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Published in:
Physical Review D

DOI:
[10.1103/PhysRevD.103.064078](https://doi.org/10.1103/PhysRevD.103.064078)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2021

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Kolář, I., Torralba, F. J. M., & Mazumdar, A. (2021). Junction conditions in infinite derivative gravity. *Physical Review D*, 103, [064078]. <https://doi.org/10.1103/PhysRevD.103.064078>

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Junction conditions in infinite derivative gravity

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(Received 18 January 2021; accepted 5 March 2021; published 29 March 2021)

The junction conditions for the infinite derivative gravity theory $R + RF(\Box)R$ are derived under the assumption that the conditions can be imposed by avoiding the “ill-defined expressions” in the theory of distributions term by term in infinite summations. We find that the junction conditions of such nonlocal theories are much more restrictive than in local theories, since the conditions comprise an infinite number of equations for the Ricci scalar. These conditions can constrain the geometry far beyond the matching hypersurface. Furthermore, we derive the junction field equations which are satisfied by the energy-momentum tensor on the hypersurface. It turns out that the theory still allows some matter content on the hypersurface (without external flux and external tension) but with a traceless energy-momentum tensor. We also discuss the proper matching condition where no matter is concentrated on the hypersurface. Finally, we explore the possible applications and consequences of our results to the braneworld scenarios and star models. Particularly, we find that the internal tension is given purely by the trace of the energy-momentum tensor of the matter confined to the brane. Consequences of the junction conditions are illustrated on two simple examples of static and collapsing stars. It is demonstrated that, even without solving the field equations, the geometry on one side of the hypersurface can be determined to a great extent by the geometry on the other side if the Ricci scalar is analytic. We further show that some usual star models in the general relativity are no longer solutions of the infinite derivative gravity.

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I. INTRODUCTION

In every physical theory, one must face the question of how to treat surfaces of discontinuity. For example, in electromagnetism, one can have charged surfaces that induce a “jump” on the value of the electric and magnetic fields [1]. The equations that relate the structure of this surface with discontinuities of the physical quantities are known as the *junction conditions*. In a gravitational context, the situation is very similar, and the junction conditions can be derived for a hypersurface that divides the spacetime into two regions. This issue was first studied in the general relativity (GR) by Lanczos [2]. Then, subsequent studies have formulated the junction conditions for timelike, spacelike, null, and general hypersurfaces, in terms of the intrinsic and extrinsic curvatures of such hypersurfaces [3–11]. Also, these junction conditions affect the field equations, allowing certain jumps in the energy-momentum tensor, first obtained for non-null hypersurfaces by Israel [12], hence acquiring the name of *Israel’s junction conditions*. Null and general hypersurfaces were later studied in Refs. [11,13].

Being such a crucial aspect of a theory, it is clear that the junction conditions are of immense importance in any modified theory of gravity. Examples of such studies include braneworld models [14,15], $f(R)$ gravity [16–18], Palatini $f(R)$ [19], $f(R)$ with torsion [20], quadratic gravity [21], Einstein-Cartan theory [22], metric-affine gravity [23], generalized scalar-tensor theories [24], and extended teleparallel gravity [25].

In this paper, we will focus on a particular nonlocal theory usually known as the infinite derivative gravity (IDG) [26,27]. For this theory, the nonlocal interaction arises due to the inclusion of operators with an infinite number of derivatives in the gravitational action. The most general action has been built around Minkowski [27], de Sitter and anti-de Sitter spaces [28], and around cosmological bouncing background [26,29]. The graviton propagator of such theories can be modified to avoid any perturbative ghosts around a specific background. Also, the nonlocal gravitational interaction has been argued to improve UV aspects gravity at the quantum level [30–35]. There were also several attempts in studying the initial value problem of IDG using diffusion equation method [36] and constructing perturbative Hamiltonian by introducing one extra time dimension [37].

At the classical level, it has been shown that the linearized IDG can yield nonsingular static solutions for

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point sources (including electromagnetic charges), p-branes (and cosmic strings), sources describing Newman-Unti-Tamburino charges [27,38–41], and spinning ring distributions [42]. It was also found that the linearized IDG prevents mini-black-hole production for small masses [43–45]. Exact solutions of the full IDG describing bouncing cosmologies and impulsive gravitational waves were obtained in Refs. [26,46,47]. An extension of IDG theories allowing a nonsymmetric connection (retaining the absence of ghosts and nonsingular linearized solutions) was proposed in Refs. [48,49].

The study of the junction conditions in nonlocal theories is of great interest. This is because one of the main aspects of the nonlocality is that the fields interact directly over a greater distance, as the nonlocal scale modifies the UV aspects of gravitational interactions. Therefore, it can be expected that, for such theories, the junction conditions will have strong influence far beyond the matching hypersurface and could significantly constrain the fields in the whole spacetime. We will show that this is indeed the case by studying some illustrative examples.

The paper is organized as follows. In Sec. II, we will briefly review distributional calculus on manifolds together with some important formulas for derivatives and distributional curvature tensors. In Sec. III, we will introduce the action of infinite derivative gravity and present the corresponding field equations. In Sec. IV, we will derive the matching conditions on the timelike hypersurface of codimension 1 between two regions of the manifold. These conditions are required for the theory to be well defined in a distributional sense. In Sec. V, we will find expressions for the singular parts and discontinuities of the energy-momentum tensor. We will write the field equations for the quantities on the junction and discuss the proper matching where no matter content is allowed on the hypersurface. In Sec. VI, we will study the consequences of our results on several illustrative examples. Particularly, we will examine the implications for the braneworld models and discuss explicit junction conditions for simple star models. The paper will be concluded with a brief summary of our results in Sec. VII.

II. MATHEMATICAL PREREQUISITES

In this section, we give a brief exposition of tensor distributions with application to spacetimes involving surfaces where the geometry is not differentiable. We present only the definitions and important results such as the distributional generalization of the curvature tensors, which are essential for the study of junction conditions. For the derivations and more details, we refer the reader to Refs. [11,21].

A. Tensor distributions

Consider a four-dimensional manifold M equipped with a metric g . Let $\mathcal{D}$ be a space of smooth tensor fields on M

with a compact support called the *test tensor fields*. A *tensor distribution* is a linear continuous functional $\chi: \mathcal{D} \rightarrow \mathbb{R}$, which maps test tensor fields $Y \in \mathcal{D}$ to real numbers,

$$\chi(Y) \equiv \langle \chi, Y \rangle. \quad (1)$$

Locally integrable tensor fields T define unique tensor distributions $\underline{T}$ by means of¹

$$\langle \underline{T}, Y \rangle \equiv \int_M g^{\frac{1}{2}} T \cdot Y, \quad T \cdot Y \equiv T^{\nu \dots}_{\mu \dots} Y^{\mu \dots}_{\nu \dots}, \quad (2)$$

where $g^{\frac{1}{2}} = \sqrt{-g} dx^4$ is the volume element and $\mu, \nu, \dots = 0, 1, 2, 3$ are the four-dimensional indices. Note that the action of $\underline{T}$ can be extended to nonsmooth test fields as well, and the action of tensor distribution is considered whenever it can be defined [11]. Components of a tensor distribution with respect to the vector frame $e_{(\mu)}$ (and dual coframe $e^{(\nu)}$), $\chi^{\nu \dots}_{\mu \dots}$, are defined by means of

$$\langle \chi^{\nu \dots}_{\mu \dots}, Y \rangle \equiv \langle \chi, Y e_{(\mu)} \dots e^{(\nu)} \dots \rangle, \quad (3)$$

where Y is a test scalar function. Consequently, we can write $\langle \chi, Y \rangle = \langle \chi^{\nu \dots}_{\mu \dots}, Y^{\mu \dots}_{\nu \dots} \rangle$.

As it is common in the theory of distributions, the operations on distributional objects are defined through the actions on test fields. In particular, the definition of the *covariant derivative of a tensor distribution* is motivated by the integration by parts formula applied to (2),

$$\langle \nabla \chi, Y \rangle \equiv -\langle \chi, \nabla \cdot Y \rangle, \quad (\nabla \cdot Y)^{\mu \dots}_{\nu \dots} \equiv \nabla_{\rho} Y^{\rho \dots \mu \dots}_{\nu \dots}. \quad (4)$$

The tensor *multiplication by a tensor field* is defined by

$$\langle T \chi, Y \rangle \equiv \langle \chi, T \cdot Y \rangle, \quad (T \cdot Y)^{\alpha \dots}_{\beta \dots} \equiv T^{\nu \dots}_{\mu \dots} Y^{\mu \dots \alpha \dots}_{\nu \dots \beta \dots}. \quad (5)$$

In general, this product is well defined if the tensor field T is smooth. However, it is possible to relax this condition, as discussed in [11].

The serious mathematical issue, however, arises with the products of distributions which cannot be associated to tensor fields such as $\delta\delta$ terms [50]. Note that it is possible to give a mathematical sense to such objects in the theories of nonlinear generalized functions, such as Colombeau algebras [51]. Unfortunately, the use of these advanced mathematical tools is quite complicated even within the theory of general relativity [52,53]. In this paper, we restrict ourselves to the theory of linear distributions.

¹In the index-free notation, we use $\cdot$ to denote contractions of adjacent indices and $\bullet$ to denote multicontractions in all indices. We also employ the musical isomorphism to raise indices of a 1-forms $\alpha^{\sharp} = g^{-1} \cdot \alpha$ and lower indices of vectors $v^{\flat} = g \cdot v$.

B. Matching hypersurface

The manifold M possesses two regions M^+ and M^- separated by a three-dimensional timelike hypersurface Σ , called the *matching hypersurface*. It is assumed that the metrics in each region, $g^\pm$, are smooth. To glue these to spacetimes together, we have to demand that the three-dimensional *induced metrics* (also known as the *first fundamental forms*) $h^\pm$ on Σ as viewed from both sides coincide,

$$h_{ab}^- = h_{ab}^+ \equiv h_{ab}, \quad (6)$$

where $a, b, \dots = 0, 1, 2$ are the three-dimensional indices. This single condition allows us to match the four-dimensional tangent space on both sides of Σ . By choosing a set of basis vector fields $e_{(a)}$ tangent to Σ , we can construct the orthonormal 1-forms $n^\pm$ whose contractions with $e_{(a)}$ vanish

$$n^\pm \cdot e_{(a)} = 0, \quad n^\pm \cdot n^{\pm\#} = 1. \quad (7)$$

Since such an orthonormal 1-form is fixed uniquely up to sign, we can drop the $\pm$ sign and denote the orthonormal 1-form pointing toward M^+ by n . The set $\{e_{(a)}, n^\# \}$ then forms a common frame of both tangent spaces with a dual coframe $\{e^{(a)}, n\}$.

We will mostly work with the four-dimensional version of the induced metric $h_{\mu\nu}$ that is related to the three-dimensional h_{ab} one through the relation

$$h_{ab} \equiv h_{\mu\nu} e_{(a)}^\mu e_{(b)}^\nu. \quad (8)$$

An important consequence of the previous analysis following from (6) is that the complete metric g is continuous² across Σ but still may have discontinuous finite derivatives. These definitions allows us to decompose an arbitrary tensor into its tangent and normal directions using the formula

$$g_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu \quad (9)$$

because $h_{\mu\nu}$ plays a role of the projector to Σ thanks to the properties $h_\nu^\mu h_\kappa^\nu = h_\kappa^\mu$, $h_\nu^\mu n^\nu = 0$.

C. Discontinuities and derivatives

The metric tensor g is not differentiable on Σ ; however, it defines a distribution $\underline{g}$ which can be differentiated in distributional sense to obtain distributional curvature tensors. In order to proceed in this direction, we need to define the *Heaviside step function* θ (and the associated distribution $\underline{\theta}$) with the jump on Σ ,

$$\theta \equiv \begin{cases} 0, & M^-, \\ \frac{1}{2}, & \Sigma, \\ 1, & M^+, \end{cases} \quad \langle \underline{\theta}, Y \rangle = \int_{M^+} g^{\frac{1}{2}} Y, \quad (10)$$

and *Dirac delta distribution* δ supported on Σ ,

$$\langle \delta, Y \rangle \equiv \int_\Sigma \mathfrak{h}^{\frac{1}{2}} Y, \quad (11)$$

where $\mathfrak{h}^{\frac{1}{2}}$ is the volume element on Σ . Definition (4) together with the divergence theorem implies

$$\nabla \underline{\theta} = n \delta. \quad (12)$$

This relation can be easily generalized to the covariant derivative of a distribution associated to an arbitrary discontinuous tensor field with finite limits and its derivatives from both sides. Consider a tensor field T (and the corresponding distribution $\underline{T}$),

$$T \equiv T^+ \theta + T^-(1 - \theta), \quad \underline{T} = T^+ \underline{\theta} + T^-(\underline{1} - \underline{\theta}). \quad (13)$$

Its covariant derivative then reads

$$\nabla \underline{T} = \nabla T^+ \underline{\theta} + \nabla T^-(\underline{1} - \underline{\theta}) + n[T] \delta, \quad (14)$$

where we introduced the square-bracket notation,

$$[T] \equiv (T^+ - T^-)|_\Sigma, \quad (15)$$

to denote the jump of T across Σ . We refer to the terms such as $[T]n\delta$ that prevent the distributions from being associated to a tensor field as the *singular parts of distributions*. Similar to (14), we can also calculate the second derivative, where one additional term appears due to the possible discontinuity of ∇T ,

$$\nabla \nabla \underline{T} = \nabla \nabla T^+ \underline{\theta} + \nabla \nabla T^-(\underline{1} - \underline{\theta}) + \nabla(n[T] \delta) + n[\nabla T] \delta. \quad (16)$$

Finally, let us mention that we will use the shorthand notation for the value of the T on Σ ,

$$T^\Sigma \equiv \frac{1}{2}(T^+ + T^-)|_\Sigma. \quad (17)$$

The discontinuity of the tensor product of A and B is then given by

$$[AB] = A^\Sigma[B] + [A]B^\Sigma. \quad (18)$$

The jump of the first derivative of a scalar field T can be written as

²This allows us to freely raise and lower indices everywhere using g .

$$[\nabla_\mu T] = n_\mu n^\nu [\nabla_\nu T] + h_\mu^\kappa \nabla_\kappa [T], \quad (19)$$

while the second derivative reads (see, e.g., Ref. [21])

$$[\nabla_\mu \nabla_\nu T] = n_\mu n^\kappa [\nabla_\kappa \nabla_\nu T] + h_\mu^\kappa \nabla_\kappa [\nabla_\nu T] - h_\mu^\rho [\Gamma_{\rho\nu}^\kappa] (\nabla_\kappa T)^\Sigma. \quad (20)$$

A more convenient form of the last formula can be found by a recursive argument (together with commutation of derivatives of scalars),

$$n_\mu n^\kappa [\nabla_\kappa \nabla_\nu T] + U_{\mu\nu} = n_\mu n^\kappa (n_\nu n^\rho [\nabla_\rho \nabla_\kappa T] + U_{\nu\kappa}) + U_{\mu\nu}, \quad (21)$$

where we denoted $U_{\mu\nu} \equiv h_\mu^\kappa \nabla_\kappa [\nabla_\nu T] - h_\mu^\rho [\Gamma_{\rho\nu}^\kappa] (\nabla_\kappa T)^\Sigma$. After inserting this identity in (20), we arrive at

$$[\nabla_\mu \nabla_\nu T] = n_\mu n_\nu n^\kappa n^\rho [\nabla_\kappa \nabla_\rho T] + n_\mu n^\kappa U_{\nu\kappa} + U_{\mu\nu}. \quad (22)$$

D. Intrinsic and extrinsic curvature

The fact that we have a metric with discontinuous derivatives across Σ means that there can be a discontinuity of the curvature tensors. That is why these tensors need to be described in a distributional sense. Starting with the metric distribution defined by the continuous metric with finite derivatives from both sides,

$$\underline{g}_{\mu\nu} = g_{\mu\nu}^+ \underline{\vartheta} + g_{\mu\nu}^- (1 - \underline{\vartheta}), \quad [g_{\mu\nu}] = 0, \quad (23)$$

we can construct distributional Christoffel symbols,

$$\underline{\Gamma}_{\mu\nu}^\kappa = \Gamma_{\mu\nu}^{\kappa+} \underline{\vartheta} + \Gamma_{\mu\nu}^{\kappa-} (1 - \underline{\vartheta}), \quad [\Gamma_{\mu\nu}^\kappa] \in \mathbb{R}, \quad (24)$$

using the standard formulas and the rule (14). As a consequence of the arbitrary jump $[\Gamma_{\mu\nu}^\kappa]$, the Riemann tensor distribution,

$$\underline{R}^\alpha_{\beta\mu\nu} = R^{+\alpha}_{\beta\mu\nu} \underline{\vartheta} + R^{-\alpha}_{\beta\mu\nu} (1 - \underline{\vartheta}) + \mathcal{R}^\alpha_{\beta\mu\nu} \delta, \quad (25)$$

contains a term proportional to δ , the singular part of Riemann distribution,

$$\mathcal{R}^\alpha_{\beta\lambda\mu} = n_\lambda [\Gamma_{\beta\mu}^\alpha] - n_\mu [\Gamma_{\beta\lambda}^\alpha]. \quad (26)$$

Unlike the induced metric, the *extrinsic curvatures* of Σ , denoted by $K_{\mu\nu}^\pm$ (also known as the *second fundamental form*), can be different when viewed from both sides. Their four- and three-dimensional versions are defined by

$$K_{\mu\nu}^\pm \equiv h_\mu^\rho h_\nu^\sigma \nabla_\rho^\pm n_\sigma, \quad K_{ab}^\pm \equiv e_{(a)}^\rho e_{(b)}^\mu \nabla_\rho^\pm n_\mu. \quad (27)$$

The Gauss-Codazzi equations also have their two versions. Particularly important are the relations

$$\begin{aligned} R^\pm - 2R_{\mu\nu}^\pm n^\mu n^\nu &= \bar{R} - (K^\pm)^2 + K_{\mu\nu}^\pm + K^{\pm\mu\nu}, \\ n^\mu R_{\mu\rho}^\pm h_\nu^\rho &= \bar{\nabla}^\mu K_{\mu\nu}^\pm - \bar{\nabla}_\nu K^\pm, \end{aligned} \quad (28)$$

where $\bar{\nabla}$ is the covariant derivative on Σ and $\bar{R}$ is the scalar curvature of Σ .

There exists an alternative expression for the singular part of Riemann tensor that relates it to the jump of the extrinsic curvature. As shown in Ref. [21], one can employ a coordinate transformation and write the jump of the Christoffel symbols as

$$[\Gamma_{\beta\lambda}^\alpha] = n_\lambda [K_{\beta\lambda}^\alpha] + n_\beta [K_{\lambda\lambda}^\alpha] - n^\alpha [K_{\beta\lambda}]. \quad (29)$$

Then, we immediately get

$$\mathcal{R}_{\alpha\beta\lambda\mu} = -n_\alpha [K_{\beta\mu}] n_\lambda + n_\alpha [K_{\beta\lambda}] n_\mu - n_\beta [K_{\alpha\lambda}] n_\mu + n_\beta [K_{\alpha\mu}] n_\lambda. \quad (30)$$

The Ricci tensor and Ricci scalar distributions can be obtained by contracting (25) and (30),

$$\begin{aligned} \underline{R}_{\beta\mu} &= R_{\beta\mu}^+ \underline{\vartheta} + R_{\beta\mu}^- (1 - \underline{\vartheta}) + \mathcal{R}_{\beta\mu} \delta, \\ \mathcal{R}_{\beta\mu} &\equiv \mathcal{R}^\rho_{\beta\rho\mu} = -[K_{\beta\mu}] - [K] n_\beta n_\mu, \end{aligned} \quad (31)$$

and

$$\underline{R} = R^+ \underline{\vartheta} + R^- (1 - \underline{\vartheta}) + \mathcal{R} \delta, \quad \mathcal{R} \equiv \mathcal{R}^\rho_\rho = -2[K], \quad (32)$$

where we denoted the trace $K \equiv K^\mu_\mu$. Consequently, the Einstein tensor reads

$$\begin{aligned} \underline{G}_{\beta\mu} &= G_{\beta\mu}^+ \underline{\vartheta} + G_{\beta\mu}^- (1 - \underline{\vartheta}) + \mathcal{G}_{\beta\mu} \delta, \\ \mathcal{G}_{\beta\mu} &\equiv -[K_{\beta\mu}] + [K] h_{\beta\mu}. \end{aligned} \quad (33)$$

In the next section, we shall use these expressions to see how the action and the field equations behave in a divided spacetime, in order to obtain the matching conditions in the boundary. Using the Gauss-Codazzi equations (28), it is possible to derive geometric relations,

$$\begin{aligned} K_{\rho\sigma}^\Sigma \mathcal{G}^{\rho\sigma} &= n^\beta n^\mu [G_{\beta\mu}] = n^\beta n^\mu [R_{\beta\mu}] - [R], \\ \bar{\nabla}^\beta \mathcal{G}_{\beta\mu} &= -n^\rho h_\mu^\sigma [G_{\rho\sigma}] = -n^\rho h_\mu^\sigma [R_{\rho\sigma}]. \end{aligned} \quad (34)$$

In GR, when Einstein's equations are imposed, these equations are known as *Israel's junction equations* [12]. It is also possible to derive distributional Bianchi identities [11],

$$\nabla_\rho \underline{R}^\alpha_{\beta\mu\nu} + \nabla_\mu \underline{R}^\alpha_{\beta\nu\rho} + \nabla_\nu \underline{R}^\alpha_{\beta\rho\mu} = 0, \quad \nabla^\beta \underline{G}_{\beta\mu} = 0. \quad (35)$$

III. INFINITE DERIVATIVE GRAVITY

Consider the following nonlocal gravity theory, which we refer to as the *infinite derivative gravity* if not stated otherwise,³

$$S = \frac{1}{2} \int_M g^{\frac{1}{2}} (R + RF(\square)R) + S_m, \quad (36)$$

where the analytic operator $F(\square)$, often referred to as the *form factor*, is defined as

$$F(\square) = \sum_{n=0}^{\infty} f_n \square^n. \quad (37)$$

We hide the parameter ℓ describing the *length scale of nonlocality* in the coefficients f_n ,

$$f_n = \ell^{2n+2} \hat{f}_n. \quad (38)$$

Furthermore, we assume that $F(\square)$ is nonpolynomial which is reflected in the presence of the infinite number of derivatives, and thus the nonlocal theory. In other words, for every $k \geq 0$, there exists $n > k$ such that $f_n \neq 0$.

Performing variations with respect to the metric, we can find the field equations (see Refs. [26,54]),

$$G_{\mu\nu} + 2R_{\mu\nu}F(\square)R - 2(\nabla_\mu \nabla_\nu - g_{\mu\nu}\square)F(\square)R - \frac{1}{2}g_{\mu\nu}RF(\square)R - R\overleftrightarrow{\Omega}_{\mu\nu}R + \frac{1}{2}g_{\mu\nu}(R\overleftrightarrow{\Omega}_\sigma^\sigma R + R\overleftrightarrow{\Theta}\square R) = T_{\mu\nu}, \quad (39)$$

where $\overleftrightarrow{\Omega}$ and $\overleftrightarrow{\Theta}$ are bilinear operators with the left/right actions defined by

$$\begin{aligned} \phi\overleftrightarrow{\Omega}_{\mu\nu}\psi &\equiv \phi\overleftrightarrow{\nabla}_\nu \frac{F(\overleftrightarrow{\square}) - F(\overleftrightarrow{\square})}{\overleftrightarrow{\square} - \overleftrightarrow{\square}} \overleftrightarrow{\nabla}_\mu \psi &\equiv \sum_{n=1}^{\infty} f_n \sum_{k=0}^{n-1} \nabla_\mu \square^k \phi \nabla_\nu \square^{n-k-1} \psi &= \sum_{k=0}^{\infty} \nabla_\mu \square^k \phi \sum_{l=0}^{\infty} f_{k+l+1} \nabla_\nu \square^l \psi, \\ \phi\overleftrightarrow{\Theta}\psi &\equiv \phi \frac{F(\overleftrightarrow{\square}) - F(\overleftrightarrow{\square})}{\overleftrightarrow{\square} - \overleftrightarrow{\square}} \psi &\equiv \sum_{n=1}^{\infty} f_n \sum_{k=0}^{n-1} \square^k \phi \square^{n-k-1} \psi &= \sum_{k=0}^{\infty} \square^k \phi \sum_{l=0}^{\infty} f_{k+l+1} \square^l \psi, \end{aligned} \quad (40)$$

and $T_{\mu\nu}$ is the usual energy-momentum tensor.

IV. MATCHING CONDITIONS

The moment one considers quadratic curvature terms in the action, problematic terms such as $\delta\delta$ may appear. These kinds of expressions are *ill defined* in the theory of distributions. Instead of dealing with the complicated nonlinear generalized functions, we adopt the pragmatic approach of Refs. [11,16,21]. It is shown that the standard theory of distributions can be used if the problematic products of distributions are avoided by imposing certain minimal conditions on spacetimes in consideration. In this section, we find such conditions.

A. Simplifying assumptions

Following Refs. [11,18,21], we need to find restrictions on various quantities in Σ so that the action and the field equations of the theory are mathematically sensible. For example, in the local limiting case $F(\square) = 1$, one is forced to impose that the jump of the trace of the extrinsic curvature is zero, $[K] = 0$; see Ref. [18]. In the infinite derivative gravity, it is not straightforward to generalize this

argument. This is due to the fact that the differential operator $F(\square)$ can in principle regularize δ distribution or deregularize a smooth function. Such situations are difficult to control.

We place a simplifying assumption on the class of form factors $F(\square)$ we want to study in a given background. Roughly speaking, we assume that the action of the operators such as⁴

$$\sum_{n=0}^{\infty} f_{n+k} \square^n, \quad \sum_{n=0}^{\infty} f_{n+k} \nabla_\mu \square^n, \quad k \geq 0, \quad (41)$$

on tensor distributions with nonzero singular parts should not give rise to tensor distributions without singular parts. On top of that, we assume that the result of such actions on perfectly smooth tensor fields should not produce tensor distributions with nonzero singular parts. This means that not only $F(\square)$ but also the one-sided action of $\overleftrightarrow{\Omega}$ and $\overleftrightarrow{\Theta}$ do not regularize any singular parts of distributions or deregularize any smooth tensor fields that appear in our discussion. These rather imprecise demands should become

³We consider $R + RF(\square)R$ theory as a simplification of the complete infinite derivative theory, which also includes quadratic terms with Ricci and Riemann tensors [27]. This simplified action was studied in the cosmological context in Ref. [26].

⁴A typical form factor for a ghost-free IDG around the Minkowski spacetime is given by $F(\square) = \frac{e^{-\ell^2 \square} - 1}{\square}$ [26,54], which does not produce a regular function when it acts on δ distribution localized at timelike hypersurface in flat Minkowski spacetime.

clearer from our construction of the junction conditions in Sec. IV C. They are necessary to assume since we want to be able to impose the conditions by avoiding ill-defined expressions in the infinite sums separately for each term only without worrying about convergence of smooth functions to distributions and vice versa.

Apart from these assumptions, we shall keep the form factor as a general one. If the results were to be applied to a

solution of the theory, one must be careful to choose a compatible form factor that renders the theory ghost free.

B. Derivatives of $\underline{R}$

The action as well as the field equations contain various terms with covariant derivatives of the Ricci scalar. After a repeated use of distributional formula (14), we arrive at

$$\begin{aligned} \nabla_{\mu_k} \cdots \nabla_{\mu_1} \underline{R} &= \nabla_{\mu_k} \cdots \nabla_{\mu_1} R^+ \underline{\theta} + \nabla_{\mu_k} \cdots \nabla_{\mu_1} R^- (\underline{1} - \underline{\theta}) \\ &+ \sum_{l=2}^{k+1} \nabla_{\mu_k} \cdots \nabla_{\mu_l} (n_{\mu_{l-1}} [\nabla_{\mu_{l-2}} \cdots \nabla_{\mu_1} R] \delta) + \nabla_{\mu_k} \cdots \nabla_{\mu_1} (\mathcal{R} \delta). \end{aligned} \quad (42)$$

Setting $(k = 2j)$ and contracting pairs of adjacent indices, we find the formula for an arbitrary power of $\square$,

$$\square^j \underline{R} = \square^j R^+ \underline{\theta} + \square^j R^- (\underline{1} - \underline{\theta}) + \sum_{l=0}^{j-1} \square^l (n^\nu [\nabla_\nu \square^{j-1-l} R] \delta) + \sum_{l=0}^{j-1} \square^l \nabla_\nu (n^\nu [\square^{j-1-l} R] \delta) + \square^j (\mathcal{R} \delta). \quad (43)$$

A different contraction (with $k = 2j + 1$) or a covariant differentiation of the last equation leads to

$$\begin{aligned} \nabla_\mu \square^j \underline{R} &= \nabla_\mu \square^j R^+ \underline{\theta} + \nabla_\mu \square^j R^- (\underline{1} - \underline{\theta}) + n_\mu [\square^j R] \delta + \sum_{l=0}^{j-1} \nabla_\mu \square^l (n^\nu [\nabla_\nu \square^{j-1-l} R] \delta) \\ &+ \sum_{l=0}^{j-1} \nabla_\mu \square^l \nabla_\nu (n^\nu [\square^{j-1-l} R] \delta) + \nabla_\mu \square^j (\mathcal{R} \delta). \end{aligned} \quad (44)$$

C. Well-defined theory in a distributional sense

Field equations (39) contain several terms that are quadratic in curvature:

$$R_{\mu\nu} F(\square) R, \quad RF(\square) R, \quad R \overset{\leftrightarrow}{\Omega}_{\mu\nu} R, \quad R \overset{\leftrightarrow}{\Omega}_\sigma R, \quad R \overset{\leftrightarrow}{\Theta} \square R. \quad (45)$$

Here, we ignored the numerical factors and the metric $g_{\mu\nu}$, which is continuous and does not affect our discussion. Due to our simplifying assumptions in Sec. IV A, the singular parts of the Ricci scalar distribution cannot be regularized due to the presence of infinite sums (41). Also, the action of operators on smooth R cannot suddenly produce singular terms. This means that the conditions can be imposed on each summand separately. Furthermore, we have to ignore all dubious cancellations between ill-defined terms, such as

$$a_1 \delta \delta + a_2 \delta \delta = (a_1 + a_2) \delta \delta,$$

and simply demand every coefficient to be zero, $a_1 = a_2 = 0$, so that no ill-defined terms ever appear (see Ref. [21]). Such cancellation, if it really occurs, could only be addressed in theory of nonlinear generalized functions.

As the conditions arising from the presence of $\overset{\leftrightarrow}{\Omega}$ and $\overset{\leftrightarrow}{\Theta}$ terms are very restrictive, it is best to start with terms involving these operators and then return to the rest. The expressions proportional to f_n of the last three terms in (45) read, respectively,

$$\sum_{k=0}^{n-1} \nabla_\mu \square^k R \nabla_\nu \square^{n-k-1} R, \quad \sum_{k=0}^{n-1} \nabla_\mu \square^k R \nabla^\mu \square^{n-k-1} R, \quad \sum_{k=0}^{n-1} \square^k R \square^{n-k} R. \quad (46)$$

These terms can make sense in the theory of distributions if the singular parts of $\nabla_\mu \square^k \underline{R}$ for $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ and singular parts of $\square^k \underline{R}$ for $0 \leq k \leq \lceil \frac{n-1}{2} \rceil$ vanish. After realizing that a nonpolynomial $F(\square)$ guarantees the existence of $f_n \neq 0$ with n larger than any arbitrary number and employing (32), (43), and (44), these requirements can be achieved by imposing

$$[K] = 0, \quad [\square^k R] = 0, \quad n^\mu [\nabla_\mu \square^k R] = 0, \quad k \geq 0. \quad (47)$$

In fact, the last condition can be strengthened with the help of (19),

$$\boxed{[K] = 0, \quad [\square^k R] = 0, \quad [\nabla_\mu \square^k R] = 0, \quad k \geq 0.} \quad (48)$$

Taking into consideration the first two conditions of (48), it is now easy to see that $F(\square)R$ does not contain the singular part. Therefore, the first two terms of (45) are well defined as well. The equations (48) are the *junction conditions of IDG*. They can be compared with local $R + R^2$ theory, $F(\square) = 1$, which is well defined whenever the trace of extrinsic curvature has to be zero, $[K] = 0$, but R can be discontinuous, $[R] \neq 0$, and its higher derivatives contain singular parts.

Since the action (36) contains just the quadratic term,

$$RF(\square)R, \quad (49)$$

which was studied above, the conditions (48) are already sufficient to make the action mathematically sensible. Let us note that also the intermediary expressions that are produced in the integration by parts, $R(\square^n R) \rightarrow (\square^n R)R$,

$$\nabla_\mu \square^k R \nabla^\mu \square^{n-k-1} R, \quad \square^k R \square^{n-k} R, \quad 0 \leq k \leq n-1, \quad (50)$$

are well defined, since these are exactly the terms that were freed from ill-defined expressions in (46).

V. JUNCTION FIELD EQUATIONS

Having obtained the junctions conditions (48), we can now ask what matter content is allowed on the hypersurface Σ . We identify the parts of the energy-momentum tensor that are associated to Σ , derive the respective junction field equations, and discuss the conditions where there is no matter concentrated on Σ .

A. Singular parts of $T_{\mu\nu}$

Following Refs. [21,55], let us consider an energy-momentum tensor distribution with a singular part proportional to δ ,

$$\underline{T}_{\mu\nu} = T_{\mu\nu}^+ \underline{\theta} + T_{\mu\nu}^- (1 - \underline{\theta}) + T_{\mu\nu} \delta, \quad (51)$$

whose coefficient can be decomposed into the tangent, tangent-normal, and normal parts,

$$\begin{aligned} T_{\mu\nu} &\equiv \tau_{\mu\nu} + \rho_\mu n_\nu + \rho_\nu n_\mu + \sigma n_\mu n_\nu, \\ \tau_{\mu\nu} &\equiv h_\mu^\rho h_\nu^\sigma T_{\rho\sigma}, \quad \rho_\mu \equiv h_\mu^\rho n^\sigma T_{\rho\sigma}, \quad \sigma \equiv n^\rho n^\sigma T_{\rho\sigma}. \end{aligned} \quad (52)$$

Here, $\tau_{\mu\nu}$ is the *energy-momentum tensor on Σ* , ρ_μ is the *external flux momentum*, and σ is the *external tension*. Assuming that the conditions (48) are satisfied, we can now return to the field equations (39). Thanks to (31) and (33), only the first two terms of (39) have nontrivial singular part,

$$\begin{aligned} \underline{G}_{\mu\nu} &= G_{\mu\nu}^+ \underline{\theta} + G_{\mu\nu}^- (1 - \underline{\theta}) + \mathcal{G}_{\mu\nu} \delta, \\ \underline{R}_{\mu\nu} F(\square) \underline{R} &= R_{\mu\nu}^+ F(\square) R^+ \underline{\theta} + R_{\mu\nu}^- F(\square) R^- (1 - \underline{\theta}) \\ &\quad + \mathcal{R}_{\mu\nu} (F(\square) R)^\Sigma. \end{aligned} \quad (53)$$

Indeed, the singular part of $\nabla_\mu \nabla_\nu F(\square) \underline{R}$ is zero since it equals $\nabla_\mu (n_\nu [F(\square) R] \delta) + n_\mu [\nabla_\nu F(\square) R] \delta = 0$. Terms with $\overleftrightarrow{\Omega}$ and $\overleftrightarrow{\Theta}$ vanish because all expressions in the products are free from singular parts. Thus, we arrive at

$$\mathcal{T}_{\mu\nu} = \mathcal{G}_{\mu\nu} + 2\mathcal{R}_{\mu\nu} (F(\square) R)^\Sigma. \quad (54)$$

Employing (31), (33), and (52), we find that the theory admits the nonzero energy-momentum tensor on Σ with vanishing external flux momentum and external tension,

$$\boxed{\tau_{\mu\nu} = -(1 + 2(F(\square) R)^\Sigma) [K_{\mu\nu}], \quad \rho_\mu = 0, \quad \sigma = 0.} \quad (55)$$

This differs from $R + R^2$ theory by vanishing external energy flux and external tension and from GR (for which $\rho_\mu = \sigma = 0$) by a different form of the energy-momentum tensor on Σ , which is $\tau_{\mu\nu} = -[K_{\mu\nu}] + h_{\mu\nu} [K]$. An interesting property of $\tau_{\mu\nu}$ in IDG is that it is necessarily traceless, $\tau \equiv \tau^\mu_\mu = 0$.

B. Jumps of $T_{\mu\nu}$

Let us find the equations for the jump of $T_{\mu\nu}$. Subtracting the field equations on both sides of Σ ,

$$\begin{aligned} T_{\mu\nu}^\pm &= G_{\mu\nu}^\pm + 2R_{\mu\nu}^\pm F(\square) R^\pm - 2(\nabla_\mu \nabla_\nu - g_{\mu\nu}^\pm \square) F(\square) R^\pm \\ &\quad - \frac{1}{2} g_{\mu\nu}^\pm R^\pm F(\square) R^\pm - R^\pm \overleftrightarrow{\Omega}_{\mu\nu} R^\pm \\ &\quad + \frac{1}{2} g_{\mu\nu}^\pm (R^\pm \overleftrightarrow{\Omega}_\sigma R^\pm + R^\pm \overleftrightarrow{\Theta} \square R^\pm), \end{aligned} \quad (56)$$

we obtain

$$[T_{\mu\nu}] = [G_{\mu\nu}] + 2[R_{\mu\nu}] (F(\square) R)^\Sigma - 2[\nabla_\mu \nabla_\nu F(\square) R]. \quad (57)$$

The jump of the second covariant derivative can be calculated using (22), where $U_{\mu\nu}$ reduces to $-h_{\mu}^{\rho}[\Gamma_{\rho\nu}^{\kappa}] \times (\nabla_{\kappa} F(\square) R)^{\Sigma}$, and the formula for the jump of Christoffel symbols (29),

$$\begin{aligned} & [\nabla_{\mu} \nabla_{\nu} F(\square) R] \\ &= n_{\mu} n_{\nu} n^{\kappa} n^{\rho} [\nabla_{\kappa} \nabla_{\rho} F(\square) R] \\ &\quad - (n_{\mu} [K_{\nu}^{\lambda}] + n_{\nu} [K_{\mu}^{\lambda}] - n^{\lambda} [K_{\mu\nu}]) (\nabla_{\lambda} F(\square) R)^{\Sigma} \\ &= -(n_{\mu} [K_{\nu}^{\lambda}] + n_{\nu} [K_{\mu}^{\lambda}] - n^{\lambda} [K_{\mu\nu}]) (\nabla_{\lambda} F(\square) R)^{\Sigma}, \end{aligned} \quad (58)$$

where the second equality comes from the trace of the first line. The energy-momentum quantities (55) arise from the normal and tangent-normal projections of the right-hand side of (57). Employing relation (58) together with (34),

$$\begin{aligned} n^{\mu} n^{\nu} [G_{\mu\nu}] &= n^{\mu} n^{\nu} [R_{\mu\nu}] = -K_{\mu\nu}^{\Sigma} [K^{\mu\nu}], \\ n^{\mu} h_{\kappa}^{\nu} [G_{\mu\nu}] &= n^{\mu} h_{\kappa}^{\nu} [R_{\mu\nu}] = \bar{\nabla}^{\mu} [K_{\mu\kappa}], \end{aligned} \quad (59)$$

we arrive at the *junction field equation of IDG*,

$$\begin{aligned} n^{\mu} n^{\nu} [T_{\mu\nu}] &= -(1 + 2(F(\square) R)^{\Sigma}) K_{\mu\nu}^{\Sigma} [K^{\mu\nu}], \\ n^{\mu} h_{\kappa}^{\nu} [T_{\mu\nu}] &= (1 + 2(F(\square) R)^{\Sigma}) \bar{\nabla}^{\mu} [K_{\mu\kappa}] \\ &\quad + 2[K_{\kappa}^{\mu}] (\nabla_{\mu} F(\square) R)^{\Sigma}. \end{aligned} \quad (60)$$

Since the divergence of (55) reads

$$\bar{\nabla}^{\mu} \tau_{\mu\kappa} = -(1 + 2(F(\square) R)^{\Sigma}) \bar{\nabla}^{\mu} K_{\mu\kappa} - 2[K_{\mu\kappa}] \bar{\nabla}^{\mu} ((F(\square) R)^{\Sigma}), \quad (61)$$

these junction field equations can be rewritten purely in terms of the energy-momentum tensor on Σ ,

$$\begin{aligned} n^{\mu} n^{\nu} [T_{\mu\nu}] &= K_{\mu\nu}^{\Sigma} \tau^{\mu\nu}, \\ n^{\mu} h_{\kappa}^{\nu} [T_{\mu\nu}] &= -\bar{\nabla}^{\mu} \tau_{\mu\kappa}. \end{aligned} \quad (62)$$

In this form, the junction field equations resemble their GR counterparts. The difference is in the expression for the energy-momentum quantities on Σ , which is given by (55) for IDG.

C. Proper matching conditions

Let us discuss a physically important scenario in which no matter content is allowed on the matching hypersurface Σ . This situation corresponds to the vanishing singular part of the energy-momentum tensor, $T_{\mu\nu} = 0$. The resulting conditions are often called the *proper junction conditions*; see Ref. [21]. The most studied examples of this kind involve the matching of the interior and exterior geometries for various static or dynamical star with models. In our specific theory, the requirement that $T_{\mu\nu} = 0$ implies that

either the jump of the extrinsic curvature vanishes [see Ref. (55)],

$$[K_{\mu\nu}] = 0, \quad (63)$$

or the $F(\square)$ -dependent expression,

$$1 + 2(F(\square) R)^{\Sigma} = 0, \quad (64)$$

is zero.

The first condition leads to the Ricci and Riemann tensors that are free of singular parts. The corresponding proper matching conditions then read

$$[K_{\mu\nu}] = 0, \quad [\square^k R] = 0, \quad [\nabla_{\mu} \square^k R] = 0, \quad k \geq 0. \quad (65)$$

It is worth noting that the extra condition (63) that arises from the proper matching is the same as in GR. This is because the term this theory introduces to the singular part of the energy-momentum tensor is directly proportional to the step of the extrinsic curvature. This is not true, for example, in modifications of GR that also involve quadratic terms of the Riemann tensor [21], where the proper matching leads to vanishing jumps (and first derivatives) of the Riemann tensor.

The second option (64) can be rewritten as

$$F(\square) R^{\pm}|_{\Sigma} = -\frac{1}{2}, \quad (66)$$

where we used $[\square^k R] = 0$. Unfortunately, this equation requires evaluation of the action of a $F(\square)$, which is highly nontrivial in a generic spacetime. However, it might be worth studying in cases where R can be decomposed into a sum of eigenfunctions of $\square$.

VI. APPLICATIONS

A. Implications for braneworld models

Braneworld models of gravity are constructed to solve the hierarchy problem and the cosmological constant problem [56,57]. The most common example is the Randall-Sundrum braneworld [58,59]. Such a model consists of a five-dimensional anti-de Sitter spacetime (called the *bulk*) with $\mathbb{Z}_2$ symmetry with respect to a flat four-dimensional hypersurface (referred to as the *brane*) representing our four-dimensional world. Following the covariant Shiromizu-Maeda-Sasaki approach [60] (see also Ref. [56] for a review), one can generalize this study to any five-dimensional bulk geometry with $\mathbb{Z}_2$ symmetry with respect to an arbitrary four-dimensional brane, and to arbitrary gravity theories in the bulk. The bulk metric is assumed to be a solution of the considered theory with a cosmological constant, and the brane field equations arise from the junction conditions of the theory. This is where

our results come into play, if the five-dimensional bulk theory is taken to be IDG.

Equation (55) has a direct physical consequence on possible cosmological braneworld models on IDG.⁵ In order to explore the properties of the brane in IDG, we write its energy-momentum tensor as⁶

$$\tau_{ab} = \tau_{ab}^m - \lambda h_{ab}, \quad (67)$$

where τ_{ab}^m represents the energy-momentum tensor of the matter fields confined to the brane and λ is the *internal tension of the brane*,⁷ which accounts for the accelerated cosmological-constant-like expansion on the brane. In the original Randall-Sundrum model in GR, it is related to the cosmological constant in the bulk.

Taking the trace of (67), we obtain

$$h^{ab}\tau_{ab} = \tau^m - 4\lambda, \quad (68)$$

where $\tau^m \equiv h^{ab}\tau_{ab}^m$. Clearly, the left-hand side is actually the trace of the total energy-momentum tensor on the brane $\tau_{\mu\nu}$ [cf. Eq. (52)],

$$\tau = \tau^m - 4\lambda, \quad (69)$$

because $\tau = g^{\mu\nu}\tau_{\mu\nu} = h^{\mu\nu}\tau_{\mu\nu} = \tau_{\mu\nu}h^{ab}e_{(a)}^\mu e_{(b)}^\nu = h^{ab}\tau_{ab}$. Finally, resorting to IDG in the bulk, we know that the trace of the total energy-momentum is zero, $\tau = 0$, which allows us to write the following condition on the internal tension:

$$\lambda = \frac{\tau^m}{4}. \quad (70)$$

Therefore, the internal tension is given purely by the trace of the stress-energy tensor of the matter confined to the brane, as is depicted in Fig. 1. The fact that the matter fields in the brane fix the value of the tension has huge consequences in the possible braneworld models with this nonlocal theory in the bulk. For example, the original Randall-Sundrum model, which assumes no matter content, cannot describe the accelerated expansion on the brane. On the other hand, generalizations of such a model could resolve the fine-tuning problem of the cosmological constant in the bulk [61], since now the tension is uniquely related to the matter content on the brane.

⁵Here, we assume that both the brane and the bulk gravity are described by the nonlocal IDG.

⁶In this section only, we use $\mu, \nu, \dots = 0, 1, 2, 3, 4$ for the bulk indices and $a, b, \dots = 0, 1, 2, 3$ for the brane indices.

⁷Do not confuse this with the external tension σ .

FIG. 1. Consequences of the junction conditions for the brane-world models with the nonlocal $R + RF(\Box)R$ theory in the five-dimensional bulk. The internal tension λ of the four-dimensional brane is given purely by the trace of the energy-momentum tensor of the matter confined to the brane $\tau^m \equiv h^{ab}\tau_{ab}^m$.

B. Junction conditions in star models

The fact that the proper junction conditions (65) impose so many constraints strongly affects the possible metrics that could potentially model physical situations by matching the geometries on the junction between M^+ and M^- . In order to illustrate this, we study two paradigmatic examples. First, we consider a generic static and spherically symmetric star whose exterior region has vanishing Ricci scalar. Then, we discuss a collapsing star with an interior region modeled by the Friedmann-Lemaître-Robertson-Walker metric (FLRW) and the spherically symmetric static exterior with vanishing Ricci scalar. It turns out that

FIG. 2. Consequences of the junction conditions in the nonlocal $R + RF(\Box)R$ theory for star models with vanishing Ricci scalar in the exterior, $R^+ = 0$. The infinite number of conditions implies that the Ricci scalar is zero also in the interior, $R^- = 0$.

in both cases the junction conditions hold only if the Ricci scalar is zero everywhere, as is illustrated schematically in Fig. 2. This implies that most of the GR solutions that describe the interior of static stars or gravitational collapse are not compatible with a vanishing Ricci scalar exterior in this nonlocal theory.

Finally, let us stress that the main purpose of this section is to demonstrate the power of the junction conditions only, not to find new nonlocal solutions, since it would require solving highly complicated field equations of IDG.

1. Static star

Let us consider a model of a static spherically symmetric star with the exterior region M^+ and the interior region M^- separated by a timelike hypersurface Σ . The geometry on both sides is described by the metrics (see, e.g., Ref. [62])

$$g^\pm = -p^\pm(r)dt^2 + q^\pm(r)d\mathbf{r}^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad (71)$$

where $p^\pm(r) > 0$ and $q^\pm(r) > 0$ are four single-variable functions. The junction hypersurface Σ is located at $r = r_0$. It is clear that one indispensable condition from the continuity of the metric is

$$p^-(r_0) = p^+(r_0), \quad q^-(r_0) = q^+(r_0). \quad (72)$$

Since we are interested in the situation without any matter concentrated on Σ , we have to ensure that the proper matching conditions (65) are satisfied ($T_{\mu\nu} = 0$).

First, we focus on the vanishing jump of the extrinsic curvature. The induced metric on Σ , which coincides from both sides, reads

$$h^- = h^+ = -p^\pm(r_0)dt^2 + r_0^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \quad (73)$$

Defining the orthonormal 1-forms, which are orthogonal to Σ and pointed toward M^+ ,

$$n^- = n^+ = \sqrt{q^\pm(r_0)}d\mathbf{r}, \quad (74)$$

we obtain the extrinsic curvature from both side of the hypersurface $r = r_0$,

$$K^\pm = \frac{-p^{\pm'}(r_0)}{2\sqrt{q^\pm(r_0)}}dt^2 + \frac{r_0}{\sqrt{q^\pm(r_0)}}d\vartheta^2 + \frac{r_0 \sin^2\vartheta}{\sqrt{q^\pm(r_0)}}d\varphi^2. \quad (75)$$

The condition $[K] = 0$ together with the continuity of q demands also the continuity of p' ,

$$p^{-'}(r_0) = p^{+'}(r_0). \quad (76)$$

Let us move on to the junction conditions involving the Ricci scalar R . Significant progress can be made if we suppose that the Ricci scalar is identically zero in the exterior region M^+ ,

$$R^+ = 0. \quad (77)$$

Under this assumption, the junction conditions for the jump of the Ricci scalar and its derivatives can be rewritten as

$$\square^k R^-|_{r=r_0} = 0, \quad \partial_r \square^k R^-|_{r=r_0} = 0, \quad k \geq 0, \quad (78)$$

where we used the fact that the Ricci scalar as well as its derivatives depend only on the coordinate r . Suppressing $\pm$ signs, we write the explicit expression for the action of the wave operator associated with the interior metric on an arbitrary r -dependent scalar field $\phi(r)$,

$$\square\phi(r) = \alpha(r)\phi'(r) + \beta(r)\phi''(r), \quad (79)$$

where we denoted

$$\alpha(r) \equiv \frac{1}{2q(r)^2} \left[\left(\frac{p'(r)}{p(r)} + \frac{4}{r} \right) q(r) - q'(r) \right], \quad \beta(r) \equiv \frac{1}{q(r)}. \quad (80)$$

Employing (79), we obtain the recursive formula for $\square^k R$,

$$\square^k R = \alpha(r)\partial_r \square^{k-1} R + \beta(r)\partial_r^2 \square^{k-1} R. \quad (81)$$

The last expression allows us to rewrite the junction conditions (78) purely in terms of partial derivatives with respect to r . The reasoning goes as follows: at $k = 0$, the Ricci scalar and its first derivative must be zero at $r = r_0$. At $k = 1$, the first condition of (78) is just a sum of first and second derivatives. Since $\beta^-(r_0) \neq 0$ (from continuity of q at r_0), the second derivative must also vanish. Going to the second condition of (78) for $k = 1$, which is a sum of first, second, and third derivatives, we find that the third derivative is zero as well thanks to $\beta^-(r_0) \neq 0$. Extending this iterative argument to any $k \geq 0$, it is now clear that the conditions (78) can be reexpressed as

$$\partial_r^k R^-|_{r=r_0} = 0, \quad k \geq 0. \quad (82)$$

Therefore, by assuming analyticity of R , we find that R vanishes throughout the whole manifold M . This gives an explicit equation for f and g that is valid in everywhere in M^- as well as in M^+ ,

$$R = \frac{r^2 q p'^2 + r p (r p' q' - 2 q (r p'' + 2 p')) + 4 p^2 (r q' + q^2 - q)}{2 r^2 p^2 q^2} = 0. \quad (83)$$

This equation (83) and its derivatives enable us to express the k th derivative of functions p^- at $r = r_0$ as

$$\partial_r^k p^-(r_0) = \Phi_k(p^-(r_0), p'^-(r_0), q^-(r_0), q^{-'}(r_0), \dots, \partial_r^{k-1} q^-(r_0)), \quad (84)$$

where we formally eliminated the dependence on $\partial_r^l p^-(r_0)$, $2 \leq l \leq k-1$, by successive substitutions. The freedom in the choice of $\partial_r^k q^-(r_0)$ for $k \geq 1$ can be translated to the statement that all the junction conditions are satisfied for an arbitrary function $q^-(r)$ and a function $p^-(r)$ that is calculated from Eq. (83).

It is important to stress that all these properties follow purely from the junction conditions. Since the theory is now well defined in the theory of distributions, we can move on to the field equations. Usually, this is an extremely difficult task, but our situation is greatly simplified thanks to the condition (83). Indeed, the field equations of the theory (39) reduce everywhere just to

$$R_{\mu\nu} = T_{\mu\nu}, \quad (85)$$

which is effectively equivalent to the Einstein field equations with vanishing scalar curvature. For this reason, such solutions with $R = 0$ everywhere (if they are regular⁸) are the same as the GR solutions, and thus they are independent of the length scale of nonlocality ℓ .

Since we are interested in star models with the vacuum outside $r > r_0$, we put $T_{\mu\nu}^+ = 0$. This is only solved by the Schwarzschild spacetime,

$$p^+(r) = 1 - \frac{2m}{r}, \quad q^+(r) = \left(1 - \frac{2m}{r}\right)^{-1}, \quad (86)$$

where m is a positive constant corresponding to mass of the star. The only possible matter allowed in the interior region $r < r_0$ (if any regular solution exists) is then given by a traceless energy-momentum tensor $T_{\mu\nu}^-$, i.e., $T^- = 0$. As a consequence, the matter inside the star cannot be modeled by common perfect fluids as is usual in GR [63]. For example, the well-known Tolman stars [64] with the Schwarzschild exterior are not solutions of IDG. Thus, if we want to describe the interior of a spherically symmetric and static star with a nontraceless energy-momentum tensor, $T^- \neq 0$, then the exterior geometry cannot have vanishing Ricci scalar R^+ .

⁸A proper treatment of singular geometries in IDG requires nonlinear generalized functions.

2. Collapsing star

In this section, we shall consider a model of a collapsing star where the interior is described by a closed⁹ FLRW metric (positive spatial curvature) in comoving coordinates,

$$g^- = -d\tau^2 + a^2(\tau)(d\lambda^2 + \sin^2 \lambda (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)), \quad (87)$$

where $a(\tau)$ is the scale factor. The junction surface is located at $\lambda = \lambda_0$. The 4-velocity of the observer comoving with the surface is $u = \partial_\tau$; see Ref. [65]. The exterior is given by the spherically symmetric static geometry (with $p^+ > 0$ and $q^+ > 0$),

$$\begin{aligned} g^+ &= -p^+(r)dt^2 + q^+(r)dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \\ &= -(p^+(r)\dot{t}^2 - q^+(r)\dot{r}^2)d\tau^2 + (-p^+(r)t'^2 \\ &\quad + q^+(r)r'^2)d\lambda^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \end{aligned} \quad (88)$$

where we performed a generic transformation $t = t(\tau, \lambda)$, $r = r(\tau, \lambda)$ in the second line (and hid the explicit dependence on τ and λ). We use the shortcuts $\dot{f} \equiv \partial_\tau f$ and $f' \equiv \partial_\lambda f$.

First, we focus on the restrictions coming from that the continuity of the metric and the continuity of the extrinsic curvature. From the inside, the induced metric reads

$$h^- = -d\tau^2 + a^2(\tau) \sin^2 \lambda_0 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (89)$$

while from the outside, it can be expressed as

$$\begin{aligned} h^+ &= -(p^+(r(\tau, \lambda_0))\dot{t}^2(\tau, \lambda_0) - q^+(r(\tau, \lambda_0))\dot{r}^2(\tau, \lambda_0))d\tau^2 \\ &\quad + r^2(\tau, \lambda_0)(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \end{aligned} \quad (90)$$

By comparing (89) with (90), it becomes clear that the continuity of h implies

$$\begin{aligned} r(\tau, \lambda_0) &= a(\tau) \sin \lambda_0, \\ \dot{t}(\tau, \lambda_0) p^+(\tau, \lambda_0) &= \sqrt{p^+(\tau, \lambda_0)(1 + q^+(\tau, \lambda_0)\dot{r}^2(\tau, \lambda_0))} \\ &\equiv \Psi(\tau). \end{aligned} \quad (91)$$

⁹The reason we focus on the closed FLRW only is because the open FLRW geometries lead to the scale factors $a(\tau) = \begin{cases} \sqrt{(\tau_* - \tau_0)(\tau - \tau_0)}, & \epsilon = 0, \\ \sqrt{(\tau - \tau_0)^2 - (\tau_* - \tau_0)^2}, & \epsilon < 0, \end{cases}$ where ϵ denotes the spatial curvature and τ_* and τ_0 are constants. These scale factors cannot represent realistic models of the collapsing star because they do not have an initial state where the star is at rest as $\dot{a}(\tau) \neq 0$.

The second expression can be used to obtain $t(\tau, \lambda_0)$ by direct integration

$$t(\tau, \lambda_0) = \int^\tau d\tilde{\tau} \frac{\Psi(\tilde{\tau})}{p^+(\tilde{\tau}, \lambda_0)}. \quad (92)$$

Following the arguments in Sec. II B, we can now argue that also $\mathbf{g}$ is continuous. Consider the orthonormal 1-form orthogonal to $\lambda = \lambda_0$ and directed toward M^+ . We immediately see that from the interior it reads

$$\mathbf{n}^- = a(\tau) \mathbf{d}\lambda. \quad (93)$$

Realizing that orthonormal 1-form (pointed toward M^+) is actually defined uniquely by the conditions $\mathbf{n} \cdot \mathbf{e}_{(a)} = 0$ and $\mathbf{n} \cdot \mathbf{n}^\sharp = 1$ with $\mathbf{e}_{(a)}$ being the vector frame tangent to the hypersurface $\lambda = \lambda_0$ [e.g., $\mathbf{e}_{(0)} \equiv \mathbf{u}$, $\mathbf{e}_{(1)} \equiv r^{-1} \partial_\vartheta$, and $\mathbf{e}_{(2)} \equiv (r \sin \vartheta)^{-1} \partial_\varphi$], we arrive at

$$\mathbf{n}^+ = \mathbf{n}^- = -\dot{r}(\tau, \lambda_0) \mathbf{d}t + \dot{t}(\tau, \lambda_0) \mathbf{d}r. \quad (94)$$

As a result, the metric $\mathbf{g}$ is continuous due to $\mathbf{g} = \mathbf{h} + \mathbf{n}\mathbf{n}$. Having defined the orthonormal 1-form, we can now calculate and compare the extrinsic curvatures from both sides of the hypersurface. For the interior side, one gets

$$\mathbf{K}^- = a(\tau) \cos \lambda_0 \sin \lambda_0 (\mathbf{d}\vartheta^2 + \sin^2 \vartheta \mathbf{d}\varphi^2), \quad (95)$$

and for the exterior,

$$\mathbf{K}^+ = -\frac{\dot{\Psi}}{\dot{r}} \mathbf{d}\tau^2 + a(\tau) \Psi(\tau) \sin \lambda_0 (\mathbf{d}\vartheta^2 + \sin^2 \vartheta \mathbf{d}\varphi^2). \quad (96)$$

By demanding that the jump of the extrinsic curvature is zero, $\mathbf{K}$, we find that $\Psi(\tau)$ is constant,

$$\Psi(\tau) = \cos \lambda_0. \quad (97)$$

Hence, from the continuity of the metric and the extrinsic curvature, we have obtained the following relations:

$$r(\tau, \lambda_0) = a(\tau) \sin \lambda_0, \quad t(\tau, \lambda_0) = \cos \lambda_0 \int^\tau \frac{d\tilde{\tau}}{p^+(\tilde{\tau}, \lambda_0)}. \quad (98)$$

Let us now focus on the remaining junction conditions, which put restrictions on the jump of the Ricci scalar. Assuming that the exterior geometry has vanishing Ricci scalar, $R^+ = 0$, we can write in comoving coordinates

$$\square^k R^-|_{\lambda=\lambda_0} = 0, \quad \partial_\tau \square^k R^-|_{\lambda=\lambda_0} = 0, \quad k \geq 0. \quad (99)$$

The Ricci scalar of the interior closed FLRW metric is given by

$$R^- = \frac{6}{a^2} (1 + \dot{a}^2 + a\ddot{a}). \quad (100)$$

Since the repeated action of the wave operator reads

$$\square^k R^- = -3 \frac{\dot{a}(\tau)}{a(\tau)} \partial_\tau \square^{k-1} R^- - \partial_\tau^2 \square^{k-1} R^-, \quad (101)$$

We can make a very similar argument to the one we did for the static case. Following these steps, one can find that the conditions (99) can be rewritten as

$$\partial_\tau^k R^-|_{\lambda=\lambda_0} = 0, \quad k \geq 0. \quad (102)$$

Again, assuming the analyticity of R , this clearly implies that the Ricci scalar must be zero not only in the exterior but also throughout the whole interior region. The equation $R^- = 0$ has the following solution,

$$a(\tau) = \sqrt{(\tau_* - \tau_0)^2 - (\tau - \tau_0)^2}, \quad (103)$$

where τ_* and τ_0 are arbitrary constants. As before, we can see that the nonlocal character of the theory causes the constraints themselves to affect the whole spacetime and determine the geometry to a great extent. In particular, they imply that the Ricci scalar must vanish everywhere, $R = 0$. Thus, even without solving the field equations, we already see that the our simple model with the closed FLRW interior and spherically symmetric static exterior of vanishing Ricci scalar (e.g., the Schwarzschild metric) does not allow bounce. A more complicated geometry in the exterior is required.

Finally, let us comment on the interior geometry with (103). The scale factor $a(\tau)$ has the maximum value at $\tau = \tau_0$ and decreases toward $\tau = \tau_*$, which corresponds to the curvature singularity as can be seen from the Kretschmann scalar,

$$R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} = \frac{24(\tau_* - \tau_0)^4}{(\tau - \tau_*)^4 (\tau - 2\tau_0 + \tau_*)^4}. \quad (104)$$

Unfortunately, due to the singular behavior, the check whether this geometry is a regular solution is highly nontrivial and goes beyond the scope of this paper. Our statement that $R = 0$ is only valid outside the point of singularity $\tau < \tau_*$. In order to check that a singular geometry is a solution of IDG, one should first include the singular point to the manifold as well and describe the Ricci scalar using nonlinear generalized functions localized at $\tau = \tau_*$.¹⁰ As a next step, one could calculate the action of $F(\square)$ on R , for instance, using the decomposition into the eigenfunctions of $\square$. Such a study is very challenging and

¹⁰See Ref. [66] for such a description of the singularity in the Schwarzschild geometry.

has never been done even for the simplest cases such as the Schwarzschild metric.

VII. CONCLUSIONS

In this paper, we found the junction conditions and junction field equations for the nonlocal infinite derivative theory $R + RF(\Box)R$. To achieve that, we relied on the theory of linear distributions to describe the potential discontinuities, and we assumed that the conditions can be imposed by avoiding ill-defined expressions term by term in the infinite sums (ignoring their regularizing and deregularizing properties). After introducing necessary tools of distributional theory and the infinite derivative gravity, we obtained the matching conditions (48) by requiring that the theory is well behaved in a distributional sense. These conditions are much more restrictive than in the general relativity or other local theories of gravity. They say that the trace of the extrinsic curvature as well as $\Box^k R$ and $\nabla\Box^k R$ for all $k \geq 0$ must be continuous across the matching hypersurface.

The junction field equations and the allowed matter content on the hypersurface get also modified in IDG; see (62) with (55). Similar to GR, the external flux and external tension are not present (which is not the case of $R + R^2$, for example), but unlike in GR, the energy-momentum tensor on the hypersurface must be traceless. From there, it also follows that the proper matching with no matter on the hypersurface requires a vanishing jump of the extrinsic curvature.

Finally, we illustrated the use of our results in several physically motivated scenarios. First, we focused on the implications for the braneworld models with five-dimensional bulk theory being IDG. It turns out that the internal tension of the brane is then given purely by the trace of the energy-momentum tensor of the matter confined to the brane. Next, we studied the consequences of the junction

conditions on two simple examples of static and collapsing stars. In particular, we found that having a spherically symmetric static exterior with zero Ricci scalar implies that the Ricci scalar in the interior region must either be nonanalytic or also vanish. These examples demonstrate that the junction condition of nonlocal theories hugely constrains the possible metrics that one can consider even before going into the field equations. Of course, a much more difficult task is to check whether the resulting geometry actually is a solution or not. To answer this question, new methods of solving nonlocal nonlinear equations in curved geometry must be developed.

To conclude, let us point out that this work can be extended by considering terms of the form $R_{\mu\nu}F_2(\Box)R^{\mu\nu}$ and $R_{\mu\nu\kappa\lambda}F_3(\Box)R^{\mu\nu\kappa\lambda}$ in the action. This will of course impose more constraints on the jumps of the curvature, possibly reaching a point where only completely smooth metrics with no matter content on the hypersurface will be allowed. An even more interesting problem is the study that goes beyond our simplifying assumptions, especially the case where the nonlocal operator can produce a singular part when it acts on a smooth function. We plan to investigate these extensions in our future works.

ACKNOWLEDGMENTS

I. K. and A. M. are supported by Netherlands Organization for Scientific Research (NWO) Grant No. 680-91-119. F. J. M. T. acknowledges financial support from NRF Grants No. 120390, Reference: BSFP190416431035; No. 120396, Reference: CSRP190405427545; No. 101775, Reference: SFH150727131568; and the financial support from the NASSP Programme—UCT node and the Van Swinderen Institute at the University of Groningen.

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