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Asynchronous and Time-Varying Proximal Type Dynamics in Multiagent Network Games

Carlo Cenedese[✉], Giuseppe Belgioioso[✉], Yu Kawano[✉], Sergio Grammatico[✉], and Ming Cao[✉]

Abstract—In this article, we study proximal type dynamics in the context of multiagent network games. We analyze several conjugations of this class of games, providing convergence results. Specifically, we look into synchronous/asynchronous dynamics with time-invariant communication network and synchronous dynamics with time-varying communication networks. Finally, we validate the theoretical results via numerical simulations on opinion dynamics.

Index Terms—Asynchronous dynamics, multiagent systems, network games, proximal dynamics, time-varying networks.

I. INTRODUCTION

A. Multiagent Decision Making Over Networks

Multiagent decision making over networks is currently a vibrant research area in the systems-and-control community, with applications in several domains, such as smart grids [1], [2], traffic and information networks [3], social networks [4], [5], consensus and flocking groups [6], and robotic [7]. Essentially, in networked multiagent systems, the states (or decisions) of some agents evolve as a result of *local decision making*, e.g., local constrained optimization, and *distributed communication* with some neighboring agents, via a communication network. The use of distributed computation and communication allows each decision maker, in short, *agent*, to keep its own data private and to exchange information with selected agents only.

Multiagent dynamics over networks embody the natural extension of distributed optimization and equilibrium seeking problems in network games. Some examples of constrained convex optimization problems, subject to homogeneous constraint sets, are studied in [8], where uniformly bounded subgradients and complete communication graphs

with uniform weights are considered; while in [9], the cost functions are assumed to be differentiable with Lipschitz continuous and uniformly bounded gradients.

Solutions for noncooperative games over networks subject to local convex constraints have been developed, e.g., in [10], under strongly convex quadratic costs and time-invariant communication graphs; in [11] and [12], with differentiable cost functions with Lipschitz continuous gradients, strictly convex cost functions, and undirected, possibly time-varying, communication graphs; and in [7], where the communication is ruled by a possibly time-varying digraph and the cost functions are assumed to be convex. In some recent works, researchers have developed algorithms to solve games over networks subject to asynchronous updates of the agents: among others, in [13] and [14], the game is subject to affine coupling constraints, and the cost functions are differentiable, while the communication graph is assumed to be undirected.

To the best of the authors' knowledge, the main works that extensively focus on multiagent network games with proximal dynamics are [2], [7], where the authors consider local convex costs and quadratic proximal terms, time-invariant, and time-varying communication graphs, but subject to some technical restrictions.

B. Contribution of this Article

Next, we highlight the novelties and main contribution of this article with respect to the literature referenced earlier.

- 1) We prove that a row-stochastic adjacency matrix describing a strongly connected graph with self-loops is an averaged operator in the Hilbert space weighted by its left Perron–Frobenius eigenvector (see Lemma 1). This result is significant itself and fundamental to generalize the work in [10].
- 2) We prove global convergence of synchronous proximal dynamics in network games under time-invariant directed communication graph (hence described by a row-stochastic adjacency matrix). This extends the results in [2] and [7].
- 3) We establish global convergence for asynchronous proximal dynamics and synchronous proximal dynamics over time-varying networks. The former setup is considered here for the first time. The latter is studied in [2] for undirected communication graph (hence doubly stochastic adjacency matrix), and in [7] via a dwell-time restriction.

C. Notation

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ denotes the set of extended real numbers. $\mathbf{0}$ ($\mathbf{1}$) denotes a matrix/vector with all elements equal to 0 (1); to improve clarity, we sometimes add the dimension of these matrices/vectors as subscripts. Given N vectors $x_1, \dots, x_N$, $\mathbf{x} := \text{col}(x_1, \dots, x_N) = [x_1^\top, \dots, x_N^\top]^\top$. $A \otimes B$ denotes the Kronecker product of the matrices A and B . For a square matrix $A = [a_{i,j}] \in \mathbb{R}^{n \times n}$, where $a_{i,j}$ is the entry in position (i, j) , its transpose is denoted by $A^\top$; $A \succ 0$ ($\succeq 0$) stands for

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a positive definite (semidefinite) matrix; $\text{diag}(A_1, \dots, A_N)$ denotes a block-diagonal matrix with the matrices $A_1, \dots, A_N$ on the diagonal. Given two vectors $x, y \in \mathbb{R}^n$ and $Q = Q^\top \succ 0$, $\langle x | y \rangle_Q = x^\top Q y$ denotes the Q -weighted inner product and $\|x\|_Q$ the Q -weighted norm; $\|A\|_Q$ denotes the Q -weighted matrix norm. A real n dimensional Hilbert space obtained by endowing $\mathcal{H} = (\mathbb{R}^n, \|\cdot\|)$ with the product $\langle \cdot | \cdot \rangle_Q$ is denoted by $\mathcal{H}_Q$.

The indicator function $\iota_C : \mathbb{R}^n \rightarrow [0, +\infty]$ of $C \subseteq \mathbb{R}^n$ is defined as $\iota_C(x) = 0$ if $x \in C$; $+\infty$ otherwise. For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\text{dom}(f) := \{x \in \mathbb{R}^n \mid f(x) < \infty\}$; $\text{prox}_f(x) = \arg \min_{y \in \mathbb{R}^n} f(y) + \frac{1}{2} \|y - x\|^2$ denotes its proximal operator.

A mapping $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is η -averaged (η -AVG) in $\mathcal{H}_Q$, with $\eta \in (0, 1)$, if $\|\mathcal{A}(x) - \mathcal{A}(y)\|_Q^2 \leq \|x - y\|_Q^2 - \frac{1-\eta}{\eta} \|(\text{Id} - \mathcal{A})(x) - (\text{Id} - \mathcal{A})(y)\|_Q^2$, for all $x, y \in \mathbb{R}^n$; $\mathcal{A}$ is nonexpansive (NE) if 1-AVG; $\mathcal{A}$ is firmly nonexpansive (FNE) if $\frac{1}{2}$ -AVG; $\mathcal{A}$ is β -cocoercive if $\beta\mathcal{A}$ is $\frac{1}{2}$ -AVG or, equivalently, FNE.

II. MATHEMATICAL SETUP AND PROBLEM FORMULATION

We consider a set of N agents (or players), where the state (or strategy) of each agent $i \in \mathcal{N} := \{1, \dots, N\}$ is denoted by $x_i \in \mathbb{R}^n$. The set $\Omega_i \subset \mathbb{R}^n$ represents all the feasible states of agent i , hence, it is used to model its local constraints, i.e., $x_i \in \Omega_i$. Throughout this article, we assume compactness and convexity of the local constraint set Ω_i .

Standing Assumption 1 (Convexity): For each $i \in \mathcal{N}$, the set $\Omega_i \subset \mathbb{R}^n$ is nonempty, compact, and convex. $\square$

We consider rational (or myopic) agents, namely, each agent i aims at minimizing a local cost function g_i that we assume convex and with the following structure.

Standing Assumption 2 (Proximal cost functions): For each $i \in \mathcal{N}$, the function $g_i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$g_i(x_i, z_i) := f_i(x_i) + \iota_{\Omega_i}(x_i) + \frac{1}{2} \|x_i - z_i\|^2 \quad (1)$$

where $\bar{f}_i := f_i + \iota_{\Omega_i} : \mathbb{R}^n \rightarrow \mathbb{R}$ is a lower semicontinuous, proximal friendly, and convex function. $\square$

We emphasize that Standing Assumption 2 requires neither the differentiability of the local cost function, nor the Lipschitz continuity and boundedness of its gradient. The *proximal-friendly* structure of $\bar{f}_i$ ensures that agent i can efficiently minimize (1) (see [15, Table 10.2] for some examples of this class of functions). In (1), the function $\bar{f}_i$ is local to agent i and models the local objective that the player would pursue if no coupling between agents is present. The quadratic term $\frac{1}{2} \|x_i - z_i\|^2$ penalizes the distance between the state of agent i and a given z_i , precisely defined later. This term is referred in the literature as *regularization* (see [16, Ch. 27]), since it makes $g_i(\cdot, z_i)$ strictly convex, even though $\bar{f}_i$ is only lower semicontinuous (see [16, Th. 27.23]).

We assume that the agents can communicate through a network structure, described by a weighted digraph. Let us represent the communication links between the agents by a weighted adjacency matrix $A \in \mathbb{R}^{N \times N}$ defined as $[A]_{ij} := a_{i,j}$. For all $i, j \in \mathcal{N}$, $a_{i,j} \in [0, 1]$ denotes the weight that agent i assigns to the state of agent j . If $a_{i,j} = 0$, then the state of agent i is independent from that of agent j . The set of agents with whom agent i communicates is denoted by $\mathcal{N}_i$. The following assumption formalizes the communication network via a digraph and the associated adjacency matrix.

Standing Assumption 3 (row stochasticity and self-loops): The communication graph is strongly connected. The matrix $A = [a_{i,j}]$ is row stochastic, i.e., $a_{i,j} \geq 0$ for all $i, j \in \mathcal{N}$, and $\sum_{j=1}^N a_{i,j} = 1$, for all $i \in \mathcal{N}$. Moreover, A has strictly positive diagonal elements, i.e., $\min_{i \in \mathcal{N}} a_{i,i} =: \underline{a} > 0$. $\square$

In our setup, the variable z_i in (1) represents the average state among the neighbors of agent i , weighted through the adjacency matrix, i.e., $z_i := \sum_{j=1}^N a_{i,j} x_j$. Therefore the cost function of agent i is $g_i(x_i, a_{i,i} x_i + \sum_{j \neq i}^N a_{i,j} x_j)$. Note that a *coupling* between the agents emerges in the local cost function, since the second argument of the g_i s depends on the strategy of (some of) the other agents.

In this article, we consider a population of rational agents that update their states/strategies, at each time instant k , according to the following myopic dynamics:

$$x_i(k+1) = \underset{y \in \mathbb{R}^n}{\text{argmin}} g_i \left(y, \sum_{j=1}^N a_{i,j} x_j(k) \right). \quad (2)$$

These dynamics are relatively simple, yet arise in diverse research areas. For instance, in [17], the authors study the Friedkin–Johnsen model, where the update rule in [17, Eq. 1] can be obtained via (2) by choosing

$$g_i(x_i, z_i) := \frac{1 - \mu_i}{\mu_i} \|x_i - x_i(0)\|^2 + \iota_{[0,1]^n}(x_i) + \|x_i - z_i\|^2 \quad (3)$$

where $\mu_i \in [0, 1]$. When $f_i = 0$, for all $i \in \mathcal{N}$, the dynamics in (2) boils down to the projected-consensus algorithm studied in [8, Eq. (3)], i.e.,

$$x_i(k+1) = \text{proj}_{\Omega_i} \left(\sum_{j=1}^N a_{i,j} x_j(k) \right). \quad (4)$$

Moreover, this structure can also be used to analyze distributed model fitting algorithm, similarly to [18, Eq. 8.3].

The collective strategy profiles that are stationary points of the dynamics in (2) are known as *network equilibria*, as formalized next.

Definition 1 (Network equilibrium [2, Def. 1]): The collective vector $\bar{x} = \text{col}(\bar{x}_1, \dots, \bar{x}_N)$ is a *network equilibrium* (NWE) if, for all $i \in \mathcal{N}$

$$\bar{x}_i = \underset{y \in \mathbb{R}^n}{\text{argmin}} g_i \left(y, \sum_{j=1}^N a_{i,j} \bar{x}_j \right). \quad (5)$$

We remark that the set of NWE directly depends on both the communication topology and on the specific weights $a_{i,j}$ of the adjacency matrix. Moreover, if there are no self loops, i.e., $a_{i,i} = 0$ for all i , then (2) are best-response dynamics and NWE correspond to Nash equilibria [2, Re. 1]. $\square$

III. PROXIMAL DYNAMICS

In this section, we study three different types of proximal dynamics, namely synchronous, asynchronous, and time-varying. While for the former two, we can study and prove convergence to an NWE, the last does not ensure convergence. Thus, we propose a modified version of the dynamics with the same equilibria of the original game.

A. Synchronous Proximal Dynamics

As a first step, we exploit the structure of the cost function g_i in (1) to rephrase the dynamics in (2) by means of the proximity operator [16, Ch. 12.4] as

$$x_i(k+1) = \text{prox}_{\bar{f}_i} \left(\sum_{j=1}^N a_{i,j} x_j(k) \right) \quad \forall k \in \mathbb{N}. \quad (6)$$

In compact form, they read as

$$\mathbf{x}(k+1) = \mathbf{prox}_F(\mathbf{A} \mathbf{x}(k)) \quad (7)$$

where the matrix $\mathbf{A} := A \otimes I_n$ represents the interactions among agents, and the operator $\mathbf{prox}_f$ is defined as

$$\mathbf{prox}_f(\mathbf{y}) := \text{col}(\text{prox}_{f_1}(y_1), \dots, \text{prox}_{f_N}(y_N)).$$

Remark 1: Definition 1 can be equivalently cast in terms of fixed points of the operator in the left-hand side of (7). In fact, a collective vector $\bar{x}$ is an NWE if and only if $\bar{x} \in \text{fix}(\mathbf{prox}_f \circ \mathbf{A})$. Under Assumptions 1 and 2, $\text{fix}(\mathbf{prox}_f \circ \mathbf{A})$ is nonempty [19, Th. 4.1.5], i.e., there always exists an NWE, thus, the convergence problem is well posed. $\square$

The following lemma shows that a row stochastic matrix A is an AVG operator in the Hilbert space weighted by a diagonal matrix Q , whose diagonal entries are the elements of the left Perron–Frobenius (PF) eigenvector of the adjacency matrix A , i.e., a vector $\bar{q} \in \mathbb{R}_{>0}^N$ s.t. $\bar{q}^\top A = \bar{q}^\top$. We always consider the normalized PF eigenvector, i.e., $q = \bar{q}/\|\bar{q}\|$.

Lemma 1 (Averagedness and left PF eigenvector): Let Assumption 3 hold true, i.e., A be row stochastic, $\underline{a} > 0$ be its smallest diagonal element and $q = \text{col}(q_1, \dots, q_N)$ its left PF eigenvector. Then, the following hold:

- i) A is η -averaged (AVG) in $\mathcal{H}_Q$, with $Q := \text{diag}(q_1, \dots, q_N)$ and $\eta \in (0, 1 - \underline{a})$;
- ii) the operator $\mathbf{prox}_f \circ \mathbf{A}$ is $\frac{1}{2-\eta}$ -AVG in $\mathcal{H}_Q$.

If A is doubly stochastic, (i) and (ii) hold with $Q = I$. $\square$

Now, we are ready to present the first result, the global convergence of the proximal dynamics in (7) to an NWE.

Theorem 1 (Convergence of proximal dynamics): For any $\mathbf{x}(0) \in \Omega$, the sequence $(\mathbf{x}(k))_{k \in \mathbb{N}}$ generated by the proximal dynamics in (7) converges to an NWE. $\square$

Remark 2: Theorem 1 extends [2, Th. 1], where the matrix A is assumed doubly stochastic [2, Ass. 1, Prop. 2]. In this case, $\mathbf{1}_N$ is the left PF eigenvector of A and the matrix Q is set as the identity matrix (see Lemma 1). $\square$

Remark 3: In [2, Sec. VIII-A], the authors study, via simulations, an application of (2) to opinion dynamics. In particular, they conjecture the convergence of the dynamics in the case of a row stochastic weighted adjacency matrix. Theorem 1 theoretically supports the convergence of this class of dynamics. $\square$

The presence of self-loops in the communication (see Assumption 3) is critical for the convergence of the dynamics in (7). In fact, by choosing $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, it is easy to construct an example in which they fail to converge.

If $\underline{a} = 0$, i.e., the self-loop requirement in Standing Assumption 3 is not met, the convergence may be restored by relaxing the dynamics in (2) via the so-called Krasnosel'skii–Mann iteration [16, Sec. 5.2]

$$\mathbf{x}(k+1) = (1 - \alpha)\mathbf{x}(k) + \alpha \mathbf{prox}_f(\mathbf{A} \mathbf{x}(k)) \quad (8)$$

where $\alpha \in (0, 1)$. These new dynamics share the same fixed points of (7), namely, the set of NWE (Remark 1).

Corollary 1: For any $\mathbf{x}(0) \in \Omega$ and for $\min_{i \in \mathcal{N}} a_{i,i} \geq 0$, the sequence $(\mathbf{x}(k))_{k \in \mathbb{N}}$ generated by the dynamics in (8) converges to an NWE. $\square$

B. Asynchronous Proximal Dynamics

The dynamics introduced in (7) assume that all the agents update synchronously. Here, we study the more realistic case in which they behave asynchronously. To model the asynchronous updates we adopt the same mathematical framework as in [13], [20], and [21]. Specifically, we assume that each agent i has a local Poisson clock with the rate τ_i and updates independently from the rest every time the clock ticks. It is convenient for the analysis to consider a virtual Poisson clock that ticks every time one of the agents updates, so it has rate $\tau = \sum_{i \in \mathcal{N}} \tau_i$.

We denote by Z_k the k th tick of the global clock, by k the time interval $[Z_{k-1}, Z_k)$ and assume that only one agent updates during each time slot. Then, we denote by $i_k \in \mathcal{N}$ the agent starting its update during the time slot k . According to this setup, at each time instant k , only one agent $i_k \in \mathcal{N}$ updates its state according to (6), while the rest keep their state unchanged, i.e.,

$$x_i(k+1) = \begin{cases} \text{prox}_{f_i} \left(\sum_{j=1}^N a_{i,j} x_j(k) \right), & \text{if } i = i_k \\ x_i(k), & \text{otherwise.} \end{cases} \quad (9)$$

Next, we derive a compact form for the dynamics above. Define H_i as the matrix of all zeros except for $[H_i]_{ii} = 1$, and also $\mathbf{H}_i := H_i \otimes I_n$. Then, we define the set $\mathbf{H} := \{\mathbf{H}_i\}_{i \in \mathcal{N}}$ as the collection of these N matrices. We denote by ζ_k an independent identically distributed (i.i.d.) random variable that takes values in $\mathbf{H}$, with $\mathbb{P}[\zeta_k = \mathbf{H}_i] = \tau_i/\tau$, for all $i \in \mathcal{N}$. If $\zeta_k = \mathbf{H}_i$, it means that agent i is updating at time k , while the rest maintain their strategies unchanged. With this notation in mind, the dynamics in (7) are modified to model asynchronous updates

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \zeta_k (\mathbf{prox}_f(\mathbf{A} \mathbf{x}(k)) - \mathbf{x}(k)). \quad (10)$$

We remark that (10) represents the natural asynchronous counterpart of the dynamics in (7). In fact, the update above is equivalent to (6) for the active agent at time $k \in \mathbb{N}$.

Each agent $i \in \mathcal{N}$ has a public and private memory. If the player is not performing an update, the strategies stored in the two memories coincide. During an update, instead, the public memory stores the strategy of the agent before the update has started, while in the private one there is the value that is modified during the computations. When the update is completed, the value in the public memory is overwritten by that in the private memory. This assumption ensures that all the reads of agent i 's public memory, performed by its neighbours $j \in \mathcal{N}_i$, are always consistent (see [22, Sec. 1.2] for technical details).

We consider the case in which the computation time for the update is not negligible, therefore, the strategies that agent i reads from each neighbor $j \in \mathcal{N}_i$ may be outdated of $\varphi_j(k) \in \mathbb{N}$ time intervals. The maximum delay is assumed uniformly upper bounded.

Assumption 4 (Bounded maximum delay): The delays are uniformly upper bounded, i.e., $\sup_{k \in \mathbb{N}} \max_{i \in \mathcal{N}} \varphi_i(k) \leq \bar{\varphi} < \infty$, for some $\bar{\varphi} \in \mathbb{N}$. $\square$

The dynamics describing the asynchronous update with delays are cast in a more compact form as

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \zeta_k (\mathbf{prox}_f(\mathbf{A} \hat{\mathbf{x}}(k)) - \hat{\mathbf{x}}(k)) \quad (11)$$

where $\hat{\mathbf{x}} = \text{col}(\hat{x}_1, \dots, \hat{x}_N)$ is the vector of possibly delayed strategies. Note that each agent i has always access to the updated value of its own strategy, i.e., $\hat{x}_i = x_i$. We stress that the dynamics in (11) coincide with (10) when no delay is present, i.e., if $\bar{\varphi} = 0$.

The following theorem claims the global convergence (in probability) of (11) to an NWE when the maximum delay $\bar{\varphi}$ is small enough.

Theorem 2 (Convergence of asynchronous dynamics): Let Assumption 4 hold true, $p_{\min} := \min_{i \in \mathcal{N}} \tau_i/\tau$ and

$$\bar{\varphi} < \frac{N\sqrt{p_{\min}}}{2(1-\underline{a})} - \frac{1}{2\sqrt{p_{\min}}}. \quad (12)$$

Then, for any $\mathbf{x}(0) \in \Omega$, the sequence $(\mathbf{x}(k))_{k \in \mathbb{N}}$ generated by (11) converges almost surely to some $\bar{x} \in \text{fix}(\mathbf{prox}_f \circ \mathbf{A})$, namely, an NWE. $\square$

Remark 4: If $\bar{\varphi} = 0$, convergence can be established also in the more general case in which multiple agents update at the exact same time instant, by relying on random block-coordinate updates for fixed-point iterations, as introduced in [23]. Specifically, one can define 2^N different operators T_j , $j \in \{1, \dots, 2^N\}$, each triggering the update of a different combination of agents. The dynamics become a particular

case of [23, Eq. 3.17] and the convergence follows from [23, Cor. 3.8]. $\square$

If the maximum delay does not satisfy (12), the convergence of the dynamics in (11) is not guaranteed. Nevertheless, it can be restored by introducing a time-varying scaling factor $\psi_k \in (0, 1)$ in the dynamics

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \psi_k \zeta_k (\mathbf{prox}_f(\mathbf{A} \hat{\mathbf{x}}(k)) - \hat{\mathbf{x}}(k)). \quad (13)$$

The next theorem proves that the modified dynamics converges, if the scaling factor is chosen small enough.

Theorem 3: Let Assumption 4 hold true and set

$$0 < \psi_k < \frac{N p_{\min}}{(2\bar{\varphi}\sqrt{p_{\min}} + 1)(1 - \underline{a})} \quad \forall k \in \mathbb{N}. \quad (14)$$

Then, for any $\mathbf{x}(0) \in \Omega$, the sequence $(\mathbf{x}(k))_{k \in \mathbb{N}}$ generated by (13) converges almost surely to some $\bar{\mathbf{x}} \in \text{fix}(\mathbf{prox}_f \circ \mathbf{A})$, namely, an NWE. $\square$

Now, if the value of $\underline{a}$ is not globally known by all the agents, one may consider a more conservative bound

$$0 < \psi_k < \frac{N p_{\min}}{(2\bar{\varphi}\sqrt{p_{\min}} + 1)} \quad \forall k \in \mathbb{N} \quad (15)$$

which is independent of $\underline{a}$. Furthermore, since at each time instant k only one agent updates its state, the agents do not need to coordinate with each other to agree on the relaxation sequence $(\psi_k)_{k \in \mathbb{N}}$. Thus, the dynamics in (13) remain distributed.

Remark 5: An interesting particular case arises when all the agents have the same update rate, i.e., $p_{\min} = 1/N$. In this case, the bound on the maximum delay becomes $\bar{\varphi} < \frac{a\sqrt{N}}{2(1-a)}$ that grows when $\underline{a}$ or the population size N increases. In (15), the scaling coefficient can be chosen as $\psi_k < \frac{\sqrt{N}}{(2\bar{\varphi} + \sqrt{N})}$. It follows that $\bar{\varphi}$ is the only global parameter that the agents must know *a priori* to compute ψ_k . $\square$

C. Time-Varying Proximal Dynamics

A challenging problem related to the (synchronous) dynamics in (7) is studying their convergence when the communication network varies over time (i.e., the associated adjacency matrix A is time dependent). In this case, the update rules in (7) become

$$\mathbf{x}(k+1) = \mathbf{prox}_f(\mathbf{A}(k) \mathbf{x}(k)) \quad (16)$$

where $\mathbf{A}(k) = A(k) \otimes I_n$, and $A(k)$ is the adjacency matrix at time instant k .

As in [2, Ass. 4, 5], we assume *persistent stochasticity* of the sequence $(\mathbf{A}(k))_{k \in \mathbb{N}}$, which can be seen as the time-varying counterpart of Standing Assumption 3.

Assumption 5 (Persistent row stochasticity and self-loops): For all $k \in \mathbb{N}$, the adjacency matrix $\mathbf{A}(k)$ is row stochastic and describes a strongly connected graph. Furthermore, there exists $\bar{k} \in \mathbb{N}$ such that, for all $k > \bar{k}$, the matrix $\mathbf{A}(k)$ satisfies $\inf_{k > \bar{k}} \min_{i \in \mathcal{N}} [A(k)]_{ii} =: \underline{a} > 0$. $\square$

The concept of NWE in Definition 1 is bound to the particular communication network considered. In the case of a time-varying communication topology, we focus on a different class of equilibria, namely, those invariant with respect to changes in the communication topology.

Definition 2 (Persistent NWE [2, Ass. 3]): A collective vector $\bar{\mathbf{x}}$ is a persistent NWE (p-NWE) of (16) if there exists some positive constant $\bar{k} > 0$, such that

$$\bar{\mathbf{x}} \in \mathcal{E} := \bigcap_{k > \bar{k}} \text{fix}(\mathbf{prox}_f(\mathbf{A}(k) \mathbf{x}(k))). \quad (17)$$

Next, we assume the existence of a p-NWE. $\square$

Assumption 6 (Existence of a p-NWE): The set of p-NWE of (16) is nonempty, i.e., $\mathcal{E} \neq \emptyset$. $\square$

We note that when the mappings $\mathbf{prox}_{\bar{f}_i}$ s have a common fixed point,¹ i.e., $\bigcap_{i \in \mathcal{N}} \text{fix}(\mathbf{prox}_{\bar{f}_i}) \neq \emptyset$, then our convergence problem boils down to the setup studied in [24]. In this case, [24, Th. 2] can be applied to the dynamics in (16) to prove convergence to a p-NWE, $\bar{\mathbf{x}} = \mathbf{1} \otimes \bar{x}$, where $\bar{x}$ is a common fixed point of the proximal operators $\mathbf{prox}_{\bar{f}_i}$ s. For example, this additional condition holds true when $\bar{f}_i = \iota_{\Omega_i}$, for all $i \in \mathcal{N}$, and $\bigcap_{i \in \mathcal{N}} \Omega_i \neq \emptyset$, namely, the constrained consensus framework considered in [8], and in the classical consensus setup [25], i.e., $\Omega_i = \mathbb{R}^n$, for all $i \in \mathcal{N}$.

However, if this additional assumption is not met, then convergence is not guaranteed. In fact, *a priori* there is no common space in which $\mathbf{prox}_f \circ \mathbf{A}(k)$ possesses the AVG propriety for every $k \in \mathbb{N}$, thus, we cannot infer the convergence of the dynamics in (16) under arbitrary switching topologies. In some special cases, a common space can be found, e.g., if all the adjacency matrices are doubly stochastic (see [2, Th. 3]).

Next, we propose modified dynamics with convergence guarantees to a p-NWE, for every switching sequence, i.e.,

$$\mathbf{x}(k+1) = \mathbf{prox}_f([I + \mathbf{Q}(k)(\mathbf{A}(k) - I)] \mathbf{x}(k)). \quad (18)$$

These new dynamics are obtained by replacing $\mathbf{A}(k)$ in (16) with $I + \mathbf{Q}(k)(\mathbf{A}(k) - I)$, where $\mathbf{Q}(k)$ is chosen as in Theorem 1. Remarkably, this key modification makes the resulting operators $(\mathbf{prox}_f \circ (I + \mathbf{Q}(k)(\mathbf{A}(k) - I)))_{k \in \mathbb{N}}$ AVG in the same space, i.e., $\mathcal{H}_I$, for all $\mathbf{A}(k)$ satisfying Assumption 3, as explained in Appendix V-E. Moreover, the change of dynamics does not lead to extra communications between the agents, since the matrix $\mathbf{Q}(k)$ is block diagonal and it does not modify the fixed points of the mapping.

The following theorem shows that the modified dynamics in (18), subject to arbitrary switching of communication topology, converge to a p-NWE, for any initial condition.

Theorem 4 (Convergence of time-varying dynamics): Let Assumptions 5, 6 hold true. Then, for any $\mathbf{x}(0) \in \Omega$, the sequence $(\mathbf{x}(k))_{k \in \mathbb{N}}$ generated by (18) converges to some $\bar{\mathbf{x}} \in \mathcal{E}$, with $\mathcal{E}$ as in (17), namely, a p-NWE of (16). $\square$

We clarify that in general, the computation of $\mathbf{Q}(k)$ associated to each $\mathbf{A}(k)$ requires global information on the communication network. Therefore, this solution is suitable for the case of switching between a finite set of adjacency matrices, for which the associated matrices $\mathbf{Q}(k)$ can be computed offline.

Nevertheless, for some network structures the left PF eigenvector is known or it can be explicitly computed locally. If the matrix $A(k)$ is symmetric, hence doubly stochastic, or if each agent i knows the weight that its neighbours assign to the information it communicates, the i th component of $q(k)$ can be computed as $\lim_{t \rightarrow \infty} [A(k)^T]_i^t x = q_i(k)$, for any $x \in \mathbb{R}^N$ [26, Prop. 1.68]. Moreover, if each agent i has the same out and in degree, denoted by $d_i(k)$, and the weights in the adjacency matrix are chosen as $[A(k)]_{ij} = \frac{1}{d_i(k)}$, then the left PF eigenvector is $q(k) := \text{col}((d_i(k) / \sum_{j=1}^N d_j(k))_{i \in \mathcal{N}})$. In other words, in this case each agent must only know its out-degree to compute its component of $q(k)$.

IV. NUMERICAL SIMULATIONS

A. Synchronous/Asynchronous Friedkin and Johnsen Model

As mentioned in Section II, the problem studied in this article arises often in the field of opinion dynamics. Next, we consider the standard Friedkin and Johnsen model, introduced in [27]. The state $x_i(k)$ of

¹Equivalently, when the cost functions $\bar{f}_i$ s have a common minimizer.

Fig. 1. Comparison between the convergence of the Friedkin and Johnsen model subject to synchronous and asynchronous update. In the latter case the different type of updates (A1)–(A4) are considered. For a fair comparison, we have assumed that the synchronous dynamics update once every N time instants.

each player represents its opinion on n independent topics at time k . An opinion is represented with a value between 0 and 1, hence, $\Omega_i := [0, 1]^n$. The opinion $[x_i]_j = 1$ if agent i completely agrees on topic j , and 0 if it disagrees. Each agent is stubborn with respect to its initial opinion $x_i(0)$ and $\mu_i \in (0, 1]$ defines how much its opinion is bound to it. Namely, $\mu_i = 0$ represents a *fully stubborn* player, while $\mu_i = 1$ a *follower*. In the following, we present the evolution of the synchronous and asynchronous dynamics (7) and (11), respectively, where the cost function is as in (3).

We considered $N = 10$ agents discussing on $n = 3$ topics. The communication network and the weights that each agent assigns to the neighbours are randomly drawn, with the only constraint of satisfying Assumption 3. The initial condition is also a random vector $x(0) \in [0, 1]^{nN}$. Half of the players are somehow stubborn $\mu = 0.1$ and the remaining are more incline to follow the opinion of the others, i.e., $\mu = 0.5$. For the asynchronous dynamics, we consider the following three scenarios.

- A1) There is no delay in the information and the probability of update is uniform between the players, hence, $\bar{\varphi} = 0$ and $p_{\min} = 1/N = 0.1$.
- A2) There is no delayed information and the agents update with different rates, so $\bar{\varphi} = 0$ and $p_{\min} = 0.0191 \neq 1/N$. In particular, we consider the case of a sensible difference between the rates to highlight the contrast with (A1) and (A3).
- A3) We consider an uniform probability of update and a maximum delay of two time instants, i.e., $p_{\min} = 1/N = 0.1$ and $\bar{\varphi} = 2$. The values of the maximum delay is chosen in order to fulfil condition (12).
- A4) We consider the same setup as in (A3) but in this case with $\bar{\varphi} = 50$. Notice that our theoretical results do not guarantee the convergence of these dynamics.

In Fig. 1, it can be seen how the convergence of the synchronous dynamics (A1) and the asynchronous ones (A3) are similar. The delay affects the convergence speed if it exceeds the theoretical upper bound in (12), as in (A4). It is worthy to notice that, even a big delay does not seem to lead to instability, while it can produce a sequence that does not converge monotonically to the equilibrium (see the zoom in Fig. 1). The slowest convergence is obtained in case of a nonuniform rate of update, i.e., (A2). From the simulation it seems that the uniform update probability produces always the fastest convergence.

B. Time-Varying DeGroot Model With Bounded Confidence

In this section, we consider a modified time varying version of the DeGroot model [28], where each agent is willing to change its

Fig. 2. Three different communication networks between which the agents switch.

Fig. 3. Evolution of the opinion $x_i(k)$ of each agent $i \in \mathcal{N}$. The light blue top and bottom regions represents the local feasible sets of positive and negative extremists, respectively.

opinion up to a certain extent. This model can describe the presence of extremists in a discussion. In the DeGroot model, the cost function $\bar{f}_i$ in (1), reduces to $\bar{f}_i = \iota_{\Omega_i}$ for all $i \in \mathcal{N}$. We consider $N = 8$ agents discussing on one topic, hence, the state x_i is a scalar. The agents belong to three different categories, which we call: *positive extremists*, *negative extremists*, and *neutralists*. This classification is based on the local feasible decision set.

- 1) *Positive (negative) extremists*: the agents agree (disagree) with the topic and are not open to drastically change their opinion, i.e., $\Omega_i = [0.75, 1]$ ($\Omega_i = [0, 0.25]$).
- 2) *Neutralists*: the agents do not have a strong opinion on the topic, so their opinions vary from 0 to 1, i.e., $\Omega_i = [0, 1]$.

Our setup is composed of two negative extremists (agents 1 and 2), four neutralists (agents 3–6), and two positive extremists (agents 7 and 8). We assume that, at each time instant, the agents can switch between three different communication network topologies, depicted in Fig. 2. The dynamics in (18) can be rewritten, for a single agent $i \in \mathcal{N}$, as

$$x_i(k+1) = \text{proj}_{\Omega_i} ((1 - q_i(k))x_i(k) + q_i(k)[A(k)]_i x(k)).$$

From this formulation, it is clear how the modification of the original dynamics (16) results into an inertia of the players to change their opinion.

We have proven numerically that Assumption 6 is satisfied and that the unique p-NWE of the game is

$$\bar{x} = [0.25, 0.25, 0.44, 0.58, 0.49, 0.41, 0.75, 0.75]^T.$$

In Fig. 3, the evolution of the players' opinion are reported, as expected, the dynamics converge to the unique p-NWE $\bar{x}$.

V. CONCLUSION AND OUTLOOK

A. Conclusion

For the class of multiagent network games, proximal type dynamics converge, provided that the communication network is strongly connected. We proved that their asynchronous counterparts also converge, and that they are robust to bounded delayed information. If each agent

has the possibility to arbitrarily choose the communications weights with its neighbors, then proximal dynamics converge even if the communication network varies over time. When the problem at hand can be recast as a proximal type dynamics, the results achieved directly provide the solution of the game, as shown in the numerical examples proposed.

B. Outlook

Although we have analyzed several different setups, this topic still presents unsolved issues that can be explored in future research. In the following, we highlight the ones that we think are the most compelling.

In the case of a time-varying graph (see Section III-C) we have supposed that at every time instant the communication networks satisfies Standing Assumption 3. It can be interesting to weak this assumption by considering jointly connected communication networks, as done in [24].

One can notice that, if the adjacency matrix is irreducible, then it is also AVG in some weighted space (see [29, Prop. 1]). The characterization of this space may lead to an extension of the results presented, as long as the matrix weighting the space preserves the distributed nature of the dynamics.

Finally, an interesting extension of this work is to replace the communication network with a signed graphs. This structure arise in several applications, e.g., Altafini opinion dynamics model [30].

APPENDIX

A. Proof of Theorem 1

First, we introduce the following auxiliary lemma.

Lemma 2: Let $P \in \mathbb{R}^{N \times N}$, $[P]_{ij} := p_{ij}$ be a nonnegative matrix that satisfies

$$P\mathbf{1}_N = \mathbf{1}_N, \iff \sum_{j \in \mathcal{N}} p_{ij} = 1 \quad (19)$$

$$q^\top P \leq q^\top \iff \sum_{i \in \mathcal{N}} q_i p_{ij} \leq q_j \quad (20)$$

where $q \in \mathbb{R}_{>0}^N$. Then, P is NE in $\mathcal{H}_Q$, with $Q = \text{diag}(q)$. $\square$

Proof: To ease the notation we adopt $\sum_j(\cdot) := \sum_{j \in \mathcal{N}}(\cdot)$ and $\sum_{j \in \mathcal{K}}(\cdot) := \sum_{j \in \mathcal{N}} \sum_{k \in (1, \dots, j-1)}(\cdot)$ in the following.

Next, we exploit (20) to compute

$$\begin{aligned} x^\top P^\top Q P x - x^\top Q x &= \sum_i q_i \left(\sum_j p_{ij} x_j \right)^2 - \sum_j q_j x_j^2 \\ &\leq \sum_i q_i \left(\left(\sum_j p_{ij} x_j \right)^2 - \sum_j p_{ij} x_j^2 \right). \end{aligned} \quad (21)$$

Notice that for all $i \in \mathcal{N}$ it holds $\sum_j p_{ij} x_j^2 = \sum_j p_{ij} (\sum_k p_{ik} x_k)^2$, furthermore this implies that

$$\left(\sum_j p_{ij} x_j \right)^2 - \sum_j p_{ij} x_j^2 = - \sum_j \sum_{k < j} p_{ij} p_{ik} (x_j - x_k)^2. \quad (22)$$

Since the matrix P is supposed nonnegative and from (21) and (22), it follows that

$$x^\top P^\top Q P x - x^\top Q x \leq - \sum_j \sum_{k < j} p_{ij} p_{ik} (x_j - x_k)^2 \leq 0.$$

Therefore, P is NE in $\mathcal{H}_Q$ from [29, Lemma 3 (ii)]. $\blacksquare$

Proof of Lemma 1 i) A can be rewrite as $A = (1 - \eta)\text{Id} + \eta B$ with $\eta \in (1 - \underline{a}, 1)$, hence, $B \geq 0$ and is row stochastic. The graph

associated to B has the same edges of that to A , so it is strongly connected and B irreducible. The PF theorem for irreducible matrix ensures the existence of a vector q satisfying (20) for B . Therefore, Lemma 2 can be applied to B , implying that B is NE in $\mathcal{H}_Q$ where $Q = \text{diag}(q) \succ 0$. By construction, this implies that the linear operator A is η -AVG in the same space [16, Def. 4.33]. This directly implies that $A = A \otimes I_n$ is η -AVG in $\mathcal{H}_Q$, with $Q = Q \otimes I_n$.

ii) From point (i), we know that A is η -AVG in $\mathcal{H}_Q$. For all $i \in \mathcal{N}$, define $q_i := [Q]_{ii}$ and recall that $\text{prox}_{\bar{f}_i}$ is FNE in $\mathcal{H}_{I_n}$ since $\bar{f}_i$ is convex from Assumption 2. Hence, from [16, Prop. 4.4. (iv)] for all $x, y \in \mathbb{R}^n$, it holds that

$$\begin{aligned} \|\text{prox}_{\bar{f}_i}(x) - \text{prox}_{\bar{f}_i}(y)\|^2 \\ \leq \langle x - y, \text{prox}_{\bar{f}_i}(x) - \text{prox}_{\bar{f}_i}(y) \rangle. \end{aligned} \quad (23)$$

The following inequality shows that prox_f is FNE in $\mathcal{H}_Q$. For all $x = \text{col}(\{x_i\}_{i=1}^N)$, $y = \text{col}(\{y_i\}_{i=1}^N) \in \mathbb{R}^{nN}$

$$\begin{aligned} \|\text{prox}_f(x) - \text{prox}_f(y)\|_Q^2 \\ = \sum_{i=1}^N q_i \|\text{prox}_{\bar{f}_i}(x_i) - \text{prox}_{\bar{f}_i}(y_i)\|^2 \\ \leq \sum_{i=1}^N q_i \langle x_i - y_i, \text{prox}_{\bar{f}_i}(x_i) - \text{prox}_{\bar{f}_i}(y_i) \rangle \\ = \langle x - y, \text{prox}_f(x) - \text{prox}_f(y) \rangle_Q. \end{aligned}$$

The second step follows from (23). It follows from [16, Prop. 4.44] that the composition $\text{prox}_f \circ A$ is AVG in $\mathcal{H}_Q$ with constant $\frac{1}{2-\eta}$, since prox_f and A are, respectively, $\frac{1}{2}$ - and η -AVG in $\mathcal{H}_Q$.

If the matrix A is doubly stochastic, then its left PF eigenvector is $\mathbf{1}$. By applying the result just attained in (ii), we conclude that $\text{prox}_f \circ A$ is $\frac{1}{2-\eta}$ -AVG in $\mathcal{H}_I$. $\blacksquare$

Proof of Theorem 1 From Lemma 1(ii), we know that the operator $\text{prox}_f \circ A$ is AVG in $\mathcal{H}_Q$ with constant $\frac{1}{2-\eta}$ and $\eta > 1 - \underline{a}$. Therefore, the convergence of the iteration in (7) follows by [16, Prop. 5.16]. $\blacksquare$

Proof of Corollary 1 The convergence follows directly from [16, Th. 5.15] and the fact that the considered A is nonexpansive in the space $\mathcal{H}_Q$, by Lemma 2. $\blacksquare$

B. Proof of Section III-B

Since the dynamics in (13) are a special case of (11), we first prove Theorem 3 and successively derive the proof of Theorem 2 exploiting a similar reasoning.

Proof of Theorem 3 From Theorem 1, we know that the operator $T := \text{prox}_f \circ A$ is η -AVG in the space $\mathcal{H}_Q$ with $\eta \in (1 - \underline{a}, 1)$. Therefore, it can be written as $T = (1 - \eta)\text{Id} + \eta \bar{T}$ where $\bar{T}$ is a suitable NE operator in $\mathcal{H}_Q$. Notice that $\text{fix}(\bar{T}) = \text{fix}(T)$. Substituting this formulation of T in (13) leads to

$$x(k+1) = x(k) + \psi_k \eta \zeta_k (\bar{T} - \text{Id}) \hat{x}(k). \quad (24)$$

Since $\bar{T}$ is NE in $\mathcal{H}_Q$, [22, Lemmas 13 and 14] can be applied to the dynamics in (24). Therefore, if we choose $\psi_k \in (0, \frac{N p_{\min}}{(2\bar{\varphi} \sqrt{p_{\min}} + 1)(1 - \underline{a})})$, the sequence generated by the dynamics in (24) are bounded and converge almost surely to a point $\bar{x} \in \text{fix}(\bar{T}) = \text{fix}(\text{prox}_f \circ A)$. The proof is completed recalling that the set of fixed points of $\text{prox}_f \circ A$ coincides with the set of NWE, by Remark 1. $\blacksquare$

Proof of Theorem 2 It is a particular case of Theorem 3 where $\psi_k = 1$ for all k . $\blacksquare$

C. Proofs of Section III-C

First, we propose two preliminary lemmas

Lemma 3: Let $P := p \in \mathbb{R}^{N \times N}$ and $[P]_{ij} = p_{ij}$ be a nonnegative, row stochastic matrix. If there exists a vector $w \in \mathbb{R}_{\geq 0}^N$ such that $w^\top P = w^\top$, then the matrix

$$\bar{P} := I + \mu \text{diag}(w)(P - I) \quad (25)$$

where $0 \leq \mu \leq \frac{1}{\max_{i \in \mathcal{N}} (1 - p_{ii}) w_i}$, is nonnegative and doubly stochastic. If $\mu < \frac{1}{\max_{i \in \mathcal{N}} (1 - p_{ii}) w_i}$, then the diagonal elements of $\bar{P}$ are positive. $\square$

Proof: To prove the first part of the lemma we have to show that $\bar{P}\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^\top \bar{P} = \mathbf{1}^\top$, hence

$$\bar{P}\mathbf{1} = \mathbf{1} - \mu w + \mu \text{diag}(w)P\mathbf{1} = \mathbf{1}. \quad (26)$$

Analogously, $\mathbf{1}^\top \bar{P} = \mathbf{1}^\top - \mu w^\top + \mu w^\top P = \mathbf{1}^\top$, where the last equality is achieved using the properties of w . So, from the relations mentioned above, we conclude that $\bar{P}$ is doubly stochastic.

The diagonal elements of $\bar{P}$ are $\bar{p}_{ii} := 1 - \mu w_i + \mu w_i p_{ii}$, the off diagonal ones are instead $\bar{p}_{ij} := \mu w_i p_{ij}$, $\forall i, j \in \mathcal{N}$. From simple calculations, it follows that if $0 \leq \mu \leq \frac{1}{\max_{i \in \mathcal{N}} (1 - p_{ii}) w_i}$ then $\bar{P}$ is a nonnegative matrix.

If $\mu < \frac{1}{\max_{i \in \mathcal{N}} (1 - p_{ii}) w_i}$ then $\bar{p}_{ii} > 0$ for all $i \in \mathcal{N}$, hence, $\bar{P}$ is doubly stochastic, with positive diagonal elements. $\blacksquare$

We show that if A satisfies Standing Assumption 3, then (25) does not change the fixed points of A .

Lemma 4: Let the matrix A satisfies Standing Assumption 3 and Q as in Theorem 1. Then, the matrix

$$\bar{A} := I + Q(A - I) \quad (27)$$

is doubly stochastic with self-loops and the following statements hold:

- 1) $\text{fix}(\bar{A}) = \text{fix}(A)$;
- 2) $\bar{A}$ satisfies Standing Assumption 3;
- 3) $\bar{A}$ is χ -AVG in $\mathcal{H}$ with $\chi \in (\max_{i \in \mathcal{N}} (1 - a_{i,i}) q_i, 1)$. $\square$

Proof: (i) The definition in (27) is equivalent to $\bar{A} = (I - Q) + QA$. Therefore, $\text{fix}(\bar{A}) = \text{fix}(A)$, since $Q \succ 0$.

(ii) The matrix $\bar{A}$ is in the form of (25) with $\mu = 1$ and, since $1 < \frac{1}{\max_{i \in \mathcal{N}} (1 - a_{i,i}(k))} \leq \frac{1}{\max_{i \in \mathcal{N}} (1 - a_{i,i}(k)) q_i}$, it satisfies the assumption of Lemma 3, thus, $\bar{A}$ is doubly stochastic with self-loops. Finally, since A satisfies Standing Assumption 3, the graph defined by $\bar{A}$ is also strongly connected, therefore, also $\bar{A}$ satisfies Standing Assumption 3.

(iii) To prove that $\bar{A}$ is χ -AVG with $\chi \in (\max_{i \in \mathcal{N}} (1 - a_{i,i}) q_i, 1)$, we can also apply [2, Lemma 9] or use the same argument as in the proof of Lemma 1. $\blacksquare$

Proof of Theorem 4 From Corollary 4, each matrix $\bar{A} := I + \mu Q(k)(A(k) - I)$ is $(\mu(k) \max_{i \in \mathcal{N}} (1 - a_{i,i}(k)) q_i(k))$ -AVG in $\mathcal{H}_I$. Since prox_f is FNE in $\mathcal{H}_I$, the operator $\text{prox}_f \circ \bar{A}(k)$ is $\frac{1}{2 - \hat{\eta}(k)}$ where $\hat{\eta}(k) := \mu(k) \max_{i \in \mathcal{N}} (1 - a_{i,i}(k)) q_i(k)$, [16, Prop. 4.44]. All the operators used in the dynamics are AVG in the same space $\mathcal{H}_I$, therefore, the global convergence follows from [16, Cor. 5.19], since all the cluster point of $(x(k))_{k \in \mathbb{N}}$ lay in $\mathcal{E}$. $\blacksquare$

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