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*Published in:*  
Indagationes Mathematicae

*DOI:*  
[10.1016/j.indag.2021.06.004](https://doi.org/10.1016/j.indag.2021.06.004)

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*Document Version*  
Publisher's PDF, also known as Version of record

*Publication date:*  
2021

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*

Evink, T., van der Heiden, G.-J., & Top, J. (2021). Two-descent on some genus two curves. *Indagationes Mathematicae*, 32(4), 883-900. <https://doi.org/10.1016/j.indag.2021.06.004>

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# Two-descent on some genus two curves

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Received 26 February 2021; received in revised form 10 June 2021; accepted 11 June 2021

Communicated by G. Cornelissen

## Abstract

For the hyperelliptic curve  $C_p$  with equation  $y^2 = x(x - 2p)(x - p)(x + p)(x + 2p)$  with  $p$  a prime number, we discuss bounds for the rank of its Jacobian over  $\mathbb{Q}$ , find many cases having 2-torsion in the associated Shafarevich–Tate group, and we present some results on rational points of  $C_p$ .

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*Keywords:* Hyperelliptic curve; Selmer group; Tate–Shafarevich group; Rational points; Rédei symbols

## 1. Introduction

In this paper we study an arbitrary prime number  $p$  the curve  $C_p$  of genus 2 defined by the equation

$$y^2 = x(x^2 - p^2)(x^2 - 4p^2). \quad (1)$$

Specifically, we start by bounding the rank of its Jacobian  $J_p$  over  $\mathbb{Q}$  in terms of the 2-Selmer group  $S^2(J_p/\mathbb{Q})$ . Next we show for three infinite sets of prime numbers  $p$  how to improve the upper bound on rank  $J_p(\mathbb{Q})$  by using a 2-Selmer group computation over  $\mathbb{Q}(\sqrt{\pm p})$  of the Jacobian of the curve  $C = C_1$  defined by  $y^2 = x(x^2 - 1)(x^2 - 4)$ . This computation applies the Rédei symbols of [16]. The improved upper bound yields cases where the Shafarevich–Tate group  $\text{III}(J_p/\mathbb{Q})$  is nontrivial. As an example: for primes  $p \equiv 23 \pmod{48}$  it turns out that  $J_p(\mathbb{Q})$  is finite and  $\text{III}(J_p/\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ .

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We also discuss the  $\mathbb{Q}$ -rational points of the curve  $C_p$ . This is easy in case the group  $J_p(\mathbb{Q})$  is finite (as occurs, for example, for all primes  $p \equiv 7 \pmod{24}$ ). A less obvious case we treat is  $p = 241$ ; the group  $J_{241}(\mathbb{Q})$  turns out to have rank 2. Using so-called ‘Two-Selmer sets’, it is shown that  $C_{241}(\mathbb{Q})$  consists of only the obvious Weierstrass points (the one at infinity and the ones with  $y = 0$ ).

Studying genus 1 curves depending on a prime number  $p$  is a very classical subject; the survey paper [12] already lists various examples; more recent ones are found, e.g., in [2,11,18]. The natural question of investigating analogous ideas in the case of genus 2 curves so far seems to have obtained less attention. The 1998 master’s thesis [8] by one of us provides a first step (not yet involving Shafarevich–Tate groups). As shown in *loc. cit.* Prop. 4.3.3 and Thm. 4.3.4, this already suffices to conclude for the curves discussed in the present paper that  $C_p(\mathbb{Q})$  consists of the 6 Weierstrass points only, whenever  $p \equiv 7 \pmod{24}$ . The recent preprint [9] studies some similar families of genus 2 curves, but with only 2 rational Weierstrass points. Again, computing the 2-Selmer group over  $\mathbb{Q}$  allow the authors to identify congruence conditions on the prime  $p$  such that the corresponding Mordell–Weil group is finite. As a consequence, for those primes the only rational points on the curve are the rational Weierstrass points.

Many results in the present paper originate from two master’s projects [6,8] (1998 resp. 2019) by the second and the first author, supervised by the third one.

## 2. Notation and results

The first step in order to obtain information on the rank of Jacobian  $J_p$  of the hyperelliptic curve  $C_p$  defined by the equation

$$y^2 = x(x^2 - p^2)(x^2 - 4p^2)$$

for a prime  $p$ , is the relatively basic computation of the 2-Selmer group of  $J_p/\mathbb{Q}$ . It fits in the well-known short exact sequence

$$0 \rightarrow J_p(\mathbb{Q})/2J_p(\mathbb{Q}) \rightarrow S^2(J_p/\mathbb{Q}) \rightarrow \text{III}(J_p/\mathbb{Q})[2] \rightarrow 0. \tag{2}$$

This Selmer group was computed in [8] (with minor corrections in [6, Appendix B]). The computation is based on the method described in [15] and uses (see [7, Section 7])

$$\#J_p(K_v)/2J_p(K_v) = |2|_v^{-2} \cdot \#J_p(K_v)[2] = |2|_v^{-2} \cdot 16$$

where  $K_v \supset \mathbb{Q}_\ell$  is a finite extension with valuation ring  $O_v$ , and

$$|2|_v = \text{vol}(2O_v)/\text{vol}(O_v) = \begin{cases} 1 & \text{if } \ell \neq 2, \\ 2^{-[K_v:\mathbb{Q}_2]} & \text{if } \ell = 2. \end{cases}$$

The result is as follows. (A calculation illustrating this type of result is the proof of Lemma 5.1.)

**Proposition 2.1.** *For a prime number  $p > 3$ , the  $\mathbb{F}_2$ -vector space  $S^2(J_p/\mathbb{Q})$  of the Jacobian  $J_p$  of the curve defined by  $y^2 = x(x^2 - p^2)(x^2 - 4p^2)$  has dimension as given in the next table.*

$p \pmod{24}$	$\dim_{\mathbb{F}_2} S^2(J_p/\mathbb{Q})$
1	8
5, 11, 13, 19	5
7	4
17, 23	6

Since all Weierstrass points on  $C_p$  are  $\mathbb{Q}$ -rational, one has  $J_p(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$ . Either by observing that in the present situation  $J_p(\mathbb{Q})[2] \hookrightarrow S^2(J_p/\mathbb{Q})$ , or using that the torsion subgroup  $J_p(\mathbb{Q})_{\text{tor}} \subset J_p(\mathbb{Q})$  yields a 4-dimensional subspace  $J_p(\mathbb{Q})_{\text{tor}}/2J_p(\mathbb{Q})_{\text{tor}}$  of  $J_p(\mathbb{Q})/2J_p(\mathbb{Q})$ , the short exact sequence (2) implies

$$\text{rank } J_p(\mathbb{Q}) + \dim_{\mathbb{F}_2} \text{III}(J_p/\mathbb{Q})[2] = \dim_{\mathbb{F}_2} S^2(J_p/\mathbb{Q}) - 4. \tag{3}$$

The group  $J_p(\mathbb{Q})_{\text{tor}}$  is as follows.

**Lemma 2.2.** *For any prime number  $p$  one has  $J_p(\mathbb{Q})_{\text{tor}} = J_p(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$ .*

**Proof.** Note that for  $p \neq 5$  one has  $\#J_p(\mathbb{F}_5) = 16$  independent of  $p$ , because over  $\mathbb{F}_5$  one has  $p^4 = 1$ , hence the equation for  $C_p$  reduces to  $y^2 = x^5 - x$ . Moreover  $\#J_5(\mathbb{F}_7) = 48$  and  $\#J_5(\mathbb{F}_{11}) = 128$ . Since for primes  $\ell \geq 3$  the reduction mod  $\ell$  map is an injective group homomorphism on rational torsion points, it follows that  $J_p(\mathbb{Q})$  has torsion subgroup as stated, for every prime  $p$ .  $\square$

Here is an immediate consequence of Proposition 2.1 together with the exact sequence (3) and Lemma 2.2:

**Corollary 2.3.** *For any prime number  $p \equiv 7 \pmod{24}$  one has  $\text{rank } J_p(\mathbb{Q}) = 0$  and  $C_p(\mathbb{Q})$  consists of only the 6 Weierstrass points of  $C_p$ .*

**Proof.** The proof of the statement about the rank is indicated above. Embedding  $C_p \subset J_p$  via  $P \mapsto [P - \infty]$  with  $\infty \in C_p$  the Weierstrass point at infinity and applying Lemma 2.2, one concludes that  $C_p \cap J_p(\mathbb{Q})$  consists of the divisor classes  $[W - \infty]$  for  $W$  any Weierstrass point on  $C_p$ , which implies the result.  $\square$

For the primes  $p \equiv 5, 11, 13, 19 \pmod{24}$  the structure of the group  $J_p(\mathbb{Q})$  is in fact also predicted by Proposition 2.1:

**Corollary 2.4.** *For any prime  $p > 3$ , assume that  $\text{III}(J_p/\mathbb{Q})$  is finite. Then  $\text{rank } J_p(\mathbb{Q}) = 1$  if  $p \equiv 5, 11, 13, 19 \pmod{24}$ , while  $\text{rank } J_p(\mathbb{Q}) \equiv 0 \pmod{2}$  otherwise.*

*In particular, if for a prime  $p \equiv 5, 11, 13, 19 \pmod{24}$  the group  $\text{III}(J_p/\mathbb{Q})$  is finite then for this prime  $J_p(\mathbb{Q}) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^4$ .*

**Proof.** By a result of Poonen and Stoll [14, §6, §8] finiteness of III and the fact that  $C_p$  contains a rational point, implies that  $\dim_{\mathbb{F}_2} \text{III}(J_p/\mathbb{Q})[2]$  is even. Hence Eq. (3) and Proposition 2.1 imply the assertions about the rank. The result follows by applying Lemma 2.2.  $\square$

The remainder of this paper deals with improvements of Proposition 2.1 and variations on Corollary 2.3. Specifically, this is possible in all remaining congruence classes (so,  $p \equiv 1, 17, 23 \pmod{24}$ ). We show the following.

**Theorem 2.5.** *Let  $p \equiv 23 \pmod{48}$  be a prime number. The Jacobian  $J_p$  of the curve corresponding to  $y^2 = x(x^2 - p^2)(x^2 - 4p^2)$  satisfies  $J_p(\mathbb{Q}) = J_p(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$  and  $\text{III}(J_p/\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ .*

**Theorem 2.6.** *Let  $p \equiv 17 \pmod{24}$  be a prime number that does not split completely in  $\mathbb{Q}(\sqrt[4]{2})$ . The Jacobian  $J_p$  of the curve corresponding to  $y^2 = x(x^2 - p^2)(x^2 - 4p^2)$  satisfies  $J_p(\mathbb{Q}) = J_p(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$  and  $\text{III}(J_p/\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ .*

**Theorem 2.7.** Let  $p \equiv 1 \pmod{24}$  be a prime number satisfying one of the conditions

- (a)  $p$  splits completely in  $\mathbb{Q}(\sqrt[4]{2})$  and not in  $\mathbb{Q}(\sqrt{1 + \sqrt{3}})$ ;
- (b)  $p \equiv 1 \pmod{48}$  and  $p$  splits completely in  $\mathbb{Q}(\sqrt{1 + \sqrt{3}})$  and not in  $\mathbb{Q}(\sqrt[4]{2})$ ;
- (c)  $p \equiv 25 \pmod{48}$  and  $p$  does not split completely in either of  $\mathbb{Q}(\sqrt[4]{2})$  and  $\mathbb{Q}(\sqrt{1 + \sqrt{3}})$ .

The Jacobian  $J_p$  of the curve corresponding to  $y^2 = x(x^2 - p^2)(x^2 - 4p^2)$  satisfies  $J_p(\mathbb{Q}) = J(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$  and  $\text{III}(J_p/\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^4$ .

Using Chebotarëv’s density theorem (see, e.g., [17]), one observes that the set of prime numbers satisfying the condition given in Theorem 2.5 has a positive Dirichlet density. The same holds for the set of primes satisfying the condition in Theorem 2.6 and for each of the three sets corresponding to Theorem 2.7(a), 2.7(b), and 2.7(c).

### 3. Rédei symbols

In this section we recall the definition and various properties of the Rédei symbol. It is a tri-linear symbol taking values in  $\mu_2$  and it satisfies a reciprocity law based on the product formula for quadratic Hilbert symbols. This reciprocity allows us to link the splitting behavior of certain primes in dihedral extensions over  $\mathbb{Q}$  of degree 8 in a non-trivial way, which functions as a useful supplement to various 2-Selmer group computations. The reciprocity of the Rédei-symbol is a recent result due to P. Stevenhagen in [16]; his text is the basis for the exposition in this section.

Let  $a, b$  be square-free integers representing non-trivial elements in  $\mathbb{Q}^*/\mathbb{Q}^{*2}$ , and suppose their local quadratic Hilbert symbols are all trivial:

$$(a, b)_p = 1, \quad \text{for all primes } p. \tag{4}$$

By the local–global principle of Hasse and Minkowski, condition (4) is equivalent to the existence of a non-zero rational solution  $(x, y, z)$  to the equation

$$x^2 - ay^2 - bz^2 = 0. \tag{5}$$

Take such a solution and put

$$\alpha = 2(x + z\sqrt{b}), \quad \beta = x + y\sqrt{a}. \tag{6}$$

Then  $F := E(\sqrt{\alpha}) = E(\sqrt{\beta})$  defines a quadratic extension of  $E = \mathbb{Q}(\sqrt{a}, \sqrt{b})$  that is normal over  $\mathbb{Q}$ , cyclic of degree 4 over  $\mathbb{Q}(\sqrt{ab})$ , and dihedral of degree 8 over  $\mathbb{Q}$  when  $\mathbb{Q}(\sqrt{ab}) \neq \mathbb{Q}$ , see [16, Lemma 5.1, Corollary 5.2]. The extension  $F$  can be twisted to  $F_t$  for  $t \in \mathbb{Q}^*$  by scaling the solution  $(x, y, z)$  to  $(tx, ty, tz)$ . By [16, Propositions 7.2, 7.3] choosing  $t$  appropriately ensures that  $F_t/E$  is unramified at all finite primes of odd residue characteristic, but in some cases ramification over 2 cannot be avoided. With  $\Delta(d) = \Delta(\mathcal{O}_{\mathbb{Q}(\sqrt{d})})$  for  $d \in \mathbb{Q}^*/\mathbb{Q}^{*2}$  denoting the discriminant, one makes the following definition.

**Definition 3.1.** Let  $K = \mathbb{Q}(\sqrt{ab})$  for non-trivial  $a, b \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ , and let  $F$  be the quadratic extension of  $E = \mathbb{Q}(\sqrt{a}, \sqrt{b})$  corresponding with a solution of (5). The extension  $F/K$  is minimally ramified if the following conditions hold:

- (a) The extension  $F/K$  is unramified over all odd primes  $p \nmid \gcd(\Delta(a), \Delta(b))$ .
- (b) The extension  $F/K$  is unramified over 2 if  $\Delta(a)\Delta(b)$  is odd, or if one of  $\Delta(a), \Delta(b)$  is  $1 \pmod{8}$ .

(c) If  $\{\Delta(a), \Delta(b)\} \equiv \{4, 5\} \pmod 8$ , take  $s \in \{a, b\}$  such that  $\Delta(s) \equiv 4 \pmod 8$ . The local biquadratic extension  $\mathbb{Q}_2(\sqrt{s}) \subset F \otimes \mathbb{Q}_2$  must have conductor 2.

Stevenhagen in [16] proposed this definition and shows that it is precisely the “right” notion allowing one to obtain a reciprocity result that generalizes classical work of Rédei; see in particular the discussion in [16, Section 6]. By [16, Lemma 7.7] it is possible to twist a given  $F$  to a suitable  $F_t$  which is minimally ramified over  $\mathbb{Q}(\sqrt{ab})$ . For convenience, a degree 8 dihedral extension of  $\mathbb{Q}$  is called minimally ramified if it is so over its subfield defined by the order 4 cyclic subgroup of the Galois group. Observe that Definition 3.1 imposes no restrictions over the prime 2 in case 2 is totally ramified in  $\mathbb{Q}(\sqrt{a}, \sqrt{b})$ .

**Definition 3.2.** For non-trivial  $a, b, c \in \mathbb{Q}^*/\mathbb{Q}^{*2}$  with local quadratic Hilbert symbols

$$(a, b)_p = (a, c)_p = (b, c)_p = 1 \tag{7}$$

for all primes  $p$  and moreover

$$\gcd(\Delta(a), \Delta(b), \Delta(c)) = 1, \tag{8}$$

set  $K = \mathbb{Q}(\sqrt{ab})$  and  $E = \mathbb{Q}(\sqrt{a}, \sqrt{b})$ , and take a corresponding  $F/K$  which is minimally ramified. Define  $[a, b, c] \in \text{Gal}(F/E) = \mu_2$  by

$$[a, b, c] = \begin{cases} \text{Art}(c, F/K) & \text{if } c > 0 \\ \text{Art}(c\infty, F/K) & \text{if } c < 0 \end{cases}$$

where  $\text{Art}(\cdot, \cdot)$  is the Artin symbol and  $c \in \mathcal{I}(\mathcal{O}_K)$  has norm  $|c_0|$  with  $c_0$  the square-free integer representing  $c$ , and  $\infty$  denotes an infinite prime of  $K$ .

If at least one of  $a, b$  and  $c$  is trivial then one sets  $[a, b, c] = 1$ .

**Proposition 3.3.** For  $a, b, c \in \mathbb{Q}^*/\mathbb{Q}^{*2}$  satisfying (7) and (8), the Rédei symbol  $[a, b, c] \in \mu_2$  is well-defined. Moreover, the symbol is tri-linear, and perfectly symmetrical in all three arguments.

**Proof (Sketch).** If  $p|c$ , then  $(c, b)_p = (c, a)_p = 1$  implies that  $p$  is either split or ramified in both  $\mathbb{Q}(\sqrt{a})$  and  $\mathbb{Q}(\sqrt{b})$ . Condition (8) implies that  $p$  cannot ramify in both, hence a prime  $\mathfrak{p}_K|p$  in  $K$  has norm  $p$  and splits in  $E$ . The prime  $\mathfrak{p}_K$  is unramified in  $F$  by the minimal ramification of  $F$ , where the parity of  $p$  determines whether this is due to condition (a) or (b), noting that (c) cannot occur when  $p = 2$  as a prime of  $K$  over  $p$  must split in  $E$ . It follows that indeed  $\text{Art}(\mathfrak{p}_K, F/K) \in \text{Gal}(F/E)$ , and as  $\text{Gal}(F/E)$  is in the center of  $\text{Gal}(F/\mathbb{Q})$  this Artin symbol is independent of  $\mathfrak{p}_K$ . When  $c < 0$  and  $K$  is real, the Artin symbol in  $F$  of any infinite prime of  $K$  measures whether  $F$  is real or complex and hence is independent of the choice of infinite prime of  $K$ . As  $[a, b, c]$  is the product of such Artin symbols we see that  $[a, b, c]$  does not depend on the choice of  $c$  or  $\infty$ . For the independence of  $F$  we refer to [16, Corollary 8.2].

The set of triples  $(a, b, c)$  in  $\mathbb{Q}^*/\mathbb{Q}^{*2}$  for which (7) and (8) hold is ‘tri-linearly closed’, and the Rédei symbol  $[a, b, c]$  is clearly linear in  $c$ , hence tri-linearity follows from the symmetry. The symmetry in the first two arguments is immediate, while the identity

$$[a, b, c] = [a, c, b]$$

is a non-trivial reciprocity depending on the product formula for quadratic Hilbert symbols in  $\mathbb{Q}(\sqrt{a})$ . The proof of this reciprocity is the subject of [16, Section 8].  $\square$

**Example 3.4.** Consider the case when  $a = b = 2$ . Then the invariant fields  $F$  and  $F'$  of the subgroups generated by  $-1 \pmod{16}$  and  $7 \pmod{16}$  inside  $\text{Gal}(\mathbb{Q}(\zeta_{16})/\mathbb{Q})$ , respectively, are two minimally ramified extensions of  $\mathbb{Q}$  which can be used to compute a Rédei symbol of the form  $[2, 2, c]$  provided that the symbol is defined, i.e. when  $\Delta(c)$  is odd and  $(2, c)_2 = 1$ , i.e. when  $c \equiv 1 \pmod{8}$ . Taking for example  $c = -p$  for a prime  $p \equiv -1 \pmod{8}$ , then as  $F$  is totally real and  $p$  splits completely in  $F$  precisely when  $p \equiv \pm 1 \pmod{16}$  we obtain

$$[2, 2, -p] = \begin{cases} 1 & \text{if } p \equiv -1 \pmod{16} \\ -1 & \text{if } p \equiv 7 \pmod{16} \end{cases}$$

Note that we get the same conclusion when using the (complex!) field  $F'$  as  $p$  splits completely in  $F'$  precisely when  $p \equiv 1, 7 \pmod{16}$ .

**Example 3.5.** Let  $p \equiv 1 \pmod{8}$  be a prime. Let  $\pi \in \mathbb{Z}[\sqrt{2}]$  be an element of norm  $p$  with conjugate  $\pi'$ . We claim that  $\pi$  is a square mod  $\pi'$  if and only if  $[2, p, p] = 1$ . To see this, note we may multiply  $\pi$  and  $\pi'$  both by a sign without affecting whether  $\pi$  is a square mod  $\pi'$  or not, and upon doing so if necessary, the field  $F = \mathbb{Q}(\sqrt{2}, \sqrt{\pi}, \sqrt{\pi'})$  becomes minimally ramified over  $\mathbb{Q}(\sqrt{2p})$  (to see this, view  $F$  as the normal closure of  $\mathbb{Q}(\sqrt{p}, \alpha)$  with  $N(\alpha)\mathbb{Q}^{*2} = 2\mathbb{Q}^{*2}$  and look 2-adically: one of  $\pm\alpha, \pm\alpha'$  is in  $(-3) \subset \mathbb{Q}^*/\mathbb{Q}^{*2}$ ). Now  $(e_p, f_p, g_p) = (2, 1, 4)$  in  $F$  over  $\mathbb{Q}$  precisely when  $[2, p, p] = 1$ . The condition on the residue class degree is equivalent to  $\pi$  being a square mod  $\pi'$ , showing the claim.

Since  $[2, p, p] = [p, p, 2]$ , this is equivalent to 2 being completely split in the quartic subfield  $E$  of  $\mathbb{Q}(\zeta_p)$ . As  $E$  corresponds with the subgroup of fourth powers in  $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) = (\mathbb{Z}/p\mathbb{Z})^*$  and  $2 \pmod{p} = \text{Frob}_p \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ , we see that 2 splits completely in  $E$  precisely when  $2 \pmod{p}$  is a fourth power, i.e. when  $p$  splits completely in  $\mathbb{Q}(\sqrt[4]{2})$ , i.e. when  $[2, -2, p] = 1$ . We thus have the identity

$$[2, p, p] = [2, -2, p].$$

With this we obtain a generalization of [18, Prop. 4.1], where it is used to prove that  $(\mathbb{Z}/2\mathbb{Z})^2 \subset \text{III}(E/\mathbb{Q})[2]$  for the elliptic curve  $E$  defined by  $y^2 = (x + p)(x^2 + p^2)$  for a prime  $p \equiv 9 \pmod{16}$  such that  $1 + \sqrt{-1} \in \mathbb{F}_p$  is a square.

**Corollary 3.6.** Let  $p \equiv 1 \pmod{8}$  be a prime, let  $\pi \in \mathbb{Z}[\sqrt{2}]$  have norm  $p$  with conjugate  $\pi'$  and let  $i \in \mathbb{F}_p$  be a primitive fourth root of unity. Consider the following statements.

- (a)  $\pi$  is a square mod  $\pi'$ .
- (b)  $1 + i$  is a square mod  $p$ .

Then the statements are equivalent when  $p \equiv 1 \pmod{16}$ , while for  $p \equiv 9 \pmod{16}$  exactly one of the statements holds.

**Proof.** Statement (a) holds when  $[2, p, p] = [2, -2, p] = 1$ , while statement (b) holds when  $[2, -1, p] = 1$  (note that  $\mathbb{Q}(\zeta_8)(\sqrt{1+i})$  is minimally ramified). The result follows because

$$[2, -2, p] \cdot [2, -1, p] = [2, 2, p] = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{16}, \\ -1 & \text{if } p \equiv 9 \pmod{16}. \quad \square \end{cases}$$

### 4. Computation of 2-Selmer groups

We start by recalling the explicit form of 2-descent that will be used. Let  $K$  be a number field and  $C$  the hyperelliptic curve defined by  $y^2 = f(x)$ , for  $f \in K[x]$  square-free and of odd degree  $2g + 1$ . We have the short exact sequence

$$0 \rightarrow J(K)/2J(K) \rightarrow S^2(J/K) \rightarrow \text{III}(J/K)[2] \rightarrow 0,$$

where  $S^2(J/K)$  and  $\text{III}(J/K)$  are respectively the 2-Selmer group and the Shafarevich–Tate group defined in terms of Galois cohomology by

$$S^2(J/K) := \ker \left( H^1(G_K, J(\overline{K})[2]) \rightarrow \prod_{\mathfrak{p}} H^1(G_{K_{\mathfrak{p}}}, J(\overline{K}_{\mathfrak{p}})) \right),$$

$$\text{III}(J/K) := \ker \left( H^1(G_K, J(\overline{K})) \rightarrow \prod_{\mathfrak{p}} H^1(G_{K_{\mathfrak{p}}}, J(\overline{K}_{\mathfrak{p}})) \right).$$

By [15, Theorems 2.1 & 2.2] one has  $H^1(G_K, J(\overline{K})[2]) \cong \ker(A^*/A^{*2} \xrightarrow{N} K^*/K^{*2})$ , where  $A = K[x]/(f(x))$  and  $N$  is induced by the norm map  $A \rightarrow K$ . This identifies  $S^2(J/K)$  with the elements in  $\ker(A^*/A^{*2} \xrightarrow{N} K^*/K^{*2})$  that are mapped, according to the commutative diagram

$$\begin{array}{ccc} J(K)/2J(K) & \xrightarrow{\delta} & A^*/A^{*2} \\ \downarrow & & \downarrow \\ J(K_{\mathfrak{p}})/2J(K_{\mathfrak{p}}) & \xrightarrow{\delta_{\mathfrak{p}}} & A_{\mathfrak{p}}^*/A_{\mathfrak{p}}^{*2}, \end{array}$$

into  $\text{im}(\delta_{\mathfrak{p}})$  for all primes  $\mathfrak{p}$  of  $K$ .

We consider the special case that  $f \in \mathcal{O}_K[x]$  is monic and completely splits, so  $f = \prod_{i=1}^{2g+1} (x - \alpha_i)$  for distinct  $\alpha_j \in \mathcal{O}_K$ . In this case  $A \xrightarrow{\sim} \bigoplus_{i=1}^{2g+1} K$  determined by  $x \mapsto (\alpha_1, \dots, \alpha_{2g+1})$ , and the norm map  $A \rightarrow K$  corresponds to multiplication  $\bigoplus_{i=1}^{2g+1} K \rightarrow K$ . Hence the kernel of the norm  $\bigoplus_{i=1}^{2g+1} K^*/K^{*2} \xrightarrow{N} K^*/K^{*2}$  consists of the ‘hyperplane’ of those  $(2g + 1)$ -tuples for which the product of all coordinates is trivial.

Let  $S$  consist of the real primes of  $K$  together with the finite primes dividing  $2\Delta(f)$ , and put  $K(S) := \{x \in K^*/K^{*2} : \text{ord}_{\mathfrak{p}}(x) \equiv 0 \pmod{2} \text{ for all finite } \mathfrak{p} \notin S\}$ . One has (compare [15, pp. 226–227])

$$S^2(J/K) \subset \ker \left( \bigoplus_{i=1}^{2g+1} K(S) \rightarrow K(S) \right), \tag{9}$$

and  $S^2(J/K)$  consists of those elements in the kernel of (9) that map into  $\text{im}(\delta_{\mathfrak{p}})$  for each  $\mathfrak{p} \in S$  in the following diagram.

$$\begin{array}{ccc} J(K)/2J(K) & \xrightarrow{\delta} & \bigoplus_{i=1}^{2g+1} K^*/K^{*2} \\ \downarrow & & \downarrow \\ J(K_{\mathfrak{p}})/2J(K_{\mathfrak{p}}) & \xrightarrow{\delta_{\mathfrak{p}}} & \bigoplus_{i=1}^{2g+1} K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2}. \end{array}$$



Here the injective homomorphism  $\delta$  and similarly  $\delta_{\mathfrak{p}}$  is given by

$$\sum_{i=1}^r [P_i] - r[\infty] \mapsto \prod_{i=1}^r (x(P_i) - \alpha_1, \dots, x(P_i) - \alpha_{2g+1}), \tag{10}$$

for  $P_1, \dots, P_r \in C(\overline{K})$  forming a  $G_K$ -orbit not containing a Weierstrass point. The  $j$ th coordinate of the  $\delta$ -image of  $[(\alpha_i, 0)] - [\infty]$  for  $i \neq j$  is  $\alpha_i - \alpha_j$ . The  $i$ th coordinate is then determined by the hyperplane condition: it equals  $\prod_{j \neq i} (\alpha_i - \alpha_j)$ . As already remarked in Section 2 the cardinality of  $J(K_{\mathfrak{p}})/2J(K_{\mathfrak{p}})$  and hence that of  $\text{im}(\delta_{\mathfrak{p}})$  are known. In practice this makes it fairly straightforward to describe explicit representants of the elements in  $\text{im}(\delta_{\mathfrak{p}})$ , for each  $\mathfrak{p} \in S$ .

The group  $K(S)$  fits in the exact sequence

$$0 \longrightarrow R_S^*/R_S^{*2} \longrightarrow K(S) \xrightarrow{\beta} \text{Cl}(R_S)[2] \longrightarrow 0$$

where  $R_S = \{0\} \cup \{x \in K^* : \text{ord}_{\mathfrak{p}}(x) \geq 0 \text{ for all finite } \mathfrak{p} \notin S\}$  is the ring of  $S$ -integers in  $K$ . Here  $\beta$  sends  $xK^{*2}$  to the class  $[IR_S]$ , where  $x\mathcal{O}_K = \mathfrak{a}I^2$  with  $\mathfrak{a}$  and  $I$  co-prime fractional ideals such that  $\mathfrak{a}$  is supported on prime ideals of  $S$  and the support of  $I$  does not contain any prime of  $S$ . This is well-known; for completeness see [6, Prop. 2.4.4]. The case of interest to us is when  $K$  has odd class number.

**Proposition 4.1.** *If  $K$  has odd class number then the map  $R_S^*/R_S^{*2} \rightarrow K(S)$  is an isomorphism. Moreover, for each finite  $\mathfrak{p} \in S$  writing  $\mathfrak{p}^{k_{\mathfrak{p}}} = (x_{\mathfrak{p}})$  with  $k_{\mathfrak{p}}$  the order of  $\mathfrak{p}$  in the class group of  $K$ , the  $x_{\mathfrak{p}}$  together with an  $\mathbb{F}_2$ -basis for  $\mathcal{O}_K^*/\mathcal{O}_K^{*2}$  form an  $\mathbb{F}_2$ -basis for  $K(S)$ .*

**Proof.** A detailed proof of this standard fact is provided in [6, Cor. 2.4.7].  $\square$

For an odd prime  $p$  write  $p^* = (-1)^{(p-1)/2}p$ , so  $\mathbb{Q}(\sqrt{p^*})$  is the quadratic subfield of the cyclotomic field  $\mathbb{Q}(\zeta_p)$ . In what follows we will compute 2-Selmer groups over these quadratic fields. One has

**Lemma 4.2.** *For any odd prime  $p$  the field  $K = \mathbb{Q}(\sqrt{p^*})$  has odd class number, and if  $K$  is real (i.e.,  $p \equiv 1 \pmod{4}$ ) then a fundamental unit of  $K$  has norm  $-1$ .*

**Proof.** For a proof using genus theory, see for example [16, Thm 2.1]. A slightly more direct argument is given in [6, Appendix A.2].  $\square$

### 5. Proofs of the rank and Shafarevich–Tate group results

Consider the genus two hyperelliptic curves

$$C/\mathbb{Q}: y^2 = f(x) := x(x^2 - 1)(x^2 - 4),$$

and, for  $p$  any prime number,

$$C_p/\mathbb{Q}: y^2 = x(x^2 - p^2)(x^2 - 4p^2).$$

Then  $C_p$  is a quadratic twist of  $C$  over both  $\mathbb{Q}(\sqrt{p})$  and  $\mathbb{Q}(\sqrt{-p})$ . Let  $J$  and  $J_p$  denote the Jacobians of  $C$  and  $C_p$ , respectively. Decomposing  $J(\mathbb{Q}(\sqrt{\pm p})) \otimes \mathbb{Q}$  into eigenspaces for the action of  $\text{Gal}(\mathbb{Q}(\sqrt{\pm p})/\mathbb{Q})$  implies the relation

$$\text{rank } J_p(\mathbb{Q}) + \text{rank } J(\mathbb{Q}) = \text{rank } J(\mathbb{Q}(\sqrt{\pm p})), \tag{11}$$

for both possibilities of the sign  $\pm$ . A quick computation (Lemma 5.1) yields  $\text{rank } J(\mathbb{Q}) = 0$ . Since for the Jacobians at hand the torsion subgroup yields a subgroup of the 2-Selmer group of dimension 4, it follows that

$$\text{rank } J_p(\mathbb{Q}) \leq \dim_{\mathbb{F}_2} S^2(J/\mathbb{Q}(\sqrt{\pm p})) - 4.$$

Using  $\mathbb{Q}(\sqrt{p})$  in case  $p \equiv 1, 17 \pmod{24}$  and  $\mathbb{Q}(\sqrt{-p})$  for  $p \equiv 23 \pmod{24}$ , it will be shown that for certain subsets of these primes the bound for  $\text{rank } J_p(\mathbb{Q})$  obtained in this way sharpens the one which follows by directly applying Proposition 2.1. Specifically, this results in proofs for Theorems 2.5–2.7.

Label the roots of  $f$  as  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (-2, -1, 0, 1, 2)$ . For a field  $F \supset \mathbb{Q}$  and a point  $(\xi, \eta) \in C(F)$  write  $D_\xi \in J(F)$  for the point corresponding to the divisor  $[(\xi, \eta)] - [\infty]$  on  $C$ . Note that although  $D_\xi$  depends on  $\eta$ , its image in the 2-Selmer group  $S^2(J/F)$  does not. The image of  $J(\mathbb{Q}[2])$  under  $\delta$  is spanned by

	$x + 2$	$x + 1$	$x$	$x - 1$	$x - 2$
$D_{-2}$	6	-1	-2	-3	-1
$D_{-1}$	1	-6	-1	-2	-3
$D_0$	2	1	1	-1	-2
$D_1$	3	2	1	-6	-1

Here  $x - \alpha_i$  denotes the map  $[P] - [\infty] \mapsto x(P) - \alpha_i$  as in (10), compare [15].

The local fields for which we need the images in  $\delta_p$  are  $\mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_3(i)$  and  $\mathbb{R}$ . Much of this was already done in [8, pp. 43–45]. One has  $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2} = \langle -1, 2, 3 \rangle, \mathbb{Q}_3^*/\mathbb{Q}_3^{*2} = \langle -1, 3 \rangle$ , for  $F = \mathbb{Q}_3(i)$  moreover  $F^*/F^{*2} = \langle 3, r \rangle$ , where  $r = 1 + i$ , and of course  $\mathbb{R}^*/\mathbb{R}^{*2} = \langle -1 \rangle$ . The local images are then spanned as follows.

$\mathbb{Q}_2$	$x + 2$	$x + 1$	$x$	$x - 1$	$x - 2$
$D_{-2}$	6	-1	-2	-3	-1
$D_{-1}$	1	-6	-1	-2	-3
$D_0$	2	1	1	-1	-2
$D_1$	3	2	1	-6	-1
$D_6$	2	-1	6	-3	1
$D_7$	1	2	-1	6	-3

$\mathbb{Q}_3$	$x + 2$	$x + 1$	$x$	$x - 1$	$x - 2$
$D_{-2}$	-3	-1	1	-3	-1
$D_{-1}$	1	3	-1	1	-3
$D_0$	-1	1	1	-1	1
$D_4$	1	-3	-1	1	3

$\mathbb{Q}_3(i)$	$x + 2$	$x + 1$	$x$	$x - 1$	$x - 2$
$D_{-2}$	3	1	1	3	1
$D_{-1}$	1	3	1	1	3
$D_i$	$r$	$r$	1	$r$	$r$
$D_{4+3i}$	$3r$	1	1	$3r$	1

$\mathbb{R}$	$x + 2$	$x + 1$	$x$	$x - 1$	$x - 2$
$D_{-1}$	1	-1	-1	-1	-1
$D_0$	1	1	1	-1	-1

**Lemma 5.1.** We have  $\text{rank } J(\mathbb{Q}) = 0$ .

**Proof.** It suffices to show  $\dim_{\mathbb{F}_2} S^2(J/\mathbb{Q}) = 4$ . Note  $\Delta(f) = 2^{10} \cdot 3^4$ , so  $S = \{2, 3, \infty\}$  and  $K(S) = \langle -1, 2, 3 \rangle$ . Then  $S^2(J/\mathbb{Q})$  injects into the 2-adic image  $\text{im } \delta_2$ , and

$$S^2(J/\mathbb{Q}) = A \oplus \delta(J(\mathbb{Q})[2])$$

where  $A$  consists of all  $x \in S^2(J/\mathbb{Q})$  with 2-adic image in the span of

$$\begin{pmatrix} 2 & -1 & 6 & -3 & 1 \\ 1 & 2 & -1 & 6 & -3 \end{pmatrix}.$$

If  $x = (e_1, \dots, e_5) \in A$ , then the 3-adic image forces  $e_3 \in \langle -1 \rangle$ , hence  $x$  is in the span of  $(1, 2, -1, 6, -3)$ . Therefore  $x$  is trivial because  $(1, 2, -1, 6, -3) \notin \text{im}(\delta_3)$ . Thus  $A = 0$  and  $S^2(J/\mathbb{Q})$  has  $\mathbb{F}_2$ -dimension 4.  $\square$

We now compute  $S^2(J/\mathbb{Q}(\sqrt{-p}))$  for  $p \equiv 23 \pmod{24}$  and  $S^2(J/\mathbb{Q}(\sqrt{p}))$  for  $p \equiv 1, 17 \pmod{24}$ . The computation follows [6, § 3.4.2-4], except that Rédei symbols are used instead of various reciprocity arguments in *loc. cit.*

Consider a prime  $p \equiv 23 \pmod{24}$  and let  $K = \mathbb{Q}(\sqrt{-p})$ . Then  $K$  is complex and both 2 and 3 split in  $K$ , so as set  $S$  of places of  $K$  needed for embedding  $S^2(J/K)$  in  $\bigoplus_{i=1}^5 K(S)$  we take the four primes dividing 6. The completion of  $K$  at a prime in  $S$  equals  $\mathbb{Q}_2$  or  $\mathbb{Q}_3$ .

Write  $\mathfrak{p}_3, \mathfrak{q}_3$  for the prime ideals in  $\mathcal{O}_K$  dividing 3 and let  $k_3$  be the order of  $[\mathfrak{p}_3]$  in  $\text{Cl}_K$ . Then  $\mathfrak{p}_3^{k_3} = (x_3)$  for some  $x_3 \in \mathcal{O}_K$ . Since  $\mathfrak{q}_3 \nmid (x_3)$  and  $K_{\mathfrak{q}_3} = \mathbb{Q}_3$ , this  $x_3$  maps to  $\pm 1$  in  $K_{\mathfrak{q}_3}^*/K_{\mathfrak{q}_3}^{*2}$ . Multiplying  $x_3$  by  $-1$  if necessary, we may and will assume that  $x_3$  is a square in  $K_{\mathfrak{q}_3}$ . The conjugate  $y_3 \in \mathcal{O}_K$  of  $x_3$  satisfies  $\mathfrak{q}_3^{k_3} = (y_3)$  and  $x_3 y_3 = 3^{k_3}$ .

Let  $\mathfrak{p}, \mathfrak{q}$  be the prime ideals in  $\mathcal{O}_K$  over 2. In the  $\mathfrak{p}$ -adic completion,  $x_3$  and  $y_3$  yield elements of  $\langle -1, 3 \rangle \subset \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$  since they are not divisible by  $\mathfrak{p}$ . By Lemma 4.2 the order  $k_3$  of  $[\mathfrak{p}_3] \in \text{Cl}_K$  is odd, so the product  $x_3 y_3$  yields  $3 \in \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$ . Hence exactly one of  $x_3, y_3$  after  $\mathfrak{p}$ -adic completion has image 1 or  $-3$  in  $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$ . As  $\text{im}_{\mathfrak{p}}(y_3) = \text{im}_{\mathfrak{q}}(x_3)$ , this implies that  $x_3$  maps into  $\langle -3 \rangle \subset \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$  for precisely one of  $\mathfrak{p}, \mathfrak{q}$ . Denote this ideal by  $\mathfrak{p}_2$ , then  $\mathfrak{p}_2$  is unramified in  $K(\sqrt{x_3})$ .

Let  $x_2 \in \mathfrak{p}_2$  be a generator for  $\mathfrak{p}_2^{k_2}$ , with  $k_2$  the order of  $[\mathfrak{p}_2]$ . As above, multiplying  $x_2$  by  $-1$  if necessary we may and will assume that  $x_2$  maps  $\mathfrak{q}_2$ -adically into  $\langle -3 \rangle \subset \mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$ , where  $\mathfrak{q}_2$  is the conjugate of  $\mathfrak{p}_2$ . Let  $y_2$  be the conjugate of  $x_2$ , so  $\mathfrak{q}_2^{k_2} = (y_2)$  and  $x_2 y_2 = 2^{k_2}$ .

Proposition 4.1 implies  $K(S) = \langle -1, x_2, y_2, x_3, y_3 \rangle$ . We collect the local images in  $K_{\mathfrak{p}}^*/K_{\mathfrak{p}}^{*2}$  of these generators, for  $\mathfrak{p} \in S = \{\mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{p}_3, \mathfrak{q}_3\}$ , as follows.

	$\mathfrak{p}_2$	$\mathfrak{q}_2$	$\mathfrak{p}_3$	$\mathfrak{q}_3$
-1	-1	-1	-1	-1
$x_2$				
$y_2$				
$x_3$			3	1
$y_3$			1	3

(12)

For  $l \in \{2, 3\}$  recall  $\text{im}_{\mathfrak{p}}(x_l) = \text{im}_{\mathfrak{q}}(y_l)$  for conjugate  $\mathfrak{p}$  and  $\mathfrak{q}$  in  $S$  and  $x_l y_l = l^{k_l}$  with  $k_l$  odd. Hence the  $2 \times 2$ -block in the table corresponding to  $x_l, y_l$  and conjugate  $\mathfrak{p}, \mathfrak{q}$  is determined by any one entry in the block.

As  $\mathfrak{p}_2$  is unramified in  $K(\sqrt{y_2})$  the ramification degree of 2 in the normal closure  $K(\sqrt{x_2}, \sqrt{y_2})$  equals 2, hence it is minimally ramified over  $\mathbb{Q}(\sqrt{-2p})$ . Hence  $y_2$  is  $\mathfrak{p}_2$ -adically a square if and only if  $[-p, 2, 2] = 1$ . Thus the top left block in (12) is determined by

the Rédei symbol  $[2, 2, -p]$ . As suggested by the coloring, the two blocks away from the diagonal are both determined by the same Rédei symbol. To see this, note that the normal closure of  $K(\sqrt{x_3})/\mathbb{Q}$  yields a minimally ramified extension of  $\mathbb{Q}(\sqrt{-3p})$ . This extension has trivial inertia degree over 3, hence  $\text{im}_{p_2}(x_3) = 1$  if and only if  $[-p, 3, 6] = 1$ . Similarly, the normal closure of  $K(\sqrt{x_2y_3})/\mathbb{Q}$  yields a minimally ramified extension of  $\mathbb{Q}(\sqrt{-6p})$ . Since  $\text{im}_{p_3}(y_3) = 1$ , this implies  $\text{im}_{p_3}(x_2) = 1$  if and only if  $[-p, 6, 3] = 1$ . Hence Table (12) is determined by the values of the two Rédei symbols  $[2, 2, -p]$  and  $[3, 6, -p]$ . Below, the four possibilities for this pair of symbols will be considered.

**Remark 5.2.** Since  $(3, 2)_3 = -1$ , the similar statement ‘ $[-p, 2, 3] = [-p, 3, 2]$ ’ cannot be used to show that the two lighter gray blocks in Table (12) are determined by the same Rédei symbol. This is the reason for the workaround with the bottom right block. However, the proof of  $[a, b, c] = [a, b, c]$  relies on the product formula for quadratic Hilbert symbols in  $\mathbb{Q}(\sqrt{a})$ ; there is nothing against using this product formula in  $\mathbb{Q}(\sqrt{-p})$ . Here the identity  $\prod_p (x_2, x_3)_p = 1$  leads to  $\text{im}_{p_2}(x_3) = 1 \Leftrightarrow \text{im}_{p_3}(x_2) = 1$ , but one still needs the symbol  $[3, 6, -p]$  to link the two blocks to splitting behavior of primes in a *fixed* (i.e., not depending on  $p$ ) number field.

For the Selmer group computations, observe that  $S^2(J/K) = A \oplus \text{im}(J(K)[2])$  for

$$A = \{(e_1, \dots, e_5) \in S^2(J/K) : e_3 \overset{p_3}{\mapsto} 1 \text{ and } e_4 \overset{p_2}{\mapsto} 1\}.$$

First consider the case  $[2, 2, -p] = [3, 6, -p] = 1$ , which means the table is as follows.

	$p_2$	$q_2$	$p_3$	$q_3$
-1	-1	-1	-1	-1
$x_2$	2	1	1	-1
$y_2$	1	2	-1	1
$x_3$	1	3	3	1
$y_3$	3	1	1	3

Let  $x = (e_1, \dots, e_5) \in A$ . The  $p_3$ -adic and  $q_3$ -adic image implies  $e_3 \in \langle x_2, -y_2 \rangle$ , and therefore  $\text{im}_{p_2}(e_3) \subset \langle -1, 2 \rangle$  and  $\text{im}_{q_2}(e_3) \subset \langle -2 \rangle$ . This removes the fifth row of the  $\mathbb{Q}_2$ -table from consideration. As  $\text{im}_{p_2}(e_4) = 1$ , one concludes  $\text{im}_{p_2}(x)$  is in the span of

$$\begin{pmatrix} 6 & 1 & -1 & 1 & -6 \end{pmatrix}, \begin{pmatrix} 6 & 3 & -2 & 1 & -1 \end{pmatrix}.$$

Together with  $\text{im}_{p_3}(e_2) \subset \langle -1 \rangle$  this gives  $e_2 \in \langle y_2, y_3 \rangle$ . Next,  $\text{im}_{q_2}(e_2) \subset \langle 2 \rangle$  and  $\text{im}_{q_2}(e_3) \subset \langle -2 \rangle$  implies  $\text{im}_{q_2}(x)$  is in the span of

$$\begin{pmatrix} 2 & 1 & 1 & -1 & -2 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 & -6 & -1 \end{pmatrix},$$

so  $e_3 \in \langle x_2 \rangle$ . Since  $n = (1, y_3, x_2, 1, x_2y_3) \in A$ , a complement inside  $A$  of  $\langle n \rangle$  is obtained by setting  $e_3 = 1$ . For  $x$  in this complement  $\text{im}_{p_2}(x)$  is trivial, hence  $e_i \in \langle y_2, x_3 \rangle$  for all  $i$ , implying  $\text{im}_{q_2}(x)$  is in the span of  $(6, 2, 1, 6, 2)$ . Then  $e_1, e_4 \in \langle y_2x_3 \rangle$  and  $e_2, e_5 \in \langle y_2 \rangle$ . A nontrivial  $\text{im}_{q_2}(x)$  can only occur for  $e_1, e_2, e_4, e_5$  all  $\neq 1$ , so this complement is at most one-dimensional. Since  $(y_2x_3, y_2, 1, y_2x_3, y_2) \in A$  one concludes that  $A$  is two-dimensional, and  $\dim_{\mathbb{F}_2} S^2(J/K) = 6$ .

In the remaining three cases (i.e.,  $[2, 2, -p]$  and  $[3, 6, -p]$  not both 1) the computation is analogous; for details see [6, § 3.4.2-4]. The results are as follows.

$[2, 2, -p]$	$[3, 6, -p]$	$\dim_{\mathbb{F}_2} S^2(J/K)$	additional generators
1	1	6	$(1, y_3, x_2, 1, x_2y_3), (y_2x_3, y_2, 1, y_2x_3, y_2)$
1	-1	6	$(-y_2y_3, y_2, 1, -y_2y_3, y_2), (1, -y_3, y_2, 1, -y_2y_3)$
-1	1	4	none
-1	-1	4	none

With this one proves [Theorem 2.5](#):

**Proof of Theorem 2.5.** Let  $p \equiv 23 \pmod{48}$  be prime. Then  $p \equiv 7 \pmod{16}$  so [Example 3.4](#) shows  $[2, 2, -p] = -1$ . The table above implies  $\dim_{\mathbb{F}_2} S^2(J/\mathbb{Q}(\sqrt{-p})) = 4$  and as a consequence  $\text{rank } J(\mathbb{Q}(\sqrt{-p})) = 0$ . Hence  $\text{rank } J_p(\mathbb{Q}) = 0$  by Eq. (11). Since  $p \equiv 23 \pmod{24}$ , [Proposition 2.1](#) yields  $\dim_{\mathbb{F}_2} S^2(J_p/\mathbb{Q}) = 6$  hence the exact sequence (3) shows  $\text{III}(J_p/\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ .  $\square$

**Remark 5.3.** Part of what is proven above is that  $\dim_{\mathbb{F}_2} S^2(J/\mathbb{Q}(\sqrt{-p}))$  for primes  $p \equiv 23 \pmod{24}$  depends only on the values of  $[2, 2, -p]$  and  $[3, 6, -p]$ . Hence instead of the provided calculations for an undetermined  $p \equiv 23 \pmod{24}$  one may take a fixed prime for each of the four possibilities for the pair of Rédei symbols, and use e.g. Magma [1] to compute the Selmer group for this prime. The smallest primes covering all cases are given in the table below.

$p$	$[2, 2, -p]$	$[3, 6, -p]$
191	1	1
47	1	-1
167	-1	1
23	-1	-1

We use Magma in this way to obtain proofs of [Theorems 2.6](#) and [2.7](#).

**Proposition 5.4.** For  $K = \mathbb{Q}(\sqrt{p})$  with  $p \equiv 17 \pmod{24}$  prime,  $\dim_{\mathbb{F}_2} S^2(J/K)$  is completely determined by the Rédei symbols  $[2, 2, p]$  and  $[2, -1, p]$ .

**Proof.** Let  $\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{R}$  be the two real embeddings of  $K$ . Take a fundamental unit  $\varepsilon \in \mathcal{O}_K^*$  with  $\sigma_1(\varepsilon) > 0$ . [Lemma 4.2](#) implies  $\varepsilon\bar{\varepsilon} = -1$ , hence there is a unique prime ideal  $\mathfrak{p}_2 \subset \mathcal{O}_K$  over 2 that is unramified in  $K(\sqrt{\varepsilon})$ . Let  $\mathfrak{q}_2$  be the conjugate of  $\mathfrak{p}_2$  and write  $\mathfrak{p}_2^k = (x_2)$  where  $k$  is the order of  $[\mathfrak{p}_2]$  in  $Cl_K$ . Multiplying  $x_2$  by  $\pm\varepsilon$  if necessary we can and will assume that  $x_2$  has positive norm and moreover  $\mathfrak{q}_2$  is unramified in  $K(\sqrt{x_2})$ . Let  $y_2$  be the conjugate of  $x_2$ , so  $x_2y_2 = 2^k$ . Put  $S = \{\mathfrak{p}_2, \mathfrak{q}_2, (3), \sigma_1, \sigma_2\}$ , then  $K(S) = \langle -1, \varepsilon, x_2, y_2, 3 \rangle$ . The table of images in  $K_v^*/K_v^{*2}$  of the generators of  $K(S)$  is as follows (as before,  $r^2 = 2i \in \mathbb{Q}_3(i)$ ).

	$\mathfrak{p}_2$	$\mathfrak{q}_2$	(3)	$\sigma_1$	$\sigma_2$
-1	-1	-1	1	-1	-1
$\varepsilon$			$r$	1	-1
$x_2$			$r$		
$y_2$			$r$		
3	3	3	3	1	1

The 3-adic images of  $\varepsilon, x_2, y_2$  follow by observing that the inertia degree of  $3\mathbb{Z}$  in the normal closures of  $K(\sqrt{x_2})$  and  $K(\sqrt{\varepsilon})$  over  $\mathbb{Q}$  equals 4. As  $\mathfrak{p}_2$  is unramified in  $K(\sqrt{\varepsilon})$  and in  $K(\sqrt{y_2})$ ,

the normal closures over  $\mathbb{Q}$  yield minimally ramified extensions. Hence  $\text{im}_{\mathfrak{p}_2}(\varepsilon) = 1 \Leftrightarrow [p, -1, 2] = 1$  and  $\text{im}_{\mathfrak{p}_2}(y_2) = 1 \Leftrightarrow [p, 2, 2] = 1$  and  $\text{im}_{\sigma_1}(x_2) = 1 \Leftrightarrow [p, 2, -1] = 1$ . Rédei reciprocity completes the proof.  $\square$

Aided by Magma for the rightmost column, one computes the following table.

$p$	$[2, 2, p]$	$[2, -1, p]$	$\dim_{\mathbb{F}_2} S^2(J/\mathbb{Q}(\sqrt{p}))$
113	1	1	6
17	1	-1	4
41	-1	1	4
89	-1	-1	6

From the above, [Theorem 2.6](#) readily follows:

**Proof of Theorem 2.6.** Take  $p \equiv 17 \pmod{24}$  prime and put  $K = \mathbb{Q}(\sqrt{p})$ . [Proposition 5.4](#) and the table above show  $\dim_{\mathbb{F}_2} S^2(J/K) = 4 \Leftrightarrow [2, 2, p][2, -1, p] = -1$ . Tri-linearity of the Rédei symbol implies that the latter condition is equivalent to  $[2, -2, p] = -1$ , which by [Example 3.5](#) means  $p$  is not completely split in  $\mathbb{Q}(\sqrt[4]{2})$ . As remarked earlier,  $\dim_{\mathbb{F}_2} S^2(J/K) = 4 \Rightarrow \text{rank } J(K) = 0 \Leftrightarrow \text{rank } J_p(\mathbb{Q}) = 0$ . [Proposition 2.1](#) and the exact sequence (3) now finish the proof.  $\square$

Lastly we cover the case  $p \equiv 1 \pmod{24}$ .

**Proposition 5.5.** For  $K = \mathbb{Q}(\sqrt{p})$  with  $p \equiv 1 \pmod{24}$  prime,  $\dim_{\mathbb{F}_2} S^2(J/K)$  is completely determined by the Rédei symbols  $[2, 2, p]$ ,  $[2, -1, p]$ ,  $[3, -2, p]$  and  $[3, 6, p]$ .

**Proof.** Let  $p \equiv 1 \pmod{24}$  be prime and put  $K = \mathbb{Q}(\sqrt{p})$ . As in the proof of [Proposition 5.4](#) let  $\sigma_1, \sigma_2 : K \hookrightarrow \mathbb{R}$  be the real embeddings, take a fundamental unit  $\varepsilon \in \mathcal{O}_K$  with  $\sigma_1(\varepsilon) > 0$ , let  $\mathfrak{p}_2$  be the prime over 2 that is unramified in  $K(\sqrt{\varepsilon})$ , and denote the conjugate of  $\mathfrak{p}_2$  by  $\mathfrak{q}_2$ . Then  $\mathfrak{p}_2^{k_2} = (x_2)$  with  $k_2 = \text{ord}([\mathfrak{p}_2])$ , where one chooses  $x_2 \in \mathcal{O}_K$  of positive norm and such that  $\mathfrak{q}_2$  is unramified in  $K(\sqrt{x_2})$ .

Let  $\mathfrak{p}_3$  be the prime over 3 that splits in  $K(\sqrt{x_2})$ , and let  $\mathfrak{q}_3$  be its conjugate. With  $k_3 = \text{ord}([\mathfrak{p}_3])$ , write  $\mathfrak{p}_3^{k_3} = (x_3)$  with  $x_3 \in \mathcal{O}_K$  of positive norm, chosen so that  $\mathfrak{p}_2$  is unramified in  $K(\sqrt{x_3})$ . For  $i \in \{2, 3\}$  let  $y_i$  be the conjugate of  $x_i$ , so  $x_i y_i = i^{k_i}$ . Put  $S = \{\mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{p}_3, \mathfrak{q}_3, \sigma_1, \sigma_2\}$ , then  $K(S) = \langle -1, \varepsilon, x_2, y_2, x_3, y_3 \rangle \subset K^*/K^{*2}$ . Information on local images of  $K(S)$  is presented in the following table.

	$\mathfrak{p}_2$	$\mathfrak{q}_2$	$\mathfrak{p}_3$	$\mathfrak{q}_3$	$\sigma_1$	$\sigma_2$
-1	-1	-1	-1	-1	-1	-1
$\varepsilon$					1	-1
$x_2$			1	-1		
$y_2$			-1	1		
$x_3$	1	3				
$y_3$	3	1				

(13)

To see this, first consider the bottom middle  $2 \times 2$  block. Note that  $[p, 6, 3] = 1$  if and only if  $\text{im}_{\mathfrak{p}_3}(y_3) = \text{im}_{\mathfrak{p}_3}(x_2 y_3) = 1$ , and similarly  $[p, 3, 6] = 1$  precisely when the equivalence  $\text{im}_{\mathfrak{q}_2}(y_3) = 1 \Leftrightarrow \text{im}_{\mathfrak{p}_3}(y_3) = 1$  holds. Since  $[p, 6, 3] = [p, 3, 6]$ , this implies  $\text{im}_{\mathfrak{q}_2}(y_3) = 1$  and moreover  $\text{im}_{\mathfrak{p}_3}(y_3) = 1$  if and only if  $[p, 6, 3] = 1$ . The choice of  $x_3$  and the equality  $x_3 y_3 = 3^{k_3}$  for  $k_3 = \text{ord}([\mathfrak{p}_3])$  odd, implies the bottom left block. The remaining assertions

about the table (in particular: the regions colored in the same shade of gray are determined by any one entry in that region) are straightforward and/or analogous to what we did in other mod 24 cases.

As in the 17 mod 24 case,  $\text{im}_{p_2}(\varepsilon) = 1 \Leftrightarrow [p, -1, 2] = 1$ , and  $\text{im}_{\sigma_1}(x_2) = 1 \Leftrightarrow [p, 2, -1] = 1$ , and  $\text{im}_{p_2}(y_2) = 1 \Leftrightarrow [p, 2, 2] = 1$ .

Finally,  $\text{im}_{p_3}(\varepsilon) = \text{im}_{p_3}(\varepsilon x_2) = 1$  precisely when  $[p, -2, 3] = 1$ . Since  $\text{im}_{p_2}(x_3) = 1$ , one has  $\text{im}_{\sigma_1}(x_3) = 1 \Leftrightarrow [p, 3, -2] = 1$ . Rédei reciprocity finishes the proof.  $\square$

Using Magma for the rightmost column results in the following table (in fact implying a stronger version of Proposition 5.5:  $\dim_{\mathbb{F}_2} S^2(J/\mathbb{Q}(\sqrt{p}))$  for the primes  $p \equiv 1 \pmod{24}$  only depends on the Rédei symbols  $[2, 2, p]$ ,  $[2, -1, p]$ , and  $[3, -2, p]$ ).

$p$	$[2, 2, p]$	$[2, -1, p]$	$[3, -2, p]$	$[3, 6, p]$	$\dim_{\mathbb{F}_2} S^2(J/\mathbb{Q}(\sqrt{p}))$
2593	1	1	1	1	8
1153	1	1	1	-1	8
337	1	1	-1	1	4
557	1	1	-1	-1	4
433	1	-1	1	1	4
97	1	-1	1	-1	4
241	1	-1	-1	1	6
193	1	-1	-1	-1	6
1321	-1	1	1	1	6
409	-1	1	1	-1	6
1129	-1	1	-1	1	4
313	-1	1	-1	-1	4
937	-1	-1	1	1	6
1033	-1	-1	1	-1	6
73	-1	-1	-1	1	4
601	-1	-1	-1	-1	4

(14)

**Proof of Theorem 2.7.** Let  $p \equiv 1 \pmod{24}$  be prime and put  $K = \mathbb{Q}(\sqrt{p})$ . Proposition 2.1 implies  $\dim_{\mathbb{F}_2} S^2(J_p/\mathbb{Q}) = 8$ , hence as in the proofs of Theorems 2.5 and 2.6 it suffices to show that  $\dim_{\mathbb{F}_2} S^2(J/K) = 4$  if  $p$  satisfies one of the conditions (a), (b), or (c) mentioned in the statement of Theorem 2.7.

Note:  $p$  splits completely in  $\mathbb{Q}(\sqrt[4]{2}) \Leftrightarrow [2, 2, p][2, -1, p] = [2, -2, p] = 1$ . Also,  $p$  splits completely in  $\mathbb{Q}(\sqrt{1 + \sqrt{3}}) \Leftrightarrow [3, -2, p] = 1$ , and  $[2, 2, p] = 1 \Leftrightarrow p \equiv 1 \pmod{16}$ . Hence condition (a) corresponds to the cases  $p \in \{73, 337, 557, 601\}$  in the table above. Condition (b) corresponds to  $p \in \{97, 433\}$  in the table, and condition (c) to  $p \in \{313, 1129\}$ . In all these cases the table shows  $\dim_{\mathbb{F}_2} S^2(J/K) = 4$ , hence the result follows by using Proposition 5.5.  $\square$

We finish this section by presenting an analogous result for elliptic curves; we restrict to  $p \equiv 1 \pmod{24}$  but in the same spirit one obtains similar statements for the other congruence classes  $p \pmod{24}$ .

**Proposition 5.6.** *Let  $E/\mathbb{Q}$  be an elliptic curve with good reduction away from 2,3 and with  $E(\mathbb{Q})[2] = E(\overline{\mathbb{Q}})[2]$ . For a prime  $p \equiv 1 \pmod{24}$ , the size of the 2-Selmer group  $S^2(E/\mathbb{Q}(\sqrt{p}))$  is determined by  $E/\mathbb{Q}$  together with the Rédei symbols*

$$[2, 2, p], [2, -1, p], [3, -2, p], [3, 6, p].$$

**Proof.** We use the notation introduced in the proof of Proposition 5.5. In particular, we make the same choices to arrive at the same  $K(S)$ . Descent yields an embedding

$$\delta: E(K)/2E(K) \hookrightarrow \ker \left( \bigoplus_{i=1}^3 K(S) \rightarrow K(S) \right)$$

and  $S^2(E/K)$  consists of the elements in  $\bigoplus_{i=1}^3 K(S)$  that locally are in the image of the corresponding maps  $\delta_v$ , for all  $v \in S = \{\sigma_1, \sigma_2, \mathfrak{p}_2, \mathfrak{q}_2, \mathfrak{p}_3, \mathfrak{q}_3\}$ . For these  $v$ , the image in  $K_v^*/K_v^{*2}$  of the chosen basis for  $K(S)$  is described in table (13), and the proof of Proposition 5.5 shows that table (13) is determined by the four given Rédei symbols, although the variant of table (14) for  $\dim_{\mathbb{F}_2} S^2(E/K)$  will in general depend heavily on  $E$ . As  $S^2(E/K)$  consists of triples of elements in  $K(S)$  that for  $v \in S$  locally are in  $\delta_v(E(K_v))$ , the result follows.  $\square$

**Remark 5.7.** The finite list of elliptic curves satisfying the conditions from Proposition 5.6 was already presented in the PhD thesis of F.B. Coghlan [5]. In fact he listed all elliptic curves over  $\mathbb{Q}$  having good reduction away from 2 and 3. Precisely 28 of these have full rational 2-torsion. In the LMFDB tables [19] contain them under the conductors  $\{24, 32, 48, 64, 72, 96, 144, 192, 288, 576\}$ .

### 6. The $\mathbb{Q}$ -rational points

Here the  $\mathbb{Q}$ -rational points of the curves  $C_p$  are briefly discussed. The proof of Corollary 2.3 shows that for primes  $p$  such that  $\text{rank } J_p(\mathbb{Q}) = 0$ , the set  $C_p(\mathbb{Q})$  consists of the Weierstrass points only. Below a less immediate case is discussed, namely a situation with  $\text{rank } J_p(\mathbb{Q}) = 2$ . We remark that in this case  $\text{rank } J_p(\mathbb{Q})$  is not strictly smaller than the genus of  $C_p$  so the standard Chabauty method does not apply.

Take the prime  $p = 241$ . Using  $\text{rank } J_p(\mathbb{Q}) \leq \dim_{\mathbb{F}_2} S^2(J/\mathbb{Q}(\sqrt{p})) - 4$ , the row  $p = 241$  in the table preceding the proof of Theorem 2.7 yields  $\text{rank } J_{241}(\mathbb{Q}) \leq 2$ . The Mumford representations

$$P = \left( x^2 - \frac{868230159329}{1782528400}x + \frac{8609056225}{4456321}, \frac{83127269153329233}{75258349048000}x - \frac{8905877454269565}{37629174524} \right),$$

$$Q = \left( x^2 - \frac{692452}{3721}x + \frac{73966756}{3721}, \frac{6990522627}{2269810}x + \frac{1284886465269}{1134905} \right)$$

turn out to define points in  $J_{241}(\mathbb{Q})$ . The homomorphism  $\delta: J_p(\mathbb{Q}) \rightarrow S^2(J_p/\mathbb{Q})$  yields  $\delta(P) = (2, p, 1, p, 2)$  and  $\delta(Q) = (1, p, p, p, p)$ . These images are independent of  $\delta(J_p(\mathbb{Q})_{\text{tor}})$  which is generated by  $(6, -p, -2p, -3p, -p)$ ,  $(p, -6, -p, -2p, -3p)$ ,  $(2p, p, 1, -p, -2p)$  and  $(3p, 2p, p, -6, -p)$ . Hence  $\text{rank } J_{241}(\mathbb{Q}) = 2$ . Moreover by Proposition 2.1 and equality (3) one concludes  $\text{III}(J_{241}/\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ .

To determine  $C_{241}(\mathbb{Q})$  the methods developed in [3] will now be used. Although this works in much greater generality, here it is only briefly recalled in the special case of the curves  $C_p$ . Consider the composition

$$C_p(\mathbb{Q}) \longrightarrow J_p(\mathbb{Q}) \xrightarrow{\delta} S^2(J_p/\mathbb{Q})$$

mapping  $(a, b) \in C_p(\mathbb{Q})$  with  $b \neq 0$  to  $(a + 2p, a + p, a, a - p, a - 2p) \in S^2(J_p/\mathbb{Q})$ . If  $s = (e_1, \dots, e_5) \in S^2(J_p/\mathbb{Q})$ , then being in the image of  $C_p(\mathbb{Q})$  implies that one has a rational point on the smooth, complete curve  $X_s/\mathbb{Q}$  corresponding to the affine equations

$$x + 2p = e_1y_1^2, \quad x + p = e_2y_2^2, \quad x = e_3y_3^2, \quad x - p = e_4y_4^2, \quad x - 2p = e_5y_5^2.$$



Here by abuse of notation  $e_j$  represents the class  $e_j \in \mathbb{Q}^*/\mathbb{Q}^{*2}$ ; the result is independent of this representing element. The curve  $X_s$  is what in [3] is called a two-cover of  $C_p$  over  $\mathbb{Q}$ . The “Two-Selmer set” of  $C_p/\mathbb{Q}$  is

$$\{s \in S^2(J_p/\mathbb{Q}) : X_s \text{ has rational points everywhere locally}\}.$$

As an example, for  $p = 241$  let  $s := \delta(P) = (2, p, 1, p, 2)$ . Among the equations for  $X_s$  one has  $x + 2p = 2y_1^2$  and  $x - p = py_4^2$ , defining the conic  $Q: 2y_1^2 - py_4^2 = 3p$ . One obtains a finite morphism  $X_s \rightarrow Q$  defined over  $\mathbb{Q}$ . Since  $Q(\mathbb{Q}_2)$  (as well as  $Q(\mathbb{Q}_3)$ ) is empty, this shows  $\delta(P)$  is not in the Two-Selmer set of  $C_p/\mathbb{Q}$ . In other words: although  $\delta(P)$  is everywhere locally (even globally!) in  $\delta_v(J_p(\mathbb{Q}_v))$ , it is not in the image of  $C_p(\mathbb{Q}_2) \subset J_p(\mathbb{Q}_2)$ .

The Magma command `TwoCoverDescent()`; computes the curves  $X_s$  corresponding to the Two-Selmer set. In our case it turns out that of the  $2^8$  elements in  $S^2(J_{241}/\mathbb{Q})$ , only the six  $\delta([W] - [\infty])$  for  $W \in C_p(\mathbb{Q})$  a Weierstrass point, are in the Two-Selmer set. We now show that for each of these six elements  $s$  one finds that  $\{R \in C_p(\mathbb{Q}) : \delta([R] - [\infty]) = s\}$  consists of only a Weierstrass point. As a consequence,  $C_{241}(\mathbb{Q}) = \{\infty, (0, 0), (\pm 241, 0), (\pm 482, 0)\}$ . We use the notation  $D_\xi$  (here for certain elements in  $J_p$ ) as introduced on page 8.

- $s := \delta(0) = (1, 1, 1, 1, 1)$ . If  $(a, b) \in C_{241}(\mathbb{Q})$  with  $b \neq 0$  would result in  $\delta$ -image  $s$ , then in particular the elliptic curve  $E_1: y^2 = x(x + p)(x + 2p)$  admits a point in  $E_1(\mathbb{Q})$  with  $x = a$  and  $y \neq 0$ . Since  $E_1(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , no such point exists.
- $s := \delta(D_{-2p}) = (6, -p, -2p, -3p, -p)$ . In this case, considering the 1st, 3rd, and 4th entry results in the elliptic curve  $E_2: y^2 = x(x + 2p)(x - p)$  satisfying  $E_2(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . Hence only the Weierstrass point  $(-482, 0) \in C_p(\mathbb{Q})$  yields  $\delta$ -image  $s$ .
- $s := \delta(D_{-p}) = (p, -6, -p, -2p, -3p)$ . Here the 2nd, 4th, and 5th entry results in the elliptic curve  $E_3: -y^2 = (x + p)(x - p)(x - 2p)$  whose only rational points are the points of order at most 2. Reasoning as before, this implies that only the Weierstrass point  $(-241, 0) \in C_p(\mathbb{Q})$  yields  $\delta$ -image  $s$ .
- $s := \delta(D_0) = (2p, p, 1, -p, -2p)$ . Using entries 1, 2, and 3 results in the elliptic curve  $E_4: 2y^2 = x(x + p)(x + 2p)$ , whose only rational points are the points of order at most 2. As above, this implies that only the Weierstrass point  $(0, 0) \in C_p(\mathbb{Q})$  yields  $\delta$ -image  $s$ .
- $s := \delta(D_p) = (3p, 2p, p, -6, -2p)$ . Here we use entries 1, 2, and 4, leading to  $E_5: -y^2 = (x + 2p)(x + p)(x - p)$ . Also here the only rational points are the points of order dividing 2. So only the Weierstrass point  $(241, 0) \in C_p(\mathbb{Q})$  yields  $\delta$ -image  $s$ .
- $s := \delta(D_{2p}) = (p, 3p, 2p, p, 6)$ . Using entries 1, 2, and 5 one obtains  $E_6: 2y^2 = (x + 2p)(x + p)(x - 2p)$ . Here as well, the only rational points are the points of order dividing 2. So  $(482, 0) \in C_p(\mathbb{Q})$  is the only rational point with  $\delta$ -image  $s$ .

This completes the determination of the rational points on  $C_{241}$ .

Note that for  $p = 5$  there are two additional points: one has  $\#C_5(\mathbb{Q}) = 8$ , where the two non-Weierstrass points are  $(20, \pm 1500)$ . Applying Chabauty’s method implies that there are no other points.

It may be possible to extend the method described here and in this way answer the question whether a prime  $p > 5$  exists such that  $\#C_p(\mathbb{Q}) > 6$ .

As a final remark, recall that the two-cover  $X := X_{(1,1,1,1,1)}$  of  $C_p/\mathbb{Q}$  corresponds to the affine model

$$x + 2p = y_1^2, \quad x + p = y_2^2, \quad x = y_3^2, \quad x - p = y_4^2, \quad x - 2p = y_5^2.$$

The maps  $y_j \mapsto -y_j$  define a group  $(\mathbb{Z}/2\mathbb{Z})^5$  in  $\text{Aut}_{\mathbb{Q}}(X)$ . Using appropriate subgroups one obtains up to isogeny the decomposition of  $\text{Jac}(X)$  over  $\mathbb{Q}$  given as follows. Let  $E_{24}: y^2 =$

$(x-1)(x^2-4)$  and  $E_{32}: y^2 = x^3 - x$  and finally  $E_{96a}: y^2 = x(x+1)(x-2)$  be elliptic curves over  $\mathbb{Q}$ . For any such  $E/\mathbb{Q}$  and any  $d \in \mathbb{Q}/\mathbb{Q}^{*2}$  we write  $E^{(d)}$  for the quadratic twist of  $E$  defined by  $d$ . Using quotients of  $X$  by suitable subgroups of  $(\mathbb{Z}/2\mathbb{Z})^5 \subset \text{Aut}_{\mathbb{Q}}(X)$  and applying [10] (see, e.g., [13, Section 2] for more details on this technique) one finds that  $\text{Jac}(X)$  is isogenous over  $\mathbb{Q}$  to the product

$$J_p \times (E_{24})^2 \times E_{24}^{(-1)} \times E_{24}^{(p)} \times E_{24}^{(-p)} \times (E_{32}^{(p)})^3 \times E_{32}^{(2p)} \times (E_{96a}^{(-2)})^2 \times (E_{96a}^{(p)})^2 \times (E_{96a}^{(-p)})^2.$$

In particular the rank of  $\text{Jac}(X)$  is determined by that of  $J_p$  and of the given twists of the three elliptic curves  $E_{24}$ ,  $E_{32}$ , and  $E_{96a}$ . Using analogs of Proposition 5.6 for various classes of primes  $p$  provides a natural approach towards bounding  $\text{rank Jac}(X)(\mathbb{Q})$ .

## Acknowledgments

Several people provided valuable suggestions during this work. We mention Peter Stevenhagen who in the context of [18] already in the 90s showed one of us relations with 4-ranks of class groups, and much more recently referred us to his work on Rédei symbols. We also mention Nils Bruin who explained to one of us the Magma implementation of the elliptic Chabauty method, originally while working on [4, Section 3] but also very relevant for the much simpler situation given in Section 6 of the present paper. Michael Stoll directed us to the theory of Two-Selmer sets and envisioned the result given in Section 6. We thank the referee of an earlier version of this text for suggesting useful improvements. Finally, Steffen Müller, Jeroen Sijsling, and Marius van der Put all showed their interest in this work and in that way encouraged us to complete it.

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