University of Groningen

Aspects of Ultraviolet Non-local Gravity

Maheshwari, Shubham

DOI:
[10.33612/diss.177814701](https://doi.org/10.33612/diss.177814701)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2021

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):
Maheshwari, S. (2021). *Aspects of Ultraviolet Non-local Gravity*. [Thesis fully internal (DIV), University of Groningen]. University of Groningen. <https://doi.org/10.33612/diss.177814701>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Aspects of Ultraviolet Non-local Gravity

Shubham Maheshwari

university of
 groningen

Aspects of Ultraviolet Non-local Gravity

PhD thesis

to obtain the degree of PhD at the
University of Groningen
on the authority of the
Rector Magnificus Prof. C. Wijmenga
and in accordance with
the decision by the College of Deans.

This thesis will be defended in public on

Friday 3 September 2021 at 16.15 hours

by

Shubham Maheshwari

born on 22 August 1993
in Jaipur, India

Supervisor

Prof. A. Mazumdar

Co-supervisor

Prof. D. Roest

Assessment Committee

Prof. E.A. Bergshoeff

Prof. M. Yamaguchi

Prof. R.H. Brandenberger

Abstract

The objective of this thesis is to introduce ultraviolet (UV) non-local gravity and discuss some new results in it. Motivated from approaches to quantum gravity like string theory, this is a higher derivative modification of General Relativity (GR) which aims to provide a softer description of gravity at short distances, while simultaneously keeping at bay the typical illnesses that accompany higher derivative theories. The derivatives appearing in the action are of infinite order, making the theory ‘non-local’. Moreover, it is ‘UV’ because the derivatives modify gravity at high energies, while recovering GR at low energies. We discuss how such theories 1) possess linearized solutions with non-divergent gravitational potentials and curvatures, 2) admit non-maximally symmetric backgrounds which are stable (free from ghosts/tachyons) under perturbations, 3) admit non-singular, isotropic and homogeneous bouncing cosmological solutions with stable perturbations and 4) admit non-singular, anisotropic and homogeneous bouncing cosmological solutions where, unlike in GR, there is no unstable growth of spacetime anisotropies during the contraction phase of the universe, and isotropy and homogeneity is recovered at late times during the expanding phase.

Contents

Publications	iv
Acknowledgements	vii
Acronyms and conventions	x
1 Introduction	1
1.1 Non-locality in physics	9
1.2 UV non-local gravity	11
2 A Point Charge in UV Non-local Gravity	15
2.1 Linearized metric solution for an electrically charged point source . .	17
2.2 Regular curvatures	21
3 Perturbations around Non-Maximally Symmetric Backgrounds	25
3.1 Perturbation theory	27
3.1.1 Covariant scalar-vector-tensor decomposition	27
3.1.2 General structure of $\delta^2 S$	28
3.2 Background spacetimes	29
3.2.1 Background equations of motion	30
3.2.2 Physical example: Near a cosmological bounce	32
3.3 Second order variation of the action	33
3.4 dS/AdS and flat spacetime limits	42
3.5 Non-local gravity around non-maximally symmetric backgrounds: Illustrative examples	45

4 An Isotropic and Homogeneous Bouncing Universe	53
4.1 Action and equations of motion	55
4.2 Perturbations	56
4.3 Isotropic and homogeneous bouncing backgrounds	59
5 An Anisotropic and Homogeneous Bouncing Universe	63
5.1 An anisotropic and homogeneous bouncing background	66
5.1.1 Matter sources for anisotropic bounces	70
5.1.2 Local versus non-local gravity	74
6 Conclusions	77
Appendices	79
6.A Reissner-Nordström metric in GR	80
6.B Working with infinite order differential equations	82
6.C Full expressions of curvatures for the linearized metric	84
6.D Full, non-linear equations of motion	87
6.E Perturbations around an arbitrary background	88
6.F Details of the second order action computation	92
6.G Linearized equations of motion around the non-maximally symmetric background in eq. (3.13)	107
6.H Commutation relations	108
6.I Simplifying the quadratic action	111
6.J 1 + 3 cosmological perturbation theory	116
6.K k -matter as a perfect fluid scalar field	118
Samenvatting	119
Bibliography	119

Publications

The work presented in this thesis is based on the following papers:

P1 L. Buoninfante, G. Harmsen, S. MAHESHWARI and A. Mazumdar, “Nonsingular metric for an electrically charged point-source in ghost-free infinite derivative gravity”, *Phys. Rev. D* 98 (2018) 084009, [arXiv:1804.09624]

P2 K. Sravan Kumar, S. MAHESHWARI and A. Mazumdar, “Perturbations in higher derivative gravity beyond maximally symmetric spacetimes”, *Phys. Rev. D* 100 (2019) 064022, [arXiv:1905.03227]

P3 K. S. Kumar, S. MAHESHWARI, A. Mazumdar and J. Peng, “Stable, nonsingular bouncing universe with only a scalar mode”, *Phys. Rev. D* 102 (2020) 024080, [arXiv:2005.01762]

P4 K. S. Kumar, S. MAHESHWARI, A. Mazumdar and J. Peng, “An anisotropic bouncing universe in non-local gravity”, *JCAP* 07 (2021) 025, [arXiv:2103.13980].

Acknowledgements

The last four years of my PhD has been quite a journey. I am finally on the last leg as I write this.

First of all, I express my deep love and gratitude to my parents, who gave me everything they had. I would not be where I am without them. Words cannot express the significance of their presence in my life. This thesis is for them.

I thank my supervisor, Anupam Mazumdar, for taking me in as a young PhD student and introducing me to a new world in academia. I am very grateful for the opportunity he gave me to work on the amazing subject of physics. I have learned a lot, and thank him for his patience, support and guidance over the years on every aspect of my PhD. His untiring enthusiasm for physics is commendable.

I am grateful to my co-supervisor, Diederik Roest, for all the fun discussions we have had on physics and otherwise. His unbridled eagerness to learn from everyone he meets is a constant source of inspiration. He always kept me on my toes. I have learnt a great deal about physics and life from him.

Along the same lines is my former master's thesis supervisor back in India, Chethan Krishnan, who gave me my first-ever serious project on physics. His practical, always-learning attitude and vast knowledge of physics were hard to catch up with at times, but always rewarding, and something that pushes me to this day.

I thank the members of my thesis' assessment committee, Eric Bergshoeff, Masahide Yamaguchi and Robert Brandenberger, for agreeing to appraise my thesis, for investing time and effort in doing so, and for their useful comments.

I would like to thank all members of VSI. In particular, I would like to thank Elisabetta Pallante for having me over as the tutor for her QFT course, which I enjoyed taking. Even though (unfortunately) we did not get to interact much, I also thank Daan Meerburg for always being very helpful and generous with his time. I also thank Annelien Blanksma and Iris de Roo-Kwant for all their help, for making VSI a comforting place to work at, and for gracefully tolerating my requests for things I could have done myself. I thank all the (former and present) PhDs and post-

docs for making VSI what it is, especially Luca Buoninfante, Marco Boers, Gideon Vos, Sravan Kumar Korumilli, Ruud Peeters, Arunesh Roy, Perseas Christodoulidis, Yongliang Hao, Jun Peng, Ceyda Şimşek, Johannes Lahnsteiner, Yangyang Guo, Alba Kalaja, David Stefanyshyn, Ivan Kolář, Georg Stettinger, Gerhard Harmsen and Pepe Maldonado Torralba. For their help in providing me the Samenvatting of this thesis, I thank, again, Marco and Gideon. Special thanks goes to Sravan for teaching me a lot of physics and how to see the bigger picture, and for being a good friend. Apologies if I missed anyone who thinks their name should have been here - I remember you!

Outside VSI, I thank all the friends I made in Groningen who made life fun beyond physics. In particular, I thank Bohuan Lin, Sivasudhan Rathnachalam, Subam Rai, Pranoti Joshi and Vibhuti Nandel. Outside Groningen, I thank Shrey Grover.

Lastly, but most importantly, I thank my family for their love, support and encouragement.

Shubham Maheshwari
Groningen
July 20, 2021

सत्यमेव जयते नानृतं सत्येन पन्था विततो देवयानः ।
येनाक्रमन्त्यृषयो ह्याप्तकामा यत्र तत् सत्यस्य परमं निधानम् ॥

- *Mundaka Upanishad, 3.1.6*

(Truth alone conquers, not untruth. Through truth, the path leading to gods is laid. Through this (path), sages whose desires have been fulfilled, travel to where that supreme abode of truth is.)

To my parents

Acronyms and conventions

UV - Ultraviolet
IR - Infrared
GR - General Relativity
IDG - Infinite Derivative Gravity
EOM - Equations of motion
FLRW - Friedmann-Lemaître-Robertson-Walker
EH - Einstein-Hilbert
SVT - Scalar-vector-tensor
dof - degree(s) of freedom
(A)dS - (anti) de Sitter

Throughout this thesis, we follow some conventions. The metric signature is $(- + + +)$. Natural units $\hbar = c = 1$ are used. M_P is the reduced Planck mass given by $M_P = \sqrt{\frac{\hbar c}{8\pi G}}$.

Greek indices $\mu, \nu, \dots$ run from 0 to 3.

Roman indices $i, j, \dots$ run from 1 to 3, unless specified otherwise.

The box operator is $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$

The Ricci scalar is denoted by R , Ricci tensor by $R_{\mu\nu}$, traceless Ricci tensor by $S_{\mu\nu}$, Riemann tensor by $R_{\mu\nu\rho\sigma}$, and Weyl tensor by $C_{\mu\nu\rho\sigma}$.

An overdot denotes derivative with respect to cosmic time t , for example, $H = \frac{\dot{a}}{a}$.

Chapter 1

Introduction

Our universe has been a consistent subject of discourse for millennia. Over the years, humans have wondered tirelessly about why nature works the way it does. They have talked about it, written poems, made imaginative guesses, used a scientific and mathematical approach, and performed various experiments and observations to get closer to the fundamental principles on which our universe operates. Of these, one phenomenon has often surprised us every time we think we have understood it - gravity. Since the times of Aristotle in the 4th century BC and Brahmagupta in the 7th century CE, there was a rudimentary understanding of the force of gravity [1][2]. Just a few centuries ago, Galileo showed that objects fall under gravity with the same acceleration, independent of their composition. This is basically a statement of what is now known as the weak equivalence principle. Soon after, Newton theorized the famous inverse square law for gravity in his treatise, *Principia* [3]. Newton's theory was later superseded by Einstein's General Relativity (GR), which marked a paradigm shift in the way we now think about space and time [4-6]. GR successfully explained Mercury's perihelion precession, a then unsolved problem for over half a century [7]. It also calculates correctly the deflection of light under the influence of gravity, which was famously verified by Eddington and his team through their observation of the solar eclipse of May 1919 [8]. The idea that gravity is not a force but rather a manifestation of the curvature of spacetime is deeply ingrained in GR. Moreover, Einstein's ideas on special relativity were not only a precursor of GR, but also drove the development of quantum field theory through its marriage with quantum mechanics. Remarkably, GR is able to explain phenomena associated with very strong gravitational fields as well, such as those corresponding to neutron stars and black holes, something which Newtonian gravity fails at. The discovery of the Hulse-Taylor binary in 1974 provided a new, astrophysical arena to test GR in the strong-field regime [9]. The observation of its orbital period decay matched with the theoretical prediction of GR of energy being carried away by gravitational radiation [10]. This was the first (indirect) detection of gravitational waves. After decades of effort, two LIGO (Laser Interferometer Gravitational-Wave Observatory) ground-based interferometers made the first direct detection of gravitational waves (GW150914) from a binary black hole merger

on September 14, 2015 [11]. On August 17, 2017, the LIGO-Virgo network made the first direct detection of gravitational waves from a binary neutron star inspiral (GW170817) [12]. Many more gravitational wave observations have been made since then. In April 2019, the Event Horizon Telescope (EHT) collaboration released the first image of a black hole, a supermassive black hole in the center of the elliptical galaxy Messier 87 [13–18]. On the cosmological side, the Λ -CDM (Cold Dark Matter) model provides an explanation of large-scale structure of galaxy distribution, CMB (Cosmic Microwave Background) and the observation that the current universe is expanding at an accelerating rate [19, 20].

Figure 1.1: First ever image of a black hole. It shows a supermassive black hole in the center of the galaxy Messier 87. The bright ring of light is emitted by gas swirling around the event horizon and falling into the black hole. Imaging black hole shadows provides us with an exciting new opportunity to test GR in the strong-field regime. Image taken from the EHT collaboration [website](#). Also see references [13–18].

It is an impressive feat of GR that it can explain gravitational physics across a wide range of distance scales and astrophysical and cosmological phenomena. After more than a hundred years since its inception, it agrees very well with all observations and experiments and has not yet been replaced by a more fundamental theory. Nevertheless, as with any physical theory, there are strong indications that we need to go beyond GR and that it is not the most fundamental theory of gravity. At the same time, from different motivational fronts, there have been several attempts to construct alternatives to GR, through the addition of extra fields like in scalar-tensor [21] theories, through higher derivatives like $f(R)$ [22] or Hořava-Lifschitz theories [23], and through models of higher dimensions [24] (For reviews of these approaches, see [25, 26]).

Let us first talk about the energy content of our universe (for the latest estimates

on cosmological parameters, see the Planck collaboration results in [27]). Assuming Λ -CDM is correct, observations tell us that about 25% of the total energy density is in some form of non-relativistic matter which does not interact electromagnetically (and is therefore ‘dark’) (for a modern account of cosmology, see [28]). One evidence for this comes from the observed flattening of galaxy rotation curves with radial distance [29, 30]. This flattening cannot be explained using the observed baryonic matter in the galaxy, but it can be if there exists a halo of dark matter around it. Nevertheless, we still do not know what dark matter fundamentally is, and we have never detected it directly. Its inference comes purely from the gravitational sector since it does not emit light (For reviews on dark matter, see [31, 32]). There have been attempts to explain the discrepancies by modifying gravity instead of assuming the existence of a new kind of matter - this is the Modified Newtonian Dynamics (MOND) paradigm [33] and its relativistic generalization, the Tensor-Vector-Scalar (TeVeS) model [34].

The majority of the energy budget of our universe, around 70%, comes from what has been dubbed dark energy (for a review on dark energy, see [35]). Unlike dark matter, dark energy does not cluster strongly. It also has a negative equation of state. The easiest way to explain dark energy is to take it as the cosmological constant Λ itself, like in the Λ -CDM model. However, this is the simplest but not the only way of explaining accelerated expansion. One may introduce a canonical (or non-canonical) scalar field called quintessence to model dark energy. Or one may modify gravity in the infrared sector, like through $f(R)$ theories or massive gravity [36]. Sticking with the cosmological constant has its own pitfalls in what has come to be known as the cosmological constant problem, where the quantum field-theoretically computed value of the vacuum energy density is completely off from its observed value: by about 120 orders of magnitude [37].

Another problem with GR is the existence of spacetime singularities under reasonable energy conditions on matter [4, 38–40]. They appear in both astrophysical and cosmological contexts: in the center of black holes formed from collapsing matter, and the initial singularity in the beginning of the universe in a Big Bang model. At such singularities, curvatures and energy densities diverge, GR breaks down, and we therefore cannot use it to explain physics at such high energy/short distance scales. Such unphysical divergences are the strongest indications that GR is not an ultraviolet (UV) complete theory of gravity. One may argue that quantum effects become important at such high energy scales, smoothening out any singularities. Although there are proposals for quantum gravity like string theory [41], loop quantum gravity [42] and causal set theory [43], we are still far away from a complete understanding of it.

Consider the Einstein's equations (without a cosmological constant) [4, 6]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M_P^2}T_{\mu\nu} \quad (1.1)$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor describing the geometry of spacetime, and $T_{\mu\nu}$ is the energy-momentum tensor of the matter sector. $T_{\mu\nu}$ is usually calculated by varying the matter action with respect to the metric

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \quad (1.2)$$

where S_m is the matter Lagrangian. In the simplest case, one does not really need to know the exact form of the matter Lagrangian in order to find a solution to Einstein's equations. One typically assumes a diagonal energy-momentum tensor with energy density and pressure components describing the collective behavior of the microscopic constituents.

Perhaps the most famous solution to Einstein's equations is the static, spherically symmetric metric describing the exterior of a spherical body [44]

$$ds^2 = - \left(1 - \frac{2Gm}{r}\right) dt^2 + \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (1.3)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on unit sphere S^2 . This is the Schwarzschild solution. It describes the most mysterious objects in our universe - black holes. There are more general black holes as well, those with electric charge and/or angular momentum. Astrophysically, all black holes are rotating and therefore the most interesting solution for practical purposes is the Kerr black hole. Charges get neutralized by absorbing outside matter, so we need not consider charged black holes for astrophysical purposes. At $r = 2Gm$ is the well-known event horizon. The event horizon is not really a violent place in principle (though it is one in astrophysical scenarios), but a curious imaginary boundary in spacetime where the causal structure flips (compared to the outside region) - anything that falls into the horizon is inevitably lead towards the center of the black hole at $r = 0$, known as the singularity. At the singularity, curvature invariants diverge. For instance, the Kretschmann scalar [6]

$$K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2m^2}{r^6} \quad (1.4)$$

blows up at $r \rightarrow 0$.

In cosmology, the most famous solution is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which describes the spacetime geometry of a universe which is evolving in time, but where the spatial sections are homogeneous and isotropic

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \quad (1.5)$$

where $a(t)$ is known as the scale factor, which determines the size of the universe at a moment in time t . For instance, if $\dot{a} > 0$, it means that the universe is expanding. Suppose that the universe consists of some matter whose energy density decays like $\rho \sim a^{-n}$. The Einstein's equations applied to the FLRW metric, known as Friedmann equations, then tell us that the scale factor of the universe evolves as $a \sim t^{2/n}$. For non-relativistic matter called dust, we have $n = 3$, while for radiation, $n = 4$. In any case, the as we go back in time, we encounter a singularity at $a = 0$ as $t \rightarrow 0$ where the energy density diverges. Again, this cosmological singularity cannot be explained within the realm of GR.

So far, we have talked about gravity as a classical phenomenon - GR is a classical field theory. Another pillar of modern physics, which was invented around the same time, is quantum field theory (QFT). QFT is the formalism which describes interactions between particles while obeying the laws of quantum mechanics and special relativity. In particular, the Standard Model (SM) is a specific kind of QFT which describes the particle content and interactions corresponding to our universe. Fermions like quarks and leptons interact through gauge boson mediators, and the non-zero vacuum expectation value of the scalar Higgs boson spontaneously breaks electroweak symmetry. Particle physics, or high energy physics, is a tool to uncover the fundamental properties of subatomic particles by colliding them at high speeds and studying the ensuing reactions and remnants. The approach to quantizing gravity at the fundamental level is called quantum gravity, of which there exist many approaches, but none are satisfactorily complete (see [45-49] for a list of the first few approaches towards quantizing gravity). The fundamental excitations of the gravitational field are called gravitons. Although a single graviton (unlike photons) has not been experimentally detected yet [50], we have already detected a coherent collection of a large number of gravitons in the form of gravitational waves. Nevertheless, there are many structural and practical issues with constructing a quantum theory of gravity [51, 52].

One may hope to uncover some hints about quantum gravity by studying semi-classical gravity first. This is the study of quantum field theory on a fixed, classical background geometry. Even in this regime, there are questions which have puzzled physicists for decades. The most famous one is the black hole information paradox [53, 54]. Hawking showed that black holes emit radiation but in seemingly such a way that any information that went inside a black hole is washed out and lost as pure radiation. In a theory which respects quantum mechanics, unitarity must be upheld at all times. This implies that pure states must evolve into pure, and not mixed, states. There is thus a conflict between two fundamental physical principles - unitarity and the equivalence principle. For the last decade or so, there has been a surge of interest in trying to better understand and possibly solve the paradox, but

there is no unanimous conclusion yet [55].

Although we do not have a good understanding of physics at the Planck scale, it does not mean that GR and quantum mechanics are inconsistent. At low energies (less than the Planck scale M_P), GR (or its higher derivative extensions) can be treated as an effective field theory (EFT), and quantum corrected observables can be reliably computed [56, 57]. In fact, GR is perhaps the best example of an EFT because the higher dimensional operators (interactions) in the derivative expansion are suppressed by the very high energy Planck scale M_P .

Consider the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g} M_P^2 R \quad (1.6)$$

where we have ignored an overall factor of $1/2$ for brevity. Let's say we expand around flat space, so the background metric is $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Using $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, we get (schematically)

$$S_{EH} = \int d^4x M_P^2 (h\partial^2 h + h(h\partial^2 h) + h^2(h\partial^2 h) + \dots) \quad (1.7)$$

where we have ignored total derivative terms and suppressed indices for brevity. Canonically normalizing the graviton field $h_{\mu\nu} \rightarrow \frac{h_{\mu\nu}}{M_P}$, we obtain an infinite series of graviton-graviton vertices suppressed by appropriate powers of M_P

$$S_{EH} = \int d^4x \left(h\partial^2 h + \frac{h}{M_P} (h\partial^2 h) + \frac{h^2}{M_P^2} (h\partial^2 h) + \dots \right) \quad (1.8)$$

The negative power of coupling $1/M_P$ shows that GR is non-renormalizable. Fourier transforming $\partial^2 \rightarrow -k^2$, we see that the perturbative expansion works well for energy scales $k \ll M_P$. As $k \sim M_P$, all interaction terms in the expansion become equally important. The theory becomes strongly coupled and leads to a breakdown of perturbation theory. GR becomes non-perturbative for $k \gtrsim M_P$. So this naive application of perturbation theory is not enough to uncover quantum gravitational physics at the Planck scale, although it is a very good approximation at low enough energies. Actually, pure GR is renormalizable at one loop (in 4 dimensions); however, gravity coupled to scalar matter is non-renormalizable at one loop with a counterterm Lagrangian which consists of the Ricci scalar squared term R^2 [58]. Pure GR is non-renormalizable at two loops, where the counterterm Lagrangian goes like (Riemann)³ with all indices contracted [59].

The necessity of including higher order in curvature terms in the gravitational action in order to perform renormalizations and make better sense of its quantum theory has a long history, dating back to (at least) the work of DeWitt and Utiyama

[60]. This also reminds us of the basic approach of constructing any EFT - given the symmetries, construct a Lagrangian with all possible operators that respect these said symmetries [61]. The higher dimensional operators are then suppressed by appropriate powers of the EFT cutoff scale. These higher dimensional operators are exactly those which hide information about UV physics. So, if we want to understand the dynamics of gravity at short distances, we must first try to extract information about UV physics from these higher dimensional operators. This idea was investigated in detail by Stelle who considered the most general fourth order theory of gravity in 4 dimensions [62, 63]

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu} \right]$$

where any possible (Riemann)² term can be effectively removed from the action using the Gauss-Bonnet identity in 4 dimensions, which says that

$$\int d^4x \sqrt{-g} (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2) \quad (1.9)$$

is a total derivative term. He showed that this theory is renormalizable and UV divergences are softened (compared to GR) because the graviton propagator decays like $1/k^4$ for high momenta (Such quadratic curvature theories were also shown to be asymptotically free [64, 65]. See also [66-69]. And see [70] for a discussion of the effective action in quantum gravity. The motivation for adding quadratic curvature terms to the gravity action also comes from considering high energy (Froissart) boundedness of gravitational cross sections [71]. Higher derivative (α') corrections also appear in the low energy effective action for string theory [72, 73]). However, this theory suffers from a ghost instability. If one performs a decomposition of the metric into its scalar, vector and tensor components, one finds that Stelle's theory propagates 8 degrees of freedom (dof) - 2 dof from the usual transverse, traceless graviton, 1 massive scalar dof and 5 dof from a massive spin-2 mode. This massive spin-2 mode is a ghost. This can be seen by finding its propagator and noticing that its residue has the opposite sign (compared to usual conventions).

Ghosts are bad degrees of freedom because they prevent us from making physical sense of a theory. They are most easily identified by their wrong sign kinetic terms. They typically appear in higher derivative field theories and gravity. There are different ways of understanding this. For example, in a classical mechanics problem with a non-degenerate¹ Lagrangian that depends on higher than one time derivative on the position variable $L(x, \dot{x}, \ddot{x})$, it can be shown that the Hamiltonian

¹Non-degeneracy means that $\frac{\partial^2 L}{\partial \ddot{x}^2} \neq 0$, which would just mean that $\ddot{x}$ cannot be eliminated by integration by parts.

of the system is linear in one of the canonical momenta [74]. This classical instability was first discussed by Mikhail Ostrogradsky and are therefore also called Ostrogradsky ghosts [75]. This implies that the Hamiltonian is unbounded from below (and above). An unbounded Hamiltonian wreaks havoc for the stability of the vacuum of any theory. Suppose we have a healthy system with a Hamiltonian bounded from below, and a ghostly system with a Hamiltonian unbounded from below. When these two systems interact and exchange energy and momentum with each other, they can do so indefinitely with the energy of the first system becoming arbitrarily high and of the ghostly system becoming arbitrarily low. Conservation of energy and momentum is satisfied because these ghost dofs have negative energy.

Ghosts are problematic in a quantum theory as well. Even if the ghostly particle does not couple to a healthy particle, they both couple to gravity. So one may have a graviton loop diagram that decays into pairs of ghostly and healthy particles, without violating energy and momentum conservation. The phase space integral of this process is divergent, unless one imposes a Lorentz non-invariant momentum cutoff on the final state phase space [76]. It can be shown that ghosts lead to negative normed states. They violate the optical theorem of QFT and therefore violate unitarity.

Consider as an example a fourth order free scalar field theory given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\phi[\Box(\Box - m^2)]\phi. \quad (1.10)$$

The propagator of this theory is obtained by inverting the kinetic term

$$\Pi \sim \frac{1}{k^2(k^2 + m^2)} \quad (1.11)$$

which can be split into two propagators using partial fractions to get

$$\Pi \sim \frac{1}{m^2} \left(\frac{1}{k^2} - \frac{1}{k^2 + m^2} \right) \quad (1.12)$$

from which we can see that the massless propagator has the opposite sign from what it should have been if it was of a healthy particle.

Apart from ghosts, one may encounter other kinds of instabilities in a system, like tachyons and gradient instabilities [77]. Tachyons appear when a Lagrangian has a negative mass squared term. For instance, if we have the theory [77]

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2, \quad (1.13)$$

the solution for ϕ in the long-wavelength limit ($k \rightarrow 0$) is $\phi(t) \sim t^{\pm mt}$, where the growing mode solution indicates an instability.

Gradient instabilities occur when one has wrong-sign spatial gradient terms in the Lagrangian for a particular degree of freedom. For instance, for the theory [77]

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2, \quad (1.14)$$

the solution for ϕ (in momentum space) is given by $\phi_k(t) \sim e^{\pm kt}$ (where $k = \sqrt{\mathbf{k}^2}$), where the growing mode solution again indicates an instability.

1.1 Non-locality in physics

As we have seen, higher derivative field theories and gravity typically propagate ghosts. This is disappointing, because higher derivative terms in the action are exactly those which hide information about UV physics, while also softening UV behavior. Is there any way of retaining the UV-stabilizing abilities of these operators without introducing any sick degrees of freedom? There are a couple of approaches of doing so, and this thesis is based on one of the methods - that is, of non-localizing the action or the equations of motion. Briefly speaking, the idea is to incorporate specific higher derivative terms in the action which do not introduce new zeros in the kinetic operator or, equivalently, do not introduce any new poles in the propagator. No new poles implies no new, and therefore no ghost, degrees of freedom in the physical spectrum.

Non-local theories have a long history that go back to the work of Max Born in [78] and Hideki Yukawa in [79-81]. Max, in the above-mentioned paper, discusses a non-local generalization of Maxwell's theory with the following equation of motion in momentum space [78]

$$e^{-p^2/M^2} p^2 A_\mu = 0 \quad (1.15)$$

where p is momentum and A_μ is the electromagnetic potential, and M is some new high energy scale at which the higher derivative terms become important (Born used different units where $M = 1$, but we keep M here explicitly). This is a specific kind of modification of Maxwell's theory such that despite the presence higher derivatives, there are no new degrees of freedom or ghosts. This is because the non-local kinetic operator $e^{-p^2/M^2} p^2$ contains the exponential function e^{-p^2/M^2} , which has no zeros in the complex p^2 plane. This follows from the Weierstrass product theorem, which says that any entire function $f(z)$ with no zeros can be written as $f(z) = e^{g(z)}$, where $g(z)$ is an entire function. So the only pole in the photon propagator appears at the usual pole at $p^2 = 0$ in the original, local Maxwell's theory. These non-local modifications also have the ability to regularize divergences. This was, in fact, one of the first motivations to consider non-local field theories. For instance, Born showed

in the same paper that the Coulomb potential of a point charge at the origin is [78]

$$\phi(r) = \frac{1}{r} \operatorname{erf}\left(\frac{Mr}{2}\right) \quad (1.16)$$

which is regular at the origin $r = 0$, and recovers the usual Coulomb $1/r$ behavior at large distances. $\operatorname{Erf}(x)$ is the error function whose leading term in its series expansion around $x = 0$ is linear in x .

Non-local theories have been studied by many authors, in different contexts, motivated from different perspectives. One comprehensive study soon after Born's paper was the work of Pais and Uhlenbeck [82]. The motivation of non-localizing the action was the hope that it may help in obtaining regular results for physical observables, and was a contender of the renormalization program of conventional QFT as we know it today. Pais and Uhlenbeck considered field equations of the type [82]

$$F(\square)\psi = \rho \quad (1.17)$$

where

$$F(\square) = \prod_{i=1}^N (\square - m_i^2) \quad (1.18)$$

for some rest masses m_i . The product in (1.18) can be finite or infinite, where the action/equation of motion is non-local in the latter case. Non-local field (scalar/gauge/gravity) theories were also studied by Pauli [83], Hayashi [84], Efimov [85–87], Kirzhnits [88], Krasnikov [89], Kuzmin [90], Tomboulis [91–93], Biswas, Mazumdar and Siegel [94], Modesto [95] and many others. More references can be found in later parts of this thesis.

An infinite series of derivatives in terms (kinetic/potential) in the action also makes its appearance in string field theory and p-adic string theory. String field theory is a description used to provide better insight into the apparent divergences in string perturbation theory in the worldsheet formalism [96–98]. p-adic string theory is a kind of string theory where the worldsheet coordinates are non-archimedean p-adic-valued [99–101]. The non-archimedean effective Lagrangian of a real scalar field ϕ is highly non-local and has the form [101]

$$\mathcal{L} = \frac{p}{p-1} \left[-\frac{1}{2} \phi p^{-\frac{1}{2}\square-1} \phi + \frac{1}{g^2} \frac{p}{p+1} \left(1 + \frac{g\phi}{p} \right)^{p+1} - \frac{\phi}{g} - \frac{1}{g^2} \frac{p}{p+1} \right] \quad (1.19)$$

for some coupling constant g and some prime number p . This effective Lagrangian of a real scalar field in real d-dimensional spacetime reproduces the tree-level non-archimedean string amplitudes. The non-locality in this theory is evident from the non-local $p^{-\frac{1}{2}\square-1}$ term which has an infinite series of derivatives.

It is possible to recast the infinite order differential equation of motion into an integral equation. For instance, using [101, 102],

$$p^{\frac{1}{2}\partial_t^2} \phi(t) = \frac{1}{\sqrt{2\pi \ln p}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\ln p}(t-t')^2} \phi(t') dt' \quad (1.20)$$

the equations of motion $p^{\frac{1}{2}\partial_t^2} = \phi^p$ can be rewritten as [101, 102]

$$\phi(t)^p = \mathcal{C}[\phi](t) \quad (1.21)$$

which is a non-linear integral equation. Remarkably, an infinite order differential equation can be solved by numerically solving its integral avatar instead. This can be done by using the fixed point iteration method generalized for functions. One then converges to a kink solution within just 3-4 iterations! The initial value problem for such non-local theories is an interesting topic in itself and has been investigated, for instance, in [102, 103]. In particular, in [103], the authors showed that, generically, an infinite order differential equation does not necessarily imply the need for an infinite number of initial conditions, and that each pole of the propagator provides two initial conditions. Also, in [102], the authors showed how having an infinite order differential equation may also come with an infinite number of constraints on the initial conditions themselves, unlike what happens in a differential equation of a finite order.

The smearing of a Dirac delta source like the regularized electrostatic potential in [1.16] is also seen for the gravitational potential in string theory, where the smearing takes place due to infinite order derivatives [104] or α' corrections [105]. This hints towards a similar resolution of curvature singularities at the full non-linear level, although more work needs to be done in this direction.

1.2 UV non-local gravity

Following suit from the infinite tower of derivatives that appears in string theory, one may propose to study a theory of gravity which is modified in the UV due to these higher derivative terms and which also recovers GR in the low energy limit. We wish to include higher derivative terms in the action in such a way that no new zeros are introduced in the kinetic matrix, or equivalently, no new poles appear in the graviton propagator apart from the usual one present in GR. For now, we will take a flat background. Let's just first consider adding all possible quadratic in curvature terms before considering the Einstein-Hilbert action. The most general quadratic in curvature part of the action is given by [106]

$$S_q = \int d^4x \sqrt{-g} R_{\mu_1 \nu_1 \lambda_1 \sigma_1} \mathcal{D}_{\mu_2 \nu_2 \lambda_2 \sigma_2}^{\mu_1 \nu_1 \lambda_1 \sigma_1} R^{\mu_2 \nu_2 \lambda_2 \sigma_2}, \quad (1.22)$$

where $\mathcal{D}$ is a differential operator containing covariant derivatives and $\eta_{\mu\nu}$. Explicitly, one finds that its most general form is [106]

$$\begin{aligned}
S_q = & \int d^4x \sqrt{-g} \left[R F_1(\square) R + R F_2(\square) \nabla_\mu \nabla_\mu R^{\mu\nu} + R_{\mu\nu} F_3(\square) R^{\mu\nu} + R_\mu^\nu F_4(\square) \nabla_\nu \nabla_\lambda R^{\mu\lambda} \right. \\
& + R^{\lambda\sigma} F_5(\square) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\lambda R^{\mu\nu} + R F_6(\square) \nabla_\mu \nabla_\nu \nabla_\lambda \nabla_\sigma R^{\mu\nu\lambda\sigma} + R_{\mu\lambda} F_7(\square) \nabla_\nu \nabla_\sigma R^{\mu\nu\lambda\sigma} \\
& + R_\lambda^\rho F_8(\square) \nabla_\mu \nabla_\sigma \nabla_\nu \nabla_\rho R^{\mu\nu\lambda\sigma} + R^{\mu_1\nu_1} F_9(\square) \nabla_{\mu_1} \nabla_{\nu_1} \nabla_\mu \nabla_\nu \nabla_\lambda \nabla_\sigma R^{\mu\nu\lambda\sigma} \\
& + R_{\mu\nu\lambda\sigma} F_{10}(\square) R^{\mu\nu\lambda\sigma} + R_{\mu\nu\lambda}^\rho F_{11}(\square) \nabla_\rho \nabla_\sigma R^{\mu\nu\lambda\sigma} \\
& + R_{\mu\rho_1\nu\sigma_1} F_{12}(\square) \nabla^{\rho_1} \nabla^{\sigma_1} \nabla_\rho \nabla_\sigma R^{\mu\rho\nu\sigma} + R_{\mu}^{\nu_1\rho_1\sigma_1} F_{13}(\square) \nabla_{\rho_1} \nabla_{\sigma_1} \nabla_{\nu_1} \nabla_\nu \nabla_\rho \nabla_\sigma R^{\mu\nu\lambda\sigma} \\
& \left. + R^{\mu_1\nu_1\rho_1\sigma_1} F_{14}(\square) \nabla_{\rho_1} \nabla_{\sigma_1} \nabla_{\nu_1} \nabla_{\mu_1} \nabla_\mu \nabla_\nu \nabla_\rho \nabla_\sigma R^{\mu\nu\lambda\sigma} \right] . \tag{1.23}
\end{aligned}$$

where $\mathcal{F}_i(\square)$ (for $i = 1, 2, \dots$) are ‘form factors’ containing only the box operator $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$. Each of these is assumed to be an analytic function of $\square$, admitting a power series expansion all the way to infinity

$$\mathcal{F}_i(\square) = \sum_{n=0}^{\infty} f_{i,n} \frac{\square^n}{M_s^{2n}} \tag{1.24}$$

where M_s ($< M_P$) is some high energy scale of ‘non-locality’ where the higher derivative terms become important. Note that we are not truncating the series at any finite order - doing so will typically introduce ghosts. The idea of non-local gravity is to add non-polynomial derivative terms in the action which help in regularizing bad UV behavior while also keeping ghosts away.

We now use the antisymmetric properties of the Riemann tensor,

$$R_{(\mu\nu)\rho\sigma} = R_{\mu\nu(\rho\sigma)} = 0 , \tag{1.25}$$

and the Jacobi identity

$$\nabla_\alpha R_{\mu\nu\beta\gamma} + \nabla_\gamma R_{\mu\nu\beta\alpha} + \nabla_\beta R_{\mu\nu\gamma\alpha} , \tag{1.26}$$

to simplify the above action. But we can do more - if we are interested in finding the graviton propagator, we need to perturb the action to second order in metric perturbation $h_{\mu\nu}$. Doing so around flat space enables us to remove the terms with covariant derivatives (because the symmetric property of successive partial derivatives and the antisymmetric properties of pairs of Riemann tensor indices combine to give zero). We are then left with [106, 107]

$$S_q = \int d^4x \left[R \mathcal{F}_1(\square) R + R_{\mu\nu} \mathcal{F}_2(\square) R^{\mu\nu} + R_{\mu\nu\lambda\sigma} \mathcal{F}_3(\square) R^{\mu\nu\lambda\sigma} \right] . \tag{1.27}$$

where we have relabeled $\mathcal{F}_3 \rightarrow \mathcal{F}_2, \mathcal{F}_{10} \rightarrow \mathcal{F}_3$ in the last equation. In fact, it can be shown that the quadratic in Riemann term with $\mathcal{F}_3$ can be eliminated using the identity [108–110]

$$R_{\mu\nu\rho\sigma}\Box^n R^{\mu\nu\rho\sigma} = 4R_{\mu\nu}\Box^n R^{\mu\nu} - R\Box^n R + \mathcal{O}(R^3) \quad (1.28)$$

which means that for the purpose of computing the quadratic action around flat space, we can put $\mathcal{F}_3 = 0$ without any loss of generality.

Adding the Einstein-Hilbert term with a cosmological constant, we may write a general form of UV non-local gravity as [107, 111, 112]

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R - \Lambda + R\mathcal{F}_1(\Box)R + R_{\mu\nu}\mathcal{F}_2(\Box)R^{\mu\nu} + R_{\mu\nu\lambda\sigma}\mathcal{F}_3(\Box)R^{\mu\nu\lambda\sigma} \right] \quad (1.29)$$

Henceforth in this thesis, when speaking about ‘non-local gravity’ or ‘UV non-local gravity’ or ‘Infinite Derivative Gravity’ (IDG), we will mean the action (1.29) (or one of its simple cases where some of the form factors $\mathcal{F}_i$ are zero). We call this UV non-local gravity because the higher derivative terms appear in the numerator, modifying the theory only at high energies, while recovering GR in the low energy limit ($M_s \rightarrow \infty$). There are also theories of non-local gravity which are non-local in the IR. They modify gravity at large distances because they have terms like $1/\Box$ [113] or $1/\Box^2$ [114]. We will not consider such models in this thesis. One of the aims of non-local gravity is to derive the structure of the form factors $\mathcal{F}_i$ such that the degrees of freedom in the theory are free from any instabilities like ghosts/tachyons. In this thesis, we will come across various examples of such specific form factors, derived after performing perturbations around a fixed background.

Before moving to the main part of the thesis, let us mention some key highlights of UV non-local gravity (or its simpler versions like non-local scalar field theory). Although finding full solutions is hard, various linearized solutions have been obtained in UV non-local gravity. It has been shown that the gravitational potential generated by a point source is regular at the origin with regular curvature invariants [107, 115]. The linearized version of a charged black hole in non-local gravity also has regular gravitational potential and curvatures [116]. The linearized metric solution to a rotating source in non-local gravity has no ring singularity [117, 118]. Non-local gravity was studied further in static setups in [119–122] and in dynamical setups in [110, 123–127]. Non-local QFTs were studied in [128–137] and at finite temperature in [138–142]. Inflation was studied in [143–145], non-singular bouncing cosmology in [94, 143, 146–150] and Hamiltonian analysis was performed in [151, 152]. Non-local gravity also shares many similar results with models of gravity inspired from non-commutative spacetime [153, 154]. It also shares features with the asymptotic safety program [155–158]. For some other references, see [159–189].

We do not claim to have provided in this thesis an exhaustive list of all the works on non-local gravity, but hope that the ones most relevant to this thesis are mentioned. The interested reader may also look at the references cited within all these works.

The thesis is organized as follows. In chapter 2, we present a new solution of linearized equations of motion of non-local gravity in the presence of a charged point source. It is shown that the corresponding gravitational potential and curvatures are regular at the origin, in contrast with their divergent behavior in GR. In chapter 3, we present a comprehensive analysis of perturbations in higher derivative gravity in general (and non-local gravity in particular) around generic spacetimes. In particular, we choose a specific class of non-maximally symmetric spacetimes and present new results of how such non-trivial backgrounds are free from ghost/tachyonic instabilities in non-local gravity. In chapter 4, we study non-singular bouncing cosmology in non-local gravity around isotropic and homogeneous spaces. We perform a full derivation of the quadratic action around a dynamical bouncing background and show that the sole scalar degree of freedom can be made free from ghost/tachyonic instabilities. In chapter 5, we show how to generalize non-singular bouncing cosmology in non-local gravity to anisotropic (but homogeneous) Bianchi-I backgrounds. We show that spacetime anisotropies grow much slower compared to GR during the contraction phase, reach a maximum at the bounce point, and then decrease rapidly as the universe asymptotes to Friedmann-Lemaître-Robertson-Walker, and ultimately to de Sitter at late times.

Chapter 2

A Point Charge in UV Non-local Gravity

It has been shown that UV non-local gravity can resolve the short-distance singularity due to a neutral point source by regularizing the $1/r$ behavior of the metric potential in the linearized limit, such that it does not blow up at the origin $r = 0$ [94, 107]. In this chapter, we aim to find the linearized gravitational response generated by a charged point source in non-local gravity. We show that the gravitational potential and curvatures (like Kretschmann scalar) are regular throughout spacetime, particularly at the origin. This example provides a glimpse into how higher derivative theories can regularize classical UV divergences.

The most general quadratic curvature gravity in 4 dimensions, which is parity invariant and torsion free around a constant curvature background can be described by the following action as [107, 112]

$$S = \int d^4x \sqrt{-g} \left[\frac{M_P^2}{2} R + R \mathcal{F}_1(\square_s) R + R_{\mu\nu} \mathcal{F}_2(\square_s) R^{\mu\nu} + R_{\mu\nu\rho\sigma} \mathcal{F}_3(\square_s) R^{\mu\nu\rho\sigma} \right] \quad (2.1)$$

where $\square_s \equiv \square/M_s^2$, the d'Alembertian operator $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$, and $\mathcal{F}_i(\square_s) = \sum_{n=0}^{\infty} f_{i,n} \square_s^n$. The new, high energy, non-local scale is given by $M_s (< M_p)$. This is where the higher order derivatives become important and modify GR at high energies. The three gravitational form factors $\mathcal{F}_i$ are constrained by the tree-level unitarity and ghost-free condition derived in Ref. [106, 107]. Let us review how to determine the propagator. Around flat space, the second order action looks like [106, 107]

$$\begin{aligned} \delta^2 S = \int d^4x \left[\frac{1}{2} h_{\mu\nu} a(\square) \square h^{\mu\nu} + h_\mu^\sigma b(\square) \partial_\sigma \partial_\nu h^{\mu\nu} + h c(\square) \partial_\mu \partial_\nu h^{\mu\nu} \right. \\ \left. + \frac{1}{2} h d(\square) \square h + h^{\lambda\sigma} \frac{f(\square)}{\square} \partial_\sigma \partial_\lambda \partial_\mu \partial_\nu h^{\mu\nu} \right] \end{aligned} \quad (2.2)$$

where we have put $M_P = 1$ for now and [106, 107]

$$\begin{aligned}
a(\Box_s) &= 1 - \frac{1}{2}\mathcal{F}_2(\Box_s)\Box_s - 2\mathcal{F}_3(\Box_s)\Box_s \\
b(\Box_s) &= -a(\Box_s) \\
c(\Box_s) &= 1 + 2\mathcal{F}_1(\Box_s)\Box_s + \frac{1}{2}\mathcal{F}_2(\Box_s)\Box_s \\
d(\Box_s) &= -c(\Box_s) \\
f(\Box_s) &= -2\mathcal{F}_1(\Box_s)\Box_s - \mathcal{F}_2(\Box_s)\Box_s - 2\mathcal{F}_3(\Box_s)\Box_s
\end{aligned} \tag{2.3}$$

Around a Minkowski background we can avoid the presence of other dynamical degrees of freedom other than the spin-2 massless graviton by demanding [106, 107]

$$2\mathcal{F}_1(\Box_s) + \mathcal{F}_2(\Box_s) + 2\mathcal{F}_3(\Box_s) = 0. \tag{2.4}$$

so that, inverting the kinetic operator, the graviton propagator is given by [106, 107]

$$\Pi(-k^2) = \frac{1}{a(-k^2)} \left(\frac{\mathcal{P}^{(2)}}{k^2} - \frac{\mathcal{P}_s^{(0)}}{2k^2} \right), \tag{2.5}$$

where $\Pi_{GR} \equiv \mathcal{P}^{(2)}/k^2 - \mathcal{P}_s^{(0)}/2k^2$ is the GR graviton propagator and $\mathcal{P}^2$, $\mathcal{P}_s^0$ are the two spin projection operators along the spin-2 and spin-0 components, respectively¹. Now, in order to have no ghosts coming from the $a(\Box_s)$ factor, its form is restricted to be that of the exponential of an entire function. This choice avoids introducing new poles in the propagator. One simple choice is [106, 107]

$$a(\Box_s) = e^{-\Box/M_s^2}. \tag{2.6}$$

Other choices of entire function have been explored in Refs. [119]. The above choice [2.6] is what we will use throughout the rest of the chapter.

In Ref. [107] it was shown that the ghost-free condition, like in Eq. (2.6), also yields regular static solutions in the linear regime, where the metric potential satisfies: $2|\Phi| < 1$. Furthermore, in Ref. [121], it was argued that this inequality can always hold true when the quadratic part of the action in Eq. (2.1) dominates over the Einstein-Hilbert term. In this chapter, we will show that a similar scenario also holds when we consider the spacetime metric generated by a charged point-source, but in this case an additional inequality, involving the electric charge, will be required in order to have bounded metric potentials.

¹Around Minkowski background, up to quadratic order in the metric perturbation, the form factor $\mathcal{F}_3(\Box_s)$ can be set to zero without any loss of generality.

2.1 Linearized metric solution for an electrically charged point source

We now want to determine the linearized spacetime metric generated by an electrically charged static point source in non-local gravity (or also known as Infinite Derivative Gravity (IDG)). In addition to the point source's mass, the electrostatic energy density in the electric field also changes the metric. The linearized field equations corresponding to the action in Eq. (2.1), with the ghost-free choice in Eq. (2.6) are given by [107]:

$$e^{-\square/M_s^2} [\square h_{\mu\nu} + (\eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \partial_\mu \partial_\nu h) - \partial_\sigma (\partial_\nu h_\mu^\sigma + \partial_\mu h_\nu^\sigma) - \eta_{\mu\nu} \square h] = -16\pi G \tau_{\mu\nu}. \quad (2.7)$$

where $\tau_{\mu\nu}$ is the electromagnetic energy-momentum tensor defined by:

$$\tau_{\mu\nu} = \frac{1}{4\pi} \left(\eta_{\rho\nu} F_{\mu\sigma} F^{\rho\sigma} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), \quad (2.8)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, with potential A_μ . The source is a static point charge, with no magnetic monopole. In spherical (t, r, θ, φ) coordinates, the non-vanishing contributions of the electromagnetic field strength are then

$$F_{10} = -F_{01} = E_r \quad \text{and} \quad E_r = \frac{Q}{r^2}, \quad (2.9)$$

where E_r is the radial component of the electric field, with the Coulomb constant $k_e = 1$ in natural (Gaussian) units. We will work in the case of weak gravitational field and static source, so that the metric solution can be written as:

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 - 2\Psi(r))(dr^2 + r^2 d\Omega^2), \quad (2.10)$$

where (t, r, θ, φ) are the isotropic coordinates, $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, and Φ and Ψ are the two unknown gravitational fields to be determined. By looking at the 00-component and trace of the linearized field equations in Eq. (2.7) we obtain, respectively:

$$\begin{aligned} e^{-\nabla^2/M_s^2} [\nabla^2 h_{00} - \partial_i \partial_j h^{ij} + \nabla^2 h] &= -16\pi G \tau_{00}, \\ e^{-\nabla^2/M_s^2} [-2\nabla^2 h + 2\partial_i \partial_j h^{ij}] &= -16\pi G \tau. \end{aligned} \quad (2.11)$$

From the metric-form in Eq. (2.10) we note that $h = 2(\Phi - 3\Psi)$, $h_{00} = -2\Phi$ and $h_{ii} = -2\Psi$, thus we obtain two differential equations for the two metric potential Φ and Ψ :

$$\begin{aligned} e^{-\nabla^2/M_s^2} \nabla^2 \Phi &= 4\pi G(\tau + 2\tau_{00}), \\ e^{-\nabla^2/M_s^2} \nabla^2 \Psi &= 4\pi G \tau_{00}. \end{aligned} \quad (2.12)$$

For Maxwell's theory in 4 dimensions, the energy momentum tensor is traceless $\tau = 0$ (Eq. (2.8)). The 00-component is given by $\tau_{00} = Q^2/8\pi r^4$, so that Eq. (2.12) reads

$$\begin{aligned} e^{-\nabla^2/M_s^2} \nabla^2 \Phi &= \frac{GQ^2}{r^4}, \\ e^{-\nabla^2/M_s^2} \nabla^2 \Psi &= \frac{GQ^2}{2r^4}. \end{aligned} \quad (2.13)$$

So we arrive at the non-local generalization of the would-be equations had we just considered GR instead. The non-locality makes its appearance due to the presence of infinite number of spatial derivatives in the form of an exponential. One may take the local GR limit by taking $M_s \rightarrow \infty$. The details of solving the differential equations above are laid out in Appendix 6.B. Here, we state the final result for the two metric potentials [116]

$$\begin{aligned} \Phi(r) &= -\frac{Gm}{r} \text{Erf}\left(\frac{M_s r}{2}\right) + \frac{GQ^2 M_s}{2r} \text{F}\left(\frac{M_s r}{2}\right), \\ \Psi(r) &= -\frac{Gm}{r} \text{Erf}\left(\frac{M_s r}{2}\right) + \frac{GQ^2 M_s}{4r} \text{F}\left(\frac{M_s r}{2}\right), \end{aligned} \quad (2.14)$$

where

$$\text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad (2.15)$$

is the Error function and

$$\text{F}(x) := e^{-x^2} \int_0^x e^{t^2} dt, \quad (2.16)$$

is the Dawson function. Note that $\Phi \neq \Psi$, as it also happens in GR (see Appendix 6.A). In the case $Q = 0$ we recover the linearized metric for a static, neutral point source in non-local gravity derived in Ref. [107], as expected. In the IR regime, $M_s r \gg 2$, the potentials in Eq. (2.14) reduce to those of the linearized Reissner-Nordström metric in GR (Eq. (6.14)). This can be seen from the asymptotic expansions (around $r = \infty$) of the Error and Dawson functions, whose leading terms are $\text{Erf}(x) \sim 1$ and $\text{F}(x) \sim 1/2x$, respectively. At the origin $r \rightarrow 0$, the metric potentials $\Phi(r)$ and $\Psi(r)$ assume non-singular values given by:

$$\begin{aligned} \lim_{r \rightarrow 0} \Phi(r) &\equiv -A = -\frac{GmM_s}{\sqrt{\pi}} + \frac{1}{4}GQ^2M_s^2, \\ \lim_{r \rightarrow 0} \Psi(r) &\equiv -B = -\frac{GmM_s}{\sqrt{\pi}} + \frac{1}{8}GQ^2M_s^2, \end{aligned} \quad (2.17)$$

Note that when $Q = 0$, we recover the case of neutral source, $A = B = GmM_s/\sqrt{\pi}$.

Figure 2.1: The gravitational potential Φ due to a static, charged point source in non-local gravity (blue line) from [2.14](#), and GR (orange line). We have chosen $G = 1$, $m = 1$ and $M_s = 0.5$ and $Q = 0.5$. Plot taken from [\[116\]](#)

In Fig.1, the potential Φ has been plotted as a function of $M_s r$ and a comparison between non-local gravity and GR has been made. We have chosen the values $G = 1$, $m = 1$, $M_s = 0.5$ and $Q = 0.5$. It is also worth mentioning that the force per unit mass, $F_g(r) = -\partial\Phi/\partial r$, goes to zero linearly as a function of r .

Since we are working in the linearized regime, we need to satisfy the following two inequalities: $2|\Phi| < 1$ and $2|\Psi| < 1$. First of all, note that it is sufficient to focus just on one of the two inequalities, as the two metric potentials are almost the same apart from a factor $1/2$ appearing in the gravitational contribution due to the charge; see Eqs. [\(2.14\)](#) [\(2.17\)](#). Moreover, for simplicity, we can drop all numerical factors and just focus on the essential parameters m , $G = 1/M_p^2$, M_s and Q .

Mathematically, the weak-field inequality can be also satisfied when the neutral and charged contributions to the gravitational potential are both very large but the modulus of their sum is still less than one. However, from a physical point of view, in order to satisfy the weak-field inequalities, we require that both contributions are

smaller than one² [107]:

$$mM_s < M_p^2, \quad (2.18)$$

$$Q^2 M_s^2 < M_p^2 \iff |Q| M_s < M_p. \quad (2.19)$$

Compared to the neutral source case, we now have the additional inequality in Eq. (2.19); together they will play a crucial role. In fact, we can make the following important observations.

- In the case of the Reissner-Nordström metric in Einstein's GR, one has to demand that the inequality

$$|Q| \leq \frac{m}{M_p} \quad (2.20)$$

always holds true in order to avoid the formation of a naked singularity. This is one example of the weak formulation of the cosmic censorship conjecture; see Refs. [4, 39, 190]. In the case of a charged source in non-local gravity, there is no need to require a cosmic censorship conjecture as the theory turns out to be singularity-free (in the linearized regime) as long as the inequalities in Eqs. (2.18)(2.19) are satisfied.

- In GR, one cannot describe the gravitational field generated by objects for which

$$|Q| > \frac{m}{M_p}, \quad (2.21)$$

such as for an electron. In non-local gravity, both inequalities in Eqs. (2.20)(2.21) are allowed. Indeed, consistently with the weak-field inequalities in Eqs. (2.18)(2.21), we can have two possible scenarios:

$$|Q| < \frac{m}{M_p} < \frac{M_p}{M_s} \quad (2.22)$$

or

$$\frac{m}{M_p} < |Q| < \frac{M_p}{M_s}. \quad (2.23)$$

The most interesting case is the one described by the second inequality, Eq. (2.23), which shows that an object whose charge is bigger than its mass (in Planck units) is at least permissible in non-local gravity³.

²As pointed out in Ref. [121], the inequality $mM_s < M_p^2$ always holds in order for the quadratic part of the gravitational action to dominate over the Einstein-Hilbert term in the UV regime. If the inequality was not satisfied, we would not be able to modify the Reissner-Nordström geometry and avoid the metric singularity at the origin.

³Note that for an elementary particle, like an electron, the Compton wavelength is much larger than its Schwarzschild radius, so in such a regime, quantum effects are not negligible and a classical theory of gravity alone cannot account for its gravitational effects.

- Finally, note that in the short-distance regime, $r < 2/M_s$, one can in principle also have repulsive gravity, when

$$Q^2 M_s > m,$$

and it is only compatible with the scenario in Eq. (2.23), and not with Eq. (2.22), as can be easily checked. However, such a repulsive contribution would be confined to such short distances, while gravity would be still attractive at larger distances as in standard GR.

2.2 Regular curvatures

In this section, we present the regular, linearized curvatures at the origin $r = 0$ for the linearized metric in Eqs. (2.10)(2.14). The full expressions are shown in Appendix 6.C

The non-zero components of the Riemann tensor are given by [116]

$$\lim_{r \rightarrow 0} R_{0101} = \frac{1}{12} G M_s^3 \left(\frac{2m}{\sqrt{\pi}} - Q^2 M_s \right); \quad (2.24)$$

the non-zero components of the Ricci tensor [116]

$$\begin{aligned} \lim_{r \rightarrow 0} R_{00} &= \frac{1}{4} G M_s^3 \left(\frac{2m}{\sqrt{\pi}} - Q^2 M_s \right), \\ \lim_{r \rightarrow 0} R_{11} &= \frac{1}{12} G M_s^3 \left(\frac{6m}{\sqrt{\pi}} - Q^2 M_s \right); \end{aligned} \quad (2.25)$$

and the Ricci scalar [116]

$$\lim_{r \rightarrow 0} R = \frac{G m M_s^3}{\sqrt{\pi}}. \quad (2.26)$$

Finally, we note that all components of the Weyl tensor tend to zero at the origin [116]

$$\lim_{r \rightarrow 0} C_{\mu\nu\rho\sigma} = 0, \quad (2.27)$$

In Fig. 2 we have plotted the C_{0101} component of the Weyl tensor for the spacetime generated by a charged point source, in both non-local gravity and in GR.

By setting $Q = 0$, we recover the curvatures for the spacetime generated by a neutral point source in non-local gravity obtained in Ref. [115], as expected. The Kretschmann scalar $\mathcal{K} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ tends to the following finite constant value at the origin [116]

$$\lim_{r \rightarrow 0} \mathcal{K} = \frac{G^2 M_s^6 (10m^2 - 6\sqrt{\pi} m Q^2 M_s + \pi Q^4 M_s^2)}{6\pi}. \quad (2.28)$$

In Fig. 3 we see explicitly that the (linearized) Kretschmann scalar for the spacetime generated by a charged point source in non-local gravity is regular at the origin, unlike the case of Reissner Nordström in GR where there is a curvature singularity.

Figure 2.2: Curvatures in non-local gravity (blue line) and GR (orange line) corresponding to a linearized metric generated by a charged point source. In the top graph, we plot the C_{0101} component of the Weyl tensor, and in the bottom graph the Kretschmann scalar $\mathcal{K}$. We have set $M_p = 1$, $m = 1$, $M_s = 0.5$ and $Q = 0.5$. We see that curvatures in non-local gravity are regular, at least in this linearized approximation. Plots taken from [\[116\]](#).

To conclude, we have found a linearized static metric solution sourced by an electric point charge in non-local gravity. The metric and curvatures are regular at the origin, unlike the counterpart case of linearized Reissner-Nordström in GR. Because of the absence of any singularity, at least in this linearized context, we do not need to invoke a cosmic censorship conjecture like for GR. Moreover, unlike the case of Reissner-Nordström in GR, in non-local gravity, we can have cases where $|Q| > m/M_p$ is satisfied. Perhaps at the full non-linear level as well, higher derivative terms may be able to resolve singularities.

Chapter 3

Perturbations around Non-Maximally Symmetric Backgrounds

The dynamics of any physical system is determined by disturbing a vacuum state slightly and studying how the disturbances evolve in space and time. This is called perturbation theory - an immensely useful tool used in all areas of physics. In most cases, it is impossible or impractical to analytically and fully solve for the dynamics of an arbitrary system. The quintessential way out is to consider a simple, symmetric background (or vacuum) and study small perturbations around it to approximate the system. In many scenarios, this perturbative approach explains many important physical phenomena. Of course, this approach works only when perturbations are 'small' (in a specific sense) such that the result of any physical observable can be expressed as a series expansion, with each term successively smaller than the previous one. Moreover, the properties of the background around which perturbations are studied also dictate their dynamics and complexity. In the context of gravity, it is easiest to study metric perturbations around flat space, the simplest and most symmetric background of all. Additionally, perturbations are also studied around dS/AdS, FLRW and black hole spacetimes. These backgrounds have many symmetries which reduce the complexity of equations of motion of the corresponding perturbations. However, it is also important to understand how perturbations evolve around less symmetric backgrounds to gain a deeper understanding of the dynamics of perturbations, given a theory.

Perturbations in non-local gravity have been studied around flat [106, 107], dS/AdS [111, 112], bouncing [143, 148-150] and inflationary backgrounds [144, 191-194]. The aim of this chapter is to study perturbations in UV non-local gravity at the quadratic level of the action around backgrounds that go *beyond* those usually studied in literature, viz. flat/dS/AdS, which are maximally symmetric spacetimes (MSS). In particular, we study the SVT modes of metric perturbation around a background which is conformally-flat, and has covariantly constant curvatures.

The chapter is organized as follows: we first introduce the action for non-local gravity that we use in this chapter. In sec. (3.1), we introduce the background field method, (covariant) SVT decomposition of metric perturbation and the general struc-

ture of the second order action. In sec.(3.2), we will discuss typical backgrounds studied in literature, like dS/AdS, and then choose a specific *non*-MSS background upon which we perform perturbations in later sections. This background is conformally-flat and satisfies covariantly constant curvature conditions. These conditions significantly reduce the convoluted perturbative expressions to be encountered in later sections, and also make contact with physically motivated examples, like a cosmological bounce. We also lay out the background equations of motion. In sec.(3.3), we segregate, organize and present second order perturbations at the level of the action, in SVT form, for the action we started with in the beginning of the chapter. They possess peculiar scalar-vector-tensor mixings, explicitly showing how perturbations around general backgrounds become difficult to analyze. In sec.(3.4), we will verify our general analysis by deriving dS/AdS and flat space limits which were studied in previous papers [111], [112]. In sec.(3.5), we analyze the physical spectrum for simple cases of the full action we started with. Finally, we conclude with some comments on possible directions for future work. Many relevant perturbative expressions and commutation relations are collected in Appendices (6.E), (6.F) and (6.H). They include perturbations and commutation relations around an *arbitrary* background and are therefore useful for future work in gravity, and higher derivative gravity in particular.

We used the *xAct* package for Mathematica for carrying out various computations presented in this chapter (<http://xact.es/>, José M. Martín-García, 2002-2018) [195, 196].

Relevant quantities like curvatures are split into background and perturbations as explained in Appendix (6.36). Overbars on curvatures indicate their values on a fixed background.

The most general action for gravity which is quadratic in Ricci scalar R , traceless Ricci tensor S^μ_ν and Weyl tensor $C^{\rho\sigma}_{\mu\nu}$ each, parity invariant, torsion-free, and which can be made ghost-free around maximally symmetric backgrounds is given by [107, 111, 112]¹

$$\begin{aligned} S &= \int d^4x \frac{\sqrt{-g}}{2} \left[M_P^2 (R - 2\Lambda) + R \mathcal{F}_1(\square_s) R + S^\nu_\mu \mathcal{F}_2(\square_s) S^\mu_\nu + C^{\rho\sigma}_{\mu\nu} \mathcal{F}_3(\square_s) C^{\mu\nu}_{\rho\sigma} \right] \\ &= \int d^4x \mathcal{L}_{EH+\Lambda} + \mathcal{L}_{R^2} + \mathcal{L}_{S^2} + \mathcal{L}_{C^2}, \end{aligned} \quad (3.1)$$

where we have defined $\square_s \equiv \square/M_s^2$, where $M_s (< M_P)$ is the scale of non-locality.

¹It is worthwhile to mention that the most general quadratic curvature action which contains torsion and generalizes Poincaré gravity was constructed recently in [197]. See also [198] for a 3D version of the action in eq.(3.1).

The traceless Ricci tensor in 4 dimensions is given by

$$S_{\mu\nu} = R_{\mu\nu} - \frac{R}{4}g_{\mu\nu}. \quad (3.2)$$

$\mathcal{F}_i(\Box_s)$ are analytic gravitational form factors which are functions of d'Alembertian and have a power series expansion:

$$\mathcal{F}_i(\Box_s) = \sum_{n=0}^{\infty} f_{i,n} \left(\frac{\Box}{M_s^2} \right)^n. \quad (3.3)$$

The coefficients go as $f_{i,n} \sim \mathcal{O}(M_P^2/M_s^2)$. In IR, for low enough momenta $k \ll M_s$, $\mathcal{L}_{EH+\Lambda}$ dominates over the higher curvature terms. One can easily consider special cases of the above general action by considering arbitrary form factors, like $\mathcal{F}_i(\Box_s) = 1$, or truncating $\mathcal{F}_i(\Box_s)$ to a finite order in $\Box_s$. The full, non-linear equations of motion (EoM), around an arbitrary background, has been derived in [199] (see Appendix (6.D)). In this chapter, we take the above action in eq.(3.1) and analyze second order perturbations around a specific (*non*-MSS) background.

3.1 Perturbation theory

We will use the well-known background field method (see [200],[201]) to study second order metric perturbations at the level of the action. We define the covariant scalar-vector-tensor (SVT) decomposition and provide the most general structure of the second order action ($\delta^2 S$) one can expect in non-local gravity (and a generic quadratic curvature gravity) around arbitrary backgrounds. Restricting to $3+1$ dimensions, we split the full metric $g_{\mu\nu}$ into background $\bar{g}_{\mu\nu}$ and perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu},$$

where $|h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$, and expand the action S up to $\mathcal{O}(h^2)$ to get $\delta^2 S$. We are interested in perturbations around vacuum solutions so we put $T_{\mu\nu} = 0$. Extremizing S to first order gives the background EoM, $\bar{E}_{\mu\nu} = 0$. We consider on-shell perturbations and therefore get rid of the linear in h term in $\delta^2 S$ by imposing $\bar{E}_{\mu\nu} = 0$. The background EoM are useful in simplifying the computation of $\delta^2 S$ as we will see later.

3.1.1 Covariant scalar-vector-tensor decomposition

We can covariantly decompose $h_{\mu\nu}$ into two scalar, one vector and one tensor (SVT) degrees of freedom [202-205]:

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \bar{\nabla}_\mu A_\nu + \bar{\nabla}_\nu A_\mu + \left(\bar{\nabla}_\mu \bar{\nabla}_\nu - \frac{1}{4} \bar{g}_{\mu\nu} \bar{\Box} \right) B + \frac{1}{4} \bar{g}_{\mu\nu} h, \quad (3.4)$$

where

$$\bar{\nabla}^\mu \hat{h}_{\mu\nu} = 0, \quad \bar{g}^{\mu\nu} \hat{h}_{\mu\nu} = 0, \quad \bar{\nabla}^\mu A_\mu = 0. \quad (3.5)$$

The first two conditions make $\hat{h}_{\mu\nu}$ transverse and traceless, respectively. The last condition ensures transversality of A_μ , while h is the trace $\bar{g}^{\mu\nu} h_{\mu\nu}$. For both GR and non-local gravity, the above decomposition produces kinetic mixings between B and h modes in $\delta^2 S$ when the background is flat or dS/AdS. This mixing can be removed by the field redefinition $\phi \equiv \bar{\square} B - h$, so that the graviton quadratic form has one tensor $\hat{h}_{\mu\nu}$ and one scalar ϕ mode, as expected [111, 112]. Following suit, we rewrite eq.(3.4) as

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \bar{\nabla}_\mu A_\nu + \bar{\nabla}_\nu A_\mu + \bar{\nabla}_\mu \bar{\nabla}_\nu B - \frac{1}{4} \bar{g}_{\mu\nu} \phi, \quad (3.6)$$

which is the decomposition we will use in this chapter. The theory has gauge redundancy expressed by the following gauge transformation:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \bar{\nabla}_\nu \xi_\mu + \bar{\nabla}_\mu \xi_\nu, \quad (3.7)$$

where the gauge transformation vector ξ_μ can be decomposed into its transverse and longitudinal parts as [206]:

$$\xi_\mu = \hat{\xi}_\mu + \partial_\mu \xi, \quad \text{where } \bar{\nabla}^\mu \hat{\xi}_\mu = 0, \quad (3.8)$$

so that the SVT modes transform as following under the gauge transformations:

$$\hat{h}_{\mu\nu} \rightarrow \hat{h}_{\mu\nu}, \quad \phi \rightarrow \phi, \quad A_\mu \rightarrow A_\mu + \hat{\xi}_\mu, \quad B \rightarrow B + 2\xi, \quad (3.9)$$

from which we see that $\hat{h}_{\mu\nu}$ and ϕ are gauge invariant while A_μ and B are not. Note that we have not yet canonically normalized $h_{\mu\nu}$, so $[h_{\mu\nu}] = 0$, $[\phi] = 0$, $[B] = -2$, $[A_\mu] = -1$ and $[\hat{h}_{\mu\nu}] = 0$. Note that in this chapter, we will use all the degrees of freedom $\hat{h}_{\mu\nu}$, A_μ , B and ϕ till our final computations without fixing the gauge.

3.1.2 General structure of $\delta^2 S$

Studying the propagating modes of the theory given by eq.(3.1) around general backgrounds is quite involved. In order to find $\delta^2 S$ for any theory, we need to compute perturbations of many quantities. We provide perturbations of curvatures around *arbitrary* backgrounds in Appendix 6.E. Decomposition of these perturbations in SVT form are given in Appendix 6.F. The second order action around

arbitrary backgrounds has the following general form [159]²:

$$\delta^2 S = \int d^4 x \sqrt{-\bar{g}} \begin{bmatrix} B & \phi & A_\rho & \hat{h}_{\mu\nu} \end{bmatrix} \begin{bmatrix} \mathcal{K}_{00} & \mathcal{K}_{01} & \mathcal{K}_{02} & \mathcal{K}_{03} \\ \mathcal{K}_{10} & \mathcal{K}_{11} & \mathcal{K}_{12} & \mathcal{K}_{13} \\ \mathcal{K}_{20} & \mathcal{K}_{21} & \mathcal{K}_{22} & \mathcal{K}_{23} \\ \mathcal{K}_{30} & \mathcal{K}_{31} & \mathcal{K}_{32} & \mathcal{K}_{33} \end{bmatrix} \begin{bmatrix} B \\ \phi \\ A_\sigma \\ \hat{h}_{\alpha\beta} \end{bmatrix}, \quad (3.10)$$

where $\mathcal{K}$ is a complicated, generalized kinetic matrix given by the sum of kinetic and effective mass matrices of all SVT modes. Its entries can be read off from the general structure of perturbations given in Appendix (6.F). From there, one can see that $\mathcal{K}$ is not diagonal in general and involves mixings between the different SVT modes. For GR and non-local gravity around dS/AdS/flat backgrounds, $\mathcal{K}$ is diagonal in ϕ and $\hat{h}_{\mu\nu}$ [111]:

$$\begin{bmatrix} B & \phi & A_\rho & \hat{h}_{\mu\nu} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{K}_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{K}_{33} \end{bmatrix} \begin{bmatrix} B \\ \phi \\ A_\sigma \\ \hat{h}_{\alpha\beta} \end{bmatrix}. \quad (3.11)$$

Next, we discuss backgrounds typically studied in perturbative analyses, and then choose a specific *non*-MSS background ($\bar{g}_{\mu\nu}$) for the purpose of this chapter.

3.2 Background spacetimes

We recall here aspects of pedagogical backgrounds one typically encounters when studying gravitational perturbations, and choose a specific background for the purpose of this chapter. The lower the symmetry of the background, the harder it is to find $\delta^2 S$, or equivalently, compute (equations of motion of) perturbations expressible in a form that is tractable to solving them, either analytically or numerically. It is useful to decompose all curvatures into their irreducible trace and traceless parts. Doing this for the Riemann tensor gives

$$\bar{R}_{\mu\nu\rho\sigma} = \underbrace{\frac{\bar{R}}{12} (\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})}_{\text{dS/AdS}} + \underbrace{\frac{1}{2} (\bar{g}_{\nu\sigma}\bar{S}_{\mu\rho} + \bar{g}_{\mu\rho}\bar{S}_{\nu\sigma} - \bar{g}_{\nu\rho}\bar{S}_{\mu\sigma} - \bar{g}_{\mu\sigma}\bar{S}_{\nu\rho})}_{\text{deviation from dS/AdS}} + \bar{C}_{\mu\nu\rho\sigma}, \quad (3.12)$$

which manifestly separates the Riemann tensor for a maximally symmetric spacetime (MSS) and deviations from it. Two commonly studied spacetimes are conformally-flat (e.g. FLRW) and Einstein ($\bar{R}_{\mu\nu} \propto \bar{g}_{\mu\nu}$) manifolds, defined by $\bar{C}_{\mu\nu\rho\sigma} = 0$ and $\bar{S}_{\mu\nu} = 0$, respectively (see FIG. 3.1).

²We have suppressed Lorentz indices in each matrix element of $\mathcal{K}$ which are needed to make the action a scalar.

Given $\bar{S}_{\mu\nu} = \bar{C}_{\mu\nu\rho\sigma} = 0$, we get $\partial_\mu \bar{R} = 0$ from the (differential) Bianchi identity, implying that the manifold is MSS with the Riemann tensor assuming the dS/AdS form highlighted in eq.(3.12). This means that the intersection of conformally-flat and Einstein manifolds is MSS as shown in FIG. 3.1. From FIG. 3.1 and eq.(3.12), it is clear that deviations from MSS can arise when $\bar{S}_{\mu\nu} \neq 0$ and/or $\bar{C}_{\mu\nu\rho\sigma} \neq 0$. In this chapter, we will focus on conformally-flat, *non*-MSS backgrounds. To be clear, the background we consider in this chapter has the following properties [159]

$$\bar{C}_{\mu\nu\rho\sigma} = 0, \quad \bar{S}_{\mu\nu} \neq 0, \quad \bar{R} \neq 0, \quad (3.13a)$$

$$\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0, \quad \partial_\mu \bar{R} = 0, \quad (3.13b)$$

where the ‘covariantly constant curvature’ conditions³ in eq.(3.13b) greatly simplify SVT perturbations of the action to be discussed in sec.(3.3). Actually, $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0$ implies $\partial_\mu \bar{R} = 0$ via the contracted Bianchi identity in eq.(3.27), so effectively we have only one condition, but we keep both of them explicit. Note that eq.(3.13b) does not imply that the background is MSS; a simple example is the Schwarzschild metric. It is easy to take the dS/AdS limit of $\delta^2 S$ around this more general background in eq.(3.13) by putting $\bar{S}_{\mu\nu} = 0$ (and $\bar{R} = 0$ for flat space limit) as we will see in sec.(3.4). The background in eq.(3.13) is not as restrictive as MSS, and we will see from the analysis of perturbations later in sec.3.5 that this is just enough to draw interesting qualitative and quantitative conclusions about the nature of gravity in less symmetric, non-trivial backgrounds, particularly scalar-vector-tensor mixing essentially because $\bar{S}_{\mu\nu} \neq 0$. In Table 3.1 we collect properties of curvatures of typical spacetimes and the background we consider in this chapter. FIG. 3.1 shows a Venn diagram of different backgrounds with our chosen background in the shaded region.

3.2.1 Background equations of motion

The full, non-linear EoM for the action in eq.(3.1) is given in Appendix (6.26). On these, we impose our background conditions $\bar{C}_{\mu\nu\rho\sigma} = 0$, $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0$ and $\partial_\mu \bar{R} = 0$ from eq.(3.13) to obtain [159]

$$\Omega \bar{S}^\mu{}_\nu = f_{2,0} \left(\frac{1}{2} \delta^\mu{}_\nu \bar{S}^{\alpha\beta} \bar{S}_{\alpha\beta} - \frac{1}{2} \bar{R} \bar{S}^\mu{}_\nu - 2 \bar{S}^{\mu\alpha} \bar{S}_{\nu\alpha} \right), \quad (3.14)$$

³Given eq.(3.13b) and $\bar{C}_{\mu\nu\rho\sigma} = 0$ from eq.(3.13a), we can succinctly write eq.(3.13b) as $\bar{\nabla}_\alpha \bar{R}_{\mu\nu\rho\sigma} = 0$. This covariantly constant curvature condition has appeared elsewhere in literature, for example, to simplify calculations of asymptotic heat kernel expansions when developing a covariant technique of computing the one-loop effective action in the presence of arbitrary background fields in curved space [207].

	$\bar{R}$	$\bar{S}_{\mu\nu}$	$\bar{R}_{\mu\nu}$	$\bar{C}_{\mu\nu\rho\sigma}$	$\bar{R}_{\mu\nu\rho\sigma}$	$\partial_\mu \bar{R}$	$\bar{\nabla}_\alpha \bar{S}_{\mu\nu}$
Maximally symmetric spacetime (MSS)	constant	0	$\bar{R}\bar{g}_{\mu\nu}/4$	0	$\bar{R}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})/12$	0	0
Einstein (Ricci flat)	$\neq 0$ (0)	0	$\bar{R}\bar{g}_{\mu\nu}/4$ (0)	$\neq 0$	$\neq \bar{R}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})/12$	$\neq 0$ (0)	0
Conformally-flat (e.g. FLRW)	$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq \bar{R}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})/12$	$\neq 0$	$\neq 0$
This chapter (eq. (3.13))	constant	$\neq 0$	$\neq 0$	0	$\neq \bar{R}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})/12$	0	0

Table 3.1: Properties of typical background spacetimes. ' $\neq 0$ ' in particular implies an arbitrary function of x^μ . Table taken from [159]

Figure 3.1: Venn diagram with our specific conformally-flat, *non*-maximally symmetric background. $\partial_\mu \bar{R} = 0$ follows from $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0$. dS/AdS limit is achieved by putting $\bar{S}_{\mu\nu} = 0$. Diagram taken from [159].

where we have defined

$$\Omega \equiv M_P^2 + 2f_{1,0}\bar{R}, \quad (3.15)$$

and used

$$\bar{R} = 4\Lambda, \quad (3.16)$$

which comes from the trace of the equations of motion given in eq. (6.33). We have a 3 parameter family of solutions characterized by $(\Lambda, f_{1,0}, f_{2,0})$. Of course, one

trivial solution is $\bar{S}_{\mu\nu} = 0$, but this is just MSS. For $\bar{S}_{\mu\nu} \neq 0$ for our background (see eq. (3.13a)), we consider non-trivial solutions of eq. (3.14), but the exact form of these solutions is irrelevant for our purposes. The background EoM in eq. (3.14) will be useful in simplifying $\delta^2 S$ as we will see later. Next, we consider an example where our background conditions in eq. (3.13) may be realized physically.

3.2.2 Physical example: Near a cosmological bounce

In bouncing cosmology, we usually deal with a flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric given by

$$ds^2 = -dt^2 + a^2 (dr^2 + r^2 d\Omega^2), \quad K = 12 \left(H^2 + \dot{H} \right)^2 + 12 \left(\frac{k}{a^2} + H^2 \right)^2, \quad (3.17)$$

where an overdot indicates differentiation with respect to cosmic time t and $H = (\dot{a}/a)$ is the Hubble factor. K is the Kretschmann scalar which can be decomposed as

$$K = \bar{R}_{\mu\nu\rho\sigma} \bar{R}^{\mu\nu\rho\sigma} = \bar{C}_{\mu\nu\rho\sigma} \bar{C}^{\mu\nu\rho\sigma} + 2\bar{S}_{\mu\nu} \bar{S}^{\mu\nu} + \frac{\bar{R}^2}{6}. \quad (3.18)$$

Since FLRW metric is conformally-flat, the Weyl tensor in eq. (3.18) vanishes. So if K is singular as $t \rightarrow 0$, it implies that $\bar{S}_{\mu\nu} \bar{S}^{\mu\nu}$ and/or $\bar{R}^2$ blow up. This is the case in Big Bang singularity where $a \sim t^\alpha$, $H \sim \alpha/t$, for $\alpha > 0$. Now, let us consider a bouncing scenario. General physical conditions at any symmetric bounce are [148]

$$H = 0, \quad \dot{H} \neq 0, \quad \ddot{H} = 0, \quad \ddot{H} \neq 0, \quad \ddot{H} = 0, \dots \implies H^{(2n)} = 0. \quad (3.19)$$

If we consider a simple bouncing scenario, which happens very slowly, we then have

$$H^{(2n+1)} \approx 0, \quad n \geq 1. \quad (3.20)$$

To realize a slow-bouncing scenario, we can Taylor-expand the Hubble factor in time as

$$H = M_s \sum_{n=0}^{\infty} c_n (M_s t)^{2n}, \quad (3.21)$$

where M_s characterizes the time scale related to bounce. The Ricci scalar and tensor for flat FLRW are

$$\bar{R} = 12H^2 + 6\dot{H}, \quad \bar{R}_0^0 = 3H^2 + 3\dot{H}, \quad \bar{R}_i^i = 3H^2 + \dot{H}. \quad (3.22)$$

At the point of slow bounce, we have $H = 0$ and $\dot{H} \approx \text{constant}$. In the limit $t \ll 1/M_s$, we have

$$\bar{R} = \bar{R}_0 M_s^2 + \bar{R}_1 M_s^2 t^2 + \dots \implies \partial_\mu \bar{R} \approx 0, \quad \square \bar{R} \approx 0 \quad \text{if} \quad \bar{R}_1 \ll \bar{R}_0. \quad (3.23)$$

The non-zero components of traceless Ricci tensor, in general, take the form

$$\bar{S}^\mu_\nu = \frac{1}{2} \dot{H} \text{diag} (3, -1, -1, -1) , \quad (3.24)$$

during the bounce. Using the approximation of a slow bouncing scenario stated in eq.(3.20) which is nearly constant, we obtain

$$\bar{\nabla}_\alpha \bar{S}^\mu_\nu \approx 0, \quad \bar{\square} \bar{S}^\mu_\nu \approx 0. \quad (3.25)$$

We see that we can satisfy our background conditions in eq.(3.13) in a bouncing scenario that occurs sufficiently slow, or in the limit $t \rightarrow 0$. Therefore, studying perturbations around backgrounds satisfying the conditions in eq.(3.13) enables us to address the stability of a bouncing universe.

3.3 Second order variation of the action

In this section, we will organize and present the second order perturbations for every term in the action given in eq.(3.1) around the background in eq.(3.13) and express them in SVT form (see eq.(3.6)):

$$\delta^2 S = \int d^4x \delta^2 \mathcal{L} = \int d^4x \delta^2 \mathcal{L}_{EH+\Lambda} + \delta^2 \mathcal{L}_{R^2} + \delta^2 \mathcal{L}_{S^2} + \delta^2 \mathcal{L}_{C^2}. \quad (3.26)$$

After obtaining perturbations in terms of $h_{\mu\nu}$ and background curvatures, a significant series of steps must be carried out in order to simplify them and express the final expressions in SVT form. We present only the final, most simplified expressions in this section, after applying our background curvature conditions in eq.(3.13) and Bianchi identities. The step-by-step procedure we have followed is outlined below:

- (i) Decompose all background curvatures into their irreducible trace and traceless parts so that all expressions are specified in terms of Ricci scalar ($\bar{R}$), traceless Ricci tensor ($\bar{S}_{\mu\nu}$) and Weyl tensor ($\bar{C}_{\mu\nu\rho\sigma}$).
- (ii) Commute covariant derivatives as much as possible so that they assume a canonical form, and end up acting on the divergenceless $\hat{h}_{\mu\nu}$ and A_μ , giving zero (see eq.(3.5)).
- (iii) Set $\bar{C}_{\mu\nu\rho\sigma} = 0$, since the background is conformally-flat (see eq.(3.13a)).
- (iv) The n th order perturbation of an arbitrary tensor shares the same index symmetry and tracelessness properties as the original tensor. For example, the first order perturbation $c^{\mu\nu}_{\rho\sigma}$ inherits the properties of the full Weyl tensor $C^{\mu\nu}_{\rho\sigma}$, in particular, it is completely traceless.

(v) Apply the contracted differential Bianchi identities:

$$\bar{\nabla}^\mu \bar{R}_{\mu\nu} = \frac{1}{2} \partial_\nu \bar{R}, \quad \bar{\nabla}^\mu \bar{S}_{\mu\nu} = \frac{1}{4} \partial_\nu \bar{R}. \quad (3.27)$$

(vi) Use the covariantly constant background curvature conditions in eq. (3.13b): $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0$, $\partial_\mu \bar{R} = 0$.

The detailed derivation of perturbations can be found in Appendix (6.F) where the above steps have been followed. In particular, we have listed perturbations first around an *arbitrary* background, then applied conformal-flatness ($\bar{C}_{\mu\nu\rho\sigma} = 0$) and finally applied the covariantly constant curvature conditions in eq. (3.13b). The general perturbations can be useful for future work when considering general or conformally-flat backgrounds (like FLRW) which do *not* satisfy eq. (3.13b).

Perturbations at the quadratic level of the action around arbitrary, non-MSS backgrounds are quite convoluted (as can be seen from explicit expressions in Appendix (6.F)) and although they tell us a great deal about the structure of possible higher order terms, some simplifying assumptions must be imposed if we are to make any headway in calculations. Our background conditions in eq. (3.13) turn out to be very helpful in simplifying calculations to a large extent. $\delta^2 S$ is not too hard to find when the background is MSS (see [111]). The reason is the appearance in perturbative expressions of many terms proportional to $\partial_\mu \bar{R}$, $\bar{S}_{\mu\nu}$ and $\bar{C}_{\mu\nu\rho\sigma}$, which are zero for MSS. In this chapter, we analyze perturbations beyond MSS because we choose to keep $\bar{S}_{\mu\nu} \neq 0$. Note that all perturbations have a common factor of $\sqrt{-g}$, which we have omitted hereon for brevity.

3.3.1 Perturbing $R - 2\Lambda$

The Einstein-Hilbert term with cosmological constant Λ in our Lagrangian in eq. (3.1) is:

$$\mathcal{L}_{EH+\Lambda} = \frac{M_P^2}{2} \sqrt{-g} (R - 2\Lambda). \quad (3.28)$$

Defining $\delta_0 \equiv \delta^2 [\sqrt{-g}(R - 2\Lambda)]$, we get (see [111]):

$$\begin{aligned} \delta^2 \mathcal{L}_{EH+\Lambda} &= \frac{M_P^2}{2} \delta_0, \\ &= \frac{M_P^2}{2} \left[\delta^2 (\sqrt{-g}) \frac{\bar{R}}{2} + \delta^2 (R) \sqrt{-g} + \delta(\sqrt{-g}) r \right]. \end{aligned} \quad (3.29)$$

where we have used $\Lambda = \bar{R}/4$ which satisfies the background trace equations of motion (see eq. (3.16)). We then first express eq. (3.29) in terms of $h_{\mu\nu}$ and then decompose the result in SVT form using eq. (3.6). δ_0 contains terms purely quadratic in

ϕ , B , A_μ and $\hat{h}_{\mu\nu}$, and mixed terms between them. Detailed calculation from scratch is given in Appendix (6.F.1). Here we present the final expressions after following the simplifying steps given before in the beginning of sec. (3.3). The purely quadratic in scalar, vector and tensor modes each, are [159]:

$$\delta_{0(BB)} = B \left[\frac{1}{8} \bar{R} \bar{S}_{\alpha\beta} \bar{\nabla}^\beta \bar{\nabla}^\alpha + \frac{1}{2} \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} \bar{\nabla}^\beta \bar{\nabla}^\alpha + \frac{1}{4} \bar{S}_{\beta\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\square} \right] B, \quad (3.30)$$

$$\delta_{0(AA)} = A^\alpha \left[\frac{1}{12} \bar{R} \bar{S}_{\alpha\beta} + \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} - \frac{1}{2} \bar{g}_{\alpha\beta} \bar{S}_{\mu\nu} \bar{S}^{\mu\nu} - \bar{S}_{\nu\mu} \bar{\nabla}^\mu \bar{\nabla}^\nu \bar{g}_{\alpha\beta} \right] A^\beta,$$

$$\delta_{0(\hat{h}\hat{h})} = \hat{h}_{\mu\nu} \left[\frac{-\bar{R} + 6\bar{\square}}{24} \right] \hat{h}^{\mu\nu}, \quad \delta_{0(\phi\phi)} = \phi \left[\frac{-\bar{R} - 3\bar{\square}}{32} \right] \phi, \quad (3.31)$$

while the 5 non-zero mix terms are [159]:

$$\delta_{0(B\phi)} = B \left[-\frac{1}{4} \bar{S}_{\alpha\beta} \bar{\nabla}^\beta \bar{\nabla}^\alpha \right] \phi, \quad \delta_{0(BA)} = B \left[\frac{1}{2} \bar{R} \bar{S}_{\alpha\beta} \bar{\nabla}^\beta + 2 \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} \bar{\nabla}^\beta \right] A^\alpha, \quad (3.32)$$

$$\delta_{0(B\hat{h})} = B \left[\frac{1}{3} \bar{R} \bar{S}^{\alpha\beta} + 2 \bar{S}^{\alpha\gamma} \bar{S}_\gamma{}^\beta - \frac{1}{2} \bar{S}^{\alpha\beta} \bar{\square} \right] \hat{h}_{\alpha\beta}, \quad \delta_{0(\phi A)} = \phi \left[-\frac{1}{2} \bar{S}_{\alpha\beta} \bar{\nabla}^\beta \right] A^\alpha, \\ \delta_{0(A\hat{h})} = A^\alpha \left[-2 \bar{S}^{\beta\gamma} \bar{\nabla}_\gamma \right] \hat{h}_{\alpha\beta}. \quad (3.33)$$

and the $\phi\hat{h}$ scalar-tensor mixing is zero. Note from eq. (3.32) - eq. (3.33) that the mode mixing terms are proportional to $\bar{S}_{\mu\nu}$, which vanish around MSS and flat backgrounds, as we will see later in sec. (3.4). But these mixings are non-zero for our background (see background EoM in eq. (3.14)).

3.3.2 Perturbing non-local R^2

The Ricci scalar squared term in our Lagrangian in eq. (3.1) is

$$\mathcal{L}_{R^2} = \frac{1}{2} \sqrt{-g} R \mathcal{F}_1(\square_s) R. \quad (3.34)$$

Let us lay out all the 10 possible $\mathcal{O}(h^2)$ perturbations in $\delta^2(\sqrt{-g} R \mathcal{F}_1(\square_s) R)$. We label⁴ each term in $\sqrt{-g} R \mathcal{F}_1(\square_s) R$ by a number $n = 0, 1$ or 2 to indicate a contribu-

⁴For example, (1100) denotes the expression which has $\mathcal{O}(h^1)$ perturbation of $\sqrt{-g}$, $\mathcal{O}(h^1)$ of R , $\mathcal{O}(h^0)$ of $\mathcal{F}_1(\square_s)$ and $\mathcal{O}(h^0)$ of R , all adding up to give $\mathcal{O}(h^2)$.

tion⁵ of $\mathcal{O}(h^n)$ [159]:

$$\begin{aligned}
(1100) &= \sqrt{-\bar{g}} \frac{h}{2} r \bar{\mathcal{F}}_1(\bar{\square}_s) \bar{R}, \\
(1010) &= \sqrt{-\bar{g}} \frac{h}{2} \bar{R} \delta(\mathcal{F}_1(\square_s)) \bar{R}, \\
(1001) &= \sqrt{-\bar{g}} \frac{h}{2} \bar{R} \bar{\mathcal{F}}_1(\bar{\square}_s) r, \\
(0110) &= \sqrt{-\bar{g}} r \delta(\mathcal{F}_1(\square_s)) \bar{R}, \\
(0101) &= \sqrt{-\bar{g}} r \bar{\mathcal{F}}_1(\bar{\square}_s) r, \\
(0011) &= \sqrt{-\bar{g}} \bar{R} \delta(\mathcal{F}_1(\square_s)) r, \\
(2000) &= \sqrt{-\bar{g}} \left(-\frac{1}{4} h_{\mu\nu} h^{\mu\nu} + \frac{h^2}{8} \right) \bar{R} \bar{\mathcal{F}}_1(\bar{\square}_s) \bar{R}, \\
(0200) &= \sqrt{-\bar{g}} \delta^2(R) \bar{\mathcal{F}}_1(\bar{\square}_s) \bar{R}, \\
(0020) &= \sqrt{-\bar{g}} \bar{R} \delta^2(\mathcal{F}_1(\square_s)) \bar{R}, \\
(0002) &= \sqrt{-\bar{g}} \bar{R} \bar{\mathcal{F}}_1(\bar{\square}_s) \delta^2(R).
\end{aligned} \tag{3.35}$$

In total, there are 9 independent terms because (0200) is equal to (0002) after integration by parts. Substituting the SVT decomposition eq.(3.6) in eq.(3.35) gives purely quadratic terms in ϕ, B, A and $\hat{h}$ ⁶ each, and additionally, mixed terms between them. There are 6 possible mix terms: $\phi B, \phi A, \phi \hat{h}, BA, B\hat{h}$ and $A\hat{h}$, just as for $\delta^2 \mathcal{L}_{EH+\Lambda}$ in sec.(3.3.1).

In $\delta^2 \mathcal{L}_{R^2}$, we collect some terms that actually turn out to be equal to $\delta^2 \mathcal{L}_{EH+\Lambda}$, as shown for maximally symmetric backgrounds in [111]. Consider the part $\delta^2 (\sqrt{-g} R) \bar{\mathcal{F}}_1(\bar{\square}_s) \bar{R}$ in $\delta^2 \mathcal{L}_{R^2}$. We see that it can be expanded as:

$$\begin{aligned}
\delta^2 (\sqrt{-g} R) \bar{\mathcal{F}}_1(\bar{\square}_s) \bar{R} &= \delta^2 (\sqrt{-g} R) f_{1,0} \bar{R} = \left[\delta^2 (\sqrt{-g}) \bar{R} + 2\sqrt{-g} \delta^2(R) \right. \\
&\quad \left. + \delta (\sqrt{-g}) r \right] f_{1,0} \bar{R},
\end{aligned} \tag{3.36}$$

where in the first equality, the only non-zero term from $\bar{\mathcal{F}}_1(\bar{\square}_s) \bar{R}$ was picked, the rest being zero from eq.(3.13b). In the second equality, the middle term has a coefficient of 2 because it occurs twice in $\delta^2 \mathcal{L}_{R^2}$, as we just noted above: (0200) = (0002). From eq.(3.36) and eq.(3.29), we notice that

$$\delta^2 (\sqrt{-g} R) \bar{\mathcal{F}}_1(\bar{\square}_s) \bar{R} = 2\delta_0 f_{1,0} \bar{R} - [\delta (\sqrt{-g}) r] f_{1,0} \bar{R}. \tag{3.37}$$

⁵ We follow the following notation: $\bar{R} = \bar{R} + r + \delta^2 R$, $S_{\mu\nu} = \bar{S}_{\mu\nu} + s_{\mu\nu} + \delta^2 S_{\mu\nu}$, and $C_{\mu\nu\rho\sigma} = \bar{C}_{\mu\nu\rho\sigma} + c_{\mu\nu\rho\sigma} + \delta^2 C_{\mu\nu\rho\sigma}$. For details, see Appendix (6.E).

⁶Indices on A_μ and $\hat{h}_{\mu\nu}$ suppressed for brevity.

We recall here for convenience $\delta^2 \mathcal{L}_{R^2}$ from [111]:

$$\begin{aligned} \delta^2 \mathcal{L}_{R^2} = & \frac{1}{2} \sqrt{-\bar{g}} \left[2 \left(\frac{h}{2} r + \frac{1}{2} \left(\frac{h^2}{8} - \frac{h_{\mu\nu} h^{\mu\nu}}{4} \right) \bar{R} + \delta^2(R) \right) f_{1,0} \bar{R} + r \mathcal{F}_1(\bar{\square}_s) r \right. \\ & \left. + \left(\frac{h}{2} \bar{R} + r \right) \delta(\mathcal{F}_1(\bar{\square}_s)) \bar{R} + \bar{R} \delta^2(\mathcal{F}_1(\bar{\square}_s)) \bar{R} + \frac{h}{2} \bar{R} (\mathcal{F}_1(\bar{\square}_s) - f_{1,0}) r + \bar{R} \delta(\mathcal{F}_1(\bar{\square}_s)) r \right]. \end{aligned} \quad (3.38)$$

Comparing eq.(3.37) and eq.(3.38), the second term in eq.(3.37) forms a part of the second last term in the second line of eq.(3.38). In the second line of eq.(3.38), the first 2 terms vanish upon using $\partial_\mu \bar{R} = 0$ from eq.(3.13b) and Appendix (6.49); the remaining 2 terms are each non-zero, but sum to zero after using $\partial_\mu \bar{R} = 0$ and integrating by parts, exactly like the MSS case as shown in [111]. Ultimately, this implies [159]

$$\delta^2 \mathcal{L}_{R^2} = \sqrt{-\bar{g}} \left[f_{1,0} \bar{R} \delta_0 + \frac{1}{2} r \mathcal{F}_1(\bar{\square}_s) r \right]. \quad (3.39)$$

In deriving this result, we just used $\bar{R} = \text{constant}$, which is true for both MSS and our background in eq.(3.13). Therefore, for $\delta^2 \mathcal{L}_{R^2}$, the only non-trivial term we have to find is $(0101) = \sqrt{-\bar{g}} r \mathcal{F}_1(\bar{\square}_s) r$. The detailed calculation from scratch is given in Appendix (6.F.2). Finally, we have terms which are purely quadratic in scalar and tensor modes [159]

$$[r \mathcal{F}_1(\bar{\square}_s) r]_{\phi\phi} = \phi \left[(\bar{R} + 3\bar{\square})^2 \frac{\mathcal{F}_1(\bar{\square}_s)}{16} \right] \phi, \quad [r \mathcal{F}_1(\bar{\square}_s) r]_{\hat{h}\hat{h}} = \hat{h}^{\alpha\beta} [\bar{S}_{\alpha\beta} \mathcal{F}_1(\bar{\square}_s) \bar{S}_{\mu\nu}] \hat{h}^{\mu\nu}, \quad (3.40)$$

and a scalar-tensor mix term [159]

$$[r \mathcal{F}_1(\bar{\square}_s) r]_{\phi\hat{h}} = \phi \left[-(\bar{R} + 3\bar{\square}) \bar{S}_{\alpha\beta} \frac{\mathcal{F}_1(\bar{\square}_s)}{2} \right] \hat{h}^{\alpha\beta} \quad (3.41)$$

with rest of the SVT perturbations zero for our chosen background in eq.(3.13). Note that the scalar-tensor $(\phi\hat{h})$ mixing in eq.(3.41) is proportional to $\bar{S}_{\mu\nu}$ and so vanishes around flat and MSS backgrounds as we see later in sec.(3.4). But it is non-zero for our background.

3.3.3 Perturbing non-local $S_{\mu\nu} S^{\mu\nu}$

The traceless Ricci tensor squared term in our Lagrangian in eq.(3.1) is

$$\mathcal{L}_{S^2} = \frac{1}{2} \sqrt{-g} S^\nu_\mu \mathcal{F}_2(\square_s) S^\mu_\nu. \quad (3.42)$$

$\delta^2 (\sqrt{-g} S^\nu_\mu \mathcal{F}_2(\square_s) S^\mu_\nu)$ has the following contributions after imposing the background conditions, eq.(3.13), and integration by parts [159]:

$$\begin{aligned}
(1100) &= \sqrt{-g} \frac{h}{2} s^\nu_\mu \mathcal{F}_2(\square_s) \bar{S}^\mu_\nu = \sqrt{-g} \frac{h}{2} f_{2,0} s^\nu_\mu \bar{S}^\mu_\nu, \\
(1010) &= \sqrt{-g} \frac{h}{2} \bar{S}^\nu_\mu \delta(\mathcal{F}_2(\square_s)) \bar{S}^\mu_\nu, \\
(1001) &= \sqrt{-g} \frac{h}{2} \bar{S}^\nu_\mu \mathcal{F}_2(\square_s) s^\mu_\nu = \sqrt{-g} \frac{1}{2} \bar{S}^\nu_\mu s^\mu_\nu \mathcal{F}_2(\square_s) h, \\
(0110) &= \sqrt{-g} s^\nu_\mu \delta(\mathcal{F}_2(\square_s)) \bar{S}^\mu_\nu, \\
(0101) &= \sqrt{-g} s^\nu_\mu \mathcal{F}_2(\square_s) s^\mu_\nu, \\
(0011) &= \sqrt{-g} \bar{S}^\nu_\mu \delta(\mathcal{F}_2(\square_s)) s^\mu_\nu, \\
(2000) &= \sqrt{-g} \left(-\frac{1}{4} h_{\alpha\beta} h^{\alpha\beta} + \frac{h^2}{8} \right) \bar{S}^\nu_\mu \mathcal{F}_2(\square_s) \bar{S}^\mu_\nu = \sqrt{-g} f_{2,0} \left(-\frac{1}{4} h_{\alpha\beta} h^{\alpha\beta} + \frac{h^2}{8} \right) \bar{S}^\nu_\mu \bar{S}^\mu_\nu, \\
(0200) &= \sqrt{-g} \delta^2(S^\nu_\mu) \mathcal{F}_2(\square_s) \bar{S}^\mu_\nu = \sqrt{-g} f_{2,0} \bar{S}^\mu_\nu \delta^2(S^\nu_\mu), \\
(0020) &= \sqrt{-g} \bar{S}^\nu_\mu \delta^2(\mathcal{F}_2(\square_s)) \bar{S}^\mu_\nu, \\
(0002) &= \sqrt{-g} \bar{S}^\nu_\mu \mathcal{F}_2(\square_s) \delta^2(S^\mu_\nu) = \sqrt{-g} f_{2,0} \bar{S}^\nu_\mu \delta^2(S^\mu_\nu),
\end{aligned} \tag{3.43}$$

giving 9 independent terms because $(0200) = (0002)$. For MSS, because $\bar{S}_{\mu\nu} = 0$, the only non-zero contribution in $\delta^2 \mathcal{L}_{S^2}$ is $(0101) = \sqrt{-g} s^\nu_\mu \mathcal{F}_2(\square_s) s^\mu_\nu$. Expressing the above second order perturbation terms (eq.(3.43)) in SVT decomposition is not straightforward. Although, most terms can be computed without much effort, the four terms (1010) , (0110) , (0011) and (0020) are non-trivial because they involve either $\delta(\mathcal{F}_2(\square_s))$ or $\delta^2(\mathcal{F}_2(\square_s))$ acting on a tensor of rank 2, which have long expressions. We now simplify these terms as much as possible [159]:

1. $(1010) = \sqrt{-g} \frac{h}{2} \bar{S}^\nu_\mu \delta(\mathcal{F}_2(\square_s)) \bar{S}^\mu_\nu$. Recall that $\mathcal{F}_2(\square_s)$ is a power series (see eq.(5.2)):

$$\begin{aligned}
\mathcal{F}_2(\square_s) &= \sum_{n=0}^{\infty} f_{2,n} \square_s^n \\
&= f_{2,0} + f_{2,1} \square_s + f_{2,2} \square_s^2 + f_{2,3} \square_s^3 + \dots,
\end{aligned}$$

This implies:

$$\begin{aligned}
\delta \mathcal{F}_2(\square_s) &= f_{2,1} \delta(\square_s) + f_{2,2} \delta(\square_s^2) + f_{2,3} \delta(\square_s^3) + \dots \\
&= f_{2,1} \delta(\square_s) + f_{2,2} \left(\delta(\square_s) \square_s + \square_s \delta(\square_s) \right) \\
&\quad + f_{2,3} \left(\delta(\square_s) \square_s^2 + \square_s \delta(\square_s) \square_s + \square_s^2 \delta(\square_s) \right) + \dots,
\end{aligned} \tag{3.44}$$

Now, using the covariantly constant curvature conditions in eq. (3.13b), we can put to zero each term in $\delta\mathcal{F}_2(\bar{\square}_s)$ which has $\bar{\square}_s$ on its rightmost end. This enables us to effectively write $\delta\mathcal{F}_2(\square_s)$ in terms of $\delta(\square_s)$ as:

$$\begin{aligned}\delta\mathcal{F}_2(\square_s) &= (f_{2,1} + f_{2,2}\bar{\square}_s + f_{2,3}\bar{\square}_s^2 + \dots) \delta(\square_s) \\ &= \left[\frac{\mathcal{F}_2(\bar{\square}_s) - f_{2,0}}{\bar{\square}_s} \right] \delta(\square_s).\end{aligned}\quad (3.45)$$

We then finally obtain [159]:

$$(1010) = \sqrt{-\bar{g}} \frac{h}{2} \bar{S}^\nu_\mu \left[\frac{\mathcal{F}_2(\bar{\square}_s) - f_{2,0}}{\bar{\square}_s} \right] \delta(\square_s) \bar{S}^\mu_\nu = \sqrt{-\bar{g}} \frac{h}{2} \bar{S}^\nu_\mu \mathcal{F}_5(\bar{\square}_s) \delta(\square_s) \bar{S}^\mu_\nu, \quad (3.46)$$

where we define

$$\mathcal{F}_4(\square_s) \equiv \left[\frac{\mathcal{F}_1(\square_s) - f_{1,0}}{\square_s} \right], \quad (3.47)$$

and

$$\begin{aligned}\delta(\square_s) \bar{S}^\mu_\nu &= \frac{1}{M_s^2} \left[-\frac{1}{6} h_\nu^\alpha \bar{R} \bar{S}^\mu_\alpha - \frac{1}{4} h_\nu^\alpha \bar{S}_{\alpha\beta} \bar{S}^{\mu\beta} + \frac{1}{12} h \bar{R} \bar{S}^\mu_\nu + \frac{1}{2} h^{\alpha\beta} \bar{S}_{\alpha\beta} \bar{S}^\mu_\nu \right. \\ &\quad - \frac{1}{6} h^{\mu\alpha} \bar{R} \bar{S}_{\nu\alpha} - \frac{3}{2} h^{\alpha\beta} \bar{S}^\mu_\alpha \bar{S}_{\nu\beta} + \frac{1}{2} h \bar{S}^{\mu\beta} \bar{S}_{\nu\beta} - \frac{1}{4} h^{\mu\alpha} \bar{S}_{\alpha\beta} \bar{S}_\nu^\beta \\ &\quad + \frac{1}{2} \bar{S}_\nu^\alpha \bar{\nabla}_\beta \bar{\nabla}_\alpha h^{\mu\beta} + \frac{1}{2} \bar{S}^{\mu\alpha} \bar{\nabla}_\beta \bar{\nabla}_\alpha h_{\nu}^\beta + \frac{1}{2} \bar{S}_\nu^\alpha \bar{\square} h^\mu_\alpha - \frac{1}{2} \bar{S}^{\mu\alpha} \bar{\square} h_{\nu\alpha} \\ &\quad \left. - \frac{1}{2} \bar{S}_\nu^\alpha \bar{\nabla}^\mu \bar{\nabla}_\beta h_{\alpha}^\beta - \frac{1}{2} \bar{S}^{\mu\alpha} \bar{\nabla}_\nu \bar{\nabla}_\beta h_{\alpha}^\beta \right].\end{aligned}\quad (3.48)$$

2. (0110) = $\sqrt{-\bar{g}} s^\nu_\mu \delta(\mathcal{F}_2(\square_s)) \bar{S}^\mu_\nu$. Just like above, here we have [159]

$$\begin{aligned}(0110) &= \sqrt{-\bar{g}} s^\nu_\mu \delta(\mathcal{F}_2(\square_s)) \bar{S}^\mu_\nu \\ &= \sqrt{-\bar{g}} s^\nu_\mu \left[\frac{\mathcal{F}_2(\bar{\square}_s) - f_{2,0}}{\bar{\square}_s} \right] \delta(\square_s) \bar{S}^\mu_\nu = \sqrt{-\bar{g}} s^\nu_\mu \mathcal{F}_5(\bar{\square}_s) \delta(\square_s) \bar{S}^\mu_\nu,\end{aligned}\quad (3.49)$$

where we have defined

$$\mathcal{F}_5(\square_s) \equiv \left[\frac{\mathcal{F}_2(\square_s) - f_{2,0}}{\square_s} \right], \quad (3.50)$$

and s^ν_μ is given in Appendix (6.64).

3. $(0011) = \sqrt{-\bar{g}} \bar{S}^\nu_\mu \delta(\mathcal{F}_2(\bar{\square}_s)) s^\mu_\nu$. Now the background $\bar{S}^\nu_\mu$ is to the left of $\delta(\mathcal{F}_2(\bar{\square}_s))$. So we use $\bar{\square}_s$ operators in eq.(3.44) on the leftmost positions to integrate by parts and put these terms to zero. We finally get [159]:

$$(0011) = \sqrt{-\bar{g}} \bar{S}^\nu_\mu \delta(\bar{\square}_s) \left[\frac{\mathcal{F}_2(\bar{\square}_s) - f_{2,0}}{\bar{\square}_s} \right] s^\mu_\nu = \sqrt{-\bar{g}} \bar{S}^\nu_\mu \delta(\bar{\square}_s) \mathcal{F}_5(\bar{\square}_s) s^\mu_\nu \quad (3.51)$$

which has a long and complicated expression and there seems to be no straightforward way to obtain a readable form for (0011).

4. $(0020) = \sqrt{-\bar{g}} \bar{S}^\nu_\mu \delta^2(\mathcal{F}_2(\bar{\square}_s)) \bar{S}^\mu_\nu$ has the most involved expression of all [159]:

$$(0020) = \sqrt{-\bar{g}} \bar{S}^\nu_\mu \left[f_{2,1} \delta^2(\bar{\square}_s) + 2\delta(\bar{\square}_s) \mathcal{F}_6(\bar{\square}_s) \delta(\bar{\square}_s) \right] \bar{S}^\mu_\nu, \quad (3.52)$$

where we have defined

$$\mathcal{F}_6(\bar{\square}_s) \equiv \left[\frac{\mathcal{F}_2(\bar{\square}_s) - f_{2,0} - f_{2,1} \bar{\square}_s}{\bar{\square}_s^2} \right]. \quad (3.53)$$

Dominant terms in the UV

The SVT decomposition of $\delta^2 \mathcal{L}_{S^2}$ in all its glory is quite complicated and we do not present it here. For a simplified analysis, in the UV limit $k \gtrsim M_s$, we consider the dominant contributions in eq.(3.43). Below we argue that (1001) and (0101) are the only contributions we need to consider in the UV limit. We assume $\bar{R}, \bar{S}_{\mu\nu} < \mathcal{O}(M_s^2)$, which is reasonable, at least for the cosmological and black hole scenarios discussed in sec.(3.2).

First of all, we neglect the local contributions in eq.(3.43), that is, which do not have the form factors $\mathcal{F}_i(\bar{\square}_s)$; in the UV limit, terms with $\mathcal{F}_i(\bar{\square}_s)$ dominate $\delta^2 S$. These are (1100), (2000), (0200) and (0002). For example, (1001) is dominant compared to (1100) because $\mathcal{F}_2(\bar{\square}_s) > f_{2,0}$. The remaining non-local contributions are (1010), (1001), (0110), (0101), (0011) and (0020). Let us compare all these terms with (0101). Suppressing indices, (0101) has the following terms schematically:

$$\begin{aligned} & h \left\{ \bar{\nabla} \bar{\nabla} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla} \bar{\nabla} + \bar{S} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla} \bar{\nabla} + \bar{R} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla} \bar{\nabla} \right\} h \\ & + h \left\{ \bar{S} \mathcal{F}_2(\bar{\square}_s) \bar{S} + \bar{R} \mathcal{F}_2(\bar{\square}_s) \bar{R} + \bar{R} \mathcal{F}_2(\bar{\square}_s) \bar{S} \right\} h, \end{aligned} \quad (3.54)$$

as can be seen from the expression for s^ν_μ in Appendix (6.64). Note that h denotes the full perturbation before SVT decomposition. Assuming $k \gtrsim M_s$ in the UV limit

and $\bar{R}, \bar{S}_{\mu\nu} < \mathcal{O}(M_s^2)$, we can neglect terms in the second line of eq.(3.54) compared to terms in the first line. In this limit, we can also deduce that

$$\frac{1}{M_s^2} \delta(\square) \bar{S}_{\mu\nu} \ll s_{\mu\nu}, \quad \frac{1}{M_s^2} \delta(\square) s_{\mu\nu} \ll \frac{1}{M_s^2} \bar{\square} s_{\mu\nu}. \quad (3.55)$$

To see this, we check contributions coming from the various dimensionful quantities. Consider the first inequality in eq.(3.55). Ignoring h , $\delta(\square) \bar{S}_{\mu\nu} / M_s^2$ given in eq.(3.48) is schematically $(\bar{S}\bar{S} + \bar{S}\bar{R} + \bar{S}\bar{\nabla}\bar{\nabla}) / M_s^2$, while $s_{\mu\nu}^\nu$ given in eq.(6.64) is schematically $(\bar{R} + \bar{\nabla}\bar{\nabla})$. Comparing these two and using the above-mentioned limit, $s_{\mu\nu}$ is dominant. The same argument can be used to see how the second inequality in eq.(3.55) comes about. Using the above reasoning, we see which terms in eq.(3.43) can be neglected:

- (i) (1010): given in eq.(3.46), is schematically written as:

$$\frac{1}{M_s^2} \bar{S}\bar{S} h \mathcal{F}_5(\bar{\square}_s) \left\{ \bar{\nabla}\bar{\nabla} + \bar{S} + \bar{R} \right\} h. \quad (3.56)$$

Comparing eq.(3.56) and (0101) in eq.(3.54), we can neglect (1010) in comparison with (0101).

- (ii) (1001): given in eq.(3.43), goes like $\sqrt{-g} \bar{S}_\mu^\nu s_\nu^\mu \mathcal{F}_s(\bar{\square}_s) h$, which we can compare with (0101) given in eq.(3.54). Counting derivatives and curvatures, we can deduce that the contribution of (1001) is of the same order as (0101).
- (iii) (0110): given in eq.(3.49) can be neglected compared to (0101) in eq.(3.54) by using the first approximation in eq.(3.55).
- (iv) (0011): given in eq.(3.51) can be similarly neglected compared to (0101), using the second approximation in eq.(3.55).
- (v) (0020): given in eq.(3.52) contains two kinds of contributions. The first one is local as it has no $\mathcal{F}_2(\bar{\square}_s)$. This can be neglected compared to (0101), which has $\mathcal{F}_2(\bar{\square}_s)$. The second part is non-local which can be neglected compared to (0101) using the first approximation in eq.(3.55).

3.3.4 Perturbing non-local $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$

Lastly, the quadratic in Weyl tensor term in our Lagrangian in eq.(3.1) is

$$\mathcal{L}_{C^2} = \frac{1}{2} \sqrt{-g} C^{\rho\sigma}_{\mu\nu} \mathcal{F}_3(\square_s) C^{\mu\nu}_{\rho\sigma}. \quad (3.57)$$

Since our background is conformally-flat, any term in $\delta^2 (\sqrt{-g} C^{\rho\sigma}_{\mu\nu} \mathcal{F}_3(\Box_s) C^{\mu\nu}_{\rho\sigma})$ containing $\bar{C}_{\mu\nu\rho\sigma}$ vanishes. We are then left with only one non-zero term:

$$(0101) = \sqrt{-\bar{g}} c^{\rho\sigma}_{\mu\nu} \bar{\mathcal{F}}_3(\Box_s) c^{\mu\nu}_{\rho\sigma}, \quad (3.58)$$

which finally becomes [159]:

$$\begin{aligned} c^{\rho\sigma}_{\mu\nu} \bar{\mathcal{F}}_3(\Box_s) c^{\mu\nu}_{\rho\sigma} &= \hat{h}_{\alpha\beta} \left\{ \left[\delta^\beta_\nu \bar{g}^{\alpha\rho} \bar{S}^\sigma_\mu + 2\delta^\alpha_\mu \bar{g}^{\beta\sigma} \bar{\nabla}_\nu \bar{\nabla}^\rho \right] \bar{\mathcal{F}}_3(\Box_s) \left[-\frac{1}{12} \delta^\nu_\sigma \delta^\lambda_\rho \bar{g}^{\gamma\mu} \bar{R} \right. \right. \\ &\quad + \frac{1}{12} \delta^\nu_\rho \delta^\lambda_\sigma \bar{g}^{\gamma\mu} \bar{R} - \frac{1}{12} \delta^\mu_\rho \delta^\lambda_\sigma \bar{g}^{\nu\gamma} \bar{R} + \frac{1}{3} \delta^\mu_\rho \delta^\nu_\sigma \bar{S}^{\gamma\lambda} - \frac{1}{2} \delta^\nu_\sigma \delta^\gamma_\rho \bar{S}^{\mu\lambda} \\ &\quad + \frac{1}{2} \delta^\nu_\rho \delta^\gamma_\sigma \bar{S}^{\mu\lambda} - \frac{1}{4} \delta^\lambda_\sigma \bar{g}^{\gamma\nu} \bar{S}^\mu_\rho + \frac{1}{4} \delta^\lambda_\rho \bar{g}^{\gamma\nu} \bar{S}^\mu_\sigma - \frac{1}{2} \delta^\mu_\rho \delta^\gamma_\sigma \bar{S}^{\nu\lambda} \\ &\quad + \frac{1}{4} \delta^\lambda_\sigma \bar{g}^{\gamma\mu} \bar{S}^\nu_\rho - \frac{1}{4} \delta^\lambda_\rho \bar{g}^{\gamma\mu} \bar{S}^\nu_\sigma - \frac{1}{4} \delta^\nu_\sigma \bar{g}^{\gamma\mu} \bar{S}^\lambda_\rho + \frac{1}{4} \delta^\nu_\rho \bar{g}^{\gamma\mu} \bar{S}^\lambda_\sigma \\ &\quad - \frac{1}{4} \delta^\mu_\rho \bar{g}^{\nu\gamma} \bar{S}^\lambda_\sigma + \frac{1}{4} \delta^\nu_\sigma \delta^\lambda_\rho \bar{g}^{\mu\gamma} \bar{\Box} - \frac{1}{4} \delta^\nu_\rho \delta^\lambda_\sigma \bar{g}^{\mu\gamma} \bar{\Box} + \frac{1}{4} \delta^\mu_\rho \delta^\lambda_\sigma \bar{g}^{\nu\gamma} \bar{\Box} \\ &\quad - \frac{1}{2} \delta^\lambda_\sigma \bar{g}^{\nu\gamma} \bar{\nabla}_\rho \bar{\nabla}^\mu + \frac{1}{2} \delta^\lambda_\sigma \bar{g}^{\mu\gamma} \bar{\nabla}_\rho \bar{\nabla}^\nu + \frac{1}{2} \delta^\lambda_\rho \bar{g}^{\nu\gamma} \bar{\nabla}_\sigma \bar{\nabla}^\mu \\ &\quad \left. \left. - \frac{1}{2} \delta^\lambda_\rho \bar{g}^{\mu\gamma} \bar{\nabla}_\sigma \bar{\nabla}^\nu \right] \right\} \hat{h}_{\gamma\lambda}, \\ &\equiv \hat{h}_{\alpha\beta} \mathcal{O}_{(C)}^{\alpha\beta\gamma\lambda} \hat{h}_{\gamma\lambda}, \end{aligned} \quad (3.59)$$

where the operator $\mathcal{O}_{(C)}^{\alpha\beta\gamma\lambda}$ is introduced for brevity; (C) indicates that it comes from $\delta^2 \mathcal{L}_{C^2}$. All other SVT contributions including mix terms are zero. Detailed calculation from scratch is given in Appendix (6.F4). MSS and flat space limits are discussed in sec. (3.4) which are greatly simplified after setting $\bar{S}_{\mu\nu} = 0$.

3.4 dS/AdS and flat spacetime limits

In this section, we verify that $\delta^2 \mathcal{L}$ in eq. (3.26) computed around our non-MSS, conformally-flat background given by eq. (3.13) reduces ⁷ to the known expressions computed before in [111] [112]. This section borrows many steps from there. Three important simplifications occur when computing $\delta^2 \mathcal{L}$ around MSS. First, all perturbations that contain at least one A_μ or one B mode go to zero since each term in these expressions is proportional to $\bar{S}_{\mu\nu}$ (see Appendix (6.F) or sec. (3.3)). This kills all terms that are purely quadratic in each A_μ or B , and also modes that are mixed with them. Second, all mode mixings between $\hat{h}_{\mu\nu}$ and ϕ vanish upon using the Riemann tensor

⁷The dS/AdS/flat space limit can also be verified from the most general, second order perturbations around *arbitrary* backgrounds given in Appendix (6.F).

for MSS in eq.(3.12), transversality constraints on SVT modes in eq.(3.5) and integration by parts. Third, the complicated piece $\delta^2 \mathcal{L}_{S^2}$ in eq.(3.42) has only one non-zero contribution coming from (0101), with the rest vanishing because $\bar{S}_{\mu\nu} = 0$.

The non-zero SVT contributions coming from $\delta^2 \mathcal{L}_{EH+\Lambda}$ are (see eq.(3.29)):

$$\begin{aligned} \delta^2 \mathcal{L}_{EH+\Lambda} &= \sqrt{-\bar{g}} \frac{M_P^2}{2} \delta_0 = \sqrt{-\bar{g}} \frac{M_P^2}{2} \left[\delta_{0(\hat{h}\hat{h})} + \delta_{0(\phi\phi)} \right] \\ &= \sqrt{-\bar{g}} \frac{M_P^2}{2} \left[\frac{1}{4} \hat{h}_{\mu\nu} \left(\bar{\square} - \frac{\bar{R}}{6} \right) \hat{h}^{\mu\nu} - \frac{1}{32} \phi (3\bar{\square} + \bar{R}) \phi \right]. \end{aligned} \quad (3.60)$$

The non-zero SVT contributions coming from $\delta^2 \mathcal{L}_{R^2}$ are (see eq.(3.39)):

$$\begin{aligned} \delta^2 \mathcal{L}_{R^2} &= \sqrt{-\bar{g}} f_{1,0} \bar{R} \left[\frac{1}{4} \hat{h}_{\mu\nu} \left(\bar{\square} - \frac{\bar{R}}{6} \right) \hat{h}^{\mu\nu} - \frac{1}{32} \phi (3\bar{\square} + \bar{R}) \phi \right] \\ &\quad + \sqrt{-\bar{g}} \frac{1}{32} \phi (3\bar{\square} + \bar{R})^2 \mathcal{F}_1(\bar{\square}_s) \phi. \end{aligned} \quad (3.61)$$

The non-zero SVT contributions coming from $\delta^2 \mathcal{L}_{S^2}$ are (see eq.(3.43)):

$$\begin{aligned} \delta^2 \mathcal{L}_{S^2} &= (0101) = \frac{1}{2} \sqrt{-\bar{g}} s^\nu_\mu \mathcal{F}_2(\bar{\square}_s) s^\mu_\nu \\ &= \sqrt{-\bar{g}} \frac{1}{2} \left[\left(\frac{1}{4} \bar{D}_\mu^\nu \phi \right) \mathcal{F}_2(\bar{\square}_s) \left(\frac{1}{4} \bar{D}_\nu^\mu \phi \right) \right] \\ &\quad + \sqrt{-\bar{g}} \frac{1}{2} \left[\frac{1}{2} \left(\bar{\square} - \frac{\bar{R}}{6} \right) \hat{h}_\mu^\nu \mathcal{F}_2(\bar{\square}_s) \frac{1}{2} \left(\bar{\square} - \frac{\bar{R}}{6} \right) \hat{h}_\nu^\mu \right] \\ &= \sqrt{-\bar{g}} \frac{1}{32} \left[\phi \mathcal{F}_2 \left(\bar{\square}_s + \frac{2\bar{R}_s}{3} \right) \left(\frac{3\bar{\square} + \bar{R}}{4} \right) \bar{\square} \phi \right] \\ &\quad + \sqrt{-\bar{g}} \frac{1}{8} \left[\left(\bar{\square} - \frac{\bar{R}}{6} \right) \hat{h}_\mu^\nu \mathcal{F}_2(\bar{\square}_s) \left(\bar{\square} - \frac{\bar{R}}{6} \right) \hat{h}_\nu^\mu \right], \end{aligned} \quad (3.62)$$

where we have defined

$$\bar{D}_\mu^\nu \equiv \bar{\nabla}_\mu \bar{\nabla}^\nu - \delta_\mu^\nu \frac{\bar{\square}}{4}, \quad \bar{D}_\mu^\mu = 0, \quad \bar{R}_s \equiv \frac{\bar{R}}{M_s^2}, \quad (3.63)$$

and used the following recursion relations applicable *only* for MSS⁸:

$$\begin{aligned} \bar{\nabla}_\mu (\bar{\square}_s)^n H^{\mu\nu} &= \left(\bar{\square}_s + \frac{5\bar{R}_s}{12} \right)^n \bar{\nabla}_\mu H^{\mu\nu}, \quad \text{where} \quad H_\mu^\mu = 0, \\ \bar{\nabla}_\mu (\bar{\square}_s)^n t^\mu &= \left(\bar{\square}_s + \frac{\bar{R}_s}{4} \right)^n \bar{\nabla}_\mu t^\mu. \end{aligned} \quad (3.64)$$

⁸It is difficult and perhaps impossible to derive recursion relations for an arbitrary background. Nevertheless, many useful commutation relations and some recursion relations for an arbitrary background are presented in Appendix (6.H).

Finally, the non-zero SVT contributions coming from $\delta^2 \mathcal{L}_{C^2}$ are (see eq.(3.58)):

$$\begin{aligned} (0101) &= \frac{1}{2} \sqrt{-\bar{g}} c^{\rho\sigma}_{\mu\nu} \bar{\mathcal{F}}_3(\bar{\square}_s) c^{\mu\nu}_{\rho\sigma} \\ &= \sqrt{-\bar{g}} \frac{1}{4} \left[\hat{h}_\mu{}^\nu \mathcal{F}_3 \left(\bar{\square}_s + \frac{\bar{R}_s}{3} \right) \left(\bar{\square} - \frac{\bar{R}}{6} \right) \left(\bar{\square} - \frac{\bar{R}}{3} \right) \hat{h}^\mu{}_\nu \right], \end{aligned} \quad (3.65)$$

where we have used the following recursion relations applicable *only* for MSS:

$$\begin{aligned} \bar{\nabla}_\nu \bar{\square}^n c^{\nu\beta\mu\alpha} &= \left(\bar{\square} + \frac{\bar{R}}{4} \right)^n \bar{\nabla}_\nu c^{\nu\beta\mu\alpha} \\ \bar{\nabla}_\alpha \bar{\square}^n V^{\beta\mu\alpha} &= \left(\bar{\square} + \frac{\bar{R}}{12} \right)^n \bar{\nabla}_\alpha V^{\beta\mu\alpha} \quad \text{where } V^{\beta\mu\alpha} \equiv \bar{\nabla}_\nu c^{\nu\beta\mu\alpha}. \end{aligned} \quad (3.66)$$

Full $\delta^2 S$ can be split into its tensor ($\delta^2 S_{\tilde{h}}$) and scalar ($\delta^2 S_{\tilde{\phi}}$) parts, and we get

$$\begin{aligned} \delta^2 S_{\tilde{h}} &= \frac{1}{2} \int d^4 x \sqrt{-\bar{g}} \tilde{h}^{\mu\nu} \left(\bar{\square} - \frac{\bar{R}}{6} \right) \left\{ 1 + \frac{2}{M_P^2} f_{1,0} \bar{R} \right. \\ &\quad \left. + \frac{1}{M_P^2} \left[\left(\bar{\square} - \frac{\bar{R}}{6} \right) \mathcal{F}_2(\bar{\square}_s) + 2 \left(\bar{\square} - \frac{\bar{R}}{3} \right) \mathcal{F}_3 \left(\bar{\square}_s + \frac{\bar{R}_s}{3} \right) \right] \right\} \tilde{h}_{\mu\nu}, \\ \delta^2 S_{\tilde{\phi}} &= -\frac{1}{2} \int d^4 x \sqrt{-\bar{g}} \tilde{\phi} \left(\bar{\square} + \frac{\bar{R}}{3} \right) \left\{ 1 + \frac{2}{M_P^2} f_{1,0} \bar{R} \right. \\ &\quad \left. - \frac{1}{M_P^2} \left[2(3\bar{\square} + \bar{R}) \mathcal{F}_1(\bar{\square}_s) + \frac{1}{2} \bar{\square} \mathcal{F}_2 \left(\bar{\square}_s + \frac{2\bar{R}_s}{3} \right) \right] \right\} \tilde{\phi}, \end{aligned} \quad (3.67)$$

where we have introduced canonically normalized fields

$$\tilde{h}_{\mu\nu} = \frac{1}{2} M_P \hat{h}_{\mu\nu}, \quad \tilde{\phi} = \sqrt{\frac{3}{32}} M_P \phi, \quad [\tilde{h}_{\mu\nu}] = [\tilde{\phi}] = 1. \quad (3.68)$$

This matches with the results obtained in [111] and verifies the MSS limit. Effectively, eq.(3.67) gives spin-2 and spin-0 components of the graviton propagator around dS/AdS backgrounds.

Now, going to the flat space limit is easy by substituting $\bar{R} = 0$ above. We get the spin-2 and spin-0 components of the propagator: $\Pi_{2(0)} = \frac{i}{p^2 \zeta_{2(0)}}$, where [111] [112]

$$\zeta_2 = 1 - \frac{2p^2}{M_P^2} [\mathcal{F}_2(-p_s^2) + 2\mathcal{F}_3(-p_s^2)], \quad \zeta_0 = - \left\{ 1 + \frac{2p^2}{M_P^2} [6\mathcal{F}_1(-p_s^2) + \frac{1}{2}\mathcal{F}_2(-p_s^2)] \right\} \quad (3.69)$$

and we have defined $p_s^2 \equiv p^2/M_s^2$. We finally get

$$\Pi_{\text{flat}} = \Pi_2 + \Pi_0 = \frac{i}{p^2} \left(\frac{1}{\zeta_2} - \frac{1}{\zeta_0} \right) \quad (3.70)$$

This matches with the result in [111] and verifies the flat space limit.

Before moving on to the next section, recall that the background curvature dependence in expressions comprising $\delta^2 S$ appears in terms of $\bar{R}$ and $\bar{S}_{\mu\nu}$. These curvatures are generated in two ways: first, when we compute the perturbation of a quantity (like Ricci scalar) to some order, and second, when covariant derivatives ($\bar{\nabla}_\mu$) are commuted across the infinite tower of $\square_s$ operators (in $\mathcal{F}_i(\square_s)$) to the other side in order to contract with corresponding covariant derivatives ($\bar{\nabla}^\mu$) to form scalar $\square$ operators. Perturbations around our background satisfying eq.(3.13) reduce to those around MSS when $\bar{S}_{\mu\nu}$ is set to zero. Naively, we might be able to separate the non-MSS and MSS parts of perturbations by separating expressions with and without $\bar{S}_{\mu\nu}$, respectively. This is not possible, however, because simple recursive relations when commuting $\bar{\nabla}^\mu$, like in eqs.(3.64) and (3.66) for MSS, do not seem to exist for non-MSS backgrounds (except for the simplest cases when $\bar{\nabla}^\mu$ operates on a low-rank tensor (see for e.g. Appendix (6.102) and (6.105)).

3.5 Non-local gravity around non-maximally symmetric backgrounds: Illustrative examples

To facilitate studying equations of motion of SVT modes in non-local gravity around our background (eq.(3.13)), we consider 3 examples, each of which has only one $\mathcal{F}_i(\square_s)$ non-zero. $\mathcal{L}_{EH+\Lambda}$ is never zero since we want the higher derivative theory to reduce to GR in IR. In two cases, we will show that the different SVT degrees of freedom can be decoupled and solutions can be obtained in the UV regime. The third case shows non-trivial mode mixing.

3.5.1 $\mathcal{L} = \mathcal{L}_{EH+\Lambda} + \mathcal{L}_{C^2}$ ($\mathcal{F}_1 = \mathcal{F}_2 = 0, \mathcal{F}_3 \neq 0$)

In this case, the action is composed of Einstein-Hilbert plus cosmological constant and quadratic in Weyl tensor parts of the full action in eq.(3.1). The background EoM in eq.(3.14) in this case reduces to $M_P^2 \bar{S}_\nu^\mu = 0$ whose solution is $\bar{S}_\nu^\mu = 0$, which is just MSS, and has been studied before in [111, 112]. We briefly recall the results

from there. The second order action can be read off from eq.(3.67) [111, 112]:

$$\delta^2 S = \int d^4 x \sqrt{-\bar{g}} \tilde{h}_{\mu\nu} \underbrace{\left[\frac{1}{2} \left(\bar{\square} - \frac{\bar{R}}{6} \right) \left[1 + \frac{2}{M_P^2} \left(\bar{\square} - \frac{\bar{R}}{3} \right) \mathcal{F}_3 \left(\bar{\square}_s + \frac{\bar{R}_s}{3} \right) \right] \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} \right]}_{\mathcal{K}_{33}} \tilde{h}_{\alpha\beta} - \underbrace{\tilde{\phi} \left[\frac{3\bar{\square} + \bar{R}}{6} \right]}_{\mathcal{K}_{11}} \tilde{\phi}, \quad (3.71)$$

with a kinetic matrix $\mathcal{K}$ diagonal in $\tilde{\phi}$ and $\tilde{h}_{\mu\nu}$ modes (see eq.(3.11)). We see in the quadratic form for $\tilde{h}_{\mu\nu}$ above that in addition to the usual pole for GR at $\bar{R}/6$ expressed by $(\bar{\square} - \bar{R}/6)$, there is a multiplicative factor which contains the higher derivative, non-local form factor $\mathcal{F}_3(\bar{\square}_s)$. By demanding this multiplicative factor to be exponential of an entire function⁹, it becomes possible to modify the graviton propagator such that there are no extra degrees of freedom, and also no ghosts. For example, for an entire function $\alpha(\bar{\square}_s)$, and [111, 112]

$$\mathcal{F}_3(\bar{\square}_s) = \frac{M_P^2}{2} \left[\frac{e^{\alpha(\bar{\square}_s - \frac{2\bar{R}_s}{3})} - 1}{\left(\bar{\square} - \frac{2\bar{R}}{3} \right)} \right], \quad (3.72)$$

which is holomorphic, we get the following tensor quadratic form:

$$S_h = \int d^4 x \frac{\sqrt{-\bar{g}}}{2} \tilde{h}_{\mu\nu} \left(\bar{\square} - \frac{\bar{R}}{6} \right) e^{\alpha(\bar{\square}_s - \frac{2\bar{R}_s}{3})} \tilde{h}^{\mu\nu}, \quad (3.73)$$

which has the usual graviton pole corresponding to dS/AdS radius, and, additionally, exponential suppression at large momenta $k \gtrsim M_s$.

3.5.2 $\mathcal{L} = \mathcal{L}_{EH+\Lambda} + \mathcal{L}_{S^2}$ ($\mathcal{F}_1 = \mathcal{F}_3 = 0, \mathcal{F}_2 \neq 0$)

In this case, the Lagrangian consists of Einstein-Hilbert plus cosmological constant and quadratic in traceless Ricci tensor parts of the full Lagrangian in eq.(3.1), and we have:

$$\delta^2 \mathcal{L} = \frac{M_P^2}{2} \delta_0 + \delta^2 \mathcal{L}_{S^2}. \quad (3.74)$$

In this case, the background EoM from eq.(3.14) is $M_P^2 \bar{S}^\mu_\nu = f_{2,0} \left(\frac{1}{2} \delta^\mu_\nu \bar{S}^{\alpha\beta} \bar{S}_{\alpha\beta} - \frac{1}{2} \bar{R} \bar{S}^\mu_\nu - 2 \bar{S}^{\mu\alpha} \bar{S}_{\nu\alpha} \right)$ which is non-trivial to solve. Furthermore, from explicit perturbative expressions given in Appendices (6.E) and (6.F) (see also sec.(3.3.3)), we

⁹According to the Weierstrass product theorem, any entire function $f(z)$ with no zeros can be written as $f(z) = e^{g(z)}$, where $g(z)$ is an entire function. We can use this to construct $\mathcal{F}_i(\bar{\square}_s)$ without poles.

see that $\delta^2 \mathcal{L}$ is highly non-trivial and possesses all the 6 possible mixings between scalar, vector and tensor modes. Therefore, in this case the generalized kinetic matrix $\mathcal{K}_{\mu\nu} \neq 0$ for all μ, ν . The full $\delta^2 \mathcal{L}$ in eq. (3.74) for all SVT modes is quite long and not very enlightening, and so we will not present them here.

However, following the discussion in sec. (3.3.3), we can greatly simplify $\delta^2 \mathcal{L}_{S^2}$ and extract the two dominant contributions in the UV, where infinite covariant derivatives dominate, i.e. (0101) and (1001). In particular, in this limit, we neglect contributions coming from $M_P^2 \delta_0$ and terms in $\delta^2 \mathcal{L}_{S^2}$ that are quadratic in background curvatures because, in this UV limit, the momenta scale as $k \gtrsim M_s$ with $\bar{R}, \bar{S}_{\mu\nu} < \mathcal{O}(M_s^2)$. In Appendix (6.F.3), we have presented the (0101) and (1001) contributions in $\delta^2 \mathcal{L}_{S^2}$ for all SVT modes. For now, let us just consider ϕ and $\hat{h}_{\mu\nu}$. There exist non-trivial SVT mode mixings in $\delta^2 \mathcal{L}$ in eq. (3.74) like $B(\mathcal{K}_{01})\phi$, $A^\mu(\mathcal{K}_{21})_\mu\phi$, $\hat{h}_{\mu\nu}(\mathcal{K}_{31})^{\mu\nu}\phi$, $\hat{h}_{\mu\nu}(\mathcal{K}_{30})^{\mu\nu}B$ and $\hat{h}_{\mu\nu}(\mathcal{K}_{32})^{\mu\nu}A^\lambda$, along with the usual purely quadratic in one SVT mode terms $\phi(\mathcal{K}_{11})\phi$ and $\hat{h}_{\mu\nu}(\mathcal{K}_{33})^{\mu\nu\rho\sigma}\hat{h}_{\rho\sigma}$. The respective kinetic matrix elements are [159]

$$\begin{aligned}
 \mathcal{K}_{01} &= \left[\frac{1}{8} \bar{S}^\nu_\mu \mathcal{F}_2(\bar{\square}_s) \bar{\square} \bar{\nabla}_\nu \bar{\nabla}^\mu \right], & \mathcal{K}_{21} &= \left[\frac{1}{2} \bar{\nabla}^\alpha \mathcal{F}_2(\bar{\square}_s) \bar{S}_{\nu\alpha} \bar{\nabla}^\nu \bar{\nabla}_\mu - \bar{\nabla}_\nu \mathcal{F}_2(\bar{\square}_s) \bar{S}^\alpha_\mu \bar{\nabla}^\nu \bar{\nabla}_\alpha \right], \\
 \mathcal{K}_{31} &= \left[\frac{1}{24} \bar{R} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla}^\nu \bar{\nabla}^\mu + \frac{1}{2} \bar{S}^{\sigma\mu} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla}^\nu \bar{\nabla}_\sigma - \frac{1}{4} \bar{S}^{\mu\nu} \bar{\square} \mathcal{F}_2(\bar{\square}_s) - \frac{1}{4} \bar{\square} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla}^\nu \bar{\nabla}^\mu \right. \\
 &\quad \left. - \frac{1}{8} \bar{S}^{\mu\nu} \mathcal{F}_2(\bar{\square}_s) \bar{\square} + \frac{1}{4} \bar{S}^{\mu\nu} \mathcal{F}_2(\bar{\square}_s) \bar{\square} \right], \\
 \mathcal{K}_{30} &= \left[-\frac{1}{4} \bar{S}^{\mu\nu} \mathcal{F}_2(\bar{\square}_s) \bar{\square}^2 \right], \\
 \mathcal{K}_{32} &= \left[\mathcal{F}_2(\bar{\square}_s) \bar{\square} \bar{S}^\nu_\rho \bar{\nabla}^\rho \delta^\mu_\lambda - \mathcal{F}_2(\bar{\square}_s) \bar{\square} \bar{S}^\mu_\rho \bar{\nabla}^\nu \delta^\rho_\lambda \right], \\
 \mathcal{K}_{11} &= \left[\frac{1}{16} \bar{\nabla}_\mu \bar{\nabla}^\nu \mathcal{F}_2(\bar{\square}_s) \bar{\nabla}_\nu \bar{\nabla}^\mu - \frac{1}{64} \mathcal{F}_2(\bar{\square}_s) \bar{\square}^2 \right], \\
 \mathcal{K}_{33} &= \left[-\frac{1}{12} \mathcal{F}_2(\bar{\square}_s) \bar{R} \bar{\square} \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} - \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\sigma\nu} \bar{\square} \bar{g}^{\mu\rho} + \frac{1}{4} \mathcal{F}_2(\bar{\square}_s) \bar{\square}^2 \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} \right]
 \end{aligned} \tag{3.75}$$

Unlike the previous case in sec. (3.5.1), here we see that decoupling the different SVT degrees of freedom at the quadratic level of the action is not an easy task, and perhaps impossible. This indicates having quadratic in (traceless) Ricci tensor terms in the action will, in general, give rise to all SVT modes being coupled to each other

when the background is not MSS^[10]. We defer the analysis of eq.(3.74) to future.

3.5.3 $\mathcal{L} = \mathcal{L}_{EH+\Lambda} + \mathcal{L}_{R^2}$ ($\mathcal{F}_2 = \mathcal{F}_3 = 0, \mathcal{F}_1 \neq 0$)

In this case, the Lagrangian consists of Einstein-Hilbert plus cosmological constant and quadratic in Ricci scalar terms of the full Lagrangian in eq.(3.1):

$$\mathcal{L} = \frac{1}{2} [M_P^2 (R - 2\Lambda) + R\mathcal{F}_1(\Box_s)R]. \quad (3.76)$$

The second order Lagrangian in this case is (see eqs.(3.29) and (3.39))^[159]:

$$\delta^2 \mathcal{L} = \frac{1}{2} [\Omega \delta_0 + r\mathcal{F}_1(\Box_s)r], \quad (3.77)$$

where $\Omega = M_P^2 + 2f_{1,0}\bar{R}$ and the background EoM from eq.(3.14) is ^[159]:

$$\Omega \bar{S}^\mu_\nu = 0. \quad (3.78)$$

If we trace back to how we obtained the background EoM in eq.(3.78) from Appendix (6.26), we notice that only the condition $\partial_\mu \bar{R} = 0$ was used, not $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0$. Now, the background EoM in eq.(3.78) has 3 possible solutions. Let us consider them one by one:

1. $\Omega \neq 0, \bar{S}_{\mu\nu} = 0$: Since $\bar{S}_{\mu\nu} = 0$, this leads us trivially to the MSS case presented in sec.(3.4) (also see ^[111]) and we get the scalar and tensor parts of the second order action as ^{[111], [112]}:

$$\begin{aligned} \delta^2 S_{\tilde{h}} &= \frac{\Omega}{2M_P^2} \int d^4x \sqrt{-\bar{g}} \tilde{h}^{\mu\nu} \left(\Box - \frac{\bar{R}}{6} \right) \tilde{h}_{\mu\nu}, \\ \delta^2 S_{\tilde{\phi}} &= -\frac{1}{2M_P^2} \int d^4x \sqrt{-\bar{g}} \tilde{\phi} \left(\Box + \frac{\bar{R}}{3} \right) \left[\Omega - 6 \left(\Box + \frac{\bar{R}}{3} \right) \mathcal{F}_1(\Box_s) \right] \tilde{\phi}. \end{aligned} \quad (3.79)$$

In order to have no extra degrees of freedom in the $\tilde{\phi}$ sector, in addition to a possible Brans-Dicke like scalar of mass m , we can choose ^{[111], [112]}

$$\left[\Omega - 6 \left(\Box + \frac{\bar{R}}{3} \right) \mathcal{F}_1(\Box_s) \right] = \left(1 - \frac{\Box}{m^2} \right)^\epsilon e^{\alpha(\Box_s)}, \quad (3.80)$$

where $\alpha(\Box_s)$ is an entire function (like in eq.(3.72)) and $\epsilon = 0$ or 1 . $\epsilon = 0$ corresponds to no extra scalar mode, while $\epsilon = 1$ corresponds to one extra Brans-Dicke scalar. Imposing $m^2 > 0$ ensures that the scalar is not tachyonic.

¹⁰Notice that when taking the MSS limit ($\bar{S}_{\mu\nu} = 0$), all mode mixing operators in eq.(3.75) manifestly vanish, except $\frac{1}{24} \bar{R} \mathcal{F}_2(\Box_s) \bar{\nabla}^\nu \bar{\nabla}^\mu - \frac{1}{4} \Box \mathcal{F}_2(\Box_s) \bar{\nabla}^\nu \bar{\nabla}^\mu$ in $(\mathcal{K}_{31})^{\mu\nu}$. This goes to zero too, after integrating by parts and using the transversality condition $\bar{\nabla}^\mu \hat{h}_{\mu\nu} = 0$.

2. $\Omega = 0, \bar{S}_{\mu\nu} \neq 0$: Since $\bar{S}_{\mu\nu} \neq 0$, we have a *non*-MSS background, as required by our background conditions in eq.(3.13). $\Omega = 0$ then determines the background Ricci scalar to be [159]:

$$\bar{R} = -\frac{M_P^2}{2f_{1,0}}. \quad (3.81)$$

The first term in eq.(3.77) vanishes because $\Omega = 0$. From eq.(3.40) and eq.(3.41), we see that $\delta^2 \mathcal{L}$ contains only ϕ and $\hat{h}_{\mu\nu}$ modes [159]:

$$\begin{aligned} \delta^2 \mathcal{L} = \frac{1}{2} \phi \left[(\bar{R} + 3\bar{\square})^2 \frac{\mathcal{F}_1(\bar{\square}_s)}{16} \right] \phi + \frac{1}{2} \hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta} \mathcal{F}_1(\bar{\square}_s) \bar{S}_{\mu\nu} \hat{h}^{\mu\nu} \\ - \frac{1}{2} \hat{h}^{\mu\nu} \left[(\bar{R} + 3\bar{\square}) \bar{S}_{\mu\nu} \frac{\mathcal{F}_1(\bar{\square}_s)}{2} \right] \phi. \end{aligned} \quad (3.82)$$

where, now, we have used the condition $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0$ (see eq.(3.13b)). We can diagonalize $\delta^2 \mathcal{L}$ using the field redefinition

$$\psi \equiv (\bar{R} + 3\bar{\square}) \frac{\phi}{4} - \bar{S}_{\mu\nu} \hat{h}^{\mu\nu}, \quad (3.83)$$

so that the quadratic action becomes [159]

$$\delta^2 S = \int d^4 x \sqrt{-\bar{g}} \frac{1}{2} \psi \mathcal{F}_1(\bar{\square}_s) \psi \quad (3.84)$$

and varying $\delta^2 S$ with respect to ψ gives us the EoM for ψ :

$$\mathcal{F}_1(\bar{\square}_s) \psi = 0. \quad (3.85)$$

This new scalar ψ is gauge-invariant as it is composed of gauge-invariant fields $\hat{h}_{\mu\nu}$ and ϕ (see eq.(3.9)). Now, the propagating degrees of freedom in the theory described by the action in eq.(3.84) depends on the form of $\mathcal{F}_1(\bar{\square}_s)$ we choose in our Lagrangian in eq.(3.76). For example, if we choose $\mathcal{F}_1(\bar{\square}_s) = 1$, then ψ has no kinetic term. Also, from the EoM in eq.(3.85), we get $\psi = 0$, which from eq.(3.83) gives the constraint equation

$$(\bar{R} + 3\bar{\square}) \frac{\phi}{4} = \bar{S}_{\mu\nu} \hat{h}^{\mu\nu}. \quad (3.86)$$

If we choose $\mathcal{F}_1(\bar{\square}_s) = \bar{\square}_s = \bar{\square}/M_s^2$, the quadratic action becomes

$$\delta^2 S = \int d^4 x \sqrt{-\bar{g}} \frac{1}{2} \tilde{\psi} \bar{\square} \tilde{\psi} \quad (3.87)$$

where $\tilde{\psi} \equiv \psi/M_s$ now has canonical dimensions $[\tilde{\psi}] = 1$. This theory describes a massless scalar in a *non*-MSS ($\bar{S}_{\mu\nu} \neq 0$) background. It is also ghost-free because the sign of the kinetic term is correct. If we choose $\mathcal{F}_1(\bar{\square}_s) = \left(1 - \frac{\bar{\square}}{m^2}\right)^\epsilon e^{\alpha(\bar{\square}_s)}$, for some entire function $\alpha(\bar{\square}_s)$, we have [159]

$$\delta^2 S = \int d^4x \sqrt{-\bar{g}} \frac{1}{2} \psi \left(1 - \frac{\bar{\square}}{m^2}\right)^\epsilon e^{\alpha(\bar{\square}_s)} \psi, \quad (3.88)$$

where $\epsilon = 0$ or 1 . The case $\epsilon = 0$ corresponds to no poles for ψ , while $\epsilon = 1$ corresponds to a pole at $\bar{\square} = m^2$. For the choice $\epsilon = 0$ and $\alpha(\bar{\square}_s) = \bar{\square}_s$, the kinetic term is similar to what one has in p-adic string theory [101].

3. $\Omega = 0, \bar{S}_{\mu\nu} = 0$: Since $\bar{S}_{\mu\nu} = 0$, we again have an MSS background, but this dS/AdS vacuum is different from that in eq.(3.79) where Ω was non-zero. Here, $\Omega = 0$, and the coefficient $f_{1,0}$ in $\mathcal{F}_1(\bar{\square}_s)$ is related to the cosmological constant via [159]

$$\bar{R} = -\frac{M_P^2}{2f_{1,0}} = 4\Lambda. \quad (3.89)$$

The second order action in this case can be obtained by putting $\Omega = 0$ in eq.(3.79), where its tensor part $\delta^2 S_{\tilde{h}}$ vanishes, leaving us with just the scalar part [159]:

$$\delta^2 S_{\tilde{\phi}} = \frac{3}{M_P^2} \int d^4x \sqrt{-\bar{g}} \tilde{\phi} \left(\bar{\square} + \frac{\bar{R}}{3}\right)^2 \mathcal{F}_1(\bar{\square}_s) \tilde{\phi} \quad (3.90)$$

which has the EoM for $\tilde{\phi}$ given by

$$\left(\bar{\square} + \frac{\bar{R}}{3}\right)^2 \mathcal{F}_1(\bar{\square}_s) \tilde{\phi} = 0. \quad (3.91)$$

As before, we can choose $\mathcal{F}_1(\bar{\square}_s) = e^{\alpha(\bar{\square}_s)}$ for some entire function $\alpha(\bar{\square}_s)$ to avoid any additional poles for $\tilde{\phi}$. In a local theory, with say $\mathcal{F}_1(\bar{\square}_s) = 1$, the EoM in eq.(3.91) reduces to that of a 4th order wave operator acting on $\tilde{\phi}$:

$$\left(\bar{\square} + \frac{\bar{R}}{3}\right)^2 \tilde{\phi} = 0. \quad (3.92)$$

In addition to modes $\tilde{\phi}_0$ which satisfy both $\left(\bar{\square} + \frac{\bar{R}}{3}\right) \tilde{\phi} = 0$ and eq.(3.92), there also exist log modes $\tilde{\phi}_{\log}$ which satisfy eq.(3.92) but not $\left(\bar{\square} + \frac{\bar{R}}{3}\right) \tilde{\phi} = 0$ [208]. We see that this dS/AdS vacuum has a different spectrum compared to the ($\Omega \neq 0, \bar{S}_{\mu\nu} = 0$) case discussed before. In that case, $\delta^2 S$ had both tensor $\tilde{h}$ and

scalar $\tilde{\phi}$ modes (see eq.(3.79)). In this case, we have only the scalar $\tilde{\phi}$ mode¹¹ that too which satisfies a different EoM given in eq.(3.91).

In this chapter, we studied perturbations at the quadratic level of the action for non-local gravity. We computed (covariant) scalar-vector-tensor perturbations $(\phi, B, A_\mu, \hat{h}_{\mu\nu})$ around conformally-flat, covariantly constant curvature backgrounds which are more general than, and go beyond maximally symmetric spacetimes. We found that around our chosen background, the different SVT modes do not decouple at the quadratic level of the action. This means that, for instance, the scalar mode ϕ is sourced by the remaining vector A_μ , tensor $\hat{h}_{\mu\nu}$ and scalar B modes. To show the consistency of computations around our *non*-maximally symmetric background, we derived dS/AdS and flat space limits, previously computed in [107, 111].

Studying quadratic order SVT perturbations at the level of the action, around our conformally-flat, covariantly constant curvature background in various limits of non-local gravity, we learn the following:

- In the case where $\mathcal{F}_1 = \mathcal{F}_2 = 0$ and $\mathcal{F}_3 \neq 0$ i.e., when the action contains only the non-local, quadratic Weyl term in addition to the usual Einstein-Hilbert plus cosmological constant term, our background equations of motion imply that the only possible background solutions are maximally symmetric dS/AdS ($\bar{S}_{\mu\nu} = 0$). So in this case, there is only one propagating degree of freedom, $\hat{h}_{\mu\nu}$, and we can construct a ghost-free propagator for it, as was shown previously in [111].
- The case with $\mathcal{F}_1 = 0$, $\mathcal{F}_2 \neq 0$ and $\mathcal{F}_3 = 0$ i.e., when the action contains non-local, quadratic in traceless Ricci tensor term in addition to the usual GR term, is quite involved. The second order action has a complicated form which was expected because of our non-trivial background that goes beyond dS/AdS. We present the framework for perturbations in this case, but for economical reasons, present the second order action with only the terms that are highly

¹¹The vacuum solution $\Omega = \bar{S}_{\mu\nu} = 0$ exists even in the local theory i.e., when $\mathcal{F}_1(\square_s) = f_{1,0}$, giving only a scalar mode but no tensor mode in the spectrum. Note that we cannot realize the solution $\Omega = 0, \bar{S}_{\mu\nu} = 0$ if we conformally transform the $R + f_{1,0}R^2$ action to its Einstein frame where we always have a tensor mode coming from the Einstein-Hilbert term and a propagating scalar minimally coupled to gravity. This is because the conformal transformation cannot always be applied. We can clearly see this by writing the local quadratic action in terms of a Lagrange multiplier χ as

$$S_{R^2} = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R + f_{1,0} R^2 \right] = \int d^4x \sqrt{-g} \left[\left(\frac{M_p^2}{2} + 2f_{1,0}\chi \right) R - \chi^2 \right]. \quad (3.93)$$

Transforming the above action into Einstein frame involves redefining metric $g_{\mu\nu} \rightarrow \left(\frac{M_p^2}{2} + 2f_{1,0}\chi \right)^{-1} g_{\mu\nu}$ which is not well-defined when $\frac{M_p^2}{2} + 2f_{1,0}\chi = 0$ [209].

dominant in the UV. Here, we see that all the SVT modes have non-local (containing $\mathcal{F}_2(\Box_s)$) kinetic terms and also non-trivial mixings with all the other SVT modes.

- In the case with $\mathcal{F}_1 \neq 0$, $\mathcal{F}_2 = 0$ and $\mathcal{F}_3 = 0$ i.e., when the action contains non-local, quadratic in Ricci scalar term along with the usual GR term, we found that the scalar B and vector A_μ modes do not appear at all. The remaining ϕ and $\hat{h}_{\mu\nu}$ modes are mixed but the second order action can be diagonalized. We have analysed all the vacuum solutions of this case and studied the spectrum of propagating modes.

In addition to the fact that SVT modes are generically coupled, we notice that their kinetic terms are not only functions of $\Box$, but also the background traceless Ricci tensor and Ricci scalar. This is similar to the case when background was dS/AdS (see [111]) where we had form factors (obtained from ghost-free conditions) depending not only on covariant derivatives, but also the background Ricci scalar (for e.g., see eq. (3.67)). In our case, form factors do depend on the background traceless Ricci tensor since our background is not MSS. In order to clearly see this, however, one must use the commutation relations presented in Appendix (6.H) to commute $\bar{\nabla}_\mu$ through $\mathcal{F}_i(\Box_s)$ and contract with $\bar{\nabla}^\mu$ on the other side of the form factors. However, these commutation relations are not recursive in most cases and it is not possible to write a closed form expression for form factors where explicit background curvature dependence is manifest. Our study indicates that in order to formulate a background independent theory of non-local gravity, we must start with form factors depending on curvatures i.e., $\mathcal{F}_i\left(\Box_s, \frac{\bar{R}}{M_s^2}, \frac{\bar{S}_{\mu\nu}}{M_s^2}\right)$ in such a way that we do not have to fix the form factors depending on any particular background. One might also consider studying perturbations using a decomposition different from the one in eq. (3.6).

The perturbations we have computed in this chapter around backgrounds beyond MSS can have wide cosmological and astrophysical applications in probing UV physics of any higher derivative theory of gravity. We illustrated with explicit computation that all *covariant* SVT modes in these backgrounds are generally coupled, indicating that gravitational waves may have non-trivial propagation with *extra* polarizations (in addition to the usual $+$ and $\times$ tensor polarizations known in GR) since the modes are coupled to each other. Study of SVT modes in modified theories of gravity is currently an active area of research in the scope of LIGO/VIRGO detections [210–213]. It will be interesting to investigate quantitative signatures of non-local gravity with respect to detection of *extra* polarizations.

Chapter 4

An Isotropic and Homogeneous Bouncing Universe

Bouncing cosmologies have been an attractive paradigm for resolving the Big Bang singularity, which is inevitable in General Relativity (GR) under reasonable assumptions [214]. They have been studied in various approaches to quantum gravity like string theory [215, 216] and loop quantum gravity [217]. For another non-singular model of the early universe motivated from string theory, called string gas cosmology, see [218–222]. Bouncing solutions are typically found by modifying the matter sector by adding fields which violate the null energy condition (NEC), and/or by modifying GR, for instance, by adding higher derivative terms to the Einstein-Hilbert (EH) action [214, 223–228]. NEC violation and higher derivative actions generally imply sicknesses in the form of ghost (fields with wrong-sign kinetic terms) and gradient instabilities. Another problem with bouncing cosmologies is the growth of vector (metric and matter) modes with time during the contraction phase, thereby invalidating perturbation theory [225]. In some bouncing models [229], and even in power-law inflation [230], scalar modes can also grow. Of central importance is then the question of finding a healthy, non-singular bouncing scenario with bounded amplitudes of perturbations.

A model of bouncing cosmology assumes that the universe did not come into existence at a finite time in the past. Instead, it purports that the universe underwent a period of contraction, shrunk to a certain minimum (but finite) size, and then expanded into its present form. This is a neat resolution of the Big Bang singularity problem. However, achieving a bounce is not straightforward and it comes with its own challenges. Let us briefly look at what happens in GR. Around FLRW, the Friedman equations are

$$\dot{H} = \frac{k}{a^2} - \frac{1}{2}(\rho + p) \quad (4.1)$$

where k = spatial curvature and ρ and p are the energy density and pressure of a perfect fluid matter component. A bounce simply means that the universe contracted in size before expanding. This means that $\dot{H} > 0$ around the bounce point. In the context of GR, in the absence of spatial curvature k , this implies that $\rho + p < 0$. This

is nothing but the violation of the NEC. There have been various attempts over the years to construct stable bouncing models. This chapter is based on one of them.

In the context of higher derivative modified gravity, quadratic scalar curvature ($R + R^2$) is an interesting model to consider when searching for non-singular solutions [231, 232]. After adding a cosmological constant to $R + R^2$ theory, one can indeed obtain bouncing solutions, but which nevertheless are plagued by the presence of negative radiation energy density $\rho < 0$. Extending the action by adding an arbitrary number of higher derivative terms typically leads to ghost degrees of freedom around, for instance, flat space and (A)dS. On the other hand, an infinite covariant derivative, non-local extension can potentially resolve the problem of ghosts, by choosing a suitable function that captures all order derivative terms in the action. In this non-local quadratic scalar curvature model, bouncing solutions were found with $\rho > 0$, around spatially flat, homogeneous and isotropic backgrounds [94, 107, 147, 148, 233, 234]. The stability of scalar perturbations around these solutions in the presence of radiation was studied in [143, 148]. One can deduce that the presence of radiation implies growth of vector perturbations during the contraction phase [194].

So far, UV non-local gravity has been constructed to be ghost-free around flat space [91, 95, 106, 107] and (A)dS [111, 112]. In an arbitrary background, inevitable mixing of scalar-vector-tensor (SVT) modes makes the task of removing ghosts challenging, but one can still remove them around specific non-maximally symmetric backgrounds [159] (see Chapter 3).

In this chapter, we find a class of non-local $R + R\mathcal{F}R$ theory with a cosmological constant which is ghost-free around homogeneous and isotropic bouncing cosmological backgrounds. $\mathcal{F}$ is the non-local term (discussed below) essential for avoiding ghosts. Only one scalar mode survives, which can be made ghost-free, and exhibits bounded, oscillatory evolution across the bounce.

The chapter is organized as follows: In Sec. (4.1), we provide the non-local action used throughout the chapter, discuss the equations of motion, and obtain conditions on the background geometry following from assuming a simplifying ansatz for the background Ricci scalar. In Sec. (4.2), we study perturbations both at the linearized level of the equations of motion, and at the quadratic level of the action, around the background chosen in the previous section. In particular, we show how the theory propagates only one scalar degree of freedom, and no vector or tensor modes, and how the kinetic operator can be chosen such that there are no extra/ghost modes. In Sec. (4.3), we choose a homogeneous and isotropic cosmological bouncing background and study the time evolution of the lone scalar mode, showing its bounded and oscillatory behavior.

4.1 Action and equations of motion

Non-local generalization of $R + R^2$ by adding an infinite series of covariant derivatives reads [94]

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R + R \mathcal{F}(\square) R - \Lambda \right] \quad (4.2)$$

where Λ is the cosmological constant, the form factor $\mathcal{F}(\square)$ is an arbitrary analytic function of d'Alembertian, $\square = g_{\mu\nu} \nabla^\mu \nabla^\nu$, containing all orders of $\square$. It has the power series expansion $\mathcal{F}(\square) = \sum_{n=0}^{\infty} f_n \square^n$, where f_n are dimensionless coefficients, dimensionless $\square_s = \square/M_s^2$, and $M_s (< M_p \sim 10^{18} \text{ GeV})$ is a new high energy scale of non-locality below the Planck scale M_p . The equations of motion (EOM) of Eq.(5.1) are [199]

$$\begin{aligned} E_\nu^\mu &\equiv - [M_p^2 + 4\mathcal{F}(\square)R] G_\nu^\mu - R \mathcal{F}(\square) R \delta_\nu^\mu \\ &\quad + 4 (\nabla^\mu \partial_\nu - \delta_\nu^\mu \square) \mathcal{F}(\square) R + 2\mathcal{K}_\nu^\mu \\ &\quad - \delta_\nu^\mu (\mathcal{K}_\sigma^\sigma + \tilde{\mathcal{K}}) - \Lambda \delta_\nu^\mu = 0 \end{aligned} \quad (4.3)$$

where $\mathcal{K}_\nu^\mu = \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (\partial^\mu \square_s^l R) (\partial_\nu \square_s^{n-l-1} R)$ and $\tilde{\mathcal{K}} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (\square_s^l R) (\square_s^{n-l} R)$. The trace of the EOM is

$$(M_p^2 - 12\square\mathcal{F}(\square)) R - 2\mathcal{K}_\mu^\mu - 4\tilde{\mathcal{K}} - 4\Lambda = 0. \quad (4.4)$$

The structure of the form factor $\mathcal{F}(\square)$ is determined by finding the second order action around a fixed background and demanding that all perturbations be ghost-free. Note that most of the upcoming computations are carried out covariantly. Solving the EOM Eq.(5.3) in full generality is non-trivial but the following ansatz for the background Ricci scalar helps in making headway [94, 143, 148]

$$\bar{\square} \bar{R} = r_1 \bar{R} + r_2 \quad (4.5)$$

where r_1 and r_2 are dimensionful constant parameters. The ansatz Eq.(5.6) is essentially the trace EOM of local $R + R^2$ theory with a cosmological constant given by $\mathcal{L} = \frac{M_p^2}{2} R + f_0 R^2 - \Lambda$ [94]. The ansatz is solved by bouncing backgrounds with cosine hyperbolic and exponential type scale factors, as we will see soon. Straightforwardly, we then obtain

$$\bar{\square}^n \bar{R} = r_1^n \left(\bar{R} + \frac{r_2}{r_1} \right) \implies \mathcal{F}(\bar{\square}) \bar{R} = \mathcal{F}_1 \bar{R} + \mathcal{F}_2 \quad (4.6)$$

where

$$\mathcal{F}_1 = \mathcal{F}(r_1) \quad \text{and} \quad \mathcal{F}_2 = \frac{r_2}{r_1} (\mathcal{F}_1 - f_0) \quad (4.7)$$

are constants. We now see why the ansatz was useful - it drastically simplifies the EOM with infinite order derivatives. Although the theory is then localized at the level of the background, the perturbations still carry information about the non-local form factor $\mathcal{F}(\Box)$ with infinite order derivatives, as we will soon see. Substituting the ansatz Eq. (5.6) in the EOM Eq. (5.3) and trace EOM Eq. (5.5) gives us the following unique conditions on the form factor $\mathcal{F}(\Box)$ [149]

$$\mathcal{F}_1 = \mathcal{F}'(r_1) = 0, \quad \mathcal{F}_2 = -\frac{M_p^2}{4}, \quad \Lambda = -\frac{M_p^4}{16f_0} \quad (4.8)$$

where $\mathcal{F}'(r_1)$ denotes the derivative of $\mathcal{F}(\Box)$ with respect to $\Box_s$, evaluated at $\Box = r_1$. These background conditions will be crucial in drastically simplifying the second order action around the background which satisfies our chosen ansatz. From Eqs. (4.7) and (5.10), we can deduce that [149]

$$f_0 < 0 \quad \text{for} \quad \Lambda > 0, \quad f_0 > 0 \quad \text{for} \quad \Lambda < 0. \quad (4.9)$$

Note that in Ref. [143, 148], bouncing solutions and linear perturbations were studied in the presence of fluid radiation, which corresponds to the case when $\mathcal{F}_1 < 0$. In this chapter, we emphasize that a bounce can be achieved in the theory satisfying the conditions in Eq. (5.10), without requiring any radiation/matter.

4.2 Perturbations

We now study perturbations and their stability around backgrounds satisfying the ansatz Eq. (5.6) in two, equivalent ways - both at the linearized level of EOM, and at the quadratic level of the action. Linearizing the full EOM Eq. (5.3) and trace EOM Eq. (5.5) around the background given by Eq. (5.6), and substituting the conditions in Eq. (5.10), we obtain, respectively [149]

$$\begin{aligned} \delta E_\nu^\mu = & \left[\left(4\bar{G}_\nu^\mu + 2\delta_\nu^\mu \bar{R} - 4(\bar{\nabla}^\mu \bar{\nabla}_\nu - \delta_\nu^\mu \bar{\Box}) \right) (\bar{\Box} - r_1) - 2(\partial_\nu \bar{R} \partial^\mu + \partial^\mu \bar{R} \partial_\nu) \right. \\ & \left. + 2\delta_\nu^\mu (\partial_\sigma \bar{R} \partial^\sigma + \bar{\Box} \bar{R}) \right] \mathcal{Z}(\bar{\Box}) \zeta = 0 \\ \delta E = & \left[\partial^\mu \bar{R} \partial_\mu + 2(r_1 \bar{R} + r_2) + 3(\bar{\Box} - r_1)^2 + (\bar{R} + 3r_1)(\bar{\Box} - r_1) \right] \mathcal{Z}(\bar{\Box}) \zeta = 0. \end{aligned} \quad (4.10)$$

where

$$\zeta = \delta(\Box) \bar{R} + (\bar{\Box} - r_1) \delta R \quad (4.11)$$

and

$$\mathcal{Z}(\bar{\square}) = \frac{\mathcal{F}(\bar{\square})}{(\bar{\square} - r_1)^2}. \quad (4.12)$$

Note that the perturbed EOM Eq.(4.10) has only one perturbation variable ζ , and we will shortly see that this leads to interesting physics. One may wonder about the absence in Eq.(4.10) of the perturbation of Einstein tensor from GR. This can be easily understood since the following coefficient of Einstein tensor in Eq.(5.3) vanishes upon using the conditions in Eq.(5.10)

$$[M_p^2 + 4\mathcal{F}(\bar{\square})\bar{R}] = 0. \quad (4.13)$$

This shows that there is no term that could produce propagating vector or tensor modes (in the context of 1 + 3 cosmological perturbation theory), at least at the linearized level. From the $i \neq j$ perturbed EOM in Eq.(4.10), we ultimately obtain the simple EOM for ζ

$$\mathcal{Z}(\bar{\square})\zeta = 0. \quad (4.14)$$

To deduce the kinetic structure of ζ and, therefore, the (background dependent) ghost-free form factor $\mathcal{F}(\bar{\square})$, we compute the second variation of the action Eq. (5.1) around Eq. (5.6), imposing further the conditions $\mathcal{F}_1 = 0$, $\mathcal{F}_2 = -\frac{M_p^2}{4}$ [149]

$$\begin{aligned} \delta^2 S = \int d^4x \sqrt{-g} \Bigg\{ & \frac{M_p^2}{4} (\delta_{GR} - \delta^{(2)} R) - \Lambda \left(\frac{h^2}{8} - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} \right) \\ & + \frac{h}{2} \bar{R} \mathcal{F}(\bar{\square}) \delta R + \frac{h}{2} \bar{R} \delta \mathcal{F}(\bar{\square}) \bar{R} + \delta R \delta \mathcal{F}(\bar{\square}) \bar{R} \\ & + \bar{R} \delta \mathcal{F}(\bar{\square}) \delta R + \bar{R} \delta^{(2)} \mathcal{F}(\bar{\square}) \bar{R} \Bigg\} \end{aligned} \quad (4.15)$$

where $\sqrt{-g} \delta_{GR} \equiv \delta^{(2)}(\sqrt{-g}R)$ and $\delta^{(2)}R$ is the second variation of R . After considerable manipulation of terms in Eq.(4.15) (see Appendix (6.I) for some useful formulas), the quadratic action remarkably simplifies to [149]

$$\delta^2 S = \int d^4x \sqrt{-g} \zeta \mathcal{Z}(\bar{\square}) \zeta. \quad (4.16)$$

As expected, at the quadratic level of the action, the presence of the sole variable, ζ , is consistent with our earlier result obtained from the linearized EOM in Eq.(4.14)¹—by varying $\delta^2 S$ with respect to ζ , we exactly recover the linearized EOM for ζ .

¹Our result Eq.(4.16) turns out to be the same as the non-local part of the quadratic action around an inflationary background satisfying $\bar{\square}\bar{R} = r_1\bar{R}$ [144]

To clearly expose the physical degrees of freedom in our theory at the level of the quadratic action, we study SVT modes in the context of (1+3) cosmological perturbation theory [235]. We fix the background $\bar{g}_{\mu\nu}$ to be spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) given by the line element $ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$, $a(t)$ being the scale factor, satisfying the ansatz Eq.(5.6). We write the full metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the perturbation. In longitudinal gauge, $h_{\mu\nu}$ has components

$$h_{00} = a^2(-2\phi), \quad h_{0i} = a^2(\hat{B}_i), \quad h_{ij} = a^2(-2\psi\delta_{ij} + 2\hat{h}_{ij}), \quad (4.17)$$

so that $h_{\mu\nu}$ has 6 degrees of freedom (2S+2V+2T) after gauge fixing² [194]. Note that $\partial^i \hat{B}_i = \partial^i \hat{h}_{ij} = \hat{h}_i{}^i = 0$. Given this decomposition, it is straightforward to see that ζ (defined in Eq.(4.11)) is a linear combination of ϕ and ψ , and therefore is effectively a scalar under $SO(3)$. For a review on (1+3) cosmological perturbation theory, see Appendix (6.J). The quadratic action Eq.(4.16) then solely has one scalar propagating degree of freedom ζ . It is easy to understand why the vector $\hat{B}_i$ and tensor mode $\hat{h}_{ij}$ are absent. The background conditions we imposed, see Eq.(5.10), lead to the removal of vector and tensor modes that could potentially come from the term proportional to M_p^2 in Eq.(4.15). All other terms in $\delta^2 S$ generate only scalars in terms of ϕ and ψ . This confirms the result we had obtained earlier from the linearized EOM.

For the theory to have no extra degrees of freedom or ghosts at the quadratic level of the action in Eq.(4.16), the kinetic operator $\mathcal{Z}(\bar{\square})$ can have at most a single zero according to the Weierstrass product theorem. For the case of $\Lambda > 0$ (see Eq.(4.9)), we therefore make the following minimal choice for $\mathcal{F}(\bar{\square})$ (see Eq.(4.12)) satisfying all the necessary conditions in Eq.(5.10) [149]

$$\mathcal{F}(\bar{\square}) = \frac{1}{M_s^6} (\bar{\square} - m^2) (\bar{\square} - r_1)^2 e^{\gamma(\bar{\square})} \quad (4.18)$$

where $m^2 \geq 0$ is some arbitrary mass scale and γ is an arbitrary entire function of $\bar{\square}/M_s^2$. This choice ensures that the kinetic term for ζ has only one zero at $\bar{\square} = m^2$. Note that the kinetic term for ζ has the correct sign to avoid ghosts. Similarly, for the case of $\Lambda < 0$, the minimal choice for $\mathcal{F}(\bar{\square})$ would be one without any zeros for $\mathcal{Z}(\bar{\square})$ [149]

$$\mathcal{F}(\bar{\square}) = \frac{1}{M_s^4} (\bar{\square} - r_1)^2 e^{\gamma(\bar{\square})}. \quad (4.19)$$

In this case, ζ contains no zeros in the kinetic operator and acts like a p -adic scalar [103].

²The longitudinal gauge removes 2 scalar and 2 vector degrees of freedom from the original total of 10 in $h_{\mu\nu}$

4.3 Isotropic and homogeneous bouncing backgrounds

We now specialize to known bouncing solutions in homogeneous and isotropic FLRW backgrounds [94, 148, 236]. They all satisfy the ansatz Eq.(5.6), are vacuum solutions of the non-local theory Eq.(5.1), and possess only a single ghost-free scalar mode (and no vector or tensor modes) around the bouncing background at the linearized level when the non-local form factor $\mathcal{F}(\bar{\square})$ is chosen appropriately as just discussed

1. Cosine hyperbolic bounce with $a(t) = a_0 \cosh(\sqrt{r_1/2}t)$. In this case, $\Lambda > 0$.
2. Exponential bounce with $a(t) = a_0 e^{\frac{\lambda}{2}t^2}$. In this case, $\Lambda > 0$ for $\lambda > 0$ and $\Lambda < 0$ for $\lambda < 0$. For $\lambda > 0$ we have contraction followed by expansion like in a usual bounce. For $\lambda < 0$, the universe reaches a maximum scale factor at $t = 0$ and then starts to decrease in size. This is also a possible way the Big Bang singularity could be resolved.
3. Cyclic universe: This scenario was numerically studied in [148]. It was found that in the case of $\Lambda < 0$, there exists a cyclic scenario where the universe undergoes multiple bounces.

Let us restrict to the case of $\Lambda > 0$ and a cosine hyperbolic bounce, in which case $\mathcal{F}(\bar{\square})$ is given by Eq.(4.18), and the corresponding quadratic action for ζ reads [149]

$$\delta^2 S = \frac{1}{M_s^6} \int d^4x \sqrt{-g} \zeta e^{\gamma(\bar{\square})} (\bar{\square} - m^2) \zeta. \quad (4.20)$$

For a stable bounce, we would require the perturbation ζ to be well behaved in time. All solutions of the non-local EOM for ζ in Eq.(4.14), for $\mathcal{F}(\bar{\square})$ given in Eq.(4.18), are captured by solutions of the local equation [149]

$$(\bar{\square} - m^2)\zeta = 0 \quad (4.21)$$

because the non-local factor $e^{\gamma(\bar{\square})}$ does not introduce any new zeros in the kinetic operator [94, 103]. Setting $a_0 = 1, r_1 = m^2 = M_s^2$ for the cosine hyperbolic bounce, the generic solution of Eq.(4.21) in Fourier space looks like [149]

$$\zeta_k(z) = c_1 H_k^{(+)}(z) + c_2 H_k^{(-)}(z), \quad (4.22)$$

where

$$H_k^{\pm} = \frac{\sqrt{e^{\sqrt{2}z}}}{1 + e^{\sqrt{2}z}} \left[\frac{\left(1 \pm i \sqrt{e^{\sqrt{2}z}}\right)^2}{1 + e^{\sqrt{2}z}} \right]^{\sqrt{\frac{2k^2}{M_s^2} + 1}}, \quad (4.23)$$

where $z = M_s t$ and $c_{1,2}$ are constants. We can easily deduce that ζ_k is real for $c_1 = c_2$. Here, we do not quantize ζ and merely wish to observe how it evolves classically through the bouncing phase. We have shown the solution ζ_k in Figure 4.1 where we can see that it is oscillatory and bounded for the chosen values of $c_{1,2}$ and k . This behaviour persists to hold for any value of k with increasing number of bounded oscillations as $k \rightarrow \infty$. We do not present here the solutions for ζ around other bouncing backgrounds mentioned earlier since the nature of perturbations remains very similar to that of cosine hyperbolic bounce.

Figure 4.1: Evolution in cosmic time $z = M_s t$ of a Fourier component of the scalar mode ζ_k for cosine hyperbolic bounce. We have set $a_0 = 1$, $r_1 = m^2 = M_s^2$, $c_1 = c_2 = 0.1M_s$, and $k = 20M_s$ (blue curve) and $k = 0$ (red curve). Plot taken from [149].

For the case of cosine hyperbolic bounce, we will have a de Sitter (inflationary) evolution as a late time attractor [94, 143, 192, 233]. We can observe from Fig. 4.1 that $\zeta \rightarrow 0$ in the late time leading to $(\bar{\square} - r_1) \delta R \rightarrow 0$. This observation confirms that we exactly recover the late time behavior of scalar fluctuations found in Ref. [143].

There is also the intriguing possibility of a phase of super inflation after cosine hyperbolic bounce, and before cosmic inflation. This makes this type of bounce an interesting framework for explaining low multipoles in the observed data for cosmic microwave background radiation [237]. It is important to note that the late time de Sitter stage is significantly different in our case compared to [143], because of the absence of any tensor modes due to our background conditions in Eq. (5.10). Note that one can take the GR limit at the level of the action in Eq. (5.1), but not at the level of our background solution satisfying EOM in Eq. (5.3) with the uniquely

tuned conditions in Eq. (5.10)³

To summarize, we have found a simple model of non-singular, homogeneous and isotropic bouncing cosmology in vacuum (but with a cosmological constant) with spatially flat geometry. Moreover, for the background satisfying certain conditions, see Eq. (5.10), there exists only a scalar perturbation at the linearized level, and no propagating vector or tensor modes. This is contrary to the case in GR and non-local gravity in general, which propagate tensor modes. We determined the curious ghost-free form factor $\mathcal{F}(\Box)$ around a dynamical background that leads to only a scalar propagating mode. Moreover, in a cosine hyperbolic bouncing scenario, we have explicitly shown the lone scalar propagating degree of freedom is stable and its evolution is oscillatory and bounded. Of course, these results are valid at the linearized level in perturbation theory and it would be interesting to see what happens at higher orders, which we leave for future discussions.

³It is possible to have special points in the theory space of a gravity model where the physical spectrum differs from the usual expectations - for example, pure R^2 theory propagates a massless scalar and tensor modes around (A)dS background which can be seen by conformally transforming pure R^2 to Einstein frame where the action turns out to be GR with a massless scalar and cosmological constant [206]. However, around flat space, the spectrum contains only a massless scalar and no tensor modes (here, the conformal transformation into Einstein frame is singular). Another example would be Critical Gravity described by $\mathcal{L} = \frac{M_P^2}{2}(R - 2\Lambda + \alpha R^{\mu\nu} R_{\mu\nu} + \beta R^2)$ - here, the tunings $\alpha = -3\beta$ and $\beta = -1/2\Lambda$ remove the massive scalar mode and make the massive spin-2 mode massless [238]. In the context of non-local theories, the theory described by action in Eq. (5.1) permits special classes of (A)dS and non-maximally symmetric covariantly constant curvature backgrounds around which the physical spectrum consists of only a scalar mode, and no tensor or vector modes [159]. In this chapter, we find a special class of dynamical backgrounds where we can have such a physical spectrum with only a scalar mode as a result of Eq. (5.10).

Chapter 5

An Anisotropic and Homogeneous Bouncing Universe

We have seen in the previous chapter that the Big Bang singularity can be avoided from the get-go if we assume that the universe was in a contracting phase before the current expanding phase, going through a bounce in the middle. If the scale factor has a finite value at the bounce point, one avoids encountering a cosmological singularity [214, 239–241]. In GR, for spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) spaces, a bounce inevitably violates the Null Energy Condition (NEC). NEC violation is typically undesirable because of ensuing ghost/gradient instabilities. Another recurrent issue with various bouncing models is the instability of the isotropic and homogeneous FLRW solution with respect to growing spacetime anisotropies. For GR in a Bianchi-I universe, for instance, the effective energy density from anisotropies grows during the contraction phase as a^{-6} , faster than matter, radiation and spatial curvature, eventually dominating and leading the solution into a Big Crunch [242, 243].

The cosmological principle states that our universe is isotropic and homogeneous at large scales. This is also confirmed from observations of the large-scale structure of galaxy distributions and the cosmic microwave background. The fact that we can safely use isotropic and homogeneous FLRW as a good approximation of our universe’s metric forces us to get anxious about any process that might make this assumption invalid. In GR in an expanding universe, spacetime anisotropies decay very quickly like a^{-6} . But for a bouncing cosmological model, this means that anisotropies can grow very quickly during the contraction phase, possibly pushing the universe into a highly anisotropic (and inhomogeneous) state. It is therefore imperative to understand the effects of spacetime anisotropies during the contraction phase of any bouncing model. They should be controlled somehow, else they may clash with the observed isotropy and homogeneity of our universe.

In this chapter, we consider a class of higher derivative, quadratic curvature gravity which aims to provide healthy bouncing solutions. The higher derivative term will be $R\mathcal{F}(\square)R$, where $\mathcal{F}(\square)$ is a power series to infinite order in $\square$ operator. As mentioned in previous chapters, this is a UV non-local generalization of the local

R^2 term, essentially needed to avoid introducing propagating ghost modes (which are common in local higher derivative gravity models)^[1]. The computations in this chapter are general and can be used whether or not $\mathcal{F}(\Box)$ is truncated at a finite order in $\Box$ - something we will use towards the end in comparing local and non-local gravity. We will provide a non-singular bouncing solution in a Bianchi-I universe and show that anisotropies grow as a^{-2} during contraction, much slowly than in GR, and then decrease during the expanding phase which approaches FLRW. We determine the matter sector required to obtain such a bounce and find that, in general, it consists of three components with energy densities decaying like a^{-2} , a^{-4} and a^{-6} . In general, it is necessary to have anisotropic pressures in the a^{-2} component, but we also outline cases where it is allowed to be a perfect fluid [245]. We also find conditions on the function $\mathcal{F}$ by imposing positivity of each component energy density and show that they can be satisfied for non-local gravity. In a local theory, on the other hand, simultaneously satisfying positivity of energy density and absence of instabilities (like tachyons) is not always possible for the same bouncing solution. Our analysis provides an example of a healthy bouncing solution with controlled growth of anisotropies.

The action of non-local generalization of $R + R^2$ theory with a cosmological constant that we will consider is [94]

$$S = \int d^4x \sqrt{-g} \left(\frac{M_p^2}{2} R - \Lambda + R \mathcal{F}(\Box) R + \mathcal{L}_m \right), \quad (5.1)$$

where $\mathcal{L}_m$ is the matter Lagrangian. We define dimensionless $\Box_s \equiv \Box/M_s^2$ and assume the form factor $\mathcal{F}(\Box)$ to be an analytic function of $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$ with a power series expansion

$$\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box_s^n \quad (5.2)$$

for dimensionless coefficients f_n . $M_s (< M_p)$ is a new high energy scale of non-locality at which higher derivative terms in $\mathcal{F}(\Box)$ become important. The rest of the chapter is based on arbitrary $\mathcal{F}(\Box)$. Towards the end, we will compare local and non-local gravity by choosing specific forms of $\mathcal{F}(\Box)$. Varying the action (5.1) gives the following equations of motion (EoM) [143, 199]

$$\begin{aligned} & - [M_p^2 + 4\mathcal{F}(\Box)R] G^\mu_\nu - R\mathcal{F}(\Box)R\delta^\mu_\nu + 4(\nabla^\mu \partial_\nu - \delta^\mu_\nu \Box) \mathcal{F}(\Box)R + 2\mathcal{K}^\mu_\nu - \delta^\mu_\nu (\mathcal{K}^\sigma_\sigma + \tilde{\mathcal{K}}) \\ & - \Lambda \delta^\mu_\nu + T^\mu_\nu = 0 \end{aligned} \quad (5.3)$$

¹Bouncing cosmology has also been studied in the context of Deser-Woodard non-local gravity [244].

where T^μ_ν is the energy-momentum tensor and

$$\mathcal{K}^\mu_\nu = \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (\partial^\mu \square_s^l R) (\partial_\nu \square_s^{n-l-1} R), \quad \tilde{\mathcal{K}} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} (\square_s^l R) (\square_s^{n-l} R). \quad (5.4)$$

The trace EoM is

$$(M_p^2 - 12\square\mathcal{F}(\square))R - 2\mathcal{K}^\mu_\mu - 4\tilde{\mathcal{K}} - 4\Lambda = -T. \quad (5.5)$$

Evidently, solving the EoM (5.3) in general is very non-trivial. However, they can be greatly simplified if the background follows some simplifying (but motivated) assumptions. The complexity of non-local EoM due to an infinite tower of $\square$ operators acting on curvatures can be reduced drastically if we take the background Ricci scalar to satisfy the following ansatz [94, 143]

$$\square R = r_1 R + r_2 \quad (5.6)$$

where $r_{1,2}$ are dimensionful constants. Using this ansatz, we can now reduce the number of $\square$ operators acting on curvatures in the EoM (5.3) since

$$\square^n R = r_1^n \left(\bar{R} + \frac{r_2}{r_1} \right) \quad \text{and} \quad \mathcal{F}(\square)R = \mathcal{F}_1 \bar{R} + \mathcal{F}_2 \quad (5.7)$$

where we have defined

$$\mathcal{F}_1 \equiv \mathcal{F}(r_1) \quad \text{and} \quad \mathcal{F}_2 \equiv \frac{r_2}{r_1} (\mathcal{F}_1 - f_0). \quad (5.8)$$

Substituting the ansatz (5.6) in the EoM (5.3), we obtain a simple expression for the energy-momentum tensor of the theory [143]

$$\begin{aligned} T^\mu_\nu = [M_p^2 + 4(\mathcal{F}_1 R + \mathcal{F}_2)] G^\mu_\nu + \Lambda \delta^\mu_\nu - 2\mathcal{F}'_1 \left[\partial^\mu R \partial_\nu R - \frac{\delta^\mu_\nu}{2} \left(g^{\sigma\rho} \partial_\sigma R \partial_\rho R + r_1 \left(R + \frac{r_2}{r_1} \right)^2 \right) \right] \\ - 4\mathcal{F}_1 [\nabla^\mu \partial_\nu R - \delta^\mu_\nu (r_1 R + r_2)] + \delta^\mu_\nu \left[\mathcal{F}_1 R^2 - \frac{r_2}{r_1} \mathcal{F}_2 \right] \end{aligned} \quad (5.9)$$

where $\mathcal{F}'_1 \equiv \mathcal{F}'(r_1)$ denotes the first derivative of $\mathcal{F}(\square)$ with respect to $\square$, evaluated at $\square = r_1$. In particular, for traceless matter with $T^\mu_\mu = 0$, substituting the ansatz (5.6) in the trace EoM (5.5) gives us the following unique conditions on $\mathcal{F}(\square)$ [143]

$$\mathcal{F}'_1 = 0, \quad \mathcal{F}_2 = -\frac{M_p^2}{4} + 3r_1 \mathcal{F}_1, \quad \Lambda = -\frac{M_p^2}{4} \frac{r_2}{r_1} \quad (5.10)$$

which substituting in (5.9) provides an even simpler expression for the energy-momentum tensor

$$T^\mu_\nu = 4\mathcal{F}_1 \left[(R + 3r_1)G^\mu_\nu - \nabla^\mu \partial_\nu R + \frac{1}{4}\delta^\mu_\nu (R^2 + 4r_1 R + r_2) \right]. \quad (5.11)$$

Everything discussed until now is covariant, that is, we did not choose any specific metric. In this chapter, we will consider cosmological spacetimes, particularly bouncing scenarios. As a specific example, for spatially flat, homogeneous and isotropic FLRW, the scale factor $a(t) = a_0 \cosh(\sqrt{\frac{r_1}{2}}t)$ describes a bouncing universe which solves the ansatz (5.6) with $r_2 = -6r_1^2$ [94, 143, 148, 234]. It is therefore also a solution of the EoM (5.3) of quadratic curvature non-local gravity (5.1). In the case of traceless matter, this cosh bounce is an exact solution of the EoM (5.11) with an energy density arising from radiation and given by $\rho_r = -27\mathcal{F}_1 r_1^2$, which implies $\mathcal{F}_1 < 0$ if we want positive energy density $\rho_r > 0$ for stability [143]. In a local $R + f_0 R^2 - \Lambda$ theory, $\mathcal{F}_1 = f_0$ and $f_0 < 0$ leads to a tachyonic instability [246]. In non-local gravity, on the other hand, it is possible to fix $\mathcal{F}_1 < 0$ without leading to any instabilities. Note that the cosh solution approaches dS asymptotically at early (as $t \rightarrow -\infty$) and late times (as $t \rightarrow +\infty$), with a constant $R = 6r_1$. Another interesting bouncing solution (in spatially flat FLRW) of the ansatz (5.6), and hence of quadratic curvature non-local gravity (5.1), is $a(t) = a_0 e^{\frac{\lambda}{2}t^2}$ with $r_1 = -6\lambda$ and $r_2 = 12\lambda^2$ [234].

5.1 An anisotropic and homogeneous bouncing background

As just mentioned, cosmological bouncing solutions which satisfy the ansatz (5.6) are already known when the background is homogeneous and isotropic FLRW. To test the robustness of these solutions, we want to see how anisotropies operate within the framework of the same ansatz. For this purpose, we choose a homogeneous but anisotropic Bianchi-I background [247]

$$ds^2 = -dt^2 + a^2 [e^{2\beta_1} dx_1^2 + e^{2\beta_2} dx_2^2 + e^{2\beta_3} dx_3^2] \quad (5.12)$$

where $a = a(t)$ is the average scale factor (geometric mean of scale factors in the three spatial directions), and where deviation from FLRW is parameterized by $\beta_i = \beta_i(t)$ which satisfy

$$\beta_1 + \beta_2 + \beta_3 = 0. \quad (5.13)$$

A quantity called shear σ^2 defined as

$$\sigma^2 = \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2 \quad (5.14)$$

essentially parameterizes the extent of anisotropic deviation of the background away from FLRW. The Ricci scalar for the anisotropic metric (5.12) is [150]

$$R = 12H^2 + 6\dot{H} + \sigma^2 = \bar{R} + \sigma^2. \quad (5.15)$$

where $H^2 = \dot{a}/a$ and $\bar{R}$ is the Ricci scalar for an FLRW metric with scale factor a . The $\square$ operator for the Bianchi-I metric (5.12) acting on a time-dependent scalar is

$$\square = -\frac{\partial^2}{\partial t^2} - 3H\frac{\partial}{\partial t}, \quad (5.16)$$

which is exactly the same as in FLRW.

Let us first review the issue of anisotropies in bouncing cosmology in GR [239, 248]. Friedmann equations in a Bianchi-I background [5.12] are [239]

$$\begin{aligned} H^2 &= \frac{1}{3}(\rho + \rho_\beta) \\ \dot{H} &= -\frac{1}{2}(\rho + p) - \frac{1}{2}\sigma^2 \end{aligned} \quad (5.17)$$

where $\rho_\beta = \sigma^2/6$ is the effective energy density coming from spacetime anisotropies. Using the above equations, and the conservation equation $\dot{\rho}/\rho = -3(1+w)H$, one can show that $\rho_\beta \propto a^{-6}$. Upon using the condition [5.13] one can further show that the anisotropic factor β_i follows a second order differential equation $\ddot{\beta}_i + 3H\dot{\beta}_i = 0$. The evolution of the universe as driven by various matter sectors is exemplified by the Friedmann equation [239]

$$H^2 = \frac{1}{3} \left[-\frac{3k}{a^2} + \frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4} + \frac{\rho_{\beta 0}}{a^6} + \cdots + \frac{\rho_{\phi 0}}{a^{3(1+w_\phi)}} \right] \quad (5.18)$$

where $\rho_{m0}, \rho_{r0}, \rho_{\beta 0}$ represent the energy densities of matter, radiation and spacetime anisotropy at the bounce point. We see that anisotropies growing like a^{-6} during the contraction phase ultimately dominate over spatial curvature, matter and radiation, unless one adds a hypothetical scalar field ϕ whose equation of state $w_\phi \gg 1$ is ultra-stiff as in the Ekpyrotic scenario [216, 249, 250]. In the absence of such an ekpyrotic scalar field, the energy density comes to be dominated by anisotropies. Neglecting matter, radiation and spatial curvature in this regime, the scale factor is solved to be $a \sim (-t)^{1/3}$, where t is negative and increasing. This scale factor gives us $a^2 e^{2\beta_i} = (-t)^{2p_i}$ which is nothing but the singular Kasner metric with some

²Note that this H is the average of the three Hubble factors in the three spatial directions. Particularly, from the metric (5.12), we can define the three scale factors in the three spatial directions as $a_i = ae^{\beta_i}$. The three Hubble factors in the three spatial directions are then $H_i = H + \dot{\beta}_i$. The average $H = \frac{1}{3}(H_1 + H_2 + H_3)$.

constants $p_i = \frac{1}{3} + \frac{b_i}{\sqrt{3}\theta_0}$ where $\theta_0 = \frac{1}{2}(b_1^2 + b_2^2 + b_3^2)$ and $\dot{\beta}_i = b_i/a^3$. This recovers the usual BKL result. We wish to avoid this in our higher derivative model of [5.1](#)

In this chapter, we generalize the known cosh bounce in quadratic curvature gravity [\(5.1\)](#) in FLRW to Bianchi-I backgrounds [\(5.12\)](#). We take the average scale factor to be cosh [\[94, 143, 148, 234\]](#)

$$a = a_0 \cosh \left(\sqrt{\frac{r_1}{2}} t \right), \quad (5.19)$$

where a_0 is the average scale factor at the bounce point $t = 0$. Henceforth, we choose a convenient normalization of $a_0 = 1$ so that the minimum value of the average scale factor is 1. Just as in FLRW, we assume the following ansatz to hold for the Bianchi-I metric [\(5.12\)](#)

$$\square R = s_1 R + s_2 \quad (5.20)$$

where $s_{1,2}$ are some dimensionful constants. Substituting [\(5.15\)](#) in [\(5.20\)](#), we obtain $s_1 = r_1, s_2 = r_2$ (where $r_{1,2}$ are the dimensionful constants in FLRW cosh bounce with $\square \tilde{R} = r_1 \tilde{R} + r_2$) and the following second order differential equation [\[150\]](#)

$$\square \sigma^2 = s_1 \sigma^2 \quad (5.21)$$

which can be solved to obtain the shear as [\[150\]](#)

$$\sigma^2 = \sigma_0^2 \operatorname{sech}^2 \left(\sqrt{\frac{r_1}{2}} t \right) = \frac{\sigma_0^2}{a^2} \quad (5.22)$$

subject to boundary conditions $\sigma^2|_{t=0} = \sigma_0^2$ and $\frac{d}{dt}(\sigma^2)|_{t=0} = 0$. The first condition is normalization, while the second is required to obtain positive σ^2 .

In GR, shear goes like $\sigma^2 \sim a^{-6}$, and the effective energy density coming from the anisotropic background $\rho_\sigma = \sigma^2/2$ therefore dominates over matter, radiation and spatial curvature when the scale factor a becomes very small [\[239\]](#). This leads the geometry during the contraction phase into a Kasner solution, and also causes BKL instability [\[242\]](#). On the other hand, in a cosh bounce [\(5.19\)](#) satisfying the ansatz [\(5.20\)](#) in a Bianchi-I universe [\(5.12\)](#), $\sigma^2 \sim a^{-2}$ increases much slower in comparison during contraction, reaches a maximum at the bounce point $t = 0$, and then decreases during the expansion phase (and, moreover, exponentially as $\sim e^{-2t}$ at late times).

From [\(5.14\)](#) and [\(5.22\)](#), we obtain (for $i = 1, 2, 3$) [\[150\]](#)

$$\dot{\beta}_i = \sigma_0 b_i \operatorname{sech} \left(\sqrt{\frac{r_1}{2}} t \right) \quad \text{or} \quad \beta_i = \sqrt{\frac{2}{r_1}} \sigma_0 b_i \operatorname{gd} \left(\sqrt{\frac{r_1}{2}} t \right) \quad (5.23)$$

where b_i are dimensionless coefficients and $\operatorname{gd}(t)$ is the Gudermannian function which is defined as $\operatorname{gd}(t) = \int_0^t \frac{dx}{\cosh(x)}$.

Figure 5.1: Time evolution of anisotropy factors $\beta_i(t)$ and $\dot{\beta}_i(t)$ given in (5.23), and the Hubble factors in the three spatial directions $H_i = H + \beta_i$. We have chosen $r_1 = 0.1$ and $\sigma_0 = 0.1$ (in Planck units). In the non-local theory (5.1), we may take $M_s^2 = r_1$. We chose $(b_1, b_2, b_3) = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$ in accordance with (5.24). In the middle plot, we see that the magnitude of background anisotropy (in each spatial direction $i = 1, 2, 3$) reaches a maximum at the bounce point $t = 0$, and then asymptotes to zero before and after, as the universe asymptotes to FLRW (and eventually to dS, as can be seen from the bottom plot where H_i approach constant values). Plots here are a slight variation of those in [150].

Any constants of integration can be absorbed into coordinate redefinitions. From (5.13) and (5.14), we get the following constraints

$$b_1 + b_2 + b_3 = 0, \quad b_1^2 + b_2^2 + b_3^2 = 1 \quad (5.24)$$

which admit an infinite number of solutions for the triplet (b_1, b_2, b_3) . Background anisotropy is quantified by $|\dot{\beta}_i|$, which from (5.23) becomes maximum at the bounce point $t = 0$, and goes to zero asymptotically (as $\sim e^{-t}$) as $t \rightarrow \pm\infty$ (see Fig. (5.1)). The universe therefore approaches isotropic and homogeneous FLRW asymptotically as $t \rightarrow \pm\infty$, along with entering a dS phase.

5.1.1 Matter sources for anisotropic bounces

Having completely fixed the anisotropic bouncing background in the previous section, we now determine the corresponding possible matter sectors which can drive such a non-singular evolution of the universe (5.19) with controlled growth of anisotropies (5.23). We show how shear anisotropy σ^2 effectively contributes to fluid energy densities and pressures. We also determine the conditions where all component energy densities are non-negative, ensuring a physically healthy bounce mechanism. Moreover, we also show that under certain conditions, it is possible to have a bounce in vacuum (but with Λ).

Without loss of generality, the energy-momentum tensor T^μ_ν for quadratic curvature non-local gravity (5.1) around a Bianchi-I cosmological background (5.12) has the diagonal form

$$T^\mu_\nu = (-\rho, p_1, p_2, p_3) \quad (5.25)$$

where p_i (for $i = 1, 2, 3$) are pressures in x, y, z directions. We will soon see that anisotropic pressures necessarily arise in some (but not all) cases of possible matter sectors. For a perfect fluid, $p_1 = p_2 = p_3$ and the energy density of each component (labeled by I) evolves as

$$\rho_I = \rho_{I0} a^{-3(1+w_I)} \quad (5.26)$$

where $I = \text{dust, radiation, or something else}$. ρ_{I0} denotes the energy density of component I at the bounce point. The zero component of energy-momentum conservation $\nabla_\mu T^\mu_\nu = 0$ in a Bianchi-I background (5.12) takes the form

$$\dot{\rho} + 3H\rho + H(p_1 + p_2 + p_3) + \dot{\beta}_1 p_1 + \dot{\beta}_2 p_2 + \dot{\beta}_3 p_3 = 0. \quad (5.27)$$

We now find the energy-momentum tensor responsible for the bounce. We use the EoM in (5.9) and substitute in it the cosh bouncing solution in a Bianchi-I background (5.12), (5.19), (5.22) and (5.23) to obtain an energy density with three components [150]

$$\rho = \frac{\rho_{k0}}{a^2} + \frac{\rho_{r0}}{a^4} + \frac{\rho_{s0}}{a^6} \quad (5.28)$$

where [150]

$$\begin{aligned}\rho_{k0} &= -\frac{1}{2}\sigma_0^2 (60\mathcal{F}_1 r_1 + 4\mathcal{F}_2 + M_P^2) - \frac{3r_1}{2} (-12\mathcal{F}_1 r_1 + 4\mathcal{F}_2 + M_P^2) \\ \rho_{r0} &= -3(\sigma_0^2 - 3r_1)^2 (\mathcal{F}_1 + \mathcal{F}_1' r_1) \\ \rho_{s0} &= 2\mathcal{F}_1' r_1 (\sigma_0^2 - 3r_1)^2\end{aligned}\quad (5.29)$$

where the first component is ‘ k -matter’ labeled by ‘ k ’ whose energy density scales as a^{-2} (just like spatial curvature in GR), second is radiation labeled by ‘ r ’, and the last is stiff matter labeled by ‘ s ’ whose energy density scales as a^{-6} . We see that anisotropic shear, parameterized by σ_0 , effectively contributes to all the three matter sectors. Note that the cosmological constant Λ is fixed to be [150]

$$\Lambda = \frac{3}{2}M_P^2 r_1 \quad (5.30)$$

in order to absorb the constant term in ρ . Pressure is anisotropic in general (due to k -matter) and given by (in each spatial direction $i = 1, 2, 3$) [150]

$$p_i = \left(\frac{p_{k0}}{a^2} + \pi_i^i \right) + \frac{p_{r0}}{a^4} + \frac{p_{s0}}{a^6} \quad (5.31)$$

where [150]

$$\begin{aligned}p_{r0} &= w_r \rho_{r0} \quad \text{with} \quad w_r = 1/3, & p_{s0} &= w_s \rho_{s0} \quad \text{with} \quad w_s = 1, \\ p_{k0} &= w_k \rho_{k0} \quad \text{with} \quad w_k = \frac{-\frac{1}{2}\sigma_0^2 (12\mathcal{F}_1 r_1 + 4\mathcal{F}_2 + M_P^2) + \frac{r_1}{2} (-12\mathcal{F}_1 r_1 + 4\mathcal{F}_2 + M_P^2)}{-\frac{1}{2}\sigma_0^2 (60\mathcal{F}_1 r_1 + 4\mathcal{F}_2 + M_P^2) - \frac{3r_1}{2} (-12\mathcal{F}_1 r_1 + 4\mathcal{F}_2 + M_P^2)},\end{aligned}\quad (5.32)$$

and the k -matter anisotropic pressure π_i^i (in each spatial direction $i = 1, 2, 3$) is given by [150]

$$\pi_i^i = \pm \sqrt{2r_1} b_i \sigma_0 (4\mathcal{F}_2 + M_P^2 + 24\mathcal{F}_1 r_1) \sqrt{\frac{1}{a^2} - \frac{1}{a^4}} \quad (+ \text{ for } t > 0, - \text{ for } t < 0) \quad (5.33)$$

where there is no summation over the index i in π_i^i . The appearance of π_i^i is a consequence of purely $\dot{\beta}_i$ and $\ddot{\beta}_i$ terms in the EoM³. The σ^2 -dependent contributions to the energy density and pressure, on the other hand, are isotropic in all spatial directions. This shows that, in general, the matter sector consists of two perfect fluids

³In GR, one obtains a second order differential equation for β_i by using the Friedmann equations and the constraint on β_i (5.13) in the presence of a perfect fluid matter sector. Essentially, the purely β_i -dependent part in $T_i^i (= G_i^i)$ put to zero gives the sought-after differential equation. In our case, we already determined β_i (5.23) by solving the Ricci scalar ansatz (5.20) for a fixed average scale factor (5.19) in a Bianchi-I background. However, this β_i does not put the purely β_i -dependent part in T_i^i (5.9) to zero, as is evident from non-zero π_i^i (5.33).

(radiation and stiff matter) and one fluid with anisotropic pressures (k -matter). Note from (5.32) that k -matter, in general, does not have the correct, constant equation of state $w_k = -1/3$ for it to be a perfect fluid, necessitating anisotropic pressures π^i_i in order to ensure energy-momentum conservation (5.27). In particular, we have the energy-momentum tensors of perfect fluid and k -matter as, respectively [251]

$$T^I_{\mu\nu} = (p_I + \rho_I)u_\mu u_\nu + p_I g_{\mu\nu}, \quad T^k_{\mu\nu} = (p_k + \rho_k)u_\mu u_\nu + p_k g_{\mu\nu} + \pi_{\mu\nu} \quad (5.34)$$

where $I = r$ or s , and u_μ is the velocity vector. The anisotropic pressure contribution $\pi^\mu_\nu = (0, \pi^1_1, \pi^2_2, \pi^3_3)$ is traceless $\pi^\mu_\mu = 0$ and transverse to the velocity vector $\pi_{\mu\nu}u^\mu = 0$. Background anisotropy, parameterized by σ_0 , does not introduce any new kinds of (effective) matter apart from those already present in the known FLRW cosh bouncing model. The FLRW limit can be taken by putting $\sigma_0 \rightarrow 0$ in (5.29), (5.32) and (5.33) [94]. We enumerate all possible matter sectors in Table (5.1), where the constraints on $\mathcal{F}_1, \mathcal{F}'_1, \mathcal{F}_2$ and σ_0 are determined by requiring that the energy density of each matter component be non-negative - $\rho_{k0} \geq 0$, $\rho_{r0} \geq 0$ and $\rho_{s0} \geq 0$.

In some special cases, however, upon constraining the parameters of the theory and solution, it is possible to turn k -matter into a perfect fluid. From (5.33), we see that imposing a constraint [150]

$$4\mathcal{F}_2 + M_P^2 + 24\mathcal{F}_1 r_1 = 0 \quad (5.35)$$

makes anisotropic pressures vanish $\pi^i_i = 0$, and recovers the correct, constant equation of state for k -matter $w_k = -1/3$, leaving us with three perfect fluids in the matter sector. Note that (for $\mathcal{F}_1 \neq 0$) it is impossible to satisfy both the perfect fluid constraint (5.35) and the second condition in (5.10) for traceless matter. This is another way of saying that a traceless matter sector driving an anisotropic cosh bounce must necessarily have anisotropic pressures (in the k -matter component). In Appendix (6.K), we determine the potential for k -matter, assuming that it can be described by a perfect fluid scalar field.

Consider as an example the most general case when all three matter components (k, r, s) are present in the model. A priori, $\mathcal{F}_1, \mathcal{F}'_1, \mathcal{F}_2, r_1$ and σ_0 are independent of each other. Relations between them are derived by invoking physical principles like positivity of energy density. Imposing this positivity for each component in (5.29), we obtain the following conditions to be satisfied by $\mathcal{F}_1, \mathcal{F}'_1$ and $\mathcal{F}_2$ (see also Table (5.1)) [150]

$$\mathcal{F}'_1 > 0, \quad \mathcal{F}_1 < -r_1 \mathcal{F}'_1 < 0, \quad \mathcal{F}_2 < -\frac{M_P^2}{4} + 3\mathcal{F}_1 r_1 \left(1 - \frac{6\sigma_0^2}{3r_1 + \sigma_0^2}\right). \quad (5.36)$$

Upon imposing the perfect fluid condition on $\mathcal{F}_2$ in (5.35), the energy density of k -matter at the bounce point becomes [150]

$$\rho_{k0} = 18\mathcal{F}_1 r_1 (3r_1 - \sigma_0^2) \quad (5.37)$$

which in turn implies a lower bound on background anisotropy $\sigma_0^2 > 3r_1$ if we wish to have $\rho_{k0} > 0$. No such bound on σ_0 exists in the general case with two perfect radiation and stiff matter fluids and one anisotropic k -matter fluid.

Let us go back to the most general case with three fluids (5.29) and (5.36). Interestingly, for the specific value $\sigma_0^2 = 3r_1$, radiation and stiff matter vanish, leaving only anisotropic k -matter with the following energy density at the bounce point [150]

$$\rho_{k0} = -\sigma_0^2(4\mathcal{F}_2 + M_P^2 + 8\mathcal{F}_1\sigma_0^2) \quad (5.38)$$

which implies

$$\mathcal{F}_2 < -\frac{M_P^2}{4} - 2\mathcal{F}_1\sigma_0^2 \quad (5.39)$$

if we require $\rho_{k0} > 0$. Moreover, if we now impose the perfect fluid condition (5.35) on $\mathcal{F}_2$ with this fixed value $\sigma_0^2 = 3r_1$, the energy densities of all fluids vanish and we obtain an anisotropic bounce in vacuum (but with Λ) (see Table (5.1)).

There exists another set of conditions where the bounce is in vacuum, but now with unconstrained σ_0^2 . Consider again the most general case (5.29). We fix $\mathcal{F}'_1 = \mathcal{F}_1 = 0$ to remove stiff matter and radiation from the model, and obtain [150]

$$\rho_{k0} = -\frac{1}{2}(4\mathcal{F}_2 + M_P^2)(3r_1 + \sigma_0^2) \quad (5.40)$$

which implies that $\mathcal{F}_2 = -\frac{M_P^2}{4}$ if we want to have a bounce in vacuum. This vacuum solution is the same one we would obtain if we start from a traceless matter theory, already satisfying the tracelessness conditions in (5.10), and then put $\mathcal{F}_1 = 0$ in (5.11).

These two vacuum solutions identify with each other for one specific tuning $\mathcal{F}'_1 = \mathcal{F}_1 = 0$, $\mathcal{F}_2 = -\frac{M_P^2}{4}$ and $\sigma_0^2 = 3r_1$. They are mutually exclusive otherwise.

It is instructive to mention the cosh bouncing model (5.19) in a Bianchi-I background (5.12) when the matter sector is traceless. For this purpose, we substitute the tracelessness conditions (5.10) in the most general EoM (5.9) to obtain the simplified EoM (5.11). In FLRW, the only matter component is radiation [148]. In a Bianchi-I background, however, we see that background anisotropy σ_0 introduces k -matter, effectively. Upon imposing the tracelessness conditions (5.10) in the general expression for ρ_{k0} and p_{k0} in (5.29) and (5.32), we obtain $\rho_{k0} = -36\mathcal{F}_1 r_1 \sigma_0^2$ and $w_k = 1/3$. Note that the condition $\mathcal{F}'_1 = 0$ in (5.10) removes stiff fluid from the model. The

a^{-6}	a^{-4}	a^{-2}	$\mathcal{F}_1$	$\mathcal{F}'_1$	$\mathcal{F}_2$	Perfect fluid possible?
✓	✓	✓	$< -r_1 \mathcal{F}'_1 < 0$	> 0	$< -\frac{M_P^2}{4} + 3\mathcal{F}_1 r_1 \left(1 - \frac{6\sigma_0^2}{3r_1 + \sigma_0^2}\right)$	if (5.35) is true; $\sigma_0^2 > 3r_1$
✗	✓	✓	< 0	0	$< -\frac{M_P^2}{4} + 3\mathcal{F}_1 r_1 \left(1 - \frac{6\sigma_0^2}{3r_1 + \sigma_0^2}\right)$	if (5.35) is true; $\sigma_0^2 > 3r_1$
✓	✗	✓	$-r_1 \mathcal{F}'_1 < 0$	> 0	$< -\frac{M_P^2}{4} + 3\mathcal{F}_1 r_1 \left(1 - \frac{6\sigma_0^2}{3r_1 + \sigma_0^2}\right)$	if (5.35) is true; $\sigma_0^2 > 3r_1$
✓	✓	✗	$< -r_1 \mathcal{F}'_1 < 0$	> 0	$-\frac{M_P^2}{4} + 3\mathcal{F}_1 r_1 \left(1 - \frac{6\sigma_0^2}{3r_1 + \sigma_0^2}\right)$	always
✗	✗	✓	0	0	$< -\frac{M_P^2}{4}$	never
✗	✓	✗	< 0	0	$-\frac{M_P^2}{4} + 3\mathcal{F}_1 r_1 \left(1 - \frac{6\sigma_0^2}{3r_1 + \sigma_0^2}\right)$	always
✓	✗	✗	$-r_1 \mathcal{F}'_1 < 0$	> 0	$-\frac{M_P^2}{4} + 3\mathcal{F}_1 r_1 \left(1 - \frac{6\sigma_0^2}{3r_1 + \sigma_0^2}\right)$	always
✗	✗	✓	-	-	$< -\frac{M_P^2}{4} - 2\mathcal{F}_1 \sigma_0^2$	never; $\sigma_0^2 = 3r_1$
✗	✗	✗	-	-	see (5.35)	vacuum; $\sigma_0^2 = 3r_1$
✗	✗	✗	0	0	$-\frac{M_P^2}{4}$	vacuum

Table 5.1: Possible matter sectors for a bounce in a Bianchi-I background for the quadratic curvature theory (5.1) following the EoM in (5.9) and the background Ricci scalar ansatz (5.20). At the top of the first three columns, a^{-6} represents stiff matter, a^{-4} represents radiation and a^{-2} represents k -matter. Inequalities arise from imposing positivity of energy density of each matter component $\rho_{k0}, \rho_{r0}, \rho_{s0} > 0$ in (5.29). Note that the inequality for $\mathcal{F}_2$ can be rewritten as an inequality for f_0 instead, using the definition of $\mathcal{F}_2$ in (5.8). A hyphen ‘-’ indicates that the value is unconstrained. The value of background anisotropy σ_0 is unconstrained, unless mentioned otherwise. The last column enumerates the cases when k -matter (and therefore the entire matter sector) can be a perfect fluid. Table taken from [150].

equation of state corresponds to radiation, even though the fluid has an energy density which decays like a^{-2} . This necessitates anisotropic pressures π_i^i in order to conserve energy-momentum (5.27), which are given by [150]

$$\pi_i^i = \pm 36b_i \mathcal{F}_1 \sigma_0 \sqrt{2r_1^3} \sqrt{\frac{1}{a^2} - \frac{1}{a^4}} \quad (+ \text{ for } t > 0, - \text{ for } t < 0). \quad (5.41)$$

5.1.2 Local versus non-local gravity

All computations until now are general, in the sense that they are applicable for both local and non-local quadratic curvature gravity. In particular, the form factor $\mathcal{F}(\Box)$ in the gravity action (5.1) was arbitrary, and we did not yet use the fact that it contains an infinite number of derivatives in an expansion like (5.2). We can now use our general results in the previous sections, apply them to specific forms of $\mathcal{F}(\Box)$, and compare local and non-local quadratic curvature gravity.

What we aim to show is that non-local gravity admits bouncing solutions with both positive matter energy density, and no instabilities (ghost/tachyonic). Both these criteria may not be satisfied simultaneously in a local theory of gravity. Since our cosh bouncing solution (5.19) is dynamical, we will look at its late time behavior instead for simplicity. At late times, the Ricci scalar for a cosh bouncing background goes to a constant given by $R = 6r_1$. Since $r_1 > 0$, this corresponds to dS. The correct form of $\mathcal{F}(\square)$ which avoids the introduction of propagating ghost or tachyonic degrees of freedom was already derived for maximally symmetric spacetimes in [111], and is given by

$$\mathcal{F}(\square) = \frac{1 + \frac{24r_1}{M_P^2} f_0 - e^{\alpha(\square_s)}}{\frac{24r_1}{M_P^2} \left(1 + \frac{\square}{2r_1}\right)} \quad (5.42)$$

where $\alpha(\square_s)$ is some entire function of $\square$ operator. This choice of $\mathcal{F}(\square)$ ensures that the spin-2 quadratic action has no new zeros apart from the one present in the GR limit. Moreover, this choice introduces no new zeros in the spin-0 quadratic action, and so there is no Brans-Dicke type scalar. The above choice of ghost-free $\mathcal{F}(\square)$ can satisfy the conditions required to ensure positive energy densities of all matter components driving a cosh bouncing universe (5.19) in a Bianchi-I background (5.12). For concreteness, we assume the simplest choice of $\alpha(\square_s) = \square/M_s^2$. Let us consider the case where all three matter components (k -matter, radiation and stiff matter) are present in the model. For $\rho_{k0} > 0$, $\rho_{r0} > 0$ and $\rho_{s0} > 0$, we need to satisfy the conditions in (5.36) (see also first row in Table (5.1)). It is easy to verify that these conditions can be satisfied for a range of (f_0, r_1, σ_0) values. One such choice is, for instance (in Planck units) $f_0 = -1698.43$, $r_1 = M_s^2 = 0.01$ and $\sigma_0 = 0.01$ ([150]).

Now consider a fourth order local theory where the form factor $\mathcal{F}(\square)$ is truncated at a finite order in derivatives, like $\mathcal{F}(\square) = f_0$. Now, we can again satisfy the conditions for positivity of energy density in the sixth row in Table (5.1), for which $\mathcal{F}_1 = f_0 < 0$, $\mathcal{F}'_1 = 0$ and $\mathcal{F}_2 = 0$ ([150]). Then, for a chosen value of σ_0^2 , these can be satisfied (in Planck units) for $0 < r_1 < \frac{5\sigma_0^2}{3}$ and $\mathcal{F}_1 = \frac{3r_1 + \sigma_0^2}{36r_1^2 - 60r_1\sigma_0^2}$. However, $f_0 < 0$ signals the presence of a tachyonic instability [246]; we cannot simultaneously satisfy positivity of energy density and absence of tachyonic instability in this example local theory.

We have shown that non-local quadratic curvature gravity admits a healthy bouncing solution in an anisotropic but homogeneous Bianchi-I background, sourced by a positive cosmological constant and positive matter energy densities. It also does not propagate ghost/tachyonic modes in the late time dS phase. Spacetime anisotropies grow slower than in GR during the contraction phase, peak to a finite value at the bounce point, and then decrease during the expansion phase as the

universe asymptotes to FLRW (and eventually dS). Driving a bounce in a Bianchi-I background necessarily requires anisotropic pressures in the k -matter sector, unless the form factor $\mathcal{F}(\square)$ in the action obeys a certain constraint. One could now compute perturbations and check if they are free from any instabilities, especially around the dynamical bouncing background. Also, it would be interesting to see what the corresponding matter sector looks like for an exponential bounce in Bianchi-I spacetimes where $a(t) \sim e^{t^2}$, another known solution of the ansatz (5.6) used to simplify the EoM (5.3). We leave these questions for future work.

Chapter 6

Conclusions

In this thesis, we have seen how UV non-local gravity can provide gravitational and cosmological solutions which avoid the usual problems of GR like singularities, and simultaneously keep instabilities at bay. In the second chapter, we saw how the gravitational response to a charged point source, with a regular potential and curvatures, provides a glimpse into how singularities could be regularized at the non-linear level. In the third chapter, we saw how even some non-trivial backgrounds in non-local gravity are stable with respect to perturbations. In the fourth chapter, we provided a toy model of a stable non-singular bouncing universe. The sole scalar degree of freedom around this dynamical bouncing background can be made free from ghost/tachyonic instabilities, and is bounded throughout the history of the universe. In the fifth chapter, we showed how spacetime anisotropies do not endanger the stability of the isotropic and homogeneous bouncing solution in non-local gravity. Any preexisting anisotropy reaches a peak at the bounce point, before being rapidly washed out as the universe asymptotes to isotropic and homogeneous FLRW, and ultimately to dS, at late times.

These results are motivating, but one must not stop here. There are many open questions in non-local field theories and gravity. There are many solutions to linearized equations of motion of non-local gravity, but full non-linear solutions are elusive. Gravity with its diffeomorphism invariance is non-linear and it is crucial to understand how gravitons self-interact in non-local gravity. There are models of bouncing cosmology in non-local gravity, like the ones presented in chapters 4 and 5, but they cannot yet be compared with the universe we live in. For completeness, it must be ensured that the non-singular bounce mechanism also resolves the standard issues of Big Bang cosmology like horizon and flatness problems, recovers standard radiation and matter domination eras before the current dark energy-dominated era. Moreover, we considered bouncing models with a non-local R^2 term, where R is the Ricci scalar. There is no reason to not include in the action other possible higher derivative non-local operators in terms of the Ricci or Weyl tensor. Also, throughout the thesis, we used perturbation theory within the formalism of background field method where there exists a fixed background. It is imperative that non-local gravity also be written in a way where background independence is manifest. A

full, non-perturbative analysis is needed here.

The fact that no other theory has been able to supersede GR since its inception more than a century ago is a testament to how robust it is. Einstein's profound insight into the fundamental nature of gravity continues to guide us as we look for ways to explain GR's own pitfalls. GR is currently going through another stringent test - with the advent of gravitational wave astronomy, we now have a new window into the strong-field, non-linear behavior of gravity. Will we observe any deviations from GR? If we do, it would perhaps be one of the most significant discoveries in modern physics. Perhaps we need a radically new idea and a very high energy/clever experiment to explain and measure physics at the Planck scale. However, this need not stop us from trying to devise approaches that bring us closer to the fundamental nature of gravity since the ultimate answer will be an amalgamation of all the ideas tried so far. This thesis is based on one such idea.

Appendices

6.A Reissner-Nordström metric in GR

In this appendix we wish to briefly review the main aspects of the Reissner-Nordström metric in GR in isotropic coordinates. The Reissner-Nordström metric is a non-vacuum solution of the full non-linear Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\tau_{\mu\nu}, \quad (6.1)$$

where $\tau_{\mu\nu}$ is the electro-magnetic energy momentum tensor

$$\tau_{\mu\nu} = \frac{1}{4\pi} \left(\eta_{\rho\nu} F_{\mu\sigma} F^{\rho\sigma} - \frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right). \quad (6.2)$$

Assuming that no magnetic monopole is present, the only components of the field strength $F_{\mu\nu}$ will be given by the electric field part; then, due to spherical symmetry there will be a radial dependence so that the only non-vanishing components are $F_{10} = -F_{01} = Q/r^2$. In Schwarzschild coordinates (t, R, θ, φ) , where R is the polar radial coordinate, the metric reads

$$ds^2 = - \left(1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2} \right) dt^2 + \left(1 - \frac{2Gm}{R} + \frac{GQ^2}{R^2} \right)^{-1} dR^2 + R^2 d\Omega^2, \quad (6.3)$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$, m and Q are the mass and charge of the source, respectively.

One can immediately notice that the Reissner-Nordström metric possesses *two* horizons that are defined as:

$$r_{\pm} = Gm \pm \sqrt{G^2 m^2 - GQ^2}, \quad (6.4)$$

where r_+ is called *outer* horizon, while r_- *inner* horizon.

We are now interested in finding the corresponding weak-field approximation for the metric in Eq. (6.3) and to do so, as it happens for the Schwarzschild metric, it is more convenient to use isotropic coordinates (t, r, θ, φ) , where r is the isotropic radial coordinate that in the case of Reissner-Nordström metric is defined by the following transformation:

$$R = r \left[\left(1 + \frac{Gm}{2r} \right)^2 - \frac{GQ^2}{4r^2} \right]. \quad (6.5)$$

In isotropic coordinates, the metric in Eq. (6.3) becomes:

$$ds^2 = - \left[\frac{1 - \left(\frac{Gm}{2r}\right)^2 + \frac{GQ^2}{4r^2}}{\left(1 + \frac{Gm}{2r}\right)^2 - \frac{GQ^2}{4r^2}} \right]^2 dt^2 + \left[\left(1 + \frac{Gm}{2r}\right) - \frac{GQ^2}{4r^2} \right]^2 (dr^2 + r^2 d\Omega^2); \quad (6.6)$$

it is evident that for $Q = 0$ we recover the Schwarzschild metric in isotropic coordinates.

In Eq. (6.6), expanding up to $\mathcal{O}(G)$ gives the metric in the linearized regime:

$$ds^2 = - \left(1 - \frac{2Gm}{r} + \frac{GQ^2}{r^2} \right) dt^2 + \left(1 + \frac{2Gm}{r} - \frac{GQ^2}{2r^2} \right) (dr^2 + r^2 d\Omega^2). \quad (6.7)$$

The linearized metric in Eq. (6.7) can be also obtained using a different procedure, i.e. by perturbing the metric around a Minkowski background:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \quad (6.8)$$

so that the linearized Einstein's equations in Eq. (6.1) are given by:

$$\begin{aligned} \square h_{\mu\nu} + (\eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \partial_\mu \partial_\nu h) \\ - \partial_\sigma (\partial_\nu h_\mu^\sigma + \partial_\mu h_\nu^\sigma) - \eta_{\mu\nu} \square h = -16\pi G \tau_{\mu\nu}. \end{aligned} \quad (6.9)$$

By working in a weak field regime and considering the case of a static source, the spacetime metric can be written as:

$$ds^2 = -(1 + 2\Phi(r))dt^2 + (1 - 2\Psi(r))(dr^2 + r^2 d\Omega^2), \quad (6.10)$$

where r is the isotropic radial coordinate; thus the only two unknowns we need to find are the two gravitational potentials Φ and Ψ . By looking at the 00-component and the trace of the linearized field equations in Eq. (6.9) we obtain:

$$\begin{aligned} \nabla^2 h_{00} - \partial_i \partial_j h^{ij} + \nabla^2 h &= -16\pi G \tau_{00}, \\ -2\nabla^2 h + 2\partial_i \partial_j h^{ij} &= -16\pi G \tau. \end{aligned} \quad (6.11)$$

Using $h = 2(\Phi - 3\Psi)$, $h_{00} = -2\Phi$ and $h_{ii} = -2\Psi$ we obtain two differential equations for the two gravitational fields Φ and Ψ :

$$\begin{aligned} \nabla^2 \Phi &= 4\pi G(\tau + 2\tau_{00}), \\ \nabla^2 \Psi &= 4\pi G \tau_{00}. \end{aligned} \quad (6.12)$$

In the case of a charged static source, the energy-momentum tensor turns out to be traceless, $\tau = 0$, while the 00-component is given by $\tau_{00} = Q^2/8\pi r^4$. We can now solve the two differential equations in Eq. (6.12) and obtain:

$$\begin{aligned}\Phi(r) &= -\frac{C_1}{r} + \frac{GQ^2}{2r^2} + C_2, \\ \Psi(r) &= -\frac{C_1}{r} + \frac{GQ^2}{4r^2} + C_2,\end{aligned}\tag{6.13}$$

where C_1 and C_2 are two integration constants whose value can be fixed by imposing boundary conditions. First of all, for $r = \infty$ we want that Φ and Ψ are zero (asymptotic flatness) which means $C_2 = 0$. C_1 can be found by noting that $Q = 0$ recovers the usual Newtonian potential generated by a neutral point mass. Thus, $C_1 = Gm$ and we get

$$\begin{aligned}\Phi(r) &= -\frac{Gm}{r} + \frac{GQ^2}{2r^2}, \\ \Psi(r) &= -\frac{Gm}{r} + \frac{GQ^2}{4r^2}.\end{aligned}\tag{6.14}$$

Notice that in the case of a charged source, $\Phi \neq \Psi$. It is clear that the linearized metric in Eq. (6.10) with the gravitational fields derived in Eq. (6.14) coincides with the metric in Eq. (6.7) which we derived instead by starting from the full non-linear Reissner-Nordström metric. It is very important to note that in the weak field approximation one has $2|\Phi|, 2|\Psi| < 1$, which means that the radial coordinate has to always be greater than the outer horizon radius, $r > r_+$.

6.B Working with infinite order differential equations

We can solve the differential equations in Eq. (2.13) by making the following field redefinitions [116]

$$\bar{\Phi} := e^{-\nabla^2/M_s^2}\Phi, \quad \bar{\Psi} := e^{-\nabla^2/M_s^2}\Psi,\tag{6.15}$$

so that the equations in (6.8) become [116]

$$\begin{aligned}\nabla^2 \bar{\Phi}(r) &= \frac{GQ^2}{r^4} \\ \nabla^2 \bar{\Psi}(r) &= \frac{GQ^2}{2r^4}.\end{aligned}\tag{6.16}$$

The structure of these equations is similar to the one we have in the case of a charged source in Einstein's GR, i.e. Reissner-Nordström, as also shown in Appendix 6.A in Eq. (6.12); thus the solution will be also of the same structure as the one in Eq. (6.13)

[116]:

$$\begin{aligned}\bar{\Phi}(r) &= -\frac{C_1}{r} + \frac{GQ^2}{2r^2} + C_2 \\ \bar{\Psi}(r) &= -\frac{C_1}{r} + \frac{GQ^2}{4r^2} + C_2.\end{aligned}\tag{6.17}$$

By requiring the boundary conditions of asymptotic flatness one gets $C_2 = 0$; while C_1 can be fixed by demanding that for $Q = 0$, one recovers the neutral point source result, which means $C_1 = Gm$. In terms of old fields Φ and Ψ , Eq. (6.15) will give [116]:

$$\begin{aligned}\Phi(r) &= -Gme^{\nabla^2/M_s^2} \left(\frac{1}{r} \right) + \frac{GQ^2}{2} e^{\nabla^2/M_s^2} \left(\frac{1}{r^2} \right) \\ \Psi(r) &= -Gme^{\nabla^2/M_s^2} \left(\frac{1}{r} \right) + \frac{GQ^2}{4} e^{\nabla^2/M_s^2} \left(\frac{1}{r^2} \right)\end{aligned}\tag{6.18}$$

We now need to compute the action of the exponential e^{∇^2/M_s^2} on the functions $1/r$ and $1/r^2$. Notice that¹ [116]:

$$\begin{aligned}e^{\nabla^2/M_s^2} \left(\frac{1}{r} \right) &= e^{\nabla^2/M_s^2} \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2} e^{-k^2/M_s^2} e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= \frac{2}{\pi} \int_0^\infty dk \frac{\sin(kr)}{kr} e^{-k^2/M_s^2} \\ &= \frac{1}{r} \text{Erf} \left(\frac{M_s r}{2} \right),\end{aligned}\tag{6.19}$$

where we have used the fact that $4\pi/k^2$ is the Fourier transform of $1/r$. Similarly, by noting that the Fourier transform of $1/r^2$ is $\frac{2\pi^2}{k} \text{sign}(k)$, one has [116]:

$$\begin{aligned}e^{\nabla^2/M_s^2} \left(\frac{1}{r^2} \right) &= e^{\nabla^2/M_s^2} \int \frac{d^3k}{(2\pi)^3} \frac{2\pi^2}{k} \text{sign}(k) e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{2\pi^2}{k} \text{sign}(k) e^{-k^2/M_s^2} e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= \int_0^\infty dk \frac{\sin(kr)}{r} e^{-k^2/M_s^2} \\ &= \frac{M_s}{r} \text{F} \left(\frac{M_s r}{2} \right),\end{aligned}\tag{6.20}$$

where $\text{F}(M_s r/2)$ is the Dawson function.

Thus, by using Eqs. (6.19)(6.20), we can now obtain the expressions for the two metric potentials in Eq. (2.14).

¹We can also consider other choices of entire functions, for example we can generalize Eq. (??) to $a(\square_s) = e^{(-\square_s)^n}$. In such a general case, the factor e^{-k^2/M_s^2} in the third equality of Eqs. (6.19)(6.20) is replaced by $e^{-k^{2n}/M_s^{2n}}$. For instance, for $n = 2$, the potentials evaluate to generalized hypergeometric functions, are regular at the origin, and recover GR in the IR.

6.C Full expressions of curvatures for the linearized metric

In this appendix, we present the curvatures corresponding to the linearized static metric arising from a charged point source derived in Eqs. (2.10)(2.14). The Ricci scalar up to $\mathcal{O}(G)$ is given by [116]

$$R = \frac{GmM_s^3 e^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}}; \quad (6.21)$$

the non-zero components of the Riemann tensor up to $\mathcal{O}(G)$ [116]

$$\begin{aligned} R_{0101} &= \frac{1}{8}G \left[M_s^3 \left(\frac{2Q^2 F\left(\frac{rM_s}{2}\right)}{r} + \frac{4me^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\ &\quad \left. + \frac{8M_s \left(Q^2 F\left(\frac{rM_s}{2}\right) + \frac{2mre^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right)}{r^3} \right. \\ &\quad \left. + Q^2 r M_s^5 F\left(\frac{rM_s}{2}\right) - \frac{16m \text{Erf}\left(\frac{rM_s}{2}\right)}{r^3} - \frac{4Q^2 M_s^2}{r^2} - Q^2 M_s^4 \right], \\ R_{0303} &= R_{0202} \sin^2(\theta) \\ &= \frac{1}{4}G \sin^2(\theta) \left[M_s \left(-\frac{2Q^2 F\left(\frac{rM_s}{2}\right)}{r} - \frac{4me^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\ &\quad \left. - Q^2 r M_s^3 F\left(\frac{rM_s}{2}\right) + \frac{4m \text{Erf}\left(\frac{rM_s}{2}\right)}{r} + Q^2 M_s^2 \right], \\ R_{1313} &= R_{1212} \sin^2(\theta) \\ &= \frac{1}{16}G \sin^2(\theta) \left[M_s \left(\frac{4Q^2 F\left(\frac{rM_s}{2}\right)}{r} + \frac{16me^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\ &\quad \left. + Q^2 r^3 M_s^5 F\left(\frac{rM_s}{2}\right) - \frac{16m \text{Erf}\left(\frac{rM_s}{2}\right)}{r} \right. \\ &\quad \left. + \frac{8mr^2 M_s^3 e^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} - Q^2 r^2 M_s^4 - 2Q^2 M_s^2 \right], \\ R_{2323} &= \frac{1}{4}Gr \sin^2(\theta) \left[M_s \left(-2Q^2 F\left(\frac{rM_s}{2}\right) - \frac{8mre^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\ &\quad \left. - Q^2 r^2 M_s^3 F\left(\frac{rM_s}{2}\right) + 8m \text{Erf}\left(\frac{rM_s}{2}\right) + Q^2 r M_s^2 \right]; \end{aligned} \quad (6.22)$$

the non-zero components of the Ricci tensor up to $\mathcal{O}(G)$ [116]

$$\begin{aligned}
R_{00} &= \frac{1}{8}GM_s^3 \left[\frac{Q^2 (r^2 M_s^2 - 2) \text{F}\left(\frac{rM_s}{2}\right)}{r} + \frac{4me^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} - Q^2 M_s \right], \\
R_{11} &= \frac{1}{4}GM_s \left[M_s \left(\frac{2mM_s e^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} + \frac{Q^2}{r^2} \right) - \frac{Q^2 (r^2 M_s^2 + 2) \text{F}\left(\frac{rM_s}{2}\right)}{r^3} \right], \\
R_{22} &= \frac{1}{16}GM_s \left[\frac{Q^2 (r^4 M_s^4 + 4) \text{F}\left(\frac{rM_s}{2}\right)}{r} + M_s \left(\frac{8mr^2 M_s e^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right. \right. \\
&\quad \left. \left. - Q^2 r^2 M_s^2 - 2Q^2 \right) \right], \\
R_{33} &= \frac{1}{16}G \sin^2(\theta) M_s \left[\frac{Q^2 (r^4 M_s^4 + 4) \text{F}\left(\frac{rM_s}{2}\right)}{r} + M_s \left(\frac{8mr^2 M_s e^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right. \right. \\
&\quad \left. \left. - Q^2 r^2 M_s^2 - 2Q^2 \right) \right];
\end{aligned} \tag{6.23}$$

the non-zero components of the Weyl tensor up to $\mathcal{O}(G)$ [116]

$$\begin{aligned}
C_{0101} &= \frac{1}{48}G \left[M_s^3 \left(\frac{12Q^2 \text{F}\left(\frac{rM_s}{2}\right)}{r} + \frac{16me^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\
&\quad \left. + \frac{1}{r^3} M_s \left(36Q^2 \text{F}\left(\frac{rM_s}{2}\right) + \frac{96mre^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\
&\quad \left. + 3Q^2 r M_s^5 \text{F}\left(\frac{rM_s}{2}\right) - \frac{96m \text{Erf}\left(\frac{rM_s}{2}\right)}{r^3} - \frac{18Q^2 M_s^2}{r^2} - 3Q^2 M_s^4 \right], \\
C_{0303} &= -C_{1313} = C_{0202} \sin^2(\theta) = -C_{1212} \sin^2(\theta) \\
&= \frac{1}{96}G \sin^2(\theta) \left[4r M_s^3 \left(-3Q^2 \text{F}\left(\frac{rM_s}{2}\right) - \frac{4mre^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\
&\quad \left. + M_s \left(-\frac{36Q^2 \text{F}\left(\frac{rM_s}{2}\right)}{r} - \frac{96me^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\
&\quad \left. - 3Q^2 r^3 M_s^5 \text{F}\left(\frac{rM_s}{2}\right) + \frac{96m \text{Erf}\left(\frac{rM_s}{2}\right)}{r} + 3Q^2 r^2 M_s^4 + 18Q^2 M_s^2 \right], \\
C_{2323} &= \frac{1}{48}Gr \sin^2(\theta) \left[4r^2 M_s^3 \left(-3Q^2 \text{F}\left(\frac{rM_s}{2}\right) - \frac{4mre^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) \right. \\
&\quad \left. + M_s \left(-36Q^2 \text{F}\left(\frac{rM_s}{2}\right) - \frac{96mre^{-\frac{1}{4}r^2 M_s^2}}{\sqrt{\pi}} \right) - 3Q^2 r^4 M_s^5 \text{F}\left(\frac{rM_s}{2}\right) \right. \\
&\quad \left. + 96m \text{Erf}\left(\frac{rM_s}{2}\right) + 3Q^2 r^3 M_s^4 + 18Q^2 r M_s^2 \right].
\end{aligned} \tag{6.24}$$

and the Kretschmann scalar up to $\mathcal{O}(G^2)$ [116]

$$\begin{aligned}
\mathcal{K} = & \frac{e^{-\frac{1}{2}r^2M_s^2}G^2}{32\pi r^6} \left[3e^{\frac{1}{2}r^2M_s^2}\pi Q^4 r^8 \text{F}\left(\frac{rM_s}{2}\right)^2 M_s^{10} \right. \\
& - 6e^{\frac{1}{2}r^2M_s^2}\pi Q^4 r^7 \text{F}\left(\frac{rM_s}{2}\right) M_s^9 \\
& + \left(32e^{\frac{1}{4}r^2M_s^2}m\sqrt{\pi}Q^2 \text{F}\left(\frac{rM_s}{2}\right) r^7 + 3e^{\frac{1}{2}r^2M_s^2}\pi Q^4 r^6 \right. \\
& + 8e^{\frac{1}{2}r^2M_s^2}\pi Q^4 \text{F}\left(\frac{rM_s}{2}\right)^2 r^6 \left. \right) M_s^8 \\
& - 4e^{\frac{1}{4}r^2M_s^2}\sqrt{\pi}Q^2 r^5 \left(7e^{\frac{1}{4}r^2M_s^2}\sqrt{\pi} \text{F}\left(\frac{rM_s}{2}\right) Q^2 + 8mr \right) M_s^7 \\
& + 4r^4 \left(18e^{\frac{1}{2}r^2M_s^2}\pi \text{F}\left(\frac{rM_s}{2}\right)^2 Q^4 + 5e^{\frac{1}{2}r^2M_s^2}\pi Q^4 \right. \\
& + 32e^{\frac{1}{4}r^2M_s^2}m\sqrt{\pi}r \text{F}\left(\frac{rM_s}{2}\right) Q^2 + 24m^2r^2 \left. \right) M_s^6 \\
& - 24e^{\frac{1}{4}r^2M_s^2}\sqrt{\pi}Q^2 r^3 \left(8mr \right. \\
& + e^{\frac{1}{4}r^2M_s^2}\sqrt{\pi} \text{F}\left(\frac{rM_s}{2}\right) \left(5Q^2 + 4mr \text{Erf}\left(\frac{rM_s}{2}\right) \right) \left. \right) M_s^5 \\
& + 4r^2 \left(40e^{\frac{1}{2}r^2M_s^2}\pi \text{F}\left(\frac{rM_s}{2}\right)^2 Q^4 + 15e^{\frac{1}{2}r^2M_s^2}\pi Q^4 \right. \\
& + 144e^{\frac{1}{4}r^2M_s^2}m\sqrt{\pi}r \text{F}\left(\frac{rM_s}{2}\right) Q^2 \\
& + 24e^{\frac{1}{2}r^2M_s^2}m\pi r \text{Erf}\left(\frac{rM_s}{2}\right) Q^2 + 128m^2r^2 \left. \right) M_s^4 \\
& - 16e^{\frac{1}{4}r^2M_s^2}\sqrt{\pi}r \left(3e^{\frac{1}{4}r^2M_s^2}\sqrt{\pi} \text{F}\left(\frac{rM_s}{2}\right) \left(5Q^2 + 8mr \text{Erf}\left(\frac{rM_s}{2}\right) \right) \right. \\
& + 4mr \left(9Q^2 + 8mr \text{Erf}\left(\frac{rM_s}{2}\right) \right) \left. \right) M_s^3 + 48 \left(5e^{\frac{1}{2}r^2M_s^2}\pi \text{F}\left(\frac{rM_s}{2}\right)^2 Q^4 \right. \\
& + 24e^{\frac{1}{4}r^2M_s^2}m\sqrt{\pi}r \text{F}\left(\frac{rM_s}{2}\right) Q^2 \\
& + 4mr \left(3e^{\frac{1}{2}r^2M_s^2}\pi \text{Erf}\left(\frac{rM_s}{2}\right) Q^2 + 8mr \right) \left. \right) M_s^2 \\
& - 384e^{\frac{1}{4}r^2M_s^2}m\sqrt{\pi} \left(3e^{\frac{1}{4}r^2M_s^2}\sqrt{\pi} \text{F}\left(\frac{rM_s}{2}\right) Q^2 + 8mr \right) \text{Erf}\left(\frac{rM_s}{2}\right) M_s \\
& \left. + 1536e^{\frac{1}{2}r^2M_s^2}m^2\pi \text{Erf}\left(\frac{rM_s}{2}\right)^2 \right].
\end{aligned} \tag{6.25}$$

6.D Full, non-linear equations of motion

Varying the action in eq.(3.1) with respect to the metric gives the eom derived in [199, 252]:

$$\begin{aligned}
M_P^2 G_\nu^\mu = & T_\nu^\mu - M_P^2 \Lambda \delta_\nu^\mu - 2S_\nu^\mu \mathcal{F}_1(\square_s) R + 2(\nabla^\mu \nabla_\nu - \delta_\nu^\mu \square) \mathcal{F}_1(\square_s) R \\
& - \frac{1}{2} R \mathcal{F}_2(\square_s) S_\nu^\mu - 2S_\beta^\mu \mathcal{F}_2(\square_s) S_\nu^\beta + \frac{1}{2} \delta_\nu^\mu S_\beta^\alpha \mathcal{F}_2(\square_s) S_\alpha^\beta \\
& + 2 \left(\nabla_\rho \nabla_\nu \mathcal{F}_2(\square_s) S^{\mu\rho} - \frac{1}{2} \square \mathcal{F}_2(\square_s) S_\nu^\mu - \frac{1}{2} \delta_\nu^\mu \nabla_\sigma \nabla_\rho \mathcal{F}_2(\square_s) S^{\sigma\rho} \right) \\
& + L_{1\nu}^\mu - \frac{1}{2} \delta_\nu^\mu (L_{1\sigma}^\sigma + \bar{L}_1) + L_{2\nu}^\mu - \frac{1}{2} \delta_\nu^\mu (L_{2\sigma}^\sigma + \bar{L}_2) \\
& + 2\Delta_\nu^\mu + 2(S_{\alpha\beta} + 2\nabla_\alpha \nabla_\beta) \mathcal{F}_3(\square_s) C_\nu^{\alpha\beta\mu} + \Omega_{3\nu}^\mu \\
& - \frac{1}{2} \delta_\nu^\mu (\Omega_{3\sigma}^\sigma + \bar{\Omega}_3) + 4\Delta_{3\nu}^\mu,
\end{aligned} \tag{6.26}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and

$$L_{1\nu}^\mu = \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} \partial^\mu R^{(l)} \partial_\nu R^{(n-l-1)}, \quad \bar{L}_1 = \sum_{n=1}^{\infty} f_{1n} \sum_{l=0}^{n-1} R^{(l)} R^{(n-l)}, \tag{6.27}$$

$$L_{2\nu}^\mu = \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \nabla^\mu S^{(l)\alpha}_\beta \nabla_\nu S^{(n-l-1)\beta}_\alpha, \quad \bar{L}_2 = \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} S^{(l)\alpha}_\beta S^{(n-l)\beta}_\alpha \tag{6.28}$$

$$\Delta_\nu^\mu = \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{2n} \sum_{l=0}^{n-1} \nabla_\beta [S^{(l)\beta}_\gamma \nabla^\mu S^{(n-l-1)\gamma}_\nu - \nabla^\mu S^{(l)\beta}_\gamma S^{(n-l-1)\gamma}_\nu] \tag{6.29}$$

$$\Omega_{3\nu}^\mu = \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} C_{\beta\lambda\sigma;\nu}^{\alpha(l)} C_\alpha^{\beta\lambda\sigma;\mu(n-l-1)}, \quad \bar{\Omega}_3 = \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} C_{\beta\lambda\sigma}^{\alpha(l)} C_\alpha^{\beta\lambda\sigma(n-l)}, \tag{6.30}$$

$$\Delta_{3\nu}^\mu = \frac{1}{M_s^2} \sum_{n=1}^{\infty} f_{3n} \sum_{l=0}^{n-1} [C^{\lambda\beta(l)}_{\sigma\alpha} C_{\lambda;\nu}^{\mu\sigma\alpha(n-l-1)} - C^{\lambda\beta(l)}_{\sigma\alpha;\nu} C_\lambda^{\mu\sigma\alpha(n-l-1)}]_{;\beta}. \tag{6.31}$$

where

$$X^{(m)} = \square_s^m X \tag{6.32}$$

Taking the trace of eq.(6.26), we get

$$-M_P^2 R = -4M_P^2 \Lambda - 6\square \mathcal{F}_1(\square_s) R - (L_1 + 2\bar{L}_1) - 2\nabla_\mu \nabla_\nu \mathcal{F}_2(\square_s) S^{\mu\nu} - (L_2 + 2\bar{L}_2) + 2\Delta. \tag{6.33}$$

6.E Perturbations around an arbitrary background

The metric is perturbed as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} . \quad (6.34)$$

In general, an expression Q (arbitrary rank tensor, indices suppressed) which depends on $g_{\mu\nu}$, has the following expansion:

$$Q = \bar{Q} + Q^{(1)} + Q^{(2)} + \dots + Q^{(n)} + \dots , \quad (6.35)$$

where the superscript in parentheses (n) denotes that the perturbation is $\mathcal{O}(h^n)$. $\bar{Q} = Q^{(0)}$ is the background value of Q . We use lowercase letters to indicate $\mathcal{O}(h)$ perturbation of the relevant quantity, while higher order perturbations are indicated explicitly by their order n . So, $q = Q^{(1)} = \delta Q$, $Q^{(2)} = \delta^2 Q$, etc.. Specifically, Q could be the inverse metric, Ricci scalar (R), Ricci tensor ($R_{\mu\nu}$), traceless Ricci tensor ($S_{\mu\nu}$), Weyl tensor ($C_{\mu\nu\rho\sigma}$) or the Levi-Civita connection. In order to set notation, we expand the curvatures up to $\mathcal{O}(h^2)$ [159]:

$$\begin{aligned} R &= \bar{R} + r + \delta^2 R , \\ R_{\mu\nu} &= \bar{R}_{\mu\nu} + r_{\mu\nu} + \delta^2 R_{\mu\nu} , \\ S_{\mu\nu} &= \bar{S}_{\mu\nu} + s_{\mu\nu} + \delta^2 \bar{S}_{\mu\nu} , \\ C_{\mu\nu\rho\sigma} &= \bar{C}_{\mu\nu\rho\sigma} + c_{\mu\nu\rho\sigma} + \delta^2 C_{\mu\nu\rho\sigma} , \end{aligned} \quad (6.36)$$

The perturbations of all other quantities involve perturbations to all orders in h . We will henceforth expand all other quantities up to $\mathcal{O}(h^2)$ at most. As an example, suppose we want to find the $\mathcal{O}(h^2)$ contribution in (QP) where Q and P are some quantities that depend on $g_{\mu\nu}$. To find $(QP)^{(2)}$, we expand each term and collect all $\mathcal{O}(h^2)$ contributions:

$$\begin{aligned} (QP)^{(2)} &= \left[Q^{(0)} + Q^{(1)} + Q^{(2)} + \dots \right] \left[P^{(0)} + P^{(1)} + P^{(2)} + \dots \right] \\ &= \underbrace{Q^{(0)}P^{(2)} + Q^{(2)}P^{(0)} + Q^{(1)}P^{(1)}}_{\text{collect}} + \underbrace{\dots \text{other terms} \dots}_{\text{discard}} . \end{aligned} \quad (6.37)$$

The inverse metric is

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu\alpha} h^\nu{}_\alpha , \quad (6.38)$$

while square root of the metric determinant is

$$\sqrt{-g} = \sqrt{-\bar{g}} \left(1 + \frac{h}{2} + \frac{h^2}{8} - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} \right) . \quad (6.39)$$

Note that $\delta g_{\mu\nu} = h_{\mu\nu}$ and $\delta g^{\mu\nu} = -h^{\mu\nu}$. All traces are taken with respect to the background metric, like $h = \bar{g}^{\mu\nu} h_{\mu\nu}$. Covariant derivatives ($\bar{\nabla}_\mu$) are defined with respect to the background $\bar{g}_{\mu\nu}$. The Levi-Civita connection is [159]

$$\Gamma_{\mu\nu}^\lambda = \bar{\Gamma}_{\mu\nu}^\lambda + \frac{1}{2} (\bar{\nabla}_\mu h^\lambda_\nu + \bar{\nabla}_\nu h^\lambda_\mu - \bar{\nabla}^\lambda h_{\mu\nu}) . \quad (6.40)$$

Expansions for curvatures are then obtained by substituting the above expansions in eq. (6.36). We use the notation in eq. (6.35) for the remaining part of Appendix (6.E). The following expansions are valid for an *arbitrary* background. We get [159]

$$\begin{aligned} R^{(1)} &= -h^{\alpha\beta} \bar{R}_{\alpha\beta} + \bar{\nabla}_\beta \bar{\nabla}_\alpha h^{\alpha\beta} - \bar{\square} h , \\ R^{(2)} &= -h^{\alpha\beta} h^{\gamma\lambda} \bar{R}_{\alpha\gamma\beta\lambda} + 2h^{\alpha\beta} h_\lambda{}^\gamma \bar{R}_{\beta\gamma\alpha}{}^\lambda + h^{\alpha\beta} \bar{\nabla}_\beta \bar{\nabla}_\alpha h - 2h^{\alpha\beta} \bar{\nabla}_\beta \bar{\nabla}_\gamma h_\alpha{}^\gamma \\ &\quad - \frac{1}{4} \bar{\nabla}_\beta h \bar{\nabla}^\beta h - \bar{\nabla}_\alpha h^{\alpha\beta} \bar{\nabla}_\gamma h_\beta{}^\gamma + \bar{\nabla}^\beta h \bar{\nabla}_\gamma h_\beta{}^\gamma + h^{\alpha\beta} \bar{\square} h_{\alpha\beta} \\ &\quad - \frac{1}{2} \bar{\nabla}_\beta h_{\alpha\gamma} \bar{\nabla}^\gamma h^{\alpha\beta} + \frac{3}{4} \bar{\nabla}_\gamma h_{\alpha\beta} \bar{\nabla}^\gamma h^{\alpha\beta} , \end{aligned} \quad (6.41)$$

$$\begin{aligned} R_{\mu\nu}^{(1)} &= \frac{1}{2} h_\nu{}^\alpha \bar{R}_{\mu\alpha} + \frac{1}{2} h_\mu{}^\alpha \bar{R}_{\nu\alpha} - h^{\alpha\beta} \bar{R}_{\mu\alpha\nu\beta} - \frac{1}{2} \bar{\square} h_{\mu\nu} + \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\alpha h_\nu{}^\alpha \\ &\quad + \frac{1}{2} \bar{\nabla}_\nu \bar{\nabla}_\alpha h_\mu{}^\alpha - \frac{1}{2} \bar{\nabla}_\nu \bar{\nabla}_\mu h , \\ R_{\mu\nu}^{(2)} &= -\frac{1}{2} h^{\beta\gamma} h_\nu{}^\alpha \bar{R}_{\mu\beta\alpha\gamma} + h_\alpha{}^\gamma h^{\alpha\beta} \bar{R}_{\mu\beta\nu\gamma} - \frac{1}{2} h^{\beta\gamma} h_\mu{}^\alpha \bar{R}_{\nu\beta\alpha\gamma} - \frac{1}{4} \bar{\nabla}_\alpha h \bar{\nabla}^\alpha h_{\mu\nu} \\ &\quad + \frac{1}{2} \bar{\nabla}^\alpha h_{\mu\nu} \bar{\nabla}_\beta h_\alpha{}^\beta + \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\beta \bar{\nabla}_\alpha h_{\mu\nu} - \frac{1}{2} \bar{\nabla}_\alpha h_{\nu\beta} \bar{\nabla}^\beta h_\mu{}^\alpha + \frac{1}{2} \bar{\nabla}_\beta h_{\nu\alpha} \bar{\nabla}^\beta h_\mu{}^\alpha \\ &\quad + \frac{1}{4} \bar{\nabla}_\alpha h \bar{\nabla}_\mu h_\nu{}^\alpha - \frac{1}{2} \bar{\nabla}_\beta h_\alpha{}^\beta \bar{\nabla}_\mu h_\nu{}^\alpha - \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\mu \bar{\nabla}_\beta h_{\nu\alpha} + \frac{1}{4} \bar{\nabla}_\mu h^{\alpha\beta} \bar{\nabla}_\nu h_{\alpha\beta} \\ &\quad + \frac{1}{4} \bar{\nabla}_\alpha h \bar{\nabla}_\nu h_\mu{}^\alpha - \frac{1}{2} \bar{\nabla}_\beta h_\alpha{}^\beta \bar{\nabla}_\nu h_\mu{}^\alpha - \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\nu \bar{\nabla}_\beta h_{\mu\alpha} + \frac{1}{2} h^{\alpha\beta} \bar{\nabla}_\nu \bar{\nabla}_\mu h_{\alpha\beta} , \\ S_{\mu\nu}^{(1)} &= \frac{1}{4} \bar{g}_{\mu\nu} h^{\alpha\beta} \bar{R}_{\alpha\beta} + \frac{1}{2} h_\nu{}^\alpha \bar{R}_{\mu\alpha} + \frac{1}{2} h_\mu{}^\alpha \bar{R}_{\nu\alpha} - \frac{1}{4} h_{\mu\nu} \bar{R} - h^{\alpha\beta} \bar{R}_{\mu\alpha\nu\beta} - \frac{1}{2} \bar{\square} h_{\mu\nu} \\ &\quad - \frac{1}{4} \bar{g}_{\mu\nu} \bar{\nabla}_\beta \bar{\nabla}_\alpha h^{\alpha\beta} + \frac{1}{4} \bar{g}_{\mu\nu} \bar{\square} h + \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\alpha h_\nu{}^\alpha + \frac{1}{2} \bar{\nabla}_\nu \bar{\nabla}_\alpha h_\mu{}^\alpha - \frac{1}{2} \bar{\nabla}_\nu \bar{\nabla}_\mu h \end{aligned} \quad (6.42)$$

$$\begin{aligned}
S_{\mu\nu}^{(2)} = & \frac{1}{4}h^{\alpha\beta}h_{\mu\nu}\bar{R}_{\alpha\beta} + \frac{1}{4}\bar{g}_{\mu\nu}h^{\alpha\beta}h^{\gamma\lambda}\bar{R}_{\alpha\gamma\beta\lambda} - \frac{1}{2}\bar{g}_{\mu\nu}h^{\alpha\beta}h_{\lambda}^{\gamma}\bar{R}_{\beta\gamma\alpha}{}^{\lambda} - \frac{1}{2}h^{\beta\gamma}h_{\nu}^{\alpha}\bar{R}_{\mu\beta\alpha\gamma} \\
& + h_{\alpha}^{\gamma}h^{\alpha\beta}\bar{R}_{\mu\beta\nu\gamma} - \frac{1}{2}h^{\beta\gamma}h_{\mu}^{\alpha}\bar{R}_{\nu\beta\alpha\gamma} - \frac{1}{4}\bar{\nabla}_{\alpha}h\bar{\nabla}^{\alpha}h_{\mu\nu} + \frac{1}{2}\bar{\nabla}^{\alpha}h_{\mu\nu}\bar{\nabla}_{\beta}h_{\alpha}^{\beta} \\
& - \frac{1}{4}h_{\mu\nu}\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h^{\alpha\beta} - \frac{1}{4}\bar{g}_{\mu\nu}h^{\alpha\beta}\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h + \frac{1}{2}h^{\alpha\beta}\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h_{\mu\nu} + \frac{1}{4}h_{\mu\nu}\bar{\square}h \\
& + \frac{1}{2}\bar{g}_{\mu\nu}h^{\alpha\beta}\bar{\nabla}_{\beta}\bar{\nabla}_{\gamma}h_{\alpha}^{\gamma} + \frac{1}{16}\bar{g}_{\mu\nu}\bar{\nabla}_{\beta}h\bar{\nabla}^{\beta}h - \frac{1}{2}\bar{\nabla}_{\alpha}h_{\nu\beta}\bar{\nabla}^{\beta}h_{\mu}^{\alpha} + \frac{1}{2}\bar{\nabla}_{\beta}h_{\nu\alpha}\bar{\nabla}^{\beta}h_{\mu}^{\alpha} \\
& + \frac{1}{4}\bar{g}_{\mu\nu}\bar{\nabla}_{\alpha}h^{\alpha\beta}\bar{\nabla}_{\gamma}h_{\beta}^{\gamma} - \frac{1}{4}\bar{g}_{\mu\nu}\bar{\nabla}^{\beta}h\bar{\nabla}_{\gamma}h_{\beta}^{\gamma} - \frac{1}{4}\bar{g}_{\mu\nu}h^{\alpha\beta}\bar{\square}h_{\alpha\beta} + \frac{1}{8}\bar{g}_{\mu\nu}\bar{\nabla}_{\beta}h_{\alpha\gamma}\bar{\nabla}^{\gamma}h^{\alpha\beta} \\
& - \frac{3}{16}\bar{g}_{\mu\nu}\bar{\nabla}_{\gamma}h_{\alpha\beta}\bar{\nabla}^{\gamma}h^{\alpha\beta} + \frac{1}{4}\bar{\nabla}_{\alpha}h\bar{\nabla}_{\mu}h_{\nu}^{\alpha} - \frac{1}{2}\bar{\nabla}_{\beta}h_{\alpha}^{\beta}\bar{\nabla}_{\mu}h_{\nu}^{\alpha} - \frac{1}{2}h^{\alpha\beta}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}h_{\beta\alpha} \\
& + \frac{1}{4}\bar{\nabla}_{\mu}h^{\alpha\beta}\bar{\nabla}_{\nu}h_{\alpha\beta} + \frac{1}{4}\bar{\nabla}_{\alpha}h\bar{\nabla}_{\nu}h_{\mu}^{\alpha} - \frac{1}{2}\bar{\nabla}_{\beta}h_{\alpha}^{\beta}\bar{\nabla}_{\nu}h_{\mu}^{\alpha} - \frac{1}{2}h^{\alpha\beta}\bar{\nabla}_{\nu}\bar{\nabla}_{\beta}h_{\mu\alpha} \\
& + \frac{1}{2}h^{\alpha\beta}\bar{\nabla}_{\nu}\bar{\nabla}_{\mu}h_{\alpha\beta},
\end{aligned} \tag{6.43}$$

$$\begin{aligned}
R_{\mu\nu\rho\sigma}^{(1)} = & \frac{1}{2}h_{\sigma}^{\alpha}\bar{R}_{\mu\alpha\nu\rho} - \frac{1}{2}h_{\rho}^{\alpha}\bar{R}_{\mu\alpha\nu\sigma} + \frac{1}{2}h_{\sigma}^{\alpha}\bar{R}_{\mu\nu\rho\alpha} - \frac{1}{2}h_{\rho}^{\alpha}\bar{R}_{\mu\nu\sigma\alpha} - \frac{1}{2}h_{\sigma}^{\alpha}\bar{R}_{\mu\rho\nu\alpha} \\
& - \frac{1}{2}h_{\nu}^{\alpha}\bar{R}_{\mu\rho\sigma\alpha} + \frac{1}{2}h_{\rho}^{\alpha}\bar{R}_{\mu\sigma\nu\alpha} + \frac{1}{2}h_{\nu}^{\alpha}\bar{R}_{\mu\sigma\rho\alpha} + \frac{1}{2}h_{\mu}^{\alpha}\bar{R}_{\nu\rho\sigma\alpha} - \frac{1}{2}h_{\mu}^{\alpha}\bar{R}_{\nu\sigma\rho\alpha} \\
& - \frac{1}{2}\bar{\nabla}_{\rho}\bar{\nabla}_{\mu}h_{\nu\sigma} + \frac{1}{2}\bar{\nabla}_{\rho}\bar{\nabla}_{\nu}h_{\mu\sigma} + \frac{1}{2}\bar{\nabla}_{\sigma}\bar{\nabla}_{\mu}h_{\nu\rho} - \frac{1}{2}\bar{\nabla}_{\sigma}\bar{\nabla}_{\nu}h_{\mu\rho}, \\
R_{\mu\nu\rho\sigma}^{(2)} = & -\frac{1}{4}\bar{\nabla}_{\alpha}h_{\nu\sigma}\bar{\nabla}^{\alpha}h_{\mu\rho} + \frac{1}{4}\bar{\nabla}_{\alpha}h_{\nu\rho}\bar{\nabla}^{\alpha}h_{\mu\sigma} + \frac{1}{4}\bar{\nabla}^{\alpha}h_{\nu\sigma}\bar{\nabla}_{\mu}h_{\rho\alpha} - \frac{1}{4}\bar{\nabla}^{\alpha}h_{\nu\rho}\bar{\nabla}_{\mu}h_{\sigma\alpha} \\
& - \frac{1}{4}\bar{\nabla}^{\alpha}h_{\mu\sigma}\bar{\nabla}_{\nu}h_{\rho\alpha} + \frac{1}{4}\bar{\nabla}_{\mu}h_{\sigma\alpha}\bar{\nabla}_{\nu}h_{\rho}^{\alpha} + \frac{1}{4}\bar{\nabla}^{\alpha}h_{\mu\rho}\bar{\nabla}_{\nu}h_{\sigma\alpha} - \frac{1}{4}\bar{\nabla}_{\mu}h_{\rho}^{\alpha}\bar{\nabla}_{\nu}h_{\sigma\alpha} \\
& + \frac{1}{4}\bar{\nabla}_{\alpha}h_{\nu\sigma}\bar{\nabla}_{\rho}h_{\mu}^{\alpha} - \frac{1}{4}\bar{\nabla}_{\nu}h_{\sigma\alpha}\bar{\nabla}_{\rho}h_{\mu}^{\alpha} - \frac{1}{4}\bar{\nabla}^{\alpha}h_{\mu\sigma}\bar{\nabla}_{\rho}h_{\nu\alpha} + \frac{1}{4}\bar{\nabla}_{\mu}h_{\sigma\alpha}\bar{\nabla}_{\rho}h_{\nu}^{\alpha} \\
& - \frac{1}{4}\bar{\nabla}_{\alpha}h_{\nu\rho}\bar{\nabla}_{\sigma}h_{\mu}^{\alpha} + \frac{1}{4}\bar{\nabla}_{\nu}h_{\rho\alpha}\bar{\nabla}_{\sigma}h_{\mu}^{\alpha} + \frac{1}{4}\bar{\nabla}_{\rho}h_{\nu\alpha}\bar{\nabla}_{\sigma}h_{\mu}^{\alpha} + \frac{1}{4}\bar{\nabla}^{\alpha}h_{\mu\rho}\bar{\nabla}_{\sigma}h_{\nu\alpha} \\
& - \frac{1}{4}\bar{\nabla}_{\rho}h_{\mu}^{\alpha}\bar{\nabla}_{\sigma}h_{\nu\alpha} - \frac{1}{4}\bar{\nabla}_{\mu}h_{\rho\alpha}\bar{\nabla}_{\sigma}h_{\nu}^{\alpha},
\end{aligned} \tag{6.44}$$

$$\begin{aligned}
C_{\mu\nu\rho\sigma}^{(1)} = & \frac{1}{6}\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}h^{\alpha\beta}\bar{R}_{\alpha\beta} - \frac{1}{6}\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}h^{\alpha\beta}\bar{R}_{\alpha\beta} - \frac{1}{4}\bar{g}_{\nu\sigma}h_{\rho}^{\alpha}\bar{R}_{\mu\alpha} + \frac{1}{4}\bar{g}_{\nu\rho}h_{\sigma}^{\alpha}\bar{R}_{\mu\alpha} \\
& - \frac{1}{2}h_{\nu\sigma}\bar{R}_{\mu\rho} + \frac{1}{2}h_{\nu\rho}\bar{R}_{\mu\sigma} + \frac{1}{4}\bar{g}_{\mu\sigma}h_{\rho}^{\alpha}\bar{R}_{\nu\alpha} - \frac{1}{4}\bar{g}_{\mu\rho}h_{\sigma}^{\alpha}\bar{R}_{\nu\alpha} + \frac{1}{2}h_{\mu\sigma}\bar{R}_{\nu\rho} \\
& - \frac{1}{2}h_{\mu\rho}\bar{R}_{\nu\sigma} - \frac{1}{4}\bar{g}_{\nu\sigma}h_{\mu}^{\alpha}\bar{R}_{\rho\alpha} + \frac{1}{4}\bar{g}_{\mu\sigma}h_{\nu}^{\alpha}\bar{R}_{\rho\alpha} + \frac{1}{4}\bar{g}_{\nu\rho}h_{\mu}^{\alpha}\bar{R}_{\sigma\alpha} - \frac{1}{4}\bar{g}_{\mu\rho}h_{\nu}^{\alpha}\bar{R}_{\sigma\alpha} \\
& + \frac{1}{6}\bar{g}_{\nu\sigma}h_{\mu\rho}\bar{R} - \frac{1}{6}\bar{g}_{\nu\rho}h_{\mu\sigma}\bar{R} - \frac{1}{6}\bar{g}_{\mu\sigma}h_{\nu\rho}\bar{R} + \frac{1}{6}\bar{g}_{\mu\rho}h_{\nu\sigma}\bar{R} + \frac{1}{2}h_{\sigma}^{\alpha}\bar{R}_{\mu\alpha\nu\rho} \\
& - \frac{1}{2}h_{\rho}^{\alpha}\bar{R}_{\mu\alpha\nu\sigma} + \frac{1}{2}\bar{g}_{\nu\sigma}h^{\alpha\beta}\bar{R}_{\mu\alpha\rho\beta} - \frac{1}{2}\bar{g}_{\nu\rho}h^{\alpha\beta}\bar{R}_{\mu\alpha\sigma\beta} + \frac{1}{2}h_{\sigma}^{\alpha}\bar{R}_{\mu\nu\rho\alpha} - \frac{1}{2}h_{\rho}^{\alpha}\bar{R}_{\mu\nu\sigma\alpha} \\
& - \frac{1}{2}h_{\sigma}^{\alpha}\bar{R}_{\mu\rho\nu\alpha} - \frac{1}{2}h_{\nu}^{\alpha}\bar{R}_{\mu\rho\sigma\alpha} + \frac{1}{2}h_{\rho}^{\alpha}\bar{R}_{\mu\sigma\nu\alpha} + \frac{1}{2}h_{\nu}^{\alpha}\bar{R}_{\mu\sigma\rho\alpha} - \frac{1}{2}\bar{g}_{\mu\sigma}h^{\alpha\beta}\bar{R}_{\nu\alpha\rho\beta} \\
& + \frac{1}{2}\bar{g}_{\mu\rho}h^{\alpha\beta}\bar{R}_{\nu\alpha\sigma\beta} + \frac{1}{2}h_{\mu}^{\alpha}\bar{R}_{\nu\rho\sigma\alpha} - \frac{1}{2}h_{\mu}^{\alpha}\bar{R}_{\nu\sigma\rho\alpha} + \frac{1}{4}\bar{g}_{\nu\sigma}\bar{\square}h_{\mu\rho} - \frac{1}{4}\bar{g}_{\nu\rho}\bar{\square}h_{\mu\sigma} \\
& - \frac{1}{4}\bar{g}_{\mu\sigma}\bar{\square}h_{\nu\rho} + \frac{1}{4}\bar{g}_{\mu\rho}\bar{\square}h_{\nu\sigma} - \frac{1}{6}\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h^{\alpha\beta} + \frac{1}{6}\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}\bar{\nabla}_{\beta}\bar{\nabla}_{\alpha}h^{\alpha\beta} \\
& + \frac{1}{6}\bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\bar{\square}h - \frac{1}{6}\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}\bar{\square}h - \frac{1}{4}\bar{g}_{\nu\sigma}\bar{\nabla}_{\mu}\bar{\nabla}_{\alpha}h_{\rho}^{\alpha} + \frac{1}{4}\bar{g}_{\nu\rho}\bar{\nabla}_{\mu}\bar{\nabla}_{\alpha}h_{\sigma}^{\alpha} \\
& + \frac{1}{4}\bar{g}_{\mu\sigma}\bar{\nabla}_{\nu}\bar{\nabla}_{\alpha}h_{\rho}^{\alpha} - \frac{1}{4}\bar{g}_{\mu\rho}\bar{\nabla}_{\nu}\bar{\nabla}_{\alpha}h_{\sigma}^{\alpha} - \frac{1}{4}\bar{g}_{\nu\sigma}\bar{\nabla}_{\rho}\bar{\nabla}_{\alpha}h_{\mu}^{\alpha} + \frac{1}{4}\bar{g}_{\mu\sigma}\bar{\nabla}_{\rho}\bar{\nabla}_{\alpha}h_{\nu}^{\alpha} \\
& + \frac{1}{4}\bar{g}_{\nu\sigma}\bar{\nabla}_{\rho}\bar{\nabla}_{\mu}h - \frac{1}{2}\bar{\nabla}_{\rho}\bar{\nabla}_{\mu}h_{\nu\sigma} - \frac{1}{4}\bar{g}_{\mu\sigma}\bar{\nabla}_{\rho}\bar{\nabla}_{\nu}h + \frac{1}{2}\bar{\nabla}_{\rho}\bar{\nabla}_{\nu}h_{\mu\sigma} + \frac{1}{4}\bar{g}_{\nu\rho}\bar{\nabla}_{\sigma}\bar{\nabla}_{\alpha}h_{\mu}^{\alpha} \\
& - \frac{1}{4}\bar{g}_{\mu\rho}\bar{\nabla}_{\sigma}\bar{\nabla}_{\alpha}h_{\nu}^{\alpha} - \frac{1}{4}\bar{g}_{\nu\rho}\bar{\nabla}_{\sigma}\bar{\nabla}_{\mu}h + \frac{1}{2}\bar{\nabla}_{\sigma}\bar{\nabla}_{\mu}h_{\nu\rho} + \frac{1}{4}\bar{g}_{\mu\rho}\bar{\nabla}_{\sigma}\bar{\nabla}_{\nu}h - \frac{1}{2}\bar{\nabla}_{\sigma}\bar{\nabla}_{\nu}h_{\mu\rho},
\end{aligned} \tag{6.45}$$

Note that $C_{\mu\nu\rho\sigma}^{(2)}$ has 144 terms and is not very enlightening and so we do not present it here.

Perturbations of mixed rank tensors can be written down once the perturbation of its fully covariant (or contravariant) counterpart is known, for instance [159]

$$\delta R_{\nu}^{\mu} = \delta(R_{\alpha\nu}g^{\alpha\mu}) = \delta R_{\alpha\nu}\bar{g}^{\alpha\mu} + \bar{R}_{\alpha\nu}\delta g^{\alpha\mu} = \delta R_{\alpha\nu}\bar{g}^{\alpha\mu} - \bar{R}_{\alpha\nu}h^{\alpha\mu}, \tag{6.46}$$

We also need perturbations of covariant derivatives. For $\nabla_{\mu}Q$, this means that the Levi-Civita connection in the definition of ∇_{μ} is expanded in h , keeping Q unperturbed. Then $\delta(\nabla_{\mu})$ means that after ∇_{μ} operates on $\bar{Q}$, it is expanded in h , and the first order is picked out. We have [159]

$$\begin{aligned}
\delta(\nabla_{\mu})Q &= \nabla_{\mu}^{(1)}Q = \nabla_{\mu}^{(1)}Q^{(0)}, \\
&= (\nabla_{\mu}Q)^{(1)} - \nabla_{\mu}^{(0)}Q^{(1)}, \\
&= \delta(\nabla_{\mu}Q) - \nabla_{\mu}\delta Q.
\end{aligned} \tag{6.47}$$

For example, for a scalar ϕ which does not depend on $g_{\mu\nu}$, we have [159]

$$\begin{aligned}\delta(\square)\phi &= \delta(g^{\mu\nu}\nabla_\mu\nabla_\nu)\phi = -h^{\mu\nu}\bar{\nabla}_\mu\partial_\nu\phi - \bar{g}^{\mu\nu}\delta\Gamma_{\mu\nu}^\lambda\partial_\lambda\phi, \\ &= \delta(\square\phi) - \bar{\square}\delta\phi.\end{aligned}\tag{6.48}$$

Similarly, for Ricci scalar R , which depends on $g_{\mu\nu}$, we have [159]

$$\begin{aligned}\delta(\square)R &= \delta(\square R) - \bar{\square}\delta R, \\ &= \frac{1}{2}\bar{\nabla}_\alpha h\bar{\nabla}^\alpha\bar{R} - \bar{\nabla}^\alpha\bar{R}\bar{\nabla}_\beta h_\alpha{}^\beta - h_{\alpha\beta}\bar{\nabla}^\beta\bar{\nabla}^\alpha\bar{R}, \\ &= 0 \quad \text{for } \bar{R} = \text{constant}.\end{aligned}\tag{6.49}$$

We can now expand covariant derivative ∇_μ , box operator $\square$ and the form factors $\mathcal{F}_i(\square_s)$ in eq.(5.2) in a series in h [159]

$$\begin{aligned}\nabla_\mu &= \nabla_\mu^{(0)} + \nabla_\mu^{(1)} + \nabla_\mu^{(2)} + \dots, \\ \square &= \square^{(0)} + \square^{(1)} + \square^{(2)} + \dots, \\ \mathcal{F}_i(\square_s) &= \mathcal{F}_i(\square_s)^{(0)} + \mathcal{F}_i(\square_s)^{(1)} + \mathcal{F}_i(\square_s)^{(2)} + \dots.\end{aligned}\tag{6.50}$$

6.F Details of the second order action computation

This appendix is self-contained. In Appendix (6.E) earlier, we had listed perturbations of all curvatures in terms of $h_{\mu\nu}$. In this appendix, we perform SVT decomposition of $\delta^2 S$ eq.(3.26) and apply our background conditions eq.(3.13). We closely follow the steps enumerated in eq.(3.3) in order to get the most simplified expressions. In particular,

- (i) We first present the most general perturbations in terms of the SVT modes ϕ, B, A_μ and $\hat{h}_{\mu\nu}$, and background curvatures $\bar{R}, \bar{R}_{\mu\nu}$ and $\bar{R}_{\mu\nu\rho\sigma}$ which are kept *arbitrary*. We also use the second Bianchi identity eq.(3.3).
- (ii) All background curvatures are then decomposed into their irreducible trace and traceless components $\bar{R}, \bar{S}_{\mu\nu}$ and $\bar{C}_{\mu\nu\rho\sigma}$. We also impose background conformal flatness $\bar{C}_{\mu\nu\rho\sigma} = 0$ eq.(3.13a). And use the Bianchi identity again.
- (iii) Covariantly constant background curvature conditions $\bar{\nabla}_\alpha\bar{S}_{\mu\nu} = 0$ and $\partial_\mu\bar{R} = 0$ in eq.(3.13b) are applied.

$$\mathbf{6.F.1} \quad \delta^2 \mathcal{L}_{EH+\Lambda} = \frac{M_P^2}{2} \delta^2 \left[\sqrt{-g} (R - 2\Lambda) \right] = \frac{M_P^2}{2} \delta_0$$

For reference, around an *arbitrary* background, we have [111]:

$$\begin{aligned} \delta^2 \left[\sqrt{-g} (R - 2\Lambda) \right] = \sqrt{-g} \left[\left(\frac{1}{4} h_{\mu\nu} \bar{\square} h^{\mu\nu} - \frac{1}{4} h \bar{\square} h + \frac{1}{2} h \bar{\nabla}_\mu \bar{\nabla}_\rho h^{\mu\rho} + \frac{1}{2} \bar{\nabla}_\mu h^{\mu\rho} \bar{\nabla}_\nu h^\nu_\rho \right) \right. \\ \left. + (h h^{\mu\nu} - 2 h^\mu_\sigma h^{\sigma\nu}) \left(\frac{1}{8} \bar{R} \bar{g}_{\mu\nu} - \frac{1}{4} \Lambda \bar{g}_{\mu\nu} - \frac{1}{2} \bar{R}_{\mu\nu} \right) \right. \\ \left. - \left(\frac{1}{2} \bar{R}_{\sigma\nu} h^\sigma_\rho h^{\nu\rho} + \frac{1}{2} \bar{R}^\sigma_{\rho\nu\mu} h^\mu_\sigma h^{\nu\rho} \right) \right]. \end{aligned}$$

(i) Performing SVT decomposition eq.(3.6) of δ_0 eq.(3.29) around an *arbitrary* background, we get the terms which are purely quadratic in one SVT mode as [159]

$$\begin{aligned} \delta_{0(BB)} = & \frac{1}{8} B \bar{R}^{\beta\gamma} \bar{\nabla}_\alpha \bar{R}_{\beta\gamma} \bar{\nabla}^\alpha B + \frac{1}{4} \Lambda B \bar{\nabla}_\alpha \bar{R} \bar{\nabla}^\alpha B - \frac{1}{8} B \bar{R} \bar{\nabla}_\alpha \bar{R} \bar{\nabla}^\alpha B - \frac{1}{8} B \bar{\nabla}^\alpha \bar{R} \bar{\square} \bar{\nabla}_\alpha B \\ & + \frac{1}{4} \Lambda B \bar{\square}^2 B - \frac{1}{8} B \bar{R} \bar{\square}^2 B - \frac{1}{8} B \bar{R}_{\alpha\beta} \bar{\nabla}^\alpha B \bar{\nabla}^\beta \bar{R} + \frac{1}{2} \Lambda B \bar{R}_{\alpha\beta} \bar{\nabla}^\beta \bar{\nabla}^\alpha B \\ & - \frac{7}{8} B \bar{R}_\alpha{}^\gamma \bar{R}_{\beta\gamma} \bar{\nabla}^\beta \bar{\nabla}^\alpha B - \frac{1}{4} B \bar{R}_{\alpha\beta} \bar{R} \bar{\nabla}^\beta \bar{\nabla}^\alpha B - \frac{1}{4} B \bar{R}^{\gamma\lambda} \bar{R}_{\alpha\gamma\beta\lambda} \bar{\nabla}^\beta \bar{\nabla}^\alpha B \\ & - \frac{1}{2} B \bar{R}_\alpha{}^{\gamma\lambda\mu} \bar{R}_{\beta\gamma\lambda\mu} \bar{\nabla}^\beta \bar{\nabla}^\alpha B - \frac{11}{8} B \bar{R}^{\beta\gamma} \bar{\nabla}^\alpha B \bar{\nabla}_\gamma \bar{R}_{\alpha\beta} + \frac{1}{2} B \bar{R}^{\beta\gamma\lambda\mu} \bar{\nabla}^\alpha B \bar{\nabla}_\gamma \bar{R}_{\alpha\beta\lambda\mu} \\ & - \frac{1}{8} B \bar{R}_{\alpha\beta} \bar{\square} \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \frac{13}{8} B \bar{R}_{\alpha\gamma} \bar{\nabla}^\gamma \bar{\square} \bar{\nabla}^\alpha B - \frac{5}{4} B \bar{R}_{\beta\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\square} B \\ & - \frac{3}{4} B \bar{\nabla}_\alpha \bar{R}_{\beta\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \frac{11}{4} B \bar{\nabla}_\beta \bar{R}_{\alpha\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\nabla}^\alpha B - \frac{7}{4} B \bar{\nabla}_\gamma \bar{R}_{\alpha\beta} \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\nabla}^\alpha B \\ & + \frac{3}{2} B \bar{R}_{\alpha\beta\gamma\lambda} \bar{\nabla}^\alpha B \bar{\nabla}^\lambda \bar{R}^{\beta\gamma} + \frac{5}{2} B \bar{R}_{\alpha\beta\gamma\lambda} \bar{\nabla}^\lambda \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\nabla}^\alpha B - \frac{1}{2} B \bar{R}_{\alpha\gamma\beta\lambda} \bar{\nabla}^\lambda \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\nabla}^\alpha B \\ & - \frac{1}{2} B \bar{R}_{\alpha\lambda\beta\gamma} \bar{\nabla}^\lambda \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\nabla}^\alpha B, \\ \delta_{0(\phi\phi)} = & -\frac{1}{8} \Lambda \phi^2 - \frac{3}{32} \phi \bar{\square} \phi, \\ \delta_{0(AA)} = & -\Lambda A^\alpha A^\beta \bar{R}_{\alpha\beta} + \frac{1}{2} A^\alpha A^\beta \bar{R}_{\alpha\beta} \bar{R} - A^\alpha A^\beta \bar{R}^{\gamma\lambda} \bar{R}_{\alpha\gamma\beta\lambda} + \frac{1}{2} A^\alpha \bar{\nabla}_\alpha A^\beta \bar{\nabla}_\beta \bar{R} \\ & + \frac{1}{2} A^\alpha A^\beta \bar{\nabla}_\beta \bar{\nabla}_\alpha \bar{R} - \Lambda A^\alpha \bar{\square} A_\alpha + \frac{1}{2} A^\alpha \bar{R} \bar{\square} A_\alpha - A^\alpha \bar{R}_{\beta\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta A_\alpha \\ & - \frac{3}{2} A^\alpha \bar{R}_{\alpha\beta\gamma\lambda} \bar{\nabla}^\lambda \bar{\nabla}^\gamma A^\beta + \frac{3}{2} A^\alpha \bar{R}_{\alpha\gamma\beta\lambda} \bar{\nabla}^\lambda \bar{\nabla}^\gamma A^\beta - \frac{3}{2} A^\alpha \bar{R}_{\alpha\lambda\beta\gamma} \bar{\nabla}^\lambda \bar{\nabla}^\gamma A^\beta, \\ \delta_{0(\hat{h}\hat{h})} = & \frac{1}{2} \Lambda \hat{h}_{\alpha\beta} \hat{h}^{\alpha\beta} + \frac{1}{2} \hat{h}_\alpha{}^\gamma \hat{h}^{\alpha\beta} \bar{R}_{\beta\gamma} - \frac{1}{4} \hat{h}_{\alpha\beta} \hat{h}^{\alpha\beta} \bar{R} + \frac{1}{2} \hat{h}^{\alpha\beta} \hat{h}^{\gamma\lambda} \bar{R}_{\alpha\gamma\beta\lambda} + \frac{1}{4} \hat{h}^{\alpha\beta} \bar{\square} \hat{h}_{\alpha\beta}, \end{aligned} \tag{6.51}$$

and six mixed terms given by [159]

$$\delta_{0(B\phi)} = \frac{1}{4}\Lambda B\bar{\square}\phi - \frac{1}{8}B\bar{\nabla}_\alpha\phi\bar{\nabla}^\alpha\bar{R} - \frac{1}{4}B\bar{R}_{\alpha\beta}\bar{\nabla}^\beta\bar{\nabla}^\alpha\phi, \quad (6.52)$$

$$\begin{aligned} \delta_{0(BA)} = & -A^\alpha B\bar{R}^{\beta\gamma}\bar{\nabla}_\alpha\bar{R}_{\beta\gamma} + \Lambda A^\alpha B\bar{\nabla}_\alpha\bar{R} - \frac{1}{2}A^\alpha B\bar{R}\bar{\nabla}_\alpha\bar{R} - \frac{1}{2}B\bar{\nabla}^\alpha\bar{R}\bar{\square}A_\alpha \\ & + 2\Lambda B\bar{R}_{\alpha\beta}\bar{\nabla}^\beta A^\alpha + 4B\bar{R}_\alpha{}^\gamma\bar{R}_{\beta\gamma}\bar{\nabla}^\beta A^\alpha - B\bar{R}_{\alpha\beta}\bar{R}\bar{\nabla}^\beta A^\alpha - 4B\bar{R}^{\gamma\lambda}\bar{R}_{\alpha\gamma\beta\lambda}\bar{\nabla}^\beta A^\alpha \\ & - 2B\bar{R}_\alpha{}^{\gamma\lambda\mu}\bar{R}_{\beta\lambda\gamma\mu}\bar{\nabla}^\beta A^\alpha - \frac{1}{2}A^\alpha B\bar{R}_{\alpha\beta}\bar{\nabla}^\beta\bar{R} + 2A^\alpha B\bar{R}^{\beta\gamma}\bar{\nabla}_\gamma\bar{R}_{\alpha\beta} \\ & - 2B\bar{R}_{\alpha\beta}\bar{\square}\bar{\nabla}^\beta A^\alpha + 2B\bar{R}_{\alpha\gamma}\bar{\nabla}^\gamma\bar{\square}A^\alpha + B\bar{\nabla}_\alpha\bar{R}_{\beta\gamma}\bar{\nabla}^\gamma\bar{\nabla}^\beta A^\alpha + 6B\bar{\nabla}_\beta\bar{R}_{\alpha\gamma}\bar{\nabla}^\gamma\bar{\nabla}^\beta A^\alpha \\ & - 6B\bar{\nabla}_\gamma\bar{R}_{\alpha\beta}\bar{\nabla}^\gamma\bar{\nabla}^\beta A^\alpha + 5A^\alpha B\bar{R}_{\alpha\beta\gamma\lambda}\bar{\nabla}^\lambda\bar{R}^{\beta\gamma} + 3B\bar{R}_{\alpha\beta\gamma\lambda}\bar{\nabla}^\lambda\bar{\nabla}^\gamma\bar{\nabla}^\beta A^\alpha \\ & - 2B\bar{R}_{\alpha\gamma\beta\lambda}\bar{\nabla}^\lambda\bar{\nabla}^\gamma\bar{\nabla}^\beta A^\alpha - 2A^\alpha B\bar{R}^{\beta\gamma\lambda\mu}\bar{\nabla}_\mu\bar{R}_{\alpha\beta\gamma\lambda}, \end{aligned} \quad (6.53)$$

$$\begin{aligned} \delta_{0(B\hat{h})} = & B\hat{h}^{\alpha\beta}\bar{R}_\alpha{}^\gamma\bar{R}_{\beta\gamma} - B\hat{h}^{\alpha\beta}\bar{R}^{\gamma\lambda}\bar{R}_{\alpha\gamma\beta\lambda} - \frac{1}{2}B\bar{R}^{\alpha\beta}\bar{\square}\hat{h}_{\alpha\beta} + B\bar{\nabla}_\beta\bar{R}_{\alpha\gamma}\bar{\nabla}^\gamma\hat{h}^{\alpha\beta} \\ & - \frac{1}{2}B\bar{\nabla}_\gamma\bar{R}_{\alpha\beta}\bar{\nabla}^\gamma\hat{h}^{\alpha\beta}, \end{aligned}$$

$$\delta_{0(\phi A)} = -\frac{1}{4}A^\alpha\phi\bar{\nabla}_\alpha\bar{R} - \frac{1}{2}\bar{R}_{\alpha\beta}\phi\bar{\nabla}^\beta A^\alpha,$$

$$\delta_{0(\phi\hat{h})} = 0, \quad \delta_{0(A\hat{h})} = -A^\alpha\hat{h}^{\beta\gamma}\bar{\nabla}_\alpha\bar{R}_{\beta\gamma} - 2A^\alpha\bar{R}^{\beta\gamma}\bar{\nabla}_\gamma\hat{h}_{\alpha\beta}. \quad (6.54)$$

(ii) Rewriting the above in terms of irreducible $\bar{R}$, $\bar{S}_{\mu\nu}$, $\bar{C}_{\mu\nu\rho\sigma}$ and imposing $\bar{C}_{\mu\nu\rho\sigma} = 0$, the terms purely quadratic in one SVT mode are then [159]

$$\begin{aligned} \delta_{0(BB)} = & \frac{1}{8}\Lambda B\bar{R}\bar{\square}B - \frac{1}{32}B\bar{R}^2\bar{\square}B - \frac{5}{8}B\bar{S}_{\beta\gamma}\bar{S}^{\beta\gamma}\bar{\square}B + \frac{1}{4}\Lambda B\bar{\nabla}_\alpha\bar{R}\bar{\nabla}^\alpha B \\ & - \frac{3}{64}B\bar{R}\bar{\nabla}_\alpha\bar{R}\bar{\nabla}^\alpha B - \frac{7}{8}B\bar{S}^{\beta\gamma}\bar{\nabla}_\alpha\bar{S}_{\beta\gamma}\bar{\nabla}^\alpha B - \frac{1}{16}B\bar{\nabla}^\alpha\bar{R}\bar{\square}\bar{\nabla}_\alpha B \\ & + \frac{1}{4}\Lambda B\bar{\square}^2 B - \frac{1}{16}B\bar{R}\bar{\square}^2 B - \frac{11}{96}B\bar{S}_{\alpha\beta}\bar{\nabla}^\alpha B\bar{\nabla}^\beta\bar{R} + \frac{1}{2}\Lambda B\bar{S}_{\alpha\beta}\bar{\nabla}^\beta\bar{\nabla}^\alpha B \\ & - \frac{11}{96}B\bar{R}\bar{S}_{\alpha\beta}\bar{\nabla}^\beta\bar{\nabla}^\alpha B + \frac{7}{8}B\bar{S}_\alpha{}^\gamma\bar{S}_{\beta\gamma}\bar{\nabla}^\beta\bar{\nabla}^\alpha B - \frac{3}{8}B\bar{S}^{\beta\gamma}\bar{\nabla}^\alpha B\bar{\nabla}_\gamma\bar{S}_{\alpha\beta} \\ & - \frac{5}{8}B\bar{S}_{\alpha\beta}\bar{\square}\bar{\nabla}^\beta\bar{\nabla}^\alpha B + \frac{17}{8}B\bar{S}_{\alpha\gamma}\bar{\nabla}^\gamma\bar{\square}\bar{\nabla}^\alpha B - \frac{5}{4}B\bar{S}_{\beta\gamma}\bar{\nabla}^\gamma\bar{\nabla}^\beta\bar{\square}B \\ & - \frac{3}{4}B\bar{\nabla}_\alpha\bar{S}_{\beta\gamma}\bar{\nabla}^\gamma\bar{\nabla}^\beta\bar{\nabla}^\alpha B + \frac{11}{4}B\bar{\nabla}_\beta\bar{S}_{\alpha\gamma}\bar{\nabla}^\gamma\bar{\nabla}^\beta\bar{\nabla}^\alpha B - \frac{7}{4}B\bar{\nabla}_\gamma\bar{S}_{\alpha\beta}\bar{\nabla}^\gamma\bar{\nabla}^\beta\bar{\nabla}^\alpha B, \end{aligned} \quad (6.55)$$

$$\delta_{0(\phi\phi)} = -\frac{1}{8}\Lambda\phi^2 - \frac{3}{32}\phi\bar{\square}\phi,$$

$$\begin{aligned}\delta_{0(AA)} = & -\frac{1}{4}\Lambda A_\alpha A^\alpha \bar{R} + \frac{1}{16}A_\alpha A^\alpha \bar{R}^2 - \Lambda A^\alpha A^\beta \bar{S}_{\alpha\beta} + \frac{1}{3}A^\alpha A^\beta \bar{R}\bar{S}_{\alpha\beta} \\ & + A^\alpha A^\beta \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} - \frac{1}{2}A_\alpha A^\alpha \bar{S}_{\beta\gamma} \bar{S}^{\beta\gamma} + \frac{1}{2}A^\alpha \bar{\nabla}_\alpha A^\beta \bar{\nabla}_\beta \bar{R} + \frac{1}{2}A^\alpha A^\beta \bar{\nabla}_\beta \bar{\nabla}_\alpha \bar{R} \\ & - \Lambda A^\alpha \bar{\square} A_\alpha + \frac{1}{4}A^\alpha \bar{R} \bar{\square} A_\alpha - A^\alpha \bar{S}_{\beta\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta A_\alpha,\end{aligned}$$

$$\delta_{0(\hat{h}\hat{h})} = \frac{1}{2}\Lambda \hat{h}_{\alpha\beta} \hat{h}^{\alpha\beta} - \frac{1}{6}\hat{h}_{\alpha\beta} \hat{h}^{\alpha\beta} \bar{R} + \frac{1}{4}\hat{h}^{\alpha\beta} \bar{\square} \hat{h}_{\alpha\beta},$$

(6.56)

with the mixed terms [159]

$$\begin{aligned}\delta_{0(B\phi)} &= \frac{1}{4}\Lambda B \bar{\square} \phi - \frac{1}{16}B \bar{R} \bar{\square} \phi - \frac{1}{8}B \bar{\nabla}_\alpha \phi \bar{\nabla}^\alpha \bar{R} - \frac{1}{4}B \bar{S}_{\alpha\beta} \bar{\nabla}^\beta \bar{\nabla}^\alpha \phi, \\ \delta_{0(BA)} &= \Lambda A^\alpha B \bar{\nabla}_\alpha \bar{R} - \frac{5}{24}A^\alpha B \bar{R} \bar{\nabla}_\alpha \bar{R} - 4A^\alpha B \bar{S}^{\beta\gamma} \bar{\nabla}_\alpha \bar{S}_{\beta\gamma} - \frac{1}{4}B \bar{\nabla}^\alpha \bar{R} \bar{\square} A_\alpha \\ &\quad + 2\Lambda B \bar{S}_{\alpha\beta} \bar{\nabla}^\beta A^\alpha + \frac{5}{4}B \bar{R} \bar{S}_{\alpha\beta} \bar{\nabla}^\beta A^\alpha + 11B \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} \bar{\nabla}^\beta A^\alpha \\ &\quad + \frac{13}{12}A^\alpha B \bar{S}_{\alpha\beta} \bar{\nabla}^\beta \bar{R} + 5A^\alpha B \bar{S}^{\beta\gamma} \bar{\nabla}_\gamma \bar{S}_{\alpha\beta} - 3B \bar{S}_{\alpha\beta} \bar{\square} \bar{\nabla}^\beta A^\alpha + 3B \bar{S}_{\alpha\gamma} \bar{\nabla}^\gamma \bar{\square} A^\alpha \\ &\quad + B \bar{\nabla}_\alpha \bar{S}_{\beta\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta A^\alpha + 6B \bar{\nabla}_\beta \bar{S}_{\alpha\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta A^\alpha - 6B \bar{\nabla}_\gamma \bar{S}_{\alpha\beta} \bar{\nabla}^\gamma \bar{\nabla}^\beta A^\alpha, \\ \delta_{0(B\hat{h})} &= \frac{1}{3}B \hat{h}^{\alpha\beta} \bar{R} \bar{S}_{\alpha\beta} + 2B \hat{h}^{\alpha\beta} \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} - \frac{1}{2}B \bar{S}^{\alpha\beta} \bar{\square} \hat{h}_{\alpha\beta} + B \bar{\nabla}_\beta \bar{S}_{\alpha\gamma} \bar{\nabla}^\gamma \hat{h}^{\alpha\beta} \\ &\quad - \frac{1}{2}B \bar{\nabla}_\gamma \bar{S}_{\alpha\beta} \bar{\nabla}^\gamma \hat{h}^{\alpha\beta}, \\ \delta_{0(\phi A)} &= -\frac{1}{4}A^\alpha \phi \bar{\nabla}_\alpha \bar{R} - \frac{1}{2}\bar{S}_{\alpha\beta} \phi \bar{\nabla}^\beta A^\alpha, \quad \delta_{0(\phi\hat{h})} = 0, \\ \delta_{0(A\hat{h})} &= -A^\alpha \hat{h}^{\beta\gamma} \bar{\nabla}_\alpha \bar{S}_{\beta\gamma} - 2A^\alpha \bar{S}^{\beta\gamma} \bar{\nabla}_\gamma \hat{h}_{\alpha\beta}.\end{aligned}$$

(6.57)

(iii) Imposing $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0, \partial_\mu \bar{R} = 0$ and substituting $\Lambda = \bar{R}/4$ which solves the background equations of motion (see eq. (3.16)), the terms purely quadratic in one

SVT mode are [159]

$$\begin{aligned}
\delta_{0(BB)} &= \frac{1}{8} B \bar{R} \bar{S}_{\alpha\beta} \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \frac{1}{2} B \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \frac{1}{4} B \bar{S}_{\beta\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta \bar{\square} B \\
\delta_{0(\phi\phi)} &= -\frac{1}{32} \bar{R} \phi^2 - \frac{3}{32} \phi \bar{\square} \phi, \\
\delta_{0(AA)} &= \frac{1}{12} A^\alpha A^\beta \bar{R} \bar{S}_{\alpha\beta} + A^\alpha A^\beta \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} - \frac{1}{2} A_\alpha A^\alpha \bar{S}_{\beta\gamma} \bar{S}^{\beta\gamma} - A^\alpha \bar{S}_{\beta\gamma} \bar{\nabla}^\gamma \bar{\nabla}^\beta A_\alpha \\
\delta_{0(\hat{h}\hat{h})} &= -\frac{1}{24} \hat{h}_{\alpha\beta} \hat{h}^{\alpha\beta} \bar{R} + \frac{1}{4} \hat{h}^{\alpha\beta} \bar{\square} \hat{h}_{\alpha\beta},
\end{aligned} \tag{6.58}$$

and the mixings are [159]

$$\begin{aligned}
\delta_{0(B\phi)} &= -\frac{1}{4} B \bar{S}_{\alpha\beta} \bar{\nabla}^\beta \bar{\nabla}^\alpha \phi, & \delta_{0(BA)} &= \frac{1}{2} B \bar{R} \bar{S}_{\alpha\beta} \bar{\nabla}^\beta A^\alpha + 2 B \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} \bar{\nabla}^\beta A^\alpha, \\
\delta_{0(B\hat{h})} &= \frac{1}{3} B \hat{h}^{\alpha\beta} \bar{R} \bar{S}_{\alpha\beta} + 2 B \hat{h}^{\alpha\beta} \bar{S}_\alpha{}^\gamma \bar{S}_{\beta\gamma} - \frac{1}{2} B \bar{S}^{\alpha\beta} \bar{\square} \hat{h}_{\alpha\beta}, \\
\delta_{0(\phi A)} &= -\frac{1}{2} \bar{S}_{\alpha\beta} \phi \bar{\nabla}^\beta A^\alpha, \delta_{0(\phi\hat{h})} = 0, & \delta_{0(A\hat{h})} &= -2 A^\alpha \bar{S}^{\beta\gamma} \bar{\nabla}_\gamma \hat{h}_{\alpha\beta}.
\end{aligned} \tag{6.59}$$

$$\mathbf{6.F.2} \quad \delta^2 \mathcal{L}_{R^2} = \delta^2 \left[\frac{1}{2} \sqrt{-g} R \mathcal{F}_1(\square_s) R \right]$$

We need to compute $\delta^2 [\sqrt{-g} R \mathcal{F}_1(\square_s) R]$ given in eq. (3.39).

(i) We just need to find perturbed Ricci scalars which, for an *arbitrary* background, are [159]

$$r_{(\phi)} = \frac{1}{4} \bar{R} \phi + \frac{3}{4} \bar{\square} \phi, \quad r_{(B)} = \frac{1}{2} \bar{\nabla}_\alpha \bar{R} \bar{\nabla}^\alpha B, \quad r_{(A)} = A^\alpha \bar{\nabla}_\alpha \bar{R}, \quad r_{(\hat{h})} = -\hat{h}^{\alpha\beta} \bar{R}_{\alpha\beta}. \tag{6.60}$$

(ii) Rewriting the above in terms of irreducible $\bar{R}, \bar{S}_{\mu\nu}, \bar{C}_{\mu\nu\rho\sigma}$ and imposing $\bar{C}_{\mu\nu\rho\sigma} = 0$ gives [159]

$$r_{(\phi)} = \frac{1}{4} \bar{R} \phi + \frac{3}{4} \bar{\square} \phi, \quad r_{(B)} = \frac{1}{2} \bar{\nabla}_\alpha \bar{R} \bar{\nabla}^\alpha B, \quad r_{(A)} = A^\alpha \bar{\nabla}_\alpha \bar{R}, \quad r_{(\hat{h})} = -\hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta}. \tag{6.61}$$

(iii) Imposing $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0, \partial_\mu \bar{R} = 0$ to above gives [159]

$$r_{(\phi)} = \frac{1}{4} \bar{R} \phi + \frac{3}{4} \bar{\square} \phi, \quad r_{(B)} = 0, \quad r_{(A)} = 0, \quad r_{(\hat{h})} = -\hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta}. \tag{6.62}$$

Finally, the non-zero terms are [159]

$$\begin{aligned}
(0101)_{\phi\phi} &= \left(\frac{1}{4} \bar{R}\phi + \frac{3}{4} \bar{\square}\phi \right) \mathcal{F}_1(\bar{\square}_s) \left(\frac{1}{4} \bar{R}\phi + \frac{3}{4} \bar{\square}\phi \right), \\
(0101)_{\hat{h}\hat{h}} &= \left(\hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta} \right) \mathcal{F}_1(\bar{\square}_s) \left(\hat{h}_{\mu\nu} \bar{S}^{\mu\nu} \right), \\
(0101)_{\phi\hat{h}} &= \left(-\frac{1}{2} \bar{R}\phi - \frac{3}{2} \bar{\square}\phi \right) \mathcal{F}_1(\bar{\square}_s) \left(\hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta} \right).
\end{aligned} \tag{6.63}$$

All other SVT perturbations are zero. The final expressions for $\delta^2 \mathcal{L}_{R^2}$ which use the above expressions are given in eq.(3.40), eq.(3.41).

$$\mathbf{6.F.3} \quad \delta^2 \mathcal{L}_{S^2} = \delta^2 \left[\frac{1}{2} \sqrt{-g} S^\nu_\mu \mathcal{F}_2(\square_s) S^\mu_\nu \right]$$

$\delta^2 [\sqrt{-g} S^\nu_\mu \mathcal{F}_2(\square_s) S^\mu_\nu]$ has perhaps the most complicated terms eq.(3.43). We present some of them here. We need s^ν_μ , which in terms of $h_{\mu\nu}$ around an *arbitrary* background is [159]

$$\begin{aligned}
s^\nu_\mu &= \frac{1}{4} \delta_\mu^\nu h^{\alpha\beta} \bar{R}_{\alpha\beta} + \frac{1}{2} h^{\nu\alpha} \bar{R}_{\mu\alpha} + \frac{1}{2} h_\mu^\alpha \bar{R}^\nu_\alpha - \frac{1}{4} h_\mu^\nu \bar{R} - h^{\alpha\beta} \bar{R}_{\mu\alpha}{}^\nu{}_\beta - h^{\nu\alpha} \bar{S}_{\mu\alpha} - \frac{1}{2} \bar{\square} h_\mu^\nu \\
&\quad - \frac{1}{4} \delta_\mu^\nu \bar{\nabla}_\beta \bar{\nabla}_\alpha h^{\alpha\beta} + \frac{1}{4} \delta_\mu^\nu \bar{\square} h + \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\alpha h^{\nu\alpha} + \frac{1}{2} \bar{\nabla}^\nu \bar{\nabla}_\alpha h_\mu^\alpha - \frac{1}{2} \bar{\nabla}^\nu \bar{\nabla}_\mu h,
\end{aligned} \tag{6.64}$$

(i) Its SVT decomposition is [159]

$$\begin{aligned}
s^\nu_\mu &= \frac{1}{4} \delta_\mu^\nu \hat{h}^{\alpha\beta} \bar{R}_{\alpha\beta} + \frac{1}{2} \hat{h}^{\nu\alpha} \bar{R}_{\mu\alpha} + \frac{1}{2} \hat{h}_\mu^\alpha \bar{R}^\nu_\alpha - \frac{1}{4} \hat{h}_\mu^\nu \bar{R} - \hat{h}^{\alpha\beta} \bar{R}_{\mu\alpha}{}^\nu{}_\beta - \hat{h}^{\nu\alpha} \bar{S}_{\mu\alpha} \\
&\quad + \frac{1}{4} \bar{S}_\mu^\nu \phi + A^\alpha \bar{\nabla}_\alpha \bar{R}_\mu^\nu - \frac{1}{4} A^\alpha \delta_\mu^\nu \bar{\nabla}_\alpha \bar{R} - \frac{1}{2} \bar{\square} \hat{h}_\mu^\nu - \frac{1}{16} \delta_\mu^\nu \bar{\square} \phi \\
&\quad - \bar{S}_{\mu\alpha} \bar{\nabla}^\alpha A^\nu + \frac{1}{2} \bar{\nabla}_\alpha \bar{R}_\mu^\nu \bar{\nabla}^\alpha B - \frac{1}{8} \delta_\mu^\nu \bar{\nabla}_\alpha \bar{R} \bar{\nabla}^\alpha B + \bar{R}_{\mu\alpha}{}^\nu{}_\beta \bar{\nabla}^\beta \bar{\nabla}^\alpha B \\
&\quad - \bar{R}_{\mu\beta}{}^\nu{}_\alpha \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \bar{R}^\nu{}_\alpha \bar{\nabla}_\mu A^\alpha - \frac{1}{4} \bar{R} \bar{\nabla}_\mu A^\nu + \frac{1}{2} \bar{R}^\nu{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B \\
&\quad + \bar{R}_{\mu\alpha} \bar{\nabla}^\nu A^\alpha - \bar{S}_{\mu\alpha} \bar{\nabla}^\nu A^\alpha - \frac{1}{4} \bar{R} \bar{\nabla}^\nu A_\mu + \frac{1}{2} \bar{R}_{\mu\alpha} \bar{\nabla}^\nu \bar{\nabla}^\alpha B \\
&\quad - \bar{S}_{\mu\alpha} \bar{\nabla}^\nu \bar{\nabla}^\alpha B - \frac{1}{4} \bar{R} \bar{\nabla}^\nu \bar{\nabla}_\mu B + \frac{1}{4} \bar{\nabla}^\nu \bar{\nabla}_\mu \phi.
\end{aligned} \tag{6.65}$$

(ii) Rewriting the above in terms of irreducible $\bar{R}, \bar{S}_{\mu\nu}, \bar{C}_{\mu\nu\rho\sigma}$ and imposing $\bar{C}_{\mu\nu\rho\sigma} = 0$, we get [159]

$$\begin{aligned} s^\nu_\mu = & \frac{1}{12} \hat{h}_\mu{}^\nu \bar{R} - \frac{1}{4} \delta_\mu{}^\nu \hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta} + \hat{h}_\mu{}^\alpha \bar{S}^\nu{}_\alpha + \frac{1}{4} \bar{S}_\mu{}^\nu \phi + A^\alpha \bar{\nabla}_\alpha \bar{S}_\mu{}^\nu - \frac{1}{2} \bar{\square} \hat{h}_\mu{}^\nu \\ & - \frac{1}{16} \delta_\mu{}^\nu \bar{\square} \phi - \bar{S}_{\mu\alpha} \bar{\nabla}^\alpha A^\nu + \frac{1}{2} \bar{\nabla}_\alpha \bar{S}_\mu{}^\nu \bar{\nabla}^\alpha B + \bar{S}^\nu{}_\alpha \bar{\nabla}_\mu A^\alpha + \frac{1}{2} \bar{S}^\nu{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B \\ & - \frac{1}{2} \bar{S}_{\mu\alpha} \bar{\nabla}^\nu \bar{\nabla}^\alpha B + \frac{1}{4} \bar{\nabla}^\nu \bar{\nabla}_\mu \phi. \end{aligned} \quad (6.66)$$

(iii) Imposing $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0, \partial_\mu \bar{R} = 0$ gives [159]

$$\begin{aligned} s^\nu_\mu = & \frac{1}{12} \hat{h}_\mu{}^\nu \bar{R} - \frac{1}{4} \delta_\mu{}^\nu \hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta} + \hat{h}_\mu{}^\alpha \bar{S}^\nu{}_\alpha + \frac{1}{4} \bar{S}_\mu{}^\nu \phi - \frac{1}{2} \bar{\square} \hat{h}_\mu{}^\nu - \frac{1}{16} \delta_\mu{}^\nu \bar{\square} \phi \\ & - \bar{S}_{\mu\alpha} \bar{\nabla}^\alpha A^\nu + \bar{S}^\nu{}_\alpha \bar{\nabla}_\mu A^\alpha + \frac{1}{2} \bar{S}^\nu{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B - \frac{1}{2} \bar{S}_{\mu\alpha} \bar{\nabla}^\nu \bar{\nabla}^\alpha B + \frac{1}{4} \bar{\nabla}^\nu \bar{\nabla}_\mu \phi. \end{aligned} \quad (6.67)$$

Let us now compute the SVT decomposition for (0101) [159]

$$\begin{aligned} (0101) = & s^\nu_\mu \mathcal{F}_2(\bar{\square}_s) s^\mu_\nu = \left[\frac{1}{12} \hat{h}_\mu{}^\nu \bar{R} - \frac{1}{4} \delta_\mu{}^\nu \hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta} + \hat{h}_\mu{}^\alpha \bar{S}^\nu{}_\alpha + \frac{1}{4} \bar{S}_\mu{}^\nu \phi - \frac{1}{2} \bar{\square} \hat{h}_\mu{}^\nu \right. \\ & - \frac{1}{16} \delta_\mu{}^\nu \bar{\square} \phi - \bar{S}_{\mu\alpha} \bar{\nabla}^\alpha A^\nu + \bar{S}^\nu{}_\alpha \bar{\nabla}_\mu A^\alpha + \frac{1}{2} \bar{S}^\nu{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B \\ & \left. - \frac{1}{2} \bar{S}_{\mu\alpha} \bar{\nabla}^\nu \bar{\nabla}^\alpha B + \frac{1}{4} \bar{\nabla}^\nu \bar{\nabla}_\mu \phi \right] \mathcal{F}_2(\bar{\square}_s) s^\mu_\nu, \\ = & \left[\frac{1}{12} \hat{h}_\mu{}^\nu \bar{R} + \hat{h}_\mu{}^\alpha \bar{S}^\nu{}_\alpha + \frac{1}{4} \bar{S}_\mu{}^\nu \phi - \frac{1}{2} \bar{\square} \hat{h}_\mu{}^\nu - \bar{S}_{\mu\alpha} \bar{\nabla}^\alpha A^\nu \right. \\ & + \bar{S}^\nu{}_\alpha \bar{\nabla}_\mu A^\alpha + \frac{1}{2} \bar{S}^\nu{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B - \frac{1}{2} \bar{S}_{\mu\alpha} \bar{\nabla}^\nu \bar{\nabla}^\alpha B \\ & \left. + \frac{1}{4} \bar{\nabla}^\nu \bar{\nabla}_\mu \phi \right] \mathcal{F}_2(\bar{\square}_s) s^\mu_\nu, \end{aligned} \quad (6.68)$$

where in the second equality, we canceled the terms to the left (only left, not right) of $\mathcal{F}_2(\bar{\square}_s)$ which contain Kronecker deltas $\delta_\mu{}^\nu$ because they hit s^μ_ν to the right of $\mathcal{F}_2(\bar{\square}_s)$ and vanish because of tracelessness. Then, we similarly expand s^μ_ν on the right, integrate by parts and get the following expressions for purely quadratic in

one SVT mode [159]

$$\begin{aligned}
(0101)_{\phi\phi} &= \phi \left[\frac{1}{16} \mathcal{F}_2(\bar{\square}_s) \bar{S}_\mu{}^\nu \bar{S}^\mu{}_\nu + \frac{1}{8} \mathcal{F}_2(\bar{\square}_s) \bar{S}_\mu{}^\nu \bar{\nabla}_\nu \bar{\nabla}^\mu \right. \\
&\quad \left. + \frac{1}{16} \bar{\nabla}_\mu \bar{\nabla}^\nu \mathcal{F}_2(\bar{\square}_s) \bar{\nabla}_\nu \bar{\nabla}^\mu - \frac{1}{64} \mathcal{F}_2(\bar{\square}_s) \bar{\square}^2 \right] \phi, \\
(0101)_{BB} &= B \left[\frac{1}{2} \bar{\nabla}^\alpha \bar{\nabla}_\mu \mathcal{F}_2(\bar{\square}_s) \bar{S}^\nu{}_\alpha \bar{S}^\mu{}_\rho \bar{\nabla}_\nu \bar{\nabla}^\rho - \right. \\
&\quad \left. \frac{1}{2} \bar{\nabla}^\alpha \bar{\nabla}_\mu \mathcal{F}_2(\bar{\square}_s) \bar{S}^\nu{}_\alpha \bar{S}_{\nu\rho} \bar{\nabla}^\mu \bar{\nabla}^\rho \right] B, \\
(0101)_{AA} &= A^\nu \left[- \bar{\nabla}^\alpha \mathcal{F}_2(\bar{\square}_s) \bar{S}_{\rho\alpha} \bar{S}^\rho{}_\mu \bar{\nabla}_\nu - \bar{\nabla}_\mu \mathcal{F}_2(\bar{\square}_s) \bar{S}_{\lambda\rho} \bar{S}^\lambda{}_\nu \bar{\nabla}^\rho \right. \\
&\quad \left. + 2 \bar{\nabla}^\alpha \mathcal{F}_2(\bar{\square}_s) \bar{S}_{\mu\alpha} \bar{S}_{\nu\rho} \bar{\nabla}^\rho \right] A^\mu, \\
(0101)_{\hat{h}\hat{h}} &= \hat{h}_{\mu\nu} \left[\frac{1}{144} \mathcal{F}_2(\bar{\square}_s) \bar{R}^2 \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} + \frac{1}{6} \mathcal{F}_2(\bar{\square}_s) \bar{R} \bar{S}^{\mu\rho} \bar{g}^{\nu\sigma} \right. \\
&\quad - \frac{1}{12} \mathcal{F}_2(\bar{\square}_s) \bar{R} \bar{\square} \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} + \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\sigma\nu} \bar{S}^{\mu\rho} \\
&\quad - \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\sigma\nu} \bar{\square} \bar{g}^{\mu\rho} + \frac{1}{4} \mathcal{F}_2(\bar{\square}_s) \bar{\square}^2 \bar{g}^{\rho\mu} \bar{g}^{\sigma\nu} \\
&\quad \left. - \frac{1}{4} \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\mu\nu} \bar{S}^{\rho\sigma} \right] \hat{h}_{\rho\sigma},
\end{aligned} \tag{6.69}$$

and the mixings are [159]

$$\begin{aligned}
(0101)_{\phi B} &= 0, \\
(0101)_{\phi A} &= A^\mu \left[\frac{1}{2} \bar{\nabla}^\alpha \mathcal{F}_2(\bar{\square}_s) \bar{S}_{\nu\alpha} \bar{\nabla}^\nu \bar{\nabla}_\mu - \frac{1}{2} \bar{\nabla}_\nu \mathcal{F}_2(\bar{\square}_s) \bar{S}^\alpha{}_\mu \bar{\nabla}^\nu \bar{\nabla}_\alpha \right] \phi, \\
(0101)_{\phi \hat{h}} &= \hat{h}_{\mu\nu} \left[\frac{1}{24} \bar{R} \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\mu\nu} + \frac{1}{24} \bar{R} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla}^\nu \bar{\nabla}^\mu + \frac{1}{2} \bar{S}^{\sigma\mu} \mathcal{F}_2(\bar{\square}_s) \bar{S}_\sigma{}^\nu \right. \\
&\quad \left. + \frac{1}{2} \bar{S}^{\sigma\mu} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla}^\nu \bar{\nabla}_\sigma - \frac{1}{4} \bar{\square} \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\mu\nu} \right. \\
&\quad \left. - \frac{1}{4} \bar{\square} \mathcal{F}_2(\bar{\square}_s) \bar{\nabla}^\nu \bar{\nabla}^\mu - \frac{1}{8} \bar{S}^{\mu\nu} \mathcal{F}_2(\bar{\square}_s) \bar{\square} \right] \phi, \\
(0101)_{BA} &= A^\mu \left[\bar{\nabla}^\alpha \mathcal{F}_2(\bar{\square}_s) \bar{S}_{\nu\alpha} \bar{S}^\nu{}_\rho \bar{\nabla}_\mu \bar{\nabla}^\rho - 2 \bar{\nabla}^\alpha \mathcal{F}_2(\bar{\square}_s) \bar{S}_{\nu\alpha} \bar{S}_{\mu\rho} \bar{\nabla}^\nu \bar{\nabla}^\rho \right. \\
&\quad \left. + \bar{\nabla}_\nu \mathcal{F}_2(\bar{\square}_s) \bar{S}^\sigma{}_\mu \bar{S}_{\sigma\rho} \bar{\nabla}^\nu \bar{\nabla}^\rho \right] B, \\
(0101)_{B\hat{h}} &= \hat{h}_{\mu\nu} \left[-\mathcal{F}_2(\bar{\square}_s) \bar{S}^{\sigma\nu} \bar{S}_{\sigma\rho} \bar{\nabla}^\mu \bar{\nabla}^\rho + \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\sigma\nu} \bar{S}^\mu{}_\rho \bar{\nabla}_\sigma \bar{\nabla}^\rho \right] B, \\
(0101)_{A\hat{h}} &= \hat{h}_{\mu\nu} \left[-\frac{1}{6} \mathcal{F}_2(\bar{\square}_s) \bar{R} \bar{S}^\nu{}_\rho \bar{\nabla}^\rho \delta^\mu{}_\lambda + \frac{1}{6} \mathcal{F}_2(\bar{\square}_s) \bar{R} \bar{S}^\mu{}_\rho \bar{\nabla}^\nu \delta^\rho{}_\lambda \right. \\
&\quad \left. - 2 \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\sigma\nu} \bar{S}_{\sigma\rho} \bar{\nabla}^\rho \delta^\mu{}_\lambda + 2 \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\sigma\nu} \bar{S}^\mu{}_\rho \bar{\nabla}_\sigma \delta^\rho{}_\lambda + \right. \\
&\quad \left. \mathcal{F}_2(\bar{\square}_s) \bar{\square} \bar{S}^\nu{}_\rho \bar{\nabla}^\rho \delta^\mu{}_\lambda - \mathcal{F}_2(\bar{\square}_s) \bar{\square} \bar{S}^\mu{}_\rho \bar{\nabla}^\nu \delta^\rho{}_\lambda \right] A^\lambda.
\end{aligned} \tag{6.70}$$

We now consider the perturbation (1001) in SVT form [159]

$$\begin{aligned}
(1001) &= \frac{1}{2} \bar{S}^\nu{}_\mu s^\mu{}_\nu \mathcal{F}_2(\bar{\square}_s) h = \frac{1}{2} \bar{S}^\nu{}_\mu \left[\frac{1}{12} \hat{h}^\mu{}_\nu \bar{R} + \hat{h}_\nu{}^\alpha \bar{S}^\mu{}_\alpha + \frac{1}{4} \bar{S}^\mu{}_\nu \phi - \frac{1}{2} \bar{\square} \hat{h}^\mu{}_\nu \right. \\
&\quad \left. - \bar{S}_{\nu\alpha} \bar{\nabla}^\alpha A^\mu - \frac{1}{2} \bar{S}_{\nu\alpha} \bar{\nabla}^\mu \bar{\nabla}^\alpha B + \bar{S}^\mu{}_\alpha \bar{\nabla}_\nu A^\alpha \right. \\
&\quad \left. + \frac{1}{2} \bar{S}^\mu{}_\alpha \bar{\nabla}_\nu \bar{\nabla}^\alpha B + \frac{1}{4} \bar{\nabla}_\nu \bar{\nabla}^\mu \phi \right] \mathcal{F}_2(\bar{\square}_s) [\bar{\square} B - \phi],
\end{aligned} \tag{6.71}$$

from which we get the terms with purely quadratic in one SVT mode [159]

$$\begin{aligned} (1001)_{\phi\phi} &= \phi \left[-\frac{1}{8} \bar{S}^\nu_\mu \bar{S}^\mu_\nu \mathcal{F}_2(\bar{\square}_s) - \frac{1}{8} \bar{\nabla}^\mu \bar{\nabla}_\nu \bar{S}^\nu_\mu \mathcal{F}_2(\bar{\square}_s) \right] \phi, \\ (1001)_{BB} &= B \left[-\frac{1}{4} \bar{S}^\nu_\mu \bar{S}_{\nu\alpha} \bar{\nabla}^\alpha \bar{\nabla}^\mu \mathcal{F}_2(\bar{\square}_s) \bar{\square} + \frac{1}{4} \bar{S}^\nu_\mu \bar{S}^\mu_\alpha \bar{\nabla}^\alpha \bar{\nabla}_\nu \mathcal{F}_2(\bar{\square}_s) \bar{\square} \right] B, \end{aligned} \quad (6.72)$$

and the mixings [159]

$$\begin{aligned} (1001)_{\phi B} &= B \left[\frac{1}{8} \bar{S}^\nu_\mu \bar{S}^\mu_\nu \mathcal{F}_2(\bar{\square}_s) \bar{\square} + \frac{1}{8} \bar{S}^\nu_\mu \mathcal{F}_2(\bar{\square}_s) \bar{\square} \bar{\nabla}_\nu \bar{\nabla}^\mu \right] \phi, \\ (1001)_{\phi A} &= A^\mu \left[-\frac{1}{2} \bar{S}^\nu_\mu \bar{S}_{\nu\alpha} \bar{\nabla}^\alpha \mathcal{F}_2(\bar{\square}_s) + \frac{1}{2} \bar{S}^\nu_\rho \bar{S}^\rho_\mu \bar{\nabla}_\nu \mathcal{F}_2(\bar{\square}_s) \right] \phi, \\ (1001)_{\phi\hat{h}} &= \hat{h}_{\mu\nu} \left[-\frac{1}{24} \bar{S}^{\mu\nu} \bar{R} \mathcal{F}_2(\bar{\square}_s) - \frac{1}{2} \bar{S}^\nu_\rho \bar{S}^{\rho\mu} \mathcal{F}_2(\bar{\square}_s) + \frac{1}{4} \bar{S}^{\mu\nu} \mathcal{F}_2(\bar{\square}_s) \bar{\square} \right] \phi, \\ (1001)_{BA} &= A^\mu \left[\frac{1}{2} \bar{S}^\nu_\mu \bar{S}_{\nu\alpha} \bar{\nabla}^\alpha \mathcal{F}_2(\bar{\square}_s) \bar{\square} - \frac{1}{2} \bar{S}^\nu_\rho \bar{S}^\rho_\mu \bar{\nabla}_\nu \mathcal{F}_2(\bar{\square}_s) \bar{\square} \right] B, \\ (1001)_{B\hat{h}} &= \hat{h}_{\mu\nu} \left[\frac{1}{24} \bar{S}^{\mu\nu} \bar{R} \mathcal{F}_2(\bar{\square}_s) \bar{\square} + \frac{1}{2} \bar{S}^\nu_\rho \bar{S}^{\rho\mu} \mathcal{F}_2(\bar{\square}_s) \bar{\square} - \frac{1}{4} \bar{S}^{\mu\nu} \mathcal{F}_2(\bar{\square}_s) \bar{\square}^2 \right] B. \end{aligned} \quad (6.73)$$

$$\mathbf{6.F.4} \quad \delta^2 \mathcal{L}_{C^2} = \delta^2 \left[\frac{1}{2} \sqrt{-g} C^{\rho\sigma}_{\mu\nu} \mathcal{F}_3(\bar{\square}_s) C^{\mu\nu}_{\rho\sigma} \right]$$

We now compute $\delta^2 [\sqrt{-g} C^{\rho\sigma}_{\mu\nu} \mathcal{F}_3(\bar{\square}_s) C^{\mu\nu}_{\rho\sigma}]$. We only present the term $(0101) = \sqrt{-g} c^{\rho\sigma}_{\mu\nu} \bar{\mathcal{F}}_3(\bar{\square}_s) c^{\mu\nu}_{\rho\sigma}$ which is relevant for our background (see eq.(3.58)). Other terms, for an *arbitrary* background, can be computed from Appendix (6.E) but we do not present them here because they are not very enlightening.

(i) Performing SVT decomposition of $c^{\rho\sigma}_{\mu\nu}$ for an *arbitrary* background, we get [159]

$$\begin{aligned} c^{\rho\sigma}_{\mu\nu(\phi)} &= \frac{1}{8} \delta_\nu^\sigma \bar{R}_\mu{}^\rho \phi - \frac{1}{8} \delta_\nu^\rho \bar{R}_\mu{}^\sigma \phi - \frac{1}{8} \delta_\mu^\sigma \bar{R}_\nu{}^\rho \phi + \frac{1}{8} \delta_\mu^\rho \bar{R}_\nu{}^\sigma \phi + \frac{1}{24} \delta_\mu^\sigma \delta_\nu^\rho \bar{R} \phi \\ &\quad - \frac{1}{24} \delta_\mu^\rho \delta_\nu^\sigma \bar{R} \phi - \frac{1}{4} \bar{R}_{\mu\nu}{}^{\rho\sigma} \phi, \end{aligned} \quad (6.74)$$

$$\begin{aligned}
c^{\rho\sigma}_{\mu\nu(B)} = & -\frac{1}{4}\delta_\nu^\sigma \bar{\nabla}_\alpha \bar{R}_\mu{}^\rho \bar{\nabla}^\alpha B + \frac{1}{4}\delta_\nu^\rho \bar{\nabla}_\alpha \bar{R}_\mu{}^\sigma \bar{\nabla}^\alpha B + \frac{1}{4}\delta_\mu^\sigma \bar{\nabla}_\alpha \bar{R}_\nu{}^\rho \bar{\nabla}^\alpha B \\
& -\frac{1}{4}\delta_\mu^\rho \bar{\nabla}_\alpha \bar{R}_\nu{}^\sigma \bar{\nabla}^\alpha B - \frac{1}{12}\delta_\mu^\sigma \delta_\nu^\rho \bar{\nabla}_\alpha \bar{R} \bar{\nabla}^\alpha B + \frac{1}{12}\delta_\mu^\rho \delta_\nu^\sigma \bar{\nabla}_\alpha \bar{R} \bar{\nabla}^\alpha B \\
& -\frac{1}{2}\delta_\nu^\rho \bar{R}_{\mu\alpha}{}^\rho{}_\beta \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \frac{1}{2}\delta_\nu^\rho \bar{R}_{\mu\alpha}{}^\sigma{}_\beta \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \frac{1}{2}\delta_\nu^\sigma \bar{R}_{\mu\beta}{}^\rho{}_\alpha \bar{\nabla}^\beta \bar{\nabla}^\alpha B \\
& -\frac{1}{2}\delta_\nu^\rho \bar{R}_{\mu\beta}{}^\sigma{}_\alpha \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \frac{1}{2}\delta_\mu^\sigma \bar{R}_{\nu\alpha}{}^\rho{}_\beta \bar{\nabla}^\beta \bar{\nabla}^\alpha B - \frac{1}{2}\delta_\mu^\rho \bar{R}_{\nu\alpha}{}^\sigma{}_\beta \bar{\nabla}^\beta \bar{\nabla}^\alpha B \\
& -\frac{1}{2}\delta_\mu^\sigma \bar{R}_{\nu\beta}{}^\rho{}_\alpha \bar{\nabla}^\beta \bar{\nabla}^\alpha B + \frac{1}{2}\delta_\mu^\rho \bar{R}_{\nu\beta}{}^\sigma{}_\alpha \bar{\nabla}^\beta \bar{\nabla}^\alpha B - \frac{1}{4}\delta_\nu^\sigma \bar{R}^\rho{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B \\
& + \frac{1}{4}\delta_\nu^\rho \bar{R}^\sigma{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B + \frac{1}{2}\bar{R}_{\nu}{}^{\rho\sigma}{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B - \frac{1}{2}\bar{R}_{\nu}{}^{\sigma\rho}{}_\alpha \bar{\nabla}_\mu \bar{\nabla}^\alpha B \\
& + \frac{1}{4}\delta_\mu^\sigma \bar{R}^\rho{}_\alpha \bar{\nabla}_\nu \bar{\nabla}^\alpha B - \frac{1}{4}\delta_\mu^\rho \bar{R}^\sigma{}_\alpha \bar{\nabla}_\nu \bar{\nabla}^\alpha B - \frac{1}{2}\bar{R}_{\mu}{}^{\rho\sigma}{}_\alpha \bar{\nabla}_\nu \bar{\nabla}^\alpha B \\
& + \frac{1}{2}\bar{R}_{\mu}{}^{\sigma\rho}{}_\alpha \bar{\nabla}_\nu \bar{\nabla}^\alpha B - \frac{1}{2}\bar{\nabla}^\alpha B \bar{\nabla}^\rho \bar{R}_{\mu\nu}{}^\sigma{}_\alpha - \frac{1}{4}\delta_\nu^\sigma \bar{R}_{\mu\alpha} \bar{\nabla}^\rho \bar{\nabla}^\alpha B \\
& + \frac{1}{4}\delta_\mu^\sigma \bar{R}_{\nu\alpha} \bar{\nabla}^\rho \bar{\nabla}^\alpha B - \frac{1}{2}\bar{R}_{\mu\alpha\nu}{}^\sigma \bar{\nabla}^\rho \bar{\nabla}^\alpha B - \bar{R}_{\mu\nu}{}^\sigma{}_\alpha \bar{\nabla}^\rho \bar{\nabla}^\alpha B \\
& + \frac{1}{2}\bar{R}_{\mu}{}^\sigma{}_{\nu\alpha} \bar{\nabla}^\rho \bar{\nabla}^\alpha B - \frac{1}{2}\bar{R}_{\nu}{}^\sigma \bar{\nabla}^\rho \bar{\nabla}_\mu B + \frac{1}{6}\delta_\nu^\sigma \bar{R} \bar{\nabla}^\rho \bar{\nabla}_\mu B + \frac{1}{2}\bar{R}_\mu{}^\sigma \bar{\nabla}^\rho \bar{\nabla}_\nu B \\
& - \frac{1}{6}\delta_\mu^\sigma \bar{R} \bar{\nabla}^\rho \bar{\nabla}_\nu B + \frac{1}{2}\bar{\nabla}^\alpha B \bar{\nabla}^\sigma \bar{R}_{\mu\nu}{}^\rho{}_\alpha + \frac{1}{4}\delta_\nu^\rho \bar{R}_{\mu\alpha} \bar{\nabla}^\sigma \bar{\nabla}^\alpha B \\
& - \frac{1}{4}\delta_\mu^\rho \bar{R}_{\nu\alpha} \bar{\nabla}^\sigma \bar{\nabla}^\alpha B + \frac{1}{2}\bar{R}_{\mu\alpha\nu}{}^\rho \bar{\nabla}^\sigma \bar{\nabla}^\alpha B + \bar{R}_{\mu\nu}{}^\rho{}_\alpha \bar{\nabla}^\sigma \bar{\nabla}^\alpha B \\
& - \frac{1}{2}\bar{R}_{\mu}{}^\rho{}_{\nu\alpha} \bar{\nabla}^\sigma \bar{\nabla}^\alpha B + \frac{1}{2}\bar{R}_{\nu}{}^\rho \bar{\nabla}^\sigma \bar{\nabla}_\mu B - \frac{1}{6}\delta_\nu^\rho \bar{R} \bar{\nabla}^\sigma \bar{\nabla}_\mu B - \frac{1}{2}\bar{R}_\mu{}^\rho \bar{\nabla}^\sigma \bar{\nabla}_\nu B \\
& + \frac{1}{6}\delta_\mu^\rho \bar{R} \bar{\nabla}^\sigma \bar{\nabla}_\nu B,
\end{aligned} \tag{6.75}$$

$$\begin{aligned}
c^{\rho\sigma}_{\mu\nu(A)} = & -\frac{1}{2}A^\alpha\delta_\nu{}^\sigma\bar{\nabla}_\alpha\bar{R}_\mu{}^\rho + \frac{1}{2}A^\alpha\delta_\nu{}^\rho\bar{\nabla}_\alpha\bar{R}_\mu{}^\sigma + \frac{1}{2}A^\alpha\delta_\mu{}^\sigma\bar{\nabla}_\alpha\bar{R}_\nu{}^\rho - \frac{1}{2}A^\alpha\delta_\mu{}^\rho\bar{\nabla}_\alpha\bar{R}_\nu{}^\sigma \\
& -\frac{1}{6}A^\alpha\delta_\mu{}^\sigma\delta_\nu{}^\rho\bar{\nabla}_\alpha\bar{R} + \frac{1}{6}A^\alpha\delta_\mu{}^\rho\delta_\nu{}^\sigma\bar{\nabla}_\alpha\bar{R} + \frac{1}{2}\bar{R}_{\nu\alpha}{}^{\rho\sigma}\bar{\nabla}^\alpha A_\mu + \frac{1}{2}\bar{R}_\nu{}^{\rho\sigma}{}_\alpha\bar{\nabla}^\alpha A_\mu \\
& -\frac{1}{2}\bar{R}_\nu{}^{\sigma\rho}{}_\alpha\bar{\nabla}^\alpha A_\mu - \frac{1}{2}\bar{R}_{\mu\alpha}{}^{\rho\sigma}\bar{\nabla}^\alpha A_\nu - \frac{1}{2}\bar{R}_\mu{}^{\rho\sigma}{}_\alpha\bar{\nabla}^\alpha A_\nu + \frac{1}{2}\bar{R}_\mu{}^{\sigma\rho}{}_\alpha\bar{\nabla}^\alpha A_\nu \\
& -\frac{1}{2}\bar{R}_{\mu\alpha\nu}{}^\sigma\bar{\nabla}^\alpha A^\rho - \frac{1}{2}\bar{R}_{\mu\nu}{}^\sigma{}_\alpha\bar{\nabla}^\alpha A^\rho + \frac{1}{2}\bar{R}_\mu{}^\sigma{}_\nu\bar{\nabla}^\alpha A^\rho + \frac{1}{2}\bar{R}_{\mu\alpha\nu}{}^\rho\bar{\nabla}^\alpha A^\sigma \\
& +\frac{1}{2}\bar{R}_{\mu\nu}{}^\rho{}_\alpha\bar{\nabla}^\alpha A^\sigma - \frac{1}{2}\bar{R}_\mu{}^\rho{}_\nu\bar{\nabla}^\alpha A^\sigma - \frac{1}{2}\delta_\nu{}^\sigma\bar{R}^\rho{}_\alpha\bar{\nabla}_\mu A^\alpha + \frac{1}{2}\delta_\nu{}^\rho\bar{R}^\sigma{}_\alpha\bar{\nabla}_\mu A^\alpha \\
& -\frac{1}{2}\bar{R}_{\nu\alpha}{}^{\rho\sigma}\bar{\nabla}_\mu A^\alpha + \frac{1}{2}\bar{R}_\nu{}^{\rho\sigma}{}_\alpha\bar{\nabla}_\mu A^\alpha - \frac{1}{2}\bar{R}_\nu{}^{\sigma\rho}{}_\alpha\bar{\nabla}_\mu A^\alpha - \frac{1}{2}\bar{R}_\nu{}^\sigma\bar{\nabla}_\mu A^\rho \\
& +\frac{1}{6}\delta_\nu{}^\sigma\bar{R}\bar{\nabla}_\mu A^\rho + \frac{1}{2}\bar{R}_\nu{}^\rho\bar{\nabla}_\mu A^\sigma - \frac{1}{6}\delta_\nu{}^\rho\bar{R}\bar{\nabla}_\mu A^\sigma + \frac{1}{2}\delta_\mu{}^\sigma\bar{R}^\rho{}_\alpha\bar{\nabla}_\nu A^\alpha \\
& -\frac{1}{2}\delta_\mu{}^\rho\bar{R}^\sigma{}_\alpha\bar{\nabla}_\nu A^\alpha + \frac{1}{2}\bar{R}_{\mu\alpha}{}^{\rho\sigma}\bar{\nabla}_\nu A^\alpha - \frac{1}{2}\bar{R}_\mu{}^{\rho\sigma}{}_\alpha\bar{\nabla}_\nu A^\alpha + \frac{1}{2}\bar{R}_\mu{}^{\sigma\rho}{}_\alpha\bar{\nabla}_\nu A^\alpha \\
& +\frac{1}{2}\bar{R}_\mu{}^\sigma\bar{\nabla}_\nu A^\rho - \frac{1}{6}\delta_\mu{}^\sigma\bar{R}\bar{\nabla}_\nu A^\rho - \frac{1}{2}\bar{R}_\mu{}^\rho\bar{\nabla}_\nu A^\sigma + \frac{1}{6}\delta_\mu{}^\rho\bar{R}\bar{\nabla}_\nu A^\sigma \\
& -\frac{1}{2}\delta_\nu{}^\sigma\bar{R}_{\mu\alpha}\bar{\nabla}^\rho A^\alpha + \frac{1}{2}\delta_\mu{}^\sigma\bar{R}_{\nu\alpha}\bar{\nabla}^\rho A^\alpha - \bar{R}_{\mu\nu}{}^\sigma{}_\alpha\bar{\nabla}^\rho A^\alpha - \frac{1}{2}\bar{R}_\nu{}^\sigma\bar{\nabla}^\rho A_\mu \\
& +\frac{1}{6}\delta_\nu{}^\sigma\bar{R}\bar{\nabla}^\rho A_\mu + \frac{1}{2}\bar{R}_\mu{}^\sigma\bar{\nabla}^\rho A_\nu - \frac{1}{6}\delta_\mu{}^\sigma\bar{R}\bar{\nabla}^\rho A_\nu + \frac{1}{2}A^\alpha\bar{\nabla}^\rho\bar{R}_{\mu\alpha\nu}{}^\sigma \\
& -\frac{1}{2}A^\alpha\bar{\nabla}^\rho\bar{R}_{\mu\nu}{}^\sigma{}_\alpha - \frac{1}{2}A^\alpha\bar{\nabla}^\rho\bar{R}_\mu{}^\sigma{}_\nu\alpha + \frac{1}{2}\delta_\nu{}^\rho\bar{R}_{\mu\alpha}\bar{\nabla}^\sigma A^\alpha - \frac{1}{2}\delta_\mu{}^\rho\bar{R}_{\nu\alpha}\bar{\nabla}^\sigma A^\alpha \\
& +\bar{R}_{\mu\nu}{}^\rho{}_\alpha\bar{\nabla}^\sigma A^\alpha + \frac{1}{2}\bar{R}_\nu{}^\rho\bar{\nabla}^\sigma A_\mu - \frac{1}{6}\delta_\nu{}^\rho\bar{R}\bar{\nabla}^\sigma A_\mu - \frac{1}{2}\bar{R}_\mu{}^\rho\bar{\nabla}^\sigma A_\nu \\
& +\frac{1}{6}\delta_\mu{}^\rho\bar{R}\bar{\nabla}^\sigma A_\nu - \frac{1}{2}A^\alpha\bar{\nabla}^\sigma\bar{R}_{\mu\alpha\nu}{}^\rho \\
& +\frac{1}{2}A^\alpha\bar{\nabla}^\sigma\bar{R}_{\mu\nu}{}^\rho{}_\alpha + \frac{1}{2}A^\alpha\bar{\nabla}^\sigma\bar{R}_\mu{}^\rho{}_\nu\alpha,
\end{aligned}
\tag{6.76}$$

$$\begin{aligned}
c^{\rho\sigma}_{\mu\nu(\hat{h})} = & \frac{1}{6}\delta_\mu^\sigma\delta_\nu^\rho\hat{h}^{\alpha\beta}\bar{R}_{\alpha\beta} - \frac{1}{6}\delta_\mu^\rho\delta_\nu^\sigma\hat{h}^{\alpha\beta}\bar{R}_{\alpha\beta} - \frac{1}{4}\delta_\nu^\sigma\hat{h}^{\rho\alpha}\bar{R}_{\mu\alpha} + \frac{1}{4}\delta_\nu^\rho\hat{h}^{\sigma\alpha}\bar{R}_{\mu\alpha} \\
& - \frac{1}{2}\hat{h}_\nu^\sigma\bar{R}_\mu^\rho + \frac{1}{2}\hat{h}_\nu^\rho\bar{R}_\mu^\sigma + \frac{1}{4}\delta_\mu^\sigma\hat{h}^{\rho\alpha}\bar{R}_{\nu\alpha} \\
& - \frac{1}{4}\delta_\mu^\rho\hat{h}^{\sigma\alpha}\bar{R}_{\nu\alpha} + \frac{1}{2}\hat{h}_\mu^\sigma\bar{R}_\nu^\rho - \frac{1}{2}\hat{h}_\mu^\rho\bar{R}_\nu^\sigma - \frac{1}{4}\delta_\nu^\sigma\hat{h}_\mu^\alpha\bar{R}^\rho_\alpha \\
& + \frac{1}{4}\delta_\mu^\sigma\hat{h}_\nu^\alpha\bar{R}^\rho_\alpha + \frac{1}{4}\delta_\nu^\rho\hat{h}_\mu^\alpha\bar{R}^\sigma_\alpha - \frac{1}{4}\delta_\mu^\rho\hat{h}_\nu^\alpha\bar{R}^\sigma_\alpha + \frac{1}{6}\delta_\nu^\sigma\hat{h}_\mu^\rho\bar{R} \\
& - \frac{1}{6}\delta_\nu^\rho\hat{h}_\mu^\sigma\bar{R} - \frac{1}{6}\delta_\mu^\sigma\hat{h}_\nu^\rho\bar{R} + \frac{1}{6}\delta_\mu^\rho\hat{h}_\nu^\sigma\bar{R} + \frac{1}{2}\hat{h}^{\sigma\alpha}\bar{R}_{\mu\alpha\nu}^\rho \\
& - \frac{1}{2}\hat{h}^{\rho\alpha}\bar{R}_{\mu\alpha\nu}^\sigma + \frac{1}{2}\delta_\nu^\sigma\hat{h}^{\alpha\beta}\bar{R}_{\mu\alpha}^\rho{}_\beta - \frac{1}{2}\delta_\nu^\rho\hat{h}^{\alpha\beta}\bar{R}_{\mu\alpha}^\sigma{}_\beta + \frac{1}{2}\hat{h}^{\sigma\alpha}\bar{R}_{\mu\nu}^\rho{}_\alpha \\
& - \frac{1}{2}\hat{h}^{\rho\alpha}\bar{R}_{\mu\nu}^\sigma{}_\alpha - \frac{1}{2}\hat{h}^{\sigma\alpha}\bar{R}_\mu^\rho{}_{\nu\alpha} - \frac{1}{2}\hat{h}_\nu^\alpha\bar{R}_\mu^\rho{}_\sigma{}_\alpha + \frac{1}{2}\hat{h}^{\rho\alpha}\bar{R}_\mu^\sigma{}_{\nu\alpha} \\
& + \frac{1}{2}\hat{h}_\nu^\alpha\bar{R}_\mu^\sigma{}_\rho{}_\alpha - \frac{1}{2}\delta_\mu^\sigma\hat{h}^{\alpha\beta}\bar{R}_{\nu\alpha}^\rho{}_\beta + \frac{1}{2}\delta_\mu^\rho\hat{h}^{\alpha\beta}\bar{R}_{\nu\alpha}^\sigma{}_\beta + \frac{1}{2}\hat{h}_\mu^\alpha\bar{R}_\nu^\rho{}_\sigma{}_\alpha \\
& - \frac{1}{2}\hat{h}_\mu^\alpha\bar{R}_\nu^\sigma{}_\rho{}_\alpha + \frac{1}{4}\delta_\nu^\sigma\hat{\square}\hat{h}_\mu^\rho - \frac{1}{4}\delta_\nu^\rho\hat{\square}\hat{h}_\mu^\sigma - \frac{1}{4}\delta_\mu^\sigma\hat{\square}\hat{h}_\nu^\rho \\
& + \frac{1}{4}\delta_\mu^\rho\hat{\square}\hat{h}_\nu^\sigma - \frac{1}{2}\bar{\nabla}^\rho\bar{\nabla}_\mu\hat{h}_\nu^\sigma + \frac{1}{2}\bar{\nabla}^\rho\bar{\nabla}_\nu\hat{h}_\mu^\sigma + \frac{1}{2}\bar{\nabla}^\sigma\bar{\nabla}_\mu\hat{h}_\nu^\rho \\
& - \frac{1}{2}\bar{\nabla}^\sigma\bar{\nabla}_\nu\hat{h}_\mu^\rho.
\end{aligned} \tag{6.77}$$

(ii) Rewriting the above in terms of irreducible $\bar{R}$, $\bar{S}_{\mu\nu}$, $\bar{C}_{\mu\nu\rho\sigma}$ and imposing $\bar{C}_{\mu\nu\rho\sigma} = 0$, we get [159]

$$c^{\rho\sigma}_{\mu\nu(\phi)} = 0,$$

$$\begin{aligned}
c^{\rho\sigma}_{\mu\nu(B)} = & \frac{1}{24}\delta_\mu^\sigma\delta_\nu^\rho\bar{\nabla}_\alpha\bar{R}\bar{\nabla}^\alpha B - \frac{1}{24}\delta_\mu^\rho\delta_\nu^\sigma\bar{\nabla}_\alpha\bar{R}\bar{\nabla}^\alpha B - \frac{1}{4}\delta_\nu^\sigma\bar{\nabla}_\alpha\bar{S}_\mu^\rho\bar{\nabla}^\alpha B \\
& + \frac{1}{4}\delta_\nu^\rho\bar{\nabla}_\alpha\bar{S}_\mu^\sigma\bar{\nabla}^\alpha B + \frac{1}{4}\delta_\mu^\sigma\bar{\nabla}_\alpha\bar{S}_\nu^\rho\bar{\nabla}^\alpha B - \frac{1}{4}\delta_\mu^\rho\bar{\nabla}_\alpha\bar{S}_\nu^\sigma\bar{\nabla}^\alpha B \\
& + \frac{1}{24}\delta_\nu^\sigma\bar{\nabla}_\mu B\bar{\nabla}^\rho\bar{R} - \frac{1}{24}\delta_\mu^\sigma\bar{\nabla}_\nu B\bar{\nabla}^\rho\bar{R} + \frac{1}{4}\delta_\nu^\sigma\bar{\nabla}^\alpha B\bar{\nabla}^\rho\bar{S}_{\mu\alpha} \\
& - \frac{1}{4}\bar{\nabla}_\nu B\bar{\nabla}^\rho\bar{S}_\mu^\sigma - \frac{1}{4}\delta_\mu^\sigma\bar{\nabla}^\alpha B\bar{\nabla}^\rho\bar{S}_{\nu\alpha} + \frac{1}{4}\bar{\nabla}_\mu B\bar{\nabla}^\rho\bar{S}_\nu^\sigma - \frac{1}{24}\delta_\nu^\rho\bar{\nabla}_\mu B\bar{\nabla}^\sigma\bar{R} \\
& + \frac{1}{24}\delta_\mu^\rho\bar{\nabla}_\nu B\bar{\nabla}^\sigma\bar{R} - \frac{1}{4}\delta_\nu^\rho\bar{\nabla}^\alpha B\bar{\nabla}^\sigma\bar{S}_{\mu\alpha} + \frac{1}{4}\bar{\nabla}_\nu B\bar{\nabla}^\sigma\bar{S}_\mu^\rho + \frac{1}{4}\delta_\mu^\rho\bar{\nabla}^\alpha B\bar{\nabla}^\sigma\bar{S}_{\nu\alpha} \\
& - \frac{1}{4}\bar{\nabla}_\mu B\bar{\nabla}^\sigma\bar{S}_\nu^\rho,
\end{aligned} \tag{6.78}$$

$$\begin{aligned}
c^{\rho\sigma}_{\mu\nu(A)} = & \frac{1}{12}A^\alpha\delta_\mu{}^\sigma\delta_\nu{}^\rho\bar{\nabla}_\alpha\bar{R} - \frac{1}{12}A^\alpha\delta_\mu{}^\rho\delta_\nu{}^\sigma\bar{\nabla}_\alpha\bar{R} - \frac{1}{2}A^\alpha\delta_\nu{}^\sigma\bar{\nabla}_\alpha\bar{S}_\mu{}^\rho + \frac{1}{2}A^\alpha\delta_\nu{}^\rho\bar{\nabla}_\alpha\bar{S}_\mu{}^\sigma \\
& + \frac{1}{2}A^\alpha\delta_\mu{}^\sigma\bar{\nabla}_\alpha\bar{S}_\nu{}^\rho - \frac{1}{2}A^\alpha\delta_\mu{}^\rho\bar{\nabla}_\alpha\bar{S}_\nu{}^\sigma - \frac{1}{12}A_\nu\delta_\mu{}^\sigma\bar{\nabla}^\rho\bar{R} + \frac{1}{12}A_\mu\delta_\nu{}^\sigma\bar{\nabla}^\rho\bar{R} \\
& + \frac{1}{2}A^\alpha\delta_\nu{}^\sigma\bar{\nabla}^\rho\bar{S}_{\mu\alpha} - \frac{1}{2}A_\nu\bar{\nabla}^\rho\bar{S}_\mu{}^\sigma - \frac{1}{2}A^\alpha\delta_\mu{}^\sigma\bar{\nabla}^\rho\bar{S}_{\nu\alpha} + \frac{1}{2}A_\mu\bar{\nabla}^\rho\bar{S}_\nu{}^\sigma \\
& + \frac{1}{12}A_\nu\delta_\mu{}^\rho\bar{\nabla}^\sigma\bar{R} - \frac{1}{12}A_\mu\delta_\nu{}^\rho\bar{\nabla}^\sigma\bar{R} - \frac{1}{2}A^\alpha\delta_\nu{}^\rho\bar{\nabla}^\sigma\bar{S}_{\mu\alpha} + \frac{1}{2}A_\nu\bar{\nabla}^\sigma\bar{S}_\mu{}^\rho \\
& + \frac{1}{2}A^\alpha\delta_\mu{}^\rho\bar{\nabla}^\sigma\bar{S}_{\nu\alpha} - \frac{1}{2}A_\mu\bar{\nabla}^\sigma\bar{S}_\nu{}^\rho,
\end{aligned}$$

$$\begin{aligned}
c^{\rho\sigma}_{\mu\nu(\hat{h})} = & -\frac{1}{12}\delta_\nu{}^\sigma\hat{h}_\mu{}^\rho\bar{R} + \frac{1}{12}\delta_\nu{}^\rho\hat{h}_\mu{}^\sigma\bar{R} + \frac{1}{12}\delta_\mu{}^\sigma\hat{h}_\nu{}^\rho\bar{R} - \frac{1}{12}\delta_\mu{}^\rho\hat{h}_\nu{}^\sigma\bar{R} \\
& - \frac{1}{3}\delta_\mu{}^\sigma\delta_\nu{}^\rho\hat{h}^{\alpha\beta}\bar{S}_{\alpha\beta} + \frac{1}{3}\delta_\mu{}^\rho\delta_\nu{}^\sigma\hat{h}^{\alpha\beta}\bar{S}_{\alpha\beta} - \frac{1}{2}\delta_\nu{}^\sigma\hat{h}^{\rho\alpha}\bar{S}_{\mu\alpha} + \frac{1}{2}\delta_\nu{}^\rho\hat{h}^{\sigma\alpha}\bar{S}_{\mu\alpha} \\
& - \frac{1}{4}\hat{h}_\nu{}^\sigma\bar{S}_\mu{}^\rho + \frac{1}{4}\hat{h}_\nu{}^\rho\bar{S}_\mu{}^\sigma + \frac{1}{2}\delta_\mu{}^\sigma\hat{h}^{\rho\alpha}\bar{S}_{\nu\alpha} - \frac{1}{2}\delta_\mu{}^\rho\hat{h}^{\sigma\alpha}\bar{S}_{\nu\alpha} \\
& + \frac{1}{4}\hat{h}_\mu{}^\sigma\bar{S}_\nu{}^\rho - \frac{1}{4}\hat{h}_\mu{}^\rho\bar{S}_\nu{}^\sigma - \frac{1}{4}\delta_\nu{}^\sigma\hat{h}_\mu{}^\alpha\bar{S}^\rho{}_\alpha + \frac{1}{4}\delta_\mu{}^\sigma\hat{h}_\nu{}^\alpha\bar{S}^\rho{}_\alpha \\
& + \frac{1}{4}\delta_\nu{}^\rho\hat{h}_\mu{}^\alpha\bar{S}^\sigma{}_\alpha - \frac{1}{4}\delta_\mu{}^\rho\hat{h}_\nu{}^\alpha\bar{S}^\sigma{}_\alpha + \frac{1}{4}\delta_\nu{}^\sigma\hat{\square}\hat{h}_\mu{}^\rho - \frac{1}{4}\delta_\nu{}^\rho\hat{\square}\hat{h}_\mu{}^\sigma \\
& - \frac{1}{4}\delta_\mu{}^\sigma\hat{\square}\hat{h}_\nu{}^\rho + \frac{1}{4}\delta_\mu{}^\rho\hat{\square}\hat{h}_\nu{}^\sigma - \frac{1}{2}\bar{\nabla}^\rho\bar{\nabla}_\mu\hat{h}_\nu{}^\sigma + \frac{1}{2}\bar{\nabla}^\rho\bar{\nabla}_\nu\hat{h}_\mu{}^\sigma + \frac{1}{2}\bar{\nabla}^\sigma\bar{\nabla}_\mu\hat{h}_\nu{}^\rho \\
& - \frac{1}{2}\bar{\nabla}^\sigma\bar{\nabla}_\nu\hat{h}_\mu{}^\rho.
\end{aligned} \tag{6.79}$$

(iii) Imposing $\bar{\nabla}_\alpha \bar{S}_{\mu\nu} = 0$, $\partial_\mu \bar{R} = 0$ to above gives [159]

$$\begin{aligned}
c^{\rho\sigma}_{\mu\nu(\phi)} &= 0, \quad c^{\rho\sigma}_{\mu\nu(B)} = 0, \quad c^{\rho\sigma}_{\mu\nu(A)} = 0. \\
c^{\rho\sigma}_{\mu\nu(\hat{h})} &= -\frac{1}{12}\delta_\nu^\sigma \hat{h}_\mu^\rho \bar{R} + \frac{1}{12}\delta_\nu^\rho \hat{h}_\mu^\sigma \bar{R} + \frac{1}{12}\delta_\mu^\sigma \hat{h}_\nu^\rho \bar{R} - \frac{1}{12}\delta_\mu^\rho \hat{h}_\nu^\sigma \bar{R} \\
&\quad - \frac{1}{3}\delta_\mu^\sigma \delta_\nu^\rho \hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta} + \frac{1}{3}\delta_\mu^\rho \delta_\nu^\sigma \hat{h}^{\alpha\beta} \bar{S}_{\alpha\beta} - \frac{1}{2}\delta_\nu^\sigma \hat{h}^{\rho\alpha} \bar{S}_{\mu\alpha} + \frac{1}{2}\delta_\nu^\rho \hat{h}^{\sigma\alpha} \bar{S}_{\mu\alpha} \\
&\quad - \frac{1}{4}\hat{h}_\nu^\sigma \bar{S}_\mu^\rho + \frac{1}{4}\hat{h}_\nu^\rho \bar{S}_\mu^\sigma + \frac{1}{2}\delta_\mu^\sigma \hat{h}^{\rho\alpha} \bar{S}_{\nu\alpha} - \frac{1}{2}\delta_\mu^\rho \hat{h}^{\sigma\alpha} \bar{S}_{\nu\alpha} \\
&\quad + \frac{1}{4}\hat{h}_\mu^\sigma \bar{S}_\nu^\rho - \frac{1}{4}\hat{h}_\mu^\rho \bar{S}_\nu^\sigma - \frac{1}{4}\delta_\nu^\sigma \hat{h}_\mu^\alpha \bar{S}^\rho_\alpha + \frac{1}{4}\delta_\nu^\rho \hat{h}_\mu^\alpha \bar{S}^\sigma_\alpha \\
&\quad + \frac{1}{4}\delta_\nu^\rho \hat{h}_\mu^\alpha \bar{S}^\sigma_\alpha - \frac{1}{4}\delta_\mu^\rho \hat{h}_\nu^\alpha \bar{S}^\sigma_\alpha + \frac{1}{4}\delta_\nu^\sigma \hat{\square} \hat{h}_\mu^\rho - \frac{1}{4}\delta_\nu^\rho \hat{\square} \hat{h}_\mu^\sigma \\
&\quad - \frac{1}{4}\delta_\mu^\sigma \hat{\square} \hat{h}_\nu^\rho + \frac{1}{4}\delta_\mu^\rho \hat{\square} \hat{h}_\nu^\sigma - \frac{1}{2}\bar{\nabla}^\rho \bar{\nabla}_\mu \hat{h}_\nu^\sigma + \frac{1}{2}\bar{\nabla}^\rho \bar{\nabla}_\nu \hat{h}_\mu^\sigma \\
&\quad + \frac{1}{2}\bar{\nabla}^\sigma \bar{\nabla}_\mu \hat{h}_\nu^\rho - \frac{1}{2}\bar{\nabla}^\sigma \bar{\nabla}_\nu \hat{h}_\mu^\rho.
\end{aligned} \tag{6.80}$$

Now we find the quadratic in $\hat{h}_{\mu\nu}$ part. There are many Kronecker delta terms in $c^{\rho\sigma}_{\mu\nu(\hat{h})}$ with free indices. Since the totally tracelessness property of the Weyl tensor is preserved at each order of perturbation, all the Kronecker delta terms in $c^{\rho\sigma}_{\mu\nu(\hat{h})}$ will hit $c^{\mu\nu}_{\rho\sigma(\hat{h})}$ and go to zero. We are then left with only a few terms in $c^{\rho\sigma}_{\mu\nu(\hat{h})}$. Assuming that it will be multiplied to $\bar{\mathcal{F}}_3(\bar{\square}_s)c^{\mu\nu}_{\rho\sigma(\hat{h})}$ later, the simplified expression is [159]

$$\begin{aligned}
c^{\rho\sigma}_{\mu\nu(\hat{h})} &= -\frac{1}{4}\hat{h}_\nu^\sigma \bar{S}_\mu^\rho + \frac{1}{4}\hat{h}_\nu^\rho \bar{S}_\mu^\sigma + \frac{1}{4}\hat{h}_\mu^\sigma \bar{S}_\nu^\rho - \frac{1}{4}\hat{h}_\mu^\rho \bar{S}_\nu^\sigma - \frac{1}{2}\bar{\nabla}^\rho \bar{\nabla}_\mu \hat{h}_\nu^\sigma \\
&\quad + \frac{1}{2}\bar{\nabla}^\rho \bar{\nabla}_\nu \hat{h}_\mu^\sigma + \frac{1}{2}\bar{\nabla}^\sigma \bar{\nabla}_\mu \hat{h}_\nu^\rho - \frac{1}{2}\bar{\nabla}^\sigma \bar{\nabla}_\nu \hat{h}_\mu^\rho.
\end{aligned} \tag{6.81}$$

Further simplifications beyond this happen when the above simplified $c^{\rho\sigma}_{\mu\nu(\hat{h})}$ is multiplied with $\bar{\mathcal{F}}_3(\bar{\square}_s)c^{\mu\nu}_{\rho\sigma(\hat{h})}$, and the index symmetries of $c^{\mu\nu}_{\rho\sigma(\hat{h})}$ are used. We ultimately get [159]

$$c^{\rho\sigma}_{\mu\nu(\hat{h})} \bar{\mathcal{F}}_3(\bar{\square}_s)c^{\mu\nu}_{\rho\sigma(\hat{h})} = \left[\hat{h}_\nu^\rho \bar{S}_\mu^\sigma + 2\bar{\nabla}^\rho \bar{\nabla}_\nu \hat{h}_\mu^\sigma \right] \bar{\mathcal{F}}_3(\bar{\square}_s) \left[c^{\mu\nu}_{\rho\sigma(\hat{h})} \right], \tag{6.82}$$

where [159]

$$\begin{aligned}
 c^{\mu\nu}_{\rho\sigma(\hat{h})} = & -\frac{1}{12}\delta^\nu_\sigma\hat{h}^\mu_\rho\bar{R} + \frac{1}{12}\delta^\nu_\rho\hat{h}^\mu_\sigma\bar{R} + \frac{1}{12}\delta^\mu_\sigma\hat{h}^\nu_\rho\bar{R} - \frac{1}{12}\delta^\mu_\rho\hat{h}^\nu_\sigma\bar{R} \\
 & -\frac{1}{3}\delta^\mu_\sigma\delta^\nu_\rho\hat{h}^{\alpha\beta}\bar{S}_{\alpha\beta} + \frac{1}{3}\delta^\mu_\rho\delta^\nu_\sigma\hat{h}^{\alpha\beta}\bar{S}_{\alpha\beta} - \frac{1}{2}\delta^\nu_\sigma\hat{h}^\alpha_\rho\bar{S}^\mu_\alpha + \frac{1}{2}\delta^\nu_\rho\hat{h}^\alpha_\sigma\bar{S}^\mu_\alpha \\
 & -\frac{1}{4}\hat{h}^\nu_\sigma\bar{S}^\mu_\rho + \frac{1}{4}\hat{h}^\nu_\rho\bar{S}^\mu_\sigma + \frac{1}{2}\delta^\mu_\sigma\hat{h}^\alpha_\rho\bar{S}^\nu_\alpha - \frac{1}{2}\delta^\mu_\rho\hat{h}^\alpha_\sigma\bar{S}^\nu_\alpha \\
 & +\frac{1}{4}\hat{h}^\mu_\sigma\bar{S}^\nu_\rho - \frac{1}{4}\hat{h}^\mu_\rho\bar{S}^\nu_\sigma - \frac{1}{4}\delta^\nu_\sigma\hat{h}^{\mu\alpha}\bar{S}_{\rho\alpha} + \frac{1}{4}\delta^\mu_\sigma\hat{h}^{\nu\alpha}\bar{S}_{\rho\alpha} \\
 & +\frac{1}{4}\delta^\nu_\rho\hat{h}^{\mu\alpha}\bar{S}_{\sigma\alpha} - \frac{1}{4}\delta^\mu_\rho\hat{h}^{\nu\alpha}\bar{S}_{\sigma\alpha} + \frac{1}{4}\delta^\nu_\sigma\hat{\square}\hat{h}^\mu_\rho - \frac{1}{4}\delta^\nu_\rho\hat{\square}\hat{h}^\mu_\sigma \\
 & -\frac{1}{4}\delta^\mu_\sigma\hat{\square}\hat{h}^\nu_\rho + \frac{1}{4}\delta^\mu_\rho\hat{\square}\hat{h}^\nu_\sigma - \frac{1}{2}\bar{\nabla}_\rho\bar{\nabla}^\mu\hat{h}^\nu_\sigma + \frac{1}{2}\bar{\nabla}_\rho\bar{\nabla}^\nu\hat{h}^\mu_\sigma \\
 & +\frac{1}{2}\bar{\nabla}_\sigma\bar{\nabla}^\mu\hat{h}^\nu_\rho - \frac{1}{2}\bar{\nabla}_\sigma\bar{\nabla}^\nu\hat{h}^\mu_\rho.
 \end{aligned} \tag{6.83}$$

It is easy to determine SVT expressions for $c^{\mu\nu}_{\rho\sigma}$ from $c^{\rho\sigma}_{\mu\nu}$ since $c^{\mu\nu}_{\rho\sigma} = c^{\gamma\lambda}_{\alpha\beta}\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\bar{g}_{\gamma\rho}\bar{g}_{\lambda\sigma} + \bar{C}^{\gamma\lambda}_{\alpha\beta}\delta(g^{\alpha\mu}g^{\beta\nu}g_{\gamma\rho}g_{\lambda\sigma})$ where the second term is zero for our conformally-flat background. So expressions for $c^{\mu\nu}_{\rho\sigma}$ are obtained from their $c^{\rho\sigma}_{\mu\nu}$ counterparts with lower (upper) indices raised (lowered).

6.G Linearized equations of motion around the non-maximally symmetric background in eq. (3.13)

Perturbations at the quadratic level of the action are equivalent to linear perturbation of the equations of motion and sometimes it is easier to study the latter. The variation of trace equations of motion eq. (6.33) under the usual conditions eq. (3.13) is [159]

$$\begin{aligned}
 -M_P^2\delta R = & -6\bar{\square}\mathcal{F}_1(\bar{\square}_s)\delta R - (\delta L_1 + 2\delta\bar{L}_1) - 2\delta(\bar{\nabla}_\mu\bar{\nabla}_\nu)\mathcal{F}_2(\bar{\square}_s)\bar{S}^{\mu\nu} \\
 & - 2\bar{\nabla}_\mu\bar{\nabla}_\nu\delta(\mathcal{F}_2(\bar{\square}_s))\bar{S}^{\mu\nu} - 2\bar{\nabla}_\mu\bar{\nabla}_\nu\mathcal{F}_2(\bar{\square}_s)\delta\bar{S}^{\mu\nu} - (\delta L_2 + 2\delta\bar{L}_2) + 2\delta\tilde{\Delta}.
 \end{aligned} \tag{6.84}$$

$\delta L_1 = \delta L_2 = 0$, because each term in their respective infinite sums have 2 background curvatures, each of which has at least one derivative acting on it. Taking a first variation always results in a derivative acting on these background curvatures (from eq. (3.13b)), resulting in zero. We have [159]

$$\delta\bar{L}_1 = \bar{R}(\mathcal{F}_1(\bar{\square}_s) - f_{1,0})\delta R \quad \delta\bar{L}_2 = \bar{S}^{\alpha\beta}(\mathcal{F}_2(\bar{\square}_s) - f_{2,0})s_{\alpha\beta}, \tag{6.85}$$

$$\delta\tilde{\Delta} = \frac{1}{M_s^2}\bar{S}^\beta_\gamma\bar{\nabla}_\beta\bar{\nabla}^\mu\mathcal{F}_4(\bar{\square}_s)s^\gamma_\mu - \frac{1}{M_s^2}\bar{S}^\gamma_\mu\bar{\nabla}_\beta\bar{\nabla}^\mu\mathcal{F}_4(\bar{\square}_s)s^\beta_\gamma, \tag{6.86}$$

$$\begin{aligned}
-2\delta(\bar{\nabla}_\mu \bar{\nabla}_\nu) \mathcal{F}_2(\bar{\square}_s) \bar{S}^{\mu\nu} &= 2f_{2,0} \bar{\nabla}_\mu \bar{\nabla}_\nu s^{\mu\nu} \\
-2\bar{\nabla}_\mu \bar{\nabla}_\nu \delta(\mathcal{F}_2(\square_s)) \bar{S}^{\mu\nu} &= -2\bar{\nabla}_\mu \bar{\nabla}_\nu \left[\frac{\mathcal{F}_2(\bar{\square}_s) - f_{2,0}}{\bar{\square}_s} \right] \delta(\square_s) \bar{S}^{\mu\nu}.
\end{aligned} \tag{6.87}$$

Collecting all terms, first variation of the trace equations of motion eq. (6.84) becomes [159]

$$\begin{aligned}
-M_P^2 \delta R &= -6\bar{\square} \mathcal{F}_1(\bar{\square}_s) \delta R - 2\bar{R} \mathcal{F}_4(\bar{\square}_s) \bar{\square}_s \delta R - 2\bar{S}^{\alpha\beta} \mathcal{F}_5(\bar{\square}_s) \bar{\square}_s s_{\alpha\beta} + 2f_{2,0} \bar{\nabla}_\mu \bar{\nabla}_\nu s^{\mu\nu} \\
&\quad - 2\bar{\nabla}_\mu \bar{\nabla}_\nu \mathcal{F}_5(\bar{\square}_s) \delta(\square_s) \bar{S}^{\mu\nu} - 2\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\mathcal{F}}_2(\bar{\square}_s) \delta S^{\mu\nu} \\
&\quad + \frac{1}{M_s^2} 2\bar{S}^\beta_\gamma \bar{\nabla}_\beta \bar{\nabla}^\mu \mathcal{F}_5(\bar{\square}_s) s^\gamma_\mu - \frac{1}{M_s^2} 2\bar{S}^\gamma_\mu \bar{\nabla}_\beta \bar{\nabla}^\mu \mathcal{F}_5(\bar{\square}_s) s^\beta_\gamma.
\end{aligned} \tag{6.88}$$

6.H Commutation relations

In a model like non-local gravity, there are many covariant derivatives, some of which are to the left of form factors $\mathcal{F}_i(\square_s)$. In order to contract them with covariant derivatives lying to the right of $\mathcal{F}_i(\square_s)$, we need to perform many commutations. These commutations, in a theory like non-local gravity with infinite number of box operators in $\mathcal{F}_i(\square_s)$, lead to generation of an infinite number of background curvatures. When the background is MSS, it is possible in some cases to obtain compact expressions after the many commutations. It is apparently not possible to obtain compact expressions when the background is arbitrary. Nevertheless, it is useful to have formulas for such commutation relations to understand the general structure of perturbative expansions of a gravity action. In this section, we will generalize the commutation relations derived in Appendix B of [111] where the background was MSS, to those when the background is *arbitrary*.

For an arbitrary scalar ϕ , we have [159]

$$\begin{aligned}
\bar{\nabla}_\mu \bar{\nabla}_\nu \phi &= \bar{\nabla}_\nu \bar{\nabla}_\mu \phi, \quad \text{provided torsion } T^\lambda_{\mu\nu} = 0, \\
\bar{\nabla}_\mu \bar{\nabla}_\alpha \bar{\nabla}_\beta \phi &= \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\nabla}_\mu \phi - \bar{R}^\lambda_{\beta\mu\alpha} \bar{\nabla}_\lambda \phi, \\
\bar{\nabla}_\mu \bar{\square} \phi &= \bar{\square} \bar{\nabla}_\mu \phi - \bar{R}_{\mu\nu} \bar{\nabla}^\nu \phi.
\end{aligned} \tag{6.89}$$

For a vector t^μ , we have [159]

$$\begin{aligned}
\bar{\nabla}_\mu \bar{\nabla}_\alpha t^\mu &= \bar{\nabla}_\alpha \bar{\nabla}_\mu t^\mu + \bar{R}_{\alpha\mu} t^\mu, \\
\bar{\nabla}_\mu \bar{\nabla}_\alpha \bar{\nabla}_\beta t^\mu &= \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\nabla}_\mu t^\mu + \bar{\nabla}_\alpha (\bar{R}_{\beta\mu} t^\mu) - \bar{R}^\lambda_{\beta\mu\alpha} \bar{\nabla}_\lambda t^\mu + \bar{R}_{\lambda\alpha} \bar{\nabla}_\beta t^\lambda, \\
\bar{\nabla}_\mu \bar{\square} t^\mu &= \bar{\square} \bar{\nabla}_\mu t^\mu + \bar{\nabla}^\mu (\bar{R}_{\mu\nu} t^\nu), \\
\bar{\nabla}_\nu \bar{\nabla}^\mu \bar{\nabla}_\rho t^\sigma &= \bar{\nabla}_\rho \bar{\nabla}^\mu \bar{\nabla}_\nu t^\sigma - \bar{\nabla}_\lambda t^\sigma (\bar{R}^\lambda_\nu{}^\mu{}_\rho + \bar{R}^\lambda_{\rho\nu}{}^\mu) + \bar{R}^\sigma{}_\lambda{}^\mu{}_\rho \bar{\nabla}_\nu t^\lambda + \bar{R}^\sigma{}_{\lambda\nu}{}^\mu \bar{\nabla}_\rho t^\lambda \\
&\quad + \bar{\nabla}^\mu (\bar{R}^\sigma{}_{\lambda\nu\rho} t^\lambda).
\end{aligned} \tag{6.90}$$

If t^μ was a transverse vector A^μ satisfying $\bar{\nabla}_\mu A^\mu = 0$, the above expressions simplify to [159]

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha A^\mu = \bar{R}_{\alpha\mu} A^\mu, \quad (6.91)$$

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha \bar{\nabla}_\beta A^\mu = \bar{\nabla}_\alpha (\bar{R}_{\beta\mu} A^\mu) - \bar{R}^\lambda_{\beta\mu\alpha} \bar{\nabla}_\lambda A^\mu + \bar{R}_{\lambda\alpha} \bar{\nabla}_\beta A^\lambda, \quad (6.92)$$

$$\bar{\nabla}_\mu \bar{\square} A^\mu = \bar{\nabla}^\mu (\bar{R}_{\mu\nu} A^\nu). \quad (6.93)$$

The last expression tells us that given a transverse vector A^μ , $\bar{\square} A^\mu$ is not a transverse vector. For symmetric (and traceless) tensor $t^{\mu\nu}$, we have [159]

$$\begin{aligned} \bar{\nabla}_\mu \bar{\nabla}_\alpha t^{\mu\nu} &= \bar{\nabla}_\alpha \bar{\nabla}_\mu t^{\mu\nu} + \bar{R}_{\lambda\alpha} t^{\lambda\nu} + \bar{R}^\nu_{\lambda\mu\alpha} t^{\mu\lambda}, \\ \bar{\nabla}_\mu \bar{\nabla}_\alpha \bar{\nabla}_\beta t^{\mu\nu} &= \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{\nabla}_\mu t^{\mu\nu} + \bar{\nabla}_\alpha (\bar{R}_{\lambda\beta} t^{\lambda\nu} + \bar{R}^\nu_{\lambda\mu\beta} t^{\mu\lambda}) - \bar{R}^\lambda_{\beta\mu\alpha} \bar{\nabla}_\lambda t^{\mu\nu} \\ &\quad + \bar{R}_{\lambda\alpha} \bar{\nabla}_\beta t^{\lambda\nu} + \bar{R}^\nu_{\lambda\mu\alpha} \bar{\nabla}_\beta t^{\mu\lambda}, \\ \bar{\nabla}_\mu \bar{\square} t^{\mu\nu} &= \bar{\square} \bar{\nabla}_\mu t^{\mu\nu} + \bar{R}^\nu_{\lambda\mu\rho} \bar{\nabla}^\rho t^{\mu\lambda} + \bar{\nabla}^\sigma (\bar{R}_{\lambda\sigma} t^{\lambda\nu} + \bar{R}^\nu_{\lambda\mu\sigma} t^{\mu\lambda}), \\ \bar{\nabla}^\sigma \bar{\nabla}_\rho \bar{\nabla}_\sigma t_{\mu\nu} &= \bar{\nabla}_\rho \bar{\square} t_{\mu\nu} + \bar{R}_{\lambda\rho} \bar{\nabla}^\lambda t_{\mu\nu} - \bar{R}^\lambda_{\mu\alpha\rho} \bar{\nabla}^\alpha t_{\lambda\nu} - \bar{R}^\lambda_{\nu\alpha\rho} \bar{\nabla}^\alpha t_{\mu\lambda}. \end{aligned} \quad (6.94)$$

For a symmetric, transverse (and traceless) tensor $T^{\mu\nu}$ satisfying $\bar{\nabla}_\mu T^{\mu\nu} = 0$, we have [159]

$$\begin{aligned} \bar{\nabla}_\mu \bar{\nabla}_\alpha T^{\mu\nu} &= \bar{R}_{\lambda\alpha} T^{\lambda\nu} + \bar{R}^\nu_{\lambda\mu\alpha} T^{\mu\lambda}, \\ \bar{\nabla}_\mu \bar{\nabla}_\alpha \bar{\nabla}_\beta T^{\mu\nu} &= \bar{\nabla}_\alpha (\bar{R}_{\lambda\beta} T^{\lambda\nu} + \bar{R}^\nu_{\lambda\mu\beta} T^{\mu\lambda}) - \bar{R}^\lambda_{\beta\mu\alpha} \bar{\nabla}_\lambda T^{\mu\nu} + \bar{R}_{\lambda\alpha} \bar{\nabla}_\beta T^{\lambda\nu} \\ &\quad + \bar{R}^\nu_{\lambda\mu\alpha} \bar{\nabla}_\beta T^{\mu\lambda}, \\ \bar{\nabla}_\mu \bar{\square} T^{\mu\nu} &= \bar{R}^\nu_{\lambda\mu\rho} \bar{\nabla}^\rho T^{\mu\lambda} + \bar{\nabla}^\sigma (\bar{R}_{\lambda\sigma} T^{\lambda\nu} + \bar{R}^\nu_{\lambda\mu\sigma} T^{\mu\lambda}), \\ \bar{\nabla}^\sigma \bar{\nabla}_\rho \bar{\nabla}_\sigma T_{\mu\nu} &= \bar{\nabla}_\rho \bar{\square} T_{\mu\nu} + \bar{R}_{\lambda\rho} \bar{\nabla}^\lambda T_{\mu\nu} - \bar{R}^\lambda_{\mu\alpha\rho} \bar{\nabla}^\alpha T_{\lambda\nu} - \bar{R}^\lambda_{\nu\alpha\rho} \bar{\nabla}^\alpha T_{\mu\lambda}. \end{aligned} \quad (6.95)$$

For a tensor $t_{\mu\nu}$, we have [159]

$$\bar{\nabla}_\rho \bar{\square} t_{\mu\nu} = \bar{\square} \bar{\nabla}_\rho t_{\mu\nu} - \bar{\nabla}^\alpha (\bar{R}^\lambda_{\mu\rho\alpha} t_{\lambda\nu} + \bar{R}^\lambda_{\nu\rho\alpha} t_{\mu\lambda}) - \bar{R}_{\lambda\sigma} \bar{\nabla}^\lambda t_{\mu\nu} - \bar{R}^\lambda_{\mu\rho\sigma} \bar{\nabla}^\sigma t_{\lambda\nu} - \bar{R}^\lambda_{\nu\rho\sigma} \bar{\nabla}^\sigma t_{\mu\lambda}. \quad (6.96)$$

Now, when the background is MSS, some of these commutations relations simplify into a compact recursion formulas. For example, we have [159]

$$\bar{\nabla}_\mu \bar{\square} \phi = \left(\bar{\square} - \frac{\bar{R}}{4} \right) \bar{\nabla}_\mu \phi \implies \bar{\nabla}_\mu \bar{\square}^n \phi = \left(\bar{\square} - \frac{\bar{R}}{4} \right)^n \bar{\nabla}_\mu \phi. \quad (6.97)$$

In eq. (6.89), in order to get a recursion relation, we must express $\bar{R}_{\mu\nu} \bar{\nabla}^\nu \phi$ in terms of $\bar{\nabla}_\mu \phi$. To that end, suppose [159]

$$\bar{\nabla}_\mu \bar{\square} \phi = \bar{\square} \bar{\nabla}_\mu \phi - \bar{R}_{\mu\nu} \bar{\nabla}^\nu \phi = \bar{\square} \bar{\nabla}_\mu \phi - x \bar{\nabla}_\mu \phi, \quad (6.98)$$

where x is some scalar to be determined. Rewriting this twice, we get

$$\bar{R}_\mu{}^\rho \bar{\nabla}_\rho \phi = x \bar{\nabla}_\mu \phi, \quad (6.99)$$

$$\bar{R}_\nu{}^\alpha \bar{\nabla}_\alpha \phi = x \bar{\nabla}_\nu \phi. \quad (6.100)$$

Multiplying both expressions above, contracting with $\bar{g}^{\mu\nu}$ and rearranging gives [159]

$$x = \sqrt{\frac{\bar{R}_{\mu\nu} \bar{R}^{\mu\nu}}{4}}, \quad (6.101)$$

so that now, we have a recursion relation for a general metric [159]:

$$\bar{\nabla}_\mu \bar{\square}^n \phi = (\bar{\square} - x)^n \bar{\nabla}_\mu \phi = \left(\bar{\square} - \sqrt{\frac{\bar{R}_{\mu\nu} \bar{R}^{\mu\nu}}{4}} \right)^n \bar{\nabla}_\mu \phi. \quad (6.102)$$

For MSS, $x = \bar{R}/4$, as expected.

Similarly, for a vector t^μ in eq. (6.90), we have [159]

$$\bar{\nabla}_\mu \bar{\square} t^\mu = \bar{\square} \bar{\nabla}_\mu t^\mu + \bar{\nabla}^\mu (\bar{R}_{\mu\nu} t^\nu) = \bar{\square} \bar{\nabla}_\mu t^\mu + y \bar{\nabla}_\mu t^\mu = (\bar{\square} + y) \bar{\nabla}_\mu t^\mu, \quad (6.103)$$

where the scalar y is given by

$$y(t^\mu) = \frac{\bar{\nabla}^\rho (\bar{R}_{\rho\sigma} t^\sigma)}{\bar{\nabla}_\alpha t^\alpha}, \quad (6.104)$$

such that we get the recursion relation [159]

$$\bar{\nabla}_\mu \bar{\square}^n t^\mu = (\bar{\square} + y)^n \bar{\nabla}_\mu t^\mu, \quad \text{where } y = \frac{\bar{R}}{4} + \frac{(\bar{\nabla}^\rho \bar{R}_{\rho\sigma}) t^\sigma}{(\bar{\nabla}^\rho t^\sigma) \bar{g}_{\rho\sigma}}. \quad (6.105)$$

For MSS, $y = \bar{R}/4$, as expected.

Given a tensor $t^{\beta\mu\alpha}$, we have [159]

$$\begin{aligned} \bar{\nabla}_\alpha \bar{\square} t^{\beta\mu\alpha} &= \bar{\square} \bar{\nabla}_\alpha t^{\beta\mu\alpha} + \bar{R}^\beta_{\lambda\alpha\rho} \bar{\nabla}^\rho t^{\lambda\mu\alpha} - \bar{R}_{\lambda\alpha} \bar{\nabla}^\lambda t^{\beta\mu\alpha} + \bar{R}^\mu_{\lambda\alpha\rho} \bar{\nabla}^\rho t^{\beta\lambda\alpha} + \bar{R}_{\lambda\rho} \bar{\nabla}^\rho t^{\beta\mu\lambda} \\ &\quad + \bar{\nabla}^\rho \left(\bar{R}^\beta_{\lambda\alpha\rho} t^{\lambda\mu\alpha} + \bar{R}^\mu_{\lambda\alpha\rho} t^{\beta\lambda\alpha} + \bar{R}_{\lambda\rho} t^{\beta\mu\lambda} \right). \end{aligned} \quad (6.106)$$

Given a tensor $t^{\mu\alpha\nu\beta}$, we have [159]

$$\begin{aligned} \bar{\nabla}_\mu \bar{\square} t^{\mu\alpha\nu\beta} &= \bar{\square} \bar{\nabla}_\mu t^{\mu\alpha\nu\beta} + \bar{R}_{\lambda\rho} \bar{\nabla}^\rho t^{\lambda\alpha\nu\beta} - \bar{R}_{\lambda\mu} \bar{\nabla}^\lambda t^{\mu\alpha\nu\beta} + \bar{R}^\alpha_{\lambda\mu\rho} \bar{\nabla}^\rho t^{\mu\lambda\nu\beta} \\ &\quad + \bar{R}^\nu_{\lambda\mu\rho} \bar{\nabla}^\rho t^{\mu\alpha\lambda\beta} + \bar{R}^\beta_{\lambda\mu\rho} \bar{\nabla}^\rho t^{\mu\alpha\nu\lambda} \\ &\quad + \bar{\nabla}^\rho \left(\bar{R}_{\lambda\rho} t^{\lambda\alpha\nu\beta} + \bar{R}^\alpha_{\lambda\mu\rho} t^{\mu\lambda\nu\beta} + \bar{R}^\nu_{\lambda\mu\rho} t^{\mu\alpha\lambda\beta} + \bar{R}^\beta_{\lambda\mu\rho} t^{\mu\alpha\nu\lambda} \right). \end{aligned} \quad (6.107)$$

6.I Simplifying the quadratic action

Simplifying the quadratic action Eq.(4.15) and the linearized EOM Eq.(4.10) involves using the formulas [149]

$$\delta\mathcal{F}(\square) = \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} f_n \square_s^l \delta(\square_s) \square_s^{n-l-1} \quad (6.108)$$

and [149]

$$\begin{aligned} \delta^{(2)}\mathcal{F}(\square) &= \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} f_n \square_s^l \delta^{(2)}(\square_s) \square_s^{n-l-1} \\ &+ 2 \sum_{n=2}^{\infty} \sum_{l=0}^{n-2} \sum_{k=0}^{n-l-2} f_n \square_s^l \delta(\square_s) \square_s^k \delta(\square_s) \square_s^{n-k-l-2}. \end{aligned} \quad (6.109)$$

Using these, the background ansatz Eq.(5.6) and conditions in Eq.(5.10), we can simplify the last five terms in the quadratic action by using the following relations [149]

$$\delta(\mathcal{F}(\square)R) = (\bar{\square} - r_1)\mathcal{Z}(\bar{\square})\zeta \quad (6.110)$$

$$\begin{aligned} \int d^4x \sqrt{-\bar{g}} \bar{R} \delta\mathcal{F}(\square) \delta R &= \int d^4x \sqrt{-\bar{g}} \left[\left(\bar{R} + \frac{r_2}{r_1} \right) \delta(\square) \mathcal{Z}(\bar{\square}) (\bar{\square} - r_1) \delta R \right. \\ &\quad \left. + \frac{h}{2} \frac{r_2}{r_1} (\mathcal{F}(\bar{\square}) - f_0) \delta R \right]. \\ \int d^4x \sqrt{-\bar{g}} \bar{R} \delta^{(2)}\mathcal{F}(\square) \bar{R} &= \int d^4x \sqrt{-\bar{g}} \left[\left(\bar{R} + \frac{r_2}{r_1} \right) \delta(\square) \mathcal{Z}(\bar{\square}) \delta(\square) \bar{R} \right. \\ &\quad + \frac{h}{2} \frac{r_2}{r_1} (\bar{\square} - r_1) \mathcal{Z}(\bar{\square}) \delta(\square) \bar{R} \\ &\quad \left. + \frac{M_p^2}{4r_1} \frac{h}{2} \left(\delta(\square) \bar{R} + \delta^{(2)}(\square) \bar{R} \right) \right]. \end{aligned} \quad (6.111)$$

Substituting Eq.(6.110) and Eq.(6.111) in Eq.(4.15), using the following identity for some scalars X and Y , performing numerous integration by parts and omitting total derivatives, we finally obtain the result in Eq.(4.16) [149]

$$\int d^4x \sqrt{-\bar{g}} Y \delta(\square) X = \int d^4x \sqrt{-\bar{g}} \left[X \delta(\square) Y - \frac{h}{2} (Y \bar{\square} X - X \bar{\square} Y) \right]. \quad (6.112)$$

Let us see the simplifications in some detail. We have

$$\delta\mathcal{F}(\square) \bar{R} = \mathcal{Z}_1(\square) \delta(\square_s) \bar{R} \quad (6.113)$$

where

$$\mathcal{Z}_1(\square) = \frac{\mathcal{F}(\square) - \mathcal{F}_1}{\square_s - s_1} = \frac{1}{\square_s - s_1} \sum_{n=1}^{\infty} f_n (\square_s^n - s_1^n) \quad (6.114)$$

where we have defined the dimensionless constant

$$s_1 \equiv \frac{r_1}{M_s^2}. \quad (6.115)$$

$\mathcal{Z}_1(\square)$ is analytic in $\square$. Note that $\mathcal{Z}_1(r_1) = \mathcal{F}'(r_1)$. To arrive at the expression (6.114), we have used the summation formula

$$\sum_{l=0}^{n-1} x^l y^{n-l-1} = \frac{x^n - y^n}{x - y} \quad (6.116)$$

We also define $\mathcal{Z}_2(\square)$. It is analytic in $\square$ and given by

$$\mathcal{Z}_2(\square) = \frac{\mathcal{F}(\square) - \mathcal{F}_1}{(\square_s - s_1)^2} = \frac{1}{(\square_s - s_1)^2} \sum_{n=1}^{\infty} f_n (\square_s^n - s_1^n) \quad (6.117)$$

Also note that $\mathcal{Z}_2(r_1) = \mathcal{F}_1''(r_1)$. Dimensionally, $[\mathcal{Z}_1] = [\mathcal{Z}_2] = 0$.

Now, let us lay out some important computations required to simplify the quadratic action (4.15). One of the complicated terms in the quadratic action is

$$\begin{aligned} \int d^4x \sqrt{-\bar{g}} \bar{R} \delta \mathcal{F}(\square) \delta R &= \int d^4x \sqrt{-\bar{g}} \left(\bar{R} + \frac{r_2}{r_1} \right) \delta(\square_s) \mathcal{Z}_1(\square) \delta R + B \\ &= \int d^4x \sqrt{-\bar{g}} \left\{ \delta(\square_s) \bar{R} \mathcal{Z}_1(\square) \delta R \right. \\ &\quad \left. - \frac{h}{2M_s^2} \left(\bar{R} + \frac{r_2}{r_1} \right) (\square - r_1) \mathcal{Z}_1(\square) \delta R \right\} + B \end{aligned} \quad (6.118)$$

where the B term is

$$\begin{aligned} B &= \int \sqrt{\bar{g}} \left(-\frac{r_2}{r_1} \right) \delta(\square_s) \sum_{n=1}^{\infty} f_n \square_s^{n-1} \delta R \\ &= \int \sqrt{\bar{g}} \frac{h}{2} \frac{r_2}{r_1} (\mathcal{F}(\square) - f_0) \delta R \end{aligned} \quad (6.119)$$

where we used (6.112).

Another complicated term in the quadratic action is

$$\int d^4x \sqrt{-\bar{g}} \bar{R} \delta^{(2)} \mathcal{F}(\square) \bar{R} = Q_1 + Q_2 \quad (6.120)$$

where

$$\begin{aligned}
Q_1 &= \int d^4x \sqrt{-\bar{g}} \bar{R} \sum_{n=2}^{\infty} \sum_{l=0}^{n-2} \sum_{k=0}^{n-l-2} f_n \square_s^l \delta(\square_s) \square_s^k \delta(\square_s) \square_s^{n-k-l-2} \bar{R} \\
&= \int d^4x \sqrt{-\bar{g}} \left(\bar{R} + \frac{r_2}{r_1} \right) \delta(\square_s) \mathcal{Z}_2(\square) \delta(\square_s) \bar{R} + C \\
Q_2 &= \int d^4x \sqrt{-\bar{g}} \bar{R} \sum_{n=1}^{\infty} \sum_{l=0}^{n-1} f_n \square_s^l \delta^{(2)}(\square_s) \square_s^{n-l-1} \bar{R} \\
&= \int d^4x \sqrt{-\bar{g}} \mathcal{F}'(r_1) \left(\bar{R} + \frac{r_2}{r_1} \right) \delta^{(2)}(\square_s) \bar{R} + D \\
&= 0 + D
\end{aligned} \tag{6.121}$$

To find Q_1 , we have used integration by parts, the ansatz (5.6), the formula for $\mathcal{Z}_2(\square)$ (6.117) and the constraint $\mathcal{F}'(r_1) = 0$ (5.10). We also used the summation formulas

$$\sum_{l=0}^{n-1} 1 = n, \quad \sum_{l=0}^{n-2} \sum_{k=0}^{n-l-2} x^k y^{n-k-2} = \frac{x^n - y^n - ny^{n-1}(x-y)}{(x-y)^2} \tag{6.122}$$

and the expression for $\mathcal{F}'(r_1)$

$$\mathcal{F}'(r_1) = \sum_{n=1}^{\infty} f_n n s_1^{n-1} \tag{6.123}$$

Note that the derivative is with respect to s_1 , not r_1 (see (6.115)). To find Q_2 , we have used integration by parts, the ansatz (5.6) and the constraint $\mathcal{F}'(r_1) = 0$ (5.10).

The term C is given by

$$\begin{aligned}
C &\stackrel{!}{=} \int d^4x \sqrt{-\bar{g}} \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} f_n \left(-\frac{r_2}{r_1} \right) \delta(\square_s) \square_s^k \delta(\square_s) s_1^{n-k-2} \bar{R} \\
&= \int d^4x \sqrt{-\bar{g}} \underbrace{\left(-\frac{r_2}{r_1} \right) \delta(\square_s)}_Y \underbrace{\sum_{n=2}^{\infty} \sum_{k=0}^{n-2} f_n \square_s^k s_1^{n-k-2} \delta(\square_s) \bar{R}}_X \\
&= \int d^4x \sqrt{-\bar{g}} \left(\frac{h}{2} \frac{r_2}{r_1} \right) \square_s \left(\sum_{n=2}^{\infty} \sum_{k=0}^{n-2} f_n \square_s^k s_1^{n-k-2} \delta(\square_s) \right) \bar{R} \\
&= \int d^4x \sqrt{-\bar{g}} \left(\frac{h}{2} \frac{r_2}{r_1} \right) \left(\sum_{n=2}^{\infty} f_n \frac{s_1 \square_s^n - \square_s s_1^n}{s_1(\square_s - s_1)} \right) \delta(\square_s) \bar{R}
\end{aligned} \tag{6.124}$$

where we have used [6.112](#) for X and Y above. Looking at the term in parentheses, we see that

$$\begin{aligned} \frac{1}{(\square_s - s_1)} \sum_{n=2}^{\infty} f_n (\square_s^n - \square_s s_1^{n-1}) &= \frac{1}{(\square_s - s_1)} \sum_{n=2}^{\infty} f_n \square_s^n - \frac{1}{(\square_s - s_1) s_1} \sum_{n=2}^{\infty} f_n \square_s s_1^n \\ &= T_1 - T_2 \end{aligned} \quad (6.125)$$

where T_1 and T_2 are

$$\begin{aligned} T_1 &= \frac{1}{(\square_s - s_1)} (\mathcal{F}(\square) - f_1 \square_s - f_0) \\ T_2 &= \frac{\square_s}{(\square_s - s_1) s_1} (\mathcal{F}(s_1) - f_1 s_1 - f_0) \end{aligned} \quad (6.126)$$

so that we finally obtain

$$T_1 - T_2 = \mathcal{Z}_1(\square) + \frac{f_0}{s_1} \quad (6.127)$$

Ultimately, we get C as

$$C = \int d^4x \sqrt{-\bar{g}} \left(\frac{h}{2} \frac{r_2}{r_1} \right) \left(\mathcal{Z}_1(\square) + \frac{f_0}{s_1} \right) \delta(\square_s) \bar{R} \quad (6.128)$$

The term D is given by

$$\begin{aligned} D &= \int d^4x \sqrt{-\bar{g}} \left(-\frac{r_2}{r_1} \right) \delta^{(2)}(\square_s) \sum_{n=1}^{\infty} f_n s_1^{n-1} \bar{R} \\ &= \int d^4x \sqrt{-\bar{g}} \left(-\frac{r_2}{r_1} \right) \delta^{(2)}(\square_s) \frac{1}{s_1} (\mathcal{F}(r_1) - f_0) \bar{R} \\ &= \int d^4x \sqrt{-\bar{g}} \left(\frac{f_0 r_2 M_s^2}{r_1^2} \right) \delta^{(2)}(\square_s) \bar{R} \end{aligned} \quad (6.129)$$

We used the summation formulas

$$\begin{aligned} \sum_{l=0}^{n-1} 1 &= n, \quad \sum_{l=0}^{n-2} \sum_{k=0}^{n-l-2} x^k y^{n-k-2} = \frac{x^n - y^n - n y^{n-1} (x - y)}{(x - y)^2} \\ \sum_{k=0}^{n-2} \square_s^k s_1^{n-k-2} &= \frac{s_1 \square_s^n - \square_s s_1^n}{\square_s s_1 (\square_s - s_1)} \end{aligned} \quad (6.130)$$

Now, recall that

$$\frac{r_2}{r_1} f_0 = \frac{M_P^2}{4} \quad (6.131)$$

so that we have

$$\begin{aligned}
B &= \int \sqrt{g} \frac{h}{2} \frac{r_2}{r_1} \mathcal{F}(\square) \delta R - \frac{M_P^2}{4} \frac{h}{2} \delta R \\
C &= \int d^4x \sqrt{-g} \frac{h}{2} \frac{r_2}{r_1} \mathcal{Z}_1(\square) \delta(\square_s) \bar{R} + \frac{M_P^2}{4s_1} \frac{h}{2} \delta(\square_s) \bar{R} \\
D &= \int d^4x \sqrt{-g} \left(\frac{M_P^2}{4s_1} \right) \delta^{(2)}(\square_s) \bar{R}
\end{aligned} \tag{6.132}$$

Finally, adding all the terms in the quadratic action in [4.15](#), we get two terms, one of which is

$$\left(\frac{M_P^2}{4s_1} \right) \int \frac{h}{2} \delta(\square_s) \bar{R} + \delta^{(2)}(\square_s) \bar{R} + \sqrt{g}^{(2)} \square_s \bar{R} \stackrel{?}{=} 0 \tag{6.133}$$

To show that this is indeed zero (total derivative), we look at the total derivative

$$A = \int \sqrt{g} \square_s R = 0 \tag{6.134}$$

which is also zero at every order in h , that is,

$$A = \underbrace{A^{(0)}}_0 + \underbrace{A^{(1)}}_0 + \underbrace{A^{(2)}}_0 + \dots = 0 \tag{6.135}$$

In particular, we look at $A^{(2)}$, which has 6 terms - 3 of which are $(110) + (020) + (200)$ (where a number n indicates a contribution of $\mathcal{O}(h^n)$) as in [6.133](#) above, and the remaining 3 terms are

$$\int \sqrt{g} \square_s R^{(2)} + \sqrt{g} \frac{h}{2} \square_s \delta R + \sqrt{g} \delta(\square_s) \delta R = 0 \tag{6.136}$$

which sum to zero - the first term is a total derivative, and the last two terms cancel each other upon integration by parts after using the formula [6.112](#). Since $A^{(2)} = 0$, the 3 terms in [6.133](#) also combine to zero.

Adding all the pieces together, ultimately, we have been able to cancel all but one term in the quadratic action [\[149\]](#):

$$S^{(2)} = \int d^4x \sqrt{-g} \frac{1}{M_s^4} \zeta \mathcal{Z}_2(\square) \zeta = \int d^4x \sqrt{-g} \zeta \mathcal{Z}(\square) \zeta \tag{6.137}$$

which matches the linearized EOM, after varying $S^{(2)}$ with respect to ζ . As desired, the quadratic action formalism matches the linearized EOM formalism. Note that things like [6.134](#) and [6.135](#) are trivial but very useful, and can help us in avoiding the computation of many terms, like $\delta^{(2)}(\square_s) \bar{R}$.

6.J 1 + 3 cosmological perturbation theory

We study an isotropic and homogeneous universe described by a spatially flat ($k = 0$) FLRW metric. The background metric in (τ, x, y, z) coordinates is then

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\tau) [-d\tau^2 + \delta_{ij} dx^i dx^j] \quad (6.138)$$

where τ is conformal time and $a(\tau)$ is the scale factor. In components,

$$\bar{g}_{00} = -a^2(\tau), \quad \bar{g}_{ij} = a^2(\tau) \delta_{ij} \quad (6.139)$$

We perturb the background metric to first order as

$$g_{\mu\nu}(\tau, \mathbf{x}) = \bar{g}_{\mu\nu}(\tau) + h_{\mu\nu}(\tau, \mathbf{x}) \quad (6.140)$$

where $\mathbf{x} = (x, y, z)$. We will henceforth drop explicit functional dependence of $a(\tau)$ on τ . The perturbed metric is [235]

$$ds^2 = a^2 \left[-(1 + 2\phi) d\tau^2 + 2B_i dx^i d\tau + (\delta_{ij} + h_{ij}) dx^i dx^j \right] \quad (6.141)$$

In component form, $h_{\mu\nu}$ can be written as [235]

$$h_{00} = a^2 (-2\phi), \quad h_{0i} = a^2 (B_i), \quad h_{ij} = a^2 (2E_{ij}) \quad (6.142)$$

where ϕ is a scalar, B_i is a 3-vector and E_{ij} is a symmetric 3-tensor. However, B_i contains a scalar mode B in addition to a pure (divergenceless) vector mode $\hat{B}_i$; we can always perform the decomposition

$$B_i = \hat{B}_i + \partial_i B \quad (6.143)$$

where $\partial^i \hat{B}_i = 0$. Similarly, E_{ij} contains, in addition to a pure (transverse and traceless (TT)) tensor mode $\hat{h}_{ij}$, a pure (divergenceless) vector mode $\hat{E}_i$ and two scalars E and ψ :

$$E_{ij} = \hat{h}_{ij} + \partial_i \hat{E}_j + \partial_j \hat{E}_i + \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) E - \psi \delta_{ij} \quad (6.144)$$

where $\partial^i \hat{E}_i = 0$ and $\partial^i \hat{h}_{ij} = 0, \hat{h}^i_i = 0$. Hatted quantities are divergenceless or transverse. Effectively, we have decomposed the ten degrees of freedom in $h_{\mu\nu}$ into four scalars ϕ, ψ, B, E with 1 dof each, two divergenceless 3-vectors $\hat{B}_i, \hat{E}_i$ with 2 dof each, and one TT 3-tensor $\hat{h}_{ij}$ with 2 dof. With this decomposition, we have clasified SVT dofs on the basis of how they transform on constant-time hypersurfaces under rotations in $SO(3)$.

The perturbations (6.142) are not independent of general coordinate transformations (GCT) and we must choose a gauge or use gauge-invariant variables to

continue our analysis. We work with perturbations that transform up to first order under GCT. We will choose the longitudinal gauge, where we have [230, 235]:

$$B = 0, \quad E = 0, \quad \hat{E}_i = 0 \quad (6.145)$$

which completely fixes the coordinate system, leaving 6 ($2S + 2V + 2T$) dofs. In this gauge then, the perturbed metric is given by [235]

$$ds^2 = a^2 \left[-(1 + 2\phi)d\tau^2 + 2\hat{B}_i dx^i d\tau + \left((1 - 2\psi)\delta_{ij} + 2\hat{h}_{ij} \right) dx^i dx^j \right] \quad (6.146)$$

Finally, in component form, $h_{\mu\nu}$ can be written as

$$h_{00} = a^2 (-2\phi), \quad h_{0i} = a^2 (\hat{B}_i), \quad h_{ij} = a^2 (-2\psi\delta_{ij} + 2\hat{h}_{ij}) \quad (6.147)$$

6.K k -matter as a perfect fluid scalar field

It is interesting to consider the simplest case of k -matter being represented by a perfect fluid, canonical scalar field in a Bianchi-I background (5.12) with the Lagrangian $\mathcal{L} = -(\partial\phi)^2/2 - V(\phi)$. See Table (5.1) for a list of cases where k -matter can be described by a perfect fluid. We assume that it is described by a homogeneous and time-dependent scalar field $\phi(t)$. Using [150]

$$\begin{aligned} \rho &= \dot{\phi}^2/2 + V(\phi), & p &= \dot{\phi}^2/2 - V(\phi), \\ \rho_k &= \rho_{k0} a^{-2}, & p_k &= w_k \rho_k, \quad \text{where } w_k = -1/3, \\ H &= \sqrt{\frac{r_1}{2}} \tanh\left(\sqrt{\frac{r_1}{2}} t\right), & \text{for } a &= \cosh\left(\sqrt{\frac{r_1}{2}} t\right), \end{aligned} \quad (6.148)$$

we can deduce an oscillatory potential given by [150]

$$V(\phi) = \frac{2\rho_{k0}}{3} \cos^2\left(\sqrt{\frac{3r_1}{4\rho_{k0}}} \phi\right). \quad (6.149)$$

The solution for $\phi(t)$ is [150]

$$\phi(t) = \sqrt{\frac{4\rho_{k0}}{3r_1}} \operatorname{gd}\left(\sqrt{\frac{r_1}{2}} t\right) \quad (6.150)$$

where $\phi(t)$ runs from $-\frac{\pi}{2} \sqrt{\frac{4\rho_{k0}}{3r_1}}$ at $t = -\infty$ to $\frac{\pi}{2} \sqrt{\frac{4\rho_{k0}}{3r_1}}$ at $t = \infty$.

Samenvatting

Het doel van dit proefschrift is om ultraviolette (UV) niet-lokale zwaartekracht te introduceren en daarin enkele nieuwe resultaten te bespreken. Gemotiveerd door benaderingswijzen van kwantumzwaartekracht zoals snaartheorie, is dit een hogere afgeleide modificatie van de algemene relativiteitstheorie (ART) die tot doel heeft een zachtere beschrijving van de zwaartekracht op korte afstanden te geven, terwijl tegelijkertijd de typische ziekten die gepaard gaan met hogere afgeleide theorieën worden geweerd. De afgeleiden die in de actie voorkomen zijn van oneindige orde, waardoor de theorie 'niet-lokaal' is. Bovendien is het 'UV' omdat de afgeleiden de zwaartekracht modificeren bij hoge energieën, terwijl ART bij lage energieën wordt teruggewonnen. We bespreken hoe dergelijke theorieën 1) gelineariseerde oplossingen bezitten met niet-divergente gravitatiepotentialen en krommingen, 2) niet-maximaal symmetrische achtergronden toelaten die stabiel zijn (vrij van spoken/tachyonen) onder verstoringen, 3) niet-singuliere, isotrope en homogene stuiterende kosmologische oplossingen toelaten met stabiele verstoringen en 4) niet-singuliere, anisotrope en homogene stuiterende kosmologische oplossingen toestaan waar, in tegenstelling tot ART, er geen onstabiele groei is van ruimtetijdanisotropieën tijdens de samentrekkingsfase van het universum, en isotropie en homogeniteit op late tijd worden hersteld tijdens de uitdijingsfase.

Bibliography

- [1] Wikipedia contributors, "Gravity — Wikipedia, the free encyclopedia." <https://en.wikipedia.org/w/index.php?title=Gravity&oldid=1012153952>, 2021.
- [2] Wikipedia contributors, "History of gravitational theory — Wikipedia, the free encyclopedia." https://en.wikipedia.org/w/index.php?title=History_of_gravitational_theory&oldid=1012154045, 2021.
- [3] G. Smith, *Newton's Philosophiae Naturalis Principia Mathematica*, in *The Stanford Encyclopedia of Philosophy* (E. N. Zalta, ed.). Metaphysics Research Lab, Stanford University, winter 2008 ed., 2008.
- [4] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2011, [10.1017/CBO9780511524646](https://doi.org/10.1017/CBO9780511524646).
- [5] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984, [10.7208/chicago/9780226870373.001.0001](https://doi.org/10.7208/chicago/9780226870373.001.0001).
- [6] S. M. Carroll, *Spacetime and Geometry*. Cambridge University Press, 7, 2019.
- [7] C. M. Will, *The Confrontation between General Relativity and Experiment*, *Living Rev. Rel.* **17** (2014) 4, [1403.7377](https://doi.org/10.1007/978-1-403-73777-1).
- [8] F. W. Dyson, A. S. Eddington and C. Davidson, *A Determination of the Deflection of Light by the Sun's Gravitational Field, from Observations Made at the Total Eclipse of May 29, 1919*, *Philosophical Transactions of the Royal Society of London Series A* **220** (Jan., 1920) 291–333.

- [9] R. A. Hulse and J. H. Taylor, *Discovery of a pulsar in a binary system*, *Astrophys. J. Lett.* **195** (1975) L51–L53.
- [10] J. H. Taylor and J. M. Weisberg, *A new test of general relativity: Gravitational radiation and the binary pulsar PS R 1913+16*, *Astrophys. J.* **253** (1982) 908–920.
- [11] VIRGO, LIGO SCIENTIFIC collaboration, B. P. Abbott et al., *Observation of Gravitational Waves from a Binary Black Hole Merger*, *Phys. Rev. Lett.* **116** (2016) 061102, [1602.03837].
- [12] LIGO SCIENTIFIC, VIRGO collaboration, B. Abbott et al., *GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral*, *Phys. Rev. Lett.* **119** (2017) 161101, [1710.05832].
- [13] EVENT HORIZON TELESCOPE collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole*, *Astrophys. J.* **875** (2019) L1, [1906.11238].
- [14] EVENT HORIZON TELESCOPE collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. II. Array and Instrumentation*, *Astrophys. J. Lett.* **875** (2019) L2, [1906.11239].
- [15] EVENT HORIZON TELESCOPE collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. III. Data Processing and Calibration*, *Astrophys. J. Lett.* **875** (2019) L3, [1906.11240].
- [16] EVENT HORIZON TELESCOPE collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole*, *Astrophys. J. Lett.* **875** (2019) L4, [1906.11241].
- [17] EVENT HORIZON TELESCOPE collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring*, *Astrophys. J. Lett.* **875** (2019) L5, [1906.11242].
- [18] EVENT HORIZON TELESCOPE collaboration, K. Akiyama et al., *First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole*, *Astrophys. J. Lett.* **875** (2019) L6, [1906.11243].
- [19] SUPERNOVA SEARCH TEAM collaboration, A. G. Riess et al., *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, *Astron. J.* **116** (1998) 1009–1038, [astro-ph/9805201].
- [20] SUPERNOVA COSMOLOGY PROJECT collaboration, S. Perlmutter et al., *Measurements of Ω and Λ from 42 high redshift supernovae*, *Astrophys. J.* **517** (1999) 565–586, [astro-ph/9812133].

- [21] C. Brans and R. H. Dicke, *Mach's principle and a relativistic theory of gravitation*, *Phys. Rev.* **124** (1961) 925–935.
- [22] T. P. Sotiriou and V. Faraoni, *$f(R)$ Theories Of Gravity*, *Rev. Mod. Phys.* **82** (2010) 451–497, [0805.1726].
- [23] P. Horava, *Quantum Gravity at a Lifshitz Point*, *Phys. Rev. D* **79** (2009) 084008, [0901.3775].
- [24] G. R. Dvali, G. Gabadadze and M. Porrati, *4-D gravity on a brane in 5-D Minkowski space*, *Phys. Lett. B* **485** (2000) 208–214, [hep-th/0005016].
- [25] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, *Modified Gravity and Cosmology*, *Phys. Rept.* **513** (2012) 1–189, [1106.2476].
- [26] S. Capozziello and M. De Laurentis, *Extended Theories of Gravity*, *Phys. Rept.* **509** (2011) 167–321, [1108.6266].
- [27] PLANCK collaboration, N. Aghanim et al., *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6, [1807.06209].
- [28] S. Dodelson, *Modern Cosmology*. Academic Press, Amsterdam, 2003.
- [29] V. C. Rubin and J. Ford, W. Kent, *Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions*, *The Astrophysical Journal* **159** (Feb., 1970) 379.
- [30] V. C. Rubin, J. Ford, W. K. and N. Thonnard, *Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 ($R=4kpc$) to UGC 2885 ($R=122kpc$)*, *The Astrophysical Journal* **238** (June, 1980) 471–487.
- [31] G. Bertone, D. Hooper and J. Silk, *Particle dark matter: Evidence, candidates and constraints*, *Phys. Rept.* **405** (2005) 279–390, [hep-ph/0404175].
- [32] J. L. Feng, *Dark Matter Candidates from Particle Physics and Methods of Detection*, *Ann. Rev. Astron. Astrophys.* **48** (2010) 495–545, [1003.0904].
- [33] M. Milgrom, *A Modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis*, *Astrophys. J.* **270** (1983) 365–370.
- [34] J. D. Bekenstein, *Relativistic gravitation theory for the MOND paradigm*, *Phys. Rev. D* **70** (2004) 083509, [astro-ph/0403694].
- [35] P. J. E. Peebles and B. Ratra, *The Cosmological Constant and Dark Energy*, *Rev. Mod. Phys.* **75** (2003) 559–606, [astro-ph/0207347].

- [36] A. Joyce, L. Lombriser and F. Schmidt, *Dark Energy Versus Modified Gravity*, *Ann. Rev. Nucl. Part. Sci.* **66** (2016) 95–122, [[1601.06133](#)].
- [37] S. Weinberg, *The Cosmological Constant Problem*, *Rev. Mod. Phys.* **61** (1989) 1–23.
- [38] R. Penrose, *Gravitational collapse and space-time singularities*, *Phys. Rev. Lett.* **14** (1965) 57–59.
- [39] R. Penrose, *Gravitational collapse: The role of general relativity*, *Riv. Nuovo Cim.* **1** (1969) 252–276.
- [40] S. W. Hawking and R. Penrose, *The Singularities of gravitational collapse and cosmology*, *Proc. Roy. Soc. Lond.* **A314** (1970) 529–548.
- [41] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007, [10.1017/CBO9780511618123](#).
- [42] A. Ashtekar and J. Lewandowski, *Background independent quantum gravity: A Status report*, *Class. Quant. Grav.* **21** (2004) R53, [[gr-qc/0404018](#)].
- [43] L. Bombelli, J. Lee, D. Meyer and R. Sorkin, *Space-Time as a Causal Set*, *Phys. Rev. Lett.* **59** (1987) 521–524.
- [44] K. Schwarzschild, *On the Gravitational Field of a Mass Point According to Einstein's Theory*, *Abh. Konigl. Preuss. Akad. Wissenschaften Jahre 1906,92, Berlin,1907* **1916** (Jan., 1916) 189–196.
- [45] S. N. Gupta, *Quantization of Einstein's gravitational field: general treatment*, *Proc. Phys. Soc. A* **65** (1952) 608–619.
- [46] R. P. Feynman, *Quantum theory of gravitation*, *Acta Phys. Polon.* **24** (1963) 697–722.
- [47] B. S. DeWitt, *Quantum Theory of Gravity. 1. The Canonical Theory*, *Phys. Rev.* **160** (1967) 1113–1148.
- [48] B. S. DeWitt, *Quantum Theory of Gravity. 2. The Manifestly Covariant Theory*, *Phys. Rev.* **162** (1967) 1195–1239.
- [49] B. S. DeWitt, *Quantum Theory of Gravity. 3. Applications of the Covariant Theory*, *Phys. Rev.* **162** (1967) 1239–1256.
- [50] F. Dyson, *Is a graviton detectable?*, *Int. J. Mod. Phys. A* **28** (2013) 1330041.

- [51] C. J. Isham, *Structural issues in quantum gravity*, in 14TH INTERNATIONAL CONFERENCE ON GENERAL RELATIVITY AND GRAVITATION (GR14), pp. 167–209, 8, 1995, [gr-qc/9510063](#).
- [52] S. Carlip, *Quantum gravity: A Progress report*, [Rept. Prog. Phys.](#) **64** (2001) 885, [gr-qc/0108040](#).
- [53] S. W. Hawking, *Particle Creation by Black Holes*, [Commun. Math. Phys.](#) **43** (1975) 199–220.
- [54] S. W. Hawking, *Breakdown of Predictability in Gravitational Collapse*, [Phys. Rev. D](#) **14** (1976) 2460–2473.
- [55] J. Polchinski, *The Black Hole Information Problem*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*, pp. 353–397, 2017, [1609.04036](#), [DOI](#).
- [56] J. F. Donoghue, *General relativity as an effective field theory: The leading quantum corrections*, [Phys. Rev. D](#) **50** (1994) 3874–3888, [gr-qc/9405057](#).
- [57] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 3, 2014.
- [58] G. 't Hooft and M. J. G. Veltman, *One loop divergencies in the theory of gravitation*, [Ann. Inst. H. Poincaré Phys. Théor. A](#) **20** (1974) 69–94.
- [59] M. H. Goroff and A. Sagnotti, *QUANTUM GRAVITY AT TWO LOOPS*, [Phys. Lett.](#) **160B** (1985) 81–86.
- [60] R. Utiyama and B. S. DeWitt, *Renormalization of a classical gravitational field interacting with quantized matter fields*, [J. Math. Phys.](#) **3** (1962) 608–618.
- [61] J. F. Donoghue, M. M. Ivanov and A. Shkerin, *EPFL Lectures on General Relativity as a Quantum Field Theory*, [1702.00319](#).
- [62] K. S. Stelle, *Renormalization of Higher Derivative Quantum Gravity*, [Phys. Rev. D](#) **16** (1977) 953–969.
- [63] K. S. Stelle, *Classical Gravity with Higher Derivatives*, [Gen. Rel. Grav.](#) **9** (1978) 353–371.
- [64] E. S. Fradkin and A. A. Tseytlin, *Renormalizable asymptotically free quantum theory of gravity*, [Nucl. Phys. B](#) **201** (1982) 469–491.

- [65] I. G. Avramidi and A. O. Barvinsky, *ASYMPTOTIC FREEDOM IN HIGHER DERIVATIVE QUANTUM GRAVITY*, *Phys. Lett. B* **159** (1985) 269–274.
- [66] E. Tomboulis, *1/N Expansion and Renormalization in Quantum Gravity*, *Phys. Lett. B* **70** (1977) 361–364.
- [67] E. Tomboulis, *Renormalizability and Asymptotic Freedom in Quantum Gravity*, *Phys. Lett. B* **97** (1980) 77–80.
- [68] I. Antoniadis and E. T. Tomboulis, *Gauge Invariance and Unitarity in Higher Derivative Quantum Gravity*, *Phys. Rev. D* **33** (1986) 2756.
- [69] E. T. Tomboulis, *Unitarity in Higher Derivative Quantum Gravity*, *Phys. Rev. Lett.* **52** (1984) 1173.
- [70] I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, *Effective action in quantum gravity*. 1992.
- [71] A. Salam and J. A. Strathdee, *Remarks on High-energy Stability and Renormalizability of Gravity Theory*, *Phys. Rev. D* **18** (1978) 4480.
- [72] B. Zwiebach, *Curvature Squared Terms and String Theories*, *Phys. Lett.* **156B** (1985) 315–317.
- [73] D. J. Gross and E. Witten, *Superstring Modifications of Einstein’s Equations*, *Nucl. Phys. B* **277** (1986) 1.
- [74] R. P. Woodard, *Ostrogradsky’s theorem on Hamiltonian instability*, *Scholarpedia* **10** (2015) 32243, [[1506.02210](#)].
- [75] M. Ostrogradsky, *Mémoires sur les équations différentielles, relatives au problème des isopérimètres*, *Mem. Acad. St. Petersburg* **6** (1850) 385–517.
- [76] J. M. Cline, S. Jeon and G. D. Moore, *The Phantom menaced: Constraints on low-energy effective ghosts*, *Phys. Rev. D* **70** (2004) 043543, [[hep-ph/0311312](#)].
- [77] A. Joyce, B. Jain, J. Khoury and M. Trodden, *Beyond the Cosmological Standard Model*, *Phys. Rept.* **568** (2015) 1–98, [[1407.0059](#)].
- [78] M. Born, *Reciprocity theory of elementary particles*, *Rev. Mod. Phys.* **21** (Jul, 1949) 463–473.
- [79] H. Yukawa, *On the radius of the elementary particle*, *Phys. Rev.* **76** (Jul, 1949) 300–301.

- [80] H. Yukawa, *Quantum Theory of Nonlocal Fields. 1. Free Fields*, *Phys. Rev.* **77** (1950) 219–226.
- [81] H. Yukawa, *Quantum Theory of Nonlocal Fields. 2: Irreducible Fields and Their Interaction*, *Phys. Rev.* **80** (1950) 1047–1052.
- [82] A. Pais and G. E. Uhlenbeck, *On Field theories with nonlocalized action*, *Phys. Rev.* **79** (1950) 145–165.
- [83] W. Pauli, *On the Hamiltonian structure of non-local field theories*, *Nuovo Cim.* **10** (1953) 648–667.
- [84] C. Hayashi, *Hamiltonian Formalism in Non-local Field Theories*, *Progress of Theoretical Physics* **10** (11, 1953) 533–548.
- [85] G. V. Efimov, *Non-local quantum theory of the scalar field*, *Commun. Math. Phys.* **5** (1967) 42–56.
- [86] G. V. Efimov, *Quantization of non-local field theory*, *Int. J. Theor. Phys.* **10** (1974) 19–37.
- [87] G. V. Efimov, M. A. Ivanov and O. A. Mogilevsky, *Electron self-energy in nonlocal field theory*, *Annals Phys.* **103** (1977) 169–184.
- [88] D. A. Kirzhnits, *Nonlocal quantum field theory*, *Usp. Fiz. Nauk* **90** (1966) 129–142.
- [89] N. V. Krasnikov, *NONLOCAL GAUGE THEORIES*, *Theor. Math. Phys.* **73** (1987) 1184–1190.
- [90] Yu. V. Kuzmin, *THE CONVERGENT NONLOCAL GRAVITATION. (IN RUSSIAN)*, *Sov. J. Nucl. Phys.* **50** (1989) 1011–1014.
- [91] E. T. Tomboulis, *Superrenormalizable gauge and gravitational theories*, [hep-th/9702146](#).
- [92] E. T. Tomboulis, *Nonlocal and quasilocal field theories*, *Phys. Rev.* **D92** (2015) 125037, [[1507.00981](#)].
- [93] E. T. Tomboulis, *Renormalization and unitarity in higher derivative and nonlocal gravity theories*, *Mod. Phys. Lett.* **A30** (2015) 1540005.
- [94] T. Biswas, A. Mazumdar and W. Siegel, *Bouncing universes in string-inspired gravity*, *JCAP* **0603** (2006) 009, [[hep-th/0508194](#)].
- [95] L. Modesto, *Super-renormalizable Quantum Gravity*, *Phys. Rev.* **D86** (2012) 044005, [[1107.2403](#)].

- [96] A. Sen, *String Field Theory as World-sheet UV Regulator*, [JHEP 10 \(2019\) 119](#), [[1902.00263](#)].
- [97] E. Witten, *Noncommutative Geometry and String Field Theory*, [Nucl. Phys. B 268 \(1986\) 253–294](#).
- [98] W. Siegel, *Introduction to string field theory*, vol. 8. 1988.
- [99] P. G. O. Freund and M. Olson, *NONARCHIMEDEAN STRINGS*, [Phys. Lett. B199 \(1987\) 186–190](#).
- [100] P. G. O. Freund and E. Witten, *ADELIC STRING AMPLITUDES*, [Phys. Lett. B199 \(1987\) 191](#).
- [101] L. Brekke, P. G. O. Freund, M. Olson and E. Witten, *Nonarchimedean String Dynamics*, [Nucl. Phys. B302 \(1988\) 365–402](#).
- [102] N. Moeller and B. Zwiebach, *Dynamics with infinitely many time derivatives and rolling tachyons*, [JHEP 10 \(2002\) 034](#), [[hep-th/0207107](#)].
- [103] N. Barnaby and N. Kamran, *Dynamics with infinitely many derivatives: The Initial value problem*, [JHEP 02 \(2008\) 008](#), [[0709.3968](#)].
- [104] W. Siegel, *Stringy gravity at short distances*, [[hep-th/0309093](#)].
- [105] A. A. Tseytlin, *On singularities of spherically symmetric backgrounds in string theory*, [Phys. Lett. B 363 \(1995\) 223–229](#), [[hep-th/9509050](#)].
- [106] T. Biswas, T. Koivisto and A. Mazumdar, *Nonlocal theories of gravity: the flat space propagator*, in *Proceedings, Barcelona Postgrad Encounters on Fundamental Physics*, pp. 13–24, 2013, [[1302.0532](#)].
- [107] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, *Towards singularity and ghost free theories of gravity*, [Phys. Rev. Lett. 108 \(2012\) 031101](#), [[1110.5249](#)].
- [108] A. O. Barvinsky and G. A. Vilkovisky, *Covariant perturbation theory. 2: Second order in the curvature. General algorithms*, [Nucl. Phys. B 333 \(1990\) 471–511](#).
- [109] L. Modesto, T. de Paula Netto and I. L. Shapiro, *On Newtonian singularities in higher derivative gravity models*, [JHEP 04 \(2015\) 098](#), [[1412.0740](#)].
- [110] V. P. Frolov and A. Zelnikov, *Head-on collision of ultrarelativistic particles in ghost-free theories of gravity*, [Phys. Rev. D 93 \(2016\) 064048](#), [[1509.03336](#)].

- [111] T. Biswas, A. S. Koshelev and A. Mazumdar, *Consistent Higher Derivative Gravitational theories with stable de Sitter and Anti-de Sitter Backgrounds*, [1606.01250](#).
- [112] T. Biswas, A. S. Koshelev and A. Mazumdar, *Gravitational theories with stable (anti-)de Sitter backgrounds*, *Fundam. Theor. Phys.* **183** (2016) 97–114, [1602.08475](#).
- [113] S. Deser and R. P. Woodard, *Nonlocal Cosmology*, *Phys. Rev. Lett.* **99** (2007) 111301, [0706.2151](#).
- [114] M. Maggiore and M. Mancarella, *Nonlocal gravity and dark energy*, *Phys. Rev. D* **90** (2014) 023005, [1402.0448](#).
- [115] L. Buoninfante, A. S. Koshelev, G. Lambiase and A. Mazumdar, *Classical properties of non-local, ghost- and singularity-free gravity*, *JCAP* **1809** (2018) 034, [1802.00399](#).
- [116] L. Buoninfante, G. Harmsen, S. Maheshwari and A. Mazumdar, *Nonsingular metric for an electrically charged point-source in ghost-free infinite derivative gravity*, *Phys. Rev. D* **98** (2018) 084009, [1804.09624](#).
- [117] A. S. Cornell, G. Harmsen, G. Lambiase and A. Mazumdar, *Rotating metric in nonsingular infinite derivative theories of gravity*, *Phys. Rev. D* **97** (2018) 104006, [1710.02162](#).
- [118] L. Buoninfante, A. S. Cornell, G. Harmsen, A. S. Koshelev, G. Lambiase, J. a. Marto et al., *Towards nonsingular rotating compact object in ghost-free infinite derivative gravity*, *Phys. Rev. D* **98** (2018) 084041, [1807.08896](#).
- [119] J. Edholm, A. S. Koshelev and A. Mazumdar, *Behavior of the Newtonian potential for ghost-free gravity and singularity-free gravity*, *Phys. Rev. D* **94** (2016) 104033, [1604.01989](#).
- [120] J. Boos, V. P. Frolov and A. Zelnikov, *Gravitational field of static p -branes in linearized ghost-free gravity*, *Phys. Rev. D* **97** (2018) 084021, [1802.09573](#).
- [121] L. Buoninfante, A. S. Koshelev, G. Lambiase, J. Marto and A. Mazumdar, *Conformally-flat, non-singular static metric in infinite derivative gravity*, *JCAP* **1806** (2018) 014, [1804.08195](#).
- [122] A. S. Koshelev, J. Marto and A. Mazumdar, *Schwarzschild $1/r$ -singularity is not permissible in ghost free quadratic curvature infinite derivative gravity*, *Phys. Rev. D* **98** (2018) 064023, [1803.00309](#).

- [123] V. P. Frolov, *Mass-gap for black hole formation in higher derivative and ghost free gravity*, *Phys. Rev. Lett.* **115** (2015) 051102, [[1505.00492](#)].
- [124] V. P. Frolov, A. Zelnikov and T. de Paula Netto, *Spherical collapse of small masses in the ghost-free gravity*, *JHEP* **06** (2015) 107, [[1504.00412](#)].
- [125] E. Kilicarslan, *PP-waves as Exact Solutions to Ghost-free Infinite Derivative Gravity*, [[1903.04283](#)].
- [126] I. Kolář and J. Boos, *Retarded field of a uniformly accelerated source in non-local scalar field theory*, [[2102.07843](#)].
- [127] S. Dengiz, E. Kilicarslan, I. Kolář and A. Mazumdar, *Impulsive waves in ghost free infinite derivative gravity in anti-de Sitter spacetime*, *Phys. Rev. D* **102** (2020) 044016, [[2006.07650](#)].
- [128] T. Biswas and N. Okada, *Towards LHC physics with nonlocal Standard Model*, *Nucl. Phys. B* **898** (2015) 113–131, [[1407.3331](#)].
- [129] S. Talaganis, T. Biswas and A. Mazumdar, *Towards understanding the ultraviolet behavior of quantum loops in infinite-derivative theories of gravity*, *Class. Quant. Grav.* **32** (2015) 215017, [[1412.3467](#)].
- [130] L. Buoninfante, G. Lambiase and A. Mazumdar, *Ghost-free infinite derivative quantum field theory*, [[1805.03559](#)].
- [131] L. Buoninfante, G. Lambiase and M. Yamaguchi, *Nonlocal generalization of Galilean theories and gravity*, *Phys. Rev. D* **100** (2019) 026019, [[1812.10105](#)].
- [132] L. Buoninfante, A. Ghoshal, G. Lambiase and A. Mazumdar, *Transmutation of nonlocal scale in infinite derivative field theories*, *Phys. Rev. D* **99** (2019) 044032, [[1812.01441](#)].
- [133] M. Frasca and A. Ghoshal, *Strongly Coupled String-inspired Infinite Derivative Non-local Yang-Mills: Diluted Mass Gap and Conformal Invariance*, [[2102.10665](#)].
- [134] M. Frasca and A. Ghoshal, *Mass Gap in Infinite Derivative Non-local Higgs: Dyson-Schwinger Approach*, [[2011.10586](#)].
- [135] A. Ghoshal, A. Mazumdar, N. Okada and D. Villalba, *Stability of infinite derivative Abelian Higgs models*, *Phys. Rev. D* **97** (2018) 076011, [[1709.09222](#)].
- [136] A. Ghoshal, A. Mazumdar, N. Okada and D. Villalba, *Non-local Non-Abelian Gauge Theory: Conformal Invariance and β -function*, [[2010.15919](#)].

- [137] M. N. Hashi, H. Isono, T. Noumi, G. Shiu and P. Soler, *Higgs Mechanism in Nonlocal Field Theories*, [*JHEP* **08** \(2018\) 064](#), [[1805.02676](#)].
- [138] T. Biswas, J. A. R. Cembranos and J. I. Kapusta, *Thermal Duality and Hagedorn Transition from p -adic Strings*, [*Phys. Rev. Lett.* **104** \(2010\) 021601](#), [[0910.2274](#)].
- [139] T. Biswas, J. A. R. Cembranos and J. I. Kapusta, *Thermodynamics and Cosmological Constant of Non-Local Field Theories from p -Adic Strings*, [*JHEP* **10** \(2010\) 048](#), [[1005.0430](#)].
- [140] T. Biswas, J. A. R. Cembranos and J. I. Kapusta, *Finite Temperature Solitons in Non-Local Field Theories from p -Adic Strings*, [*Phys. Rev.* **D82** \(2010\) 085028](#), [[1006.4098](#)].
- [141] T. Biswas, J. Kapusta and A. Reddy, *Thermodynamics of String Field Theory Motivated Nonlocal Models*, [*JHEP* **12** \(2012\) 008](#), [[1201.1580](#)].
- [142] J. Boos, V. P. Frolov and A. Zelnikov, *On thermal field fluctuations in ghost-free theories*, [[1904.07917](#)].
- [143] T. Biswas, A. S. Koshelev, A. Mazumdar and S. Yu. Vernov, *Stable bounce and inflation in non-local higher derivative cosmology*, [*JCAP* **1208** \(2012\) 024](#), [[1206.6374](#)].
- [144] A. S. Koshelev, K. Sravan Kumar and A. A. Starobinsky, *R^2 inflation to probe non-perturbative quantum gravity*, [*JHEP* **03** \(2018\) 071](#), [[1711.08864](#)].
- [145] A. S. Koshelev, K. Sravan Kumar, A. Mazumdar and A. A. Starobinsky, *Non-Gaussianities and tensor-to-scalar ratio in non-local R^2 -like inflation*, [*JHEP* **06** \(2020\) 152](#), [[2003.00629](#)].
- [146] T. Biswas, R. Brandenberger, A. Mazumdar and W. Siegel, *Non-perturbative Gravity, Hagedorn Bounce & CMB*, [*JCAP* **0712** \(2007\) 011](#), [[hep-th/0610274](#)].
- [147] A. S. Koshelev, *Stable analytic bounce in non-local Einstein-Gauss-Bonnet cosmology*, [*Class. Quant. Grav.* **30** \(2013\) 155001](#), [[1302.2140](#)].
- [148] T. Biswas, T. Koivisto and A. Mazumdar, *Towards a resolution of the cosmological singularity in non-local higher derivative theories of gravity*, [*JCAP* **1011** \(2010\) 008](#), [[1005.0590](#)].
- [149] K. S. Kumar, S. Maheshwari, A. Mazumdar and J. Peng, *Stable, nonsingular bouncing universe with only a scalar mode*, [*Phys. Rev. D* **102** \(2020\) 024080](#), [[2005.01762](#)].

- [150] K. S. Kumar, S. Maheshwari, A. Mazumdar and J. Peng, *An anisotropic bouncing universe in non-local gravity*, [JCAP **07** \(2021\) 025](#), [[2103.13980](#)].
- [151] S. Talaganis and A. Teimouri, *Hamiltonian Analysis for Infinite Derivative Field Theories and Gravity*, [[1701.01009](#)].
- [152] I. Kolar and A. Mazumdar, *Hamiltonian for scalar field model of infinite derivative gravity*, [Phys. Rev. D **101** \(2020\) 124028](#), [[2003.00590](#)].
- [153] P. Nicolini, A. Smailagic and E. Spallucci, *Noncommutative geometry inspired Schwarzschild black hole*, [Phys. Lett. B **632** \(2006\) 547–551](#), [[gr-qc/0510112](#)].
- [154] P. Nicolini, *Noncommutative Black Holes, The Final Appeal To Quantum Gravity: A Review*, [Int. J. Mod. Phys. A **24** \(2009\) 1229–1308](#), [[0807.1939](#)].
- [155] S. Weinberg, *ULTRAVIOLET DIVERGENCES IN QUANTUM THEORIES OF GRAVITATION*, in *General Relativity: An Einstein Centenary Survey*, pp. 790–831. 1980.
- [156] M. Niedermaier and M. Reuter, *The Asymptotic Safety Scenario in Quantum Gravity*, [Living Rev. Rel. **9** \(2006\) 5–173](#).
- [157] M. Reuter and F. Saueressig, *Quantum Einstein Gravity*, [New J. Phys. **14** \(2012\) 055022](#), [[1202.2274](#)].
- [158] L. Bosma, B. Knorr and F. Saueressig, *Resolving Spacetime Singularities within Asymptotic Safety*, [Phys. Rev. Lett. **123** \(2019\) 101301](#), [[1904.04845](#)].
- [159] K. Sravan Kumar, S. Maheshwari and A. Mazumdar, *Perturbations in higher derivative gravity beyond maximally symmetric spacetimes*, [Phys. Rev. D **100** \(2019\) 064022](#), [[1905.03227](#)].
- [160] A. Conroy, A. Mazumdar and A. Teimouri, *Wald Entropy for Ghost-Free, Infinite Derivative Theories of Gravity*, [Phys. Rev. Lett. **114** \(2015\) 201101](#), [[1503.05568](#)].
- [161] A. S. Koshelev and A. Tokareva, *Non-local self-healing of Higgs inflation*, [Phys. Rev. D **102** \(2020\) 123518](#), [[2006.06641](#)].
- [162] J. Edholm, *Infinite Derivative Gravity: A finite number of predictions*, Ph.D. thesis, Lancaster U., 2019. [[1904.10248](#)]. [10.17635/lancaster/thesis/518](#).
- [163] J. Edholm, *Revealing infinite derivative gravity’s true potential: The weak-field limit around de Sitter backgrounds*, [Phys. Rev. D **97** \(2018\) 064011](#), [[1801.00834](#)].

- [164] J. Edholm and A. Conroy, *Newtonian Potential and Geodesic Completeness in Infinite Derivative Gravity*, *Phys. Rev. D* **96** (2017) 044012, [[1705.02382](#)].
- [165] J. Edholm, *UV completion of the Starobinsky model, tensor-to-scalar ratio, and constraints on nonlocality*, *Phys. Rev. D* **95** (2017) 044004, [[1611.05062](#)].
- [166] A. S. Koshelev, K. Sravan Kumar, L. Modesto and L. Rachwał, *Finite quantum gravity in dS and AdS spacetimes*, *Phys. Rev. D* **98** (2018) 046007, [[1710.07759](#)].
- [167] L. Buoninfante, G. Lambiase, Y. Miyashita, W. Takebe and M. Yamaguchi, *Generalized ghost-free propagators in nonlocal field theories*, *Phys. Rev. D* **101** (2020) 084019, [[2001.07830](#)].
- [168] L. Buoninfante, G. Lambiase and M. Yamaguchi, *Enlarging local symmetries: A nonlocal Galilean model*, *Int. J. Geom. Meth. Mod. Phys.* **17** (2020) 2040009.
- [169] G. Calcagni, L. Modesto and P. Nicolini, *Super-accelerating bouncing cosmology in asymptotically-free non-local gravity*, *Eur. Phys. J. C* **74** (2014) 2999, [[1306.5332](#)].
- [170] F. Briscese, L. Modesto and S. Tsujikawa, *Super-renormalizable or finite completion of the Starobinsky theory*, *Phys. Rev. D* **89** (2014) 024029, [[1308.1413](#)].
- [171] G. Calcagni and L. Modesto, *Nonlocality in string theory*, *J. Phys. A* **47** (2014) 355402, [[1310.4957](#)].
- [172] G. Calcagni and L. Modesto, *Nonlocal quantum gravity and M-theory*, *Phys. Rev. D* **91** (2015) 124059, [[1404.2137](#)].
- [173] L. Modesto and L. Rachwał, *Super-renormalizable and finite gravitational theories*, *Nucl. Phys. B* **889** (2014) 228–248, [[1407.8036](#)].
- [174] L. Modesto and I. L. Shapiro, *Superrenormalizable quantum gravity with complex ghosts*, *Phys. Lett. B* **755** (2016) 279–284, [[1512.07600](#)].
- [175] L. Modesto, *Super-renormalizable or finite Lee-Wick quantum gravity*, *Nucl. Phys. B* **909** (2016) 584–606, [[1602.02421](#)].
- [176] L. Modesto, L. Rachwał and I. L. Shapiro, *Renormalization group in super-renormalizable quantum gravity*, *Eur. Phys. J. C* **78** (2018) 555, [[1704.03988](#)].
- [177] L. Modesto and L. Rachwał, *Nonlocal quantum gravity: A review*, *Int. J. Mod. Phys. D* **26** (2017) 1730020.

- [178] G. Calcagni, L. Modesto and G. Nardelli, *Initial conditions and degrees of freedom of non-local gravity*, [JHEP 05 \(2018\) 087](#), [[1803.00561](#)].
- [179] G. Calcagni, L. Modesto and G. Nardelli, *Non-perturbative spectrum of non-local gravity*, [Phys. Lett. B 795 \(2019\) 391–397](#), [[1803.07848](#)].
- [180] F. Brischese and L. Modesto, *Cutkosky rules and perturbative unitarity in Euclidean nonlocal quantum field theories*, [Phys. Rev. D 99 \(2019\) 104043](#), [[1803.08827](#)].
- [181] M. Asorey, J. L. Lopez and I. L. Shapiro, *Some remarks on high derivative quantum gravity*, [Int. J. Mod. Phys. A 12 \(1997\) 5711–5734](#), [[hep-th/9610006](#)].
- [182] V. P. Frolov and A. Zelnikov, *Radiation from an emitter in the ghost free scalar theory*, [Phys. Rev. D 93 \(2016\) 105048](#), [[1603.00826](#)].
- [183] J. Boos, V. P. Frolov and A. Zelnikov, *Quantum scattering on a delta potential in ghost-free theory*, [Phys. Lett. B 782 \(2018\) 688–693](#), [[1805.01875](#)].
- [184] J. Boos, V. P. Frolov and A. Zelnikov, *Probing the vacuum fluctuations in scalar ghost-free theories*, [Phys. Rev. D 99 \(2019\) 076014](#), [[1901.07096](#)].
- [185] J. Boos, V. P. Frolov and A. Zelnikov, *Ghost-free modification of the Polyakov action and Hawking radiation*, [Phys. Rev. D 100 \(2019\) 104008](#), [[1909.01494](#)].
- [186] J. Boos, J. Pinedo Soto and V. P. Frolov, *Ultrarelativistic spinning objects in nonlocal ghost-free gravity*, [Phys. Rev. D 101 \(2020\) 124065](#), [[2004.07420](#)].
- [187] V. P. Frolov, *Nonlocal scalar field in an external potential: The WKB approximation*, [Phys. Rev. D 103 \(2021\) 025020](#), [[2011.11700](#)].
- [188] I. Kolář and A. Mazumdar, *NUT charge in linearized infinite derivative gravity*, [Phys. Rev. D 101 \(2020\) 124005](#), [[2004.07613](#)].
- [189] I. Kolář, T. Málek and A. Mazumdar, *Exact solutions of non-local gravity in class of almost universal spacetimes*, [[2103.08555](#)].
- [190] R. M. Wald, *Gravitational collapse and cosmic censorship*, pp. 69–85, 10, 1997, [[gr-qc/9710068](#)], [DOI](#).
- [191] N. Barnaby, T. Biswas and J. M. Cline, *p-adic Inflation*, [JHEP 04 \(2007\) 056](#), [[hep-th/0612230](#)].
- [192] T. Koivisto, *Dynamics of Nonlocal Cosmology*, [Phys. Rev. D 77 \(2008\) 123513](#), [[0803.3399](#)].

- [193] A. S. Koshelev, K. Sravan Kumar and P. Vargas Moniz, *Inflation from string field theory*, [1604.01440](#).
- [194] B. Craps, T. De Jonckheere and A. S. Koshelev, *Cosmological perturbations in non-local higher-derivative gravity*, *JCAP* **1411** (2014) 022, [[1407.4982](#)].
- [195] D. Brizuela, J. M. Martin-Garcia and G. A. Mena Marugan, *xPert: Computer algebra for metric perturbation theory*, *Gen. Rel. Grav.* **41** (2009) 2415–2431, [[0807.0824](#)].
- [196] T. Nutma, *xTras : A field-theory inspired xAct package for mathematica*, *Comput. Phys. Commun.* **185** (2014) 1719–1738, [[1308.3493](#)].
- [197] A. de la Cruz-Dombriz, F. J. M. Torralba and A. Mazumdar, *Non-singular and ghost-free infinite derivative gravity with torsion*, [[1812.04037](#)].
- [198] A. Mazumdar and G. Stettinger, *New massless and massive infinite derivative gravity in three dimensions and perturbations around Minkowski and in (A)dS*, [[1811.00885](#)].
- [199] T. Biswas, A. Conroy, A. S. Koshelev and A. Mazumdar, *Generalized ghost-free quadratic curvature gravity*, *Class. Quant. Grav.* **31** (2014) 015022, [[1308.2319](#)].
- [200] L. F. Abbott, *Introduction to the Background Field Method*, *Acta Phys. Polon.* **B13** (1982) 33.
- [201] S. M. Christensen and M. J. Duff, *Quantizing Gravity with a Cosmological Constant*, *Nucl. Phys.* **B170** (1980) 480.
- [202] P. Van Nieuwenhuizen, *On ghost-free tensor lagrangians and linearized gravitation*, *Nucl. Phys.* **B60** (1973) 478–492.
- [203] S. Deser, *Covariant decomposition of symmetric tensors and the gravitational cauchy problem*, *Annales de l'I.H.P. Physique théorique* **7** (1967) 149–188.
- [204] J. W. York, Jr., *Covariant decompositions of symmetric tensors in the theory of gravitation*, *Ann. Inst. H. Poincaré Phys. Theor.* **21** (1974) 319–332.
- [205] G. W. Gibbons and M. J. Perry, *Quantizing Gravitational Instantons*, *Nucl. Phys.* **B146** (1978) 90–108.
- [206] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Lust and A. Riotto, *Aspects of Quadratic Gravity*, *Fortsch. Phys.* **64** (2016) 176–189, [[1505.07657](#)].

- [207] I. G. Avramidi, *The Covariant Technique for Calculation of One Loop Effective Action*, *Nucl. Phys.* **B355** (1991) 712–754.
- [208] E. A. Bergshoeff, O. Hohm, J. Rosseel and P. K. Townsend, *Modes of Log Gravity*, *Phys. Rev.* **D83** (2011) 104038, [[1102.4091](#)].
- [209] B. Whitt, *Fourth Order Gravity as General Relativity Plus Matter*, *Phys. Lett.* **145B** (1984) 176–178.
- [210] C. Molina, P. Pani, V. Cardoso and L. Gualtieri, *Gravitational signature of Schwarzschild black holes in dynamical Chern-Simons gravity*, *Phys. Rev.* **D81** (2010) 124021, [[1004.4007](#)].
- [211] J. L. Blazquez-Salcedo, C. F. B. Macedo, V. Cardoso, V. Ferrari, L. Gualtieri, F. S. Khoo et al., *Perturbed black holes in Einstein-dilaton-Gauss-Bonnet gravity: Stability, ringdown, and gravitational-wave emission*, *Phys. Rev.* **D94** (2016) 104024, [[1609.01286](#)].
- [212] V. Cardoso, T. Igata, A. Ishibashi and K. Ueda, *Massive tensor field perturbations on extremal and near-extremal static black holes*, [[1904.05109](#)].
- [213] LIGO SCIENTIFIC, VIRGO collaboration, B. P. Abbott et al., *Search for Tensor, Vector, and Scalar Polarizations in the Stochastic Gravitational-Wave Background*, *Phys. Rev. Lett.* **120** (2018) 201102, [[1802.10194](#)].
- [214] R. Brandenberger and P. Peter, *Bouncing Cosmologies: Progress and Problems*, *Found. Phys.* **47** (2017) 797–850, [[1603.05834](#)].
- [215] M. Gasperini and G. Veneziano, *The Pre - big bang scenario in string cosmology*, *Phys. Rept.* **373** (2003) 1–212, [[hep-th/0207130](#)].
- [216] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, *The Ekpyrotic universe: Colliding branes and the origin of the hot big bang*, *Phys. Rev. D* **64** (2001) 123522, [[hep-th/0103239](#)].
- [217] A. Ashtekar, *Loop Quantum Cosmology: An Overview*, *Gen. Rel. Grav.* **41** (2009) 707–741, [[0812.0177](#)].
- [218] R. H. Brandenberger and C. Vafa, *Superstrings in the Early Universe*, *Nucl. Phys. B* **316** (1989) 391–410.
- [219] A. Nayeri, R. H. Brandenberger and C. Vafa, *Producing a scale-invariant spectrum of perturbations in a Hagedorn phase of string cosmology*, *Phys. Rev. Lett.* **97** (2006) 021302, [[hep-th/0511140](#)].

- [220] R. H. Brandenberger, *String Gas Cosmology: Progress and Problems*, *Class. Quant. Grav.* **28** (2011) 204005, [[1105.3247](#)].
- [221] R. H. Brandenberger, *String Gas Cosmology*, **8**, 2008, [[0808.0746](#)].
- [222] T. Battefeld and S. Watson, *String gas cosmology*, *Rev. Mod. Phys.* **78** (2006) 435–454, [[hep-th/0510022](#)].
- [223] E. I. Buchbinder, J. Khoury and B. A. Ovrut, *New Ekpyrotic cosmology*, *Phys. Rev. D* **76** (2007) 123503, [[hep-th/0702154](#)].
- [224] C. Lin, R. H. Brandenberger and L. Perreault Levasseur, *A Matter Bounce By Means of Ghost Condensation*, *JCAP* **1104** (2011) 019, [[1007.2654](#)].
- [225] T. J. Battefeld and R. Brandenberger, *Vector perturbations in a contracting universe*, *Phys. Rev. D* **70** (2004) 121302, [[hep-th/0406180](#)].
- [226] M. Gasperini, M. Giovannini and G. Veneziano, *Perturbations in a nonsingular bouncing universe*, *Phys. Lett. B* **569** (2003) 113–122, [[hep-th/0306113](#)].
- [227] C. Cartier, J.-c. Hwang and E. J. Copeland, *Evolution of cosmological perturbations in nonsingular string cosmologies*, *Phys. Rev. D* **64** (2001) 103504, [[astro-ph/0106197](#)].
- [228] M. Bojowald and G. M. Hossain, *Cosmological vector modes and quantum gravity effects*, *Class. Quant. Grav.* **24** (2007) 4801–4816, [[0709.0872](#)].
- [229] B. Xue, D. Garfinkle, F. Pretorius and P. J. Steinhardt, *Nonperturbative analysis of the evolution of cosmological perturbations through a nonsingular bounce*, *Phys. Rev. D* **88** (2013) 083509, [[1308.3044](#)].
- [230] N. Deruelle, M. Sasaki, Y. Sendouda and A. Youssef, *Inflation with a Weyl term, or ghosts at work*, *JCAP* **1103** (2011) 040, [[1012.5202](#)].
- [231] A. A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, *Phys. Lett. B* **91** (1980) 99–102.
- [232] A. A. Starobinsky and H. J. Schmidt, *On a general vacuum solution of fourth-order gravity*, *Class. Quant. Grav.* **4** (1987) 695–702.
- [233] T. Biswas, T. Koivisto and A. Mazumdar, *Could our Universe have begun with Negative Lambda?*, [[1105.2636](#)].
- [234] A. S. Koshelev and S. Yu. Vernov, *On bouncing solutions in non-local gravity*, *Phys. Part. Nucl.* **43** (2012) 666–668, [[1202.1289](#)].

- [235] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, *Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions*, *Phys. Rept.* **215** (1992) 203–333.
- [236] A. S. Koshelev and S. Yu. Vernov, *Cosmological Solutions in Nonlocal Models*, *Phys. Part. Nucl. Lett.* **11** (2014) 960–963, [[1406.5887](#)].
- [237] T. Biswas and A. Mazumdar, *Super-Inflation, Non-Singular Bounce, and Low Multipoles*, *Class. Quant. Grav.* **31** (2014) 025019, [[1304.3648](#)].
- [238] H. Lu and C. Pope, *Critical Gravity in Four Dimensions*, *Phys. Rev. Lett.* **106** (2011) 181302, [[1101.1971](#)].
- [239] D. Battfeld and P. Peter, *A Critical Review of Classical Bouncing Cosmologies*, *Phys. Rept.* **571** (2015) 1–66, [[1406.2790](#)].
- [240] Y.-F. Cai, D. A. Easson and R. Brandenberger, *Towards a Nonsingular Bouncing Cosmology*, *JCAP* **08** (2012) 020, [[1206.2382](#)].
- [241] Y.-F. Cai, *Exploring Bouncing Cosmologies with Cosmological Surveys*, *Sci. China Phys. Mech. Astron.* **57** (2014) 1414–1430, [[1405.1369](#)].
- [242] V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, *Oscillatory approach to a singular point in the relativistic cosmology*, *Adv. Phys.* **19** (1970) 525–573.
- [243] Y.-F. Cai, R. Brandenberger and P. Peter, *Anisotropy in a Nonsingular Bounce*, *Class. Quant. Grav.* **30** (2013) 075019, [[1301.4703](#)].
- [244] C.-Y. Chen, P. Chen and S. Park, *Primordial bouncing cosmology in the Deser-Woodard nonlocal gravity*, *Phys. Lett. B* **796** (2019) 112–116, [[1905.04557](#)].
- [245] J. D. Barrow and K. Yamamoto, *Anisotropic Pressures at Ultra-stiff Singularities and the Stability of Cyclic Universes*, *Phys. Rev. D* **82** (2010) 063516, [[1004.4767](#)].
- [246] H.-J. Schmidt, *Fourth order gravity: Equations, history, and applications to cosmology*, *eConf* **C0602061** (2006) 12, [[gr-qc/0602017](#)].
- [247] T. S. Pereira, C. Pitrou and J.-P. Uzan, *Theory of cosmological perturbations in an anisotropic universe*, *JCAP* **0709** (2007) 006, [[0707.0736](#)].
- [248] B. Xue, *Nonsingular Bouncing Cosmology*, Ph.D. thesis, Princeton U., 2013.

- [249] A. Ijjas, W. G. Cook, F. Pretorius, P. J. Steinhardt and E. Y. Davies, *Robustness of slow contraction to cosmic initial conditions*, [JCAP **08** \(2020\) 030](#), [[2006.04999](#)].
- [250] W. G. Cook, I. A. Glushchenko, A. Ijjas, F. Pretorius and P. J. Steinhardt, *Supersmoothing through Slow Contraction*, [Phys. Lett. B **808** \(2020\) 135690](#), [[2006.01172](#)].
- [251] G. F. R. Ellis and H. van Elst, *Cosmological models: Cargese lectures 1998*, [NATO Sci. Ser. C **541** \(1999\) 1–116](#), [[gr-qc/9812046](#)].
- [252] A. Conroy, T. Koivisto, A. Mazumdar and A. Teimouri, *Generalized quadratic curvature, non-local infrared modifications of gravity and Newtonian potentials*, [Class. Quant. Grav. **32** \(2015\) 015024](#), [[1406.4998](#)].