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Hyperbolic Voronoi Percolation

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Hyperbolic Voronoi Percolation

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Introduction

Percolation theory is often introduced as a model for liquid passing through a porous medium where we model the medium as a random structure. We generally model the liquid as a deterministic system. Depending on the structure of the medium, the liquid may or may not pass through.

Broadbent and Hammersley [8] introduced the model in the following way: Start with a structure of “abstract crystals,” the liquid can flow between two crystals if they are connected to each other. We introduce randomness to the medium by damming the connections between crystals randomly.

In the language of graph theory, we fix some $p \in [0, 1]$ and a graph $G = (V, E)$ where the set of vertices V are the crystals (also known as sites/atoms in the language of percolation theory) and edges of E (also known as bonds) are independently kept with probability p (or considered open with probability p). Since the random subgraph is determined by which edges are kept, we refer to this process as *bond percolation* with the associated independent bond measure $\mathbb{P}_p$ on sub-graphs of G . If V is infinite, we say that *percolation* occurs if the resulting subgraph contains an infinite connected subgraph. The *critical probability* is defined as

$$p_c := \inf\{p : \mathbb{P}_p(\text{percolation}) > 0\}.$$

The critical value p_c will depend upon the choice of G . Variations of this model are also natural, such as *site percolation*: keep vertices independently with probability p and consider the induced subgraph. Another variation is to allow some dependence among the edges or vertices kept, *dependent percolation*.

Since the introduction of percolation theory, determining p_c for various graphs has been a top priority. In the case of bond percolation in the square lattice $\mathbb{Z}^2$, the twenty-year gap between results of Harris [13]: $p_c \geq 1/2$ and Kesten [16]: $p_c = 1/2$ hint at the difficulty of these problems. But such challenges invite help from other areas of mathematics, for example: Boolean functions $\longrightarrow$ sharpness of the crossing probabilities [9] and for some critical exponents, statistical mechanics $\longrightarrow$ mean-field behaviour [12].

So far we have treated the underlying graph G as fixed. However with some care, we can allow G to be randomly generated as well. In the words of Broadbent and Hammersley [8], it is reasonable to consider “a structure ... which is homogeneous in the large though it may possess local variations.”

In this thesis, we will consider percolation on the Voronoi tessellation generated by a homogeneous Poisson point process on the hyperbolic plane $\mathbb{H}^2$. That is, with each point of a constant intensity Poisson process on $\mathbb{H}^2$ we associate its Voronoi cell – which is the set of all points of the hyperbolic plane that are closer to it than to any other point of the Poisson process – and we colour each cell black with probability p and white with probability $1 - p$, independently of the colours of all other cells. We refer the reader to Chapter 1 for detailed definitions and some background on the hyperbolic plane, hyperbolic Poisson point processes and their Voronoi tessellations. Figure 2 shows a computer simulation of Voronoi percolation in the hyperbolic plane, rendered in the Poincaré disk representation of $\mathbb{H}^2$. (Note that in all depictions of hyperbolic Voronoi percolation in the thesis the black cells are rendered light blue to aid visibility.)

Figure 1: Computer simulation of Voronoi percolation in the hyperbolic plane, shown in the Poincaré disk representation of $\mathbb{H}^2$. ($p = 1/2, \lambda = 1$)

We say that *percolation* occurs if there is an infinite connected cluster of black cells. For each intensity $\lambda > 0$ of the underlying Poisson process, the *critical probability* is defined as

Figure 2: Computer simulation of Voronoi percolation in the hyperbolic plane, shown in the half-plane representation of $\mathbb{H}^2$. ($p = 1/2, \lambda = 1$)

$$p_c(\lambda) := \inf\{p : \mathbb{P}_{\lambda,p}(\text{percolation}) > 0\}.$$

We will let $p_u(\lambda)$ be the infimum of the set $p \in [0, 1]$ such that is a unique infinite black cluster, almost surely. Clearly, $p_c(\lambda) \leq p_u(\lambda)$.

In an influential paper, Benjamini and Schramm [5] showed that

$$0 < p_c(\lambda) = 1 - p_u(\lambda) < 1/2$$

for all λ and they conjectured that $p_c(\lambda)$ tends to $1/2$ as $\lambda \rightarrow \infty$. Among other things, we prove this conjecture in this thesis.

The results of Benjamini and Schramm [5] include that $p_c(\lambda)$ tends to zero as $\lambda \searrow 0$ and that for any $p \in (p_c, 1 - p_c)$ there are infinitely many infinite black clusters, almost surely. These results demonstrate profound differences with Voronoi percolation on the ordinary, Euclidean plane. In that case, the critical probability was shown to equal $1/2$ in the work of Zvavitch [27] and Bollobás and

Figure 3: Three examples under changing intensity, shown in the half-plane representation of $\mathbb{H}^2$: From left to right $\lambda = 1/40$, $\lambda = 1$, $\lambda = 40$ (with $p = 1/2$)

Riordan [6], no matter the intensity of the underlying Poisson process. (That the precise value of the intensity is irrelevant in the Euclidean setting is easily seen by considering the effect of dilations $x \mapsto \rho x$ on a homogeneous Poisson process and its Voronoi tessellation: we are only changing the magnification, not the adjacencies between cells.) What is more, in the Euclidean case there is almost surely at most one infinite cluster for any value of p and λ .

Additionally, in [5], Benjamini and Schramm also ask what are the asymptotics of $p_c(\lambda)$ as $\lambda \searrow 0$. In this thesis, we show $p_c(\lambda) \approx \pi\lambda/3$ as $\lambda \searrow 0$.

The content of this thesis is joint work with Tobias Müller.

Thesis Structure

In the first chapter, we introduce fundamental aspects of hyperbolic geometry, Voronoi diagrams, and percolation theory and then show some preliminary results, including

- The combinatorial structure of the Voronoi cells is invariant under changing the metric from the hyperbolic to the Euclidean.
- most distances between adjacent elements of $\mathcal{Z}$ are in the interval $(r_\lambda - w, r_\lambda + w)$ where r_λ depends on λ and w is a constant depending on “most.”

In the second chapter, we demonstrate

- $\lim_{\lambda \rightarrow \infty} p_c(\lambda) = 1/2$, confirming a conjecture of Benjamini and Schramm.

In the third chapter, we compute the expected number of neighbours for the *typical cell*, a certain “average” cell for $\lambda \searrow 0$. In the fourth chapter, we show

- $p_c(\lambda) = (1 + o_\lambda(1)) \frac{\pi\lambda}{3}$ as $\lambda \searrow 0$.

A portion of Chapter 1 and the entirety of Chapter 2 appear in [11].