^a)}{d(R)} = \frac{T(R)d(G)}{d(R)} \quad (\text{A.7})$$

where $d(G) = d(\mathrm{adj}) = N^2 - 1$, $d(F) = d(\bar{F}) = N$, $T(F) = T(\bar{F}) = \frac{1}{2}$, and $T(\mathrm{adj}) = N$.

A.1.2 Euclidean spacetime

Throughout this thesis we are working in Euclidean spacetime (or simply *space*), and the metric is denoted by $\delta_{\mu\nu}$, which is simply the unit matrix with $\mu, \nu = 1, \dots, 4$ in 4 spacetime dimensions (we will comment on the d -dependence in appendix B). Gamma matrices have slightly different properties in Euclidean space. The defining anticommutation relation remains the same as in Minkowski spacetime, albeit now with $\delta_{\mu\nu}$ instead of $\eta_{\mu\nu}$ as metric:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \quad (\text{A.8})$$

The change with respect to Minkowski spacetime is that all gamma matrices are now Hermitian:

$$\gamma_\mu^\dagger = \gamma_\mu \quad (\text{A.9})$$

In the chiral basis we explicitly have:

$$\gamma_\mu = \begin{pmatrix} 0 & \rho_\mu \\ \rho_\mu^\dagger & 0 \end{pmatrix}, \quad (\text{A.10})$$

with $\mu = 1, 2, 3, 4$, where the vector ρ_μ is defined as $\rho_\mu = (i\vec{\sigma}, \mathbf{1})$, with σ_i the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.11})$$

The definition of γ_5 is:

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \frac{1}{4!} \varepsilon_{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad (\text{A.12})$$

with $\varepsilon_{\mu\nu\rho\sigma}$ the totally anti-symmetric tensor with $\varepsilon_{1234} = 1$. It obeys $\{\gamma_5, \gamma_\mu\} = 0$.

For the Euclidean d -dimensional space- and momentum integrals we use the abbreviations:

$$\int_x \equiv \int d^d x, \quad \int_p \equiv \int \frac{d^d p}{(2\pi)^d} \quad (\text{A.13})$$

A.2 $SU(N)$ and matter content

The group elements $\Lambda(x)$ from (A.2) act on the matter and gauge fields which are in representations of the group $SU(N)$, see [101, 217] for more details on representations. The gauge field $A_\mu^a(x)$ is in the adjoint representation, and the fermions can be in any nontrivial representation R . In QCD the (anti-)fermions are in the (anti-)fundamental representation of $SU(3)$.

Being more specific: we are considering $SU(N)$ gauge theories with $N_f(R)$ number of massless spin- $\frac{1}{2}$ Dirac fermions $\psi_i(x)$ in representation R of $SU(N)$, with $i = 1, \dots, d(R)$, where $d(R)$ is the dimension of the representation R . We will usually omit the index i and argument R , and simply refer to ‘ N_f flavors’ for brevity. The $\bar{\psi}(x)$ fields are in the dual of the representation carried by $\psi(x)$. We are not considering scalar fields.

A.2.1 Transformation properties of the fields

As mentioned above, the gauge field $A_\mu(x) = A_\mu^a(x)T^a$ is in the adjoint representation. Under a gauge transformation $\Lambda(x)$ it transforms as:

$$A'_\mu(x) = \Lambda A_\mu \Lambda^{-1} - \frac{1}{g_0} (\partial_\mu \Lambda) \Lambda^{-1} = \Lambda A_\mu \Lambda^{-1} + \frac{1}{g_0} \Lambda \partial_\mu \Lambda^{-1} \quad (\text{A.14})$$

with g_0 the bare gauge coupling¹. When in the (anti-)fundamental representation, the (anti-)fermions transform as:

$$\psi'(x) = \Lambda(x)\psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x)\Lambda^{-1}(x) \quad (\text{A.15})$$

The infinitesimal transformation laws follow by expanding (A.2) for $\omega^a(x) \ll 1$:

$$\delta A_\mu(x) = [\omega(x), A_\mu(x)] - \frac{1}{g_0} \partial_\mu \omega(x) \quad (\text{A.16})$$

and:

$$\delta\psi(x) = \omega(x)\psi(x), \quad \delta\bar{\psi}(x) = -\bar{\psi}(x)\omega(x) \quad (\text{A.17})$$

Covariant derivatives transform as:

$$D'_\mu = \Lambda D_\mu \Lambda^{-1} \quad (\text{A.18})$$

They act on fields in representation R as:

$$D_\mu = \partial_\mu + g_0 A_\mu^a T_R^a \quad (\text{A.19})$$

When acting on (anti-)fundamental fermions the covariant derivatives take on the form:

$$\begin{aligned} D_\mu \psi(x) &= (\partial_\mu + g_0 A_\mu(x)) \psi(x) \\ \bar{\psi}(x) \overleftarrow{D}_\mu &= \partial_\mu \bar{\psi}(x) - g_0 \bar{\psi}(x) A_\mu(x) \end{aligned} \quad (\text{A.20})$$

and when acting on fields in the adjoint representation, since $(T_{adj}^a)^{bc} = -f^{abc}$ which follows from the Jacobi identity, they take on the form:

$$D_\mu = \partial_\mu + g_0 [A_\mu, \cdot] \quad (\text{A.21})$$

The gauge field strength $F_{\mu\nu}(x) = F_{\mu\nu}^a(x) T^a$ is given by the commutator of the covariant derivatives. Explicitly it reads:

$$F_{\mu\nu}(x) = \frac{1}{g_0} [D_\mu, D_\nu] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + g_0 [A_\mu(x), A_\nu(x)] \quad (\text{A.22})$$

¹We are working with *canonical normalization* of the gauge field and the gauge coupling shows up in gauge transformations and covariant derivatives. There is an alternative called *Wilsonian normalization* where the gauge coupling is absent in gauge transformation laws and covariant derivatives, but the kinetic term has a normalization factor proportional to $1/g_0^2$. We will briefly discuss it at the end of section 2.1.2.

It is in the adjoint representation and under a gauge transformation it transforms according to:

$$F'_{\mu\nu}(x) = \Lambda(x)F_{\mu\nu}(x)\Lambda^{-1}(x) \quad (\text{A.23})$$

which for infinitesimal transformation parameter $\omega^a(x) \ll 1$ reads:

$$\delta F_{\mu\nu}(x) = [\omega(x), F_{\mu\nu}(x)] \quad (\text{A.24})$$

In section 2.1.2 present the most general Lagrangian consistent with the $SU(N)$ gauge symmetry.

A.3 Chiral basis for the fermions

When working with massless fermions it is often useful to explicitly work with the chiral basis of the Dirac matrices given in (A.10), which indicates that we can separate the Dirac spinor ψ into two Weyl spinors ψ_L and ψ_R [97]:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad (\text{A.25})$$

The subscripts L and R refer to the chirality (left, right) of the spinors. Using γ_5 from (A.12) we construct the projection operators:

$$P_L = \frac{1}{2}(1 - \gamma_5) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \frac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad (\text{A.26})$$

so that we have $P_L\psi = \psi_L$ and $P_R\psi = \psi_R$. Although the fact that in Euclidean spacetime $\bar{\psi}$ and ψ are not related via hermitian conjugation as in Minkowski spacetime (see e.g. [218] and references therein), we can nevertheless define $\bar{\psi}_L = \bar{\psi}P_R$ and $\bar{\psi}_R = \bar{\psi}P_L$. Note that:

$$P_L^2 = P_L, \quad P_R^2 = P_R, \quad P_LP_R = P_RP_L = 0 \quad (\text{A.27})$$

Consider the Lagrangian of N_f massless free fermions:

$$\mathcal{L} = \bar{\psi}_i \not{\partial} \psi_i, \quad i = 1, \dots, N_f \quad (\text{A.28})$$

If we now apply $\psi = (P_L + P_R)\psi$, $\bar{\psi} = \bar{\psi}(P_L + P_R)$, we find that the left and righthanded fields are decoupled:

$$\mathcal{L} = \bar{\psi}_{i,L} \not{\partial} \psi_{i,L} + \bar{\psi}_{i,R} \not{\partial} \psi_{i,R} \quad (\text{A.29})$$

Now, it is immediately apparent that the Lagrangian has two global $U(N_f)$ symmetry groups, labeled $U(N_f)_L$ and $U(N_f)_R$, with group actions:

$$\begin{aligned} U(N_f)_L : \quad \psi_i &\rightarrow (e^{P_L \omega_L^a T^a})_{ij} \psi_j, & \bar{\psi}_i &\rightarrow \bar{\psi}_j (e^{-P_R \omega_L^a T^a})_{ji} \\ U(N_f)_R : \quad \psi_i &\rightarrow (e^{P_R \omega_R^a T^a})_{ij} \psi_j, & \bar{\psi}_i &\rightarrow \bar{\psi}_j (e^{-P_L \omega_R^a T^a})_{ji} \end{aligned} \quad (\text{A.30})$$

where $\omega_{L,R}^a$ are the group parameters and T^a are the generators of the unitary group $U(N_f)$. The group action of the product group $U(N_f)_L \times U(N_f)_R$ is therefore (omitting the flavor indices):

$$U(N_f)_L \times U(N_f)_R : \quad \psi \rightarrow e^{(P_L \omega_L^a + P_R \omega_R^a) T^a} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-(P_R \omega_L^a + P_L \omega_R^a) T^a} \quad (\text{A.31})$$

Note that the Baker-Campbell-Hausdorff formula tells us in general:

$$e^X e^Y = e^Z, \quad Z = X + Y + \frac{1}{2}[X, Y] + \dots \quad (\text{A.32})$$

however since $P_L P_R = P_R P_L = 0$ all higher order contributions vanish.

If we now change the basis and associate the following parameter combinations to new parameters:

$$\omega_V^a = \frac{1}{2}(\omega_L^a + \omega_R^a), \quad \omega_A^a = \frac{1}{2}(\omega_R^a - \omega_L^a) \quad (\text{A.33})$$

and explicitly plug in the expressions for the projection operators from (A.26), then we find for the transformations of ψ and $\bar{\psi}$:

$$\psi \rightarrow e^{(\omega_V^a + \gamma_5 \omega_A^a) T^a} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{(-\omega_V^a + \gamma_5 \omega_A^a) T^a} \quad (\text{A.34})$$

Using this basis, we label the flavor groups $U(N_f)_V \times U(N_f)_A$, with subscripts V and A referring to the corresponding conserved currents: the *vector* current in the case of $U(N_f)_V$ and the *axial* vector current for $U(N_f)_A$.

A.4 Noether's theorem

Noether's first theorem states that for a global continuous symmetry group G of dimension $d(G)$, for each generator T^a , with $a = 1, \dots, d(G)$, there exists an associated conserved current. We briefly review it here.

We collectively denote the fields $A_\mu(x)$, $\psi(x)$ and $\bar{\psi}(x)$ by $\phi_i(x)$. Under an infinitesimal transformation of the group G with parameters $\alpha^a(x) = \alpha^a$, $a = 1, \dots, d(G)$, the fields transform as:

$$\phi'_i(x) = \phi_i(x) + \alpha^a (\delta \phi_i)^a(x) \quad (\text{A.35})$$

We assume that G is a symmetry of the theory. This implies that the action must be invariant, which in turn implies that the Lagrangian must be invariant up to a total derivative, i.e. the Lagrangian transforms as:

$$\mathcal{L}'(\phi'_i, x) = \mathcal{L}(\phi_i, x) + \alpha^a (\delta \mathcal{L}(\phi_i, x))^a = \mathcal{L}(\phi_i, x) + \alpha^a \partial_\mu Y_\mu^a(x) \quad (\text{A.36})$$

for some $Y_\mu^a(x)$. We can explicitly write the variation of the Lagrangian under the variation of the fields from (A.35), and employing the Euler-Lagrange equations:

$$\frac{\delta \mathcal{L}}{\delta \phi_i} - \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} = 0 \quad (\text{A.37})$$

we find:

$$\alpha^a (\delta \mathcal{L}(\phi_i, x))^a = \alpha^a \partial_\mu \left((\delta \phi_i)^a \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} \right) \quad (\text{A.38})$$

where we implicitly sum over i . Then, the conserved currents are given by:

$$j_\mu^a(x) = (\delta \phi_i)^a \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_i)} - Y_\mu^a(x) \quad (\text{A.39})$$

which satisfy:

$$\partial_\mu j_\mu^a(x) = 0 \quad (\text{A.40})$$

We can further define charges that are conserved in Euclidean time x_4 :

$$Q^a = \int d^3x j_4^a(x) \quad (\text{A.41})$$

where $d^3x = dx_1 dx_2 dx_3$. Conservation of the charges is now implied by (A.40):

$$\partial_4 Q^a = \int d^3x \partial_4 j_4^a(x) = - \int d^3x \partial_i j_i^a(x) = 0 \quad (\text{A.42})$$

where $i = 1, 2, 3$, and in the last equality we assume that $j_i^a(x)$ vanishes sufficiently fast at spatial infinity. The conserved charges Q^a can be identified with the generators T^a of the symmetry group G .

Appendix B

Dimensional regularization

B.1 General properties

Dimensional regularization relies on the basic fact that the convergence properties of multi-dimensional integrals in general depend on the number of dimensions d .

By letting the spacetime dimension assume irrational values $d \neq 4$, we render all loop-integrals appearing in perturbation theory finite analytic functions of d in a unique way. Divergences manifest themselves as poles of the type $(4-d)^{-k}$ in the limit of $d \rightarrow 4$ with $k \in \mathbb{N}$. We will be using $d = 4 - 2\varepsilon$, and consider the limit $\varepsilon \rightarrow 0$. Proofs for properties such as uniqueness are provided in [106].

B.1.1 Useful properties in d -dimensions

We present some useful properties in d -dimensions that will be used throughout the thesis in several calculations.

First of all, when switching to d -dimensions, Lorentz indices run from 1 to d . This implies:

$$\delta_{\mu\nu}\delta_{\mu\nu} = d \tag{B.1}$$

In section B.1.3 we will discuss the implications for the Dirac matrices.

Euclidean momentum integrals in d -dimensions (recall that $\int_p \equiv \int \frac{d^d p}{(2\pi)^d}$)

are evaluated as:

$$\int_p \frac{(p^2)^\alpha}{(p^2 + m^2)^\beta} = (4\pi)^{-\frac{d}{2}} m^{d+2\alpha-2\beta} \frac{\Gamma(\alpha + \frac{d}{2}) \Gamma(\beta - \alpha - \frac{d}{2})}{\Gamma(\frac{d}{2}) \Gamma(\beta)} \quad (\text{B.2})$$

Dimensional regularization now works as follows: we choose d such that the left-hand side of (B.2) is finite for given α, β . Next, we use the right-hand side of (B.2) to analytically continue this result to any value of d . Subsequently we can set $d = 4 - 2\varepsilon$ and expand around $\varepsilon = 0$ in order to parametrize the divergences as $\frac{1}{\varepsilon}$ poles, see section B.1.2.

Another useful property is that scaleless integrals vanish in d -dimensions:

$$\int_p (p^2)^\alpha = 0 \quad (\text{B.3})$$

for any $\alpha \in \mathbb{R}$.

The final property we will cover is very useful when calculating higher-loop integrals. It can be used to reduce them to easier integrals. It is the integration by parts property:

$$\int_p \frac{\partial f(p)}{\partial p_\mu} = 0 \quad (\text{B.4})$$

We will use it in appendix C.3 to show the finiteness of certain integrals.

There are more properties listed in [106], but since we will not use them throughout the thesis we will not list them here.

B.1.2 Extracting UV divergences

Let's consider the simplest integral that contains a UV divergence in $d \leq 4$ but is convergent in $d < 4$:

$$\int_p \frac{1}{(p^2 + m^2)^2} = (4\pi)^{-\frac{d}{2}} (m^2)^{\frac{d}{2}-2} \Gamma\left(2 - \frac{d}{2}\right) \quad (\text{B.5})$$

which is obtained using (B.2), and we keep $m \neq 0$. In order to extract the UV divergence in $d = 4$, we take $d = 4 - 2\varepsilon$ and expand around $\varepsilon = 0$:

$$\int_p \frac{1}{(p^2 + m^2)^2} \stackrel{d \rightarrow 4}{=} \frac{1}{(4\pi)^2} \left(\frac{1}{\varepsilon} - \log(m^2) + \log(4\pi e^{-\gamma_E}) + \mathcal{O}(\varepsilon) \right) \quad (\text{B.6})$$

The correspondence between the ε pole and an explicit UV-cut-off Λ at one-loop order is given by:

$$\frac{1}{\varepsilon} \Longleftrightarrow \log \frac{\Lambda^2}{\mu^2} \quad (\text{B.7})$$

B.1.3 Dirac matrices

Traces and contractions

In d dimensions the traces of Dirac matrices remain the same as in 4 dimensions:

$$\text{tr}[1] = 4 \quad (\text{B.8})$$

$$\text{tr}[\gamma_\mu \gamma_\nu] = 4\delta_{\mu\nu} \quad (\text{B.9})$$

$$\text{tr}[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma] = 4(\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}) \quad (\text{B.10})$$

while the trace of any odd number of γ 's vanishes. However, the contractions of Dirac matrices now become:

$$\gamma_\mu \gamma_\mu = d \quad (\text{B.11})$$

$$\gamma_\mu \gamma_\nu \gamma_\mu = -(d-2)\gamma_\nu \quad (\text{B.12})$$

$$\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\mu = 4\delta_{\nu\rho} - (4-d)\gamma_\nu \gamma_\rho \quad (\text{B.13})$$

$$\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu = -2\gamma_\sigma \gamma_\rho \gamma_\nu + (4-d)\gamma_\nu \gamma_\rho \gamma_\sigma \quad (\text{B.14})$$

Defining γ_5

In 4 Euclidean spacetime dimensions γ_5 is given by:

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \frac{1}{4!} \varepsilon_{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \quad (\text{B.15})$$

with $\varepsilon_{\mu\nu\rho\sigma}$ the totally anti-symmetric Lorentz-invariant tensor with $\varepsilon_{1234} = 1$. When generalizing to d -dimensions it turns out we still have to use this expression for γ_5 , however now with [106, 181]:

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } (\mu\nu\rho\sigma) \text{ is an even permutation of } (1234) \\ -1 & \text{if } (\mu\nu\rho\sigma) \text{ is an odd permutation of } (1234) \\ 0 & \text{otherwise} \end{cases} \quad (\text{B.16})$$

in order to stay consistent.

B.2 ε -expansion of the evolved 2-point vector correlator at $\mathcal{O}(g^2)$

Here we provide the explicit ε -expansion of $\Pi_{\mu\nu,2}^{V(\varepsilon)}(t, x)$ in (5.54), which is given by (we temporarily replace $x - y \rightarrow x$ for brevity):

$$\Pi_{\mu\nu,2}^{V(\varepsilon)}(t, x) = (\text{I.1})^{(\varepsilon)} + (\text{III.1}')^{(\varepsilon)} + \text{finite contributions} \quad (\text{B.17})$$

where $(\text{I.1})^{(\varepsilon)}$ and $(\text{III.1}')^{(\varepsilon)}$ are the contributions given in (5.37) and (5.43), evaluated in $d = 4 - 2\varepsilon$ dimensions. The remaining finite contributions consist of all terms other than (I.1) and (III.1') in (5.30).

The explicit expressions in d -dimensions are given by:

$$(\text{I.1})^{(d)} = \frac{\text{tr}(T^a T^a)}{(4\pi)^{\frac{3d}{2}}} \frac{16 \Gamma\left(\frac{d}{2}\right) \Gamma\left(3 - \frac{d}{2}\right)}{d(d-2)(4-d)} t^{2-d} (x^2)^{-\frac{d}{2}} \text{tr}[\gamma_\mu \not{x} \gamma_\nu \not{x}] \\ \times \gamma\left(\frac{d}{2}, \frac{x^2}{8t}\right) {}_1F_1\left(d-2, 1 + \frac{d}{2}, -\frac{x^2}{8t}\right) \quad (\text{B.18})$$

$$(\text{III.1}')^{(d)} = \frac{\text{tr}(T^a T^a)}{(4\pi)^{\frac{3d}{2}}} \frac{2^{3+\frac{3d}{2}}}{(4-d)} t^{2-\frac{d}{2}} (x^2)^{-d} \text{tr}[\gamma_\mu \not{x} \gamma_\nu \not{x}] \gamma\left(\frac{d}{2}, \frac{x^2}{8t}\right)^2 \quad (\text{B.19})$$

while the leading order contribution in d -dimensions is given by:

$$\Pi_{\mu\nu,0}^{V(d)}(t, x) = \frac{d(R)}{(4\pi)^d} 2^{2d-2} (x^2)^{-d} \text{tr}[\gamma_\mu \not{x} \gamma_\nu \not{x}] \gamma\left(\frac{d}{2}, \frac{x^2}{8t}\right)^2 \quad (\text{B.20})$$

where $d(R)$ is the dimension of the fermion representation R . We first provide the ε -expansions of the lower incomplete gamma function and the confluent hypergeometric function ${}_1F_1$. For the former we obtain:

$$\gamma\left(\frac{d}{2}, z\right) = \gamma(2, z) + \frac{d}{d\varepsilon} \gamma(2 - \varepsilon, z) \Big|_{\varepsilon=0} \varepsilon + \mathcal{O}(\varepsilon^2) \\ = \gamma(2, z) - \int_0^z dt t \log(t) e^{-t} \varepsilon + \mathcal{O}(\varepsilon^2) \\ = \gamma(2, z) + \left[\gamma(1, z) - \text{Ein}(z) + \gamma(2, z) \log(z) \right] \varepsilon + \mathcal{O}(\varepsilon^2) \quad (\text{B.21})$$

where we used the definition in (4.40), and $\text{Ein}(z)$ is the complementary exponential integral:

$$\text{Ein}(z) = \int_0^z dt \frac{1 - e^{-t}}{t} \quad (\text{B.22})$$

For the confluent hypergeometric function we use the following properties:

$${}_1F_1(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} z^n \quad (\text{B.23})$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol, and the Kummer's transformation:

$${}_1F_1(a, b, -z) = e^{-z} {}_1F_1(b-a, b, z) \quad (\text{B.24})$$

Further, the relation between the confluent hypergeometric function and the lower incomplete gamma function is given by:

$${}_1F_1(1, 1+s, z) = s z^{-s} e^z \gamma(s, z) \quad (\text{B.25})$$

and we will use the following expressions for the derivatives of the confluent hypergeometric function with respect to its parameters a, b [219]:

$$\begin{aligned} \frac{d}{da} {}_1F_1(a, b, z) \\ = \frac{a}{b} z \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{a+m_1} \frac{(a+1)_{m_1+m_2}}{(2)_{m_1+m_2} (b+1)_{m_1+m_2}} z^{m_1+m_2} \end{aligned} \quad (\text{B.26a})$$

$$\begin{aligned} \frac{d}{db} {}_1F_1(a, b, z) \\ = -\frac{a}{b} z \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{b+m_1} \frac{(a+1)_{m_1+m_2}}{(2)_{m_1+m_2} (b+1)_{m_1+m_2}} z^{m_1+m_2} \end{aligned} \quad (\text{B.26b})$$

Using these properties, we derive:

$$\begin{aligned} {}_1F_1\left(d-2, 1+\frac{d}{2}, -z\right) &= e^{-z} {}_1F_1(1+\varepsilon, 3-\varepsilon, z) \\ &= \frac{2}{z^2} \gamma(2, z) + e^{-z} \frac{d}{d\varepsilon} {}_1F_1(1+\varepsilon, 3-\varepsilon, z) \Big|_{\varepsilon=0} \varepsilon \\ &= \frac{2}{z^2} \gamma(2, z) + \frac{4}{z^2} \sum_{n=0}^{\infty} \frac{\gamma(n+3, z)}{\Gamma(n+4)} \frac{n+2}{n+1} \varepsilon + \mathcal{O}(\varepsilon^2) \end{aligned} \quad (\text{B.27})$$

where in the first equality we use (B.24), and to obtain the third equality we first employ (B.26), then combine (B.23) and (B.25) to rewrite the sum in m_2 .

Plugging the expansions into (B.18) and (B.19), we obtain:

$$\begin{aligned} (\text{I.1})^{(\varepsilon)} &= -C_2(R) \frac{2}{(4\pi)^2} \Pi_{\mu\nu,0}^{V(\varepsilon)}(t, x) \left\{ \frac{1}{\varepsilon} + \log(t) + \gamma \left(2, \frac{x^2}{8t} \right)^{-1} \right. \\ &\times \left[\gamma \left(1, \frac{x^2}{8t} \right) - \text{Ein} \left(\frac{x^2}{8t} \right) + 2 \sum_{n=0}^{\infty} \frac{\gamma \left(n+3, \frac{x^2}{8t} \right)}{\Gamma(n+4)} \frac{n+2}{n+1} \right] + \text{finite parts} \Big\} \end{aligned} \quad (\text{B.28})$$

$$(\text{III.1}')^{(\varepsilon)} = -C_2(R) \frac{4}{(4\pi)^2} \Pi_{\mu\nu,0}^{V(\varepsilon)}(t, x) \left\{ \frac{1}{\varepsilon} + \log(t) + \text{finite parts} \right\} \quad (\text{B.29})$$

where we employed $\text{tr}(T^a T^a) = -C_2(R)d(R)$, and adding them we have for (B.17):

$$\begin{aligned} \Pi_{\mu\nu,2}^{V(\varepsilon)}(t,x) = & -C_2(R) \frac{2}{(4\pi)^2} \Pi_{\mu\nu,0}^{V(\varepsilon)}(t,x) \left\{ \frac{3}{\varepsilon} + 3 \log(t) + \gamma \left(2, \frac{x^2}{8t} \right)^{-1} \right. \\ & \left[\gamma \left(1, \frac{x^2}{8t} \right) - \text{Ein} \left(\frac{x^2}{8t} \right) + 2 \sum_{n=0}^{\infty} \frac{\gamma \left(n+3, \frac{x^2}{8t} \right)}{\Gamma(n+4)} \frac{n+2}{n+1} \right] + \text{finite parts} \Big\} \\ & + \text{finite contributions} \end{aligned} \quad (\text{B.30})$$

For completeness we also explicitly present $\Pi_{\mu\nu,0}^{V(\varepsilon)}$, though the $\mathcal{O}(\varepsilon)$ term will not contribute in the final expression for $\Pi_{R,\mu\nu}^V$ at $\mathcal{O}(g^2)$ in (5.54):

$$\begin{aligned} \Pi_{\mu\nu,0}^{V(\varepsilon)}(t,x) = & \Pi_{\mu\nu,0}^V(t,x) \left(1 + 2 \left\{ \log(x^2) + \log \left(\frac{x^2}{8t} \right) \right. \right. \\ & \left. \left. + \gamma \left(\frac{x^2}{8t} \right)^{-1} \left[\gamma \left(1, \frac{x^2}{8t} \right) - \text{Ein} \left(\frac{x^2}{8t} \right) \right] + \text{finite parts} \right\} \varepsilon + \mathcal{O}(\varepsilon^2) \right) \end{aligned} \quad (\text{B.31})$$

with $\Pi_{\mu\nu,0}^V$ given in (5.32).

To summarize, we have for (5.54):

$$\begin{aligned} \Pi_{R,\mu\nu}^V(t,x;g(\mu),\mu) = & \left(1 - \frac{g^2(\mu)}{(4\pi)^2} C_2(R) \left\{ 6 \log(t\mu^2) + 2\gamma \left(2, \frac{x^2}{8t} \right)^{-1} \right. \right. \\ & \times \left[\gamma \left(1, \frac{x^2}{8t} \right) - \text{Ein} \left(\frac{x^2}{8t} \right) + 2 \sum_{n=0}^{\infty} \frac{\gamma \left(n+3, \frac{x^2}{8t} \right)}{\Gamma(n+4)} \frac{n+2}{n+1} \right] \\ & \left. \left. + \text{finite terms} \right\} \right) \Pi_{\mu\nu,0}^V(t,x) + \dots \end{aligned} \quad (\text{B.32})$$

where here the finite terms include the finite parts from the ε -expansions in (B.28) and (B.29), while the ellipsis stands for $\mathcal{O}(g^4)$ contributions and additional $\mathcal{O}(g^2)$ finite terms.

We add that at $t > 0$ the short-distance limit of the $\mathcal{O}(g^2)$ contributions computed here is nonsingular and vanishes. This is shown using:

$$\frac{d}{dz} \gamma(a,z) = z^{a-1} e^{-z}, \quad \frac{d}{dz} \text{Ein}(z) = \frac{1 - e^{-z}}{z} \quad (\text{B.33})$$

which both follow from the definitions in (4.40) and (B.22), and employing l'Hopital's rule to find:

$$\lim_{z \rightarrow 0} \gamma(2, z)^{-1} \left(\gamma(1, z) - \text{Ein}(z) + 2 \sum_{n=0}^{\infty} \frac{\gamma(n+3, z)}{\Gamma(n+4)} \frac{n+2}{n+1} \right) = -\frac{1}{2} \quad (\text{B.34})$$

which then combines with the vanishing limit of $\Pi_{\mu\nu,0}^V(t, x)$.

B.3 Integrals from fermionic contribution to $\langle E \rangle$

The integrals are given by:

$$I_1 = \int_0^t ds \int_{p,q} e^{-2t(p^2+q^2)-(2t-s)2p \cdot q} \frac{p \cdot q}{p^2 q^2} \quad (\text{B.35})$$

$$I_2 = \int_0^t ds \int_{p,q} e^{-2t(p^2+q^2)-(2t-s)2p \cdot q} \frac{1}{(p+q)^2} \quad (\text{B.36})$$

$$I_3 = \int_0^t ds \int_{p,q} e^{-2t(p^2+q^2)-(2t-s)2p \cdot q} \frac{p \cdot q}{p^2(p+q)^2} \quad (\text{B.37})$$

$$I_4 = \int_0^t ds \int_{p,q} e^{-2t(p^2+q^2)-(2t-s)2p \cdot q} \frac{(p \cdot q)^2}{p^2 q^2 (p+q)^2} = \frac{1}{2} I_1 - I_3 \quad (\text{B.38})$$

$$I_5 = \int_0^t ds_1 \int_0^t ds_2 \int_{p,q} e^{-2t(p^2+q^2)-(2t-s_1-s_2)2p \cdot q} \quad (\text{B.39})$$

$$I_6 = \int_0^t ds_1 \int_0^t ds_2 \int_{p,q} e^{-2t(p^2+q^2)-(2t-s_1-s_2)2p \cdot q} \frac{p \cdot q}{p^2} \quad (\text{B.40})$$

$$I_7 = \int_0^t ds_1 \int_0^t ds_2 \int_{p,q} e^{-2t(p^2+q^2)-(2t-s_1-s_2)2p \cdot q} \frac{(p \cdot q)^2}{p^2 q^2} \quad (\text{B.41})$$

$$I_8 = \int_0^t ds \int_0^s du \int_{p,q} e^{-2tp^2-2sq^2-(s-u)2p \cdot q} \quad (\text{B.42})$$

$$I_9 = \int_0^t ds \int_0^s du \int_{p,q} e^{-2tp^2-2sq^2-(s-u)2p \cdot q} \frac{(p \cdot q)^2}{p^2 q^2} = \frac{1}{2} I_2 + \frac{1}{2} I_3 \quad (\text{B.43})$$

$$I_{10} = \int_0^t ds \int_{p,q} e^{-2sp^2-2tq^2} \frac{1}{p^2} \quad (\text{B.44})$$

$$I_{11} = \int_0^t ds \int_{p,q} e^{-2sp^2-2tq^2-s2p \cdot q} \frac{1}{p^2} \quad (\text{B.45})$$

where we related different integrals by using:

$$p \cdot q = \frac{1}{2}((p+q)^2 - p^2 - q^2), \quad p \cdot q e^{s2p \cdot q} = \frac{1}{2} \frac{d}{ds} e^{s2p \cdot q} \quad (\text{B.46})$$

Performing the integrals, we obtain:

$$I_1 = -\frac{(4\pi)^{-d} t^{2-d}}{2(d-2)(4-d)} \left\{ \frac{(2^{5-d} - 2^3)}{(d-2)} \Gamma\left(\frac{d}{2}\right) \Gamma\left(3 - \frac{d}{2}\right) + 3^{2-\frac{d}{2}} {}_2F_1\left(1, 1, 3 - \frac{d}{2}, \frac{3}{4}\right) \right\} \quad (\text{B.47})$$

$$I_5 = -2^{2-2d} (4\pi)^{-d} t^{2-d} \left\{ \beta\left(\frac{1}{4}, 1 - \frac{d}{2}, 1 - \frac{d}{2}\right) - 4\beta\left(\frac{1}{4}, 2 - \frac{d}{2}, 1 - \frac{d}{2}\right) - 2\beta\left(\frac{1}{2}, 1 - \frac{d}{2}, 1 - \frac{d}{2}\right) + 4\beta\left(\frac{1}{2}, 2 - \frac{d}{2}, 1 - \frac{d}{2}\right) - \frac{2^d}{(d-2)(4-d)} {}_2F_1\left(1 - \frac{d}{2}, \frac{d}{2}, 3 - \frac{d}{2}, \frac{1}{4}\right) \right\} \quad (\text{B.48})$$

$$I_6 = (4\pi)^{-d} t^{2-d} \left\{ \frac{3^{1-\frac{d}{2}}}{2(d-2)(d-3)} - \frac{2^{4-2d}}{(d-3)} \beta\left(\frac{1}{4}, 2 - \frac{d}{2}, 1 - \frac{d}{2}\right) + \frac{3^{-1-\frac{d}{2}}}{2(d-2)} \left[3F_1\left(2, \frac{d}{2}, \frac{d}{2}, 3, \frac{1}{3}, -1\right) - F_1\left(3, \frac{d}{2}, \frac{d}{2}, 4, \frac{1}{3}, -1\right) \right] - \frac{2^{1-2d} 3}{(d-2)} \left[\beta\left(\frac{1}{4}, 1 - \frac{d}{2}, 1 - \frac{d}{2}\right) - \beta\left(\frac{1}{2}, 1 - \frac{d}{2}, 1 - \frac{d}{2}\right) \right] \right\} \quad (\text{B.49})$$

$$I_7 = (4\pi)^{-d} t^{2-d} \left\{ \frac{2^{2-d}}{(d-2)^2} - \frac{(2^2 - 2^{3-d})}{(d-2)^2(4-d)} \Gamma\left(\frac{d}{2}\right) \Gamma\left(3 - \frac{d}{2}\right) + \frac{3^{2-\frac{d}{2}}}{2(d-2)(4-d)} {}_2F_1\left(1, 1, 3 - \frac{d}{2}, \frac{3}{4}\right) \right\} \quad (\text{B.50})$$

$$I_8 = (4\pi)^{-d} t^{2-d} \left\{ \frac{2^{4-2d}}{(d-3)} \beta\left(\frac{1}{4}, 2 - \frac{d}{2}, 1 - \frac{d}{2}\right) - \frac{2^{1-d}}{(d-3)} {}_2F_1\left(\frac{1}{2}, \frac{d}{2}, \frac{3}{2}, \frac{1}{4}\right) \right\} \quad (\text{B.51})$$

$$I_{10} = (4\pi)^{-d} t^{2-d} \frac{2^{3-d}}{(4-d)(d-2)} \quad (\text{B.52})$$

$$I_{11} = (4\pi)^{-d} t^{2-d} \left\{ \frac{3^{1-\frac{d}{2}}}{(4-d)(d-3)} {}_2F_1 \left(1, 3-d, 3-\frac{d}{2}, \frac{1}{4} \right) - \frac{3^{1-\frac{d}{2}}}{(d-3)(d-2)} \right\} \quad (\text{B.53})$$

where we use the incomplete beta function, the hypergeometric function and the Appell hypergeometric function, whose integral representations are given by:

$$\beta(x, a, b) = \int_0^x dt t^{a-1} (1-t)^{b-1} = \frac{x^a}{a} {}_2F_1(a, 1-b, a+1, x) \quad (\text{B.54})$$

$${}_2F_1(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \frac{t^{b-1} (1-t)^{c-b-1}}{(1-zt)^a} \quad (\text{B.55})$$

$$F_1(a, b_1, b_2, c, z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt \frac{t^{a-1} (1-t)^{c-a-1}}{(1-z_1 t)^{b_1} (1-z_2 t)^{b_2}} \quad (\text{B.56})$$

Next we set $d = 4 - 2\varepsilon$ and expand in ε . The following identities have been derived using results from [220]:

$$\left. \frac{d}{dc} {}_2F_1 \left(1, 1, c, \frac{3}{4} \right) \right|_{c=1} = -8 \log 2 \quad (\text{B.57})$$

$$\left. \frac{d}{db} {}_2F_1 \left(1, b, 1, \frac{1}{4} \right) \right|_{b=-1} = \frac{3}{2} \log 2 - \frac{3}{4} \log 3 \quad (\text{B.58})$$

$$F_1 \left(2, 2, 2, 3, \frac{1}{3}, -1 \right) = \frac{9}{16} \log 3 \quad (\text{B.59})$$

$$\beta \left(\frac{1}{2}; 1 - \frac{d}{2}, 1 - \frac{d}{2} \right) = \frac{2}{\varepsilon} - 2 + \mathcal{O}(\varepsilon) \quad (\text{B.60})$$

$$\beta \left(\frac{1}{4}; 1 - \frac{d}{2}, 1 - \frac{d}{2} \right) = \frac{2}{\varepsilon} - \frac{14}{3} - 2 \log 3 + \mathcal{O}(\varepsilon) \quad (\text{B.61})$$

$$\beta \left(\frac{1}{2}; 2 - \frac{d}{2}, 1 - \frac{d}{2} \right) = \frac{1}{\varepsilon} + 1 + \mathcal{O}(\varepsilon) \quad (\text{B.62})$$

$$\beta \left(\frac{1}{4}; 2 - \frac{d}{2}, 1 - \frac{d}{2} \right) = \frac{1}{\varepsilon} + \frac{1}{3} - \log 3 + \mathcal{O}(\varepsilon) \quad (\text{B.63})$$

Employing these expressions, together with the usual gamma function expansions, we obtain:

$$I_1 = -\frac{1}{8}t^{2-d}(4\pi)^{-d} \left\{ \frac{1}{\varepsilon} + 1 + 4 \log 3 - 6 \log 2 + \mathcal{O}(\varepsilon) \right\} \quad (\text{B.64})$$

$$I_5 = \frac{1}{16}t^{2-d}(4\pi)^{-d} \left\{ \frac{1}{\varepsilon} + 1 + 4 \log 2 - \log 3 + \mathcal{O}(\varepsilon) \right\} \quad (\text{B.65})$$

$$I_6 = -\frac{1}{16}t^{2-d}(4\pi)^{-d} \left\{ \frac{1}{\varepsilon} + 1 - 2 \log 3 + 4 \log 2 + \mathcal{O}(\varepsilon) \right\} \quad (\text{B.66})$$

$$I_7 = \frac{1}{16}t^{2-d}(4\pi)^{-d} \left\{ \frac{1}{\varepsilon} + 2 - 14 \log 2 + 8 \log 3 + \mathcal{O}(\varepsilon) \right\} \quad (\text{B.67})$$

$$I_8 = \frac{1}{16}t^{2-d}(4\pi)^{-d} \left\{ \frac{1}{\varepsilon} + 1 + 4 \log 2 - 2 \log 3 + \mathcal{O}(\varepsilon) \right\} \quad (\text{B.68})$$

$$I_{10} = \frac{1}{8}t^{2-d}(4\pi)^{-d} \left\{ \frac{1}{\varepsilon} + 1 + 2 \log 2 + \mathcal{O}(\varepsilon) \right\} \quad (\text{B.69})$$

$$I_{11} = \frac{1}{8}t^{2-d}(4\pi)^{-d} \left\{ \frac{1}{\varepsilon} + 1 + 4 \log 2 - \log 3 + \mathcal{O}(\varepsilon) \right\} \quad (\text{B.70})$$

Plugging these expansions into (6.50) — optionally supplemented with (6.74) — we arrive at the total fermionic contribution (6.53) resp. (6.76).

Appendix C

Supplementary material for chapter 5

C.1 Derivation of (5.42) for (III.1)+(III.2)

In this appendix we derive (5.42), which reads:

$$(III.1) + (III.2) = (III.1') + (III.2') - (II.1) \quad (C.1)$$

We start with:

$$\begin{aligned} (III.1) + (III.2) &= 2 \langle (\bar{\chi}_2 \gamma_\mu \chi_0)_x (\bar{\chi}_0 \gamma_\nu \chi_0)_y \rangle_0 + (x \leftrightarrow y; \mu \leftrightarrow \nu) \\ &= 2 \int_0^t ds \langle (e^{(t-s)\Delta_x} \{ \bar{\chi}_0(s) B_{\rho,1}(s) B_{\rho,1}(s) \} \gamma_\mu \chi_0)_x (\bar{\chi}_0 \gamma_\nu \chi_0)_y \rangle_0 \\ &\quad + 8 \int_0^t ds \int_0^s du \langle (e^{(t-s)\Delta_x} \{ \partial_\sigma^x [e^{(s-u)\Delta_x} \{ (\partial_\rho^x \bar{\chi}_0(u)) B_{\rho,1}(u) \}] B_{\sigma,1}(s) \} \\ &\quad \times \gamma_\mu \chi_0)_x (\bar{\chi}_0 \gamma_\nu \chi_0)_y \rangle + (x \leftrightarrow y; \mu \leftrightarrow \nu) \end{aligned}$$

associated to the diagrams (III.1) and (III.2) in figure 5.6. After performing the Wick contractions, we obtain:

$$\begin{aligned} (III.1) + (III.2) &= -2d \operatorname{tr}(T^a T^a) \int_0^t ds \int_z \operatorname{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu e^{(t-s)\Delta_x} \right. \\ &\quad \times \left. \left\{ S(\bar{y}_t - \bar{x}_s) \delta^{(d)}(x - z) D(\bar{x}_s - \bar{z}_s) \right\} \right] - \dots \end{aligned}$$

$$\begin{aligned} & \dots - 8\text{tr}(T^a T^a) \int_0^t ds \int_0^s du \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(x - z) \right. \right. \\ & \quad \left. \left. \times e^{(s-u)\Delta_x} \partial_\rho^x \left\{ D(\bar{x}_u - \bar{z}_s) \partial_\rho^x S(\bar{y}_t - \bar{x}_u) \right\} \right\} \right] + (x \leftrightarrow y; \mu \leftrightarrow \nu) \end{aligned} \quad (\text{C.2})$$

We then employ the relation:

$$\begin{aligned} & \int_0^s du \frac{d}{du} \left(e^{(s-u)\Delta_x} \left\{ D(\bar{x}_u - \bar{z}_s) S(\bar{y}_t - \bar{x}_u) \right\} \right) \\ & = - \int_0^s du e^{(s-u)\Delta_x} \Delta_x \left\{ D(\bar{x}_u - \bar{z}_s) S(\bar{y}_t - \bar{x}_u) \right\} \\ & \quad + \int_0^s du e^{(s-u)\Delta_x} \left\{ \Delta_x D(\bar{x}_u - \bar{z}_s) S(\bar{y}_t - \bar{x}_u) \right\} \\ & \quad + \int_0^s du e^{(s-u)\Delta_x} \left\{ D(\bar{x}_u - \bar{z}_s) \Delta_x S(\bar{y}_t - \bar{x}_u) \right\} \\ & = -2 \int_0^s du e^{(s-u)\Delta_x} \left\{ \partial_\rho^x D(\bar{x}_u - \bar{z}_s) \partial_\rho^x S(\bar{y}_t - \bar{x}_u) \right\} \end{aligned} \quad (\text{C.3})$$

to rewrite the second term in (C.2) as:

$$\begin{aligned} & -8\text{tr}(T^a T^a) \int_0^t ds \int_0^s du \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu \right. \\ & \quad \left. \times e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(x - z) e^{(s-u)\Delta_x} \partial_\rho^x \left\{ D(\bar{x}_u - \bar{z}_s) \partial_\rho^x S(\bar{y}_t - \bar{x}_u) \right\} \right\} \right] \\ & = -8\text{tr}(T^a T^a) \int_0^t ds \int_0^s du \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu \right. \\ & \quad \left. \times e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(x - z) e^{(s-u)\Delta_x} \left\{ D(\bar{x}_u - \bar{z}_s) \Delta_x S(\bar{y}_t - \bar{x}_u) \right\} \right\} \right] \\ & \quad + 4\text{tr}(T^a T^a) \int_0^t ds \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu \right. \\ & \quad \left. \times e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(x - z) D(\bar{x}_s - \bar{z}_s) S(\bar{y}_t - \bar{x}_s) \right\} \right] \\ & \quad - 4\text{tr}(T^a T^a) \int_0^t ds \int_z \text{tr} \left[\gamma_\mu S(\bar{x}_t - \bar{y}_t) \gamma_\nu \right. \\ & \quad \left. \times e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(x - z) e^{s\Delta_x} \left\{ D(x - \bar{z}_s) S(\bar{y}_t - x) \right\} \right\} \right] \end{aligned} \quad (\text{C.4})$$

The last term in (C.4) exactly cancels (II.1) in (5.39), after noting that:

$$\begin{aligned}
& \int_z e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(x-z) e^{s\Delta_x} \left\{ D(x-\bar{z}_s) S(\bar{y}_t-x) \right\} \right\} \\
&= \int_z \int_{x'} e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(x-z) e^{s\Delta_x} \left\{ \delta^{(d)}(x-x') D(x'-\bar{z}_s) S(\bar{y}_t-x') \right\} \right\} \\
&= \int_{x'} e^{(t-s)\Delta_x} \left\{ \int_z \delta^{(d)}(x-z) \delta^{(d)}(\bar{x}_s-x') D(x'-\bar{z}_s) S(\bar{y}_t-x') \right\} \quad (C.5) \\
&= \int_{x'} S(\bar{y}_t-x') e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(\bar{x}_s-x') D(x'-\bar{x}_s) \right\} \\
&= \int_z S(\bar{y}_t-z) e^{(t-s)\Delta_x} \left\{ \delta^{(d)}(\bar{x}_s-z) D(z-\bar{x}_s) \right\}
\end{aligned}$$

Then, the first term in (C.4) is III.2' in (5.44), after employing $\Delta_x S(\bar{y}_t-\bar{x}_u) = -\not{\partial}_x \delta^{(d)}(\bar{y}_t-\bar{x}_u)$, and the second term in (C.4) combines with the first term in (C.2) to give III.1' in (5.43). Hence, the relation in (C.1) follows.

C.2 Momentum space expressions

We provide the Type I-IV contributions to the gradient flow evolved 2-point vector correlator in terms of momentum-space integrals:

$$\begin{aligned}
(I.1) &= -2\text{tr}(T^a T^a) \int_{p,q,k} e^{i(p+q)(x-y)} e^{-2t(p^2+q^2)} \\
&\quad \times \frac{1}{(p^2)^2 q^2 k^2 (p+k)^2} \text{tr}[\gamma_\mu \not{p} \gamma_\rho \not{k} \gamma_\rho \not{p} \gamma_\nu \not{q}] \quad (C.6)
\end{aligned}$$

$$\begin{aligned}
(I.2) &= \text{tr}(T^a T^a) \int_{p,q,k} e^{i(p+q)(x-y)} e^{-t(p^2+q^2+k^2+(p+q+k)^2)} \\
&\quad \times \frac{1}{p^2 q^2 k^2 (p+k)^2 (p+q+k)^2} \text{tr}[\gamma_\mu (\not{p} + \not{q} + \not{k}) \gamma_\rho \not{p} \gamma_\nu \not{p} \gamma_\rho \not{k}] \quad (C.7)
\end{aligned}$$

$$\begin{aligned}
(II.1) &= 8\text{tr}(T^a T^a) \int_0^t ds \int_{p,q,k} e^{i(p+q)(x-y)} e^{-2t(p^2+q^2)-2sk \cdot (p+k)} \\
&\quad \times \frac{1}{p^2 q^2 (p+k)^2} \text{tr}[\gamma_\mu \not{q} \gamma_\nu \not{p}] \quad (C.8)
\end{aligned}$$

$$\begin{aligned}
(\text{II.2}) &= 4\text{tr}(T^a T^a) \int_0^t ds \int_{p,q,k} e^{i(p+q)(x-y)} \\
&\quad \times e^{-t(p^2+q^2+k^2+(p+q+k)^2)+2sq \cdot (p+k)} \\
&\quad \times \frac{1}{p^2 q^2 k^2 (p+k)^2} \text{tr}[\gamma_\mu \not{p} \gamma_\nu (\not{p} \not{q} \not{k} + \not{k} \not{q} \not{p})]
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
(\text{III.1}') &= -4(d-2)\text{tr}(T^a T^a) \int_0^t ds \int_{p,q,k} e^{i(p+q)(x-y)} e^{-2t(p^2+q^2)-2sk^2} \\
&\quad \times \frac{1}{p^2 q^2 k^2} \text{tr}[\gamma_\mu \not{p} \gamma_\nu \not{p}]
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
(\text{III.2}') &= 16\text{tr}(T^a T^a) \int_0^t ds \int_0^s du \int_{p,q,k} e^{i(p+q)(x-y)} \\
&\quad \times e^{-2t(p^2+q^2)-2sk^2-2(s-u)p \cdot k} \frac{1}{q^2 k^2} \text{tr}[\gamma_\mu \not{p} \gamma_\nu \not{p}]
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
(\text{III.3}) &= -8\text{tr}(T^a T^a) \int_0^t ds_1 \int_0^t ds_2 \int_{p,q,k} e^{i(p+q)(x-y)} \\
&\quad \times e^{-2t(p^2+q^2+k^2)-2(t-s_1)p \cdot k+2(t-s_2)q \cdot k} \frac{q \cdot p}{p^2 q^2 k^2} \text{tr}[\gamma_\mu \not{p} \gamma_\nu \not{p}]
\end{aligned} \tag{C.12}$$

$$\begin{aligned}
(\text{IV.1}) &= -8\text{tr}(T^a T^a) \int_0^t ds_1 \int_0^t ds_2 \int_{p,q,k} e^{i(p+q)(x-y)} \\
&\quad \times e^{-t(p^2+q^2+k^2+(p+q+k)^2)+2s_1 q \cdot (p+k)-2s_2 k \cdot (p+k)} \\
&\quad \times \frac{k \cdot q}{q^2 k^2 (p+k)^2} \text{tr}[\gamma_\mu \not{p} \gamma_\nu \not{k}]
\end{aligned} \tag{C.13}$$

$$\begin{aligned}
(\text{IV.2}) &= -8\text{tr}(T^a T^a) \int_0^t ds_1 \int_0^t ds_2 \int_{p,q,k} e^{i(p+q)(x-y)} \\
&\quad \times e^{-2t(k^2+(p+q+k)^2)+2(s_1+s_2)q \cdot (p+k)} \frac{1}{k^2 (p+k)^2} \text{tr}[\gamma_\mu \not{p} \gamma_\nu \not{k}]
\end{aligned} \tag{C.14}$$

C.3 Finiteness of (III.2')

It is convenient in this case to work directly with momentum-space integrals. The contribution (III.2') then reads:

$$(III.2') = 16\text{tr}(T^a T^a) \int_{p,q} e^{i(p+q)(x-y)} e^{-2t(p^2+q^2)} \frac{1}{q^2} \text{tr}[\gamma_\mu \not{q} \gamma_\nu \not{p}] \mathcal{I}(p) \quad (C.15)$$

with:

$$\begin{aligned} \mathcal{I}(p) &= \int_0^t ds \int_0^s du \int_k \frac{1}{k^2} e^{-2sk^2 - 2(s-u)p \cdot k} \\ &= \int_0^t ds \int_0^s du \int_k \frac{1}{k^2} e^{-2sk^2 - 2up \cdot k} \end{aligned} \quad (C.16)$$

where in the last equality we let $u \rightarrow s - u$. We first perform the integral in k in $\mathcal{I}(p)$, by inserting the identity:

$$1 = \frac{1}{d} \frac{\partial k_\mu}{\partial k_\mu} \quad (C.17)$$

then using integration by parts (IBP), while rewriting the resulting $p \cdot k$ appearing in the numerator as $-\frac{1}{2} \frac{d}{du}$, and using that in dimensional regularization:

$$\int_k \frac{\partial}{\partial k_\mu} f(k) = 0 \quad (C.18)$$

Subsequently, IBP in u yields:

$$\begin{aligned} \mathcal{I}(p) &= \mathcal{I}_1(p) + \mathcal{I}_2(p) + \mathcal{I}_3(p) \\ \mathcal{I}_1(p) &\equiv \frac{3}{d} \int_0^t ds \int_0^s du \int_k \frac{1}{k^2} e^{-2sk^2 - 2up \cdot k} = \frac{3}{d} \mathcal{I}(p) \\ \mathcal{I}_2(p) &\equiv \frac{4}{d} \int_0^t ds \int_0^s du \int_k s e^{-2sk^2 - 2up \cdot k} \\ \mathcal{I}_3(p) &\equiv -\frac{1}{d} \int_0^t ds s \int_k \frac{1}{k^2} e^{-2sk^2 - 2sp \cdot k} \end{aligned} \quad (C.19)$$

Next we perform IBP in k in $\mathcal{I}_3$, and we express the appearing $p \cdot k$ in the numerator as $-k^2 - \frac{1}{2} \frac{d}{ds}$, and perform IBP in s in the $\frac{d}{ds}$ term. This yields:

$$\begin{aligned} \mathcal{I}_3(p) &= \frac{1}{d(d-4)} t^2 \int_k \frac{1}{k^2} e^{-2tk^2 - 2tp \cdot k} \\ &\quad - \frac{2}{d(d-4)} \int_0^t ds s^2 \int_k e^{-2sk^2 - 2sp \cdot k} \end{aligned} \quad (C.20)$$

Next we focus on $\mathcal{I}_2$. Using the same methods, we find:

$$\mathcal{I}_2(p) = -\frac{2}{d}t \int_0^t du \int_k \frac{1}{k^2} e^{-2tk^2 - 2up \cdot k} - 2\mathcal{I}_3(p) + \frac{2}{d}\mathcal{I}(p) \quad (\text{C.21})$$

Plugging everything back in, we find:

$$\begin{aligned} \mathcal{I}(p) &= \frac{2}{(d-5)}\mathcal{I}_{2a}(p) + \frac{1}{(d-5)(d-4)}(\mathcal{I}_{3a}(p) + \mathcal{I}_{3b}(p)) \\ \mathcal{I}_{2a}(p) &\equiv - \int_0^t du \int_k \frac{1}{k^2} e^{-2tk^2 - 2up \cdot k} \\ \mathcal{I}_{3a}(p) &\equiv -t^2 \int_k \frac{1}{k^2} e^{-2tk^2 - 2tp \cdot k} \\ \mathcal{I}_{3b}(p) &\equiv 2 \int_0^t ds s^2 \int_k e^{-2sk^2 - 2sp \cdot k} \end{aligned} \quad (\text{C.22})$$

The term containing $\mathcal{I}_{2a}$ is finite in $d = 4$, due to the damping effect of e^{-2tk^2} . Next we show that the terms with the $\frac{1}{(d-4)}$ prefactor cancel in $d = 4$, rendering this term finite. For $\mathcal{I}_{3a}$ we use Schwinger parametrization:

$$\begin{aligned} \mathcal{I}_{3a}(p) &= -t^2 \int_0^\infty dv \int_k e^{-(2t+v)k^2 - 2tp \cdot k} \\ &= -(4\pi)^{-d/2} t^2 \int_0^\infty dv (2t+v)^{-d/2} e^{\frac{t^2}{2t+v} p^2} \\ &= (-1)^{-d/2} (4\pi)^{-d/2} t^{4-d} (p^2)^{1-d/2} \gamma\left(\frac{d}{2} - 1, -\frac{tp^2}{2}\right) \\ &\stackrel{d \rightarrow 4}{=} \frac{1}{(4\pi)^2} \frac{1}{p^2} \gamma\left(1, -\frac{tp^2}{2}\right) \end{aligned} \quad (\text{C.23})$$

while for $\mathcal{I}_{3b}$ we have:

$$\begin{aligned} \mathcal{I}_{3b}(p) &= (4\pi)^{-d/2} 2^{1-d/2} \int_0^t ds s^{2-d/2} e^{\frac{1}{2}s p^2} \\ &= (-1)^{1-d/2} (4\pi)^{-d/2} 2^{4-d} (p^2)^{d/2-3} \gamma\left(3 - \frac{d}{2}, -\frac{tp^2}{2}\right) \\ &\stackrel{d \rightarrow 4}{=} -\frac{1}{(4\pi)^2} \frac{1}{p^2} \gamma\left(1, -\frac{tp^2}{2}\right) \end{aligned} \quad (\text{C.24})$$

and we see that these terms cancel when $d \rightarrow 4$. Thus, we conclude that $\mathcal{I}(p)$ is finite, which implies that (III.2') is finite. See also [58] for an automated implementation of IBP reduction for loop integrals involving flow-time integrations.

C.4 Nonconservation of the evolved vector current at the leading order

We explicitly derive the classical nonconservation of the vector current evolved by the gradient flow:

$$\partial_\mu J_\mu(t, x) \neq 0 \quad (\text{C.25})$$

at leading order in the bare gauge coupling g_0 . At this order, the evolved current reads (see (5.4)):

$$J_{\mu,0}(t, x) = \bar{\chi}_0(t, x) \gamma_\mu \chi_0(t, x) = e^{t\Delta_x} \bar{\psi}(x) \gamma_\mu e^{t\Delta_x} \psi(x) \quad (\text{C.26})$$

with the exponentials-of-Laplacian acting only on what is directly to the right of them. Taking into account the commutativity of the derivative and the exponential-of-Laplacian, the derivative of $J_{\mu,0}$ in (C.26) yields (see also (4.15) and (4.26)):

$$\begin{aligned} \partial_\mu J_{\mu,0}(t, x) &= e^{t\Delta} \{ \partial_\mu \bar{\psi}(x) \} \gamma_\mu \chi_0(t, x) + \bar{\chi}_0(t, x) \gamma_\mu e^{t\Delta} \{ \partial_\mu \psi(x) \} \\ &= e^{t\Delta} \left\{ \bar{\psi}(x) \overleftarrow{\partial} \right\} \chi_0(t, x) + \bar{\chi}_0(t, x) e^{t\Delta} \{ \not{\partial} \psi(x) \} \end{aligned} \quad (\text{C.27})$$

Next we employ the classical equations of motion for $\bar{\psi}$ and ψ :

$$(\not{\partial} + g_0 \not{A}) \psi = 0, \quad \bar{\psi} \left(\overleftarrow{\not{\partial}} - g_0 \not{A} \right) = 0 \quad (\text{C.28})$$

to arrive at:

$$\partial_\mu J_{\mu,0}(t, x) = g_0 \left(e^{t\Delta} \{ \bar{\psi}(x) \not{A}(x) \} e^{t\Delta} \psi(x) - e^{t\Delta} \bar{\psi}(x) e^{t\Delta} \{ \not{A}(x) \psi(x) \} \right) \quad (\text{C.29})$$

As a final step, we employ the Fourier transforms for $\bar{\psi}$, ψ and A_μ :

$$\bar{\psi}(x) = \int_p e^{ipx} \tilde{\bar{\psi}}(p), \quad A_\mu(x) = \int_k e^{ikx} \tilde{A}_\mu(k), \quad \psi(x) = \int_q e^{iqx} \tilde{\psi}(q) \quad (\text{C.30})$$

yielding:

$$\begin{aligned} \partial_\mu J_{\mu,0}(t, x) &= g_0 \int_{p,q,k} e^{i(p+q+k)x} \left(e^{-t(p+k)^2 - tq^2} - e^{-tp^2 - t(k+q)^2} \right) \tilde{\bar{\psi}}(p) \tilde{A}(k) \tilde{\psi}(q) \\ &= g_0 \int_{p,q,k} e^{i(p+q+k)x} e^{-t(p+k)^2 - tq^2} \left(\tilde{\bar{\psi}}(p) \tilde{A}(k) \tilde{\psi}(q) - \tilde{\bar{\psi}}(q) \tilde{A}(k) \tilde{\psi}(p) \right) \end{aligned} \quad (\text{C.31})$$

which does not vanish at $t > 0$, while we recover the conservation of the nonevolved current at $t = 0$.

Appendix D

Supplementary material for chapter 6

D.1 $\langle E \rangle$ results for general N

Here we collect the results for $\langle E \rangle$ for the different gradient flow cases for general N and N_f :

$$\langle E \rangle_i = \frac{3(N^2 - 1)}{32\pi t^2} \alpha(\mu) \{1 + k_i(\mu^2 t) \alpha(\mu) + \mathcal{O}(\alpha^2)\} \quad (\text{D.1})$$

with $\alpha(\mu) = \frac{g^2(\mu)}{4\pi}$ the renormalized $\overline{\text{MS}}$ coupling, see (6.13), and:

$$k_{YM}(\mu^2 t) = \frac{1}{4\pi} \left\{ N \left(\frac{11}{3} L + \frac{52}{9} - 3 \log 3 \right) - N_f \left(\frac{2}{3} L + \frac{4}{9} - \frac{4}{3} \log 2 \right) \right\} \quad (\text{D.2a})$$

$$k_{NM}(\mu^2 t) = \frac{1}{4\pi} \left\{ N \left(\frac{11}{3} L + \frac{52}{9} - 3 \log 3 \right) - N_f \left(\frac{2}{3} L + \frac{22}{9} - \log 3 \right) \right\} \quad (\text{D.2b})$$

$$k_{sNM}(\mu^2 t) = \frac{1}{4\pi} \left\{ N \left(\frac{11}{3} L + \frac{52}{9} - 3 \log 3 \right) - N_f \left(\frac{2}{3} L + \frac{22}{9} + 4 \log 2 - 3 \log 3 \right) \right\} \quad (\text{D.2c})$$

where $L = \log(8\mu^2 t) + \gamma_E$.

Rewriting (6.13) in terms of α we have:

$$\alpha_0 = \mu^{2\varepsilon} (4\pi e^{-\gamma_E})^{-\varepsilon} \alpha(\mu) Z_\alpha(\alpha, \varepsilon) \quad (\text{D.3})$$

with $Z_\alpha(\alpha, \varepsilon) = Z_g^2(g, \varepsilon)$ from (6.14). The renormalization group (RG) invariant coupling $\alpha(q)$ is implicitly defined by integrating the $d = 4 - 2\varepsilon$ beta function (for a general discussion and explanation see section 3.4):

$$\int_{\alpha(\mu)}^{\alpha(q)} \frac{d\alpha}{\alpha\beta(\alpha, \varepsilon)} = \log \left(\frac{q^2}{\mu^2} \right) \quad (\text{D.4})$$

where:

$$\beta(\alpha, \varepsilon) = \frac{d \log \alpha}{d \log \mu^2} = -\varepsilon + \beta(\alpha), \quad \beta(\alpha) = -\frac{d \log Z_\alpha}{d \log \mu^2} = -\frac{\beta_0}{4\pi} \alpha(\mu) + \dots \quad (\text{D.5})$$

with β_0 given below in (D.8). The one-loop perturbative expression for the RG invariant coupling $\alpha(q)$ in $d = 4$ then reads:

$$\alpha(q) = \alpha(\mu) - \frac{\beta_0}{4\pi} \alpha^2(\mu) \log \left(\frac{q^2}{\mu^2} \right) + \mathcal{O}(\alpha^3) \quad (\text{D.6})$$

We write (D.1) in terms of the RG invariant coupling $\alpha(q)$ by inverting (D.6), and subsequently setting $q = (8t)^{-1/2}$ and $N = 3$ we obtain (6.81) and (6.82).

Equation (D.1) can be expressed in terms of the fundamental scale of the nonevolved theory Λ_{QCD} by inserting the 2-loop universal RG improved UV asymptotic expression for the coupling [8], see (3.29):

$$\alpha(q) = \frac{4\pi}{\beta_0 \log \left(\frac{q^2}{\Lambda_{\text{QCD}}^2} \right)} \left\{ 1 - \frac{\beta_1}{\beta_0^2} \frac{\log \log \left(\frac{q^2}{\Lambda_{\text{QCD}}^2} \right)}{\log \left(\frac{q^2}{\Lambda_{\text{QCD}}^2} \right)} \right\} + \mathcal{O} \left(\log \left(\frac{q^2}{\Lambda_{\text{QCD}}^2} \right)^{-3} \right) \quad (\text{D.7})$$

with:

$$\beta_0 = \frac{11}{3} N - \frac{2}{3} N_f, \quad \beta_1 = \frac{34}{3} N^2 - \frac{10}{3} N N_f - \frac{N^2 - 1}{N} N_f \quad (\text{D.8})$$

thus yielding:

$$\begin{aligned}
 \langle E \rangle_i &= \frac{3(N^2 - 1)}{8t^2} \frac{1}{\beta_0 \log \left(\frac{q^2}{\Lambda_{\text{QCD}}^2} \right)} \\
 &\times \left\{ 1 + \frac{1}{\beta_0 \log \left(\frac{q^2}{\Lambda_{\text{QCD}}^2} \right)} \left(4\pi k_i(q^2 t) - \frac{\beta_1}{\beta_0} \log \log \left(\frac{q^2}{\Lambda_{\text{QCD}}^2} \right) \right) \right\} \quad (\text{D.9}) \\
 &+ \mathcal{O} \left(\log \left(\frac{q^2}{\Lambda_{\text{QCD}}^2} \right)^{-3} \right)
 \end{aligned}$$

Again setting $N = 3$ and $q = (8t)^{-1/2}$ we obtain (6.83).

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