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Rationality, Imitation, and Rational Imitation in Spatial Public Goods Games

Alain Govaert , Pouria Ramazi , and Ming Cao

Abstract—In both economic and evolutionary theories of games, two general classes of evolution can be identified: 1) dynamics based on myopic optimization and 2) dynamics based on imitations or replications. The collective behavior of structured populations governed by these dynamics can vary significantly. Particularly in social dilemmas, myopic optimizations typically lead to Nash equilibrium payoffs that are well below the optimum, e.g., *the tragedy of the commons*, whereas imitations can hinder equilibration while allowing higher cooperation levels and payoffs. Motivated by economic and behavioral studies, in this article, we investigate how the benefits of the two dynamics can be combined in an intuitive decision rule, *rational imitation*, that is to mimic successful others only if it earns you a higher payoff. In contrast to purely rational (best-response) or purely imitative decision rules, the combination in *rational imitation* dynamics both guarantees *finite time convergence* to an imitation equilibrium profile on arbitrary networks and can facilitate *high levels of cooperation* for small public goods multipliers.

Index Terms—Convergence, cooperation, generalized Nash equilibrium, imitation, n -player games, public goods game, rationality, spatial games.

I. INTRODUCTION

IMITATION and rational selection are two seemingly paradoxically coexisting behaviors that are often observed in real-life decision-making processes. For example, companies can make investment decisions based on deliberate benefit-to-cost analysis or simply decide to invest because a successful competitor has done so already [2]–[4]. Likewise, the adoption of a product can be motivated by others, or because it provides some immediate benefit for an “innovator” [5]. Indeed, whereas

rationality is often coupled to *innovation*, imitation is often linked to *social learning* [6]. Also in the literature of game theory, myopic best response dynamics are known as *innovative* dynamics because they can introduce actions that were not played before [7]. Clearly, dynamics based on imitations do not share this innovative feature and, hence, can have significantly different equilibrium profiles than those processes based on best responses.

The rationality of best responses is in line with economic theories that suggest *absolute optimization* is a natural result of evolutionary forces [8]. For these types of dynamics, the long-run collective behavior of noncooperative network games, in which players interact exclusively with a fixed group of neighbors, has been studied extensively. This has resulted in the identification of several classes of games that converge to a pure Nash equilibrium under a variety of (myopic) best response processes [9]–[13], and brought forth a number of algorithms that ensure convergence to an equilibrium [14]–[16]. More biologically inspired imitation dynamics on spatial structures defined by graphs have, amongst others, been studied in [7], [17]–[19]. One of the key drivers of *evolutionary graph theory* is to identify conditions and mechanisms under which *cooperation* can emerge and evolve in social dilemmas in which the individual incentives are contradictory to the benefit of the system as a whole [20]. In such social dilemmas, myopic optimizations tend to generate outcomes with payoffs that are far from the system optimum: a situation known as the tragedy of the commons [21]. Spatial structure in the game-play interactions potentially overcomes this problem by means of *network reciprocity* through which cooperators can succeed by forming network clusters in which they help each other [20]. A simple rule of thumb dictates that cooperation can indeed succeed in spatial social dilemmas under imitation-based dynamics whenever the benefit-to-cost ratio exceeds the average degree in the network [22]. However, for innovative rational dynamics in which players can introduce new actions at any given time, network reciprocity is far less effective in promoting cooperative actions, and the simple rule-of-thumb does not hold. The main reason is that, within a cluster of cooperators, the myopic best response of a player is to *defect*. Hence, the clusters can break down relatively easily if players can best respond and introduce new actions.

The convergence properties related to decision-making processes based on myopic optimizations are well understood. However, because the mechanisms for the evolution of cooperation tend to be less effective, more often than not at the equilibrium of a social dilemma, the players need to be satisfied

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with relatively low payoffs. And even though imitation-based dynamics can lead to better outcomes, the mathematical study of their dynamics on spatial games is a challenging problem: first, because there typically exists a multitude of possible nontrivial outcomes [17] and second, because the existing optimization techniques used in the analysis of rational best responses are not applicable. Recent literature showed convergence of imitation dynamics in population potential games on networks whose behavior is described by ODEs [23]–[25]. In finite networks, however, imitations in pairwise matrix-games can easily *prevent* the decision process to converge to an equilibrium at which all players are satisfied with their decision [26]—even in potential games and when some of the players best respond [27]. To the best of our knowledge, the convergence of imitation dynamics in games with group-wise interactions on finite networks remains unexplored. Thus, it remains unknown whether the appealing high cooperation level in social dilemmas under imitation dynamics can be combined with the well-known convergence of best response dynamics in potential games on finite networks [13]. To this end, we study the effects of rationality and imitation on the asymptotic behavior and cooperation levels of a general groupwise social dilemma model known as the spatial public goods game [19]. Motivated by economic and behavioral research that supports the idea that relative performance [28]–[31] and rational social learning [32] are decisive in decision-making processes, our intuitive decision rule, called *rational imitation*, guarantees convergence, and can effectively facilitate cooperation for relatively small public goods multipliers. Under such rational imitations, successful neighbors are imitated only if they are expected to be an improvement, not necessarily the best [33], in terms of one’s success. This combination leads to beneficial dynamic features in public goods games and spatial prisoner’s dilemmas that cannot be explained only by best responses or imitations. We find that every network converges under best-response and rational imitation, but the same *does not* hold for unconditional imitation. We then focus on some of those specific network structures that equilibrate under all three dynamics and compare equilibrium cooperation levels. For several regular networks—in which persistent oscillations exist under unconditional imitation dynamics—we find the conditions under which cooperation clusters are sustained at a rational imitation equilibrium but break down under best response. This holds for support for more complex networks as shown in simulations. Our findings contribute to the convergence analysis of unconditional imitation dynamics in groupwise potential games, and, in particular, the public goods game. We show that the commonly studied ring and star networks are guaranteed to converge, but that the generalization of convergence to arbitrary networks is impossible without adjustments to the dynamics. More important, our results contribute to the development of learning dynamics for network games that possess both the beneficial properties of imitation dynamics in social dilemmas and the convergence properties of best response dynamics in finite networks. This opens the door to the design of novel control methods for complex network systems that have to rely on cooperation for the maintenance of publicly available goods.

The rest of this article is organized as follows. In Section II, a general model for spatial public goods games is described. The spatial dynamics of rationality and imitation are formally defined in Sections II-B and II-C. The convergence properties of best response rational and unconditional imitation dynamics in spatial public goods games are studied in Section III. In Section IV, the effect of rational imitation on the level of cooperation at an equilibrium is discussed and compared to unconditional and best response dynamics. Finally, Section V concludes this article.

I. Notation

To emphasize the i th element of $\mathbf{x} \in \mathbb{R}^n$, we write $\mathbf{x} = (x_i, \mathbf{x}_{-i})$, where $\mathbf{x}_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$. The cardinality of a set $\mathcal{A}$ is denoted by $|\mathcal{A}|$. Given a nonempty finite set $\mathcal{B}$ with cardinality m , the set valued function $\max^k(\mathcal{B})$, where $k \leq m$, returns the k -highest values in the set $\mathcal{B}$. The power set of a nonempty set $\mathcal{B}$ is denoted by $2^{\mathcal{B}}$.

II. RATIONALITY AND IMITATION UNDER SPATIAL PUBLIC GOODS GAME

We study a network of decision-making players who participate in a spatial public goods game and revise their decisions over a discrete course of time.

A. Spatial Public Goods Games

Consider a simple undirected network $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, where the node set $\mathcal{V} = \{1, \dots, n\}$ represents players. Denote the set of neighbors of $i \in \mathcal{V}$ by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Corresponding to each player i , the graph structure defines a *group* of players $\tilde{\mathcal{N}}_i = \mathcal{N}_i \cup \{i\}$, known as the *neighborhood* of player i , that play a *public goods game (PGG)* referred to as the game *centered around player i* . Therefore, in total, every player i participates in $|\mathcal{N}_i| + 1$ games [19]: one centered around herself and $|\mathcal{N}_i|$ centered around her neighbors. Every player i chooses an action $a_i \in \{0, 1\}$ that is either to *cooperate* ($a_i = 1$), namely, to contribute a fixed amount $c_i > 0$ to a publicly available resource called the *public good*, or *defect* ($a_i = 0$), namely, to contribute nothing. The player employs this same action in all of the games that she participates in [19]. This applies to the cases when players do not have the cognitive capabilities to discriminate between co-players, or when there is only one public good and the contribution scales with the degree of the player. Cooperators and defectors profit equally from the public good: all players participating in a game centered around player $j \in \mathcal{V}$, evenly share the *production* of that game defined by $p_j : \{0, 1\}^{|\tilde{\mathcal{N}}_j|} \rightarrow \mathbb{R}$, which is a function of the actions of the players in the neighborhood $\tilde{\mathcal{N}}_j$, denoted by the vector $\mathbf{a}_j$. The local payoff of an player i upon the participation in this game is, hence

$$\pi_{ij}(\mathbf{a}_j) = \frac{p_j(\mathbf{a}_j)}{|\tilde{\mathcal{N}}_j| + 1} - c_i a_i \quad \forall i \in \tilde{\mathcal{N}}_j. \quad (1)$$

Production functions are, typically, nondecreasing in the number of cooperators, reflecting that contributions increase the public good [34]–[36]. We relax this monotonicity assumption, allowing us to study the maintenance of artificially scarce goods

known as *club goods*. The total payoff of player i is a weighted summation of the local payoffs she earns at each of the $|\mathcal{N}_i| + 1$ games

$$\pi_i(\mathbf{a}) = \sum_{j \in \mathcal{N}_i} \lambda_j \pi_{ij}(\mathbf{a}_j) \quad (2)$$

where $\lambda_j \in \mathbb{R}$, $j \in \mathcal{V}$, represents the relative importance of the game centered around player j , and $\mathbf{a} = (a_1, \dots, a_n)^\top \in \{0, 1\}^n$ is the collective action profile of all of the players.

In an alternative spatial representation, the group structures are determined by a bipartite graph $\mathbb{B} = (\mathcal{M}, \mathcal{V}, \mathcal{K})$, with the player set $\mathcal{V}$, a set of nonempty group structures $\mathcal{M} \subset 2^\mathcal{V}$ and edge set $\mathcal{K} \subset \mathcal{V} \times \mathcal{M}$ [37]. The biadjacency matrix of $\mathbb{B}$, denoted by $\mathbf{B} = [b_{ji}] \in \mathbb{R}^{|\mathcal{M}| \times |\mathcal{V}|}$, is defined such that $b_{ji} = 1$ if and only if $(j, i) \in \mathcal{K}$ for all $i \in \mathcal{V}$, $j \in \mathcal{M}$, and zero otherwise. So the j^{th} row of the biadjacency matrix $\mathbf{B}$ determines which players interact in the PGG played in group j . Hence, the number of players in a group $j \in \mathcal{M}$ equals $|\sum_{i \in \mathcal{V}} b_{ji}| > 0$. The payoff obtained in group $j \in \mathcal{M}$ and the total payoff of player $i \in \mathcal{V}$ are, thus

$$\pi_{ij}(\mathbf{a}_j) = \frac{p_j(\mathbf{a}_j)}{|\sum_{i \in \mathcal{V}} b_{ji}|} - c_i a_i, \quad \pi_i(\mathbf{a}) = \sum_{j \in \mathcal{M}} b_{ji} \pi_{ij}(\mathbf{a}_j). \quad (3)$$

Such a representation may be used when the group structures are obtained from data that does not contain information about the entire complex social network, but just the groups that each individual belongs to.

Example 1 (Homogeneous linear public goods game):

The simplest and most widely studied production function scales linearly with the number of cooperators in the game and every player has the same contribution $c > 0$, i.e.

$$p_i(\mathbf{a}) = rc \sum_{j \in \mathcal{N}_i} a_j, \quad 1 < r < n. \quad (4)$$

Here, r is the *public good multiplier* that can be seen as the benefit-to-cost ratio of the game. It is worth to mention that, even though the parameters c and r are the same for every player, a nonregular network structure introduces asymmetries in the players' payoffs and contributions as detailed in [19], and therefore, this corresponding sequential dynamics can lead to complex behaviors. $\square$

B. Best Response, Unconditional Imitation, and Rational Imitation Update Rules

Each player is associated with an action at time $t = 0$. At every time $t \in \mathbb{Z}_{\geq 0}$, a single player i becomes active to update her action at time $t + 1$ to one in her *feasible action set* $\mathcal{F}_i^x$: $\mathbf{a} \rightarrow \{0, 1\}$ that is determined by some *update rule* x and the action profile at time t

$$a_i(t + 1) \in \mathcal{F}_i^x(\mathbf{a}(t)) \quad \forall t.$$

This results in a dynamical decision-making process, which is referred to as an *asynchronous spatial public goods game*. We consider three update rules: *best response*, *unconditional imitation*, and *rational imitation*. The best-response update rule, denoted by b , defines the feasible action set as those actions

that maximize player i 's payoff against the remaining players' actions. That is

$$\mathcal{F}_i^b(\mathbf{a}) := \operatorname{argmax}_{z \in \{0, 1\}} \pi_i(z, \mathbf{a}_{-i}). \quad (5)$$

In the literature, the feasible action set in (5) is commonly referred to as the *best response set* of player i for opponent action profile $\mathbf{a}_{-i}$. The unconditional imitation rule, denoted by u , defines the feasible set as the actions of the top h_i highest-earners in player i 's neighborhood, $h_i \in \{1, \dots, |\mathcal{N}_i|\}$

$$\mathcal{F}_i^u(\mathbf{a}) := \left\{ a_j \mid j \in \operatorname{argmax}_{j \in \mathcal{N}_i}^{h_i} \pi_j(\mathbf{a}) \right\}. \quad (6)$$

When $h_i = 1$, unconditional imitation recovers “imitate-the-best” decisions [17], where only the most successful players can be imitated. An unconditional imitation may be seen as an *irrational* decision since players do not take into account their *expected* payoff change that results from imitating their neighbors' actions. Arguably, imitation becomes more rational when the expected payoff change is taken into account. Under *rational imitation*, players seek to improve their payoffs, but are restricted to only copy their neighbors' actions. In other words, (similar to unconditional imitation) players copy their neighbors' actions yet (similar to best-response) only if it improves their own payoffs. More specifically, the rational imitation update rule, denoted by r , defines the feasible set as

$$\mathcal{F}_i^r(\mathbf{a}) := \begin{cases} a_i & \text{if } \mathcal{B}_i(\mathbf{a}) = \emptyset \\ \mathcal{B}_i(\mathbf{a}) & \text{otherwise} \end{cases}$$

where $\mathcal{B}_i(\mathbf{a})$ is the set of *better responses* within the feasible action set of unconditional imitation, i.e.

$$\mathcal{B}_i(\mathbf{a}) := \{y \in \mathcal{F}_i^u(\mathbf{a}) \mid \pi_i(y, \mathbf{a}_{-i}) > \pi_i(a_i, \mathbf{a}_{-i})\}.$$

Note that a rational imitation differs from a “relative best response” [30], [31], [33], in the sense that a rational imitation does not require the imitated action to be a best reply over a constrained action set.

For any of the update rules, it may happen that the feasible action set includes both cooperation and defection. We assume that then the active player will stick to her previous action.

Assumption 1: The active player i will keep her action if the update rule puts forward both of the possible actions. That is, given the feasible set $\mathcal{F}_i$

$$\mathcal{F}_i(\mathbf{a}(t)) = \{0, 1\} \Rightarrow a_i(t + 1) = a_i(t).$$

For best response and rational imitation, this assumption ensures that players will only switch if there is a *profitable deviation*, i.e., they earn more by switching. As for unconditional imitation, the assumption implies that imitation occurs only if the active player's action is not played by any of the highest earners.

C. Asynchronous Dynamics

The players' activation sequence $\{i_t\}_{t=0}^\infty$ together with the imitation update rule govern the evolution of the player's actions over time, resulting in *asynchronous PGG dynamics*. Namely,

for every time $t \in \mathbb{Z}_{\geq 0}$, there exists a unique player $i_t \in \mathcal{V}$ such that the collective action dynamics satisfy

$$\mathbf{a}(t+1) \in (\mathcal{F}_{i_t}(\mathbf{a}(t)), \mathbf{a}_{-i_t}(t)) \quad (7)$$

with $\mathcal{F}_{i_t}$ denoting the feasible set. We assume that the activation sequence is *persistent*; that is, each player becomes active infinitely many times [26]. In the long-run, the dynamics either reach an equilibrium action profile in which all players are satisfied with their decisions *or* undergo perpetual oscillations in which a subset of the players do not reach a satisfactory decision and, not necessarily periodically, imitate each other's actions indefinitely [17]. Given an update rule $\mathbf{x}$, the action profile $\mathbf{a}^* \in \{0, 1\}^n$ is an *equilibrium* if

$$\mathbf{a}_i^* \in \mathcal{F}_i^{\mathbf{x}}(\mathbf{a}^*) \quad \forall i \in \mathcal{V}.$$

For best response ($\mathbf{x} = \mathbf{b}$) the equilibrium is *Nash*. We refer to the action profile $\mathbf{a}^*$ as an *unconditional imitation* and *rational imitation* equilibrium when the corresponding update rules are in force. It is worth noting that a rational imitation equilibrium corresponds to a generalized Nash equilibrium, where the constraints over the action sets are determined by the relative performance of the players. In the following section, we study the asymptotic behavior of the sequential PGG dynamics under all three update rules.

III. FINITE TIME CONVERGENCE

A. Best Response And Rational Imitation

Both best response and rational imitation dynamics ensure that whenever a player switches actions, her payoff increases. For these update rules, potential game theory can be used to guarantee the action profile converges in finite time.

Proposition 1 (Finite time convergence under rational imitation): Under Assumption 1, every asynchronous spatial PGG with network or bipartite group structures governed by the rational imitation update rule reaches a rational imitation equilibrium in finite time. $\square$

Proof: It can easily be verified that

$$\psi_i(\mathbf{a}) = \frac{p_i(\mathbf{a})}{|\mathcal{N}_i| + 1} - \sum_{j \in \mathcal{N}_i} c_j a_j \quad (8)$$

is an exact potential function for the local public goods game centered around player $i \in \mathcal{V}$. As a consequence, $P(\mathbf{a}) := \sum_{i=1}^n w_i \psi_i(\mathbf{a})$, is a potential function for the aggregated payoff function (2). To see this, let $\mathbf{a} = (a_i, \mathbf{a}_{-i})$ and $\mathbf{a}' = (a'_i, \mathbf{a}_{-i})$ with $a'_i \neq a_i$. Then

$$\begin{aligned} \pi_j(\mathbf{a}) - \pi_j(\mathbf{a}') &= \sum_{l \in \mathcal{N}_j} w_l (\pi_{jl}(a_j, \mathbf{a}_{-j}) - \pi_{jl}(a'_j, \mathbf{a}_{-j})) \\ &= \sum_{i=1}^n w_i (\psi_i(a_j, \mathbf{a}_{-j}) - \psi_i(a'_j, \mathbf{a}_{-j})) \end{aligned}$$

where we used the fact that for any $k \notin \mathcal{N}_j$, $\psi_k(a_j, \mathbf{a}_{-j}) - \psi_k(a'_j, \mathbf{a}_{-j}) = 0$. It follows that the spatial PGG is an exact potential game. Assumption 1 guarantees that, for every $h = (h_i)_{i \in \mathcal{V}}$, the rational imitation dynamics generate improvement paths and standard potential game theory arguments imply

finite time convergence. For the bipartite representation of the PGG the proof from the local potential $\psi_j(\mathbf{a}) = \frac{p_j(\mathbf{a})}{|\sum_{i \in \mathcal{V}} b_{ji}|} - \sum_{i \in \mathcal{V}} b_{ji} c_i a_i$, and the potential function for the complete payoffs as $\sum_{j \in \mathcal{M}} \psi_j(\mathbf{a})$.

The above theorem shows that for a general class of PGGs, i.e., with heterogeneous contributions and arbitrary production functions, the rational imitation dynamics are guaranteed to converge to a rational imitation equilibrium in finite time. The proof of Theorem 1 also allows us to formulate the following analogous result for best response dynamics.

Corollary 1 (Finite time convergence under best response): Under Assumption 1, any asynchronous spatial PGG with network or bipartite group structures governed by the best response update rule reaches a Nash equilibrium in finite time. $\square$

It is worth noting that the proof of Theorem 1 also implies that for general classes of spatial PGGs, the stationary distribution of log-linear learning can be characterized analytically [13, Theorem 6.1]. In Section IV, we will discuss how the equilibria of rational imitation can significantly differ from the long-run behavior of best response, and unconditional imitation dynamics. Before doing so, let us take a closer look at the convergence properties of the unconditional imitation dynamics for the spatial public goods game in which the group structures are determined by the neighborhoods of a graph.

B. Unconditional Imitation

The behavior of decision processes based on unconditional imitations is a challenging open problem because the generated paths in the combined action profile are not necessarily *improvement paths*: by copying the action of a successful neighbor, a player may decrease its payoff, even if all other players do not change their actions. A second complicating factor, specific to groupwise interactions, is that imitations are limited to direct neighbors whereas the payoffs of the players also depend on *2-hop* neighbors. This creates an asymmetry in the spatial structure: the interaction graph that determines one's payoff and the replacement graph, that determines one's feasible action set are, in general, not equal.

Indeed, for arbitrary spatial structures, convergence is not guaranteed under unconditional imitation. Even for the relatively simple homogeneous linear PGG in Example 1, imitating the best performing neighbors can lead to persistent oscillations. The 2-regular tree in Fig. 1 is a simple example of persistent oscillations of unconditional imitation dynamics. For the public goods multiplier $r = \frac{5}{2}$, best response dynamics of the linear homogeneous PGG converge to the unique full-defection Nash equilibrium. In fact, in order for cooperation to exist at the Nash equilibrium of any 2-regular tree, it is required that $r \geq \frac{8}{3}$ (see Section IV-C). Under unconditional imitation dynamics with $h_i = 1$ for all $i \in \mathcal{V}$, the action profile has persistent oscillations with a large number of cooperators (Fig. 1). Thus, even though players cooperate, they are not necessarily *satisfied* with their decision and keep changing their actions. Under *rational imitation* with $h_i = 1$ for all $i \in \mathcal{V}$, the dynamics converge to the action profile in Fig. 1(a) which, in fact, is a rational imitation equilibrium that coincides with a *generalized Nash equilibrium* [38] in which all players select a *relative best*

Fig. 1. Persistent imitation oscillations in a spatial PGG on a 2-regular tree under sequential unconditional imitation dynamics with parameters $c = 1$ and $r = 2.5$. Green vertices represent a cooperator, red vertices represent defectors. The square indicates the unique deviator.

response that maximizes their payoffs over their constrained action sets that are determined by the relative success of their neighbors [33]. Thus, in the spatial public goods game it is not the rationality of selfish players that is necessarily detrimental to cooperation levels, but rather the players' ability to maximize *absolutely* by innovating rationally and introducing defection in their local game interactions.

Through simulations, we find persistent oscillations also occur in lattices and more complex spatial structures, such as random and scale-free networks. For the purpose of comparing equilibrium cooperation levels under best response and rational imitation dynamics, in Appendix VI-A we show that for star and ring networks unconditional imitation dynamics do converge to an imitation equilibrium. Although these networks are special cases in which all three dynamics converge, in the next sections we will show that their analysis also provides relevant insight into the long-run cooperation levels of the three dynamics on more general network topologies.

IV. SUSTAINING COOPERATION CLUSTERS: AN EQUILIBRIUM COMPARISON

To illustrate the various effects of network reciprocity under the different dynamics, we now compare the conditions under which cluster(s) of cooperators can exist at an equilibrium profile of a homogeneous linear spatial PGG. We provide several examples that indicate how the combination of rational decisions and imitations in rational imitation lead to high levels of cooperation, while ensuring convergence of the overall network dynamics. Throughout this section, it is assumed Assumption 1 holds.

A. Star Networks

We start with analyzing star networks that have been shown to be *efficient networks* for the provision of public goods [39]. We first consider the case where the central player forms a cluster of cooperators with the p cooperating leaf players (i.e., noncentral players). The remaining number of defecting leaf players is indicated by q . For compact notation, we define $\gamma := \frac{p+q+1}{p+\frac{1}{2}q-\frac{1}{2}}$.

The following Lemma characterizes the equilibrium profiles of unconditional imitation.

Lemma 1: Consider a linear homogeneous PGG played on a star network. Assume that initially the central player is cooperating and there are $p \geq 0$ cooperating and $q \geq 1$ defecting leaf players. Then under imitate-the-best dynamics:

- 1) if $r < \gamma$, then the network will reach the full-defection imitation equilibrium;
- 2) if $r = \gamma$, the network is already at a mixed unconditional imitation equilibrium;
- 3) if $r > \gamma$, then the network will reach the full-cooperation imitation equilibrium. $\square$

Proof: We separately prove each of the three cases.

Case 1: $r < \gamma$. It follows from (9) that $\pi_1 < \pi_{l_d}$. In case there are no cooperating leafs in the network, the central player switches to defection in the next time step and the networks reaches the full-defection equilibrium. Now consider the case when there is at least one cooperating leaf in the network, i.e., $p \geq 1$. Then since $q \geq 1$, it holds that

$$3 < 3p + q \Rightarrow \frac{p + q + 1}{p + \frac{1}{2}q - \frac{1}{2}} < 4 \Rightarrow r < 4.$$

Thus, from (9) it follows that $\pi_{l_c} < \pi_{l_d}$. Now since $\pi_1 < \pi_{l_d}$, it can be concluded that only the central player may switch at the next time step. Due to the persistent activation assumption, there exists some time that the central player becomes active, and the first time when that happens, she will switch to defection. Then in view of Lemma 4, the network will reach the full-defection equilibrium state.

Case 2: $r = \gamma$. Then $\pi_1 = \pi_{l_d}$. Hence, neither the central player nor any of the defecting leafs will switch actions at the next time step. Trivially the same holds for every cooperating leaf, resulting in an equilibrium state.

Case 3: $r > \gamma$. Then $\pi_1 > \pi_{l_d}$. Hence, since the cooperating leafs do not switch actions, the first time that a defecting leaf is active, she will switch to cooperation. So the new number of cooperating and defecting leafs will be $\bar{p} = p + 1$ and $\bar{q} = q - 1$.

Then the condition $r > \frac{p+q+1}{p+\frac{1}{2}q-\frac{1}{2}}$ for the new state becomes

$$r > \frac{\bar{p} + \bar{q} + 1}{\bar{p} + \frac{1}{2}\bar{q} - \frac{1}{2}} = \frac{p + q + 1}{p + \frac{1}{2}q}$$

which holds since $r > \frac{p+q+1}{p+\frac{1}{2}q-\frac{1}{2}}$. Hence, again a defecting leaf will switch to cooperation. Therefore, eventually, the network will reach a full-cooperation equilibrium state.

The following result is a best-response version of Lemma 1, where we assign $\phi := \frac{p+q+1}{1+p+q+\frac{1}{2}(p+q)}$ and $\theta := \frac{2}{\frac{1}{2}+\frac{1}{p+q+1}}$ for compact notation.

Lemma 2: Consider a linear homogeneous PGG played on a star network. Assume that initially the central player is cooperating and there are $p \geq 0$ cooperating and $q \geq 1$ defecting leaf players. Then under best response dynamics:

- 1) if $r < \phi$, the network will reach the full-defection Nash equilibrium;
- 2) if $\phi \leq r < \theta$, the network converges to a Nash equilibrium with the central player as the unique cooperator;
- 3) if $r = \theta$, the network is already at a mixed Nash equilibrium;
- 4) if $r > \theta$, the network will reach the full-cooperation Nash equilibrium. $\square$

Proof: The proof follows by substituting $d_i = p + q$ in the best response conditions in (12) and (13) in Appendix VI-B.

An important insight from Lemma 2 is that the different degrees in the star network *promotes* cooperation for the central player, but *hinders* cooperation for all leaf players. More specifically, cooperation is a best-response in the PGG centered around the central player if $r \geq p + q + 1$, whereas it is a best-response in the PGG centered around any of the leaf players if $r \geq 2$, which is a weaker condition. Because the central player is involved in many of the “smaller” public goods games centered around the leaves, her lower bound of the cooperation condition in the entire network game is reduced (see Appendix VI-B and Section IV-C for best response conditions in general networks). Indeed, if the degree of the central player goes to infinity, then cooperation becomes the best response for the central player if $r \geq 2$, which is exactly the cooperation condition for the PGG centered around the leaf players. The reverse is true for leaf players: because they only participate in two PGGs (one centered around themselves and one centered around the central player), their lower bound of the cooperation condition significantly increases due to their interaction with the central player. In the extreme case, when the degree of the central player goes to infinity, their cooperation condition goes to $r \geq 4$, making it twice as difficult for leaf players to cooperate. It follows that under best response, cooperation clusters can easily break down: the leaf players have a much stronger incentive to switch to defection than the central player.

The following illustrates how the downfall of a cooperation cluster is prevented under rational imitation when the feasible action set of the central player is maximally restricted, as in “imitate the best” dynamics. Without loss of generality, assume the central player is labeled as i and the leaf players as $j \neq i$.

Lemma 3: Let $h_i = 1$ and $h_j \in \{1, 2\}$ for all $j \neq i$ and let $\epsilon > 0$ be an arbitrarily small number. Consider a linear homogeneous PGG played on a star network. Assume that initially the central

player is cooperating and there are $p \geq 0$ cooperating and $q \geq 1$ defecting leaf players. Then, under rational imitation dynamics:

- 1) if $r < \min\{\phi, \gamma\}$, the network will reach the full-defection rational imitation equilibrium;
- 2) if $\min\{\phi, \gamma + \epsilon\} \leq r \leq \theta$, the network is already at a mixed rational imitation equilibrium;
- 3) if $r > \theta$, the network will reach the full-cooperation Nash equilibrium. $\square$

Proof: The statement follows from the fact that as long as the central player only has cooperation in her feasible action set, or has no incentive to deviate to defection, then the cooperating leaf players are at a rational imitation equilibrium, and the defecting leaf players will only switch to cooperation if it is profitable.

Lemma 3 does not hold when $h_i > 1$ or $r = \gamma$ because defection can then still be contained in the central player’s feasible action set. In this case, the minimum bounds in the first two statements of Lemma 3 must be adjusted to ϕ . If the initial number of cooperating leafs p is large, this is a clear improvement over best response dynamics but worse than unconditional imitation. More importantly, Lemma 3 shows that if $h_i = 1$, in *almost all* cases rational imitation sustains a cooperation cluster *no worse* than unconditional imitation in Lemma 1. The main advantage of the latter is that *full cooperation* is achieved for lower values of r since defecting leafs will rationally imitate cooperation only if it is profitable to do so, while unconditional imitations occur even if the switch to cooperation is detrimental. The ability of unconditional imitation to grow cooperation clusters faster than rational imitation is also observed in the simulations shown in Fig. 2. This raises the question if it is possible for rational imitation to achieve *higher* cooperation levels than unconditional imitation. The answer is yes. From Lemma 4, we already know that when the central player is defecting then, under imitate-the-best dynamics, the network converges to full defection for every $r \geq 1$. This is true for every $h_i \geq 1$, because the defecting central player is the most successful player in the network for every $p \geq 1$ and will, therefore, never switch to cooperation and induce a downfall of cooperation. This is *not* the case under rational imitation. For $r > \theta$ cooperation is a profitable deviation for the central player and will be included in her feasible action set if she takes into account the action of at least one of her cooperating neighbors, which due to the network structure and homogeneity of the payoff parameters r and c is satisfied whenever $h_i > 2$. Once the central player has switched to cooperation, the cluster can be sustained and the downfall to full defection (that occurs under unconditional imitation) is averted.

B. Ring and Lattice Networks

The factors that allow rational imitation to efficiently facilitate cooperation in the PGG are as follows: (i) restrictions on the action sets; (ii) improvement-seeking players; and (iii) the network structure. The last one may not always be present, and readers may tend to confine the promoting effect of rational imitation solely to the special network structure of the star. However, rational imitation can efficiently sustain cooperation even in the absence of beneficial network structures. This can be illustrated by considering commonly studied regular networks like the ring and lattice [17], [22], [35].

Fig. 2. Simulations of spatial linear public goods game (same legend for all subfigures). The graphs show the fraction of cooperators at an equilibrium as a function of the public goods multiplier for the three dynamics. The abbreviations in the legends are UI (Unconditional Imitation), RI (Rational Imitation), BR (Best Response), SD (Standard Deviation). The average degrees are: scale-free (10), Torus (4), Random (5.5), Bipartite (7). The cooperators are initially clustered in the scale-free and bipartite networks. In the torus network, the location of the cooperators at the initial condition is random with an initial fraction of 50%.

In the PGG, these networks have no beneficial network effect due to their homogeneous degree: cooperation is a best response if and only if the public goods multiplier is at least as high as the size of any local PGG ($r = 3$ for a ring and $r = 5$ for a lattice). However, both unconditional imitation and rational imitation with $h_i = 1$ for all $i \in \mathcal{V}$ can sustain clusters of cooperators at significantly lower public good multipliers. The main reason is that the feasible action sets of cooperators at the boundary only contain cooperation whenever their most successful cooperating neighbors have a strictly higher payoff than their most successful defecting neighbors. Using this approach, it can be shown that a cluster formed by a single neighborhood of cooperators on a ring can be sustained under rational imitation whenever $r > \frac{9}{4}$ and on a lattice whenever $r > \frac{25}{7}$. Thus, compared to best response, rational imitation is 25% more effective at sustaining a single neighborhood of cooperators on a ring, and 28% more effective on a lattice. When clusters increase in size this beneficial effect of imitation increases. For example, in a ring network, a cluster of four or more cooperators, surrounded by at least two defectors can be sustained under rational imitation whenever $r > 1.8$, which is a 40% reduction compared to best response. Now because in this case, the cooperating neighbor's payoff is already at a maximum, the lower bound on r cannot be further reduced in a ring network. However, as the degree of the players increases, more benefit can be achieved: a cooperator cluster of four overlapping neighborhoods on a lattice can be sustained with a more than 60% reduction in the public goods multiplier ($r > \frac{25}{13}$ against $r > 5$).

It is always possible to derive similar conditions as above for general network topologies. Although the principle remains

the same, asymmetry in the network introduces heterogeneity in payoffs that increases the number of payoff comparisons that have to be made in order to obtain the conditions. For this reason, we limited our analysis to specialized networks. The following sections provide a more general perspective to these results.

C. Best Response Conditions in General Networks and the Spatial Prisoner's Dilemma

The proof of Theorem 1 ensures the existence of a Nash equilibrium in the spatial public goods game on any network. For the linear public goods game, the set of Nash equilibrium profiles can also be conveniently characterized in terms of r and the degree distribution of the network. In particular, cooperation is the best response for i if and only if

$$r \geq r_i^* := \frac{1 + d_i}{\frac{1}{1+d_i} + \sum_{j=1}^N A_{ij} \frac{1}{1+d_j}}$$

where A_{ij} is the ij th element of the adjacency matrix (as in Appendix VI-B). This simple condition holds for general networks and results in a deterministic fraction of cooperators at the Nash equilibrium that, aside from indifference, is unique. Cooperation exists at the Nash equilibrium if and only if $r \geq \min_i r_i^*$. In comparison, our analysis on star, ring, and lattice networks shows that for rational imitation equilibrium profiles, this requirement for the existence of cooperation is often not necessary and—depending on the initial condition—cooperation exists at the rational imitation equilibrium for much smaller public goods multipliers. We show this by simulations on scale-free, torus, bipartite and random networks in Fig. 2.

However, it must also be noted that when r increases such that there exists a player i satisfying $r_i^* < r$, best response dynamics may lead to higher cooperation levels compared to both imitation dynamics. The reason is that while $r > r_i^*$ is sufficient for player i to cooperate under best response dynamics, it is not sufficient under imitation dynamics because cooperation can be excluded from player i 's feasible action set (e.g., as in a cluster of defectors). In these cases, cooperation is hindered by the noninnovative nature of (rational) imitation dynamics. This, however, does not falsify our findings that rational imitation sustains cooperation at smaller values of r than best response.

In the commonly studied pairwise alternative, the so-called Spatial Prisoner's Dilemma (SPD) game [17], [40], [41], defection strictly dominates cooperation in every pairwise interaction. As a consequence, all networks converge to full defection under best response dynamics [40], [42]. Here, the cooperation promoting effect of rational imitation is even more pronounced than in the public goods game. Using the same analysis for the public goods game, it is not difficult to find conditions under which cooperation exists at an imitation equilibrium in the SPD under rational imitation. For example, on a ring network, rational imitation sustains clusters of three or more cooperators surrounded by two or more defectors whenever $2R > T + P$; that is, two times the reward for mutual cooperation (R) is strictly larger than the sum of the temptation to defect (T) and the punishment for mutual defection (P).

D. Simulations

In this section, we support our findings on more complex network structures by simulations. The simulations are obtained for $h_i = 1$ for all players in a homogeneous linear PGG with $c = 1$. By varying r , we are interested in how the public goods parameter influences the number of cooperators at an equilibrium. Imitate-the-best dynamics often do not converge to an imitation equilibrium. In this case, we let the action profile evolve for 10^4 time steps and determine the fraction of cooperators by the number of cooperators at the final time step $t = 10^4$. Cooperation levels are determined by the asymptotic fraction of cooperators in the network. The plots are obtained by averaging over 100 random activation sequences. In scale-free networks ($n = 100$), clusters of cooperators around high degree hubs are sustained under rational imitation and unconditional imitation but fall apart under best response. For increasing values of r , the cooperation clusters grow fast under unconditional imitation, while rational imitation "stabilizes" the cluster for a large range of public goods multipliers (as seen from the flat line in the top-left graph in Fig. 2). This emergent behavior is in line with our analysis of star networks in Lemmas 1 and 3. Similar behavior is observed in lattice networks with periodic boundaries such that the spatial structure corresponds to a torus. As our analysis showed, in these networks, rational imitation can sustain cooperation clusters at much lower values of r than the best response that requires $r \geq 5$ for cooperation to exist at the Nash equilibrium. Compared to scale-free networks (in which hubs largely influence outcomes), the homogeneous degree and random placement of cooperators in the torus network lead to a larger standard deviation in the mean fraction of cooperators.

Though, again, unconditional imitation more easily leads to cooperation levels close to one. This behavior also occurs in the SPD, and while the lower cooperation levels under rational imitation (compared to unconditional imitation) can be detrimental for the overall welfare, they better represent cooperation in human experiments that tend to converge to relatively low but *nonzero* cooperation levels [41], [43]. In the random network, cooperation exists at the Nash equilibrium for relatively small r as the minimum r_i^* is about 2.9. However, as in our analysis on stars, rings, and lattices, initial cooperation clusters are effectively sustained for much smaller values of r , see the bottom right graph in Fig. 2. If instead cooperators are placed randomly in the network, both unconditional and rational imitation dynamics can lead to worse cooperation levels than the best response for $r > 2.9$. Finally, the simulation in the bipartite network ($n = 28$) shows the interesting phenomenon that if the cooperators are initially clustered in one of the disjoint sets of the bipartite network, then for *small values* of r , rational imitations tend to result in *higher cooperation levels* than unconditional imitation. This situation occurs when the initial growth of the cooperator cluster under unconditional imitation backfires and allows a few very successful defectors to take over the entire network due to the random activation sequence, which is also the reason behind the large standard deviation. This does not occur for rational imitation because the initial cluster is already an equilibrium for a wide range of public good multipliers simply because the initial switches to cooperation are not profitable. Hence the lower standard deviation compared to unconditional imitation. These examples show that it is not always the irrationality of imitations that allows cooperation to exist, but rather the player's ability to observe, via the relative success of others, when their own defective motives become detrimental to their *own* success.

V. CONCLUSION

We have shown that for a general class of asynchronous spatial PGGs, rational imitation (and best response) dynamics converge to an equilibrium regardless of the spatial structure, heterogeneity in the payoff parameters, and representation of group structures. When the rationality of having incentives to deviate is lost, as in unconditional imitation dynamics, the convergence of the decision process can only be guaranteed for specific structures such as ring and star networks. Convergence fails already on simple structures like lattices and trees. Thus, even when players only imitate more successful neighbors (a concept called *payoff monotonicity* [7], [44]), in general, the players are not expected to reach a satisfactory decision. However, an important aspect of imitation dynamics remains that even in the absence of convergence and mechanisms such as punishment, reward, and voluntary participation [45] and multistep payoff prediction [40], it allows for high levels of *cooperative* actions to exist in the long run behavior of a social dilemma game (Lemma 1, Fig. 1, and [19]). This cooperation promoting feature of unconditional imitation is often absent for best response dynamics because the temptation to defect breaks down cooperation clusters in the network. Through various examples and simulations, we have shown how rational imitation dynamics, overcomes this issue and efficiently facilitates cooperation by combining

the rationality of best responses with the relative performance conditions of imitation dynamics. This provides new insights into the design of solutions to the tragedy of the commons. And although our aim has not been to fit cooperation levels of rational imitation to experimental research, we believe the proposed dynamics provide important insights for experimentalists. The reason is that first, the dynamics are intuitive and tractable for commonly studied games such as the PGG and PD; and second, rational imitation shows how relative-performance considerations and rationality result in stable cooperation levels below the often overly optimistic cooperation levels of unconditional imitation, but above the pessimistic zero cooperation levels of best response. Several directions for future research can be identified. First, it is interesting to apply direct and incentive-based control [26] to rational imitation dynamics to construct desired equilibrium profiles. Second, it is of interest to investigate how nonlinear production functions affect cooperation levels under rational imitation. And finally, for future research, it is interesting to investigate rational imitation in other competitive games and study the effect of cooperation-promoting mechanisms such as the multistep horizon payoff prediction and voluntary participation under rational imitation.

VI. APPENDICES

A. Convergence of Unconditional Imitation Dynamics

We restrict our analysis to “*imitate the best*” unconditional imitation dynamics, i.e., $h_i = 1$ for all $i \in \mathcal{V}$. It can be easily shown that if the network is fully connected (complete), then the linear homogeneous PGG converges to full defection for any initial action profile in which defectors exist. This is in line with experimental results in [46], which indicate that focusing on the success of others leads to selfish behaviors in complete network games. This observed highly defective tendency does not necessarily occur in more complex spatial structures. We proceed to one of the minimally connected network structures: a star. Because in the homogeneous linear PGG, the players’ payoffs are only scaled by the contribution value c as indicated in (4), without loss of generality, in the remainder of this section we assume $c = 1$.

Lemma 4: Let $h_i = 1$ for all $i \in \mathcal{V}$. Consider a linear homogeneous PGG played on a star network. If the central player defects at some time instance, then the action profile generated by unconditional imitation dynamics will reach the full-defection unconditional imitation equilibrium in finite time. $\square$

Proof: Let player 1 represents the central player and players l_c and l_d represent the cooperating and defecting leaves, respectively. From (1), (2), and (4), the payoffs in

$$\begin{aligned}\pi_1 &= \frac{(a_1 + p)r}{n} + \frac{a_1 + 1}{2}pr + \frac{a_1}{2}qr - (n + 1)a_1 \\ \pi_{l_c} &= \frac{(a_1 + p)r}{n} + \frac{1 + a_1}{2}r - 2 \\ \pi_{l_d} &= \frac{(a_1 + p)r}{n} + \frac{a_1}{2}r.\end{aligned}\quad (9)$$

We prove by induction on m defined as the number of cooperating leaves when the central player is defecting. The result is trivial for $m = 0$, i.e., when all the players are defecting. Assume the result holds for $m = p - 1$, $p \geq 1$. Consider some time k_0 when the central player is defecting and there are p cooperating leaves in the network, i.e., $m = p$. First note that if the central player ever becomes active, she will not have an incentive to switch to cooperation because $\pi_1 > \pi_{l_c}$ for all $p \geq 1$. Also, if the active player at k_0 is a defecting leaf, she will not switch since her only neighbor is the central player who is also defecting. Hence, the state at $k_0 + 1$ will be the same as the initial state. So consider the first time $k_1 \geq k_0$ when a cooperating leaf is active. This time exists due to the persistent activation assumption. From (1), (2), and (4), the payoffs at k_1 of the active player l_c and her neighbor, player 1, are given by

$$\left. \begin{aligned}\pi_1 &= \frac{pr}{n} + \frac{1}{2}pr \\ \pi_{l_c} &= \frac{pr}{n} + \frac{1}{2}r - 2.\end{aligned} \right\} \Rightarrow p \geq 1\pi_1 > \pi_{l_c}.$$

Therefore, the cooperating leaf will switch to defection at $k_1 + 1$, resulting in a new state where the central player is still defecting and there are $p - 1$ cooperating leaves. This is the case with $m = p - 1$, which completes the proof.

Proposition 2 (Finite time convergence in star networks):

Let $h_i = 1$ for all $i \in \mathcal{V}$. Under imitate-the-best dynamics, every linear homogeneous PGG played on a star network reaches an unconditional imitation equilibrium in finite time. $\square$

Proof: The proof follows from Lemmas 1 and 4.

After establishing convergence of a completely connected and a minimally connected network, we now proceed to a network that has a regular and close-to-minimal connectivity: a ring. Part of the result requires the persistent assumption to be slightly stronger.

Assumption 2 (Pairwise persistence): For any pair of players $i, j \in \mathcal{V}$ and each time k , there exists some finite time $k' > k$ such that i and j are activated consecutively at k' and $k' + 1$. $\square$

Although stronger than the persistent activation assumption, the pairwise persistent activation assumption still holds almost surely in most stochastic settings, particularly when players are activated independently, e.g., according to Poisson clocks or by a stochastic process where at each time step, one random player becomes active to alter its current action [47].

Proposition 3 (Finite time convergence in ring networks):

Let $h_i = 1$ for all $i \in \mathcal{V}$. Consider a linear homogeneous PGG played on a ring network. If the public goods multiplier r belongs to the interval $[0, \frac{9}{2}]$ the imitate-the-best dynamics reach an imitation equilibrium in finite time. The same holds for $r > \frac{9}{2}$, but when activation is pairwise persistent. $\square$

We first show that the behavior of the homogeneous linear PGG under the imitate the best dynamics, although depending on the public-goods multiplier r , is the same for different values of r in certain ranges. In order to do this, we first introduce the notation $\alpha(k)_{|r=g}$, which denotes the state vector at time k , given that $r = g \in \mathbb{R}_{\geq 0}$.

Lemma 5: Given a ring network, its initial action vector and activation sequence, for every public goods multiplier r_1 and r_2

taken from one of the following intervals

$$\left(0, \frac{9}{5}\right), \left(\frac{9}{5}, \frac{9}{4}\right), \left(\frac{9}{4}, \frac{9}{3}\right), \left(\frac{9}{3}, \frac{9}{2}\right), \left(\frac{9}{2}, \frac{9}{1}\right), \left(\frac{9}{1}, \infty\right)$$

it holds that for all time $k \geq 0$ $\mathbf{a}(k)|_{r=r_1} = \mathbf{a}(k)|_{r=r_2}$.

Proof: The proof can be found in [1].

Proof of Proposition 3

For ring networks consisting of 5 players or fewer, the result can be verified by exhausting all the cases. We prove the statement for ring networks consisting of more than 5 players. Due to the ring topology, π_i is determined by the actions $a_{i-2}, a_{i-1}, a_i, a_{i+1}$ and a_{i+2} . By following the same argument for each of the neighbors of player i , i.e., $i-1$ and $i+1$, we conclude that the payoffs of player i and her neighbors are determined by the vector

$$\mathbf{s}^i = \begin{pmatrix} a_{i-3} & a_{i-2} & a_{i-1} & a_i & a_{i+1} & a_{i+2} & a_{i+3} \end{pmatrix}. \quad (10)$$

Therefore, $\mathbf{a}_i(t+1)$ is completely determined by the actions in $\mathbf{s}^i(t)$. Based on Lemma 5, we prove the theorem for each of the following cases:

Case 1: $0 < r < \frac{9}{5}$. In view of Lemma 5, we only need to prove the result for just one value of r in this range, say $r = 1$. Consider the function $n_{1111}(\mathbf{a}) : \{0, 1\}^n \rightarrow \mathbb{Z}_{\geq 0}$ defined as the number of 4 consecutive cooperators in the whole network:

$$n_{1111}(\mathbf{a}) = |\{j \in \mathcal{V} \mid a_j = a_{j+1} = a_{j+2} = a_{j+3} = 1\}|$$

where $|\mathcal{X}|$ denotes the cardinality of the set $\mathcal{X}$. We show that n_{1111} never decreases over time, i.e., at every time $K \geq 0$

$$\Delta n_{1111}(K) = n_{1111}(\mathbf{a}(K+1)) - n_{1111}(\mathbf{a}(K)) \geq 0. \quad (11)$$

Let player i be active at K . Then all consecutive quadruple actions that may change in number at $K+1$ are

$$(a_{i-3}, a_{i-2}, a_{i-1}, a_i), (a_{i-2}, a_{i-1}, a_i, a_{i+1}) \\ (a_{i-1}, a_i, a_{i+1}, a_{i+2}), (a_i, a_{i+1}, a_{i+2}, a_{i+3}).$$

Therefore, it suffices to show that the number of these four quadruples equaling (1,1,1,1) at $K+1$ is not less than the number of quadruples at K . It is worth to mention that if the action that receives the maximum payoff in the neighborhood is not unique, then the binary action set implies that the player will not switch. Stack all 2^7 different states of $\mathbf{s}^i(K)$ and $\mathbf{s}^i(K+1)$ in two $2^7 \times 7$ binary-matrices S^- and S^+ so that for every row $j = 1, 2, 3, \dots, 2^7$, if $\mathbf{s}^i(K) = S_j^-$, then $\mathbf{s}^i(K+1) = S_j^+$ where X_j represents the j th row of matrix X . For every $j = 1, 2, 3, \dots, 2^7$, delete the j th rows of S^+ and S^- , if they are the same, to obtain S_0^+ and S_0^- . Then the rows of S_0^- represent all possible values of $\mathbf{s}^i(K)$ that will result in player i switching her action. One can check that the number of 4 consecutive 1's in every row of S_0^+ is **no less than** that in the same row in S_0^- . This implies that the number of quadruples (1,1,1,1) in $\mathbf{s}^i(K+1)$ is no less than $\mathbf{s}^i(K)$, regardless of what value $\mathbf{s}^i(K)$ takes. Consequently, (11) holds. Hence, whenever a player switches, the function n_{1111} either increases or remains constant. Since n_{1111} is bounded, this yields the existence of some time k_1 at which n_{1111} becomes fixed and never changes afterward. Consider the matrices S_1^- and S_1^+ that are obtained from S_0^- and S_0^+ after deleting each row j from both of them if the

number of (1,1,1,1)s in the j th row of S_0^- is less than that in S_0^+ . Since n_{1111} is fixed after k_1 , no switching that results in a change in the number of quadruples (1,1,1,1) may take place after k_1 . Hence, for $k \geq k_1$, $\mathbf{s}^i(k)$ equals one of the rows of S_1^- and $\mathbf{s}^i(k+1)$ equals the corresponding row in S_1^+ . Now one can check that the number of quadruples (1,1,0,1) in every row of S_1^+ is **no less than** that in the same row in S_1^- . Hence, the function n_{1101} defined as the number of (1,1,0,1)s in the network never decreases after k_1 . Therefore, similar to the argument above, there exists some finite time k_2 when n_{1101} becomes fixed and never changes afterwards. So after k_2 , the number of quadruples (1,1,1,1) and (1,1,0,1) will remain constant.

Next, we obtain S_2^- and S_2^+ by deleting all rows j from both S_1^- and S_1^+ where the number of (1,1,0,1)s in the j th row of S_1^+ is more than that in S_1^- . Then by following the above process, one can show the existence of some time k_5 after which the number of each of the quadruples (1,1,1,1), (1,1,0,1), (1,0,1,1), (0,1,1,1), (0,1,0,1), and (1,1,1,0) becomes fixed. Correspondingly, we obtain S_5^- and S_5^+ . Then one can check that the number of quadruples (0,0,0,0) in every row of S_5^+ is **no more than** that in the same row in S_5^- . Hence, the function $-n_{0000}$ where n_{0000} is defined as the number of quadruples (0,0,0,0) in the network never decreases after k_5 . Therefore, there exists some finite time k_7 , after which n_{0000} remains constant.

Following this approach and by using consecutively the functions $n_{0011}, n_{0110}, n_{1100}, n_{1001}, -n_{0010}, -n_{0100}, -n_{1000}$, and $-n_{0001}$, one can show the existence of some time k_{15} after which, the number of each corresponding quadruple becomes fixed. Moreover, we obtain S_{15}^- and S_{15}^+ as explained above, yet this time they both become an empty matrix. This implies that after k_{15} , no more switches of actions may take place in the network. Hence, the network will reach a stationary state at k_{15} , which must be an equilibrium due to the persistent activation assumption.

Case 2: $\frac{9}{5} \leq r \leq \frac{9}{4}$. In view of Lemma 5, we only need to prove the result for just one value of r in this range, say $r = 2$. Similar to the previous case and by using the exact same potential-like functions, one can show that the network reaches an equilibrium.

Case 3: $\frac{9}{4} \leq r \leq \frac{9}{3}$. This case can be proven exactly as case 1.

Case 4: $\frac{9}{3} \leq r \leq \frac{9}{2}$. In view of Lemma 5, we only need to prove the result for just one value of r in this range, say $r = 3.5$. Similar to the previous case and by using consecutively the functions $n_{1110}, n_{1101}, n_{1011}, n_{0111}, n_{0101}, n_{0011}, n_{1100}, n_{1010}, n_{0110}, -n_{1111}, n_{1001}, n_{0010}, n_{0100}$, and $-n_{0000}$, one can show the existence of some time k_{14} when the network reaches an equilibrium.

Case 5: $\frac{9}{2} \leq r \leq 9$. This case can be proved similar to case 6.

Case 6: $r > 9$. The proof can be found in [1].

B. Best Responses in Linear Homogeneous Spatial PGGs

Let A_{ij} be the ij th element of the adjacency matrix $\mathbf{A}$ of the simple undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$, and let $d_i = \sum_{j=1}^n A_{ij}$ be the degree of $i \in \mathcal{V}$. For player i the difference in cooperation

and defection payoffs is

$$\pi_i^C - \pi_i^D = r_i \left(\frac{c_i}{1+d_i} + \sum_{j=1}^N A_{ij} \frac{c_i}{1+d_j} \right) - (1+d_i)c_i.$$

It follows that player i 's best response is cooperation if and only if $r_i \geq \frac{1+d_i}{\frac{1}{1+d_i} + \sum_{j=1}^N A_{ij} \frac{1}{1+d_j}}$. A typical example is the star network. The central player i in the network has a degree equal to $d_i > 1$, while all other players have a degree equal to 1. Cooperation is the best response for the focal player if and only if

$$r \geq r_i^* = \frac{1+d_i}{\frac{1}{1+d_i} + \frac{1}{2}d_i} \quad (12)$$

and for leaf players if and only if

$$r \geq r_j^* = \frac{2}{\frac{1}{2} + \frac{1}{1+d_i}}. \quad (13)$$

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