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Nonlinear Analysis of an Improved Swing Equation

Pooya Monshizadeh, Claudio De Persis, Nima Monshizadeh, and Arjan J van der Schaft

Abstract—In this paper, we investigate the properties of an improved swing equation model for synchronous generators. This model is derived by omitting the main simplifying assumption of the conventional swing equation. We carry out a nonlinear analysis for the stability and frequency regulation and provide region of attraction estimates for two scenarios. First we study the case that a synchronous generator is connected to a constant load. Second, we inspect the case of the single machine connected to an infinite bus. Finally, the different behaviors of the conventional and improved swing equations are depicted by simulations.

I. INTRODUCTION

Driven by environmental and technical motivations, restructuring the classical power networks has been under vast attention during the recent decades. Among the goals are decreasing energy losses by moving towards distributed generation and preventing fault propagation through building up smart *microgrids*. Microgrids are small power network areas which can be seen as single entities from the large power grids. In such small-scale networks, the energy consumption and production uncertainty increases to a great extent according to the fewer number of consumers and unpredictable power injections of renewables such as wind turbines and solar panels. Designing controllers for the electrical sources under such perturbations and abrupt power variations, calls for rethinking about the accuracy of the models that were (mostly) valid for the classical electrical systems. Despite the extensive advances in extracting energy from renewables, synchronous generators are still the main supplier of the implemented microgrids (see e.g. [1]). Under unpredictable changes in loads and renewable generations, and to assure of the stability and frequency regulation of the power grid, it appears crucial to investigate the accuracy of the electrical and dynamical models of these machines. While a large number of articles have exploited the classical *swing equation* as the model for the synchronous generator, a few have recently brought up doubts about its accuracy and validity, and proposed models of higher accuracy [2]–[8]. Although some promising models are provided [3], [6], the use of such models in power networks is rather complicated.

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In this paper, we provide a nonlinear analysis for an improved yet easy-to-use swing equation to fulfill higher accuracy in analysis and modeling of the synchronous generator that can also be exploited in small-scale networks. We first explicate the contradicting assumptions in obtaining the conventional swing equation and elaborate deriving the improved model. In Section III, as the first scenario, we look into the properties of the model when the generator is connected to a constant load and provide an estimate of the region of attraction through nonlinear analysis. Further in Section IV, the second scenario which is mainly referred to as *Single Machine Infinite Bus* (SMIB) is investigated. We provide analytical estimates of the region of attraction for both conventional and improved swing models in this case. Finally, simulations are provided as a verification of the results.

II. CONVENTIONAL AND IMPROVED SWING EQUATIONS

The mechanical dynamics of the synchronous generator reads as

$$J\dot{\omega} + D_d(\omega - \omega^*) = \tau_m - \tau_e, \quad (1)$$

where $J \in \mathbb{R}^+$ is the total moment of inertia of the turbine and generator rotor ($kg m^2$), $\omega \in \mathbb{R}^+$ is the rotor shaft velocity (mechanical rad/s), $\omega^* \in \mathbb{R}^+$ is the angular velocity associated with the nominal frequency ($(2\pi)60Hz$), $\tau_m \in \mathbb{R}^+$ is the net mechanical shaft torque (Nm), $\tau_e \in \mathbb{R}^+$ is the counteracting electromagnetic torque (Nm), and $D_d \in \mathbb{R}^+$ is the damping-torque coefficient (Nms)¹. The mechanical rotational loss due to friction is ignored. Bearing in mind that $\tau_m = \frac{P_m}{\omega}$ and $\tau_e = \frac{P_e}{\omega}$ we can model the synchronous generator as

$$J\omega\dot{\omega} + D_d\omega(\omega - \omega^*) = P_m - P_e, \quad (2)$$

where P_m and P_e are the mechanical (physical input) and electrical (physical output) power respectively.

In the conventional swing equation, $J\omega$ and $D_d\omega$ are approximated with constants $M = J\omega^*$ and $A = D_d\omega^*$.

$$M\dot{\omega} + A(\omega - \omega^*) = P_m - P_e, \quad (3)$$

where $M \in \mathbb{R}$ is the angular momentum associated with the nominal frequency, and $A \in \mathbb{R}$ is the (new) damping coefficient. In other words, the swing equation is derived supposing that $\omega = \omega^*$. Such an assumption is in contradiction with the proof of stability and frequency regulation of the synchronous generator. Therefore we adhere to the equation

¹Despite the general misuse, D_d accounts for the damping torque generated by the damper windings (amortisseur) only when the generator is connected to an infinite bus [9]. In the case of a single generator connected to a load, the damping term refers to a proportional torque governor.

(2) and endeavor to investigate the stability and frequency regulation through a nonlinear approach. This model has been first suggested by [2], where however, the stability of the synchronous generator (connected to an infinite bus) is analyzed with the small-signal (linearization) analysis. Consistently with [2], we refer to (2) as the *improved swing equation*.

III. SYNCHRONOUS GENERATOR CONNECTED TO A CONSTANT LOAD

In this scenario, we assume that both the injected and extracted power (P_m and P_e) are constant.

A. Stability

Let $\bar{\omega} \in \mathbb{R}$ be an equilibrium of (2). Then, clearly

$$D_d \bar{\omega}(\bar{\omega} - \omega^*) = P_m - P_e. \quad (4)$$

The equation above admits two real solutions only if

$$P_m - P_e > -\frac{1}{4} D_d \omega^{*2}, \quad (5)$$

which will be a standing assumption in this section. Under this assumption, we obtain the following two equilibria for system (2):

$$\bar{\omega}_s = \frac{\omega^* + \sqrt{\Delta}}{2}, \quad \bar{\omega}_u = \frac{\omega^* - \sqrt{\Delta}}{2}, \quad (6)$$

where

$$\Delta := \omega^{*2} + 4 \frac{P_m - P_e}{D_d}. \quad (7)$$

Note that $\Delta > 0$ by (5). Observe that

$$\bar{\omega}_s > \bar{\omega}_u, \quad (8)$$

$$\bar{\omega}_u = \omega^* - \bar{\omega}_s. \quad (9)$$

First, we show that $\bar{\omega}_s$ is locally stable and $\bar{\omega}_u$ is locally unstable. For the moment, we assume that the set $\mathbb{R}^+$ is positive invariant for the system (2), and $\omega(0) > 0$. We will relax this assumption later, once a Lyapunov argument is provided. This means that ω^{-1} is well defined for the interval of the definition of the solutions. Now, let dynamics (2) be rewritten as $\dot{\omega} = f(\omega)$, where

$$f(\omega) = \frac{1}{J} \left(-D_d(\omega - \omega^*) + \frac{P_m - P_e}{\omega} \right).$$

We have

$$\frac{\partial f}{\partial \omega} = \frac{1}{J} \left(-D_d - \frac{P_m - P_e}{\omega^2} \right).$$

By using (4) and (9), we find that

$$\frac{\partial f}{\partial \omega} = \frac{D_d}{J} \left(\frac{\bar{\omega}_s \bar{\omega}_u}{\omega^2} - 1 \right). \quad (10)$$

Bearing in mind inequality (8), it is easy to check that $\frac{\partial f}{\partial \omega} > 0$ around $\omega = \bar{\omega}_u$, and $\frac{\partial f}{\partial \omega} < 0$ around $\omega = \bar{\omega}_s$. Hence $\omega = \bar{\omega}_u$ is repulsive. Now, the following theorem addresses the stability of the equilibrium $\bar{\omega}_s$ in (6).

Fig. 1. Region of attraction of system (2) for $\omega = \bar{\omega}_s$ is estimated by Ω_s .

Theorem 1 Consider the candidate Lyapunov function

$$V(\omega) = \frac{1}{2} J(\omega - \bar{\omega}_s)^2$$

with $\bar{\omega}_s$ given by (6). Let Δ be given by (7) and assume that (5) holds. Then the solutions of the system (2) starting from any initial condition in the set $\Omega_s = \{\omega \in \mathbb{R}^+ : V(\omega) \leq \frac{1}{2} J \Delta\}$ except for $\omega(0) = \bar{\omega}_u$ converge asymptotically to the equilibrium $\omega = \bar{\omega}_s$.

Proof: We have

$$\dot{V} = \omega^{-1}(\omega - \bar{\omega}_s) \left(P_m - P_e - D_d \omega(\omega - \omega^*) \right).$$

The fact that ω stays away from zero will be made clear later. According to (4) this leads to

$$\begin{aligned} \dot{V} &= \omega^{-1}(\omega - \bar{\omega}_s) \left(D_d \bar{\omega}_s(\bar{\omega}_s - \omega^*) - D_d \omega(\omega - \omega^*) \right) \\ &= -\omega^{-1} D_d(\omega - \bar{\omega}_s) \left(\omega^2 - \omega \omega^* + \bar{\omega}_s \omega^* - \bar{\omega}_s^2 \right) \\ &= -D_d \omega^{-1}(\omega - \bar{\omega}_s) \left((\omega - \bar{\omega}_s)(\omega + \bar{\omega}_s - \omega^*) \right) \\ &= -D_d(\omega - \bar{\omega}_s)^2 \left(1 - \frac{\bar{\omega}_u}{\omega} \right), \end{aligned}$$

where we have used $\bar{\omega}_u = \omega^* - \bar{\omega}_s$. Hence $\dot{V}$ is negative semi-definite on the set $\Omega'_s = \{\omega \in \mathbb{R}^+ : \omega \geq \bar{\omega}_u\}$. Below, we show that $\Omega_s \subseteq \Omega'_s$.

For any $\omega \in \Omega_s$ we have

$$\frac{1}{2} J(\omega - \bar{\omega}_s)^2 \leq \frac{1}{2} J(\omega^{*2} + 4 \frac{P_m - P_e}{D_d})$$

which yields

$$(\omega - \bar{\omega}_s)^2 \leq (2\bar{\omega}_s - \omega^*)^2$$

by using (6). By (5) we have $2\bar{\omega}_s - \omega^* \geq 0$. Therefore

$$\omega^* - 2\bar{\omega}_s \leq \omega - \bar{\omega}_s \leq 2\bar{\omega}_s - \omega^*$$

and hence

$$\bar{\omega}_u \leq \omega \leq 2\bar{\omega}_s - \bar{\omega}_u. \quad (11)$$

The left hand side of (11) gives $\Omega_s \subseteq \Omega'_s$.

Now we show that ω remains in the positive half-line $\mathbb{R}^+$. By (6) and (7), in the case $P_m < P_e$, we have $\bar{\omega}_u > 0$ and together with (11) we obtain $\omega > 0$. Therefore in this case,

the point $\omega = 0$ does not belong to the interval specified by (11). In cases $P_m > P_e$ or $P_m = P_e$, the point $\omega = 0$ lies within the boundaries (11). Figure 1 illustrates these situations. Since $(\omega - \bar{\omega}_s)^2 = 2JV(\omega)$, and $V(\omega)$ is strictly decreasing in Ω_s , $(\omega - \bar{\omega}_s)^2$ is decreasing as well, showing that ω stays away from $\omega = 0$ as time goes by. This proves that $\omega(t) > 0, \forall t > 0$.

Notice that Ω_s is closed and bounded by (11). By invoking LaSalle's invariance principle, the solutions of the system starting in Ω_s asymptotically converge to the set of points where

$$-D_d(\omega - \bar{\omega}_s)^2(1 - \frac{\bar{\omega}_u}{\omega}) = 0. \quad (12)$$

Equality (12) results in $\omega = \bar{\omega}_s$, which completes the proof. ■

Remark 1 Note that the Lyapunov function $V(\omega) = \frac{1}{2}J(\omega - \bar{\omega}_s)^2$ is a shifted version of the actual kinetic energy of the physical system, whereas the commonly used Lyapunov function for the swing equation, $\frac{1}{2}M(\omega - \bar{\omega}_s)^2 = \frac{1}{2}J\omega^*(\omega - \bar{\omega}_s)^2$, does not have a physical interpretation.

B. Considering Mechanical Losses

We add the viscous damping coefficient D_m accounting for the mechanical losses ($N\ m\ s$), modifying the dynamics (1) as

$$J\dot{\omega} + D_m\omega + D_d(\omega - \omega^*) = \tau_m - \tau_e, \quad (13)$$

and similarly (2) is modified as

$$J\omega\dot{\omega} + D_m\omega^2 + D_d\omega(\omega - \omega^*) = P_m - P_e, \quad (14)$$

which can be rewritten as

$$J\omega\dot{\omega} + (D_m + D_d)\omega(\omega - \frac{D_d}{D_m + D_d}\omega^*) = P_m - P_e. \quad (15)$$

Defining $D := D_m + D_d$ and $\tilde{\omega}^* := \frac{D_d}{D_m + D_d}\omega^*$ we obtain

$$J\omega\dot{\omega} + D\omega(\omega - \tilde{\omega}^*) = P_m - P_e, \quad (16)$$

which is analogous to the improved swing equation (2). Therefore the stability results extend to the case with mechanical losses.

C. Incremental Passivity Property

To facilitate the control design, in this section we investigate the incremental passivity of the system associated to (2) (see Remark 2 later on)

$$\begin{aligned} J\omega\dot{\omega} + D_d\omega(\omega - \omega^*) &= u + P_m - P_e \\ y &= \frac{\omega - \omega^*}{\omega}. \end{aligned} \quad (17)$$

Let the triple $(\bar{u}, \bar{\omega}, \bar{y})$ be an input-state-output solution of (17) with

$$\begin{aligned} D_d\bar{\omega}(\bar{\omega} - \omega^*) &= \bar{u} + P_m - P_e \\ \bar{y} &= \frac{\bar{\omega} - \omega^*}{\bar{\omega}}. \end{aligned} \quad (18)$$

Assume that

$$\bar{u} + P_m - P_e > -\frac{1}{4}D_d\omega^{*2}. \quad (19)$$

Then the dynamics (17) possesses the equilibria

$$\bar{\omega}_s = \frac{\omega^* + \sqrt{\Delta}}{2}, \quad \bar{\omega}_u = \frac{\omega^* - \sqrt{\Delta}}{2} \quad (20)$$

where with a little abuse of the notation

$$\Delta := \omega^{*2} + 4\frac{\bar{u} + P_m - P_e}{D_d}. \quad (21)$$

Now, we have the following proposition:

Proposition 1 Consider the candidate Lyapunov function

$$W(\omega) = \frac{1}{2}J(\omega - \bar{\omega}_s)^2\frac{\omega^*}{\bar{\omega}_s}$$

and assume that (19) holds. Then, $W(\omega)$ computed along any solution to (17) satisfies

$$\dot{W} < (y - \bar{y})(u - \bar{u}), \quad (22)$$

as long as the solution stays within the set $\Omega_k = \{\omega \in \mathbb{R}^+ : W(\omega) \leq \frac{1}{2}J\Delta\frac{\omega^*}{\bar{\omega}_s}\}$. This amounts to an incremental passivity property with respect to $(\bar{u}, \bar{\omega}, \bar{y})$.

Proof: We have

$$\begin{aligned} \dot{W} &= \left(\frac{1}{\omega}\right)\left(\frac{\omega^*}{\bar{\omega}_s}\right)(\omega - \bar{\omega}_s)\left(u + P_m - P_e - D_d\omega(\omega - \omega^*)\right. \\ &\quad \left.- \bar{u} - P_m + P_e + D_d\bar{\omega}_s(\bar{\omega}_s - \omega^*)\right) \\ &= \left(\frac{\omega - \omega^*}{\omega} - \frac{\bar{\omega}_s - \omega^*}{\bar{\omega}_s}\right)(u - \bar{u}) \\ &\quad - D_d\left(\frac{\omega^*(\omega - \bar{\omega}_s)}{\omega\bar{\omega}_s}\right)\left((\omega - \bar{\omega}_s)(\omega + \bar{\omega}_s - \omega^*)\right) \\ &= (y - \bar{y})(u - \bar{u}) \\ &\quad - D_d(\omega - \bar{\omega}_s)^2\left(1 - \frac{\bar{\omega}_u}{\omega}\right)\left(\frac{\omega^*}{\bar{\omega}_s}\right). \end{aligned} \quad (23)$$

Hence the inequality (22) holds as long as the solutions evolve in the set $\Omega'_k = \{\omega \in \mathbb{R}^+ : \omega \geq \bar{\omega}_u\}$. The fact that $\Omega_k \subseteq \Omega'_k$ follows analogously to the proof of Theorem 1. ■

Remark 2 Defining $y = \omega^{-1}(\omega - \omega^*)$ as in (17), results in a dimensionless output. Therefore, yu has the dimension of physical power, which is consistent with the energy dissipation inequality (22). This does not hold for the conventional swing equation where ω is taken as the output and thus yu does not have a meaningful physical dimension.

D. Frequency Regulation

In this section we investigate the frequency regulation of the system by an integral controller. Motivated by Proposition 1, we propose the controller for system (17) as

$$\begin{aligned} \dot{\xi} &= y = \frac{\omega - \omega^*}{\omega} \\ u &= -\xi. \end{aligned} \quad (24)$$

Note that for the purpose of frequency regulation, the solution (equilibrium) of interest for the system (17), (24), is given by

$$\bar{u} = -\bar{\xi} = -(P_m - P_e), \quad \bar{\omega} = \omega^*, \quad \bar{y} = 0, \quad (25)$$

which trivially satisfies the inequality (19). In addition, for the solution above, the quantities in (20) and (21) are computed as $\bar{\omega}_s = \omega^*$, $\bar{\omega}_u = 0$, $\Delta = \omega^{*2}$. Therefore, the dissipation equality (23) in this case reduces to

$$\dot{W} = (y - \bar{y})(u - \bar{u}) - D_d(\omega - \bar{\omega}_s)^2, \quad (26)$$

as long as the solutions stay within the set $\Omega_k = \{\omega \in \mathbb{R}^+ : W(\omega) \leq \frac{1}{2}J\omega^{*2}\}$ which confines ω within the interval $\omega \in [0, 2\omega^*]$. Now, the following theorem establishes the convergence of the solutions to the desired equilibrium associated with the nominal frequency regulation.

Theorem 2 Consider the candidate Lyapunov function

$$U(\xi, \omega) = W_c(\xi) + W(\omega)$$

where $W_c = \frac{1}{2}(\xi - \bar{\xi})^2$ and $W = \frac{1}{2}J(\omega - \omega^*)^2$. Then the solutions of the closed loop system (17), (24), starting from any initial condition in the set $\mathcal{O} = \{(\xi, \omega) \in \mathbb{R}^2 : U \leq \frac{1}{2}J\omega^{*2}\}$ and different from $(\xi, \omega) = (P_m - P_e, 0)$ converge asymptotically to the equilibrium $(\xi, \omega) = (P_m - P_e, \omega^*)$.

Proof: First note that $\dot{W}_c = \omega^{-1}(\omega - \omega^*)(\xi - \bar{\xi})$. Hence, by (26) we find that

$$\dot{U} = -D_d(\omega - \omega^*)^2. \quad (27)$$

as long as the solutions belong to the set $\mathcal{O}$. As the right hand side of the above equality is nonpositive, we conclude that the set $\mathcal{O}$ is forward invariant along the solution (ξ, ω) , and thus (27) is valid for all time. Now, observe that U is radially unbounded and has a strict minimum at $(\xi, \omega) = (P_m - P_e, \omega^*)$. Then by invoking LaSalle's invariance principle, the solutions converge to the largest invariant subset of $\mathcal{O}$ in which $\omega = \omega^*$ and $\bar{u} = P_e - P_m$. The latter shows that all the solutions on the invariant set $\mathcal{O}$, and not starting from $(\xi, \omega) = (P_m - P_e, 0)$ converge asymptotically to the equilibrium $(\xi, \omega) = (P_m - P_e, \omega^*)$. Note that the solutions stay away from the point $(\xi, \omega) = (P_m - P_e, 0)$, since as $(\xi, \omega) \rightarrow (P_m - P_e, 0^+)$, we have $\dot{\omega} \rightarrow +\frac{D_d}{J}\omega^*$. We can rewrite the set $\mathcal{O}$ as

$$\left(\frac{\omega - \omega^*}{\omega^*}\right)^2 + \left(\frac{\xi - \bar{\xi}}{\sqrt{J}\omega^*}\right)^2 \leq 1. \quad (28)$$

This set guarantees ω to remain in the positive half-line $\mathbb{R}^+$ for ω (see Figure 2). ■

IV. SYNCHRONOUS GENERATOR CONNECTED TO AN INFINITE BUS

In this scenario we consider the case of SMIB, where the synchronous generator is connected to an infinite bus through an inductive link. The infinite bus is a node with fixed voltage and frequency (60Hz).

Fig. 2. Region of attraction for the system (17), (24) is estimated with the oval characterized by (28).

A. Stability of the Improved Swing Model

It can be shown that the power delivered by the synchronous generator to the bus is $\gamma \sin(\delta)$ where δ is the voltage angle relative to the infinite bus, and $\gamma := \frac{V_g V_b}{X}$. Here, the voltages (V_g, V_b) and the reactance of the line (X) are assumed to be constant. The dynamics of an SMIB system modeled with the improved swing equation (2) is

$$\begin{aligned} \dot{\delta} &= \omega - \omega^* \\ J\omega\dot{\omega} + D_d\omega(\omega - \omega^*) &= P_m - \gamma \sin(\delta), \end{aligned} \quad (29)$$

where the mechanical input $P_m > 0$ is considered constant. It is easy to check that $(\delta, \omega) = (\arcsin \frac{P_m}{\gamma}, \omega^*)$ is the equilibrium of the system (29).

Theorem 3 Consider the candidate Lyapunov function

$$V(\delta, \omega) = V_k(\omega) + V_p(\delta),$$

where

$$V_k(\omega) = \frac{1}{2}J(\omega - \omega^*)^2$$

and

$$V_p(\delta) = \frac{\gamma}{\omega^*}[-\cos \delta + \cos \bar{\delta} - (\delta - \bar{\delta}) \sin \bar{\delta}]$$

are associated with the kinetic and potential energy and $\bar{\delta} = \arcsin \frac{P_m}{\gamma}$. Assume that

$$\omega^* > \sqrt{\frac{\gamma}{D_d}} \quad (30)$$

and

$$\frac{P_m}{\gamma} < \frac{2}{\pi}. \quad (31)$$

Let $c := \min(c_k, c_p)$, where $c_k := \frac{1}{2}J(\omega^* - \frac{\gamma}{D_d\omega^*})^2$ and $c_p := V_p(\frac{\pi}{2})$. Then the solutions of the system (29) starting from any initial condition in the set $\Omega = \{(\delta, \omega) \in [-\pi, \pi] \times \mathbb{R} : V(\delta, \omega) \leq c\}$ converge asymptotically to the equilibrium $(\delta, \omega) = (\arcsin \frac{P_m}{\gamma}, \omega^*)$.

Proof: Observe that $V_p(\delta)$ has a minimum at $\bar{\delta}$ and is convex within the set $\Omega_p = \{\delta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$. We here show that $\Omega \subset \Omega_p$. Having $V_k(\omega) > 0$, the inequality $V(\delta, \omega) < c$ results in $V_p(\delta) < c_p$ and reads as

$$\cos \delta > \left(\frac{\pi}{2} - \delta\right) \sin \bar{\delta}. \quad (32)$$

The inequality (32) contains a unique subset of Ω_p . More precisely, under the criterion $0 < \bar{\delta} < \arcsin \frac{2}{\pi}$ as a result of (31), there exists $\delta^- \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that (32) holds for all $\delta \in [\delta^-, \frac{\pi}{2}]$. Figure 3 shows an example interval for δ that satisfies (32) and consequently the energy function V_P remains convex ($\Omega \subset \Omega_p$).

Now, it remains to prove that $\dot{V} < 0$. We have

$$\begin{aligned}\dot{V} &= -D_d(\omega - \omega^*)^2 - \frac{\gamma}{\omega}[(\omega - \omega^*)(\sin \delta - \sin \bar{\delta})] \\ &\quad + \frac{\gamma}{\omega^*}[(\omega - \omega^*)(\sin \delta - \sin \bar{\delta})] \\ &= -(\omega - \omega^*)^2 \left(D_d - \frac{\gamma}{\omega \omega^*}(\sin \delta - \sin \bar{\delta}) \right).\end{aligned}$$

Note that $\sin \delta - \sin \bar{\delta} < 1$. Hence $\dot{V} < 0$ on the set $\Omega_k = \{\omega \in \mathbb{R} : \omega > \frac{\gamma}{D_d \omega^*}\}$. Here we show that $\Omega \subset \Omega_k$. In the set Ω we have $V(\delta, \omega) < c$ which leads to $V_k(\omega) < c_k$. Therefore

$$(\omega - \omega^*)^2 < (\omega^* - \frac{\gamma}{D_d \omega^*})^2,$$

and since $\omega^* > \sqrt{\frac{\gamma}{D_d}}$,

$$\frac{\gamma}{D_d \omega^*} < \omega < 2\omega^* - \frac{\gamma}{D_d \omega^*}. \quad (33)$$

The left hand side of the inequality (33) shows that $\Omega \subset \Omega_k$. Observe that U has a strict minimum at $\omega = \omega^*$ and $\delta = \bar{\delta}$, and the solutions are bounded to $\frac{\gamma}{D_d \omega^*} < \omega < 2\omega^* - \frac{\gamma}{D_d \omega^*}$ and $\delta^- < \delta < \frac{\pi}{2}$. By invoking LaSalle's invariance principle, the solutions converge to the largest invariant subset of Ω for (29) s.t. $\omega = \omega^*$. On this set, the solution to (29) satisfy $\delta = \arcsin \frac{P_m}{\gamma}$. This shows that all the solutions on the invariant set converge asymptotically to the equilibrium $(\arcsin \frac{P_m}{\gamma}, \omega^*)$. This completes the proof. ■

Remark 3 The assumption $\frac{P_m}{\gamma} < \frac{2}{\pi}$ results in $\bar{\delta} < \arcsin \frac{2}{\pi}$ which means that at steady state the angle between the synchronous generator and the infinite bus should be less than about 40° . In practice, the steady state angle is much lower even for large machines (see e.g. [10]). Note that this assumption is made merely for characterizing the region of attraction, otherwise it is not necessary for the proof of local stability since V is convex around the equilibrium ($\frac{\partial^2 V}{\partial \delta^2}|_{\delta=\bar{\delta}} = \cos \bar{\delta} > 0$).

B. Comparison with the Swing Equation

Here we compare the results of the improved model with to swing equation. The dynamics of a synchronous generator modeled with the swing equation and connected to an infinite bus is

$$\begin{aligned}\dot{\delta} &= \omega - \omega^* \\ M\dot{\omega} + A(\omega - \omega^*) &= P_m - \gamma \sin(\delta),\end{aligned} \quad (34)$$

where the mechanical input $P_m > 0$ is considered constant. It is straightforward to see that the equilibrium of the system (34) is $(\delta, \omega) = (\arcsin \frac{P_m}{\gamma}, \omega^*)$.

Fig. 3. The estimate of the system (29) region of attraction ($V_P(\delta) < c_p$) for $\bar{\delta} = \arcsin 0.1$ confines δ in $[0.4\pi - 0.5\pi]$ (green). On such an interval $V_P(\delta)$ remains positive and convex.

Corollary 1 Consider the candidate Lyapunov function $V(\delta, \omega) = V_k(\omega) + V_p(\delta)$, where

$$V_k(\omega) = \frac{1}{2}M(\omega - \omega^*)^2$$

and

$$V_p(\delta) = \gamma[-\cos \delta + \cos \bar{\delta} - (\delta - \bar{\delta}) \sin \bar{\delta}].$$

Assume that $\frac{P_m}{\gamma} < \frac{2}{\pi}$ and let $c := V_p(\frac{\pi}{2})$. Then the solutions of the SMIB system described by the swing equation (34) starting from any initial condition in the set $\Omega = \{(\delta, \omega) \in [-\pi, \pi] \times \mathbb{R} : V(\delta, \omega) \leq c\}$ converge asymptotically to the equilibrium $(\delta, \omega) = (\arcsin \frac{P_m}{\gamma}, \omega^*)$.

Proof: The proof is similar to the proof of the Theorem 3 and therefore omitted. Note that here, $\dot{V} = -A(\omega - \omega^*)^2$ and hence to prove stability, the condition $V < V_p(\frac{\pi}{2})$ suffices. ■

Corollary 1 and Theorem 3 characterize similar estimates for region of attraction of both models (improved swing and the conventional one) if $c_k \geq c_p$ (and as a result $c = c_p$).

V. SIMULATION

Here we provide examples depicting the mismatch between the behavior suggested by the swing equation and the improved version. In all simulations, the parameters are set as follows: $M = 0.2$, $A = 0.04$, $\omega^* = (2\pi)60$, and $\gamma = 2$. Note that $J = \frac{M}{\omega^*}$ and $D_d = \frac{A}{\omega^*}$.

A. Constant Load: Example 1

As described in Section III, the steady state value of the frequency differs for the swing equation and the improved swing model under similar constant loads. Figure 4(a) illustrates this issue. In this example, we set $P_m = 1$ pu and $P_e = 2$ pu, and start from the initial condition $f(0) = 60$ Hz. The steady state value of the frequency is 56.02Hz for the swing equation and 55.72Hz for the improved swing model.

B. Constant Load: Example 2

An estimate of the region of attraction for the improved swing equation is provided in Section III (Ω_s in Theorem 1). Figure 4(b) illustrates a solution that initiates just out of the domain and becomes unstable. However, the conventional swing equation is falsely depicting that the system remains stable. Here, we set $P_m = 1$ pu and $P_e = 4.65$ pu, and start from $f(0) = 24$ Hz. According to (11), the estimate of region of attraction allows for $f > 25$ Hz.

Fig. 4. Simulation results for the examples of Section V(A-D).

C. Constant Load: Example 3

According to the analysis in Section III, condition (5) should hold for the improved swing equation, so that there exist a steady state frequency value. Figure 4(c) shows that the system becomes unstable if the inequality (5) is violated. Note that the stability of the swing equation does not necessitate such a condition. In this example we adjust $P_m = 1\text{pu}$ and $P_e = 4.90\text{pu}$ such that Δ as defined in (7) possesses a negative value.

D. SMIB: Different Behavior

The behaviors of both systems, conventional and improved, are similar when connected to an infinite bus. However still with some specific initial conditions, the systems act quite differently. Figure 4(d) illustrates an example of this different behavior. Note that here, the initial condition is outside the region of attraction estimate for both models.

E. SMIB: Region of Attraction

Figure 5 illustrates the phase portrait of the system (29) and the Lyapunov function $V(\delta, \omega)$ level sets. It is verified that our estimate of the domain of attraction is not very conservative. All solutions (dotted green) within the Lyapunov level set $\{V(\delta, \omega) = c\}$ (dark blue) converge to the equilibrium $(\delta, \omega) = (\frac{\pi}{6}, 2\pi \times 60)$. Here, $c_p < c_k$, and hence $c = c_p = V_p(\frac{\pi}{2})$. Note that there are solutions outside the estimate of region of attraction that still converge, however the solutions further away from the estimate of domain of attraction diverge from the equilibrium.

VI. CONCLUSION AND FUTURE WORK

We have investigated the properties of an improved swing equation. Modeling the synchronous generator by this equation, two scenarios are considered in this paper. First, the

Fig. 5. The estimate of the domain of attraction for system (29) (SIMB).

stability of a single generator connected to a constant load is shown and frequency regulation is achieved through the proposed integral controller. In the second scenario, the synchronous machine is connected to an infinite bus. As a contribution with respect to [2], where similar dynamics are investigated through linearization, here a nonlinear Lyapunov analysis is provided to prove stability and frequency regulation. Finally, simulations are carried out to show that the swing equation model gives rise to a behavior that does not match with the one suggested by the improved swing equation. Future works include considering the analysis of voltage dynamics and multi-machine systems by exploiting incremental energy functions [11].

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