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Positive-operator-valued measure optimization of classical correlations

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We study the problem of optimization over positive-operator-valued measures to extract classical correlation in a bipartite quantum system. The proposed method is applied to binary states only. Moreover, to illustrate this method, an explicit example is studied in detail.

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I. INTRODUCTION

Quantum computing constitutes a rich research area of physics and computing at the same time. It is believed that the expected power of a quantum computer is derived from genuine quantum resources. Entanglement, and *correlations* in general, are typical quantum resources. However, not all correlations have pure quantum nature. Generically, total correlations are “mixture” of classical and quantum correlations. An important issue is to know to what extent classical correlations are used in teleportation protocols and quantum algorithms. For example, if one is able to determine the classical part of correlations then by the optimal measurement one can extract some information in classical form leaving the quantum state with less entropy. For this procedure to be useful it should be done while retaining the ability to regenerate the source state exactly from the classical measurement result and the postmeasurement state of the quantum system. This has been studied in [1]. A possible application of this is to send the postmeasurement state through a noiseless channel while sending the classical information through a more robust channel.

Quantifying correlations implies measurement in most cases, which is a nontrivial task for quantum states. A general measurement strategy is described by a positive-operator-valued measure (POVM) that decomposes the unity in the Hilbert space of the particle under measurement. For the different measures of correlations (or information) one has to find the optimal POVM, which can be projective (orthogonal or nonorthogonal) or nonprojective, and then the number of elements in the optimal POVM. POVM optimization is studied in different contexts in quantum information theory such as accessible information, the cost function, fidelity, etc. Davies’ theorem [2] gives an upper bound for the maximum number of elements of an optimal POVM needed to attain the *accessible information* (or the mutual information) of an ensemble of states in a d -dimensional Hilbert space; this bound is d^2 . The upper bound has been reduced to $d(d+1)/2$ for states that are real [3]. There were attempts to reduce this bound to d elements, but Shor [4] has found an ensemble of states in a three-dimensional Hilbert space that was optimized with a six-element POVM. This eliminates the possibility of further reduction of the upper bound. Simi-

lar POVM optimization was made for the *fidelity* in [5], where for an ensemble of quantum states in two-dimensional Hilbert the optimal POVM was found to have three elements, which fits the $d(d+1)/2$ bound. In the present paper we present a POVM optimization for the classical correlations. The study leads to two important results: for arbitrary binary states projective POVM’s are found to be sufficient to optimize classical correlation, which resembles the cases studied in [2,3]; on the other hand the optimal POVM is found to have at most 4 ($=2^2$) elements with indications that the number can be reduced to 3 [$=2(2+1)/2$] for real states. Moreover, for specific states the optimal POVM is shown to be orthogonal. Another important property that is shown in the paper is the possibility of decomposing the mutual information contained in a given bipartite system into classical correlation and quantum discord.

The paper is organized as follows. The definitions of POVM and classical correlation and their physical and mathematical properties are given in the first two sections of the paper. Then we show the natural decomposition of the mutual information into classical correlation and quantum discord. The remaining part of the paper is devoted to the complete optimization procedure, including numerical simulation, of the classical correlation for a particular state illustrating the need for the quantum discord in order to match the mutual information.

II. POVM

Let $\mathcal{B}=\{B_i\}$ be a POVM; then B_i should be a positive-valued operator, to preserve the positivity of the measurement outcome probability. Each POVM “set” should decompose the unity, to get probability completeness, i.e.,

$$\sum B_i = 1. \quad (1)$$

For later use it should be mentioned that the set of all POVM’s is convex, i.e., the *segment* joining two POVM’s is a POVM. In simpler terms, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two POVM’s and $0 < \lambda < 1$ (a probability), then

$$\lambda \mathcal{B}_1 + (1 - \lambda) \mathcal{B}_2$$

is indeed a POVM.

III. CLASSICAL CORRELATION

Unfortunately, there is no unique measure of classical correlations. Different measures are found in the literature [6–11]. Two of the most relevant requirements that should be satisfied by any classical correlation are (I) the classical correlations of product states ($\rho_{AB} = \rho_A \otimes \rho_B$) should be zero; (II) Classical correlation should not be affected by local unitary transformations, which correspond simply to a change of basis. Other important conditions are listed in [8,9]. Comparison between different measures is found in the literature. For example, property II has been investigated recently by Synak and Horodecki [12] for the measure proposed in [9], called the classical information deficit (Δ_{cl}^-). It is found that Δ_{cl}^- does increase under local operations. Moreover, it was shown [12] that Δ_{cl}^- is bounded above by the classical correlation measure proposed in [8]. The only case where Δ_{cl}^- is monotone under local operation is when it coincides with the classical correlation of [8]. This comparison can be considered to be in favor of the Henderson and Vedral measure. However, there is no evidence that the other measures of classical correlations violate some of the expected physical properties of a correlation. So one can choose a classical correlation measure out of the proposed measures. Our choice is the Henderson and Vedral measure.

Given the bipartite state ρ_{AB} a possible measure of the classical correlation between subsystems A and B is [8]

$$C_B(\rho_{AB}) = \max_B \left[S(\rho_A) - \sum_i p_i S(\rho_A^i) \right], \quad (2)$$

where $\rho_A = \text{tr}_B(\rho_{AB})$ is the reduced density matrix. The von Neumann entropy is $S(\rho) = -\text{tr}(\rho \log \rho)$. $\mathcal{B}$ is a POVM and the sum over i runs over all its elements B_i . The conditional density matrix ρ_A^i is the density matrix of A after performing the measurement B_i on B :

$$\rho_A^i = \frac{\text{tr}_B(B_i \rho_{AB})}{\text{tr}_{AB}(B_i \rho_{AB})}. \quad (3)$$

The probability of A being in the state ρ_A^i is $p_i = \text{tr}_{AB}(B_i \rho_{AB})$.

The correlation measure C_B has a simple physical interpretation: if A and B are not correlated then the marginal entropy of A [$S(\rho_A)$] and the residual entropy of A after a POVM measurement on B [$\sum p_i S(\rho_A^i)$] should coincide to give $C_B = 0$, since for an uncorrelated system AB , A is not affected by a POVM measurement on B . Moreover, note that

$$\rho_A = \sum_i p_i \rho_A^i;$$

hence for a given POVM $\mathcal{B}$ the combination

$$S(\rho_A) - \sum_i p_i S(\rho_A^i)$$

is closely related to the *entropy defect* defined by Levitin (see [13] and references therein). So the classical correlation C_B can be seen as the maximum average decrease in the entropy of the system A when a state ρ_A^i (after a measurement B_i is performed on B) is specified compared with the situation when only the mixture of states ρ_A is known.

The correlation $C_A(\rho_{AB})$ is obtained from C_B by making the replacement $A \leftrightarrow B$ in the above formulas. It is evident that the measure $C_{A,B}$ are not manifestly symmetric under the exchange of the roles of A and B , whereas, one would expect that the classical correlation is a measure of how strongly the two subsystems are correlated no matter which subsystem is used to extract such a correlation. Hence, until a formal proof of an explicit symmetry of this measure, if it exists, the above proposed measure cannot be considered as a universal measure of the existing classical correlations.

IV. POVM CHOICE AND CONVEXITY

In what follows we consider binary states exclusively. Extension to higher dimensions is not straightforward.

A. Convexity

An important property that helps in reducing the range of exploration in the set of all POVM's is the concavity [14] of the classical correlation measure. If one is able to show that the classical correlation is convex then (see [2]) for binary states one can consider POVM's of rank 1 elements that can be taken to be proportional to the one-dimensional projectors.

Proof. For binary states the von Neumann entropy is concave, i.e.,

$$S(\lambda \rho_1 + (1 - \lambda) \rho_2) \geq \lambda S(\rho_1) + (1 - \lambda) S(\rho_2) \quad (4)$$

for all $0 < \lambda < 1$.

To study the concavity of the classical measure it suffices to study the variation of the functional $F(\rho_A, \mathcal{B})$ defined as

$$F(\rho_A, \mathcal{B}) = -p_B S(\rho_A^B), \quad (5)$$

where the compact notation $p_B = \text{tr}_{AB}(\mathcal{B} \rho_{AB})$ and $\rho_A^B = \text{tr}_B(\mathcal{B} \rho_{AB})/p_B$ is used for all elements B_i of the POVM $\mathcal{B}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be two POVM's and let $\mathcal{G}$ be their combination

$$\mathcal{G} = \lambda \mathcal{C} + (1 - \lambda) \mathcal{D}, \quad (6)$$

for $0 < \lambda < 1$. $\mathcal{G}$ is a POVM since, as mentioned before, the set of all POVM's is convex.

It is straightforward to show that

$$p_A^{\mathcal{G}} = \lambda p_A^{\mathcal{C}} + (1 - \lambda) p_A^{\mathcal{D}}, \quad (7)$$

$$\rho_A^{\mathcal{G}} = \lambda \frac{p_A^{\mathcal{C}}}{p_A^{\mathcal{G}}} \rho_A^{\mathcal{C}} + (1 - \lambda) \frac{p_A^{\mathcal{D}}}{p_A^{\mathcal{G}}} \rho_A^{\mathcal{D}}. \quad (8)$$

Using these equations and the fact that the entropy is a concave function, it can be shown that

$$F(\rho_A, \mathcal{G}) \leq \lambda F(\rho_A, \mathcal{C}) + (1 - \lambda) F(\rho_A, \mathcal{D}). \quad (9)$$

Therefore, F is convex over the convex set of all POVM's. Hence the maximum of F occurs for an *extremal* POVM [15]. For binary states it was explicitly shown recently [16]

that extreme POVM's are of rank 1.¹ Hence, to optimize F and consequently the classical correlation, since $S(\rho_A)$ is independent of the POVM performed, one can consider the special class of POVM with rank-1 elements. Hence we are left with projective measurements only.²

An important consequence of this result is presented in the next section.

B. Quantum discord and mutual information

As mentioned in [12] the quantum mutual information $I(A:B)$ of a bipartite system AB can be decomposed into an information deficit (or work deficit) Δ and the classical information deficit Δ_{cl} :³

$$I = \Delta_{cl} + \Delta. \quad (10)$$

So naturally one should be concerned with an analogous decomposition of the mutual information when using a different measure of classical correlation.

By analyzing different quantum states, the authors of [8] found that the estimated classical correlation and the relative entropy of entanglement (E_{RE}) do not add up to give the von Neumann mutual information between the two subsystems, i.e.,

$$[C_B(\rho_{AB})]_{\text{optimized}} + E_{RE} < I(\rho_{A:B}). \quad (11)$$

It was argued that either the mutual information is not the best quantity to measure the total correlations, or a more elaborate choice of the POVM may saturate the total correlations. By considering all possible POVM's for a given state, we show that the optimal POVM cannot saturate the total correlations. This implies that either the mutual information is not a good measure of total correlations, which is highly improbable, or one has to change the measure C_B of classical correlation. The third alternative, which renders the definition of classical correlations compatible with the mutual information, is the possibility of having a different definition of quantum correlations which is different from E_{RE} . A possible candidate is the quantum discord, which was first defined in [17]. It is the result of the difference between classical and quantum conditional entropies. In contrast to classical conditional entropy, quantum conditional entropy is a measurement dependent quantity (see further [18]). The quantum discord is defined as [17]

¹For higher dimensional Hilbert spaces there are higher rank extreme POVM's [16].

²It should be mentioned that for generic cases one can use the Naimark theorem to get the projective POVM only without being restrictive, but this is done at the cost of extending the original Hilbert space.

³The previously mentioned classical information deficit Δ_{cl} is a one-way quantity, i.e., for a system with two parties A and B we allow communication from A to B only, or vice versa. This asymmetric definition is analogous to the seemingly asymmetric property of the classical correlations where one has to perform measurements on one of the two parties.

$$\delta(A:B) = I(A:B) - J(A:B)_{\{\Pi_i^B\}}. \quad (12)$$

J is the information gained about B as a result of the set of measurements $\{\Pi_i^B\}$:

$$J(A:B)_{\{\Pi_i^B\}} = S(A) - S(A|\{\Pi_i^B\}) = S(A) - \sum_i^* p_i S(\rho_i^A), \quad (13)$$

where *p_i and $^*\rho_i^A$ were defined originally as the special case of p_i and ρ_i^A when the choice of the set of measurement is restricted to one-dimensional projectors Π_i^B . The above definition of the quantum discord reflects clearly the inherited measurement dependence of the quantum conditional entropy. For the quantum discord to be zero one has to find at least one measurement for which it is zero. Therefore the minimum of the quantum discord is the relevant quantum correlation. By focusing on perfect measurements of B , defined by a set of one-dimensional projectors, one can easily check that the set of measurements that minimizes the quantum discord, i.e., maximizes J , is exactly the same POVM set that optimizes the classical correlations for binary states. This follows from the definition of both quantities, and the result of the previous section on the optimization of classical correlation using projective measurements only. Hence

$$\max[J(A:B)_{\{\Pi_i^B\}}] = C_B(\rho_{AB}). \quad (14)$$

Therefore

$$I(A:B) = C_B + \min_{\{\Pi_i^B\}} \delta(A:B), \quad (15)$$

i.e., for binary states the classical correlation and the quantum discord add up to give the mutual information.

C. What is next?

Limiting the domain of exploration in the set of all POVM's to projective POVM leads to important simplification. But the game is not over. What would be the optimal projective POVM? How many elements there are in the optimal POVM? Corollary 1 of [16] implies that POVM's with five or more elements acting on a two-dimensional Hilbert space are not extreme, and hence cannot optimize the classical correlation.⁴ So we are left with POVM's having two, three, or four elements. In the next section we illustrate, through an example, the steps that can be followed to complete the optimization procedure.

V. AN EXAMPLE

Consider a system AB in a state ρ_{AB} with probability p to be in a state ρ_1 and $(1-p)$ to be in another state ρ_2 . A possible example is the following state studied in [8] :

$$p|0\rangle\langle 0| \otimes |0\rangle\langle 0| + (1-p)|+\rangle\langle +| \otimes |+\rangle\langle +|, \quad (16)$$

where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. To enlarge the "library" of computed classical correlations we will not use the above state but rather a slightly modified version of it, namely,

$$\rho_{AB} = p|1\rangle\langle 1| \otimes |1\rangle\langle 1| + (1-p)|+\rangle\langle +| \otimes |+\rangle\langle +|. \quad (17)$$

The classical correlation of the state in Eq. (16) was calcu-

lated in [8] using optimization over all *orthogonal* measurements, which was not a justified restriction.

For the state given in Eq. (17), it is useful to use the Pauli matrices decomposition. Hence the measurement operator B_i can be written as

$$B_i = |b_i| \left(\frac{1 + \vec{\sigma} \cdot \hat{b}_i}{2} \right). \quad (18)$$

The projection direction is set by the unit vector $\hat{b}_i$. We should further require the completeness relation [Eq. (1)]

$$\sum \vec{b}_i = \vec{0}, \quad \sum |b_i| = 2. \quad (19)$$

The state ρ_{AB} in Eq. (17) becomes

$$\rho_{AB} = \frac{p}{2} |1\rangle\langle 1| \otimes (1 - \sigma_z) + \frac{1-p}{2} |+\rangle\langle +| \otimes (1 + \sigma_x). \quad (20)$$

It is straightforward to show that the density ρ_A^i of subsystem A, after the measurement B_i on B, is

$$\rho_A^i = \alpha_i |1\rangle\langle 1| + (1 - \alpha_i) |+\rangle\langle +| \quad (21)$$

where

$$\alpha_i = \frac{p(1 - \hat{b}_i \cdot \hat{e}_z)}{p(1 - \hat{b}_i \cdot \hat{e}_z) + (1-p)(1 + \hat{b}_i \cdot \hat{e}_x)}. \quad (22)$$

Hence the entropy of the subsystem A after the measurement B_i of the subsystem B is

$$S(\rho_A^i) = h(\omega_i^{(1)}), \quad (23)$$

where as usual $h(x) = -x \log(x) - (1-x) \log(1-x)$ is the Shannon entropy function and

$$\omega_i^{(1),(2)} = \frac{1}{2} [1 \pm \sqrt{\alpha_i^2 + (1 - \alpha_i)^2}] \quad (24)$$

are the eigenvalues of ρ_A^i .

The probability p_i to have the subsystem A in the state ρ_A^i is

$$p_i = \text{tr}_{AB}(B_i \rho_{AB}) = \frac{|b_i|}{2} [p(1 - \hat{b}_i \cdot \hat{e}_z) + (1-p)(1 + \hat{b}_i \cdot \hat{e}_x)]. \quad (25)$$

Note that the same procedure can be applied for the state given in Eq. (16). One has to replace σ_z by $-\sigma_z$ in Eq. (20). This change of sign will propagate to all the subsequent equations.

A. Choice of basis

The previous section seems to underline a choice of the basis for projection. One can argue that the component $(\hat{b}_i)_y$ does not show up in the conditional density matrix ρ_A^i , hence we can set it to zero in the projection operator. In other words measurement can be performed in the xz plane. In general, this is not correct; it was shown in [16] (see Sec. III)

that an extreme POVM with four outcomes ($n=4$) cannot have its four vectors $\hat{b}_i$ coplanar. On the other hand, it is easy to show that the entropy

$$\sum p_i S(\rho_A^i)$$

is maximum, i.e., it is equal to the entropy of the reduced density matrix ρ_A ,

$$\sum p_i S(\rho_A^i) = S(\rho_A),$$

for a projective measurement in the y direction, since the state of B is completely random in that direction. However, for classical correlations the minimum of $\sum p_i S(\rho_A^i)$ is needed not its maximum. Therefore, it is justified to use measurements such that $(\hat{b}_i)_y = 0$. Hence $(\hat{b}_i)_z = \cos \theta_i$ and $(\hat{b}_i)_x = \sin \theta_i$. But with this assumption the $n=4$ extreme POVM is excluded. This should be investigated further to see whether it is a general property of real states, to which the present state belongs, i.e., whether optimal POVM's for classical correlation of real binary states have at most three outcomes in analogy with the $d(d+1)/2$ bound found in [3]. In fact, the proof of [3] (see Lemma 5 and Theorem 1 there) relies on the convexity of the accessible information and some of its scaling properties under group transformation. This seems to be equally valid for the classical correlation. This will be studied in a future work. But until such a rigorous proof is given the present "exclusion" of the $n=4$ case should be considered as a reasonable assumption.

B. Optimization

1. Method 1: Lagrange multipliers

The determination of the classical correlation C_B requires finding a POVM that minimizes $\sum p_i S(\rho_A^i)$, given the constraint equations (19). Hence *a priori* we have to explore all possible POVM's and then pick up the one that minimizes the considered sum. The result of Sec. IV A has limited the domain of exploration to projective POVM's only. The optimization problem can be done using the Lagrange multipliers method. It is more convenient to recast the problem as

$$\min \quad \sum p_i S(\rho_A^i) = \sum r_i f(\theta_i),$$

$$\text{subject to} \quad \sum r_i - 2 = 0,$$

$$\sum r_i \cos \theta_i = 0,$$

$$\sum r_i \sin \theta_i = 0, \quad (26)$$

where $r_i = |b_i|$ and

$$f(\theta_i) = \frac{1}{2} [p(1 - \cos \theta_i) + (1-p)(1 + \sin \theta_i)] h(\omega_i^{(1)}(\theta_i)).$$

Let $\lambda_{1,2,3}$ be the Lagrange multipliers associated with the above constraints. Hence the minimization problem is equivalent to finding the values of θ_i , r_i , $\lambda_{1,2,3}$ satisfying

$$f(\theta_i) + \lambda_1 + \lambda_2 \cos \theta_i + \lambda_3 \sin \theta_i = 0, \quad (27)$$

$$r_i \left(\frac{df(\theta_i)}{d\theta_i} - \lambda_2 \sin \theta_i + \lambda_3 \cos \theta_i \right) = 0, \quad (28)$$

$$\sum r_i = 2, \quad (29)$$

$$\sum r_i \cos \theta_i = 0, \quad (30)$$

$$\sum r_i \sin \theta_i = 0. \quad (31)$$

This optimization should be done for $n=2,3$.

2. Orthogonal projective measurements

For a two-element POVM ($n=2$) we recover the orthogonal projective measurement,⁵ since $\vec{b}_1 + \vec{b}_2 = 0$. Hence $\theta_2 = \theta_1 + \pi$. The *best* measurement, i.e., the optimal measurement among the $n=2$ POVM's only, is obtained when $f(\theta_1) = f(\theta_2) = f(\theta_1 + \pi)$. Therefore, the best set of measurement is a two-shot measurements with opposite directions. This should be solved numerically for each p . The result is plotted in Fig. 1. Our result matches well with the result of [8]. This implies that although our state is slightly different from the state they have studied the two states have the same classical correlations. However, as mentioned before, this is just the orthogonal measurement; nonorthogonal measurements should be considered, which was not done in [8].

3. Method 2: Monte Carlo simulation $n=3$

Finding the minimum by the method presented above (method 1) for $n=3$ seems to be a nontrivial task; however, it is possible to use a different method like the Monte Carlo simulation (MCS) method. We generate random events that correspond to the set of the three angles $\{\theta_i\}_{i=1}^3$, while from Eq. (26) the three r_i 's are no longer independent variables. Thus our strategy is to compute the classical correlations for each randomly generated event. For each event a numerical value is obtained for the classical correlation; for a given p , this is represented by the large number of points (forming long continuous strips) in Fig. 1. In Fig. 1 we plot the simulation outputs for 10^6 events. Clearly the classical correlation predicted for the best orthogonal projective measurement case constitutes the upper limit of all the results obtained for the random events. Hence the optimal POVM, among all possible POVM's, is the two-outcome orthogonal projective POVM. This result can be linked to the remark found in the

⁵If one takes $b_1=b_2=1$, it is straightforward to check that the optimal POVM measurement is indeed projective, i.e., $B_1 = |\psi_1\rangle\langle\psi_1|$ and $B_2 = |\psi_2\rangle\langle\psi_2|$, with the bases of measurement parametrized by θ as

$$\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)|1\rangle, \sin\left(\frac{\theta}{2}\right)|0\rangle - \cos\left(\frac{\theta}{2}\right)|1\rangle.$$

FIG. 1. We plot as a function of p the classical correlation C_B (lower curve) evaluated using the best orthogonal POVM, and the mutual information $I(A:B)$ (higher curve). The vertical strips are sets of large numbers of dots representing the obtained value of the classical correlation for the different randomly generated events.

conclusion of [13] that one should dare conjecture the following: orthogonal measurement is optimal whenever the number of states (the two states $|1\rangle\langle 1| \otimes |1\rangle\langle 1|$ and $|+\rangle\langle +| \otimes |+\rangle\langle +|$ in the present case) does not exceed the dimensionality of the state space.

4. Mutual information

To understand the role of the classical correlation and its compatibility with the mutual information, the mutual information is evaluated using the formula

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \quad (32)$$

For the present state of the system AB the mutual information can be easily calculated to obtain

$$I(A:B) = 2h\left(\frac{1 + \sqrt{p^2 + (1-p)^2}}{2}\right) - h\left(\frac{1 + \sqrt{1 + 3p^2 - 3p}}{2}\right). \quad (33)$$

$I(A:B)$ is plotted in Fig. 1 in addition to the classical correlation obtained by the optimal orthogonal projective measurement and the one generated by MCS. Since ρ_{AB} is separable, then its relative entropy of entanglement is zero. Therefore, from our general POVM analysis, and as shown in Fig. 1, no POVM can lead to classical correlation that saturates, when added to the relative entropy of entanglement, the mutual information between the two subsystems. This confirms the necessity to use the quantum discord as a quantum counterpart for the classical correlation used in this paper rather than the relative entropy of entanglement.

VI. CONCLUSIONS

By considering a generic POVM we show that classical correlations of binary states are optimized via projective POVM's. It is found that the classical correlation and the quantum discord add up to give the mutual quantum information.

This work should be considered as a first step toward a generalization of the POVM optimization for states in higher dimensions and to see whether rank-1 POVM elements are still the optimal choice for classical correlation. Moreover,

there is at present strong evidence for the possibility of an experimental realization of nonprojective POVM's using optical devices as was recently proposed in [19]. This gives a new dimension to POVM optimization.

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