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Chapter 10

Strong Structural Controllability of Networks

Nima Monshizadeh

Abstract In this chapter, we discuss strong structural controllability and strong targeted controllability of networks from a unified viewpoint. The problem of strong structural controllability accounts for controllability of the whole family of matrices carrying the structure of an underlying graph. By looking into a certain infection process identified by a coloring rule, topological characterizations for strong structural properties of the network is provided. In particular, the strong structurally reachable subspace is translated into the derived set of a given leader set. Moreover, the set of leaders rendering the network strongly structurally controllable are characterized by zero forcing sets. Then, the minimum number of leaders to achieve strong structural controllability is given by the zero forcing number of the graph. Motivated by the fact that network controllability is neither always feasible nor necessary, we discuss the problem of (strong) targeted controllability where controllability is only required for a subset of the nodes in the network. We illustrate graph theoretic sufficient conditions guaranteeing strong targeted controllability of the network.

10.1 Preliminaries

For a given simple directed graph G , the vertex set of G is a nonempty set and is denoted by V . The arc set of G , denoted by E , is a subset of $V \times V$, and $(i, i) \notin E$ for all $i \in V$. The cardinality of a given set V is denoted by $|V|$. Also we sometimes use $|G|$ to denote in short the cardinality of V . We call vertex j an out-neighbor of vertex i if $(i, j) \in E$.

For $V = \{1, 2, \dots, n\}$ and $V' = \{v_1, v_2, \dots, v_r\} \subseteq V$, we define the $n \times r$ matrix $P(V; V') = [P_{ij}]$ by:

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$$P_{ij} = \begin{cases} 1 & \text{if } i = v_j \\ 0 & \text{otherwise.} \end{cases} \quad (10.1)$$

10.2 Problem Formulation

We consider the following dynamics evolving on a directed graph $G = (V, E)$:

$$\dot{x} = Xx + Uu \quad (10.2)$$

where $x \in \mathbb{R}^{|G|}$ is the state, $u \in \mathbb{R}^m$ is the input, and $U = P(V; V_L)$ for some given leader set $V_L \subseteq V$. Note that the matrix X represents the *coupling* in accordance with G , and reveals how information is exchanged throughout the network.

Recall that controllability is the ability of an external input to steer a system from any initial state to any other final state in a finite time. Here, we are primarily interested in “strong structural controllability” of systems of the form (10.2). Roughly speaking, by “structural” we refer to properties which are identified by the graph G rather than a particular realization of the system (10.2). To formalize this, we define the *qualitative class* of G as the family of matrices compatible with the structure of G :

$$\mathcal{Q}(G) = \{X \in \mathbb{R}^{|G| \times |G|} : \text{for } i \neq j, X_{ij} \neq 0 \Leftrightarrow (j, i) \in E\}$$

Then, for a given leader set V_L , we call the system (10.2) *strongly structurally controllable* if the pair (X, U) is controllable for all $X \in \mathcal{Q}(G)$. In that case, we write $G; V_L$ is controllable. Note that the term “strong” is used to distinguish with the case of “weak structural controllability” which amounts to the existence of a controllable pair (X, U) , $X \in \mathcal{Q}(G)$.

Another problem of interest is “minimal leader selection” the goal of which is to choose a leader set V_L with minimum cardinality such that the pair (X, U) is controllable. A strong structural variation of this problem is to select a leader set V_L with minimum cardinality such that (X, U) is controllable for all $X \in \mathcal{Q}(G)$. We denote the cardinality of such minimal leader set by $\ell_{\min}(G)$, i.e.,

$$\ell_{\min}(G) = \min_{V_L \subseteq V(G)} \{|V_L| : (G; V_L) \text{ is controllable}\}. \quad (10.3)$$

For simplicity, we use calligraphic notation in this chapter to denote the image of a matrix induced by a subset $V' \subseteq V$. More precisely, $\mathcal{V}'$ denotes, in short, the subspace $\text{im } P(V; V')$.

In the next section, we briefly review the state of the art in controllability as well as structural controllability of systems of the form (10.2).

10.3 Review: Controllability of Networks

One main line of research within the context of controllability of networks has been devoted to the case where $X = L$ in (10.2) with L being the Laplacian matrix of an undirected graph G . This line of research has been initiated by [15] and further developed by [14]. Motivated by the fact that algebraic conditions do not provide much insight into network controllability, topological translation of controllability properties in terms of certain graph partitions has been pursued by some authors [8, 17]. These partitions in fact provide a partial characterization of the reachable subspace associated with the pair (L, U) . An extension to the controlled invariant subspace has also been reported in [10]. Characterization of controllability of the pair (L, U) in terms of graph automorphisms is provided in [1].

In addition, a (minimum) leader selection for rendering the pair (L, U) controllable has been investigated for particular classes of undirected graphs [12, 13, 17].

Another instance of systems that has been studied in the context of controllability corresponds to the case where $X = A$ in (10.2) with A being the adjacency matrix of an undirected graph, see, e.g., [4].

In complex networks, the weights of the communications are typically unknown or partially known. Hence, it is interesting to investigate a controllability property which depends on the structure of the graph/network rather than a particular realization. In fact, this property, known as structural controllability deals with a family of pairs (X, U) rather than a particular instance and asks whether the family contains a controllable pair (weak structural controllability [7]) or all members of the family are controllable (strong structural controllability [11]). The latter is also the main focus of the present chapter. For a more general look at controllability of structured systems see, e.g., [2, 6, 9].

10.4 Strong Structural Controllability

First, we recap the notion of the reachability subspace in time-invariant linear systems.

10.4.1 Reachability Subspace

Consider, the system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (10.4)$$

with state space $\mathbb{R}^n$. For a given initial state x_0 and input function u , we denote the resulting state trajectory of the system by $x_u(t, x_0)$. The smallest A -invariant subspace containing the image of the input matrix U is denoted by $\langle A \mid \text{im } B \rangle$. This

subspace, called the *reachable subspace*, consists of all points in the state space that can be reached from the origin in finite time by choosing an appropriate input function, i.e., all points $x_1 \in \mathbb{R}^n$ for which there exists $T > 0$ and u such that $x_1 = x_u(T, 0)$. It is well known that the system is controllable if and only if the reachable subspace $\langle A \mid \text{im } B \rangle$ is equal to the entire state space $\mathbb{R}^n$. In turn, this is equivalent to the condition

$$\text{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

10.4.2 Strong Structurally Reachable Subspace

Recall that we are interested in structural controllability properties of systems of the form (10.2). To incorporate these structural properties, we consider all the points in the state space which can be reached by applying appropriate input signals to the nodes in the leader set V_L , irrespective of the choice of $X \in Q(G)$. These points constitute a subspace which we call *strong structurally reachable subspace*. By definition, this subspace is equal to

$$\bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle.$$

Moreover, it is easy to observe that the strong structurally reachable subspace provides a geometric characterization for strong structural controllability, i.e.,

$$(G; V_L) \text{ is controllable} \Leftrightarrow \bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle = \mathbb{R}^{|G|} \quad (10.5)$$

Note that the geometric characterization (10.5) by itself does not provide much insight to the strong structural controllability property. In particular, we would like to “visualize” a network enjoying this property in comparison with the one which lacks such a property. Ultimately, we would like to draw some conclusions on possible minimal leader selections guaranteeing strong structural controllability. To this end, we use the notion of zero forcing sets.

10.4.3 Zero Forcing Sets

In this section, we recap the notion of zero forcing sets together with the terminology that will be used later in this chapter. For more details see, e.g., [5].

Let G be a graph, and suppose that each vertex is colored either white or black. Consider the following coloring rule:

●: If u is a black vertex and exactly one out-neighbor v of u is white, then change the color of v to black.

The following terminology will be used when we apply the color-change rule above to a graph G :

- If the color-change rule is applied to $u \in V$ to change the color of $v \in V$, we say u *forces* or *infects* v , and write $u \rightarrow v$.
- Given a coloring set $C \subseteq V$, i.e., C indexes the initially black vertices of G , the *derived set* of C is denoted by $D(C)$, and is the set of black vertices obtained by applying the color-change rule until no more changes are possible.
- The set $Z(G) \subseteq V$ is a *zero forcing set* (ZFS) for G if $D(Z(G)) = V$.
- The *zero forcing number* $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z(G) \subseteq V$.

Figure 10.1 illustrates the coloring rule, where vertices 1, 2, and 5 are initially colored black. As vertex 1 has only one white out-neighbor, $1 \rightarrow 3$. Similarly, $5 \rightarrow 7$. Consequently, $2 \rightarrow 4$, and then $3 \rightarrow 9$. Therefore, the derived set of $\{1, 2, 5\}$ is equal to $\{1, 2, 3, 4, 5, 7, 9\}$, and clearly $\{1, 2, 5\}$ is not a zero forcing set. It is easy to observe that the set $\{1, 2, 5, 6\}$ constitutes a minimal zero forcing set, and thus the zero forcing number is equal to 4 in this case.

Fig. 10.1 An example of the coloring rule

10.4.4 Topological Characterization

The coloring rule and zero forcing sets discussed in the previous subsection generate a chain of forces/infections in the graph, starting from a leader set V_L . Next, we investigate how controllability properties of the network is affected by this forcing/infection process. More precisely, let $v \in V_L$, $w \notin V_L$ and suppose that $v \rightarrow w$. Then, we examine the effect of this infection on the reachable subspace, on the strong structurally reachable subspace, and ultimately we draw conclusions on strong structural controllability of (10.2).

First, we look into the reachability subspace, i.e., $\langle X \mid \mathcal{V}_L \rangle$ for any given $V_L \subseteq V$ and $X \in Q(G)$. We *claim* that

$$\text{im } P(V; V'_L) \subseteq \langle X \mid \mathcal{V}_L \rangle. \quad (10.6)$$

where $V'_L = V_L \cup \{w\}$ and P is given by (10.1). Without loss of generality, assume that $V_L = \{1, 2, \dots, m\}$, $v = m$, and $w = m + 1$. Then, the matrix X can be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix} \quad (10.7)$$

where the diagonal blocks/elements X_{11} , X_{22} , X_{33} , and X_{44} correspond to the vertices in $V_L \setminus \{v\}$, the vertex v , the vertex w , and the rest of the vertices, respectively.

Let $\xi \in \mathbb{R}^n$ be a vector in $\langle X \mid \mathcal{V}_L \rangle^\perp$. Clearly, we have $\xi^T X^{k-1} P(V; V_L) = 0$ for each $k \in \mathbb{N}$. We write $\xi = \text{col}(\xi_1, \xi_2, \xi_3, \xi_4)$ compatible with the partitioning of X . Note that $P(V; V'_L)$ now reads as

$$P(V; V'_L) = \begin{bmatrix} I_{m-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From the equality $\xi^T P(V; V_L) = 0$, we obtain that $\xi_1 = 0$ and $\xi_2 = 0$. Then, the equality $\xi^T X P(V; V_L) = 0$ yields

$$\begin{bmatrix} \xi_3^T & \xi_4^T \end{bmatrix} \begin{bmatrix} X_{31} & X_{32} & X_{33} \\ X_{41} & X_{42} & X_{43} \end{bmatrix} = 0 \quad (10.8)$$

Observe that, since $v \rightarrow w$, the vertex v has exactly one out-neighbor in $V \setminus V_L$, and thus we have $X_{32} \neq 0$ and $X_{42} = 0$. Therefore, by (10.8), we obtain that the scalar ξ_3 is equal to zero. Clearly, $\xi = \text{col}(0, 0, 0, \xi_4)$ is orthogonal to the subspace $\text{im } P(V; V'_L)$. Hence, the subspace inclusion (10.6) holds.

Now, by repeating the argument above, we conclude that

$$\mathcal{D}(V_L) \subseteq \langle X \mid \mathcal{V}_L \rangle \quad (10.9)$$

where $\mathcal{D}(V_L) = \text{im } P(V; D(V_L))$. This yields

$$\langle X \mid \mathcal{D}(V_L) \rangle \subseteq \langle X \mid \mathcal{V}_L \rangle \quad (10.10)$$

On the other hand, noting that $\mathcal{V}_L \subseteq \mathcal{D}(V_L)$, we obtain that

$$\langle X \mid \mathcal{V}_L \rangle \subseteq \langle X \mid \mathcal{D}(V_L) \rangle.$$

This together with (10.10) yields

$$\langle X \mid \mathcal{V}_L \rangle = \langle X \mid \mathcal{D}(V_L) \rangle \quad (10.11)$$

for any given leader set V_L . The equality above reveals the fact that the reachable subspace is invariant under the infection process. Consequently, the strong structurally reachable subspace enjoys this invariance property as well, i.e.

$$\bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle = \bigcap_{X \in Q(G)} \langle X \mid \mathcal{D}(V_L) \rangle \quad (10.12)$$

The aforementioned invariance properties can be used to provide a topological translation of the strong structurally reachable subspace. In particular, observe that

$$\mathcal{D}(V_L) \subseteq \bigcap_{X \in Q(G)} \langle X \mid \mathcal{D}(V_L) \rangle = \bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle \quad (10.13)$$

Now, we show that $\mathcal{D}(V_L)$ is indeed equal to the strong structurally reachable subspace. Clearly, by (10.13), it remains to show that

$$\bigcap_{X \in Q(G)} \langle X \mid \mathcal{D}(V_L) \rangle \subseteq \mathcal{D}(V_L) \quad (10.14)$$

We define the set Δ as

$$\Delta = \{\delta \in \mathbb{R}^n : \delta_i = 0 \Leftrightarrow i \in D(V_L)\} \quad (10.15)$$

Let δ be a vector in Δ . Without loss of generality, let $D(V_L) = \{1, 2, \dots, d\}$. Then, δ can be written as $\text{col}(0_d, \delta_2)$ where each element of $\delta_2 \in \mathbb{R}^{n-d}$ is nonzero. Let the matrix X be partitioned accordingly as

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

Clearly, we have

$$\delta^T X = \delta_2^T \begin{bmatrix} X_{21} & X_{22} \end{bmatrix}$$

Observe that nonzero elements of X_{21} correspond to the arcs from the vertices in $D(V_L)$ to the vertices in $V \setminus D(V_L)$. Hence, by the coloring rule, each column of X_{21} is either identically zero or contains at least two nonzero elements. We choose these nonzero elements, if any, such that $\delta_2^T X_{21} = 0$. Noting that the diagonal elements of X_{22} are free parameters, we conclude that, for any vector $\delta \in \Delta$, there exists a matrix $X \in Q(G)$ such that $\delta^T X = 0$. Therefore, we obtain that

$$\delta \in \langle X \mid \mathcal{D}(V_L) \rangle^\perp$$

for some matrix $X \in Q(G)$. Now, let $\xi \in \mathbb{R}^n$ be a vector in $\bigcap_{X \in Q(G)} \langle X \mid \mathcal{D}(V_L) \rangle$. Clearly, we have $\delta^T \xi = 0$ which yields $\delta_2^T \xi_2 = 0$, by writing $\xi = \text{col}(\xi_1, \xi_2)$. As this conclusion holds for an arbitrary choice of $\delta \in \Delta$, we obtain that $\xi_2 = 0$, and thus $\xi \in \mathcal{D}(V_L)$ which proves (10.14). Hence, we conclude that

$$\bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle = \mathcal{D}(V_L). \quad (10.16)$$

By (10.16), the strong structurally reachable subspace is topologically equivalent to the subspace induced by the derived set of V_L , i.e., $\mathcal{D}(V_L)$. An important special case of (10.16) is obtained for $\mathcal{D}(V_L) = \mathbb{R}^{|G|}$, or equivalently $V_L = V$. This results in an exact topological translation of strong structural controllability:

$$(G; V_L) \text{ is controllable} \iff V_L \text{ is a zero forcing set of } G \quad (10.17)$$

In addition, by a simple cardinality argument, we obtain that

$$\ell_{\min}(G) = Z(G). \quad (10.18)$$

Note that the equalities (10.16)–(10.18) provide a topological characterization of structural controllability properties of systems of the form (10.2). In particular, the derived set of V_L determines the strong structurally reachable subspace, there is a one-to-one correspondence between zero forcing sets and sets of leaders rendering the network strongly structurally controllable, and the minimum number of leaders required is given by the zero forcing number of the graph.

10.4.5 Leader Selection

An important question in the context of controllability of dynamical networks is minimal leader selection, i.e., to choose a leader set, with minimum cardinality, such

that the network is controllable. The research effort in this direction has been devoted mostly to the case where X is equal to the Laplacian matrix of an undirected graph, in particular undirected path, cycle, complete, and circulant graphs [12, 14, 17]. Minimum leader selections rendering the network weakly structurally controllable is also discussed in [7].

The equality (10.18) reveals the fact that our knowledge about minimal leader selection for strong structural controllability of networks is closely related to the knowledge we have on the zero forcing number of graphs. In fact, for any graph whose zero forcing number is known or can be computed, we immediately obtain the minimum number of leaders for controllability, and, in principle, a minimal leader selection scheme. To illustrate this, we mention next few examples.

Note that an undirected graph can be identified by a corresponding directed graph whose arc set is symmetric, see [11] for more details. Then, clearly, either of the two boundary vertices in an undirected path graph P forms a zero forcing set, and thus $\ell_{\min}(P) = 1$. For an undirected cycle graph C , any two neighboring vertices constitute a minimal zero forcing set and hence $\ell_{\min}(C) = 2$. Similarly, for an undirected complete graph K with n vertices, we have $\ell_{\min}(K) = n - 1$. For a directed cycle graph, merely one vertex suffices for strong structural controllability.

The *path cover number* of G , denoted by $P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all the vertices of G ; such a set of paths realizing $P(G)$ is called a *minimal path cover*. For acyclic graphs, it has been shown that the zero forcing number is equal to the path cover number of the graph. Moreover, the initial points of the vertex disjoint paths realizing a minimal path cover form a minimal zero forcing set [5]. Therefore, by selecting those initial points as leaders, we obtain a minimal leader selection scheme for strong structural controllability of acyclic networks.

However, computing a minimal zero forcing set for general graphs with cycles is very difficult. Hence, determining zero forcing number for certain subclasses of graphs, as well as finding suboptimal (non-minimal) zero forcing sets for general directed graphs are interesting problems in the context of strong structural controllability of dynamical networks.

10.5 Strong Targeted Controllability

Network controllability is not always present in complex networks, or it may ask for considerable number of nodes to be directly controlled which is not always feasible. Besides, in certain cases steering the entire network to any arbitrary state may not be necessary, and instead the interest is to drive a subset of the network to a desired state. Following [3], we refer to this problem as *targeted controllability* of networks. By the results discussed in the previous section, we investigate the targeted controllability problem in a strong structural sense. Note that targeted controllability is essentially an “output controllability” problem which is recapped next.

10.5.1 Output Controllability

Consider again the system (10.4) with an additional output equation

$$y(t) = Cx(t) \quad (10.19)$$

where the output $y(t)$ takes its values in the output space $\mathbb{R}^p$. Denote the output trajectory corresponding to the initial state x_0 and input function u by $y_u(t, x_0)$. The system (10.4)–(10.19) is then called *output controllable* if for any $x_0 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^p$ there exists an input function u and a $T > 0$ such that $y_u(T, x_0) = y_1$. We also say that the triple (A, B, C) is output controllable meaning that the system (10.4), (10.19) is output controllable. It is well known (see ,e.g., [16, Exc. 3.22]) that (A, B, C) is output controllable if and only if the rank condition

$$\text{rank} \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \end{bmatrix} = p.$$

holds. In turn this is equivalent to the condition

$$C\langle A \mid \text{im } B \rangle = \mathbb{R}^p,$$

,i.e., the image under C of the reachable subspace is equal to the output space $\mathbb{R}^p$. Obviously, this condition is equivalent to $\ker C + \langle A \mid \text{im } B \rangle = \mathbb{R}^n$. Finally, by taking orthogonal complements, the latter holds if and only if

$$\text{im } C^\top \cap \langle A \mid \text{im } B \rangle^\perp = \{0\}.$$

10.5.2 Topological Characterization

In this subsection, we discuss the “strong targeted controllability” problem for systems of the form (10.2) with an additional output equation:

$$\dot{x} = Xx + Uu \quad (10.20a)$$

$$y = Hx \quad (10.20b)$$

where $X \in Q(G)$, $U = P(V; V_L)$ for some given leader set $V_L \subseteq V$, and $H = P^T(V; V_T)$ for some given *target set* $V_T \subseteq V$. For a given leader set V_L and a target set V_T , we call the system (10.20) *strongly targeted controllable* if the triple (X, U, H) is output controllable for all $X \in Q(G)$. In that case, we also write as $(G; V_L; V_T)$ is targeted controllable.

Note that strong targeted controllability is basically a strong structural output controllability property, where the output of the network can be steered to any desired state in $\mathbb{R}^{|V_T|}$, irrespective of the choice of $X \in Q(G)$.

Let $\mathcal{V}_T = \text{im } P(V; V_T)$, $|V_T| = p$, and $|V| = n$. Then, by Sect. 10.5.1, $(G; V_L; V_T)$ is targeted controllable if and only if

$$\text{rank } [HU \quad HXU \quad HX^2U \quad \dots \quad HX^{n-1}U] = p \text{ for all } X \in Q(G),$$

which is equivalent to

$$H \langle X \mid \mathcal{V}_L \rangle = \mathbb{R}^p \text{ for all } X \in Q(G),$$

and thus to

$$\mathcal{V}_T \cap \langle X \mid \mathcal{V}_L \rangle^\perp = \{0\} \text{ for all } X \in Q(G). \quad (10.21)$$

Note that targeted controllability captures the structural controllability as a special case, namely by setting $H = I$, or equivalently $V_T = V$. Hence, investigating targeted controllability of networks is a more general, and thus more challenging problem. Next, by using the results elaborated in the previous section, we discuss topological conditions under which the network is strongly targeted controllable.

Observe that $(G; V_L; V_T)$ is targeted controllable if

$$\mathcal{V}_T \subseteq \bigcap_{X \in Q(G)} \langle X \mid \mathcal{V}_L \rangle \quad (10.22)$$

Note that indeed (10.22) implies (10.21), which results in targeted controllability. Therefore, by (10.16), we conclude that

$$V_T \subseteq D(V_L) \implies (G; V_L; V_T) \text{ is targeted controllable.} \quad (10.23)$$

This means that the state components corresponding to the vertices in the derived set of V_L can be steered to any desired point in $\mathbb{R}^d$ with $d = |D(V_L)|$, by applying appropriate inputs to the vertices in V_L .

Consider the graph depicted in Fig. 10.2, and let $V_L = \{1, 2\}$. It is easy to observe that by the color-change rule the derived set of V_L is obtained as $D(V_L) = \{1, 2, 3, 4\}$. By (10.23), we have that $(G; V_L; V_T)$ is targeted controllable for any

$$V_T \subseteq \{1, 2, 3, 4\}. \quad (10.24)$$

However, this is not necessary as one can show that $(G; V_L; V_T)$ is targeted controllable with

$$V_T = \{1, 2, 3, 4, 5, 6, 7\}. \quad (10.25)$$

Fig. 10.2 The graph $G = (V, E)$

Next, we discuss how the condition (10.23) can be sharpened. To this end, we define the subgraph $G' = (V, E')$ with

$$E' = \{(i, j) : i \in D(V_L) \text{ and } j \in V_T\}. \quad (10.26)$$

Hence, the set E' captures all the arcs from the vertices in $D(V_L)$ to the vertices in V_T . To avoid confusion, let $D(V_L)$ be denoted by V'_L . Then, by $D'(V'_L)$ we denote the derived set of V'_L in the graph G' . Note that the set $D'(V'_L)$ is in fact constructed as a result of the following steps:

1. Compute the set $D(V_L)$, that is the derived set of V_L in the graph $G = (V, E)$. This means that the vertices in V_L are initially colored black, and we apply the color-change rule based on the arc set E .
2. Construct the subgraph $G' = (V, E')$ from G , with E' given by (10.26)
3. Compute the set $D'(V'_L)$, that is derived set of V'_L in G' . This means that the vertices in $V'_L = D(V_L)$ are initially colored black, and we apply the color-change rule based on the arc set E' .

Noting that $D(V_L) \subseteq D'(V'_L)$, without loss of generality, assume that $D(V_L) = \{1, 2, \dots, d\}$ and $D'(V'_L) = \{1, 2, \dots, d, d+1, \dots, d+e\}$. Consider the condition (10.21). Let $X \in Q(G)$ and let ξ be a vector in the subspace $\mathcal{V}_T \cap \langle X \mid \mathcal{V}_L \rangle^\perp$. Hence, by (10.11),

$$\xi \in \mathcal{V}_T \cap \langle X \mid \mathcal{D}(V_L) \rangle^\perp. \quad (10.27)$$

Note that $\xi \in \mathbb{R}^n$ can be written as $\xi = \text{col}(\xi_1, \xi_2, \xi_3)$, where $\xi_1 \in \mathbb{R}^d$, $\xi_2 \in \mathbb{R}^e$, and $\xi_3 \in \mathbb{R}^{n-d-e}$. Compatible with ξ , let the matrix X be partitioned as

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}. \quad (10.28)$$

Now assume that $V_T = D'(V'_L)$, and thus

$$\mathcal{V}_T = \mathcal{D}'(V'_L)$$

where $\mathcal{D}'(V'_L) = \text{im } P(V; D'(V'_L))$. Then, clearly $\xi \in \mathcal{D}'(V'_L)$, and hence $\xi_3 = 0$. By (10.27), we have

$$\xi^T X^{k-1} P(V; D(V_L)) = 0 \quad (10.29)$$

for each $k \in \mathbb{N}$. The equality $\xi^T P(V; D(V_L)) = 0$ yields $\xi_1 = 0$. Then, by (10.29) with $k = 2$, we obtain that

$$\xi_2^T X_{21} = 0. \quad (10.30)$$

Now, observe that the matrix

$$X' = \begin{bmatrix} 0 & 0 & 0 \\ X_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

belongs to the qualitative class $\mathcal{Q}(G')$, where the partitioning is compatible to (10.28). Therefore, by (10.9), we have

$$\mathcal{D}'(V'_L) \subseteq \langle X' \mid \mathcal{V}'_L \rangle = \langle X' \mid \mathcal{D}(V_L) \rangle \quad (10.31)$$

where $\mathcal{V}'_L = \text{im } P(V; V'_L)$. The subspace inclusion (10.31) yields

$$\mathcal{D}'(V'_L) = \text{im} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \subseteq \text{im} \begin{bmatrix} I & 0 \\ 0 & X_{21} \\ 0 & 0 \end{bmatrix} = \langle X' \mid \mathcal{D}(V_L) \rangle$$

where the partitioning is again compatible to (10.28), and we have used the fact that $(X')^k = 0$ for $k > 1$. This implies that X_{21} is full row rank. Hence, the equality (10.30) results in $\xi_2 = 0$ which in turn implies targeted controllability of $(G; V_L; V_T)$ by (10.21). Therefore, we conclude that

$$V_T = D'(V'_L) \implies (G; V_L; V_T) \text{ is targeted controllable,} \quad (10.32)$$

which obviously can be restated as

$$V_T \subseteq D'(V'_L) \implies (G; V_L; V_T) \text{ is targeted controllable.} \quad (10.33)$$

Noting that $D(V_L) \subseteq D'(V'_L)$, the condition (10.23) can be replaced by the sharper condition (10.33) to deduce strong targeted controllability of the network.

As an example consider again the graph depicted in Fig. 10.2 with $V_L = \{1, 2\}$. Recall that the derived set of V_L in G is given by $D(V_L) = \{1, 2, 3, 4\}$ in this case. Let V_T be given by

Fig. 10.3 The subgraph $G' = (V, E')$

$$V_T = \{1, 2, 3, 4, 5, 6\}$$

Then, Fig. 10.3 shows the subgraph $G' = (V, E')$ with E' given by (10.26). It is easy to observe that $D'(V'_L)$ is obtained as $D'(V'_L) = \{1, 2, 3, 4, 5, 6\}$. Noting that $V_T = D'(V'_L)$, we conclude that $(G; V_L; V_T)$ is targeted controllable by (10.32). It is worth mentioning that the sufficient condition (10.33) is not tight, as evident by (10.25). In fact, the vertices in $D(V_L)$ do not infect vertex 7 in G' .

10.6 Conclusions and Outlook

We have considered the problem of controllability of the network for a family of matrices carrying the structure of an underlying directed graph. This family of matrices is called the qualitative class, and as observed, there is a one-to-one correspondence between zero forcing sets and the set of leaders rendering the network controllable for all matrices in the qualitative class. As illustrated, this provides a bridge connecting the results available in graph theory for zero forcing sets/number to (minimal) leader selection schemes for strong structural controllability of dynamical networks. As minimal zero forcing sets for general graphs with cycles are very difficult to compute, finding suboptimal (non-minimal) zero forcing sets, or equivalently suboptimal (non-minimal) leader selection schemes for strong structural controllability of networks is an interesting problem for future research.

We have also studied the case where the network is not strongly structurally controllable, yet we are interested in controllability properties in some parts of the network identified by a target set. We have discussed topological sufficient conditions guaranteeing strong targeted controllability of the network. There are still important questions to be addressed in this direction. One notable problem is to establish a tractable topological necessary and sufficient condition verifying strong targeted controllability. Then, given a leader set, a problem of interest is to characterize a

target set, with maximum cardinality, such that the network is strongly targeted controllable. Another interesting question is the “dual” problem, i.e., to select a set of leaders, with minimum cardinality, such that the network is strongly targeted controllable for a given target set.

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