Synchronization in directed complex networks using graph comparison tools

Liu, Hui; Cao, Ming; Wu, Chai Wah; Lu, Jun-an; k tse, chi

Published in:
IEEE Transactions on Circuits and Systems I - Regular papers

DOI:
10.1109/TCSI.2015.2395632

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2015

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license. More information can be found on the University of Groningen website: https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment.

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Download date: 14-09-2023
Synchronization in Directed Complex Networks Using Graph Comparison Tools

Hui Liu, Member, IEEE, Ming Cao, Member, IEEE, Chai Wah Wu, Fellow, IEEE, Jun-An Lu, and Chi K. Tse, Fellow, IEEE

Abstract—This paper proposes lower bounds for the coupling strengths of oscillators in directed networks to guarantee global synchronization. The novel idea of graph comparison from spectral graph theory is employed so that the combinatorial features of a given network can be fully utilized to simplify computations. For large networks that can be decomposed into a set of smaller strongly connected components, the comparison can be carried out at the local level as well. To validate theoretical analysis, examples are provided to demonstrate how to apply the proposed methodologies to typical directed complex networks.

Index Terms—Coupling strength allocation, directed complex networks, spectral graph theory, synchronization.

I. INTRODUCTION

THE STUDY OF the conditions for synchronization in various complex network has been a central research topic in recent decades [1]–[7]. It has numerous applications in man-made and biological systems, such as coordination in robot networks, synchrony in power-grid networks, flocking in birds and social insects, synchronous spiking in neural networks, and so forth. Several influential systematic approaches have been proposed for this research topic. A general approach, called the master stability function method, was developed in [2] to study the local synchronization problem for linearly coupled chaotic systems. A systematic framework for the study of synchronization of nonlinear dynamical systems with diffusive couplings was developed in [4]. It has been shown that synchronization takes place in a directed network under sufficiently strong couplings provided that the interaction directed graph contains a directed spanning tree [4]. However, it is still not easy to determine the lower bound of coupling strength for synchronization in a directed complex network. Another critical observation is that graph combinatorial features that are associated with the network topologies are essential for identifying synchronization conditions [8]. A new general method, called connection graph method, was proposed in [9] to study the global synchronization in undirected graphs. And similar results were extended for directed but balanced graphs [10] and for generalized directed graphs [11]. In particular, the lengths of all the paths passing through chosen edges in a network have been used to allocate coupling strengths to achieve global synchronization in the network [9], [11].

We have designed in [12], [13] new coupling-strength allocation methods for undirected networks using recently reported tools in spectral graph theory [14]. The idea of graph comparison [14], [15] turns out to be especially useful. In this paper, we further develop our methodologies by looking at general directed networks. However, since the theory developed in [14], [15] mainly focuses on undirected graphs, it is challenging to extend the results in [13] to that of directed graphs. To deal with this problem, we symmetrize the network coupling graphs and find another equivalent synchronization criterion based on the symmetrized graphs. We prove that the synchronization conditions given in [11] for allocating coupling strengths can be explained by comparing a symmetrized graph with the corresponding modified complete graph whose edges are weighted according to pairwise node imbalance in the original directed graph.

The rest of the paper is organized as follows. In Section II, we formulate the system model. Then several graphical synchronization criteria are proposed using graph comparison techniques and the methodologies of coupling strength allocation in networks are obtained accordingly, which are presented in Section III. We deal with decomposable networks in Section IV. Finally, concluding remarks are given in Section V.

II. PROBLEM SETUP AND PRELIMINARIES

A. Model

We consider a network of \( n > 1 \) coupled identical oscillators whose dynamics are described by

\[
\dot{x}_i = f(x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t) P x_j, \quad i = 1, \ldots, n, \tag{1}
\]
where $x_i \in \mathbb{R}^d$ is the state of the $i$th oscillator, $f(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denotes the identical self-dynamics of each oscillator, the diagonal $(0, 1)$-matrix $P \in \mathbb{R}^{d \times d}$ determines through which components of the states that the oscillators are coupled together, and $\varepsilon_{ij}(t) \geq 0 (i \neq j)$ describes the time-varying strength of the coupling from oscillator $j$ to $i$ at time $t$. The couplings between the oscillators can be conveniently described by a weighted graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}, \varepsilon(t))$ with the vertex set $\mathcal{V} = \{1, \ldots, n\}$, the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weight function $\varepsilon : \mathcal{E} \rightarrow \mathbb{R}$. There is an edge from vertex $j$ to $i$ if and only if $\varepsilon_{ij}(t) > 0$ and the weights $\varepsilon_{ij}$ may change with time. Since the couplings between the oscillators are in general unidirectional, the network is directed and thus $\mathcal{G}(t)$ is directed as well. Let $L_{\mathcal{G}(t)} \in \mathbb{R}^{n \times n}$ be the Laplacian matrix [16] of the graph $\mathcal{G}(t)$. Then the $ij$th entry of $L_{\mathcal{G}(t)}$ is $-\varepsilon_{ij}(t)$ for $1 \leq i, j \leq n$.

Definition 1: [4] The synchronization manifold $\mathcal{M}$ of system (1) is defined as the linear subspace $\{x : x_i = x_j, \forall i, j\}$ where $x = (x_1^T, x_2^T, \ldots, x_n^T)^T$.

It is easy to see that each point in the manifold $\mathcal{M}$ is indeed an equilibrium point of (1).

System (1) has been used widely to study under what conditions the coupled oscillators can approach global and complete synchronization, where for any initial condition, the coupled oscillators are globally synchronized when their coupling strengths increase but remain upper bounded. Such networks include a majority of the coupled limit-cycle or chaotic oscillators [9]–[11], [17]. Networks from the second class incline to lose their locally stable synchronization behaviors when the coupling strengths increase too much. Pecora et al. [2] proposed the master stability function method to analyze this phenomenon and provided such a standard example of $x$-coupled Rössler oscillators. A recent paper [18] also studies local stability of various bounded or even empty synchronization regions in complex dynamical networks. However, the limitation of these results lies in that only local stability is revealed.

Our method in this paper and the previous work [13] is only applicable to the first class of complex dynamical networks admitting global and complete synchronization. When studying such networks, it is common to make the following standard technical assumption about the self-dynamics $f(\cdot)$.

Assumption 1: For any two vectors $x_i, x_j \in \mathbb{R}^d$, it is assumed that

$$
(x_j - x_i)^T \left[ (f(x_j) - f(x_i)) - \varepsilon P(x_j - x_i) \right] \leq -c \|x_j - x_i\|^2
$$

holds for some $c > 0$, when $\varepsilon$ exceeds a positive threshold $a$. Here the constant $a$ is determined by both individual dynamics $f$ and the projection matrix $P$. This assumption is equivalent to the assumption that has been made in [9]–[11], see the appendix in [13]. Assumption 1 implies that two unidirectionally coupled oscillators with the projection matrix $P$ are always able to get synchronized when their coupling strength $\varepsilon$ exceeds the threshold $a$ [11].

C. Wu-Chua Result and Its Variation

In the following, we introduce a general synchronization criterion for networks with time-varying dynamics. Some notations are explained first. For a matrix $A \in \mathbb{R}^{n \times n}$, we say $A > 0$ (resp. $A > 0$) if $x^T Ax$ is positive (resp. non-negative) for all nonzero $x \in \mathbb{R}^n$. We use $\mathcal{W}_c$ to denote the set of irreducible, symmetric matrices that have zero row sums and non-positive off-diagonal elements.

Lemma 1: (Minor rephrasing of [19, Theorem 2] and a result in [4, Ch. 4]) Let $Y(t)$ be a $d$-by-$d$ time-varying matrix and $A$ a $d$-by-$d$ symmetric, positive definite matrix such that $(y - z)^T V (f(y, t) + Y(t) y - f(z, t) - Y(t) z) \leq -c \|y - z\|^2$ for some $c > 0$ and all $y, z, t$. Then system (1) synchronizes globally if there exists a $n$-by-$n$ matrix $U \in \mathcal{W}_c$ such that

$$
(U \otimes V) \left( L_{\mathcal{G}(t)} \otimes (-P) - I_n \otimes Y(t) \right) \leq 0
$$

for all $t$, where $\otimes$ denotes the Kronecker product [20]. Now we present a synchronization criterion in terms of graphical properties by applying Lemma 1.

Theorem 1: Under Assumption 1, (1) synchronizes globally if there exists a connected and undirected graph $\mathcal{G}_0$ with the same vertex set of graph $\mathcal{G}(t)$ such that

$$
L_{\mathcal{G}_0}, L_{\mathcal{G}(t)} - a L_{\mathcal{G}_0} \geq 0, \quad \text{for all } t.
$$

Proof: Assumption 1 on the self-dynamics $f(\cdot)$ is implied by the condition that $(y - z)^T V (f(y, t) + Y(t) y - f(z, t) - Y(t) z) \leq -c \|y - z\|^2$ when we set $Y(t) = -\varepsilon P, V = I_d$. Observing that $-P \preceq 0$, we have that the condition (2) is satisfied under the settings $Y(t) = -\varepsilon P, V = I_d$ and $U = L_{\mathcal{G}_0}$ if $L_{\mathcal{G}_0}, L_{\mathcal{G}(t)} - a L_{\mathcal{G}_0} \geq 0$. This completes the proof.

Theorem 1 has also been introduced in our previous work [13] where the case of undirected topologies is considered. However, here we emphasize that the graph $\mathcal{G}(t)$ can be directed. Actually, an undirected graph can be considered as a special directed graph by replacing each undirected edge by two directed edges with opposite orientations. Hence, directed graphs are more general than undirected ones. In what follows, we look into how to apply spectral graph theory to the synchronization problem that we have just set up.

III. GRAPHICAL SYNCHRONIZATION CRITERIA USING GRAPH COMPARISON

A. Graphical Comparison Inequalities

In this paper, we intend to introduce tools from spectral graph theory to study the synchronization problem. To do so, we need to define some notations. Let $A$ and $B$ be $n$-dimensional real matrices. We say $A \geq B$ if $A - B \succeq 0$, i.e., $A - B$ is positive definite, in other words all eigenvalues of $(A - B)^T (A - B)$ are positive. Similarly, we say $A \succeq B$ if $A - B$ is semi-definite. We further apply this notation to undirected graphs.

Definition 2: For two undirected graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ with the same vertex set $\mathcal{V} = \{1, \ldots, n\}$, we say

$$
\mathcal{H}_1 \succeq \mathcal{H}_2
$$

if their Laplacian matrices satisfy $L_{\mathcal{H}_1} \succeq L_{\mathcal{H}_2}$.
We use $K_n$ to denote the unweighted, undirected complete graph with $n$ vertices. We investigate the graph theoretic condition for synchronization in directed complex dynamical networks. We take graph $G_i$ in Theorem 1 to be the complete graph $K_n$. Therefore, the complete synchronization of (1) is guaranteed if

$$L_{K_n} \geq \alpha L_{K_n}. \quad (4)$$

Note that $L_{K_n} = nE - J$ where $J$ is the $n$-by-$n$ all-one matrix. From (4), one has

$$nL_{G(t)} - J > \alpha L_{K_n}. \quad (5)$$

In the case where $G(t)$ is undirected, $(5)$ can be further reduced to $G(t) \geq (a/n)K_n$. However, in this paper we are looking at the more challenging scenario where the networks are directed. We will use the following property of directed graphs.

**Lemma 2:** For a directed graph $G(t)$, the condition $nL_{G(t)} - J > \alpha L_{K_n}$ is equivalent to $(1/n^2) \{L_{G(t)} + L_{G(t)}^T\} - (1/2) \{JL_{G(t)} + J^T_{G(t)}\} \geq \alpha L_{K_n}$.

**Proof:** i) $\Rightarrow$ From $nL_{G(t)} - J > \alpha L_{K_n}$, it follows that $nL_{G(t)}^T - J > \alpha L_{K_n}$. From these two inequalities, we have $(1/n^2) \{L_{G(t)} + L_{G(t)}^T\} - (1/2) \{JL_{G(t)} + J^T_{G(t)}\} \geq \alpha L_{K_n}$. ii) $\Leftarrow$ From $(1/n^2) \{L_{G(t)} + L_{G(t)}^T\} - (1/2) \{JL_{G(t)} + J^T_{G(t)}\} \geq \alpha L_{K_n}$, we have

$$z^T \left( \frac{n}{2} \{L_{G(t)} + L_{G(t)}^T\} - \frac{1}{2} \{JL_{G(t)} + J^T_{G(t)}\} - \alpha L_{K_n} \right) z \geq 0$$

for all $z \in \mathbb{R}^n$. It implies then that $nL_{G(t)}^T - J > \alpha L_{K_n}$.

With Theorem 1 and Lemma 2 at hand, we have arrived at a general graph theoretic synchronization criterion.

**Theorem 2:** Suppose that Assumption 1 holds, and that the graph $G(t)$ contains a directed spanning tree [8]. The synchronization manifold of (1) is globally stable if

$$\frac{1}{2} \{L_{G(t)} + L_{G(t)}^T\} - \frac{1}{2n} \{JL_{G(t)} + J^T_{G(t)}\} \geq \frac{a}{n} \alpha L_{K_n}, \quad \text{for all } t. \quad (6)$$

In the following, we will show how to interpret (6) from the perspective of graph comparisons. Here and in what follows, we sometimes use graph $G$ as an abbreviation for $G(t)$, and coupling $g_{ij}$ for $g_{ij}(t)$. All conditions and criteria for the couplings are understood to hold for all times $t$, even if $t$ is not explicitly specified. Let $D^e_i$ denote the vertex unbalance [11] of vertex $i$, namely $D^e_i = \sum_{k \neq i} e_{ki} - \sum_{k \neq i} e_{ik} + \sum_{k \neq i} e_{ki} - \sum_{k \neq i} e_{ki}$, which is the difference between the out-degree and in-degree of vertex $i$. It holds that

$$JL_G = 1 \otimes \left[ -\sum_{k = 1}^n e_{k1} - \sum_{k = 1}^n e_{k2} \ldots - \sum_{k = 1}^n e_{kn} \right]$$

$$= 1 \otimes \left[ D^e_1 \ D^e_2 \ldots \ D^e_n \right]$$

And one has

$$I_{G(t)}^T J = -1^T \otimes \left[ D^e_1 \ D^e_2 \ldots \ D^e_n \right]^T.$$

It follows that the matrix $-\{JL_G + L_{G(t)}^T\}$ is

$$\begin{bmatrix}
D^e_1 + D^e_2 & \ldots & D^e_n + D^e_n \\
D^e_1 & 2D^e_2 & \ldots & D^e_n \\
\vdots & \vdots & \ddots & \vdots \\
D^e_1 & D^e_2 & \ldots & 2D^e_n 
\end{bmatrix},$$

where the $(i,j)$th entry is $D^e_i + D^e_j$ for $i, j = 1, \ldots, n$. Since the sum of the out-degrees of all the vertices in $G$ is equal to the sum of the in-degrees of all the vertices, we have $\sum_{i=1}^n D^e_i = 0$. The $i$th row-sum of the matrix $-\{JL_G + L_{G(t)}^T\}$ is then $nD^e_i + \sum_{k=1}^n D^e_k - nD^e_i$ for $i = 1, \ldots, n$. Let the $n \times n$ matrix

$$\Delta = \text{diag} \{nD^e_1, nD^e_2, \ldots, nD^e_n\}.$$

Thus, the matrix $(a/n)L_{K_n} + (1/2n) \{JL_G + L_{G(t)}^T\} + (1/2n) \Delta$ is symmetric and has zero row sums. Since the $i$th row-sum of the matrix $-\{JL_G + L_{G(t)}^T\}$ is $-\sum_{k=1}^n e_{ki} - D^e_i$ for $i = 1, \ldots, n$, we know that the matrix $(1/2)(\{JL_G + L_{G(t)}^T\} + (1/2n) \Delta$ is symmetric and has zero row sums and non-positive off-diagonal entries. Now we are ready to compare the two symmetric matrices $(a/n)L_{K_n} + (1/2n) \{JL_G + L_{G(t)}^T\} + (1/2n) \Delta$ and $(1/2)(\{JL_G + L_{G(t)}^T\} + (1/2n) \Delta$. From (6), we have

$$\frac{1}{2} \{L_{G(t)} + L_{G(t)}^T\} + \frac{1}{2n} \Delta \geq \frac{a}{n} \{L_{K_n} + \frac{1}{2n} \{JL_{G(t)} + L_{G(t)}^T\} \} + \frac{1}{2n} \Delta . \quad (7)$$

The two graphical comparison inequalities (6) and (7) will be useful later on as we further develop spectral graph theoretic conditions in this section.

**B. Coupling Strength Allocation in General Directed Graphs**

In order to apply more tools from spectral graph theory, we first introduce an equivalent definition of the Laplacian matrix of graph. Following [15], the elementary Laplacian $L_{(u,v)}$ is defined as the Laplacian of the graph containing just the edge of unit weight between vertices $u$ and $v$. Then for an undirected graph $H(t) = (\mathcal{V}, \mathcal{E}, \varepsilon(t))$ consisting of the vertex set $\mathcal{V}$, the edge set $\mathcal{E}$, and the weight function $\varepsilon : \mathcal{E} \rightarrow \mathbb{R}$, its Laplacian matrix has the form

$$L(H(t)) \Delta = \sum_{\{u,v\} \in \mathcal{E}, u > v} \varepsilon_{(u,v)}(t) \cdot L_{(u,v)}. \quad (8)$$

Moreover, we say graph $H$ is unweighted if the weights $\varepsilon_{(u,v)} = 1$ for all $u \neq v$.

In order to better understand the graphical condition given by the inequality (6), now we interpret Theorem 2 using the elementary Laplacians.

**Theorem 3:** Suppose that Assumption 1 holds, and that the graph $G(t)$ contains a directed spanning tree. The synchronization manifold of (1) is globally stable if

$$\sum_{j > i} \left( \frac{\varepsilon_{ij} + \varepsilon_{ji}}{2} - \frac{D^e_i + D^e_j}{2n} \right) L_{(i,j)} \geq \frac{a}{n} \sum_{j > i} L_{(i,j)}; \quad (9)$$

or equivalently if

$$\sum_{j > i} \left( \frac{\varepsilon_{ij} + \varepsilon_{ji}}{2} - \frac{D^e_i + D^e_j}{2n} \right) L_{(i,j)} \geq \frac{a}{n} \sum_{j > i} \left( 1 + \frac{D^e_i + D^e_j}{2n} \right) L_{(i,j)}; \quad (10)$$

for all $t$. 
Proof: Using elementary Laplacians, the inequality (9) can be obtained straightforwardly from the graphical condition (6); the inequality (10) follows from the graphical criterion (7).

In the following, we introduce two graphical inequalities taken from [15].

Lemma 3: [15] Let $c_1, \ldots, c_{n-1} > 0$. It holds that

$$c \left( \sum_{i=1}^{n-1} c_i L_{i(i+1)} \right) \geq L_{(1,n)}$$

where $c = \sum_{i=1}^{n-1} (1/c_i)$.

If we take $c_1 = c_2 = \ldots = c_{n-1} = 1$, then Lemma 3 becomes the following result.

Lemma 4: [15] It holds that

$$(n-1) \left( \sum_{i=1}^{n-1} L_{i(i+1)} \right) \geq L_{(1,n)}$$

The spectral graph theory discussed in [15] mainly focuses on undirected graphs. It has been demonstrated in our previous work [13] that tools in spectral graph theory are powerful in utilizing flexibly topological features of a given network. However, the results developed in [13] can be applied only to undirected networks and are thus not general enough. Motivated by [11], what we propose to do next is to symmetrize the graph $G$ first, and then construct synchronization criteria on the symmetrized graph using spectral graph theory. To be more specific, for any pair of unidirectionally coupled vertices $i$ and $j$, we replace the directed edge between them by an undirected edge with the weight $\varepsilon_{ij}/2$ that is half of the original coupling strength; for any bi-directionally coupled pair of vertices $i$ and $j$, we replace the two edges between them by an undirected edge with the coupling strength $(\varepsilon_{ij} + \varepsilon_{ji})/2$. Let $G^s$ be the obtained symmetrized graph from $G$. One can then check that the Laplacian matrix of $G^s$ is $L_{G^s} = (1/2) \left( L_G + L_G^T \right) + (1/2n) \Delta$.

For the symmetrized graph $G^s$, consider a set of paths $P = \{ P_{ij} | i, j = 1, \ldots, n, j > i \}$, one path for each pair of distinct vertices $i$ and $j$. Now we use Theorems 2 and 3 to construct graph theoretic conditions for the synchronization of network (1). We use $G^s$ to denote the set of all the edges of $G^s$ and assume that there are altogether $\sum_{k=1}^{m} \varepsilon_k$ edges that are labeled by $1, \ldots, m$. In the following theorem, we show that lower bounds on the coupling strengths $\varepsilon_k^{(s)}$, $k = 1, \ldots, m$, can be constructed to guarantee that the inequality (7) holds.

Theorem 4: Suppose that Assumption 1 holds, and the graph $G$ contains a directed spanning tree. The synchronization manifold of network (1) is globally stable if

$$\varepsilon_k^{(s)} \geq \frac{a}{n} b_k,$$

where $b_k = \sum_{j>i, k \in P_{ij}} w(P_{ij})$ is the sum of the weighted lengths $w(P_{ij})$ of all those paths $P_{ij}$ in $P$ that contain the edge $k$ that belongs to the symmetrized graph $G^s$ and the weighted path length $w(P_{ij})$ is defined by

$$w(P_{ij}) = \begin{cases} |P_{ij}| \chi \left( \frac{1 + D^+_k + D^-_k}{2a} \right), & \text{edge } (i,j) \notin E(G^s); \\ 1 + D^+_k + D^-_k, & \text{edge } (i,j) \in E(G^s); \\ \end{cases}$$

where for $z \in \mathbb{R}$, the function $\chi(z) = z$ if $z > 0$, $\chi(z) = 0$ otherwise.

Proof: Since the two matrices $L_{G^s}$ and $(a/n) L_{K_n} + (1/2a) \left( JL_{G^s} + L_{G^s}^T J + (1/2a) \Delta \right)$ are symmetric and have zero row and column sums, we can compare them as follows.

$$a \sum_{i,j} \left( L_{K_n} + \frac{1}{2a} \left( JL_{G^s} + L_{G^s}^T J + \frac{1}{2a} \Delta \right) \right) L_{(i,j)}$$

$$= a \sum_{j>i} \left( \chi \left( \frac{1 + D^+_k + D^-_k}{2a} \right) L_{(i,j)} \right)$$

$$= a \sum_{j>i} \chi \left( \frac{1 + D^+_k + D^-_k}{2a} \right) L_{(i,j)}$$

In the last two terms, for edge $(i,j) \in E(G^s)$, we keep just the term $(a/n) \left( 1 + (D^+_k + D^-_k)/2a \right) L_{(i,j)}$; for edge $(i,j) \notin E(G^s)$, we choose a path $P_{ij}$ in $G^s$ that connects $i$ and $j$. Then one can apply Lemma 4 by comparing the sum of all the Laplacian matrices $L_k$, $k \in P_{ij}$, of all the edges along this chosen path and the Laplacian matrix $L_{(i,j)}$ of the single edge $(i,j)$, which leads to $L_{(i,j)} \leq |P_{ij}| \sum_{k \in P_{ij}} L_k$. Applying this inequality to each $L_{(i,j)}$ where $(i,j) \notin E(G^s)$, one obtains that the right-hand side of the inequality (13) is less than or equal to

$$a \sum_{j>i} \left( \chi \left( \frac{1 + D^+_k + D^-_k}{2a} \right) \right) \sum_{k \in P_{ij}} L_k$$

$$= a \sum_{j>i} \left( \chi \left( \frac{1 + D^+_k + D^-_k}{2a} \right) \right) \sum_{k \in P_{ij}} L_k$$

For the term (14), we sum up all the weights for $L_k$ where edge $k \in E(G^s)$. Then we obtain

$$|P_{ij}| \chi \left( \frac{1 + D^+_k + D^-_k}{2a} \right) \sum_{k \in P_{ij}} L_k$$

where $b_k = \sum_{j>i, k \in P_{ij}} w(P_{ij})$ has been defined in Theorem 4. And the last inequality holds trivially when $\varepsilon_k^{(s)} > (a/n) b_k$ for each edge $k$. Therefore, the constructed coupling strengths $\varepsilon_k^{(s)}$, $k = 1, \ldots, m$, guarantee that (7) holds. Thus we complete the proof.

Remark 1: Theorem 4 is similar to Theorem 1 obtained by Belykh et al. in [11], except for some minor differences in (11) and (12). Since we aim to propose several useful graphical comparison synchronization criteria and develop a new methodology to study network synchronization, we interpret Belykh's result [10] from a different viewpoint of spectral graph theory. Our approach utilizes more the features of the graphs associated with the networks. In addition, using graph comparison, we have provided a much simpler proof compared with those in [10], [11].
Remark 2: If $G$ is directed and balanced, i.e., $D_i^t = 0$ for $i = 1, \ldots, n$, then the graphical synchronization criterion (6) becomes

$$\frac{1}{2} \left( L_{G_{1t}} + L_{G_{1t}^T} \right) \geq \frac{a}{n} L_{K_n}.$$  

And (10) accordingly becomes

$$\sum_{j \geq i} \varepsilon_{ij} + \varepsilon_{ji} L_{G_{ij}} \geq \frac{a}{n} \sum_{j > i} L_{G_{ij}}.$$  

From Theorem 4, it follows that network (1) can be asymptotically synchronized if $\varepsilon_{ij} > \lbrack b_k/n \rbrack a$ for $k = 1, \ldots, n$, where $b_k = \sum_{j > i, k \in \mathcal{P}_{ij}} |\mathcal{P}_{ij}|$. The result then becomes the same as [10, Theorem 1] in which the connection graph stability method on directed graphs with node balance is discussed.

Theorem 4 can be used to find a set of coupling strengths to realize global synchronization in a network. We describe below an algorithm to achieve this goal.

Algorithm 1 Coupling strength allocation in a directed network

Step 1. Determine the vertex unbalance $D_i^t$ for each vertex.
Step 2. Symmetrize $G$ to obtain the undirected graph $G^*$.  
Step 3. Compare $G^*$ with the corresponding complete graph $K_n$. For any pair of vertices $i, j$, choose a path $\mathcal{P}_{ij}$ in $G^*$. Here, we prefer to choose the shortest paths.
Step 4. For those paths $\mathcal{P}_{ij}$ whose lengths are greater than 1, assign the weight $1 + (D_i^t + D_j^t)/2a$ if $1 + (D_i^t + D_j^t)/2a > 0$, and zero otherwise. For those paths $\mathcal{P}_{ij}$ whose lengths equal 1, assign the weight $1 + (D_i^t + D_j^t)/2a$.
Step 5. For each edge $k$ in $G^*$ write down the inequality (11):  
$$\varepsilon_k \geq \frac{(a/n) b_k}{e(G^*_k)},$$  
where $b_k = \sum_{j > i, k \in \mathcal{P}_{ij}} w(\mathcal{P}_{ij})$.
Step 6. Solve for the solutions to the obtained set of inequalities, which gives possible combinations of coupling strengths.

Remark 3: Similar ideas in this algorithm have been discussed in [11]. The main differences lie in Steps 4 and 5 where we have used graph comparison techniques. Following [11], we call this algorithm the generalized connection graph method and use the abbreviation GCGM in the rest of the paper.

C. Illustrative Examples

Now we give several simple examples to show how to apply Theorem 4.

Example 1: We consider one example as the directed ring graph with $n$ vertices, shown in Fig. 1(a). Suppose that the coupling strength for each edge is $\varepsilon(t)$. In the following, we use Theorem 2 to determine the lower bound for $\varepsilon(t)$ to synchronize the network. Note that the graph (a) is directed and balanced. And one has $D_i^t = 0$ for all $i = 1, \ldots, n$. Then (6) can be reduced to (1/2) $\left( L_{G_{1t}} + L_{G_{1t}^T} \right) \geq \frac{a}{n} L_{K_n}$. We symmetrize the directed graph by replacing each directed edge by an undirected edge with the coupling strength $\varepsilon(t)/2$, which is shown in Fig. 1(b). Our problem becomes to determine $\varepsilon(t)$ such that $L_{G_{1t}} \geq \frac{a}{n} L_{K_n}$. One can use the technique of graph comparison to determine the $\varepsilon(t)$. However, we choose to use an easier approach to solve the issue, since the undirected and unweighted ring graph $\mathbb{R}_n$ is a special graph and the eigenvalues of its Laplacian matrix can be explicitly written as $2 - 2 \cos(2\pi k/n)$ for $0 \leq k \leq n/2$ [15]. One can check that the second smallest eigenvalue is $4 \sin^2(\pi/k/n)$ when $k = 1$. From Theorem 3 in our previous work [13], to make $L_{G_{1t}^*} \geq \frac{a}{n} K_n$ is equivalent to make $\lambda_2(G_{1t}^*) > a$, i.e., $(\varepsilon(t)/2) \cdot 4 \sin^2(\pi/k/n) > a$. It follows that $\varepsilon(t) \geq (a/2 \sin^2(\pi/n))$ for all times $t$.

Example 2: We consider the graph shown in Fig. 2(a). Assume that the coupling strengths for edges $(4,1), (4,2), (4,3)$ are $\varepsilon_1(t)$, and the couplings for edges $(1,2), (2,3), (3,1)$ are $\varepsilon_2(t)$. We use this example to show how to apply graph comparison to allocate coupling strengths for synchronization in a directed graph. Following the steps below, we aim to make the graphical comparison inequality (10) to hold for the graph (a).

S1. Calculate the node unbalance $D_i^t$ for each node $i = 1, 2, 3, 4$. Here we have $D_1^t = D_2^t = D_3^t = -\varepsilon_1(t), D_4^t = 3\varepsilon_2(t)$.
S2. Symmetrize the graph by replacing each directed edge by the undirected edge with half weight. The symmetrized graph obtained is shown in Fig. 2(b).
S3. According to inequality (10), compare the symmetrized graph with the modified complete graph $K_4$ whose weights of edges $(i, j)$ are $(a/n) + (D_i^t + D_j^t)/2n$ for all $j > i$. The modified complete graph is shown in Fig. 2(c).
S4. In order to guarantee $G_{(b)} \geq G_{(c)}$, we have the inequalities below from the graph comparison of the two graphs (b) and (c):

$$\varepsilon_1(t)/2 > a/4 + \varepsilon_1(t)/4;$$
$$\varepsilon_2(t)/2 > a/4 - \varepsilon_1(t)/4.$$  
Therefore, one has $\varepsilon_1(t) > a$ and $\varepsilon_2(t) > a - \varepsilon_1(t)/2$ (i.e., $\varepsilon_2(t) > 0$) for all times $t$. From Theorem 3, these $\varepsilon_1(t), \varepsilon_2(t)$ guarantee the synchronization in the network (a).

Example 3: We consider the graph shown in Fig. 3(a). Suppose the coupling strength for edge $(1,2)$ is $\varepsilon_1$, and the couplings for $(2,3)$ and $(3,2)$ are $\varepsilon_2$. This example helps us understand how to apply graph comparison to obtain the bounds for coupling strengths in Theorem 4. Following the steps below, we aim to make the graphical inequality (10) to hold for the graph (a).

S1. Calculate the node unbalance $D_i^t$ for each node $i = 1, 2, 3$. We have $D_1^t = \varepsilon_1, D_2^t = -\varepsilon_1, D_3^t = 0$.
S2. Symmetrize the graph by replacing each directed edge by the undirected edge with half weight. The symmetrized graph obtained is shown in Fig. 3(b).
S3. According to inequality (10), compare the symmetrized graph with the modified complete graph $K_4$ whose weights
of edges \( \{i, j\} \) are \( \{a/n\} + (D_{ij}^r + D_{ij}^s)/2n \) for all \( j > i \).

The modified complete graph is shown in Fig. 3(c).

S4. We use the technique of graph comparison to make \( G_{(B)} \geq G_{(C)} \). For edge (1,3) that does not exist in the graph (b), we choose the candidate path (1,2,3) in the graph (b) for connecting vertices 1 and 3. Note that \( L_{(1,3)} \leq 2L_{(1,2)} + 2L_{(1,3)} \) from Lemma 4. Then we have

\[
L_{G_{(B)}} = \frac{a}{3} L_{(1,2)} + \left( \frac{a}{3} - \frac{d_1}{6} \right) L_{(2,3)} + \left( \frac{a}{3} + \frac{d_1}{6} \right) L_{(1,3)}
\]

\[
= \frac{a}{3} L_{(1,2)} + \left( \frac{a}{3} - \frac{d_1}{6} \right) L_{(2,3)} + \left( \frac{a}{3} + \frac{d_1}{6} \right) (2L_{(1,2)} + 2L_{(2,3)})
\]

\[
= \left( a + \frac{d_1}{3} \right) L_{(1,2)} + \left( a + \frac{d_1}{6} \right) L_{(2,3)}.
\]

The terms in the last row only depend on the combinations of Laplacians of the edges in the graph (b), i.e., \( L_{(1,2)}, L_{(2,3)} \). Graph (d) in Fig. 3 shows the combinations. It follows that \( G_{(D)} \geq G_{(C)} \).

S5. In order to make \( G_{(B)} \geq G_{(C)} \), we turn to make \( G_{(B)} \geq G_{(D)} \). Thus we obtain the following inequalities for edges (1,2) and (2,3):

\[
\varepsilon_{1/2} > a + \varepsilon_1/3;
\]

\[
\varepsilon_2 > a + \varepsilon_1/6.
\]

Therefore, one has \( \varepsilon_1 > 6a \) and \( \varepsilon_2 > a + \varepsilon_1/6 \). These \( \varepsilon_1, \varepsilon_2 \) guarantee that \( G_{(B)} \geq G_{(C)} \), i.e., inequality (10) holds. It returns the synchronization in the graph (a) in Fig. 3.

Remark 4: The results of Examples 2–3 become the same when using Algorithm 1 and Belykh et al.’s algorithm in [11]. However, steps 4 and 5 in our algorithm are simpler and easier to be carried out in calculations.

In the next section, we discuss in more detail a new systematic way to allocate coupling strengths for large networks with local structures.

IV. NETWORKS WITH LOCAL STRUCTURES

Although GCGM uses the combinatorial features of graphs and sometimes can greatly simplify computations, it still has two shortcomings:

1) The computational complexity of counting paths grows exponentially as the size of the network increases.
2) As the number of inequalities obtained in step 5 increases, it becomes more difficult, sometimes impossible, to find a solution in step 6.

To address these two shortcomings, we improve the results by looking more carefully at the networks’ local structures and thus apply graph comparison only locally. To do so, we need to decompose graphs.

Definition 3: The Frobenius normal form of the Laplacian matrix of a directed graph \( G \) is:

\[
L_G = M \begin{bmatrix} B_1 & B_{12} & \ldots & B_{1k} \\ B_2 & \vdots & \ddots & B_{2k} \\ \vdots & \ddots & \ddots & \vdots \\ B_k & \cdots & \cdots & B_k \end{bmatrix} M^T \tag{15}
\]

where \( M \) is a permutation matrix and \( B_i \), \( i \), are square irreducible matrices.

Lemma 5: [4] The matrices \( H_k \) in (15) are uniquely determined by \( L_G \) although their ordering can be arbitrary as long as they follow a partial order induced by \( < \) that is defined by \( B_i < B_j \iff B_{ij} \neq 0 \).

The uniqueness of the matrices \( B_i \) can be seen from the fact that these matrices correspond to the strongly connected components of graph \( G \). The decomposition of a Laplacian matrix into its Frobenius normal form is thus equivalent to the decomposition of \( G \) into its strongly connected components. The partial order in Lemma 5 leads to the definition of condensation directed graphs as follows.

Definition 4: [21] The condensation directed graph of a directed graph \( G \) is constructed by assigning a vertex to each strongly connected component of \( G \) and an edge between two vertices if and only if there exists an edge of the same orientation between the corresponding strongly connected components of \( G \).
The construction of condensation graphs can be done in linear time using standard graph searching algorithms [22]. We give an example of a directed graph and its corresponding condensation graph in Fig. 4. The condensation graph has the following property.

**Lemma 6:** [4] The condensation directed graph of $G$ contains a directed spanning tree if and only if $G$ contains a directed spanning tree.

The following synchronization criterion can be derived directly from Theorem 4.20 and Corollary 4.21 in [4].

**Lemma 7:** Under Assumption 1, if the graph $G$ contains a directed spanning tree, then the network (1) synchronizes for sufficiently large coupling strength.

The idea of graph decomposition motivates us to design the following algorithm to obtain the sets of coupling strengths for global synchronization using only local topological information. To avoid notational confusion and distinguish from the coupling strengths obtained by GCGM, we use notation $\delta$ instead of $\varepsilon$ to denote coupling strengths in the rest of this section.

---

**Algorithm 2 Coupling strength allocation in a decomposable directed network**

**Step 1.** Decompose $G$ into its $k$, $k \leq n$, strongly connected components $C_1, C_2, \ldots, C_k$ and the partial ordering is given by Lemma 5.

**Step 2.** For $C_k$, use the GCGM algorithm in Section III to obtain a lower bound of the coupling strength $d_k$ to synchronize the systems corresponding to the vertices in $C_k$.

**Step 3.** In descending order for $i = k - 1, \ldots, 1$, treat $C_i$ one by one. Replace all those vertices in $C_{i+1}, \ldots, C_k$, by a single vertex 0. And keep the edges between $C_i$ and $C_{i+1}, \ldots, C_k$.

Thus we arrive at an凝ensed component $\tilde{C}_i$. Use the GCGM algorithm to obtain a lower bound of the coupling strength $d_i$ for synchronization in $\tilde{C}_i$.

**Step 4.** We obtain the lower bounds of coupling strengths $d_i$ for $i = 1, \ldots, k$ in the network.

In the following, we will prove that the coupling strength allocation in the above algorithm can guarantee synchronization in network (1).

**Proof of the Synchronization of Network (1) Under Algorithm 2:** Without loss of generality, we assume that $\mathbf{M}$ is an identity matrix in the Frobenius normal form of $L_G$. Let $x \triangleq [x_1^T, x_2^T, \ldots, x_n^T]^T$. Let $\tilde{x}_i$ be the part of the state vector $x$ corresponding to $B_i$ for $i = 1, \ldots, k$. For vector $\tilde{x}_i = [\tilde{x}_{i1}^T, \tilde{x}_{i2}^T, \ldots, \tilde{x}_{i1}^T]^T$, we use $F(\tilde{x}_i)$ to denote $[F^T(\tilde{x}_{i1}), F^T(\tilde{x}_{i2}), \ldots, F^T(\tilde{x}_{ik})]^T$. Then the dynamics of $[\tilde{x}_{i1}^T, \ldots, \tilde{x}_{ik}^T]$ can be written as (16).

\[
\begin{bmatrix}
\tilde{x}_{11} \\
\tilde{x}_{21} \\
\vdots \\
\tilde{x}_{k1} \\
\end{bmatrix}
= 
\begin{bmatrix}
F(\tilde{x}_1) \\
F(\tilde{x}_2) \\
\vdots \\
F(\tilde{x}_k) \\
\end{bmatrix}
\begin{bmatrix}
B_1 & B_{12} & \cdots & B_{1k} \\
B_2 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
B_{k-1} & \cdots & \cdots & B_k \\
\end{bmatrix}
\otimes P 
\begin{bmatrix}
\tilde{x}_{11} \\
\tilde{x}_{21} \\
\vdots \\
\tilde{x}_{k1} \\
\end{bmatrix}
\]  
(16)

From (16), the state equation for $\tilde{x}_k$ is

\[
\dot{\tilde{x}}_k = F(\tilde{x}_k) + (B_k \otimes P)\tilde{x}_k; 
\]  
(17)

and the state equation for $\tilde{x}_{k-1}$ is

\[
\dot{\tilde{x}}_{k-1} = F(\tilde{x}_{k-1}) + (B_{k-1} \otimes P)\tilde{x}_{k-1} + (B_{k-1,k} \otimes P)\tilde{x}_k. 
\]  
(18)

For the component $C_k$, we obtain a lower bound of the coupling strength $d_k$ to synchronize the oscillators in (17) by GCGM. Suppose $x_s \in \mathbb{R}^d$ is the synchronous state for the oscillators in (17), and then the dynamics for $x_s$ can be described by

\[
\dot{x}_s = f(x_s) + \phi_s(t), 
\]  
(19)

where $\phi_s(t) \to 0$ as $t \to \infty$. Let 1 be the column vector of all ones with a proper dimension. Then we have $(\hat{x}_k - 1 \otimes x_s) \to 0$ as $t \to \infty$. Let $\phi_k(t) \triangleq (B_{k-1,k} \otimes P)(\hat{x}_{k-1} - 1 \otimes x_s)$. Then (18) can be rewritten as

\[
\dot{\hat{x}}_{k-1} = F(\hat{x}_{k-1}) + (B_{k-1,k} \otimes P)\hat{x}_{k-1} \\
+ (B_{k-1,k} \otimes P)(1 \otimes x_s) + \phi_k(t). 
\]  
(20)

where $\phi_k(t)$ satisfies $\lim_{t \to \infty} \phi_k(t) = 0$.

Now we replace all those vertices in $C_k$ by a single vertex 0 and collect their dynamics into that of a single system whose state is $x_s$. Keeping the edges between $C_k$ and $C_{k-1}$, we arrive at a condensed component $\tilde{C}_{k-1}$ whose vertex set consists of the vertex 0 and the vertices in the component $C_{k-1}$. The dynamics of the condensed component $\tilde{C}_{k-1}$ are described by (19) and

\[
\dot{\hat{x}}_{k-1} = F(\hat{x}_{k-1}) + (H_{k-1,k} \otimes P)\hat{x}_{k-1} + (H_{k-1,k} \otimes P)(1 \otimes x_s). 
\]  
(21)

According to the graph decomposition, there are edge(s) from the vertex 0 to the vertices in $C_{k-1}$, but no edge in the opposite
direction. Thus, the states of the oscillators in $\hat{C}_{k-1}$ will synchronize to $x_k$ if the coupling strengths in $\hat{C}_{k-1}$ are big enough. For the component $\mathcal{C}_{k-1}$, we can obtain a lower bound of coupling strength $d_{k-1}$ by GCGM such that the oscillators in $\hat{C}_{k-1}$ synchronize. From Section III, such coupling strength condition for graph $\hat{C}_{k-1}$ guarantees that $L_{\mathcal{K}} \geq aL_{\mathcal{K}}$, where $\mathcal{K}$ is the complete graph with the same vertex set of graph $\hat{C}_{k-1}$, which further guarantees that (2) in Lemma 1 holds. In addition, we have shown that $\lim_{t \to \infty} \phi_k(t) = 0$. Hence we are ready to use Theorem 4.4 in [4]. From it, one has that the oscillators coupled by the dynamics (19) and (20) synchronize; to be more specific, the states of those oscillators asymptotically converge to $x_k$. Now we can replace all those vertices in $\hat{C}_{k-1}$, $\mathcal{C}_k$ by a single vertex 0 and collect their dynamics into that of a single system whose state is $x_k$.

Then, we repeat the same operations for the components $\mathcal{C}_i$, for $i = k - 2, \ldots, 1$ in descending order. We treat $\mathcal{C}_0$ one by one, replace all those vertices in $\hat{C}_{i+1}, \ldots, \hat{C}_k$, by a single vertex 0, and keep the edges between $\hat{C}_i$ and $\hat{C}_{i+1}, \ldots, \hat{C}_k$. Thus we arrive at a condensed component $\hat{C}_i$ whose vertex set consists of the vertex 0 and the vertices in $\hat{C}_i$. Using the GCGM algorithm to obtain a lower bound for the coupling strength $d_i$ for synchronization in $\hat{C}_i$, we guarantee that the states of the oscillators in $\hat{C}_i$ asymptotically synchronize to $x_i$.

Finally, we get all the lower bounds $d_k, d_{k-1}, \ldots, d_1$ for the components $\mathcal{C}_k, \mathcal{C}_{k-1}, \ldots, \mathcal{C}_1$. Under these coupling strengths, the states of the oscillators of the whole network asymptotically synchronize.

From the proof above, one can see that the oscillators' states of the network (1) asymptotically synchronize to $x_k$. Note that $x_k$ is the synchronous state for the oscillators in (17) in which $B_k$ describes the connections between the oscillators in the root part of the whole directed network. This tells us that the synchronous state of the whole network can be concisely described by the synchronous state of the network's root part. In fact, the oscillators from the root part do not receive any information flows from the oscillators outside, i.e., the oscillators in the root part are not influenced by the oscillators outside but induce the latter to asymptotically synchronize to themselves (the oscillators in the root part). Therefore, we have the following proposition about the asymptotic synchronous state of a directed complex network.

**Proposition 1:** Suppose that the graph $\mathcal{G}(t)$ contains a directed spanning tree and is decomposable. The synchronous state of the network (1) only depends on the asymptotic synchronous state of the network's root part.

In the following, we use two examples to show the effectiveness of the proposed algorithm based on network decompositions.

**Example 4:** We consider the directed network $\mathcal{G}$ on the left of Fig. 4. We follow all the four steps in Algorithm 2. First, we decompose $\mathcal{G}$ into $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5$ as shown in Fig. 4. And thus we have the partial orderings $B_1 < B_2 < B_3 < B_4$ and $B_3 < B_4 < B_5$. The condensation graph is shown on the right of Fig. 4. For $\mathcal{C}_5$, one can calculate that the lower bound for coupling strength $d_5 > (6/5)a$ using Theorem 4. For $\mathcal{C}_4$, we obtain $\hat{C}_4$ shown in Fig. 5, followed by the coupling strength between the components $\mathcal{C}_3$ and $\mathcal{C}_4$ satisfying $d_4 > a$ and the coupling within $\mathcal{C}_4$ satisfying $d_2 > d_4 > 0$ from Example 2. For the condensed components $\hat{C}_3$, $\hat{C}_2$, and $\hat{C}_1$, we obtain $d_2 > 6a$ and $d_1 > a + d_2/6$ from Example 3. Finally, we go back to the original network on the left of Fig. 4. Putting the bounds for $d_1$ to $d_5$ together, we conclude that the global synchronization of the network associated with $\mathcal{G}$ can be realized when the coupling strengths $d_5 > \frac{6}{5}a, d_4 > a, d_3 > 0, d_2 > 6a, d_1 > a + d_2/6$.

Using this example, we compare the complexities of Algorithms 1 and 2, and explain the statements 1) and 2) made at the beginning of Section IV. The computational complexity mainly comes from step 3 and step 5 in Algorithm 1. First, we compare the numbers of the chosen paths in step 3 for the directed graph shown on the left of Fig. 4 and for its decomposed graph with local structures shown in Fig. 5. One needs to choose $C_4^2 + 17$ paths to connect all the pairs of vertices in the original directed graph, while $C_4^2 + C_2^2 + C_2^2 + C_2^2 + C_3^2 = 21$ paths to all the pairs of vertices in the graph components in Fig. 5. In addition, choosing a path to connect a pair of vertices in a smaller-size graph usually requires less computation time. Second, we compare the complexities to solve the sets of inequalities obtained by the two algorithms. Since there are 17 edges in the original directed graph or its corresponding symmetrized graph, one obtains a set of 17 linear inequalities according to step 5 in Algorithm 1. In the decomposed graph, one obtains 5 independent sets of linear inequalities with the dimensions 5, 6, 2, 2, 2, respectively. The difficulties to search for solutions to lower dimensional inequality sets are also smaller.

**Example 5:** We use a simple example to show the benefit of network decompositions in the proposed algorithm. We consider the directed path with $n$ vertices shown in Fig. 6(a). We aim to determine the coupling strength $d$ for each edge that guarantees synchronization in the network. As one choice, one can follow the coupling allocation method in Section III. However, a better choice is to use Algorithm 2. Note that the directed path is decomposable. According to Algorithm 2, the directed path with $n$ vertices can be decomposed into $n - 1$ condensed components each of which is the two unidirectionally coupled vertices shown in Fig. 6(c). Thus, we only need to determine the coupling strength for the graph (c). From Assumption 1, the two unidirectionally coupled oscillators can be synchronized when $d > a$. Thus, the directed path shown by the graph (a) can be synchronized when the coupling strength $d$ for each edge exceeds $a$, as long as $n$ is a finite positive integer.

In comparison, the case of undirected paths is quite different. We cannot use graph decomposition for undirected paths. Below we also provide the bounds of coupling strengths for the synchronization in undirected paths. According to [15], the Laplacian matrix of the undirected path graph with $n$ vertices has eigenvalues $2 - 2\cos(\pi k/n)$ for $0 \leq k \leq n - 1$. Then the second smallest eigenvalue is $4 \sin^2(\pi/2n)$ when $k = 1$. Combining Theorem 2 and Theorem 3 in the previous work [1], we
know that the network can achieve synchronization when the edge weight is larger than $(a/\lambda_2) = (a/4 \sin^2(\pi/2n))$. From this expression, the undirected path with $n = 2$ can get synchronized when its coupling strength is greater than $a/2$; however, the lower bound for the coupling strength increases almost at the speed of $O(n^2)$ as $n$ increases. In short, the lower bounds of coupling strengths for the synchronization in directed and undirected paths are different apparently.

V. CONCLUSIONS

In this paper we have presented new ways to allocate coupling strengths using the tool of graph comparison from spectral graph theory in order to achieve synchronization in directed complex networks. The main idea is to keep the symmetrized Laplacian matrices associated with the given networks dominating that of the complete graphs with the same vertex sets via graph comparison. By exploiting the symmetrization operation, we have dealt with the challenge that the Laplacian matrices associated with directed graphs are not guaranteed to be positive semi-definite anymore. The obtained results using graph combinatorial features can simplify the computation. In addition, the proposed algorithms can be applied to large but decomposable networks.

In our future study, we are interested in studying how the assigned coupling strengths affect the semistability [23] of the network. We also look forward to applying the proposed allocation strategies to practical engineered complex networks, such as optimal topology design in communication networks and synchronization of generators in electric power grids.

REFERENCES


received an IBM Research outstanding technical achievement award and an IEEE Region 1 Technological Innovation Award for this work. He is one of the main contributors and pioneers of the emerging fields of chaotic communication systems and synchronization in coupled networks of dynamical systems and has written two books on these subjects. His other research interests include cellular neural networks, nonlinear dynamics of circuits and systems, peer-to-peer multimedia delivery, multimedia security, and algebraic graph theory. He has published over 100 papers and was granted over 60 U.S. patents. Dr. Wu has served as chair and technical program committee member for several international conferences. He was an Associate Editor of IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS and served on the CAS Board of Governors. He has been serving as an IEEE EAB Program Evaluator since 2006. He is a member of the American Mathematical Society.

**Jun-An Lu** received the B.Sc. degree in geophysics from Peking University, Beijing, China, and the M.Sc. degree in applied mathematics from Wuhan University, Wuhan, China, in 1968 and 1982, respectively. He is currently a Professor with the School of Mathematics and Statistics, Wuhan University. His research interests include complex networks, nonlinear systems, chaos control, and scientific and engineering computing. He has published over 200 journal papers in the above fields. He received the Second Prize of the Natural Science Award from the Hubei Province, China in 2006, the First Prize of the Natural Science Award from the Ministry of Education of China in 2007, the Second Prize of the State Natural Science Award from the State Council of China in 2008, and the First Prize of the Natural Science Award from the Hubei Province, China, in 2013.

**Chi K. Tse** (M’90–SM’97–F’06) received the B.Eng. (Hons.) degree in electrical engineering and the Ph.D. degree from the University of Melbourne, Australia, in 1987 and 1991, respectively. He is presently Chair Professor of Electronic Engineering at the Hong Kong Polytechnic University, Hong Kong. From 2005 to 2012, he was the Head of Department of Electronic and Information Engineering at the same university. His research interests include complex network applications, power electronics, and nonlinear systems. Currently Dr. Tse serves as Editor-in-Chief for the IEEE Circuits and Systems Magazine and IEEE Circuits and Systems Society Newsletter. He was/is an Associate Editor for the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—PART I from 1999 to 2001 and again from 2007 to 2009. He has also been an Associate Editor for the IEEE TRANSACTIONS ON POWER ELECTRONICS since 1999. He is on the Editorial Board of the International Journal of Circuit Theory and Applications and International Journal of Bifurcation and Chaos. Dr. Tse received the Best Paper Award from IEEE TRANSACTIONS ON POWER ELECTRONICS in 2001 and the Best Paper Award from International Journal of Circuit Theory and Applications in 2003. In 2005 and 2011, he was selected and appointed as IEEE Distinguished Lecturer. In 2007, he was awarded the Distinguished International Research Fellowship by the University of Calgary, Canada. In 2009 and 2013, he and his co-inventors won the Gold Medal at the International Exhibition of Inventions of Geneva, Switzerland. In 2013 and 2015, he was awarded the Gledden Fellowship and the Distinguished International Fellowship by the University of Western Australia, Australia.