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DC power flow feasibility for constant-power loads

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DC Power flow feasibility

for constant-power loads

Mark Jeeninga

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The research described in this dissertation has been carried out at the Engineering and Technology Institute Groningen, and at the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, the Netherlands.

disc

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DC power flow feasibility for constant-power loads

Proefschrift

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Rijksuniversiteit Groningen
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to my parents,
Bert and Valantina

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Ah yes, the acknowledgments. Time to reminisce about the past 4+ years of PhD life. The individual nature of PhD projects in mathematics makes our work seem somewhat solitary. Luckily we in Groningen are blessed with some amazing research institutes and colleagues to support us in our individual quests, making us never feel truly alone.

I have always been convinced that one should always take on an opportunity to push one's boundaries, and never avoid a struggle. And so, at the end of the PhD project, part of me is glad that it has been a struggle at times, and that I have grown both professionally and personally. Spending many long nights to meet deadlines, contemplating new approaches, or perhaps wasting time while being too tired to realize it, have given me a strange feeling of enjoyment and achievement. However, some personal events have made the past years tough as well, which, from time to time, made me feel that I bit off more than I could chew. And then there was this thing that went viral despite it not being very popular, although I seem to have forgotten the details. The PhD journey has been quite eventful to say the least, but fortunately it all worked out in the end.

Now, with that out of the way, on to the actual acknowledgments!

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Mark Jeeninga

29 March 2021

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1 | Introduction

Arguably, the difference between tinkering and engineering is having a model. One of the earliest of such models is Kirchhoff's model for current flow in an electrical circuit, published in 1845, which sprouted the development of circuit theory and network analysis. Over the last two centuries, further developments in these topics have significantly improved the understanding and analysis of electricity, its models, and its engineering challenges. Together with the many breakthroughs in (power) engineering, these advancements have brought about immense prosperity, and undoubtedly helped shape the world we know today. A world without electricity is simply unthinkable.

And so, on the flip side, our reliance on these technologies is ever-increasing. Unfortunately, reports of (nation-wide) blackouts, frequency desynchronization, and other large-scale power grid failures are still making headlines today, albeit rarely. Such events are a reminder that new methods to improve the operation of the power grid are always necessary, both for large-scale and small-scale systems. In particular, a good understanding of the fundamental limits of a power grid is essential to guarantee its safe operation.

In the classical picture of a power grid, the electricity is generated at power plants, which rely on fossil fuels or nuclear energy, and are able to plan their energy production. Renewable energy sources, such as photo-voltaic cells, wind turbines, and hydroelectric dams, are a relatively new player in the power grid. In contrast, the production of renewable sources depend on weather conditions, and as a result, their expected production may depend on meteorological predictions. The inclusion of these sources therefore increase the uncertainty in the power generation, and hence asks for more flexibility from the power grid than the classical energy sources.

About half a century after Kirchhoff's 1845 paper marks the end of the war of the currents: the infamous conflict between Edison's direct current (DC) and Westinghouse's alternating current (AC) paradigm. With the war of the currents being won by Westinghouse, the vast majority of our power grid relies on the AC paradigm.

1.1 Control of DC power grids

In this thesis we study DC power grids. Small-scale DC power grids are found in the form of closed DC systems such as airplanes, ships, and spacecraft, to name a few. Over the past decades there has also been an increasing interest in using also DC power grids on a large scale, in particular in the context of microgrids [13, 37, 56, 63], and high-voltage direct current (HVDC) power transmission [64]. There exist many approaches to

model such power grids, for example by the Brayton-Moser framework [4, 12, 32], by a port-Hamiltonian framework [1, 5, 61], or, more elementary, by Kirchhoff's and Ohm's laws [63], to name a few.

The primary control objective of the controllers in these power grids is to distribute power from sources to loads. Secondary control objectives are also considered, such as current-sharing [13, 55] and power-sharing [15] among sources in the power grid. Various control design methodologies can be found in the literature, such as Lyapunov based approaches [15] and passivity based approaches [13, 32] amongst others, to achieve these control objectives.

Many different load models are studied in the literature. Common (time-independent) examples include (combinations of) constant-impedance, constant-current, constant-power, or more general exponential load models [23, 58]. The non-linear nature of the flow of power due to constant-power loads and exponential load models lead to the problem of power flow feasibility, as it may happen that there does not exist an equilibrium to the system dynamics that satisfies the demand of the loads. The purpose of this thesis is to study this phenomenon for constant-power loads.

1.2 Power flow feasibility

A power grid often has operational limits in their components, which should be considered in their modeling and operation. For example, power lines may have limits in terms of the currents flowing through them, and sources and loads can have limits in terms of the current injection, voltage potentials or power injection at their buses. Violations of such limits may lead to failures of these components.

A more fundamental and elusive limit of a power grid originates from the transportation of power in a power grid at steady state, which is referred to as *power flow*. We say that the power flow is *feasible* when the load demands in the power grid can be met by the supply of the sources. However, in certain situations the demands cannot be met by the supply, in which case we say that the power flow is *unfeasible*. This is for example the case when the total power injected in a power grid is less than the total power demand in the grid. The problem of power flow feasibility is also known as long-term voltage stability analysis [33].

In a power grid, control schemes are in place to stabilize the power grid and steer the system to an equilibrium state that satisfies the operational limits and such that the load demands are satisfied by the sources. However, if a power grid has unfeasible power flow, then there does not exist such an equilibrium state for which the load demands are satisfied by the sources. The result is that these control schemes may exhibit unintended and erratic behavior, and can violate operational limits. This may lead to phenomenon

such as voltage collapse, failure of power lines, failures at substations, and (rolling) blackouts.

Power flow feasibility is typically a load-driven phenomenon [58]. Power flow is therefore also known as *load flow*. In other words, the feasibility of power flow depends on the characteristics of the loads which are present. Typical (academic) examples of load models are constant-impedance, constant-current, and constant-power loads. Unfeasibility of the power flow can often be attributed to the presence of loads that behave like constant-power loads. Examples of such loads for DC power grids include tightly regulated power converters and motor drives [22]. Dealing with such constant-power loads has remained an important open problem in the literature.

In this work we study this fundamental limit of DC power grids, and more specifically we study the power flow feasibility of DC power grids with constant-power loads. It is noted that there are several types of power grids for which the power flow equations are equivalent to or well-approximated by the power flow of DC power grids with constant-power loads. Indeed, the paper [17] shows how the active power flow problem for a lossless AC power grid may be approximated by a DC power flow grid. The papers [51,52] show a similar result for the reactive power flow problem for a lossless AC power grid. See also [7,41] for other examples.

1.3 Operating points

The feasibility of power flow is captured by the so-called *power flow equation*. This equation match the power supply and demand at the load nodes, and is typically nonlinear. We refer to any feasible solution of this equation as an *operating point*, which represents an equilibrium state of the system. The nonlinearity of the power flow equation implies that no operating point may exist, in which case the power flow is unfeasible. On the other hand, if the power flow is feasible, then multiple operating points may exists. This inspires the question which operating point is the “correct” one to select, and for the control scheme to converge to.

There are several properties which make an operating point desirable to select. First, we like the operating point to minimize the dissipation in the power lines. Second, it is desirable for an operating point that the voltage potential are the largest among all operating points for each individual node. Such an operating point is called a *high-voltage solution*. Third, we prefer that a small increase of a constant-power load leads to a small decrease of the voltage potentials of the operating point, which is a property we refer to as *long-term voltage stability*. A fundamental question is whether or under which conditions there exist operating points which satisfy any or all of these desirable properties. Partial answers to this question were given in [41,52].

1.4 Thesis outline and contributions

In this thesis we are concerned with the power flow feasibility of DC power grids with constant-power loads. At its core, this thesis presents a “bottom-up” approach to study this feasibility problem, and is mostly self-contained. To achieve this, we have included Chapter 2, which collects a range of definitions and results in matrix theory, graph theory, and circuit theory that are necessary to study this problem.

The DC power flow problem is formally defined in Chapter 3. We furthermore show that, if the problem is feasible, there may exist multiple solutions to this problem, which are referred to as operating points. This motivates the problem of selecting the “correct” operating point. We discuss three desirable properties of such operating points. The definitions in this chapter will be used throughout the thesis, and are based on [27]. The chapter presents two academic examples which illustrate the difficulties of this feasibility problem, and includes a detailed analysis of a dynamical DC power grid to motivate the problem of power flow feasibility.

Chapter 4 forms the keystone of our analysis of the power flow problem. Its contributions are also found in [27]. In this chapter we give a full characterization of the constant power demands for which the power flow is feasible. We show that the three desirable properties of operating points belong to a unique operating point, which extends the work of [41, 52]. Moreover, we prove that there is a one-to-one correspondence between a feasible vector of power demands and a desirable operating point. We show that this operating point can be found by solving an initial value problem. The set of feasible power demands is shown to be closed and convex, and the feasibility of a constant-power load can be verified by a necessary and sufficient LMI condition. This result extends [3] by proving that the necessary condition in [3] is also sufficient. In this chapter we also obtain a parametrization of the set of desirable operating points.

In Chapter 5 we continue the analysis of Chapter 4, and consider the case where the constant power demands are nonnegative. This case corresponds to a more classical setup where loads cannot supply power to the grid. For such loads we derive an alternative necessary and sufficient condition, which is (to some extent) cheaper to compute than its counterpart in Chapter 4. In addition we derive a second parametrization of the set of desirable operating points, and establish a duality between this parametrization and the parametrization in Chapter 4. We also obtain two parametrizations of the boundary of the set of feasible power demands, a parametrization of the boundary of this set in the nonnegative orthant, and a geometric characterization of the nonnegative feasible power demands. These results are found in [28].

In Chapter 6 we return to the case where constant power demands can also be negative, and study sufficient conditions for power flow feasibility. The advantages of

such conditions are that they are cheaper to compute than, for example, the necessary and sufficient LMI condition from Chapter 4. We prove two sufficient conditions, and show that a vector of power demand is feasible if it is element-wise dominated by a feasible vector of power demands. Furthermore we show how our conditions generalize the sufficient conditions in [7, 52]. Moreover, we show how these conditions can be recovered from our conditions. The contents of this chapter are in [28] as well.

Chapter 7 deals with the feasibility of the interconnection of multiple DC microgrids with constant-power loads. A microgrid has the feature to connect and disconnect from the main power grid, and instead operate in “island mode”. We formulate a “plug-and-play” certificate, along with a test, to guarantee that the power flow of a power grid remains feasible after a new microgrid is connected. This certificate relies on the block Cholesky decomposition, and to facilitate this approach we introduce the concept of a virtual power grid. We show that, if the power flow in this virtual power grid is feasible, then the power flow of the physical power grid is also feasible. This approach was presented in [29].

Finally, in Chapter 8 we consider uncertainty in conductances of the line. First, we show that for DC power grids with constant-power loads, and at least two sources, there is a phenomenon akin to Braess’ paradox. In this context, the paradox states that, when a new line is added to a power grid, or when the conductance of a line in the power grid is increased, it can happen that the power flow is no longer feasible after this change. We also prove that this phenomenon is not specific to any particular power grid topology, and that it can occur in any power grid with at least two sources. In the second half of this chapter we consider power grids where the line conductances are not known, and only upper and lower bounds are known, in an attempt to combat our analogue of Braess’ paradox. We give a sufficient condition for the feasibility of the power flow in all power grids that satisfy the bounds on the line conductances. The contents of this chapter corresponds to [26].

1.5 Publications

Conference publications

- C3** M. Jeeninga, A.J. van der Schaft, C. De Persis, “Plug-and-play Solvability of the Power Flow Equations for Interconnected DC Microgrids with Constant Power Loads”, in *Proceedings of the IFAC World Congress 2020*, Berlin, Germany, 2020.
- C2** P. Wijnbergen, M. Jeeninga, S. Trenn, “On stabilizability of switched Differential Algebraic Equations”, in *Proceedings of the IFAC World Congress 2020*, Berlin, Germany, 2020.
- C1** M. Jeeninga, A.J. van der Schaft, C. De Persis, “Graph theoretic formulae for the determinant and adjugate of Matrices carrying Graph Structure”, in *Proceedings of the IFAC Workshop on Distributed Estimation and Control in Networked Systems*, Groningen, The Netherlands, pp. 259–264, 2018.

Peer-reviewed extended abstracts

- EA2** M. Jeeninga, C. De Persis, A.J. van der Schaft, “Characterizing all feasible power injections of DC power grids with constant-power loads”, *Extended abstract accepted for presentation at the International Symposium on Mathematical Theory of Networks and Systems*, Cambridge, United Kingdom, 2021.
- EA1** M. Jeeninga, C. De Persis, A.J. van der Schaft, “A Scheme for Computing the Transfer Function of Power Grids with Grounded Capacitors”, *Extended abstract for the International Symposium on Mathematical Theory of Networks and Systems*, Hong Kong, 2018.

Preprints

- P4** M. Jeeninga, C. De Persis, A.J. van der Schaft, “Braess’ paradox for power flow feasibility and parametric uncertainties in DC power grids with constant-power loads”, *under preparation*, 2021.
- P3** P. Wijnbergen, M. Jeeninga, B. Besselink, “Nonlinear spacing policies for vehicle platoons: a geometric approach to decentralized control”, *submitted to Systems & Control Letters*, 2020.
- P2** M. Jeeninga, C. De Persis, A.J. van der Schaft, “DC power grids with constant-power loads—Part II: nonnegative power demands, sufficient conditions for feasibility,

and high-voltage solutions”, *Provisionally accepted to IEEE Transactions on Automatic Control*, preprint available at *ArXiv:2011.09333*, 2020.

- P1** M. Jeeninga, C. De Persis, A.J. van der Schaft, “DC power grids with constant-power loads—Part I: A full characterization of power flow feasibility, long-term voltage stability and their correspondence”, *Provisionally accepted to IEEE Transactions on Automatic Control*, preprint available at *ArXiv:2010.01076*, 2020.

1.6 Notation

We denote by $\mathbb{R}$ the set of real numbers. All vectors and matrices are have real entries.

For a vector $x = \begin{pmatrix} x_1 & \cdots & x_k \end{pmatrix}^\top$ we define

$$[x] := \text{diag}(x_1, \dots, x_k).$$

We let $\mathbf{1}$ and $\mathbf{0}$ denote the all-ones and all-zeros vector, respectively, and let I denote the identity matrix. We let their dimensions follow from their context. We let e_i denote the i -th column of I .

All vector and matrix inequalities are taken to be element-wise. We write $x \not\leq y$ if $x \leq y$ and $x \neq y$.

We define $\mathbf{max}(a, b) \in \mathbb{R}^n$ as the vector obtained by taking the element-wise maximum of $a, b \in \mathbb{R}^n$, i.e.,

$$\mathbf{max}(a, b)_i := \max(a_i, b_i).$$

For a square matrix A we let $\rho(A)$ denote the spectral radius of A . The p -norm of a vector x is denoted by $\|x\|_p$.

The Schur complement of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to the principal submatrix D is denoted by

$$M/D := A - BD^{-1}C.$$

We define $\mathbf{n} := \{1, \dots, n\}$. The submatrix of an $n \times m$ matrix A with rows and columns indexed by $\alpha \subseteq \mathbf{n}$ and $\beta \subseteq \mathbf{m}$, respectively, is denoted by $A_{[\alpha, \beta]}$. The same notation $v_{[\alpha]}$ is used for subvectors of a vector v . We let α^c denote the set-theoretic complement of $\alpha \subseteq \mathbf{n}$ with respect to $\mathbf{n}$.

For a set $S \subseteq \mathbb{R}^n$, the notation $\text{int}(S)$, $\text{cl}(S)$, ∂S and $\text{conv}(S)$ is used for the interior, closure, boundary and convex hull of S , respectively.

2 | Matrices, graphs, and resistive circuits

The purpose of this chapter is to lay a theoretical foundation for the subsequent chapters. We start by listing several results in matrix theory and algebraic graph theory. Next, we look at resistive circuits from the perspective of graphs and matrices.

2.1 Matrix theory

This section gives an overview of the matrix theory that is used in this work, and is based on [24]. We refer to [24] for a more extensive treatment of this topic. On occasion we include a proof, for illustration or whenever the exact formulation is not found in the literature.

2.1.1 Irreducible matrices

Definition 2.1. A square matrix is *reducible* if it is permutation-similar to a block upper-triangular matrix. That is, $A \in \mathbb{R}^{n \times n}$ is reducible if there exists a nonempty $\alpha \subseteq n$ such that $A_{[\alpha, \alpha^c]} = 0$.

A square matrix is *irreducible* if it is not reducible. Put differently, $A \in \mathbb{R}^{n \times n}$ is irreducible if $A_{[\alpha, \alpha^c]} \neq 0$ for all nonempty $\alpha \subseteq n$.

Note that the (ir)reducibility of a matrix does not depend on its diagonal entries.

If a square matrix $A \in \mathbb{R}^{n \times n}$ is reducible, it is permutation similar to the matrix

$$\begin{pmatrix} A_{[\alpha_1, \alpha_1]} & \cdots & A_{[\alpha_1, \alpha_k]} \\ & \ddots & \vdots \\ 0 & & A_{[\alpha_k, \alpha_k]} \end{pmatrix}$$

where $\alpha_1, \dots, \alpha_k$ partition n , and where the matrices $A_{[\alpha_i, \alpha_i]}$ are irreducible. The matrices $A_{[\alpha_i, \alpha_i]}$ are principal submatrices of A , and are known as the *irreducible components* of A . The eigenvalues of A are the union of the eigenvalues of the irreducible components $A_{[\alpha_i, \alpha_i]}$.

2.1.2 Z-matrices

Definition 2.2. A *Z-matrix* is a square matrix with nonpositive off-diagonal entries. More formally, $A \in \mathbb{R}^{n \times n}$ is a Z-matrix if $A_{ij} \leq 0$ for all $i \neq j$.

A negated Z-matrix is also known as a Metzler matrix.

Definitions 2.2 and 2.1 imply the following characterization of the irreducible Z-matrices.

Lemma 2.3. *A square matrix $A \in \mathbb{R}^{n \times n}$ is an irreducible Z-matrix if and only if $A_{[\alpha, \alpha^c]} \not\leq 0$ for all nonempty $\alpha \subseteq n$.*

The proof is left as an exercise to the reader. Lemma 2.3 implies that the sum of irreducible Z-matrices is an irreducible Z-matrix.

Lemma 2.4. *A sum of irreducible Z-matrices is again an irreducible Z-matrix.*

Similar to irreducible matrices, a matrix is a Z-matrix irrespective of its diagonal entries. More generally we have the following result.

Lemma 2.5. *Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix, irreducible, or both. Let $v, w, u \in \mathbb{R}^n$ where $v, w > 0$, then $[v]A[w] + [u]$ is also a Z-matrix, irreducible, or both, respectively.*

The proof is left as an exercise to the reader.

At the center of this thesis lies the following result on irreducible Z-matrices. A proof is included since, frankly, we were not able to find a proof for this fact in the literature. The proof relies on the celebrated Perron-Frobenius Theorem [25].

Lemma 2.6. *Let A be an irreducible Z-matrix. There is a unique eigenvalue r of A with smallest (i.e., “most negative”) real part. The eigenvalue r is real and simple. A corresponding eigenvector v is unique up to scaling, and can be chosen such that $v > 0$.*

Proof. Let B be a nonnegative matrix and s a scalar such that $A = sI - B$. Irreducibility is independent of the diagonal elements of a matrix. Hence, since A is irreducible, so is B . Let $\rho(B)$ denote the spectral radius of B . By the Perron-Frobenius Theorem [25] (or see [24, Thm. 4.8]), $\rho(B)$ is a simple eigenvalue of B , and there exists a positive eigenvector v , unique up to scaling, so that $Bv = \rho(B)v$. Hence, $r = s - \rho(B)$ is a simple eigenvalue of A , where the corresponding eigenvector v is positive and unique up to scaling. $\square$

Throughout this work we refer to the eigenvalue r and eigenvector v in Lemma 2.6 of an irreducible Z-matrix as the *Perron root* and a *Perron vector*, respectively. Lemma 2.6 states that the Perron root is a strict lower-bound for the real part of any of the other eigenvalues of an irreducible Z-matrix.

Lemma 2.7. *The Perron root and Perron vector of an irreducible Z-matrix A are continuous in the entries of A .*

Proof. All eigenvalues, and any eigenvector corresponding to a simple eigenvalue, are continuous in the elements of the corresponding matrix (see [45, 3.1.2 - 3.1.3]). $\square$

Lemma 2.8. *Let A be a reducible Z-matrix. There is an eigenvalue r of A which is real and has smallest (i.e., “most negative”) real part.*

Proof. Each principal submatrix of a Z-matrix is again a Z-matrix. If a Z-matrix A is reducible, then each irreducible component $A_{[\alpha_i, \alpha_i]}$ of A is an irreducible Z-matrix which has a Perron root r_i . We let $r := \min_{i \in k} r_i$. Since the eigenvalues of A are the union of the eigenvalues of the irreducible components $A_{[\alpha_i, \alpha_i]}$, it follows from Lemma 2.6 that r is a real eigenvalue and is the eigenvalue with smallest (i.e., “most negative”) real part. $\square$

We refer to the eigenvalue described in Lemma 2.8 as the *Perron root* of the reducible Z-matrix. Hence Perron root of a Z-matrix (reducible or irreducible) is a tight lower-bound for the real part of an eigenvalue of that matrix.

2.1.3 Stable matrices

We use the following definitions for the stability of a matrix.

Definition 2.9. A matrix is *Hurwitz stable* (*Hurwitz anti-stable*) if all its eigenvalues have negative (positive) real part.

Hence, a matrix A is Hurwitz stable if $-A$ is Hurwitz anti-stable, and vice versa.

Definition 2.10. A matrix is *Hurwitz semi-stable* (*Hurwitz semi-anti-stable*) if all its nonzero eigenvalues have negative (positive) real part, and if (i) the matrix is nonsingular, or (ii) the matrix has zero as a semi-simple eigenvalue.

Any *Hurwitz stable* (*Hurwitz anti-stable*) is also *Hurwitz semi-stable* (*Hurwitz semi-anti-stable*). We remark that also other (inequivalent) definitions of a Hurwitz semi-stable can be found in the literature.

2.1.4 M-matrices

Definition 2.11. A square matrix A is an *M-matrix* if there exists a nonnegative matrix B and a scalar s such that $A = sI - B$ and $s \geq \rho(B)$.

Note from Definition 2.11 that the off-diagonal entries of an M-matrix are nonpositive. Hence, an M-matrix is also a Z-matrix. And so, an M-matrix has a Perron root. In addition, an irreducible M-matrix has Perron vector.

The Perron root of a Z-matrix determines if it is also an M-matrix.

Lemma 2.12 ([44]). *A Z-matrix is an M-matrix if and only if its Perron root is nonnegative.*

Proof. If the matrix is irreducible, then taking $r \geq 0$ in the proof of Lemma 2.6 implies that $s \geq \rho(B)$. If the matrix is reducible, the result follows similarly from studying the irreducible components of the matrix. $\square$

The singularity of an M-matrix is also determined by the Perron root.

Lemma 2.13 ([24, Thm. 5.8]). *An M-matrix is singular if and only if its Perron root is zero. Moreover, if an irreducible M-matrix is singular, its kernel is one-dimensional and is spanned by any Perron vector.*

Proof. Let r denote the Perron root of the M-matrix A . The Perron root r is a lower bound of all the real part of all other eigenvalues, and since it is nonnegative by Lemma 2.12, it follows that if r is positive, then the matrix is nonsingular since all eigenvalues of the matrix have positive real part. If r is zero, then A is singular since r is an eigenvalue of A . Moreover, if A is irreducible, let v denote a Perron vector of A , then $Av = rv = 0$, and hence v lies in the kernel of A . The kernel is spanned by v since r is a simple eigenvalue when A is irreducible. $\square$

Lemma 2.12 and Lemma 2.13 imply the following corollary.

Corollary 2.14. *An M-matrix is nonsingular if and only if its Perron root is positive.*

The inverse of an irreducible M-matrix is positive, which is a defining property of such matrices.

Lemma 2.15 ([24, Thm. 5.12]). *A nonsingular irreducible Z-matrix is an M-matrix if and only if its inverse is a positive matrix.*

There are many other equivalent properties for a Z-matrix to be an M-matrix—see for example [6, 24, 46]. Here we list a few of these equivalences.

Lemma 2.16 ([46]). *Let A be a nonsingular Z-matrix. The following are equivalent:*

1. *A is an M-matrix;*
2. *A is Hurwitz anti-stable;*
3. *There exists a vector $d > 0$ such that $A[d] + [d]A^\top$ is positive definite.*

The fact whether or not a symmetric Z-matrix is an M-matrix can also be determined by the definiteness of the matrix.

Lemma 2.17. *Let A be a symmetric Z-matrix. The following are equivalent:*

1. *A is an M-matrix;*

2. A is positive semi-definite;
3. A is Hurwitz semi-anti-stable.

Proof. If A is symmetric, then all its eigenvalues are real and semi-simple. Hence, A is positive semi-definite if and only if A is Hurwitz semi-anti-stable. The eigenvalues are nonnegative if and only if the Perron root is nonnegative, since the Perron root of a matrix is a tight lower bound for (the real part of) its eigenvalues. By Lemma 2.13, nonnegativity of the Perron root is equivalent to A being an M-matrix. $\square$

Similar to Lemma 2.5, an irreducible M-matrix is closed under certain operations concerning diagonal matrices.

Lemma 2.18 ([30]). *If A is an M-matrix, and $d > 0$, then $A[d]$ are $[d]A$ are M-matrices.*

Lemma 2.19. *If A is an irreducible M-matrix, and $d \geq 0$, then $A + [d]$ is a nonsingular irreducible M-matrix;*

Proof. If A is singular this follows from [36, Cor. 3.9]. If A is nonsingular, this is implied by [46, Thm. 1, A_3]. $\square$

2.1.5 Graphs and Laplacian matrices

In this work we consider undirected simple graphs with positive edge-weights, which we refer to simply as a graph. Formally, such graphs are defined as follows.

Definition 2.20. *An undirected weighted simple graph, from now on a graph, is a collection of n nodes and distinct unordered pairs $\{i, j\}$ of edges where $i, j \in \mathbf{n}$ and $i \neq j$, to which positive edge weights $w_{ij} = w_{ji} \in \mathbb{R}$ are assigned.*

We write $i \sim j$ if $\{i, j\}$ is an edge in the graph, and $i \not\sim j$ otherwise. We will occasionally also consider loopy graphs, which we formally define as follows.

Definition 2.21. *An undirected weighted graph with freely-weighted loops, from now on a loopy graph, is a collection of n nodes and distinct unordered pairs $\{i, j\}$ of edges where $i, j \in \mathbf{n}$, to which nonzero edge weights $w_{ij} = w_{ji} \in \mathbb{R}$ are assigned. These weights are positive if $i \neq j$.*

The difference between a graph and a loopy graph is that the latter may include edges that connect a node to itself. Such an edge is known as a *loop*. Note that any graph is a loopy graph, and any loopy graph without loops is a graph. For the purpose of this text we allow the weight of a loop to be negative, which is uncommon compared to the standard graph theory literature.

The Laplacian matrix

To each graph Γ we assign a matrix $L_\Gamma \in \mathbb{R}^{n \times n}$, called the *Laplacian matrix*, which is defined element-wise by

$$(L_\Gamma)_{ij} = \begin{cases} \sum_{k:k \sim i} w_{ik} & \text{if } i = j \\ -w_{ij} & \text{if } i \neq j \text{ and } i \sim j \\ 0 & \text{if } i \neq j \text{ and } i \not\sim j \end{cases}$$

The Laplacian matrix gives a complete description of a graph. The zero/nonzero structure of the off-diagonal entries of a Laplacian matrix encodes the edges in the graph. The nonzero entries also encode the weight of the edges, since $w_{ij} = -(L_\Gamma)_{ij}$. We extend the definition of w_{ij} and put $w_{ij} = 0$ if $i \not\sim j$, so that $w_{ij} = -(L_\Gamma)_{ij}$ holds for all $i \neq j$.

The Laplacian matrix is symmetric, and is an example of a Z-matrix. It satisfies

$$(L_\Gamma x)_i = \sum_{k:k \sim i} w_{ik}(x_i - x_k), \quad (2.1)$$

for some vector x , from which we observe that L_Γ is singular by taking $x = \mathbf{1}$. In fact, each symmetric Z-matrix A such that the rows sum to zero (i.e., $A\mathbf{1} = 0$) is a Laplacian matrix associated to some graph. Some calculation shows that

$$y^\top L_\Gamma x = \sum_{i,j:i > j, i \sim j} w_{ij}(x_i - x_j)(y_i - y_j), \quad (2.2)$$

for some vectors x and y . By taking $x = y$ we observe that

$$x^\top L_\Gamma x = \sum_{i,j:i > j, i \sim j} w_{ij}(x_i - x_j)^2 \geq 0, \quad (2.3)$$

and hence L_Γ is positive semi-definite.

We say that a pair of distinct nodes $i, j \in \mathbf{n}$ in a (loopy) graph are *path-connected* if there exists a path of edges between that connect the two nodes. Formally this means that we can find distinct nodes $i = v_0, \dots, v_k = j$ such that $v_{l-1} \sim v_l$ for all $l \in \mathbf{k}$. We say that a (loopy) graph is *connected* if any pair of distinct nodes are path-connected.

The connectedness of a graph can also be observed from the Laplacian matrix. Note from (2.1) that $(L_\Gamma x)_i$ vanishes if and only if x_i equals x_k for all $k \sim i$. Hence, x lies in the kernel of L_Γ if and only if x_i and x_j are equal whenever i and j are path-connected. It follows that Γ is connected if and only if the entries of any vector in the kernel of L_Γ are equal, which means that $\ker L_\Gamma = \text{span}\{\mathbf{1}\}$.

Whenever a graph is not connected we can partition the graph into *connected components*. Each connected component corresponds to a nonempty subset $\alpha \subseteq \mathbf{n}$, which satisfies that if $i \in \alpha$ then $j \in \alpha$ if and only if i and j are path-connected. The collection

of connected components $\alpha_1, \dots, \alpha_k$ partitions the node set $\mathbf{n}$. By definition there are no edges between two nodes in distinct connected components. This in particular means that $(L_\Gamma)_{[\alpha_i, \alpha_j]} = 0$ whenever $i \neq j$. It follows that L_Γ is permutation similar to the block-diagonal matrix

$$\begin{pmatrix} (L_\Gamma)_{[\alpha_1, \alpha_1]} & & 0 \\ & \ddots & \\ 0 & & (L_\Gamma)_{[\alpha_k, \alpha_k]} \end{pmatrix}, \quad (2.4)$$

and that L_Γ is reducible. Conversely, if L_Γ is reducible, then $(L_\Gamma)_{[\alpha, \alpha^c]} = 0$ some nonempty $\alpha \subseteq \mathbf{n}$, which means that any $i \in \alpha$ cannot be path-connect to a node $j \in \alpha^c$, and that the graph is not connected. This shows that L_Γ is irreducible if and only if Γ is connected.

The above observations are captured in the following lemma.

Lemma 2.22. *Consider a graph Γ and its Laplacian matrix L_Γ . The following are equivalent:*

1. *The graph Γ is connected;*
2. *The kernel of L_Γ is spanned by $\mathbb{1}$;*
3. *The matrix L_Γ is irreducible.*

So far we know that for a connected graph Γ the Laplacian matrix L_Γ is a singular irreducible Z-matrix with kernel $\text{span}\{\mathbb{1}\}$. It follows from Lemma 2.12 that L_Γ is a singular M-matrix, and that 0 and $\mathbb{1}$ are respectively the Perron root and a Perron vector of L_Γ .

The loopy Laplacian matrix and grounded Laplacian matrix

To each loopy graph Γ we assign a matrix $L_\Gamma \in \mathbb{R}^{n \times n}$, called the *loopy Laplacian matrix*, which is defined element-wise by

$$(L_\Gamma)_{ij} = \begin{cases} w_{ii} + \sum_{k: k \sim i} w_{ik} & \text{if } i = j \text{ and } i \sim i \\ \sum_{k: k \sim i} w_{ik} & \text{if } i = j \text{ and } i \not\sim i \\ -w_{ij} & \text{if } i \neq j \text{ and } i \sim j \\ 0 & \text{if } i \neq j \text{ and } i \not\sim j \end{cases},$$

and extends the definition of a Laplacian matrix. Note that L_Γ is a symmetric Z-matrix. For loopy graphs we use the same notion of path-connectedness and connectedness as for graphs. Recall that path-connectedness requires that the nodes $v_0, \dots, v_k$ are distinct, and so it follows that loops do not affect path-connectedness of nodes and connectedness of a loopy graph. Hence, adding or removing loops does not alter these properties. Similarly, recall that irreducibility of a matrix is independent of diagonal entries. Based on Lemma 2.22 and (2.7) we deduce the following.

Lemma 2.23. Consider a loopy graph Γ and its loopy Laplacian matrix L_Γ . The following are equivalent:

1. The loopy graph Γ is connected;
2. The kernel of $L_\Gamma - [L_\Gamma \mathbb{1}]$ is spanned by $\mathbb{1}$;
3. The matrix $L_\Gamma - [L_\Gamma \mathbb{1}]$ is irreducible;
4. The matrix L_Γ is irreducible.

A special class of the loopy Laplacian matrices are the grounded Laplacian matrices.

Definition 2.24. We refer to L_Γ as a *grounded Laplacian matrix* if each node in Γ either has a loop or is path-connected to a node with a loop (or both), and if the weights of all loops are positive.

Similar to (2.2), it can be verified that

$$y^\top L_\Gamma x = \sum_{i:i \sim i} w_{ii} x_i y_i + \sum_{i,j:i > j, i \sim j} w_{ij} (x_i - x_j)(y_i - y_j)$$

for vectors x, y . By taking $x = y$ it follows that

$$x^\top L_\Gamma x = \sum_{i:i \sim i} w_{ii} x_i^2 + \sum_{i,j:i > j, i \sim j} w_{ij} (x_i - x_j)^2. \quad (2.5)$$

From (2.5) we deduce that if L_Γ is a grounded Laplacian matrix then L_Γ is a positive definite M-matrix.

Lemma 2.25. A grounded Laplacian matrix L_Γ is a nonsingular M-matrix and is positive definite.

Proof. Let x be a vector such that $x^\top L_\Gamma x = 0$. It follows that both terms in (2.5) vanish. Since the first sum in (2.5) vanishes, we have $x_i = 0$ whenever node i has a loop. Since the second sum in (2.5) vanishes we have $x_i = x_j$ whenever $i \sim j$. Since, by our definition of a grounded Laplacian matrix, the nodes have a loop or are path-connected to a node with a loop, we deduce that $x = \mathbb{0}$. Hence L_Γ is positive definite. Lemma 2.17 implies that L_Γ is a nonsingular M-matrix. $\square$

The loopy Laplacian matrix gives a complete description of a loopy graph. Similar to the Laplacian matrix, the offdiagonal entries of L_Γ encode the edges between distinct nodes in the loopy graph. The vector $L_\Gamma \mathbb{1}$ encodes if there are loops in the loopy graph, along with their weights. Indeed, the loopy Laplacian matrix satisfies

$$(L_\Gamma x)_i = w_{ii} x_i + \sum_{k:k \sim i} w_{ik} (x_i - x_k) \quad (2.6)$$

for a vector x when $i \sim i$, in which case we have $w_{ii} = (L_\Gamma \mathbb{1})_i$. We put $w_{ii} = 0$ if $i \not\sim i$ so that $w_{ii} = (L_\Gamma \mathbb{1})_i$ holds for all i .

We may obtain a graph $\hat{\Gamma}$ from a loopy graph Γ by removing the loops in Γ . The corresponding Laplacian matrix is given by

$$L_{\hat{\Gamma}} = L_\Gamma - [L_\Gamma \mathbb{1}]. \quad (2.7)$$

Any symmetric Z-matrix A is the loopy Laplacian matrix of some loopy graph, where $A\mathbb{1}$ describe the (possibly negative) weights of the loops, and where the off-diagonal elements of A describe the edges between distinct nodes. Moreover, a symmetric Z-matrix A is a grounded Laplacian matrix of some loopy graph with at least one loop and positive loop weights if and only if $A_{[\sigma, \sigma]} \mathbb{1} \geq 0$ for each irreducible component $A_{[\sigma, \sigma]}$. In particular, we have the following result.

Lemma 2.26. *Let Γ be a connected graph. Each principal submatrix of L_Γ is a grounded Laplacian matrix, and is therefore a positive definite nonsingular M-matrix.*

Proof. The principal submatrix $(L_\Gamma)_{[\alpha, \alpha]}$ is a symmetric Z-matrix, and hence the loopy Laplacian matrix of some loopy graph $\hat{\Gamma}$. Let $\sigma \subseteq \alpha$ such that $(L_\Gamma)_{[\sigma, \sigma]}$ is an irreducible component of $(L_\Gamma)_{[\alpha, \alpha]}$. We examine the weights of the loops. Lemma 2.22 states that $\mathbb{1}$ spans the kernel of L_Γ , and so

$$0 = (L_\Gamma \mathbb{1})_{[\sigma]} = (L_\Gamma)_{[\sigma, \sigma]} \mathbb{1} + (L_\Gamma)_{[\sigma, \sigma^c]} \mathbb{1}.$$

Lemma 2.22 and Lemma 2.3 imply that $(L_\Gamma)_{[\sigma, \sigma^c]} \mathbb{1} \leq 0$, and so $(L_\Gamma)_{[\sigma, \sigma]} \mathbb{1} \geq 0$. It follows that $(L_\Gamma)_{[\sigma, \sigma]} \mathbb{1} \geq 0$, which means that each connected component of $\hat{\Gamma}$ has a loop, and all loops have a positive weight. Thus, by Lemma 2.25, $(L_\Gamma)_{[\alpha, \alpha]}$ is a positive definite nonsingular M-matrix. $\square$

2.2 Resistive circuits

In this section we give a brief introduction to resistive circuits. The contents of this section are based on [16, 18, 59, 60].

2.2.1 Current flow in resistive circuits

In this work we consider resistive circuits, which are networks of linear resistors. We follow a nodal approach to analyze the current flow (and power flow) in such systems. These systems are static since no dynamical components are present.

A resistive circuit is composed of n nodes that represent the points at which linear resistors are connected. We let G_{ij} denote the conductance of the resistor between nodes

i and j . A resistive circuit corresponds to a graph, where the edges and edge weights in the graph correspond to the resistors and their conductance, respectively. We again write $i \sim j$ if nodes i and j are connected by a resistor.

At each node i in the circuit we assign a *voltage potential* V_i , which is a positive scalar. Conceptually, the voltage potential at a node represents the potential difference between the node and ground. The voltage potentials at the nodes are collected in the vector $V \in \mathbb{R}^n$ where $V > 0$.

Throughout this text we consider *open* resistive circuits, where nodes can exchange current with external systems by adjusting the voltage potential at the node. At each node i we assign the *current injection* $\mathcal{I}_i$, which represents the currents that are injected into the circuit by an external system. That is, $\mathcal{I}_i$ is positive if the current at the node flows into the circuit, and negative if the current flows out the circuit. The current injection at the nodes are collected in the vector $\mathcal{I} \in \mathbb{R}^n$.

A node is a *boundary node* if it can exchange current with an external system, by controlling V_i . Such nodes are also known as *buses* in the literature. The current flow in the circuit is controlled by the voltage potentials at the boundary nodes. Conversely, a node is an *internal node* if it is not connected to an external system, in which case we have $\mathcal{I}_i = 0$, and V_i is determined by Kirchhoff's current law.

The relation between the voltage potentials and currents are described by Ohm's law and Kirchhoff's current law. Ohm's law dictates that the current flowing from node i to node j through the resistor between the nodes is equal to the product of the potential difference over the line and its conductance, and hence is given by

$$G_{ij}(V_i - V_j).$$

Kirchhoff's current law dictates that the total current flowing into a node is equal to the total current flowing outward the node. In light of the above, this means that

$$\mathcal{I}_i = \sum_{k:k \sim i} G_{ik}(V_i - V_k). \quad (2.8)$$

We observe that (2.8) defines a linear map from V to $\mathcal{I}$. The matrix which represents this linear map is denoted by $Y \in \mathbb{R}^{n \times n}$, and is known as the *Kirchhoff matrix*. By comparing (2.8) and (2.1) it becomes apparent that Y coincides with the Laplacian matrix of the graph that corresponds to the resistive circuit. Hence, the contents of Section 2.1.5 extends to Kirchhoff matrices and resistive circuits. In particular, the Kirchhoff matrix is given element-wise by

$$Y_{ij} = \begin{cases} \sum_{k:k \sim i} G_{ik} & \text{if } i = j \\ -G_{ij} & \text{if } i \neq j \text{ and } i \sim j \\ 0 & \text{if } i \neq j \text{ and } i \not\sim j \end{cases} \quad (2.9)$$

2.2.2 Open-circuit voltages and Kron reduction

In the remainder of this section we consider resistive circuits which are connected (as a graph). We proceed by showing how internal nodes may be eliminated from a circuit.

By putting (2.8) in vector form we obtain the nodal voltage-current relation

$$\mathcal{I} = YV. \quad (2.10)$$

We let $\beta \subseteq n$ denote the boundary nodes in the grid, whereas the internal nodes are denoted by β^c .

For internal nodes we have that $\mathcal{I}_i = 0$ for $i \in \beta^c$, which means that

$$0 = \mathcal{I}_{[\beta^c]} = (YV)_{[\beta^c]} = Y_{[\beta^c, \beta^c]} V_{[\beta^c]} + Y_{[\beta^c, \beta]} V_{[\beta]}. \quad (2.11)$$

Lemma 2.26 implies that $Y_{[\beta^c, \beta^c]}$ is nonsingular. Solving for $V_{[\beta^c]}$ in (2.11) therefore yields

$$V_{[\beta^c]} = -Y_{[\beta^c, \beta^c]}^{-1} Y_{[\beta^c, \beta]} V_{[\beta]}. \quad (2.12)$$

Definition 2.27. The *open-circuit voltages* are the voltage potentials at the internal nodes given by (2.12).

Recall that we have defined voltages potentials as positive scalars, and so we therefore need to verify that $V_{[\beta^c]}$ in (2.12) is a positive vector.

Lemma 2.28. *The open-circuit voltages are positive.*

Proof. Since the resistive circuit is connected, we know that Y is an irreducible Z-matrix, and so $Y_{[\beta^c, \beta]} \not\leq 0$ by Lemma 2.3. If $Y_{[\beta^c, \beta^c]}$ is irreducible, then $Y_{[\beta^c, \beta^c]}^{-1}$ is positive by Lemma 2.26 and Lemma 2.15. Since $V_{[\beta]}$ is positive we have $Y_{[\beta^c, \beta]} V_{[\beta]} \not\leq 0$. The product of a positive matrix and a nonpositive nonzero vector is negative, and it follows that $V_{[\beta^c]} > 0$ in (2.12). If $Y_{[\beta^c, \beta^c]}$ is reducible, let $\alpha_1, \dots, \alpha_k$ partition β^c such that $Y_{[\alpha_i, \alpha_i]}$ are the irreducible components of $Y_{[\beta^c, \beta^c]}$. Since $Y_{[\beta^c, \beta^c]}$ is symmetric, it is block diagonal and it follows from (2.12) that $V_{[\alpha_i]} = Y_{[\alpha_i, \alpha_i]}^{-1} Y_{[\alpha_i, \beta]} V_{[\beta]}$. Similar to the irreducible case we have that the inverse of $Y_{[\alpha_i, \alpha_i]}^{-1}$ is positive and that $Y_{[\alpha_i, \beta]} V_{[\beta]} \not\leq 0$. It follows that $V_{[\alpha_i]} > 0$ for all $i \in k$, and so $V_{[\beta^c]} > 0$. $\square$

From the above we conclude that the open-circuit voltages are the unique voltage potentials at the internal nodes such that $\mathcal{I}_{[\beta^c]} = 0$. Note that the open-circuit voltages are linear in the voltage potentials at the boundary nodes.

When we look at the rows of (2.10) corresponding to the boundary nodes and substitute (2.12), we obtain the voltage-current relation

$$\begin{aligned}\mathcal{I}_{[\beta]} &= (YV)_{[\beta]} = Y_{[\beta,\beta]}V_{[\beta]} + Y_{[\beta,\beta^c]}V_{[\beta^c]} \\ &= (Y_{[\beta,\beta]} - Y_{[\beta,\beta^c]}Y_{[\beta^c,\beta^c]}^{-1}Y_{[\beta^c,\beta]})V_{[\beta]} = Y/Y_{[\beta^c,\beta^c]}V_{[\beta]}.\end{aligned}$$

The Schur complement $Y/Y_{[\beta^c,\beta^c]}$ is again a Kirchhoff matrix of a connected resistive circuit [59], and represents the nodal voltage-current relation from the perspective of the boundary nodes. The above process of eliminating internal nodes is known as *Kron reduction*, and the circuit corresponding to $Y/Y_{[\beta^c,\beta^c]}$ is known as the *Kron-reduced circuit* corresponding to Y and the set of internal nodes β^c .

Definition 2.29. *Kron reduction* is the process of eliminating all internal nodes in an open resistive circuit, to obtain an alternative circuit of smaller dimension that has an equivalent nodal voltage-current relation from the perspective of the boundary nodes.

When studying open resistive circuit, the process of Kron reduction shows that it suffices to study circuits in which all nodes are boundary nodes. Hence, in the remainder of this work we will consider only such resistive circuits. Kron reduction itself, however, will remain instrumental in the analysis and interpretation of open resistive circuits.

2.2.3 Resistive shunts

One example of a static component that can be connected to an open circuit is the ground. Connecting a node i to the ground corresponds to setting $V_i = 0$. We refer to such a node as a *grounded node*. The resistor between a grounded node and an other node is known as a *resistive shunt*. If we let $\alpha \subseteq \mathbf{n}$ index the grounded nodes, then we have $V_{[\alpha]} = \mathbf{0}$, and it follows from (2.10) that

$$I_{[\alpha^c]} = (YV)_{[\alpha^c]} = Y_{[\alpha^c,\alpha^c]}V_{[\alpha^c]} + Y_{[\alpha^c,\alpha]}V_{[\alpha]} = Y_{[\alpha^c,\alpha^c]}V_{[\alpha^c]}. \quad (2.13)$$

Hence, the voltage-current relation of the non-grounded nodes is fully described by the matrix $Y_{[\alpha^c,\alpha^c]}$, which by Lemma 2.26 is a grounded Laplacian matrix.

Conversely, each grounded Laplacian matrix corresponds to a resistive circuit with resistive shunts, which motivates the name “grounded Laplacian matrix”. Indeed, consider a grounded Laplacian matrix L_Γ , then the vector $L_\Gamma \mathbf{1}$ corresponds to the loops in the graph Γ and their weights. Since L_Γ is a grounded Laplacian matrix there exists at least one loop in the graph and all loop weights are positive. This means that $L_\Gamma \mathbf{1} \gneq \mathbf{0}$. The matrix

$$\begin{pmatrix} L_\Gamma & -L_\Gamma \mathbf{1} \\ -(L_\Gamma \mathbf{1})^\top & \mathbf{1}^\top L_\Gamma \mathbf{1} \end{pmatrix} \quad (2.14)$$

is a Laplacian matrix corresponding to a graph, and has the grounded Laplacian matrix L_Γ as a principal submatrix. Similar to (2.13), the voltage-current relation $\mathcal{I} = L_\Gamma V$ corresponds to the resistive circuit described by (2.14), where the last row and column correspond to a grounded node.

The discussion above implies that increasing a diagonal entry of a (grounded) Laplacian matrix corresponds to adding a resistive shunt to the circuit, which results in a grounded Laplacian matrix. This is captured in the following lemma.

Lemma 2.30. *Consider a resistive circuit (with resistive shunts) and its (grounded) Laplacian matrix L_Γ . Let D be a nonnegative nonzero diagonal matrix. The matrix $L_\Gamma + D$ is a grounded Laplacian matrix corresponding to the resistive circuit obtained by adding a resistive shunt at node i with conductance D_{ii} for all i such that $D_{ii} \neq 0$.*

3 | The DC power flow problem with constant-power loads

Throughout this thesis we study DC power grids that consist of nodes (buses), which are either loads or sources, and are interconnected by power lines. Source nodes are voltage controlled buses which provide power to the power grid, and represent generators such as power plants. Load nodes are voltage controlled buses which generically extract power from the power grid. We are interested in the conditions under which the power demand of the load nodes are satisfied in the long-term time scale. For this reason we study the power balance at the load nodes at steady state, which is described by the power flow equations. Note that, at steady state, the line dynamics do not contribute to this power balance. We therefore model the DC power grid as a resistive circuit.

The aim of this chapter is to formally introduce the DC power flow problem, and to define and derive the necessary models and notions of its analysis. In addition, we give some pointers to the subjects which are studied in the subsequent chapters. We use the nodal analysis of resistive circuits that was introduced in Section 2.2 to model the DC power grids at steady state. The inclusion of operational limits complicates the study of power flow feasibility. Hence, for the sake of simplicity, we do not consider operational limits on lines, currents or voltage potentials in the power grid throughout this thesis. We will briefly revisit this choice in Section 4.7, where we study the inclusion of operational limits.

This chapter is structured as follows. In Section 3.1 we derive the steady-state model of power grid which interconnects the sources and loads. In Section 3.2 we introduce the model of the loads and formulate the DC power flow problem. Section 3.3 considers the solutions of this problem, and lists several properties we would like these solutions to have. Section 3.4 studies the maximal amount of power that can be transported over the grid, and characterizes when this is the case. The concepts of this chapter are illustrated by two academic examples, which are found in Section 3.5. We conclude the chapter in Section 3.6, which describes a DC power grid with dynamical components. Here we motivate why it is important to study the DC power flow problem, and show what could go wrong if the power flow in such a power grid is not feasible. We will refer back to the examples in Section 3.5 and Section 3.6 throughout this thesis.

3.1 The DC power grid model at steady state

We consider a DC power grid with n load nodes and m source nodes, which may be connected by (power) lines. We assume that the nodes and lines form a connected graph. Following Section 2.2, we write $i \sim j$ if there exists a line between node i and node j , and $i \not\sim j$ otherwise. The conductance of the line between node i and node j is denoted by $G_{ij} = G_{ji}$, which is a positive real number. The Kirchhoff matrix $Y \in \mathbb{R}^{(n+m) \times (n+m)}$ describes both the topology and the conductances of the lines in the power grid—see (2.9) for an element-wise description of Y .

We partition Y according to whether nodes are loads (L) or sources (S):

$$Y = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix}. \quad (3.1)$$

The matrices Y_{LL} and Y_{SS} are positive definite by Lemma 2.26. Following the same partition, let

$$V = \begin{pmatrix} V_L \\ V_S \end{pmatrix} \in \mathbb{R}^{n+m}$$

denote the vector of voltage potentials at the load and source nodes. All voltage potentials are assumed to be positive, and hence $V > 0$.

We let $\mathcal{I} \in \mathbb{R}^{n+m}$ denote the electric current injected into the power grid at the nodes. The power that a node provides to the power grid is given by the vector

$$P = [V]\mathcal{I} \in \mathbb{R}^{n+m}. \quad (3.2)$$

Kirchhoff's current law together with Ohm's law (2.10) state that

$$\begin{pmatrix} \mathcal{I}_L \\ \mathcal{I}_S \end{pmatrix} = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix} \begin{pmatrix} V_L \\ V_S \end{pmatrix}, \quad (3.3)$$

and therefore

$$\begin{pmatrix} P_L \\ P_S \end{pmatrix} = \begin{bmatrix} V_L \\ V_S \end{bmatrix} \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix} \begin{pmatrix} V_L \\ V_S \end{pmatrix}. \quad (3.4)$$

The *total dissipated power* in the lines is derived in [59] as

$$R(V_L, V_S) := V^\top Y V \geq 0. \quad (3.5)$$

Alternatively, it is observed from (2.3) that $V^\top Y V$ corresponds to the sum of power dissipated by the lines, which for each line is given by $G_{ij}(V_i - V_j)^2$. In addition, by summing over the rows of (3.4) we observe that

$$R(V_L, V_S) = \mathbf{1}^\top P_L + \mathbf{1}^\top P_S,$$

which means that the total dissipated power in the grid is equal to the sum of the total power injected at the loads and the total power injected at the sources.

3.2 The power flow problem

Throughout this thesis we consider V_L as a variable of the system, whereas the Kirchhoff matrix Y and the voltages at the sources V_S are fixed.¹ We therefore write $\mathcal{I}_L = \mathcal{I}_L(V_L)$ and $P_L = P_L(V_L)$. In particular we have

$$\mathcal{I}_L(V_L) = Y_{LL}V_L + Y_{LS}V_S \quad (3.6)$$

and

$$P_L(V_L) = [V_L]Y_{LL}V_L + [V_L]Y_{LS}V_S. \quad (3.7)$$

Two important notions for the power flow of a DC power grid are the *source-injected currents* and the *open-circuit voltages*, which we define next.

The source-injected currents

Definition 3.1. We define the *source-injected currents* by

$$\mathcal{I}_L^* := -Y_{LS}V_S = -\mathcal{I}_L(\mathbb{0}), \quad (3.8)$$

which correspond to the currents injected into the loads by the sources when $V_L = \mathbb{0}$.

Lemma 3.2. The source-injected currents $\mathcal{I}_L^*$ form a nonzero nonnegative vector (i.e., $\mathcal{I}_L^* \geq \mathbb{0}$).

Proof. The Kirchhoff matrix Y is an irreducible Z-matrix, and so $Y_{LS} \leq 0$ by Lemma 2.3. Since $V_S > \mathbb{0}$ we have $\mathcal{I}_L^* \geq 0$. $\square$

Remark 3.3. A natural interpretation of the source-injected currents follows from rewriting (3.6) into

$$\mathcal{I}_L(V_L) + \mathcal{I}_L^* = Y_{LL}V_L. \quad (3.9)$$

By comparing with (2.13), the voltage-current relation (3.9) of the load nodes corresponds to a resistive circuit with resistive shunts, described by the grounded Laplacian matrix Y_{LL} , with nonnegative nonzero current injections $\mathcal{I}_L^*$ from external current sources.

The open-circuit voltages

Recall that an internal node is a node at which the injected current is zero. When the power injection $P(V_L)_i$ of a load node i is zero, we have that also $I(V_L)_i = 0$ by (3.2) and since $(V_L)_i > 0$, in which case this node may be regarded as internal nodes. Similar to the open-circuit voltages of (2.12) and Definition 2.27 we define the following.

¹In Chapter 7 and Chapter 8 we consider variations in Y as well.

Definition 3.4. The *open-circuit voltages* V_L^* are the voltage potentials at the loads given by

$$V_L^* := -Y_{LL}^{-1}Y_{LS}V_S = Y_{LL}^{-1}\mathcal{I}_L^*. \quad (3.10)$$

The properties of the open-circuit voltages of Definition 2.27 carry over to the open-circuit voltages V_L^* , and hence the following holds.

Lemma 3.5. *The open-circuit voltages V_L^* are positive (i.e., $V_L^* > 0$), and are the unique voltage potentials at the loads such that $\mathcal{I}_L(V_L) = 0$, and are the unique voltage potentials at the loads such that $P_L(V_L) = 0$.*

Proof. Lemma 2.28 proves that $V_L^* > 0$, and in Section 2.2.2 it was argued that V_L^* is the unique vector of voltage potentials such that $\mathcal{I}_L(V_L) = 0$. Lastly, since $P_L(V_L) = [V_L]\mathcal{I}_L(V_L)$ and $V_L > 0$, it follows that $P_L(V_L) = 0$ if and only if $\mathcal{I}_L(V_L) = 0$. $\square$

If we substitute (3.10) into (3.4), we obtain the equation

$$P_L(V_L) = [V_L]Y_{LL}(V_L - V_L^*), \quad (3.11)$$

which describes the power injected into the load nodes, as a function of the voltage potentials at the loads, and in terms of the open-circuit voltages V_L^* and the grounded Laplacian matrix Y_{LL} .

In this thesis we consider *constant-power loads*, which is to say that each load demands a fixed quantity of power from the power grid. We let $P_c \in \mathbb{R}^n$ denote the vector of constant power demands at the loads. Note that we do not impose any sign restrictions on P_c and that, in principle, load nodes could also demand negative power, in which case the loads provide constant power to the grid.

By equating the power demand with the power injection at the load nodes, we obtain the power balance

$$P_L(V_L) + P_c = 0. \quad (3.12)$$

Note that power demand and power injection have opposite signs, and that indeed the vectors in (3.12) should be summed. The substitution of (3.11) in (3.12) yields the *DC power flow equation for constant-power loads*:

$$[V_L]Y_{LL}(V_L - V_L^*) + P_c = 0. \quad (3.13)$$

Definition 3.6. Given Y and V_S , we say that the power flow equations (3.13) are *feasible* for a vector of power demands P_c if there exists a vector of voltage potentials V_L which satisfies (3.13). We say that V_L is an *operating point* associated to P_c if V_L satisfies (3.13) for P_c .

Recall that throughout Definition 3.6 we require that $V_S > 0$ and $V_L > 0$.

Definition 3.7. We say that a vector of power demands P_c is *feasible* if (3.13) is feasible for P_c . The set of *feasible power demands* is given by

$$\begin{aligned}\mathcal{F} &:= \{ P_c \mid \text{Eq. (3.13) is feasible for } P_c \} \\ &= \{ P_c \mid \exists V_L > 0 \text{ such that } P_c = [V_L]Y_{LL}(V_L^* - V_L) \}.\end{aligned}\quad (3.14)$$

Since (3.13) is a quadratic equation in V_L , the existence of a solution to (3.13) for a given P_c is not guaranteed. Furthermore, (3.13) may have multiple solutions, and multiple operating points for a single P_c may exist.

Our goal is to characterize all constant power demands P_c such that (3.13) is feasible, which is precisely the set $\mathcal{F}$. This is formalized in the following problem statement.

Problem 3.8. Consider a DC power grid with Kirchhoff matrix Y and voltage potentials at the sources $V_S > 0$. Let V_L^* be given by (3.10). For which power demands at the loads $P_c \in \mathbb{R}^n$ does there exist an operating point $V_L > 0$ of voltage potentials at the loads which satisfies (3.13)?

We make a few observations. First, note that Problem 3.8 is not affected by the lines between sources, since the matrix Y_{SS} does not appear in (3.13). More specifically, since (3.13) only depends on Y_{LL} , V_L^* , P_L and V_L , it follows that Problem 3.8 only depends on Y_{LL} and V_L^* , or on Y_{LL} and $\mathcal{I}_L^*$ by (3.10). Second, if the graph formed by loads and the lines between them is not connected, then the matrix Y_{LL} is permutation similar to a block diagonal matrix with multiple blocks, as was observed in the proof of Lemma 2.28. It follows that (3.13) can be analyzed for each block separately. Hence, without loss of generality, we make the assumption that the graph formed by the loads and the lines between them is connected.

Assumption 3.9. The load nodes and the lines between loads form a connected graph, or equivalently by Lemma 2.23, the matrix Y_{LL} is irreducible.

3.3 Desirable operating points

For a feasible power demand P_c there may be multiple operating points which satisfy (3.13). We are generically interested in the following two criteria to determine a desirable operating point.

Long-term voltage stable operating points

First, we desire that the selected operating point is such that a small increase in a single power demand leads to a small decrease in all voltage potentials [52, 58]. Note that each

vector V_L of voltage potentials at the loads is associated by (3.13) to a vector of constant power demands P_c , given by

$$P_c(V_L) = [V_L]Y_{LL}(V_L^* - V_L). \quad (3.15)$$

We remark that (3.15) should be interpreted as the vector of constant power demand which are satisfied by the vector V_L of voltage potentials at steady state, and that constant power demands do not depend on the voltage potentials at the loads. By virtue of (3.15) the Jacobian of P_c at an operating point $\tilde{V}_L$, which is denoted by $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$, is well-defined and is given by

$$\frac{\partial P_c}{\partial V_L}(\tilde{V}_L) = [Y_{LL}(V_L^* - \tilde{V}_L)] - [\tilde{V}_L]Y_{LL}. \quad (3.16)$$

We use the following definition from [52].²

Definition 3.10. An operating point $\tilde{V}_L$ associated to $\tilde{P}_c$ is *long-term voltage stable* if the Jacobian of $P_c(V_L)$ at $\tilde{V}_L$ is nonsingular, and its inverse is a matrix with negative elements (i.e.,³ $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^{-1} = \frac{\partial V_L}{\partial P_c}(\tilde{P}_c) < 0$). The set of all long-term voltage stable operating points is defined by

$$\mathcal{D} := \left\{ V_L \left| \begin{array}{l} \exists \tilde{P}_c \text{ such that } V_L \text{ is a long-term voltage} \\ \text{stable operating point associated to } \tilde{P}_c \end{array} \right. \right\}.$$

We remark that there are many equivalent definitions and names for long-term voltage stability. For example, [52] refers to operating points described by Definition 3.10 as “locally voltage stable”, whereas [41] uses the term “voltage-regularity” for the same set. We refer to Remark 4.5 for a more detailed discussion. Long-term voltage stability also has a interpretation for certain DC power grids with dynamic components. Indeed, in Section 3.6 we present a simple model of a dynamical DC power grid with constant-power loads for which the long-term voltage stable operating points coincide with the stable equilibria of that power grid.

Similar to Definition 3.10 we define the notion of long-term voltage semi-stability:

Definition 3.11. An operating point $\tilde{V}_L$ associated to $\tilde{P}_c$ is *long-term voltage semi-stable* if for every $\varepsilon > 0$ there exists a long-term voltage stable operating point $\hat{V}_L$ associated to some $\hat{P}_c$ such that $\|\tilde{V}_L - \hat{V}_L\|_2 < \varepsilon$. Consequently, the set of all long-term voltage semi-stable operating points equals $\text{cl}(\mathcal{D})$, the closure of $\mathcal{D}$.

²In [52] this property is referred to as local voltage stability, whereas we prefer the term long-term voltage stability.

³The equality $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^{-1} = \frac{\partial V_L}{\partial P_c}(\tilde{P}_c)$ holds locally and follows from the Inverse Function Theorem, see e.g. [49].

We emphasize that it is *a priori* not clear that for each feasible P_c there exists a corresponding long-term voltage semi-stable operating point, or that such an operating point is unique to P_c .

Dissipation-minimizing operating points

Second, it is desirable that an operating point V_L associated to P_c minimizes $R(V_L, V_S)$, the total power dissipated in the lines.

Definition 3.12. Given P_c , an operating point $\tilde{V}_L$ associated to $\tilde{P}_c$ is *dissipation-minimizing* if for all operating points V_L associated to $\tilde{P}_c$ we have $R(\tilde{V}_L, V_S) \leq R(V_L, V_S)$. Put differently, $\tilde{V}_L$ is a solution to the optimization problem

$$\begin{aligned} & \underset{V_L > 0}{\text{minimize}} && R(V_L, V_S) \\ & \text{subject to} && P(V_L) = -P_c, \end{aligned} \tag{3.17}$$

where V_S is given.

For dissipation-minimizing operating points the following proposition applies.

Proposition 3.13. Given $\tilde{P}_c \in \mathcal{F}$, an operating point $\tilde{V}_L$ associated to $\tilde{P}_c$ is *dissipation-minimizing* if and only if it maximizes $V_L^\top \mathcal{I}_L^*$ among all operating points V_L associated to $\tilde{P}_c$, where $\mathcal{I}_L^*$ are the source-injected currents defined in (3.8).

Proof. We define $\mathcal{S}$ to be the set of all $V_L > 0$ which satisfies (3.13). We write the total dissipated power $R(V_L, V_S) = V^\top Y V$ in terms of the partitioning in (3.1) and substitute (3.10) and (3.8). This results in

$$\begin{aligned} R(V_L, V_S) &= V_L^\top (Y_{LL} V_L + Y_{LS} V_S) + V_S^\top (Y_{SS} V_S + Y_{SL} V_L) \\ &= V_L^\top Y_{LL} (V_L - V_L^*) + V_S^\top Y_{SS} V_S - V_L^\top \mathcal{I}_L^*. \end{aligned} \tag{3.18}$$

For any $V_L \in \mathcal{S}$, multiplying (3.13) by $\mathbf{1}^\top$ shows that

$$V_L^\top Y_{LL} (V_L - V_L^*) = -\mathbf{1}^\top P_c. \tag{3.19}$$

We are interested in minimizing $R(V_L, V_S)$ over all $V_L \in \mathcal{S}$. By using (3.18) and substituting (3.19), for all $V_L \in \mathcal{S}$ we have

$$R(V_L, V_S) = \underbrace{-\mathbf{1}^\top P_c + V_S^\top Y_{SS} V_S}_{\text{fixed}} - V_L^\top \mathcal{I}_L^*. \tag{3.20}$$

Only the last term in (3.20) depends on V_L , whereas the other terms are fixed. Thus, minimizing $R(V_L, V_S)$ over all $V_L \in \mathcal{S}$ is equivalent to maximizing $V_L^\top \mathcal{I}_L^*$ over all $V_L \in \mathcal{S}$. $\square$

It was shown in [41] that for each feasible vector of power demands there exists an operating point associated to a given P_c which element-wise dominates all other operating points associated to P_c . Such an operating point is referred to as the *high-voltage solution* to (3.13) (see also [52]). We formalize this notion by the following definition.

Definition 3.14. An operating point $\tilde{V}_L$ associated to the power demands P_c is a *high-voltage solution* if $\tilde{V}_L \geq V_L$ for all V_L associated to P_c , and is a *strict high-voltage solution* if $\tilde{V}_L > V_L$ for all $V_L \neq \tilde{V}_L$ associated to P_c .

It follows from Proposition 3.13 that a high-voltage solution is always dissipation-minimizing.

Corollary 3.15. *If the operating point $\tilde{V}_L$ associated to the power demands P_c is a high-voltage solution, then $\tilde{V}_L$ is dissipation-minimizing. Moreover, if $\tilde{V}_L$ is a strict high-voltage solution, then $\tilde{V}_L$ is the unique dissipation-minimizing operating point.*

Proof. Let V_L be an operating point associated to P_c . If $\tilde{V}_L$ is a high-voltage solution, then $\tilde{V}_L - V_L \geq 0$. We have $(\tilde{V}_L - V_L)^\top \mathcal{I}_L^* \geq 0$ since $\mathcal{I}_L^* \geq 0$ by Lemma 3.2, and hence $\tilde{V}_L^\top \mathcal{I}_L^* \geq V_L^\top \mathcal{I}_L^*$. Proposition 3.13 implies that $\tilde{V}_L$ is dissipation-minimizing. Similarly, if $\tilde{V}_L$ is a strict high-voltage solution, then $\tilde{V}_L - V_L > 0$. We have $(\tilde{V}_L - V_L)^\top \mathcal{I}_L^* > 0$ since $\mathcal{I}_L^* \geq 0$ and $\mathcal{I}_L^* \neq 0$ by Lemma 3.2, and hence $\tilde{V}_L^\top \mathcal{I}_L^* > V_L^\top \mathcal{I}_L^*$. Proposition 3.13 implies that $\tilde{V}_L$ is the unique dissipation-minimizing operating point. $\square$

Note that Definitions 3.10 and 3.11 describe local properties of an operating point, while Definitions 3.12 and 3.14 are global properties that concern all operating points associated to P_c . It is *a priori* not clear how dissipation-minimizing operating points and long-term voltage semi-stable operating points are related, nor is it clear when a feasible vector of power demands has a (possibly unique) long-term voltage semi-stable operating point or when it has a strict high-voltage solution. Some partial answers to these questions are known—see [41, 52]—but a full characterization is lacking. This fundamental question will be answered in Chapter 4. Indeed, in Section 4.6 we show that for each feasible vector of power demands there exists a unique long-term voltage semi-stable operating point associated to P_c , which is a strict high-voltage solution. This proves that the aforementioned notions coincide due to Corollary 3.15.

3.4 The maximizing feasible power demand

One measure for how well the power flows through a power grid is given by the maximal amount of power that the grid can transport. This quantity is defined as follows.

Definition 3.16. For a feasible power demand $P_c \in \mathcal{F}$, the *total feasible power demand* is the sum $\mathbb{1}^\top P_c$ of the power demands at the loads.

Definition 3.17. A *maximizing feasible power demand* is a feasible power demand $P_{\max} \in \mathcal{F}$ that maximizes the total feasible power demand. Thus for all $P_c \in \mathcal{F}$ it satisfies

$$\mathbb{1}^\top P_c \leq \mathbb{1}^\top P_{\max}.$$

The maximizing feasible power demand is the power demand for which the total power demand in the grid is maximized. There is a unique vector of power demands with this property. Hence we can speak of *the* maximizing feasible power demand.

Lemma 3.18. *There is a unique maximizing feasible power demand $P_{\max} \in \mathcal{F}$. It is given by*

$$P_{\max} = \frac{1}{4}[V_L^*]\mathcal{I}_L^* \succeq \mathbb{0}. \quad (3.21)$$

The unique operating point corresponding to $P_{\max}$ is $\frac{1}{2}V_L^$.*

Proof. Let P_c be feasible, and let V_L be an associated operating point. Recall from (3.19) that the total feasible power demand $\mathbb{1}^\top P_c$ satisfies

$$\mathbb{1}^\top P_c = -V_L^\top Y_{LL}(V_L - V_L^*).$$

By completing the squares we find that

$$\mathbb{1}^\top P_c = -(V_L - \frac{1}{2}V_L^*)^\top Y_{LL}(V_L - \frac{1}{2}V_L^*) + \frac{1}{4}V_L^{*\top} Y_{LL} V_L^*.$$

Since Y_{LL} is positive definite, it follows that

$$\mathbb{1}^\top P_c \leq \frac{1}{4}V_L^{*\top} Y_{LL} V_L^*, \quad (3.22)$$

with equality if and only if $V_L = \frac{1}{2}V_L^*$. This implies that equality in (3.22) holds if and only if

$$P_c = -P_L(\frac{1}{2}V_L^*) = \frac{1}{4}[V_L^*]Y_{LL}V_L^* = \frac{1}{4}[V_L^*]\mathcal{I}_L^*,$$

where we have substituted (3.10). The above implies that there is a unique $P_{\max}$ given by (3.21), and corresponds to the unique operating point $\frac{1}{2}V_L^*$. By Lemma 3.5 and Lemma 3.2 we have $V_L^* > \mathbb{0}$ and $\mathcal{I}_L^* \succeq \mathbb{0}$, which implies that $P_{\max} \succeq \mathbb{0}$. $\square$

Remark 3.19. If a load node i does not share a line with a source node, then the i -th row of Y_{LS} is zero, and $(\mathcal{I}_L^*)_i = (-Y_{LS}V_S)_i = 0$, meaning that there is no direct current injection from a source at load i . In such a case we have $P_{\max,i} = 0$ by (3.21).

Definition 3.17 and Lemma 3.18 are summarized in the following proposition.

Proposition 3.20. *If P_c is a vector of power demands such that the power flow is feasible, then*

$$\mathbb{1}^\top P_c \leq \mathbb{1}^\top P_{\max}, \quad (3.23)$$

with equality if and only if $P_c = P_{\max}$. In addition, the quantity $\mathbb{1}^\top P_{\max} = \frac{1}{4}V_L^{\top} Y_{LL} V_L^*$ is the maximal total power demand which can be supplied by the power grid.*

The inequality (3.23) describes a closed half-space in the space of power demands, and is a necessary condition for the feasibility of (3.13). This condition coincides with the inclusion

$$\mathcal{F} \subseteq \{ P_c \mid \mathbb{1}^\top P_c \leq \mathbb{1}^\top P_{\max} \}. \quad (3.24)$$

We observe that (3.24) generalizes (3.27) for $n \geq 1$. Since there is a unique maximizing feasible power demand by Lemma 3.18, equality in (3.23) only holds for $P_{\max}$, and the inclusion in (3.24) is strict for $n > 1$.

The converse of (3.23) states that, if P_c is such that $\mathbb{1}^\top P_c > \mathbb{1}^\top P_{\max}$, then no solution to (3.13) exists. The existence of $P_{\max}$ therefore shows that the DC power flow equations with constant-power loads are not always feasible.

3.5 Two academic examples

In this section we explore the intricacies of Problem 3.8 by considering two simple examples. We focus on building some intuition for the sets $\mathcal{F}$ and $\mathcal{D}$, which were defined in Definition 3.7 and Definition 3.10, respectively. We first consider the simplest case of a DC power grid with constant-power loads.

Example 3.21 (Single load case). Consider a DC power grid with a single load and a single source (*i.e.*, $n = m = 1$), as depicted in Figure 3.1. The corresponding graph of $P_L(V_L)$ is given in Figure 3.2. Figure 3.3 depicts the relation between P_c and V_L . In this example we let $V_S > 0$ be arbitrary and recall that the open-circuit voltages $V_L^* > 0$ are defined by (3.10). Since $n = 1$, it follows that (3.13) is scalar-valued. By taking (3.13) and completing the squares we find

$$Y_{LL}(V_L - \tfrac{1}{2}V_L^*)^2 - \tfrac{1}{4}Y_{LL}(V_L^*)^2 + P_c = 0.$$

Since $n = 1$, Y_{LL} is a positive scalar, and it follows that

$$V_L = \tfrac{1}{2}V_L^* \pm \sqrt{Y_{LL}^{-1}(\tfrac{1}{4}Y_{LL}(V_L^*)^2 - P_c)}. \quad (3.25)$$

We deduce from (3.25) that (3.13) has a real solution for $n = 1$ if and only if

$$P_c \leq \tfrac{1}{4}Y_{LL}(V_L^*)^2. \quad (3.26)$$

The set of all feasible power demands is therefore given by

$$\mathcal{F} = \{ P_c \mid P_c \leq \tfrac{1}{4}Y_{LL}(V_L^*)^2 \}. \quad (3.27)$$

If equality holds in (3.26), it follows from (3.25) that there is precisely one operating point, given by $V_L = \tfrac{1}{2}V_L^*$. In the case that (3.26) is strict, we see that there are two

Figure 3.1: A schematic depiction of a power grid with a single load node and a single source node ($n = m = 1$), where G_{12} is the conductance of the line between the nodes.

Figure 3.2: A plot of $P_L(V_L)$ against V_L for the power grid in Figure 3.1 with $G_{12} = 3 \Omega^{-1}$ and $V_S = V_2 = 1$ V. The red point indicates the voltage such that the power that the grid transports is maximized. The thick purple half-line corresponds to $\mathcal{D}$, the set of all long-term voltage stable operating points.

solutions, and that the positive branch of (3.25) leads to a higher voltage potential at the load. Hence, the positive branch of (3.25) is the high-voltage solution, which is also dissipation-minimizing by Corollary 3.15. In addition, the positive branch decreases when P_c increases, and so it is also the long-term voltage stable solution. We have $\mathcal{D} = \{ V_L \mid V_L > \frac{1}{2}V_L^* \}$.

In Example 3.21 we observe from (3.26) that (3.13) is not always feasible for each P_c for $n = 1$. By studying the maximal total amount of power that can be transported to the

Figure 3.3: A plot of V_L against P_c for the power grid in Figure 3.1 with $G_{12} = 3 \Omega^{-1}$ and $V_S = V_2 = 1$ V. The thick blue half-line corresponds to all feasible power demands at the load node. The red square indicates the maximal power that the load can drain from the power grid. There are multiple operating points if $0 < P_c < 0.75$. The Jacobian $\frac{\partial V_L}{\partial P_c}$ is not defined in the red point. The corresponding operating point is long-term voltage semi-stable, but not long-term voltage stable. The solid curve depicts a one-to-one correspondence between the feasible power demands and the long-term voltage semi-stable operating points.

Figure 3.4: A schematic depiction of a power grid with two load nodes ($n = 2, m = 1$).

load nodes, it was shown in (3.24) that the same is true for $n > 1$. This is illustrated for $n = 2$ in the following example, to which we will refer throughout this thesis.

Example 3.22 (two loads, one source case). Consider the DC power grid with two loads ($n = 2$) and one source ($m = 1$) depicted in Figure 3.4. Figure 3.5 gives the feasible power demands when we let $G_{13} = 3 \Omega^{-1}$, $G_{23} = 2 \Omega^{-1}$ and $V_S = 1$ V, and vary the conductance

Figure 3.5: Plots of $\mathcal{F}$, the set of feasible power demands, for the power grid in Example 3.22, which is depicted in Figure 3.4 where $G_{13} = 3 \Omega^{-1}$, $G_{23} = 2 \Omega^{-1}$ and $V_S = 1 \text{ V}$, and with different values of G_{12} . The dashed line is the set of the points for which equality in (3.23) holds.

G_{12} . It can be shown that $V_L^* = \mathbb{1} \text{ V}$.

First, we disregard the red line between node 1 and node 2 (*i.e.*, $1 \not\sim 2$), or equivalently take $G_{12} = 0 \Omega^{-1}$. The absence of the red line implies that Y_{LL} is (block) diagonal, and Problem 3.8 reduces to two copies of Example 3.21. From (3.26) it follows that P_c is feasible if and only if $P_{c,1} \leq 0.75$ and $P_{c,2} \leq 0.5$, which corresponds to the yellow rectangle in Figure 3.5. Next, we consider the red line between loads 1 and 2. We observe from the same figure that increasing G_{12} to $1 \Omega^{-1}$ and $2 \Omega^{-1}$ will result in a larger set of feasible power demands, as indicated by the green and blue areas, respectively. The dashed line are the points for which equality in (3.23) holds. We note that these sets lie below the dashed line, and intersect the line only at the point $P_{\max}$, which illustrates (3.24).

Figure 3.6 relates the sufficient conditions of [52] and [7] to the feasible power demands of the DC power grid depicted in Figure 3.4. Figures 3.5 and 3.6 suggest some properties of the set of feasible power demands $\mathcal{F}$, which we will prove in Chapter 4:

- The set $\mathcal{F}$ is convex (see also [19]);
- Each hyperplane which is tangent to the boundary of $\mathcal{F}$ (such as the dashed line in Figure 3.5) gives a necessary condition for feasibility (See also [3]);

Figure 3.6: A plot of the set $\mathcal{F}$ for the power grid in Example 3.22, which is depicted in Figure 3.4, where $G_{12} = 2 \Omega^{-1}$, $G_{13} = 3 \Omega^{-1}$, $G_{23} = 2 \Omega^{-1}$ and $V_S = 1 \text{ V}$. The yellow area is the set described by the sufficient condition in [52]. The red area is the sufficient condition from [7]. The boundaries of the two conditions intersect in the red point.

In addition, we observe the following properties, which we will prove in Chapter 6:

- If $y \in \mathcal{F}$ and $\hat{y} \leq y$, then also $\hat{y} \in \mathcal{F}$;
- The convex hull of the points on the boundary of $\mathcal{F}$ lead to a sufficient condition for a power demand to be feasible. In particular, the convex hull of $\mathbf{0}$, $P_{\max}$ and the points where the axes intersect the boundary of $\mathcal{F}$ forms a polyhedral subset of $\mathcal{F}$. The interior of this set describes the sufficient condition from [52]; See also Figure 3.6.
- The ball with the smallest radius such that it touches the boundary of the condition in [52] is contained in $\mathcal{F}$. This subset describes the sufficient condition of [3]; see again Figure 3.6.

We also observe in Figure 3.5 that $P_{\max}$ does not change when G_{12} is changed, and increasing G_{12} leads to nested⁴ sets $\mathcal{F}$ of feasible power demands. We remark that this is not true in general. This phenomenon is analyzed in Chapter 8, and is similar to the Braess' paradox.

⁴Nested with respect to inclusion.

3.6 A motivating example of a dynamical power grid

We close this chapter by considering a DC power grid with *dynamical* components. By using the definitions in this chapter we motivate why the steady-state analysis of DC power grids with constant-power loads is useful and necessary.

There is a multitude of power grid models for which the presence of constant-power loads requires us to consider Problem 1. We refer to [41, 52] for examples of such models. In this section, however, we consider a single, simple model of a power grid with dynamical components. Our goal is to get a flavor for the problems that may arise due to constant-power loads.

In this section we consider DC power grids with resistive and inductive lines. We let l denote the number on lines in the grid. The conductances and inductances of the lines are positive, and are collected in the vectors $G = (G_1 \ \dots \ G_l)^\top \in \mathbb{R}^l$ and $L = (L_1 \ \dots \ L_l)^\top \in \mathbb{R}^l$, respectively. The currents that flow through the lines are represented by the vector $\mathcal{I}^{\text{lines}} \in \mathbb{R}^l$. We let the positive flow direction in the lines be in accordance with the numbering of the nodes. Using the notation of Definition 2.20, we define the *incidence matrix* $B \in \{-1, 0, 1\}^{(n+m) \times l}$ of the grid in an element-wise fashion by

$$B_{ik} = \begin{cases} 1 & \text{if } k = \{i, j\} \text{ for some } j, \text{ and } i < j \\ -1 & \text{if } k = \{i, j\} \text{ for some } j, \text{ and } i > j, \\ 0 & \text{otherwise} \end{cases}$$

where we understand that the notation $k = \{i, j\}$ means that line k is the line between nodes i and j . Each load consists of a shunt capacitor and a constant-power load component. The shunt capacitances are positive and are collected in the vector $C = (C_1 \ \dots \ C_n)^\top \in \mathbb{R}^n$, and the constant-power demands are represented by $P_c \in \mathbb{R}^n$ as before. The voltage potentials over these components are given by the vector $V_L \in \mathbb{R}^n$. We regard V_S , the voltage potentials at the sources, as inputs to the system, and assume that they are constant. Figure 3.7 depicts an example of this RLC circuit with two loads and a single source.

The power grid of Figure 3.7 is a dynamical instance of the static power grid of Figure 3.4. The dynamics of the former power grid are given by the nonlinear system

$$\begin{cases} [L]\dot{\mathcal{I}}^{\text{lines}}(t) &= -[G]^{-1}\mathcal{I}^{\text{lines}}(t) - B_{[L,l]}^\top V_L(t) - B_{[S,l]}^\top V_S \\ [C]\dot{V}_L(t) &= B_{[L,l]}\mathcal{I}^{\text{lines}}(t) - [V_L(t)]^{-1}P_c \end{cases}. \quad (3.28)$$

Figure 3.7: A schematic depiction of a power grid with RL-lines, two loads with a constant-power component and a shunt capacitor, and a single voltage source.

At steady state we have $\dot{\mathcal{I}}^{\text{lines}} = \mathbb{0}$ and $\dot{V}_L = \mathbb{0}$, and by studying (3.28) this means that

$$\begin{aligned} \mathbb{0} &= B_{[L,l]}[G]B_{[L,l]}^\top V_L + B_{[L,l]}[G]B_{[S,l]}^\top V_S + [V_L]^{-1}P_c \\ &= (B[G]B^\top)_{LL}V_L + (B[G]B^\top)_{LS}V_S + [V_L]^{-1}P_c. \end{aligned} \quad (3.29)$$

It can be verified that $B[G]B^\top$ equals the Kirchhoff matrix Y associated to the power grid. By substituting this equality in (3.29) we deduce that

$$\mathbb{0} = Y_{LL}V_L + Y_{LS}V_S + [V_L]^{-1}P_c.$$

We multiply by $[V_L]$ and use the notation (3.10) of the open-circuit voltages to obtain

$$\mathbb{0} = [V_L]Y_{LL}(V_L - V_L^*) + P_c,$$

which coincides with the DC power flow equations (3.13). Hence, the feasibility of Problem 3.8 is necessary for the existence of an equilibrium of the system (3.28).

Next we investigate the stability of the equilibria. We do so by studying the linearization of (3.28), which is described by its Jacobian. The Jacobian J of (3.28) at an equilibrium $(\tilde{\mathcal{I}}^{\text{lines}\top} \quad \tilde{V}_L^\top)^\top$ is given by

$$J(\tilde{\mathcal{I}}^{\text{lines}}, \tilde{V}_L) := \begin{pmatrix} -[L]^{-1}[G]^{-1} & -[L]^{-1}B_{[L,l]}^\top \\ [C]^{-1}B_{[L,l]} & -[C]^{-1}[\tilde{V}_L]^{-2}[P_c] \end{pmatrix}. \quad (3.30)$$

We want to know when the Jacobian (3.30) is Hurwitz stable, in which case the equilibrium is locally asymptotically stable. Since the principal submatrix $-[L]^{-1}[G]^{-1}$ of J is Hurwitz stable, it follows by the Haynsworth inertia additivity formula [62, Section 0.10] that $J(\tilde{\mathcal{I}}^{\text{lines}}, \tilde{V}_L)$ is Hurwitz stable if and only if the matrix

$$[C]^{-1}[\tilde{V}_L]^{-2}[P_c] - [C]^{-1}B_{[L,l]}[G]B_{[L,l]}^\top = [C]^{-1}[\tilde{V}_L]^{-1} \left([\tilde{V}_L]^{-1}[P_c] - [\tilde{V}_L]Y_{LL} \right) \quad (3.31)$$

Figure 3.8: The trajectories of the power that is drained by the loads for the four cases in Example 3.23. The red and blue squares are the power demands in the different cases. The black circle is the maximizing feasible power demand. This point lies on the boundary of the set $\mathcal{F}$, the set of feasible power demands, which is depicted in blue and corresponds to the same sets in Figure 3.5 and Figure 3.6. The black square is the all-zero power demand.

is Hurwitz stable. Since $\tilde{V}_L$ is an equilibrium of the system, it satisfies

$$P_c = [\tilde{V}_L] Y_{LL} (V_L^* - \tilde{V}_L),$$

and substitution of this relation in (3.31) yields

$$[C]^{-1} [\tilde{V}_L]^{-1} \left([\tilde{V}_L] [Y_{LL} (V_L^* - \tilde{V}_L)] - [\tilde{V}_L] Y_{LL} \right) = [C]^{-1} [\tilde{V}_L]^{-1} \frac{\partial P_c}{\partial V_L} (\tilde{V}_L), \quad (3.32)$$

where we have also substituted (3.16). It will be shown in Proposition 4.3 and Remark 4.5 of Chapter 4 that (3.32) is Hurwitz stable if and only if $\tilde{V}_L$ is long-term voltage stable. In conclusion, this means that an equilibrium $\left(\tilde{\mathcal{I}}^{\text{lines}\top} \tilde{V}_L^\top \right)^\top$ of (3.28) is locally asymptotically stable if and only if $\tilde{V}_L$ is long-term voltage stable, and hence $\tilde{V}_L \in \mathcal{D}$.

Our analysis of (3.28) is illustrated by the following example, where we consider four instances of the power grid in Figure 3.7. We relate the power flow feasibility of the steady-state power grid to the existence of and convergence to an equilibrium of the dynamics (3.28).

Example 3.23. We consider the power grid depicted in Figure 3.7, where we the lines have inductance $L_1 = L_2 = L_3 = 1$ H, and conductances $G_1 = G_{12} = 2 \Omega^{-1}$, $G_2 = G_{13} = 3 \Omega^{-1}$, $G_3 = G_{23} = 2 \Omega^{-1}$. Furthermore we let $V_S = 1$ V. Note that the source voltages and the conductances of the power grid coincide with the power grid corresponding to Figure 3.6.

Figure 3.9: The trajectories of the voltage potential differences over the loads for the four cases in Example 3.23. The blue square corresponds to the desired equilibrium for Case 2–4, which lies in $\mathcal{D}$ and is therefore a long-term voltage stable operating point. The black square corresponds to the open-circuit voltages for which the power drain is zero, and the black point corresponds to the voltage potentials corresponding to the maximizing feasible power demand $P_{\max}$ (c.f. Figure 3.8).

The same is true for Figure 3.5 with $G_{12} = 2 \Omega^{-1}$. Hence, the set $\mathcal{F}$ of feasible power demands of these power grids coincide.

We consider four cases in which we choose different values for the capacitance and constant-power component of both loads. The dynamics of the power grid are described by (3.28), and we consider the initial conditions $\mathcal{I}(0) = [G]B_{[L,l]}^\top V_L(t) + [G]B_{[S,l]}^\top V_S$ and $V_L(0) = V_L^*$, which corresponds to an equilibrium if $P_c = 0$. This choice of initial conditions means that (3.28) models the sudden activation of the constant-power load.

Case 1: We choose $P_c = \begin{pmatrix} 0.8 & 0.6 \end{pmatrix}^\top$ W and $C_1 = C_2 = 8$ F. The vector P_c lies outside $\mathcal{F}$ (see the red square in Figure 3.8), meaning that Problem 3.8 is not feasible, and that the system (3.28) has no equilibria. The trajectories of the voltage potentials and power drains at the loads are depicted in red in Figures 3.8, 3.9 and 3.10. We observe in Figure 3.9 and Figure 3.10 that the voltage potentials drop rapidly and do not converge to an equilibrium. This phenomena is known as *voltage drop* [58].

Case 2: We lower the constant power demands by choosing $P_c = \begin{pmatrix} 0.6 & 0.5 \end{pmatrix}^\top$ W, and keep $C_1 = C_2 = 8$ F. The vector P_c lies inside $\mathcal{F}$ (see the blue square Figure 3.8), meaning that Problem 3.8 is feasible, and that the system (3.28) has an equilibrium. The corresponding trajectory are depicted in green in Figures 3.8, 3.9 and 3.11. We

Figure 3.10: The voltage potential differences V_L over the loads over time in Case 1 and Case 4 of Example 3.23. We observe that $V_{L,1}$ rapidly drops to zero in both cases, which is known as voltage drop.

observe that the trajectories converge to an equilibrium, depicted in blue. Note that this equilibrium lies in $\mathcal{D}$, which means that it is a long-term voltage stable operating point.

Case 3: We keep the constant power demands $P_c = \begin{pmatrix} 0.6 & 0.5 \end{pmatrix}^\top$ W, and lower the shunt capacitances by choosing $C_1 = C_2 = 6$ F. The equilibria are the same as in Case 2. The corresponding trajectories are depicted in magenta in Figures 3.8, 3.9 and 3.11. We observe in Figure 3.11 that the decrease in capacitance has decreased the dampening of the oscillations, but that the system still converges. Note that the trajectory of V_L in Figure 3.9 does not have to stay within $\mathcal{D}$ to converge to a point in $\mathcal{D}$.

Case 4: We keep the constant power demands $P_c = \begin{pmatrix} 0.6 & 0.5 \end{pmatrix}^\top$ W, and lower the shunt capacitances further, by choosing $C_1 = C_2 = 4$ F. The equilibria are the same as in Case 2 and Case 3. The corresponding trajectory are depicted in yellow in Figures 3.8, 3.9 and 3.10. We observe in Figure 3.11 that voltage drop occurs, even though there exists an equilibrium of the system. The further decrease of the dampening by the capacitances resulted in the destabilization of the power grid.

From Cases 2–4 in Example 3.23 we observe that decreasing shunt capacitances C_i can destroy the convergence to an equilibrium and cause voltage drop. Similarly, it can occur that an increase in the inductances of the lines destabilizes the power grid. This shows that voltage drop not equivalent to the non-existence of an equilibrium to (3.28).

Recall that we have assumed that the voltage potentials V_S at the sources are constant. In a more practical setting one may consider a proper control scheme with V_S as inputs

Figure 3.11: The voltage potential differences V_L over the loads over time in Case 2 and Case 3 of Example 3.23. We observe that V_L converges in both cases to the desired equilibrium, which is depicted in blue. Due to the decreased dampening in Case 3 we observe more extreme oscillations.

to guarantee the convergence to a (desirable) equilibrium, if it exists. Other control objectives may also be considered, such as power sharing and current sharing among the nodes. Furthermore, one may consider controllers at the loads as well. The topic of designing such control strategies, both for the system (3.28) and other dynamical DC power grid, is beyond the scope of this thesis. We refer the reader to [?, 13] Nevertheless, at steady state it is necessary that Problem 3.8 is feasible to ensure the existence of an equilibrium, and it is the hope of the author is that the contents of this thesis might aid in the development of such control strategies to ensure that Problem 3.8 is feasible.

4 | A geometric framework for power flow feasibility

In this chapter we develop a geometric framework for DC power flow feasibility for constant-power loads, which unifies known results on power flow feasibility in the literature. The novelty of our approach is that we combine results in matrix theory, convex analysis, and initial value problems to analyze DC power flow. In contrast to other approaches, we do not explicitly rely on fixed point analysis or iterative methods.

The main objective of this chapter is to prove that there is a one-to-one correspondence between $\mathcal{F}$ and $\mathcal{D}$, and to give a method to compute the desired operating point. In addition, we prove that $\mathcal{F}$ is convex and present a full geometric characterization of $\mathcal{F}$ as an intersection of half-spaces. Moreover, we prove a necessary and sufficient LMI condition for the feasibility of a vector of power demands, and a similar condition for feasibility under small perturbation. To establish these results, we derive a parametrization of the set $\mathcal{D}$ of long-term voltage stable operating points. Finally, we show that the notions of desirable operating points defined in Section 3.3 describe one and the same operating point.

The main results of this chapter are summarized as follows.

- M1.** We give a parametrization of $\mathcal{D}$, its closure and its boundary, which establishes a constructive method to describe the long-term voltage (semi-)stable operating points (Theorem 4.7).
- M2.** For each vector of power demand that lies on the boundary of $\mathcal{F}$ there exists a unique corresponding operating point which solves the power flow equations. Moreover, these operating points form the boundary of $\mathcal{D}$ (Corollary 4.20).
- M3.** There is a one-to-one correspondence between the feasible power demands $\mathcal{F}$ and the long-term voltage semi-stable operating points $\text{cl}(\mathcal{D})$. This means that if the power flow equations are feasible, then there exists a unique long-term voltage semi-stable operating point that solves the power flow equations. This operating point can be found by solving an initial value problem (Theorem 4.17).
- M4.** We give a novel and insightful proof for the fact that the set $\mathcal{F}$ is closed and convex. Consequently, $\mathcal{F}$ is the intersection of all supporting half-spaces of $\mathcal{F}$. We describe all such half-spaces, which gives a complete geometric characterization of $\mathcal{F}$ (Theorem 4.18).

- M5.** We prove a necessary and sufficient LMI condition for the feasibility of the power flow equations, and a necessary and sufficient LMI condition for the feasibility of the power flow equations under small perturbations of the power demands (Theorem 4.22).
- M6.** We show that the long-term voltage stable operating point is a strict high-voltage solution (Theorem 4.23). Consequently, the operating points associated to a feasible power demand which are either long-term voltage stable, a high-voltage solution, or dissipation-minimizing, are one and the same (Theorem 4.25).

Throughout this chapter we will refer to these main results by their labels **M1**, **M2**, etc.

Relation to the current literature

Before we begin our analysis, we explain how these results are related to the literature, and to what extent these results are, to the best of the authors' knowledge, novel.

Regarding **M3**, it was shown in [41] that if the power flow equations are feasible, then there “almost surely” exists an operating point which is long-term voltage stable, and that it is the unique long-term voltage stable operating point if all power demands are positive, or all are negative. By studying long-term voltage *semi*-stable operating points, we show that for each feasible vector of power demands there always exists a unique long-term semi-stable operating point.

Regarding **M4**, the convexity of $\mathcal{F}$ was already shown in [19] (see also [20,21]), and follows from an analysis of the convexity of the numerical range of non-homogeneous quadratic maps. Our approach to prove convexity is different from and less general than the one proposed by [19], and is a byproduct of the proof of **M3**, the one-to-one correspondence between $\mathcal{F}$ and $\text{cl}(\mathcal{D})$. We believe our proof for convexity to be simpler.

Regarding **M5**, our contribution is a necessary and sufficient condition for the feasibility of power demands under small perturbations. In [3] a similar condition was shown to be sufficient for a more general system with constant-power loads at steady state. It was shown in [3] to also be necessary whenever $\mathcal{F}$ is closed convex, as is the case here.

Regarding **M6**, it was shown in [41] that, if the power flow equations are feasible, then there exists a high-voltage solution, *i.e.*, an operating point that element-wise dominates all other operating points which satisfy the power flow equations, and that this operating point is “almost surely” long-term voltage stable. We show that the element-wise domination is strict, and that this operating point always coincides with the unique long-term voltage semi-stable operating point. This shows the algorithm proposed in [41] converges to the unique long-term voltage semi-stable operating point wherever the power flow equations are feasible.

Organization of this chapter

This chapter is structured as follows.

In Section 4.1 we show that every operating point is uniquely associated to a Z-matrix: the Jacobian of P_c at that operating point. Moreover, we show that an operating point is long-term voltage stable if and only if the Jacobian of P_c is a nonsingular M-matrix.

Section 4.2 uses this characterization to obtain a parametrization of the set of long-term voltage stable operating points $\mathcal{D}$ (M1). In particular, we parametrize the boundary of $\mathcal{D}$ by a set Λ_1 .

In Section 4.3 we study the convex hull of $\mathcal{F}$ and show that Λ_1 also parametrizes the boundary of $\text{conv}(\mathcal{F})$. This establishes a one-to-one correspondence between the boundary of $\mathcal{D}$ and the boundary of $\text{conv}(\mathcal{F})$ (M2).

Our main results are stated in Section 4.4, where we prove that for each feasible power demand there exists a unique long-term voltage semi-stable operating point (M3). In addition we present an explicit method for computing this operating point, and show that the set of feasible power demands is closed and convex (M4).

In Section 4.5 we prove that the LMI condition in [3] is necessary and sufficient for the feasibility of a vector of power demands, and present a similar necessary and sufficient LMI for the feasibility of a vector of power demands under small perturbation (M5).

Section 4.6 focuses on the long-term voltage semi-stable operating points. We show that any such operating point is a strict high-voltage solution. As a consequence, the notions of long-term voltage stable operation points, dissipation-minimizing operation points and (strict) high-voltage solutions coincide (M6).

Section 4.7 revisits the assumption that there are no operational limits for the DC power grids, and studies how such limits may be included and how they affect the power flow feasibility problem.

Section 4.8 concludes the chapter and lists some suggestions for future research.

A generalization of the results in this chapter

Remark 4.1. All results presented in this chapter rely solely on the fact that Y_{LL} is an irreducible nonsingular M-matrix (see Lemma 2.25 and Assumption 3.9) and that the source-injected currents $\mathcal{I}_L^*$ are nonzero and nonnegative (see Lemma 3.2). This means that the results in this chapter also hold for the more general case where Y_{LL} is substituted for any nonsingular irreducible (possibly non-symmetric) M-matrix, and $\mathcal{I}_L^*$ is substituted for any nonzero nonnegative vector. In this general setting we are able to characterize the power flow feasibility of DC power grids with a broad class of the so-called ZIP-loads, which are loads with a constant-impedance, constant-current and constant-power component. We elaborate on this in the final section of this chapter.

We have chosen not to present the contents of this chapter in terms of this more general setting, in favor of making the notation and interpretation of the formulas and results coherent with the other chapters of the thesis. In particular we will refer back to properties of the maximizing feasible power demand (see Section 3.4) to illustrate our methods. An analogue of the maximizing power demand need not exist in this more general case, since (the analogue of) the maximal total feasible power demand may be unbounded. We emphasize that this generalization applies to results in this chapter only.

4.1 Relating operating points to the Jacobian of P_c

The Jacobian of P_c at $\tilde{V}_L$ is given by

$$\frac{\partial P_c}{\partial V_L}(\tilde{V}_L) = [Y_{LL}(V_L^* - \tilde{V}_L)] - [\tilde{V}_L]Y_{LL}. \quad (4.1)$$

Recall that a vector $\tilde{V}_L$ qualifies as a vector of voltage potentials only if $\tilde{V}_L > \mathbb{0}$. The following lemma shows that the matrix (4.1) is an irreducible Z-matrix if (and only if) $\tilde{V}_L > \mathbb{0}$, and that each such matrix is unique for $\tilde{V}_L > \mathbb{0}$.

Lemma 4.2. *The matrix $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ (i.e., the Jacobian of $-P_c$ at $\tilde{V}_L$) is an irreducible Z-matrix if and only if $\tilde{V}_L > \mathbb{0}$. The map $\tilde{V}_L \mapsto \frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is injective for $\tilde{V}_L > \mathbb{0}$.*

Proof. ($\Rightarrow$): Let $i \in \mathbf{n}$ and note that we have

$$\left(-\frac{\partial P_c}{\partial V_L}(x)\right)_{[i, i^c]} = ([x]Y_{LL} - [Y_{LL}(V_L^* - x)])_{[i, i^c]} = ([x]Y_{LL})_{[i, i^c]} = x_i(Y_{LL})_{[i, i^c]}. \quad (4.2)$$

If $x_i = 0$, then $\left(-\frac{\partial P_c}{\partial V_L}(x)\right)_{[i, i^c]} = 0$ by (4.2), which violates the irreducibility of the $-\frac{\partial P_c}{\partial V_L}(x)$. Hence $x_i \neq 0$. Since Y_{LL} is an irreducible Z-matrix, we have $(Y_{LL})_{[i, i^c]} \not\leq 0$. It follows that there exists at least one negative element in $(Y_{LL})_{[i, i^c]}$. Hence, there exists a $j \neq i$ so that $(Y_{LL})_{ij} < 0$. It follows from (4.2) that

$$\left(-\frac{\partial P_c}{\partial V_L}(x)\right)_{ij} = x_i(Y_{LL})_{ij}. \quad (4.3)$$

The right-hand side of (4.3) is nonzero, $(Y_{LL})_{ij}$ is negative, and the left-hand side of (4.3) is nonpositive since $-\frac{\partial P_c}{\partial V_L}(x)$ is a Z-matrix. This implies that x_i is positive. Hence $x > \mathbb{0}$.

($\Leftarrow$): The matrix Y_{LL} is an irreducible Z-matrix. Since $x > \mathbb{0}$, it follows by Lemma 2.5 that $[x]Y_{LL} - [Y_{LL}(V_L^* - x)]$ is also an irreducible Z-matrix.

Let $x, z > \mathbb{0}$ satisfy $\frac{\partial P_c}{\partial V_L}(x) = \frac{\partial P_c}{\partial V_L}(z)$. By (4.3) this implies that for all i there exists a j such that $x_i(Y_{LL})_{ij} = z_i(Y_{LL})_{ij}$. Since $(Y_{LL})_{ij} \neq 0$, it follows that $x_i = z_i$, and hence $x = z$. $\square$

Lemma 4.2 states that the Jacobian of P_c at an operating point $\tilde{V}_L > \mathbb{0}$ is unique to $\tilde{V}_L$. This implies that operating points are uniquely identified by properties of the associated Jacobian matrix. The following result identifies all long-term voltage stable operating points (see Definition 3.10) by means of properties of the associated Jacobian.

Proposition 4.3. *The set $\mathcal{D}$ of long-term voltage stable operating points equals*

$$\mathcal{D} = \left\{ \tilde{V}_L > \mathbb{0} \mid -\frac{\partial P_c}{\partial V_L}(\tilde{V}_L) \text{ is a nonsingular M-matrix} \right\}.$$

Proof. Let $\tilde{V}_L > \mathbb{0}$ be an operating point associated to some vector of power demands $\tilde{P}_c$. Recall from Definition 3.10 that $\tilde{V}_L$ is long-term voltage stable if $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is nonsingular and $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^{-1} = \frac{\partial V_L}{\partial P_c}(\tilde{P}_c)$ is a matrix with negative elements. Since $\tilde{V}_L > \mathbb{0}$, $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is an irreducible Z-matrix by Lemma 4.2. It follows by Lemma 2.15 that $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^{-1} > \mathbb{0}$ if and only if $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is a nonsingular M-matrix. $\square$

Recall from Definition 3.11 that an operating point is long-term voltage semi-stable if it lies in the closure of $\mathcal{D}$. Proposition 4.3 implies the following characterization of such operating points.

Corollary 4.4. *The closure and boundary of $\mathcal{D}$ satisfies*

$$\begin{aligned} \text{cl}(\mathcal{D}) &= \left\{ \tilde{V}_L > \mathbb{0} \mid -\frac{\partial P_c}{\partial V_L}(\tilde{V}_L) \text{ is an M-matrix} \right\}; \\ \partial\mathcal{D} &= \left\{ \tilde{V}_L > \mathbb{0} \mid -\frac{\partial P_c}{\partial V_L}(\tilde{V}_L) \text{ is a singular M-matrix} \right\}. \end{aligned}$$

The proof follows directly from Lemma 2.12 and Lemma 2.13.

Remark 4.5. Many equivalent characterizations of long-term voltage stable operating points may be derived from Proposition 4.3. Indeed, references such as [6, 24, 46] list numerous equivalent conditions for when a Z-matrix is a nonsingular M-matrix. In particular, it follows from Lemma 2.5 and Lemma 2.16 that if $\tilde{V}_L$ is positive, and hence $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is a Z-matrix by Lemma 4.2, then $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is a nonsingular M-matrix if and only if $[\tilde{V}_L]^{-1} \frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is Hurwitz stable. This shows that $\tilde{V}_L > \mathbb{0}$ is long-term voltage stable if and only if $[\tilde{V}_L]^{-1} \frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is Hurwitz stable. The latter property coincides with the definition of voltage-regularity found in [41]. Similarly it follows that $\tilde{V}_L > \mathbb{0}$ is long-term voltage stable if and only if $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is Hurwitz stable. Analogously, in light of the dynamical example of Section 3.6, $\tilde{V}_L > \mathbb{0}$ is long-term voltage stable if and only if $[C]^{-1}[\tilde{V}_L]^{-1} \frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is Hurwitz stable. Furthermore, by Lemma 2.17 it can be shown that $\tilde{V}_L > \mathbb{0}$ is long-term voltage semi-stable if and only if $\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is Hurwitz semi-stable. In summary, long-term voltage (semi-)stability of an operating point coincides with Hurwitz (semi-)stability of the Jacobian of P_c at the operating point.

4.2 A parametrization of $\mathcal{D}$

Proposition 4.3 and Corollary 4.4 allow us to deduce a parametrization for the set $\mathcal{D}$ of all long-term voltage stable operating points. Such a parametrization gives a constructive method to determine where such operating points lie in the voltage domain, as opposed to testing at which operating points of interest the Jacobian of P_c is Hurwitz stable (see Remark 4.5).

We introduce the following definitions. For a vector $\lambda \in \mathbb{R}^n$ we introduce the $n \times n$ matrix

$$h(\lambda) := \frac{1}{2}([\lambda]Y_{LL} + Y_{LL}^\top[\lambda]). \quad (4.4)$$

Note that $h(\mathbb{1}) = Y_{LL}$ since Y_{LL} is symmetric, and that

$$x^\top h(\lambda)x = x^\top [\lambda]Y_{LL}x = \lambda^\top [x]Y_{LL}x. \quad (4.5)$$

In addition we define the set

$$\Lambda := \{ \lambda \mid h(\lambda) \text{ is positive definite} \}. \quad (4.6)$$

Note that $\mathbb{1} \in \Lambda$ since $h(\mathbb{1}) = Y_{LL}$ is positive definite by Lemma 2.25. The set Λ is studied in Appendix A. In particular, Lemma A.1 shows that Λ is convex, and Lemma A.2 shows that Λ lies in the positive orthant.

The following theorem extends Lemma 4.2, and allows us to parametrize the sets $\mathcal{D}$, $\text{cl}(\mathcal{D})$, and $\partial\mathcal{D}$.

Lemma 4.6. *Let $r \in \mathbb{R}$ and $\lambda \in \mathbb{R}^n$ such that $r \geq 0$ and $\lambda > \mathbb{0}$. The Jacobian $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^\top$ is an irreducible M-matrix with Perron root r and Perron vector λ if and only if $h(\lambda)$ is positive definite (i.e., $\lambda \in \Lambda$) and $\tilde{V}_L$ satisfies*

$$\tilde{V}_L = \frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbb{1}), \quad (4.7)$$

in which case we have $\tilde{V}_L > \mathbb{0}$.

Proof. ($\Rightarrow$): The matrix Y_{LL} is an irreducible Z-matrix and $\lambda > \mathbb{0}$, and so $h(\lambda)$ is an irreducible Z-matrix by Lemma 2.5 and Lemma 2.4. We let s and $v > \mathbb{0}$ denote respectively the Perron root and a Perron vector of $h(\lambda)$. The matrix $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^\top$ is an M-matrix, and hence a Z-matrix. By Lemma 4.2 we have $\tilde{V}_L > \mathbb{0}$. Using the fact that (r, λ) is an eigenpair to $-\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^\top$, we observe that

$$\begin{aligned} r\lambda &= -\frac{\partial P_c}{\partial \tilde{V}_L}(\tilde{V}_L)^\top \lambda = -([Y_{LL}(V_L^* - \tilde{V}_L)] - Y_{LL}^\top[\tilde{V}_L])\lambda \\ &= -[\lambda]Y_{LL}V_L^* + [\lambda]Y_{LL}\tilde{V}_L + Y_{LL}^\top[\lambda]\tilde{V}_L \\ &= -[\lambda]\mathcal{I}_L^* + 2h(\lambda)\tilde{V}_L. \end{aligned}$$

By rearranging the terms, it follows that

$$[\lambda]\mathcal{I}_L^* + r\lambda = 2h(\lambda)\tilde{V}_L. \quad (4.8)$$

Multiplying (4.8) by $v^\top$ results in

$$v^\top([\lambda]\mathcal{I}_L^* + r\lambda) = 2v^\top h(\lambda)\tilde{V}_L = 2sv^\top\tilde{V}_L. \quad (4.9)$$

Since $\tilde{V}_L > 0$, $v > 0$, $\lambda > 0$, $r \geq 0$ and $\mathcal{I}_L^* \geq 0$, it follows that the left hand side of (4.9) is positive. Since $v^\top\tilde{V}_L$ is also positive, it follows from (4.9) that the Perron root s is positive. Hence, $h(\lambda)$ is a nonsingular M-matrix by Corollary 2.14, and (4.7) follows from (4.8). Since $h(\lambda)$ is a symmetric nonsingular M-matrix, it is positive definite and $\lambda \in \Lambda$.

($\Leftarrow$): If $\lambda \in \Lambda$, then $\lambda > 0$ by Lemma A.2. The rest of the proof follows by reversing the steps of the " $\Rightarrow$ "-part. $\square$

Lemma 4.6 allows for an explicit parametrization of the set $\mathcal{D}$ by Λ , and without relying on properties of the Jacobian of P_c . Note that (4.7) is invariant under scaling of λ , and hence the vectors λ may be normalized. For this purpose we define

$$\Lambda_1 := \Lambda \cap \{ \lambda > 0 \mid \|\lambda\|_1 = 1^\top \lambda = 1 \}, \quad (4.10)$$

which is a convex set, as it is the intersection of convex sets. Appendix A lists several properties of the closure of Λ_1 .

Theorem 4.7 (M1). *The set $\mathcal{D}$ of all long-term voltage stable operating point, its closure $\text{cl}(\mathcal{D})$ and its boundary $\partial\mathcal{D}$ are parametrized by*

$$\begin{aligned} \mathcal{D} &= \left\{ \frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbb{1}) \mid \lambda \in \Lambda_1, r > 0 \right\}; \\ \text{cl}(\mathcal{D}) &= \left\{ \frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbb{1}) \mid \lambda \in \Lambda_1, r \geq 0 \right\}; \\ \partial\mathcal{D} &= \left\{ \frac{1}{2}h(\lambda)^{-1}[\lambda]\mathcal{I}_L^* \mid \lambda \in \Lambda_1 \right\}. \end{aligned}$$

Furthermore, the map

$$(\lambda, r) \mapsto \frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbb{1}) \quad (4.11)$$

from $\Lambda_1 \times \mathbb{R}_{\geq 0}$ to $\text{cl}(\mathcal{D})$ is a bicontinuous map, and the sets $\mathcal{D}$, $\text{cl}(\mathcal{D})$ and $\partial\mathcal{D}$ are simply connected.

Proof. Lemma 2.12, Lemma 2.13 and Corollary 2.14 state that a Z-matrix is a (nonsingular/singular) M-matrix if and only if its Perron root r is nonnegative (positive/zero). Proposition 4.3 and Corollary 4.4 together with Lemma 4.6 imply that the vector (4.7) with $\lambda \in \Lambda$ lies in $\mathcal{D}$, $\text{cl}(\mathcal{D})$ or $\partial\mathcal{D}$ if and only if r in (4.7) satisfies respectively $r > 0$, $r \geq 0$ or $r = 0$.

The map (4.11) is a continuous bijection from $\Lambda_1 \times \mathbb{R}_{\geq 0}$ to $\text{cl}(\mathcal{D})$, which follows from Lemma 4.6 and Corollary 4.4. The inverse of the map (4.11) is described taking $x \in \text{cl}(\mathcal{D})$

and computing the Perron vector $\lambda > \mathbb{0}$ and the Perron root r of $-\frac{\partial P_c}{\partial V_L}(x)$. By Lemma 2.7, the Perron root and Perron vector of $-\frac{\partial P_c}{\partial V_L}(x)$ are continuous in x . Hence the inverse of the map (4.11) is also continuous.

The set $\Lambda_1 \times \mathbb{R}_{\geq 0}$ is convex, and is therefore simply connected, which is a topological property. Topological properties are preserved by bicontinuous maps, and thus $\text{cl}(\mathcal{D})$ is also simply connected. The same holds for $\Lambda_1 \times \mathbb{R}_{> 0}$ and $\Lambda_1 \times \{0\}$, and hence $\mathcal{D}$ and $\partial\mathcal{D}$ are simply connected. $\square$

Along with its theoretical contribution, Theorem 4.7 allows for a practical computation of the set $\mathcal{D}$ of long-term voltage stable operating points. Indeed, we obtain an approximation for $\mathcal{D}$ by computing a fine mesh for Λ_1 and applying the map (4.11) to all points λ in the mesh for different values of $r > 0$. Note that our method works well for small scale examples, but that computing a sufficiently fine mesh for Λ_1 for power grids with a large number of loads is computationally more intensive. The set $\mathcal{D}$ in Figure 3.9 was obtained using this method.

4.3 The convex hull of $\mathcal{F}$ and its boundary

In order to study the set $\mathcal{F}$ of feasible power demands, we will be studying its convex hull and its boundary. In particular, we show that there is a one-to-one correspondence between points in the boundary of $\mathcal{D}$ and points in $\partial \text{conv}(\mathcal{F})$, the boundary of the convex hull of $\mathcal{F}$.

To simplify notation, we define for $\lambda \in \Lambda$ the map

$$\varphi(\lambda) := \frac{1}{2}h(\lambda)^{-1}[\lambda]\mathcal{I}_L^*. \quad (4.12)$$

Note that $\varphi(\lambda)$ is invariant under scaling of λ . Also, since $\mathbb{1} \in \Lambda$ we have

$$\varphi(\mathbb{1}) = \frac{1}{2}h(\mathbb{1})^{-1}[\mathbb{1}]\mathcal{I}_L^* = \frac{1}{2}Y_{LL}^{-1}\mathcal{I}_L^* = \frac{1}{2}V_L^* \in \partial\mathcal{D},$$

which are the voltage potentials corresponding to $P_{\max}$, by Lemma 3.18.

The boundary of $\mathcal{D}$ is by definition the set of the operating points which are long-term voltage semi-stable, but not long-term voltage stable. It follows from Theorem 4.7 that $\partial\mathcal{D}$ satisfies

$$\partial\mathcal{D} = \{ \varphi(\lambda) \mid \lambda \in \Lambda_1 \} = \varphi(\Lambda_1). \quad (4.13)$$

To study the convex hull of $\mathcal{F}$ and its boundary, we make use of the following two identities involving $P_c(V_L)$.

Lemma 4.8. *For $x, z \in \mathbb{R}^n$ we have*

$$P_c(x + z) = P_c(x) + \frac{\partial P_c}{\partial V_L}(x)z - [z]Y_{LL}z. \quad (4.14)$$

Proof. We write out the formula using (3.15) to obtain

$$\begin{aligned}
P_c(x+z) &= [x+z]Y_{LL}(V_L^* - x - z) \\
&= [x]Y_{LL}(V_L^* - x) + [z]Y_{LL}(V_L^* - x) - [x]Y_{LL}z - [z]Y_{LL}z \\
&= P_c(x) + [Y_{LL}(V_L^* - x)]z - [x]Y_{LL}z - [z]Y_{LL}z \\
&= P_c(x) + \frac{\partial P_c}{\partial V_L}(x)z - [z]Y_{LL}z,
\end{aligned}$$

where we have substituted the formula for the Jacobian of P_c (4.1). $\square$

We remark that (4.14) equals the second-order Taylor approximation of $P_c(x+z)$ at x . The approximation is exact since the components of $P_c(x)$ are polynomials of degree 2 in x_i .

The matrix $h(\lambda)$ for $\lambda \in \Lambda$ is positive definite by definition, and therefore induces the vector norm

$$\|x\|_{h(\lambda)} := \sqrt{x^\top h(\lambda)x}. \quad (4.15)$$

This vector norm is related to $\lambda^\top P_c(x)$ for $\lambda \in \Lambda$ by the following lemma.

Lemma 4.9. *Let $\lambda \in \Lambda$. For each $x \in \mathbb{R}^n$ we have*

$$\lambda^\top P_c(x) = \|\varphi(\lambda)\|_{h(\lambda)}^2 - \|\varphi(\lambda) - x\|_{h(\lambda)}^2. \quad (4.16)$$

Moreover, we have

$$\lambda^\top P_c(x) \leq \|\varphi(\lambda)\|_{h(\lambda)}^2 = \lambda^\top P_c(\varphi(\lambda)), \quad (4.17)$$

with equality if and only if $x = \varphi(\lambda)$. Consequently, we have $P_c(x) = P_c(\varphi(\lambda))$ if and only if $x = \varphi(\lambda)$.

Proof. Using (3.15), (3.10), (4.12) and (4.5), we verify that

$$\begin{aligned}
\lambda^\top P_c(x) &= \lambda^\top [x]Y_{LL}V_L^* - \lambda^\top [x]Y_{LL}x \\
&= x^\top [\lambda]\mathcal{I}_L^* - x^\top [\lambda]Y_{LL}x \\
&= 2x^\top h(\lambda)\varphi(\lambda) - x^\top h(\lambda)x \\
&= \varphi(\lambda)^\top h(\lambda)\varphi(\lambda) - \varphi(\lambda)^\top h(\lambda)\varphi(\lambda) + 2x^\top h(\lambda)\varphi(\lambda) - x^\top h(\lambda)x \\
&= \|\varphi(\lambda)\|_{h(\lambda)}^2 - \|\varphi(\lambda) - x\|_{h(\lambda)}^2.
\end{aligned}$$

Eq. (4.17) follows from (4.16) since $\|\varphi(\lambda) - x\|_{h(\lambda)} \geq 0$, with equality if and only if $x = \varphi(\lambda)$. Thus, equality in (4.17) holds if and only if $x = \varphi(\lambda)$. Finally, if $P_c(x) = P_c(\varphi(\lambda))$, then $\lambda^\top P_c(x) = \lambda^\top P_c(\varphi(\lambda))$ therefore $x = \varphi(\lambda)$. $\square$

Note how Lemma 4.9, and in particular (4.17), generalize Proposition 3.20. Indeed, by taking $\lambda = \mathbb{1} \in \Lambda$ we obtain $P_c(\varphi(\mathbb{1})) = P_c(\frac{1}{2}V_L^*) = P_{\max}$, and hence we recover Proposition 3.20. Note also that (4.17) describes a closed half-space, which we formally define as follows.

For a vector ν such that $\|\nu\|_1 = 1$ and a scalar s we define the closed half-space

$$H(\nu, s) := \{ y \mid \nu^\top y \leq s \},$$

which has as boundary the hyperplane

$$\partial H(\nu, s) = \{ y \mid \nu^\top y = s \}.$$

The vector ν is normal to the boundary of the half-space and points outwards.

Definition 4.10. A half-space $H(\nu, s)$ is said to *support* a set S if $S \subseteq H(\nu, s)$ and $\text{cl}(S) \cap \partial H(\nu, s)$ is nonempty. *I.e.*, for a given ν , s is the smallest number so that $S \subseteq H(\nu, s)$. A point in $\text{cl}(S) \cap \partial H(\nu, s)$ is a *point of support*.

If the half-space $H(\nu, s)$ supports $\mathcal{F}$ and $\tilde{P}_c \in \mathcal{F}$ is a point of support, then $\nu^\top \tilde{P}_c$ maximizes $\nu^\top P_c$ for all $P_c \in \mathcal{F}$. For example, let $\nu = \frac{1}{n}\mathbb{1}$, then the corresponding supporting half-space is given by (3.24). The vector $P_{\max}$ is the unique point of support, as was shown in Lemma 3.18. See also Figure 3.5, in which the dashed line corresponds to the boundary of this half-space.

In order to obtain a geometric description of the convex hull of $\mathcal{F}$ we aim to apply the following proposition.

Proposition 4.11 ([47, Cor. 11.5.1]). *Let the set S be a subset of $\mathbb{R}^n$, then*

$$\text{cl}(\text{conv}(S)) = \bigcap_{\text{support of } S} H(\nu, s),$$

where the intersection is taken over all half-spaces $H(\nu, s)$ that support S .

To apply Proposition 4.11 to $\mathcal{F}$, we identify all supporting half-spaces of the set $\mathcal{F}$. We will simultaneously identify the supporting half-spaces of the image of P_c , which is given by

$$\text{im } P_c := \{ y \mid \exists x \in \mathbb{R}^n : P_c(x) = y \} \quad (4.18)$$

and satisfies the inclusion $\mathcal{F} \subseteq \text{im } P_c$ (cf. (3.14)).

Theorem 4.12. *Let λ be a vector such that $\|\lambda\|_1 = 1$ and s be a scalar. A half-space $H(\lambda, s)$ supports $\mathcal{F}$ if and only if $\lambda \in \Lambda_1$ and $s = \|\varphi(\lambda)\|_{h(\lambda)}^2$. The point $P_c(\varphi(\lambda)) \in \mathcal{F}$ is the unique point of support. Moreover, the supporting half-spaces of the set $\mathcal{F}$ and the image of P_c coincide.*

Proof. ($\Leftarrow$): The half-space $H(\lambda, s)$ with $\lambda \in \Lambda_1$ and $s = \|\varphi(\lambda)\|_{h(\lambda)}^2$ is given by

$$H(\lambda, s) = \{ y \mid \lambda^\top y \leq \|\varphi(\lambda)\|_{h(\lambda)}^2 \}.$$

Since $\lambda \in \Lambda_1$, Lemma 4.9 states that (4.17) holds for all $x \in \mathbb{R}^n$. This implies that $P_c(x) \in H(\lambda, s)$ for all $x \in \mathbb{R}^n$, and thus $\text{im } P_c \subseteq H(\lambda, s)$. To show that $P_c(\varphi(\lambda))$ is the unique point of support, we show that

$$\text{cl}(\text{im } P_c) \cap \partial H(\lambda, s) = \{P_c(\varphi(\lambda))\}. \quad (4.19)$$

Let $y \in \text{cl}(\text{im } P_c) \cap \partial H(\lambda, s)$, then there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \in \mathbb{R}^n$ such that

$$\lim_{k \rightarrow \infty} P_c(x_k) = y. \quad (4.20)$$

Since $y \in \partial H(\lambda, s)$, multiplying (4.20) by $\lambda^\top$ yields

$$\lim_{k \rightarrow \infty} \lambda^\top P_c(x_k) = \lambda^\top y = s = \|\varphi(\lambda)\|_{h(\lambda)}^2. \quad (4.21)$$

It follows from rearranging (4.21) and applying (4.16) that

$$0 = \lim_{k \rightarrow \infty} (\|\varphi(\lambda)\|_{h(\lambda)}^2 - \lambda^\top P_c(x_k)) = \lim_{k \rightarrow \infty} \|\varphi(\lambda) - x_k\|_{h(\lambda)}^2.$$

Hence $\lim_{k \rightarrow \infty} x_k = \varphi(\lambda)$, and so (4.19) holds. This proves that $H(\lambda, s)$ supports $\text{im } P_c$, and that $P_c(\varphi(\lambda))$ is a point of support. The same is true for $\mathcal{F}$ since $\mathcal{F} \subseteq \text{im } P_c$ and $P_c(\varphi(\lambda)) \in \mathcal{F}$.

($\Rightarrow$): Let $x > 0$ be a vector. Let λ be such that $\|\lambda\|_1 = 1$ and $\lambda \notin \Lambda_1$, which means that $h(\lambda)$ is not positive definite. We will show that there exists a vector $v \geq 0$ such that $\lambda^\top P_c(x + tv)$ for scalars $t \geq 0$ is not bounded from above. Since $x + tv > 0$ for all $t \geq 0$, this implies that the hyperplane $H(\lambda, s)$ does not contain $\mathcal{F}$ for any scalar s . The same holds for $\text{im } P_c$ since $\mathcal{F} \subseteq \text{im } P_c$. Lemma 4.8 yields

$$P_c(\hat{x} + tv) = P_c(\hat{x}) + t \frac{\partial P_c}{\partial V_L}(\hat{x})v - t^2[v]Y_{LL}v. \quad (4.22)$$

We multiply (4.22) by $\lambda^\top$ and use (4.5), which implies

$$\lambda^\top P_c(\hat{x} + tv) = \lambda^\top P_c(\hat{x}) + t \lambda^\top \frac{\partial P_c}{\partial V_L}(\hat{x})v - t^2 v^\top h(\lambda)v. \quad (4.23)$$

If $\lambda_i < 0$ for some i , then $e_i^\top h(\lambda)e_i = (Y_{LL})_{ii}\lambda_i < 0$. Hence, taking $v = e_i \geq 0$ in (4.23) describes a parabola in t which is not bounded from above. Thus $\lambda^\top P_c(x)$ is not bounded from above for $t \geq 0$.

If $\lambda \geq 0$ and $\lambda \notin \text{cl}(\Lambda_1)$ then the matrix $h(\lambda)$ has a negative eigenvalue by Lemma A.4. Let r be the eigenvalue of $h(\lambda)$ with the smallest (*i.e.*, most negative) real part. Since

$\lambda \geq 0$, it follows that $h(\lambda)$ is a Z-matrix. The matrix $h(\lambda)$ is block diagonal, where each block corresponds to an irreducible component of $h(\lambda)$. Let $h(\lambda)_{[\alpha, \alpha]}$ be the irreducible component that corresponds to the negative eigenvalue r . The matrix $h(\lambda)_{[\alpha, \alpha]}$ is an irreducible Z-matrix with Perron root r and Perron vector $w > 0$. Let v in (4.23) be such that $v_{[\alpha]} = w$ and $v_{[\alpha^c]} = 0$, then $v^\top h(\lambda)v = rw^\top w < 0$. It follows that (4.23) describes a parabola in t which is not bounded from above. Thus $\lambda^\top P_c(x)$ is not bounded from above for $t \geq 0$.

Finally, suppose $\lambda \in \partial\Lambda_1$, which implies by Lemma A.5 that $\lambda > 0$ and that $h(\lambda)$ is an irreducible M-matrix. The matrix $h(\lambda)$ is singular since $\lambda \notin \Lambda_1$. Let $v > 0$ in (4.23) be a Perron vector of $h(\lambda)$. Lemma 2.13 states that v spans the kernel of $h(\lambda)$, and so $v^\top h(\lambda)v = 0$. By Lemma A.6 we know that $\lambda^\top \frac{\partial P_c}{\partial V_L}(x)v > 0$. This implies that (4.23) describes a half-line for $t \geq 0$ which is not bounded from above. Hence, $\lambda^\top P_c(x)$ is not bounded from above for $t \geq 0$. $\square$

To simplify notation, we define for $\lambda \in \Lambda$ the half-spaces

$$H_\lambda := H(\lambda, \|\varphi(\lambda)\|_{h(\lambda)}^2). \quad (4.24)$$

The definition of H_λ is invariant under scaling of λ . Theorem 4.12 states that H_λ for $\lambda \in \Lambda$ are all supporting half-spaces of $\mathcal{F}$. Proposition 4.11 therefore allows us to give a direct formula for the closure of the convex hull of $\mathcal{F}$.

Corollary 4.13. *The closure of the convex hull of $\mathcal{F}$ is the intersection of all half-spaces H_λ where $\lambda \in \Lambda$, and is equal to the closure of the convex hull of $\text{im } P_c$. I.e.,*

$$\text{cl}(\text{conv}(\mathcal{F})) = \text{cl}(\text{conv}(\text{im } P_c)) = \bigcap_{\lambda \in \Lambda_1} H_\lambda. \quad (4.25)$$

Now that we have identified the closure of the convex hull of $\mathcal{F}$, we may also identify the boundary of this set.

Theorem 4.14. *The map $\lambda \mapsto P_c(\varphi(\lambda))$ for $\lambda \in \Lambda_1$ is one-to-one and parametrizes the boundary of $\text{conv}(\mathcal{F})$. Moreover, the set $\text{conv}(\mathcal{F})$ is closed.*

Proof. The half-spaces H_λ for $\lambda \in \Lambda_1$ are all supporting half-spaces of $\text{conv}(\mathcal{F})$, which follows from (4.25) of Corollary 4.13. Theorem 4.12 proves that $P_c(\varphi(\lambda))$ is a point of support to H_λ , and that it is unique in the case of $\mathcal{F}$. Theorem 2.15 of [57] states that all boundary points of a convex set are a point of support associated to some supporting half-space. This implies that $P_c(\partial\mathcal{D}) \subseteq \partial\text{conv}(\mathcal{F})$. We prove equality by showing that there are no other points of support.

Let $y \in \text{cl}(\text{conv}(\mathcal{F}))$. Then there exists a sequence $\{y_k\}_{k \in \mathbb{N}} \in \text{conv}(\mathcal{F})$ such that $\lim_{k \rightarrow \infty} y_k = y$. This means that for $k \in \mathbb{N}$ there exists $x_k, z_k \in \mathbb{R}^n$ and scalars θ_k such

that $y_k = \theta_k P_c(x_k) + (1 - \theta_k) P_c(z_k)$ and $0 < \theta_k < 1$. Suppose there exists $\tilde{\lambda} \in \Lambda_1$ so that $y \in \partial H_{\tilde{\lambda}}$. Hence, y is a point of support associated to $H_{\tilde{\lambda}}$. We define $s := \|\varphi(\lambda)\|_{h(\lambda)}^2$ and observe that

$$\lim_{k \rightarrow \infty} \tilde{\lambda}^\top (\theta_k P_c(x_k) + (1 - \theta_k) P_c(z_k)) = \lim_{k \rightarrow \infty} \tilde{\lambda}^\top y_k = \tilde{\lambda}^\top y = s.$$

Lemma 4.9 implies that for all k we have $\tilde{\lambda}^\top P_c(x_k) \leq s$, with equality if and only if $x_k = \varphi(\tilde{\lambda})$, and the same holds for z_k . This implies that

$$\tilde{\lambda}^\top (\theta_k P_c(x_k) + (1 - \theta_k) P_c(z_k)) \leq s. \quad (4.26)$$

In order to converge to equality in (4.26) as $k \rightarrow \infty$, we require that either $x_k \rightarrow \varphi(\tilde{\lambda})$ and $z_k \rightarrow \varphi(\tilde{\lambda})$, $x_k \rightarrow \varphi(\tilde{\lambda})$ and $\theta_k \rightarrow 1$, or $z_k \rightarrow \varphi(\tilde{\lambda})$ and $\theta_k \rightarrow 0$. In all cases it follows that $y = \lim_{k \rightarrow \infty} y_k = P_c(\varphi(\tilde{\lambda}))$. Hence $P_c(\varphi(\tilde{\lambda}))$ is the unique point of support.

Lemma 4.9 implies that if $P_c(\varphi(\lambda_1)) = P_c(\varphi(\lambda_2))$, then $\varphi(\lambda_1) = \varphi(\lambda_2)$. Hence the map $\lambda \rightarrow P_c(\varphi(\lambda))$ is a one-to-one correspondence between Λ_1 and $\partial \text{conv}(\mathcal{F})$, and $\lambda \rightarrow P_c(\varphi(\lambda))$ for $\lambda \in \Lambda_1$ parametrizes $\partial \text{conv}(\mathcal{F})$.

Note the inclusion

$$\partial \text{conv}(\mathcal{F}) = P_c(\partial \mathcal{D}) \subseteq \mathcal{F} \subseteq \text{conv}(\mathcal{F}),$$

which implies that $\text{conv}(\mathcal{F})$ is closed. $\square$

Theorem 4.14 uncovers a valuable correspondence between Λ_1 and the boundaries of $\mathcal{D}$ and $\text{conv}(\mathcal{F})$.

Corollary 4.15. *The sets Λ_1 , $\partial \mathcal{D}$ and $\partial \text{conv}(\mathcal{F})$ are in one-to-one correspondence. In particular, P_c is a one-to-one map from $\partial \mathcal{D}$ to $\partial \text{conv}(\mathcal{F})$.*

Proof. It follows from Theorem 4.7 that $\varphi(\lambda)$ is a one-to-one map from Λ_1 to $\partial \mathcal{D}$, and Theorem 4.14 states that $\lambda \rightarrow P_c(\varphi(\lambda))$ is a one-to-one map from Λ_1 to $\partial \text{conv}(\mathcal{F})$. Hence, P_c is a one-to-one map from $\partial \mathcal{D}$ to $\partial \text{conv}(\mathcal{F})$. $\square$

Corollary 4.15 states that a vector of power demands $\tilde{P}_c$ that lies on the boundary the convex hull of $\mathcal{F}$ corresponds uniquely to an operating point $\tilde{V}_L$ which is long-term voltage semi-stable but not long-term voltage stable. The pair $(\tilde{P}_c, \tilde{V}_L)$ corresponds to a unique $\lambda \in \Lambda_1$, and the corresponding hyperplane ∂H_λ intersects $\text{conv}(\mathcal{F})$ only in the unique point of support $\tilde{P}_c$. Hence, ∂H_λ is the tangent plane at $\tilde{P}_c$ of the boundary of $\text{conv}(\mathcal{F})$. This is observed again in Figure 3.5 for $\tilde{P}_c = P_{\max}$, $\tilde{V}_L = \frac{1}{2} V_L^*$ and $\lambda = \frac{1}{n} \mathbf{1}$, and the same holds for all points on the boundary of $\text{conv}(\mathcal{F})$.

4.4 A one-to-one correspondence between $\mathcal{F}$ and $\text{cl}(\mathcal{D})$

In this section we use Corollary 4.15 to prove that P_c is a one-to-one mapping from $\text{cl}(\mathcal{D})$ to $\text{conv}(\mathcal{F})$, and that therefore $\mathcal{F}$ is convex. This means each feasible power demand is uniquely associated to a long-term voltage semi-stable operating point. This operating point can be found by solving an initial value problem. The following lemma is instrumental in proving these results.

Lemma 4.16. *Let $\widehat{V}_L \in \mathcal{D}$, define $\widehat{P}_c := P_c(\widehat{V}_L) \in P_c(\mathcal{D})$ and let $\widetilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. There exists a unique path $\gamma : [0, 1] \rightarrow \mathcal{D}$ so that the convex combination of $\widehat{P}_c$ and $\widetilde{P}_c$ is described by*

$$P_c(\gamma(\theta)) = \theta \widetilde{P}_c + (1 - \theta) \widehat{P}_c. \quad (4.27)$$

for $0 \leq \theta \leq 1$. The path γ solves the initial value problem

$$\dot{\gamma}(\theta) = \left(\frac{\partial P_c}{\partial V_L}(\gamma(\theta)) \right)^{-1} (\widetilde{P}_c - \widehat{P}_c) \quad (4.28)$$

with initial value $\gamma(0) = \widehat{V}_L$. We have $\widetilde{P}_c = P_c(\gamma(1))$.

Proof. First we show that if a path $\gamma : [0, T] \rightarrow \mathcal{D}$ satisfies (4.28) with $\gamma(0) = \widehat{V}_L \in \mathcal{D}$, then (4.27) holds. Indeed, note that the matrix $\frac{\partial P_c}{\partial V_L}(\gamma(\tau))$ is invertible for $0 \leq \tau \leq T$ since $\gamma(\tau) \in \mathcal{D}$, and note for $0 \leq \theta \leq T$ that by the fundamental theorem of calculus we have

$$P_c(\gamma(\theta)) = P_c(\gamma(0)) + \int_0^\theta \frac{\partial P_c}{\partial V_L}(\gamma(\tau)) \dot{\gamma}(\tau) d\tau. \quad (4.29)$$

Substitution of (4.28) in (4.29) yields

$$P_c(\gamma(\theta)) = P_c(\gamma(0)) + \int_0^\theta (\widetilde{P}_c - \widehat{P}_c) d\tau. \quad (4.30)$$

Eq. (4.27) follows from (4.30) since $P_c(\gamma(0)) = P_c(\widehat{V}_L) = \widehat{P}_c$.

To complete the proof it remains to show that a solution $\gamma(\theta) \in \mathcal{D}$ to (4.28) for $\theta \in [0, 1]$ exists and that this solution is unique. Let the map $\psi : \mathcal{D} \rightarrow \mathbb{R}^n$ be defined by

$$\psi(z) := \left(\frac{\partial P_c}{\partial V_L}(z) \right)^{-1} (\widetilde{P}_c - \widehat{P}_c).$$

The map $\psi(z)$ is continuously differentiable since $\frac{\partial P_c}{\partial V_L}(z)$ is invertible for $z \in \mathcal{D}$. Corollary 8.17 of [31] states that the initial value problem (4.28) has a unique solution $\gamma : (-\varepsilon, \varepsilon) \rightarrow B$ for some $\varepsilon > 0$, where B is an open neighborhood of $\widehat{V}_L$ which is contained in $\mathcal{D}$. Since $\psi(z)$ is continuous at all $z \in \mathcal{D}$, the solution γ can be extended to a maximal interval of existence. Indeed, by the Theorem 8.33 of [31] we extend γ so that either (i) $\gamma(\theta) \rightarrow \partial\mathcal{D}$,

or (ii) $|\gamma(\theta)_i| \rightarrow \infty$ for some $i \in \mathbf{n}$, as $\theta \rightarrow \omega$ where $\omega \in \mathbb{R}_{>0} \cup \{+\infty\}$. We will treat cases (i) and (ii) separately.

Case (i): Let $x \in \partial D$ such that $\gamma(\theta) \rightarrow x$ as $\theta \rightarrow \omega$ and let $y := P_c(x)$. By continuity of $P_c(V_L)$ it follows that $P_c(\gamma(\theta)) \rightarrow y$ as $\theta \rightarrow \omega$. Since $\gamma(\theta) \in \mathcal{D}$ for $0 \leq \theta < \omega$, the first part of this proof showed that (4.27) holds for $0 \leq \theta < \omega$. Suppose $\omega = 1$, then taking the limit $\theta \rightarrow \omega$ in (4.27) implies that $\tilde{P}_c = P_c(\gamma(\omega)) = y$, which lies on the boundary of $\mathcal{F}$. This contradicts the fact that $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. Suppose $1 > \omega$, then (4.27) implies that $P_c(\gamma(\omega)) = y$ is a convex combination of $\hat{P}_c$ and $\tilde{P}_c$. Since $\hat{V}_L \notin \partial \mathcal{D}$ it follows from Theorem 4.14 that $\hat{P}_c \notin \partial \text{conv}(\mathcal{F})$. Since $\hat{P}_c \in \mathcal{F}$ we therefore have $\hat{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. Let $\lambda \in \Lambda_1$ such that $P_c(\varphi(\lambda)) = y$, which exists by Theorem 4.14, and define $s := \|\varphi(\lambda)\|_{h(\lambda)}^2 = \lambda^\top P_c(\varphi(\lambda)) = \lambda^\top y$. Note that $\lambda^\top \hat{P}_c < s$ and $\lambda^\top \tilde{P}_c < s$ since $\hat{P}_c, \tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. But since y is a convex combination of $\hat{P}_c$ and $\tilde{P}_c$, this would imply that $\lambda^\top y < s$, which is a contradiction. We conclude that $1 < \omega$, and in particular $\gamma(\theta) \in \mathcal{D}$ for $0 \leq \theta \leq 1$.

Case (ii): We will show that $\omega = +\infty$ and that $P_c(\gamma(\theta))$ describes a half-line for $0 \leq \theta < \infty$. Let $\lambda \in \Lambda_1$. Note that $|\gamma(\theta)_i| \rightarrow \infty$ implies that also $|\varphi(\lambda)_i - \gamma(\theta)_i| \rightarrow \infty$. Therefore also $\|\varphi(\lambda) - \gamma(\theta)\|_{h(\lambda)} \rightarrow \infty$. It follows from Lemma 4.9 that $\lambda^\top P_c(\gamma(\theta)) \rightarrow -\infty$. This holds for all $\lambda \in \Lambda_1$ and so $P_c(\gamma(\theta))$ does not intersect the boundary of $\mathcal{F}$ for $0 \leq \theta < \omega$. The first part of this proof showed that (4.27) holds for $0 \leq \theta < \omega$, which describes a half-line in θ . Since $\lambda > 0$ by Lemma A.2, it follows from $\lambda^\top P_c(\gamma(\theta)) \rightarrow -\infty$ that $P_c(\gamma(\theta))_j \rightarrow -\infty$ for some $j \in \mathbf{n}$. As a result, (4.27) implies that $\omega = +\infty$. In particular it follows that $\tilde{P}_c$ lies on the half-line and that $\gamma(\theta) \in \mathcal{D}$ for $0 \leq \theta \leq 1$.

To show uniqueness, we note again that $\psi(z)$ is continuously differentiable. Corollary 8.17 of [31] states that (4.28) has a unique solution in an open neighborhood around *any* given initial value in $\mathcal{D}$. Taking any point $\gamma(\theta)$ with $0 \leq \theta \leq 1$ as an initial value shows that the solution γ is unique at each point, and hence is unique in $\mathcal{D}$.

Since (4.27) holds for $0 \leq \theta \leq 1$, (4.27) implies that $\tilde{P}_c = P_c(\gamma(1)) \in P_c(\mathcal{D})$. $\square$

The next theorem forms the cornerstone of this thesis, and shows that for each feasible vector of power demands we can find a unique long-term voltage semi-stable operating point in a constructive manner, by solving an initial value problem.

Theorem 4.17 (M3). *There is a one-to-one correspondence between the long-term voltage semi-stable operating points $\text{cl}(\mathcal{D})$ and the feasible power demands. I.e., for each $\tilde{P}_c \in \mathcal{F}$ there exists a unique $\tilde{V}_L \in \text{cl}(\mathcal{D})$ which satisfies $\tilde{P}_c = P_c(\tilde{V}_L)$, implying that $\mathcal{F} = P_c(\text{cl}(\mathcal{D}))$. More explicitly, $\tilde{V}_L$ is obtained by solving the initial value problem*

$$\dot{\gamma}(\theta) = \left(\frac{\partial P_c}{\partial V_L}(\gamma(\theta)) \right)^{-1} \tilde{P}_c \quad (4.31)$$

for $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ with initial value $\gamma(0) = V_L^*$, where the solution γ exists, is unique and satisfies $\gamma(1) = \tilde{V}_L$.

Proof. Note that $\tilde{P}_c \in \mathcal{F} \subseteq \text{conv}(\mathcal{F})$ and that $V_L^* \in \mathcal{D}$. Suppose $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. By taking $\hat{V}_L = V_L^*$ and $\hat{P}_c = P_c(V_L^*) = 0$ in Lemma 4.16, there is a unique $\gamma : [0, 1] \rightarrow \mathcal{D}$ which solves (4.31) with $\gamma(0) = V_L^*$ and which satisfies $\tilde{P}_c = P_c(\gamma(1))$. Hence we take $\tilde{V}_L := \gamma(1) \in \mathcal{D}$. To show that there is a unique $\tilde{V}_L \in \mathcal{D}$ such that $\tilde{P}_c = P_c(\tilde{V}_L)$, suppose that we have $\tilde{V}'_L \in \mathcal{D}$ such that $\tilde{P}_c = P_c(\tilde{V}'_L)$. Then by Lemma 4.16 there is a unique $\gamma' : [0, 1] \rightarrow \mathcal{D}$ which solves

$$\dot{\gamma}'(\theta) = \left(\frac{\partial P_c}{\partial V_L}(\gamma'(\theta)) \right)^{-1} (P_c(V_L^*) - \tilde{P}_c) \quad (4.32)$$

with $\gamma'(0) = \tilde{V}'_L$, and satisfies $P_c(\gamma'(1)) = P_c(V_L^*) = 0$. Recall from Proposition 4.3 that $\gamma'(1) > 0$ since $\gamma'(1) \in \mathcal{D}$. Since $0 = P_c(\gamma'(1)) = [\gamma'(1)]Y_{LL}(V_L^* - \gamma'(1))$ and Y_{LL} is nonsingular, this implies that $\gamma'(1) = V_L^*$. But now note that $\hat{\gamma}(\theta) := \gamma'(1 - \theta) + V_L^* - \tilde{V}'_L$ is a solution to (4.31), since $P_c(V_L^*) = 0$ in (4.32) and since $\hat{\gamma}(0) = V_L^*$. Since (4.31) has a unique solution, it follows that $\hat{\gamma} = \gamma$, and in particular $\tilde{V}_L = \gamma(1) = \hat{\gamma}(1) = \tilde{V}'_L$, which proves that $\tilde{V}_L$ is unique. Alternatively, if $\tilde{P}_c \in \partial \text{conv}(\mathcal{F})$ then by Corollary 4.15 there exists a unique $\tilde{V}_L \in \partial \mathcal{D}$ such that $\tilde{P}_c = P_c(\tilde{V}_L) \in P_c(\text{cl}(\mathcal{D}))$. It follows from Lemma 4.9 that there is no other operating point $\tilde{V}'_L$ such that $\tilde{P}_c = P_c(\tilde{V}'_L)$. The operating point $\tilde{V}_L$ is also obtained by the initial value problem (4.31), which follows from taking the limit $\tilde{P}_c \rightarrow \partial \mathcal{D}$ for $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$. $\square$

Figure 4.1 illustrates the result of the initial value problem (4.31) of Theorem 4.17 by applying it to Example 3.23 for a feasible power demand. In addition we plug in an unfeasible power demand, and see what happens in this case.

The next theorem proves that the set $\mathcal{F}$ of feasible power demands is closed and convex, and gives a geometric characterization of $\mathcal{F}$ in terms of the closed half-spaces H_λ .

Theorem 4.18 (M4). *The set $\mathcal{F}$ of feasible power demands is closed and convex. Moreover, the set $\mathcal{F}$ is the intersection of all half-spaces H_λ with $\lambda \in \Lambda_1$ (see (4.24)), and $\mathcal{F}$ coincides with the image of P_c . I.e.,*

$$\mathcal{F} = P_c(\text{cl}(\mathcal{D})) = \text{cl}(\text{conv}(\mathcal{F})) = \bigcap_{\lambda \in \Lambda_1} H_\lambda = \text{im } P_c. \quad (4.33)$$

Proof. We will first prove convexity. By definition we have $P_c(\text{cl}(\mathcal{D})) \subseteq \mathcal{F} \subseteq \text{conv}(\mathcal{F})$. Hence it suffices to show that $\text{conv}(\mathcal{F}) \subseteq P_c(\text{cl}(\mathcal{D}))$. Let $\tilde{P}_c \in \text{conv}(\mathcal{F})$. If $\tilde{P}_c \in \partial \text{conv}(\mathcal{F})$, then Corollary 4.15 implies that there exists $\tilde{V}_L \in \partial \mathcal{D}$ such that $\tilde{P}_c = P_c(\tilde{V}_L) \in P_c(\text{cl}(\mathcal{D}))$. Alternatively, if $\tilde{P}_c \in \text{int}(\text{conv}(\mathcal{F}))$, then by Lemma 4.16 there exists a path $\gamma : [0, 1] \rightarrow \mathcal{D}$

Figure 4.1: Plots of the power domain (left) and the voltage domain (right) of the loads corresponding to the four cases in Example 3.23. The blue square in the left plot is the feasible power demand $\tilde{P}_c = \begin{pmatrix} 0.6 & 0.5 \end{pmatrix}^\top$, which coincides with Case 2–4 in Example 3.23. The solution γ to the initial value problem (4.31) is plotted on the right in blue. The blue trajectory on the left is the trajectory in $\mathcal{F}$ that corresponds to $P_c(\gamma)$. The blue dot in the right plot corresponds to the long-term voltage stable operating point belonging to $\tilde{P}_c$.

The red square in the left plot is the power demand $\tilde{P}_c = \begin{pmatrix} 0.8 & 0.6 \end{pmatrix}^\top$, which coincides with Case 1 in Example 3.23, and is not feasible. Applying the initial value problem (4.31) results in the red trajectory of the voltages, which meets the boundary of $\mathcal{D}$, after which the initial value problem becomes ill-defined and numerically unstable. Naturally, the corresponding trajectory in the left-hand plot in red does not converge to the red square.

such that $\gamma(0) = V_L^* \in \mathcal{D}$ and (4.27) holds. In particular, (4.27) implies that $\tilde{P}_c = P_c(\gamma(1)) \in P_c(\mathcal{D})$. Thus $\text{conv}(\mathcal{F}) \subseteq P_c(\text{cl}(\mathcal{D}))$, and thus $\mathcal{F}$ is convex. Corollary 4.13, Theorem 4.14 and the convexity of $\mathcal{F}$ further imply that

$$\bigcap_{\lambda \in \Lambda_1} H_\lambda = \text{cl}(\text{conv}(\mathcal{F})) = \text{conv}(\mathcal{F}) = \mathcal{F}. \quad (4.34)$$

Finally we show that $\mathcal{F} = \text{im } P_c$. Note that $\mathcal{F} \subseteq \text{im } P_c$ by definition. Corollary 4.13 proves that $\text{cl}(\text{conv}(\text{im } P_c)) = \text{cl}(\text{conv}(\mathcal{F}))$. We therefore have

$$\text{cl}(\text{conv}(\text{im } P_c)) = \text{cl}(\text{conv}(\mathcal{F})) = \mathcal{F} \subseteq \text{im } P_c.$$

Since $\text{im } P_c \subseteq \text{cl}(\text{conv}(\text{im } P_c))$, we have $\mathcal{F} = \text{im } P_c$. $\square$

Remark 4.19. Theorem 4.18 shows that the image of P_c coincides with $\mathcal{F}$. This means that, if nonpositive voltage potentials would be permitted, then any feasible power demand that is satisfied by nonpositive voltage potentials can also be satisfied by positive voltage potentials. Hence, from a theoretical standpoint, the restriction to positive voltage potentials does not make the set of feasible power demands more conservative.

Due to the convexity of $\mathcal{F}$, Corollary 4.15 implies that $\partial\mathcal{D}$ and $\partial\mathcal{F}$ in are one-to-one correspondence. Moreover, Lemma 4.9 implies that there are no vectors $x \notin \partial\mathcal{D}$ such that $P_c(x) \in \partial\mathcal{F}$. This implies the following corollary.

Corollary 4.20 (M2). *For each $\tilde{P}_c$ on the boundary of $\mathcal{F}$ there exist a unique $\tilde{V}_L \in \mathbb{R}^n$ that satisfies (3.13). All such $\tilde{V}_L$ satisfy $\tilde{V}_L > \mathbf{0}$ and form the boundary of $\mathcal{D}$. Hence, there is a one-to-one correspondence between $\partial\mathcal{D}$ and $\partial\mathcal{F}$.*

Theorem 4.17 and Corollary 4.20 immediately imply that there is a one-to-one correspondence between the set $\mathcal{D}$ of long-term voltage stable operating points and the power demands which are feasible under small perturbation, by which we mean that such a power demand $\tilde{P}_c$ is feasible and does not lie on the boundary of $\mathcal{F}$ (i.e., $\tilde{P}_c \in \text{int}(\mathcal{F})$). Consequently, if a power demand is feasible under small perturbation, then there exists a unique long-term voltage stable operating point which satisfies the power flow equation.

Corollary 4.21. *There is a one-to-one correspondence between the long-term voltage stable operating points $\mathcal{D}$ and the feasible power demands under small perturbations $\text{int}(\mathcal{F})$.*

4.5 A necessary and sufficient LMI condition for feasibility

We conclude this section by restating the geometric characterization of $\mathcal{F}$ in Theorem 4.18 in terms of an LMI condition. In the context of Problem 3.8, [3] presents a necessary LMI condition for the feasibility of power demands, and states that the LMI condition is also necessary when the set of feasible power demands is closed and convex, as was shown in Theorem 4.18. The next theorem recovers this result and extends the result for power demands which are feasible under small perturbation.

Theorem 4.22 (M5). *A vector $\tilde{P}_c$ of power demands is feasible (i.e., $\tilde{P}_c \in \mathcal{F}$) if and only if there does not exist a positive vector $\nu \in \mathbb{R}^n$ such that the $(n+1) \times (n+1)$ matrix*

$$\begin{pmatrix} [\nu]Y_{LL} + Y_{LL}^\top[\nu] & [\nu]\mathcal{I}_L^* \\ ([\nu]\mathcal{I}_L^*)^\top & 2\nu^\top \tilde{P}_c \end{pmatrix} = 2 \begin{pmatrix} h(\nu) & \frac{1}{2}[\nu]\mathcal{I}_L^* \\ \frac{1}{2}([\nu]\mathcal{I}_L^*)^\top & \nu^\top \tilde{P}_c \end{pmatrix} \quad (4.35)$$

is positive definite. Similarly, $\tilde{P}_c$ is feasible under small perturbation (i.e., $\tilde{P}_c \in \text{int}(\mathcal{F})$) if and only if there does not exist a positive vector $\nu \in \mathbb{R}^n$ such that (4.35) is positive semi-definite.

Proof. We will prove the logical transposition.

($\Leftarrow$): Without loss of generality we assume that $\|\nu\|_1 = 1$. If (4.35) is positive semi-definite, then $h(\nu)$ is positive semi-definite. It follows from Lemmas A.4 and A.5 that

$h(\nu)$ is an irreducible M-matrix. Let $v > 0$ be a Perron vector of $h(\nu)$. Suppose that $h(\nu)$ is singular, then $h(\nu)v = 0$ by Lemma 2.12. However, note that for $t \in \mathbb{R}$ we have

$$\begin{pmatrix} tv \\ 1 \end{pmatrix}^\top \begin{pmatrix} h(\nu) & \frac{1}{2}[\nu]\mathcal{I}_L^* \\ \frac{1}{2}([\nu]\mathcal{I}_L^*)^\top & \nu^\top \tilde{P}_c \end{pmatrix} \begin{pmatrix} tv \\ 1 \end{pmatrix} = tv^\top [\nu]\mathcal{I}_L^* + \nu^\top \tilde{P}_c,$$

which is a nonconstant line in t since $v^\top [\nu]\mathcal{I}_L^* > 0$, and is not bounded from below. This contradicts the assumption that (4.35) is positive semi-definite. Hence $h(\nu)$ must be positive definite and $\nu \in \Lambda_1$. Alternatively, if (4.35) is positive definite, then $h(\nu)$ is positive definite. If $h(\nu)$ is positive definite, then by the Haynsworth inertia additivity formula [62, Section 0.10], (4.35) is positive definite (semi-definite) if and only if

$$\nu^\top \tilde{P}_c - \frac{1}{4}([\nu]\mathcal{I}_L^*)^\top h(\nu)^{-1} [\nu]\mathcal{I}_L^* > (\geq) 0. \quad (4.36)$$

Using (4.12) and (4.15), we note that (4.36) is equivalent to

$$\nu^\top \tilde{P}_c > (\geq) \frac{1}{4}([\nu]\mathcal{I}_L^*)^\top h(\nu)^{-1} [\nu]\mathcal{I}_L^* = \|\varphi(\nu)\|_{h(\nu)}^2. \quad (4.37)$$

Theorem 4.18 implies that $\tilde{P}_c$ is not feasible if and only if there exists $\lambda \in \Lambda$ such that $\tilde{P}_c \notin H_\lambda$, or equivalently, $\lambda^\top \tilde{P}_c > \|\varphi(\lambda)\|_\lambda^2$. Thus, if (4.35) is positive definite, then the strict inequality in (4.37) holds and $\tilde{P}_c$ is not feasible. Moreover, if equality in (4.37) holds then

$$\nu^\top \tilde{P}_c = \|\varphi(\nu)\|_{h(\nu)}^2 = \nu^\top P_c(\varphi(\nu)).$$

Lemma 4.9 implies that $\tilde{P}_c = P_c(\varphi(\nu))$, and thus $\tilde{P}_c \in \partial\mathcal{F}$ by Theorem 4.14. Thus, if (4.35) is positive semi-definite, then $\tilde{P}_c \notin \mathcal{F}$ or $\tilde{P}_c \in \partial\mathcal{F}$, and therefore $\tilde{P}_c \notin \text{int}(\mathcal{F})$.

($\Rightarrow$): The converse is obtained by reversing the steps. $\square$

Theorem 4.22 presents a necessary and sufficient LMI conditions for the feasibility (under small perturbation) of a DC power grid with constant-power loads. A more common formulation of Theorem 4.22 as an LMI condition can be obtained by replacing $[\nu]$ by a positive definite diagonal matrix D , and replacing $\nu^\top \tilde{P}_c$ by $\mathbb{1}^\top D\tilde{P}_c$ (cf. [3]).

4.6 Desirable operating points

In this section we show that for each feasible vector of power demands the different Definitions 3.11, 3.12 and 3.14 of desirable operating points in Section 3.3 identify the same unique operating point. It was shown in [41] that for each feasible power demand there exists a unique operating point which is a high-voltage solution, and that this operating point is “almost surely” long-term voltage stable. In addition, [41] states that

this operating point is the unique long-term voltage stable operating point if all power demands have the same sign. The next theorem sharpens these results by showing that the long-term voltage stable operating point associated to a feasible vector of power demands is a strict high-voltage solution.

Theorem 4.23 (M6). *Let $\tilde{P}_c$ be a feasible vector of power demands (i.e., $\tilde{P}_c \in \mathcal{F}$). Let $\tilde{V}_L \in \text{cl}(\mathcal{D})$ be such that $\tilde{V}_L$ is an operating point associated to $\tilde{P}_c$ (i.e., $\tilde{P}_c = P_c(\tilde{V}_L)$), which exists and is unique by Theorem 4.17. Suppose there exists a vector $\tilde{V}'_L \in \mathbb{R}^n$ such that $\tilde{V}'_L \neq \tilde{V}_L$ and $\tilde{P}_c = P_c(\tilde{V}'_L)$, then $\tilde{V}'_L < \tilde{V}_L$. Hence, $\tilde{V}_L$ is a strict high-voltage solution. Moreover, $\frac{1}{2}(\tilde{V}'_L + \tilde{V}_L)$ lies on the boundary of $\mathcal{D}$.*

Proof. If $\tilde{P}_c \in \partial\mathcal{F}$, then by Corollary 4.20 the operating point $\tilde{V}_L \in \partial\mathcal{D}$ is the unique operating point associated to $\tilde{P}_c$. Hence a second operating point $\tilde{V}'_L$ does not exist. The uniqueness of $\tilde{V}_L$ implies that $\tilde{V}_L$ is a high-voltage solution and is dissipation-minimizing.

If $\tilde{P}_c \in \text{int}(\mathcal{F})$, then by Corollary 4.21 we have $\tilde{V}_L \in \mathcal{D}$. We define the vectors $v := \frac{1}{2}(\tilde{V}_L + \tilde{V}'_L)$ and $\mu := \frac{1}{2}(\tilde{V}_L - \tilde{V}'_L)$, and the line $\gamma(\theta) := v + \theta\mu$. Note that $\gamma(1) = \tilde{V}_L$ and $\gamma(-1) = \tilde{V}'_L$. Since $\tilde{V}_L \in \mathcal{D}$ and $\tilde{V}'_L \notin \text{cl}(\mathcal{D})$ we have $\tilde{V}_L \neq \tilde{V}'_L$, and so $\mu \neq 0$. Lemma 4.9 implies that

$$P_c(\gamma(\theta)) = P_c(v + \theta\mu) = P_c(v) + \theta \frac{\partial P_c}{\partial V_L}(v)\mu - \theta^2[\mu]Y_{LL}\mu. \quad (4.38)$$

Since $\tilde{P}_c = P_c(\gamma(1)) = P_c(\gamma(-1))$, it follows from (4.38) that

$$\frac{\partial P_c}{\partial V_L}(v)\mu = 0. \quad (4.39)$$

We therefore have

$$P_c(\gamma(\theta)) = P_c(v) - \theta^2[\mu]Y_{LL}\mu, \quad (4.40)$$

which describes a half-line contained in $\mathcal{F}$. Note also that $P_c(\gamma(\theta)) = P_c(\gamma(-\theta))$ and $\gamma(\theta) \neq \gamma(-\theta)$ if $\theta \neq 0$, which shows that the map $P_c(V_L)$ gives rise to a two-to-one correspondence between the line $\gamma(\theta)$ and the half-line (4.40) for $\theta \neq 0$. The line $\gamma(\theta)$ crosses the boundary of $\mathcal{D}$ since $\gamma(1) \in \mathcal{D}$ and $\gamma(-1) \notin \mathcal{D}$. Let $\hat{\theta}$ be such that $\gamma(\hat{\theta}) \in \partial\mathcal{D}$. Corollary 4.20 implies that there does not exist $\hat{V}_L \neq \gamma(\hat{\theta})$ such that $P_c(\gamma(\hat{\theta})) = P_c(\hat{V}_L)$. Hence, due to the two-to-one correspondence between $\gamma(\theta)$ and (4.40) for $\theta \neq 0$, we conclude that $\hat{\theta} = 0$ and $\gamma(0) = v \in \partial\mathcal{D}$. Corollary 4.4 implies that $-\frac{\partial P_c}{\partial V_L}(v)$ is a singular M-matrix. Note that μ lies in the kernel of $-\frac{\partial P_c}{\partial V_L}(v)$ due to (4.39), and it follows from Lemma 2.12 that $\pm\mu > 0$ and that μ spans the kernel of $-\frac{\partial P_c}{\partial V_L}(v)$. Since $\gamma(\theta)$ intersects $\partial\mathcal{D}$ only when $\theta = 0$, and since $\gamma(1) \in \mathcal{D}$ and $\gamma(-1) \notin \text{cl}(\mathcal{D})$, it follows that $\gamma(\theta) \in \mathcal{D}$ if and only if $\theta > 0$. However, if $\mu < 0$ then $\gamma(\theta) = v + \theta\mu$ is a negative vector for sufficiently large θ , which contradicts that all vectors in $\text{cl}(\mathcal{D})$ are positive. We conclude that $\mu > 0$, which by definition of μ implies that $\tilde{V}_L > \tilde{V}'_L$. The operating point $\tilde{V}_L$ is a strict high-voltage solution by Definition 3.14. $\square$

Remark 4.24. Theorem 4.23 shows that the operating point $\frac{1}{2}(\tilde{V}_L + \tilde{V}'_L)$ lies on boundary of $\mathcal{D}$, and shows that the line through the operating points $\tilde{V}'_L$ and $\tilde{V}_L$ is mapped by $P_c(V_L)$ to a half-line through $\tilde{P}_c$ in $\mathcal{F}$, which gives rise to a two-to-one correspondence between this line and the half-line in $\mathcal{F}$, except at the base point of the half-line. These observations are, to some extent, also found in [39], which analyzes a more general quadratic map.

We conclude by proving that the different types of operating points defined in Section 3.3 describe one and the same operating point.

Theorem 4.25 (M6). *Consider a power grid with Kirchhoff matrix Y and source voltages V_S . Let $\tilde{P}_c$ be a feasible vector of power demands, and let $\tilde{V}_L$ be an associated operating point (i.e., $\tilde{P}_c = P_c(\tilde{V}_L)$). The following statements are equivalent:*

- i) $\tilde{V}_L$ is long-term voltage semi-stable (i.e., $\tilde{V}_L \in \text{cl}(\mathcal{D})$);
- ii) $\tilde{V}_L$ is the unique long-term voltage semi-stable operating point associated to $\tilde{P}_c$;
- iii) $\tilde{V}_L$ is dissipation-minimizing;
- iv) $\tilde{V}_L$ is the unique dissipation-minimizing operating point associated to $\tilde{P}_c$;
- v) $\tilde{V}_L$ is a high-voltage solution;
- vi) $\tilde{V}_L$ is a strict high-voltage solution.

Proof. Theorem 4.17 guarantees the existence and uniqueness of a long-term voltage semi-stable operating point $\hat{V}_L$ associated to $\tilde{P}_c$. It therefore suffices to show that $\hat{V}_L$ is the unique operating point which satisfies statements iii)-vi) individually. Theorem 4.23 implies that $\hat{V}_L$ is a (strict) high-voltage solution. Note that there exists at most one high-voltage solution, since $\tilde{V}_L \leq \tilde{V}'_L$ and $\tilde{V}'_L \leq \tilde{V}_L$ imply that $\tilde{V}_L = \tilde{V}'_L$. Corollary 3.15 implies that $\hat{V}_L$ is the unique dissipation-minimizing operating point. $\square$

Theorem 4.25 shows that the desirable operating points defined in Section 3.3 coincide, and that we may speak of a single desired operating point.

Definition 4.26. An operating point $\tilde{V}_L$ associated to $\tilde{P}_c$ is the *desired operating point* if $\tilde{V}_L$ satisfies one (and therefore all) of the equivalent statements in Theorem 4.25.

Remark 4.27. In [41] it was shown that for a feasible vector of power demands the algorithm proposed in [41] converges to a high-voltage solution. By Theorem 4.25 this means that this algorithm converges to the unique long-term voltage semi-stable operating point associated to these power demands.

4.7 Notes on operational limits and the source voltages

Recall that we made the standing assumption that there are no operational limits on the components in the power grid. Before we close this chapter we investigate how the addition of operational limits would affect the power flow feasibility problem. In this section we also give some insight into how power flow feasibility is affected by changes in the voltage potentials V_S at the sources, or similarly, by changes in the source-injected currents $\mathcal{I}_L^*$.

Components in a (dynamical) power grid are commonly subjected to operational limits. When such a limit is violated, measures are taken to protect the components, for example by tripping an overloaded line, or by disconnection a load. Naturally it is in our interest to keep within these limits, both in dynamical and the steady-state case. In this section we continue our steady-state analysis of DC power grids, while taking these operational limits into account.

We consider three types of operational limits. First, we consider an upper bound for P_S , the power supplied by the sources (see (3.4)). These upper bounds are collected in the nonnegative vector $\bar{P}_S \in \mathbb{R}^m$, and should satisfy

$$P_S(V_L) \leq \bar{P}_S. \quad (4.41)$$

Second, we consider lower and upper bounds for V_L , the voltage potentials at the loads. We collect these bounds in the vectors $\underline{V}_L$ and $\bar{V}_L$ respectively, and we desire that

$$\underline{V}_L \leq V_L \leq \bar{V}_L. \quad (4.42)$$

Lastly, the dissipated power in a line, which is given by $G_{ij}(V_i - V_j)^2$ for the line $\{i, j\}$, should be upper bounded by $\bar{P}_{ij}$, meaning that

$$G_{ij}(V_i - V_j)^2 \leq \bar{P}_{ij} \text{ for all } i \sim j. \quad (4.43)$$

Note that the bounds 4.42 on the dissipated power in a line is equivalent to the inequality

$$-\sqrt{\frac{\bar{P}_{ij}}{G_{ij}}} \leq V_i - V_j \leq \sqrt{\frac{\bar{P}_{ij}}{G_{ij}}}.$$

Figure 4.2 depicts the regions in the voltage domain that these operational limits describe.

We are interested in solving Problem 3.8 such that the operational limits (4.41)-(4.43) are satisfied, while also minimizing the dissipation in the lines. This means that we want

Figure 4.2: Two plots of the voltage domain for the power grid in Example 3.22 with $G_{12} = 2$. We take $P_c = \begin{pmatrix} 0.6 & 0.5 \end{pmatrix}^\top$ and $V_S = 1$, which corresponds to the dynamical power grid in Case2–4 in Example 3.23 and Figure 3.9. The blue area corresponds to the set $\mathcal{D}$ of long-term voltage stable operating points. We take as operational limits $\bar{P}_S = 4$, $\underline{V}_L = \begin{pmatrix} 0.2 & 0.2 \end{pmatrix}^\top$, $\bar{V}_L = \begin{pmatrix} 0.8 & 0.8 \end{pmatrix}^\top$, $\bar{P}_{12} = 1$ and $\bar{P}_{13} = \bar{P}_{23} = 1.73$. The gray and yellow areas in the left plot correspond to (4.41) and (4.42), respectively. The magenta, green and red areas in the right plot corresponds to the (4.43) for the lines $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ respectively. The blue and magenta squares are two operating points corresponding to P_c . Both operating points satisfy the operational limits, and the blue square is the long-term voltage stable operating point and is the unique solution to (4.44).

Figure 4.3: The same plots as in Figure 4.2, but now for $V_S = 1.1$. We observe that the long-term voltage stable operating point, depicted by the blue square, does not lie in the yellow rectangle and therefore does not satisfy (4.42). It is therefore not a solution to (4.44). The operating point depicted by the magenta square does satisfy the operational limits and is the only solution to (4.44). It is however not long-term voltage stable.

to solve the following minimization problem:

$$\begin{aligned}
& \underset{V_L > 0}{\text{minimize}} && R(V_L, V_S) \\
& \text{subject to} && P_L(V_L) = -P_c \\
& && P_S(V_L) \leq \bar{P}_S \\
& && \underline{V}_L \leq V_L \leq \bar{V}_L \\
& && G_{ij}(V_i - V_j)^2 \leq \bar{P}_{ij} \text{ for all } i \sim j,
\end{aligned} \tag{4.44}$$

where V_S is given.

The minimization problem (4.44) is equivalent to the minimization problem (3.17), but that in addition V_L should satisfy our operational limits (4.41)-(4.43). Theorem 4.25 implied that, if the minimization problem (3.17) is feasible, then the long-term voltage stable operating points is also dissipation-minimizing, and is the unique solution to (3.17). This means that if this solution to (3.17) also satisfies (4.41)-(4.43), then it is the unique solution to (4.44). This is the case in Figure 4.2.

It should be noted that it may happen that the long-term voltage stable solution does not satisfy the operational limits, while an other operating point does. This case can be observed in Figure 4.3. Hence, a minimizers of (4.44) is not necessarily long-term voltage stable. In the context of the dynamical power grid in Section 3.6 this means that there may exist an equilibrium which satisfies the operational limits, but that none of the equilibria are locally asymptotically stable.

Figure 4.2 and Figure 4.3 depict the voltage domain corresponding to the same power grid with the same loads, but with different source voltages. It is observed that increasing the source voltages can also increase the long-term voltage stable operating point corresponding to a vector of constant-power demand, and that this may lead to a violation of the operational limits. On the other hand it is intuitive that increasing the currents that are injected into the loads will improve the power flow in a power grid. The following proposition gives a formal argument for these two phenomena.

Proposition 4.28. *The set $\text{cl}(\mathcal{D})$ of long-term voltage semi-stable operating points is strictly monotone increasing in the entries of $\mathcal{I}_L^*$. Also, any increase in the entries of $\mathcal{I}_L^*$ makes the power flow feasibility strictly less conservative.*

Proof. Theorem 4.7 states that for each point in $\text{cl}(\mathcal{D})$ can be written as

$$\frac{1}{2}h(\lambda)^{-1}[\lambda](\mathcal{I}_L^* + r\mathbb{1}) \tag{4.45}$$

for a unique $\lambda \in \Lambda_1$ and $r \geq 0$. The vector λ is positive and $h(\lambda)$ is an irreducible M-matrix, which implies that $h(\lambda)^{-1}[\lambda]$ is a positive matrix. Hence (4.45), and therefore $\text{cl}(\mathcal{D})$, is strictly monotone increasing in the elements of $\mathcal{I}_L^*$.

Figure 4.4: Two plots of the set $\mathcal{F}$ in the power domain (left) and the set $\mathcal{D}$ (right) in the voltage domain corresponding to Example 3.22 with $G_{12} = 2$, for different values of V_S . In blue are the sets that correspond to the case $V_S = 1$, whereas in yellow are the sets that correspond to $V_S = 1.1$. It is observed in the plot on the left that in the set $\mathcal{F}$ becomes less conservative after an increase in V_S . In the plot on the right we see that the elements in the set $\mathcal{D}$ increase after an increase in V_S . These observations illustrate Proposition 4.28. The plot on the right also gives a comparison of the sets $\mathcal{D}$ in Figure 4.2 and Figure 4.3.

Similarly, by Theorem 4.12 and Theorem 4.18, a point $\tilde{P}_c$ lies in $\mathcal{F}$ if and only if

$$\lambda^\top \tilde{P}_c \leq \|\varphi(\lambda)\|_{h(\lambda)}^2 = \frac{1}{4} \mathcal{I}_L^*{}^\top [\lambda] h(\lambda)^{-1} [\lambda] \mathcal{I}_L^* \quad (4.46)$$

for all $\lambda \in \Lambda_1$. As before, the matrix $[\lambda] h(\lambda)^{-1} [\lambda]$ is positive. This implies that an increase in an entry of $\mathcal{I}_L^*$ implies that (4.46) holds with a strict inequality for all $\lambda \in \Lambda_1$. Thus, P_c lies in the interior of $\mathcal{F}$ after this increase. $\square$

Note that Proposition 4.28 also applies for increased in any of the voltage potentials at the sources. This follows from the observation that $\hat{\mathcal{I}}_L^* \leq \tilde{\mathcal{I}}_L^*$ whenever $\hat{V}_S \leq \tilde{V}_S$ due to 3.8 and the fact that $-Y_{LS}$ is a nonnegative matrix.

Figure 4.4 illustrates Proposition 4.28 by a comparison of the sets $\mathcal{F}$ and $\mathcal{D}$ corresponding to Figure 4.2 and Figure 4.3.

Remark 4.29. Proposition 4.28 implies that any vector of constant-power demands becomes feasible by a sufficiently large increase of the source-injected currents at any load, or similarly, by a sufficiently large increase of the voltage potentials at any source. This is interesting in the context of the dynamical example of Section 3.6 for the following reason. If any of the voltage potentials at the sources is not bounded from above, then a sufficient large choice of this voltage potential will ensure that the power flow is feasible. This, in a sense, trivializes the control design from the point of power flow feasibility. In a more practical context there do exist operational limits on the source voltages. In this

case, Proposition 4.28 implies that power flow feasibility is fundamentally limited by the largest choices of voltage potentials at the sources.

4.8 Conclusion and discussion

In this chapter we have studied the power flow feasibility of DC power grids with constant-power loads, and have presented a framework for the analysis of this feasibility problem. Specifically, we have presented a geometric characterization of the feasible power demands in terms of half-spaces, along with necessary and sufficient LMI conditions to check if a vector of power demands is feasible (under small perturbation). In addition, we have given a novel proof for the convexity of the set of feasible power demands. More importantly, we proved that there exists a one-to-one correspondence between the feasible power demands and the long-term voltage semi-stable operating points. This shows that for each feasible power demand there exists a unique operating point which is long-term voltage semi-stable and satisfies the power flow equations. This operating point can be found by solving an initial value problem. The existence and uniqueness of this operating point proves that long-term (semi-)stability can be guaranteed for each feasible power demand. Furthermore, we showed that there exists a one-to-one correspondence between the feasible power demands under small perturbations and the long-term voltage stable operating points. Finally, we showed that the operating points corresponding to a power demand which are long-term voltage semi-stable, dissipation-minimizing, or a (strict) high-voltage solution, are one and the same.

Discussion and future research directions

As was explained in Remark 4.1, the approach and results in this chapter can be extended naturally to the case where Y_{LL} is any (possibly asymmetric) nonsingular irreducible M-matrix, and where $\mathcal{I}_L^*$ is any nonnegative nonzero vector. In particular this means that Y_{LL} can be substituted for any grounded Laplacian. This allows us to also study the more general ZIP-loads, which are loads that satisfy

$$P_L(V_L) + [V_L][Z_c]V_L + [V_L]\mathcal{I}_c + P_c = 0, \quad (4.47)$$

where the vectors Z_c , $\mathcal{I}_c$ and P_c represent respectively the constant-impedance, constant-current and constant-power components of the ZIP-loads. The contents of this chapter apply to the subset of ZIP-loads for which $Y_{LL} + [Z_c]$ is positive definite, $Y_{LS}V_S + \mathcal{I}_c$ is a nonpositive nonzero vector, and where there again are no sign-restrictions on P_c . Note that the load (4.47) is quadratic in the voltage potential at a load, which allows for a

second-order approximation of the load characteristic. For ZIP-loads the power flow equations are described by

$$[V_L]Y_{LL}V_L + [V_L]Y_{LS}V_S + [V_L][Z_c]V_L + [V_L]\mathcal{I}_c + P_c = 0.$$

By rewriting this equation, we observe that

$$[V_L](Y_{LL} + [Z_c])V_L + [V_L](Y_{LS}V_S + \mathcal{I}_c) + P_c = 0$$

which is of the same form as (3.13). Hence, the results of this chapter apply to this more general case if we substitute Y_{LL} by $Y_{LL} + [Z_c]$, and substitute $\mathcal{I}_L^*$ by $-(Y_{LS}V_S + \mathcal{I}_c)$, by Remark 4.1. It would be interesting to investigate general class of ZIP-loads as well, meaning that Z_c and $\mathcal{I}_c$ are unrestricted.

The case where $\mathcal{I}_L^* = 0$ can be studied by taking the limit $\mathcal{I}_L^* \rightarrow 0$ throughout this chapter. This case correspond to a resistive circuit with shunts, where there is no current injection at the loads (see (3.9)), and can be applied to power grids where the source *powers* (and not the source voltages) are fixed, and where all loads are constant-power loads. The case where $\mathcal{I}_L^* = 0$ was also studied in [50], although its contents are somewhat disconnected from the power systems literature. Some aspects of the analysis in [50] may be recovered by taking the limit $\mathcal{I}_L^* \rightarrow 0$ throughout this chapter and the next chapter.

The papers [7, 52] give a first-order approximation of the desired operating point. Higher-order approximations of the desired operating point may be obtained approximating the solution of the initial value problem in Theorem 4.17 by evaluating its Maclaurin approximations at $\theta = 1$. Further study of the quality of these approximations may be a viable research direction.

The optimal power flow problem [14, 42] is a active research topic which formulates the power flow in a power grid as an optimization problem, and studies its feasibility. A special case of this optimization problem is the DC optimal power flow problem [9, 35], which is given by

$$\begin{aligned} & \underset{V_L > 0, V_S > 0}{\text{minimize}} && R(V_L, V_S) \\ & \text{subject to} && P_L(V_L, V_S) \leq -P_c \\ & && P_S(V_L, V_S) \leq \bar{P}_S \\ & && \underline{V}_L \leq V_L \leq \bar{V}_L \\ & && \underline{V}_S \leq V_S \leq \bar{V}_S \\ & && G_{ij}(V_i - V_j)^2 \leq \bar{P}_{ij} \text{ for all } i \sim j. \end{aligned} \tag{4.48}$$

This optimization problem is fundamentally different from the problem (4.44). Note for instance that (4.48) also minimizes over the voltages at the source, in contrast to (4.44), where source voltages are given. Also, the inequality constraint $P_L(V_L) \leq -P_c$ in

(4.48), which demands that a load drains a *minimal amount* of power, is different from the equality constraint in $P_L(V_L, V_S) = -P_c$ in (4.44). It is unclear how the results in this thesis carry over to the DC optimal power flow problem. It would be interesting to study if and/or how we can guarantee that a solution to (4.48) is also long-term voltage stable.

The result ALT1 of [2] shows that the unfeasibility of the LMI condition in Theorem 4.22 is equivalent to the feasibility of its “alternative”, which is again a linear problem. It would be interesting to study if the operational limits (4.41)-(4.43) can be included as linear constraints on the solution to this alternative, which again results in a linear problem.

Finally, a more ambitious direction of research is the generalization of this approach to complex line conductances, and to the power flow of AC power grids at “steady state”. However, it is known that the set $\mathcal{F}$ is not convex when Y_{LL} is complex [21, 40], which indicates that the geometric characterization of Theorem 4.18 does not extend to such power grids.

5 | The feasibility of nonnegative constant power demands

In this chapter we study the feasibility of nonnegative power demands (*i.e.*, power demands $\tilde{P}_c$ such that $\tilde{P}_c \geq \mathbb{0}$). Recall from Chapter 3 that we consider constant-power loads that can both drain power or inject power. However, practical applications of DC power grids often deal with constant-power loads that do not inject power into the network, in which case the power demands are nonnegative. The goal of this chapter is to extend and refine the result of Chapter 4 for nonnegative power demands. In particular we show that the necessary and sufficient LMI condition for the feasibility of a vector of power demands $\tilde{P}_c \in \mathbb{R}^n$ (Theorem 4.22) can be refined, leading to a (nonlinear) condition which is cheaper to compute.

This chapter is structured as follows. We first identify the operating points corresponding to a nonnegative power demand (Lemma 5.1). In addition we present a refinement for the geometric characterization of Theorem 4.18 (Lemma 5.3), which motivates us to study the boundary of $\mathcal{F}$ in more detail. To study this boundary we deduce an alternative parametrization of $\mathcal{D}$ (Theorem 5.5), which is in a sense dual to the parametrization in Theorem 4.7. We subsequently give a parametrization of the boundary of $\mathcal{F}$ (Theorem 5.7). This parametrization gives rise to a parametrization of the boundary of $\mathcal{F}$ in the nonnegative orthant (Theorem 5.11). We then reformulate the geometric characterization (Corollary 5.13), and refine the necessary and sufficient LMI condition of Theorem 4.22 for nonnegative power demands (Theorem 5.14).

5.1 Nonnegative power demands and their operating points

We are interested in the nonnegative feasible power demands, which are described by the set $\mathcal{F} \cap \mathcal{N}$, where

$$\mathcal{N} := \{ \nu \in \mathbb{R}^n \mid \nu \geq \mathbb{0} \}$$

denotes the nonnegative vectors. The next lemma characterizes the operating points which correspond to a nonnegative power demand.

Lemma 5.1. *A feasible power demand $\tilde{P}_c$ is nonnegative (*i.e.*, $\tilde{P}_c \in \mathcal{F} \cap \mathcal{N}$) if and only if the operating points $\tilde{V}_L$ associated to $\tilde{P}_c$ satisfy $Y_{LL}\tilde{V}_L \leq Y_{LL}V_L^* = \mathcal{I}_L^*$.*

Figure 5.1: A plot of the voltage domain for the power grid in Example 3.22 with $G_{12} = 2$. The blue area corresponds to the set $\mathcal{D}$ of long-term voltage stable operating points. The red area depicts the operating points corresponding to a nonnegative power demand. The yellow area corresponds to the vectors in $\mathcal{M}$, and contains the set $\text{cl}(\mathcal{D})$. The black operating points correspond to the black power demands in Figure 6.1.

Proof. Since operating points are assumed to be positive, we have $\tilde{V}_L > \mathbb{0}$. Hence

$$\tilde{P}_c = P_c(\tilde{V}_L) = [\tilde{V}_L]Y_{LL}(V_L^* - \tilde{V}_L) \geq \mathbb{0}$$

if and only if $Y_{LL}(V_L^* - \tilde{V}_L) = \mathcal{I}_L^* - Y_{LL}\tilde{V}_L \geq \mathbb{0}$, where we used (3.10). $\square$

Figure 5.1 illustrates the location of these operating points in the voltage domain.

Lemma 5.1 shows that all operating points corresponding to a positive power demand lie in the polyhedral set

$$\left\{ \tilde{V}_L \in \mathbb{R}^n \mid \tilde{V}_L > \mathbb{0}, Y_{LL}\tilde{V}_L \leq \mathcal{I}_L^* \right\}. \quad (5.1)$$

Note that equality holds in $Y_{LL}\tilde{V}_L \leq \mathcal{I}_L^*$ if and only if $\tilde{V}_L = V_L^*$, which corresponds to the power demand $\tilde{P}_c = \mathbb{0}$. The next result shows that the vector of open-circuit voltages V_L^* element-wise strictly dominates all operating points corresponding to a nonzero nonnegative power demand.

Corollary 5.2. *Let $\tilde{P}_c \neq \mathbb{0}$ be a nonnegative feasible power demand, then any operating point $\tilde{V}_L$ associated to $\tilde{P}_c$ satisfies $\tilde{V}_L < V_L^*$. Hence, (5.1) is bounded.*

Proof. The matrix Y_{LL} is an irreducible M-matrix, and hence its inverse is positive by Lemma 2.15. By Lemma 5.1 we have $Y_{LL}(V_L^* - \tilde{V}_L) \geq \mathbb{0}$. Since $P_c(V_L^*) = \mathbb{0}$ and $\tilde{P}_c \neq \mathbb{0}$ it

follows that $\tilde{V}_L \neq V_L^*$ and therefore $Y_{LL}(V_L^* - \tilde{V}_L) \not\geq 0$. Multiplying this inequality by the positive matrix Y_{LL}^{-1} implies that $V_L^* - \tilde{V}_L > 0$. $\square$

Using Theorem 4.18 we present a geometric characterization of $\mathcal{F} \cap \mathcal{N}$.

Lemma 5.3. *The set $\mathcal{F} \cap \mathcal{N}$ is closed, convex, bounded, and is the intersection over all $\lambda \in \Lambda_1$ of the half-spaces H_λ for which $P_c(\varphi(\lambda))$ is nonnegative, i.e.,*

$$\mathcal{F} \cap \mathcal{N} = \mathcal{N} \cap \bigcap_{\lambda \in \Lambda_1: P_c(\varphi(\lambda)) \geq 0} H_\lambda.$$

Proof. The set $\mathcal{F} \cap \mathcal{N}$ is the intersection of closed convex sets, and is therefore closed and convex. It follows from (3.23) that $\mathcal{F} \cap \mathcal{N} \subseteq H_1 \cap \mathcal{N}$. The set $\mathcal{F} \cap \mathcal{N}$ is bounded since $H_1 \cap \mathcal{N}$ is bounded. It follows from Theorem 4.18 that

$$\mathcal{F} \cap \mathcal{N} = \mathcal{N} \cap \bigcap_{\lambda \in \Lambda_1} H_\lambda.$$

Since $\mathcal{F} \cap \mathcal{N}$ is closed and convex, it coincides with the intersection of its supporting half-spaces (see Section 4.3). Theorem 4.12 identifies all supporting half-spaces of $\mathcal{F}$, and in particular shows that $P_c(\varphi(\lambda))$ is the unique point of support associated to the half-space H_λ . By definition, H_λ is also a supporting half-space for $\mathcal{F} \cap \mathcal{N}$ if and only if $P_c(\varphi(\lambda)) \in \mathcal{F} \cap \mathcal{N}$, which is equivalent to $P_c(\varphi(\lambda)) \geq 0$. $\square$

The power demands $P_c(\varphi(\lambda))$ for $\lambda \in \Lambda_1$ describe the boundary of $\mathcal{F}$ (see Corollary 4.20 and Theorem 4.7). Lemma 5.3 characterizes all nonnegative feasible power demands in terms of the boundary in the nonnegative orthant (i.e., $\partial F \cap \mathcal{N}$). In its current form, this requires the identification of all λ such that $\lambda \in \Lambda$ and $P_c(\varphi(\lambda)) \geq 0$, which is a nontrivial computational problem. In the remainder of this chapter we deduce an alternative parametrization of the boundary of $\mathcal{F}$ in the nonnegative orthant. This parametrization leads to a more constructive description of all such λ .

5.2 An alternative parametrization of $\mathcal{D}$

In order to parametrize the boundary of $\mathcal{F}$ in the nonnegative orthant, we study the set $\mathcal{D}$ of long-term voltage stable operating points in more detail. In Section 4.2 we have parametrized the set $\mathcal{D}$ and its boundary by means of the set Λ_1 (see Theorem 4.7). In the following we present an alternative parametrization of $\mathcal{D}$, which is dual to the parametrization in Theorem 4.7, in the sense that we parametrize $\mathcal{D}$ by the (right) Perron vector of $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ instead of its transpose $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^\top$.

We introduce the following definitions. For a vector $\mu \in \mathbb{R}^n$ we introduce the notation

$$g(\mu) := [\mu]Y_{LL} + [Y_{LL}\mu]. \quad (5.2)$$

Note that $g(\mu)$ is linear in μ , and that for any vector v we have $g(\mu)v = g(v)\mu$. By using (4.1) and (3.10) we observe that

$$\begin{aligned} \frac{\partial P_c}{\partial V_L}(\tilde{V}_L) &= [Y_{LL}V_L^*] - [\tilde{V}_L]Y_{LL} - [Y_{LL}\tilde{V}_L] \\ &= [\mathcal{I}_L^*] - g(\tilde{V}_L). \end{aligned} \quad (5.3)$$

Analogous to Λ we define the set

$$\mathcal{M} := \{ \mu \mid g(\mu) \text{ is a nonsingular M-matrix} \}.$$

Appendix B lists several properties of the set $\mathcal{M}$. In particular, Lemma B.1 shows that $\mathcal{M}$ is an open cone which lies in the positive orthant, and that $\mathcal{M}$ is simply connected.

Similar to Lemma 4.6, we relate the Perron root and Perron vector of the Jacobian of P_c to the set $\mathcal{M}$.

Lemma 5.4. *Let $r \in \mathbb{R}$ and $\mu \in \mathbb{R}^n$ such that $r \geq 0$ and $\mu > 0$. The Jacobian $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is an irreducible M-matrix with Perron root r and Perron vector μ if and only if $g(\mu)$ is an M-matrix (i.e., $\mu \in \mathcal{M}$) and $\tilde{V}_L$ satisfies*

$$\tilde{V}_L = g(\mu)^{-1}[\mu](\mathcal{I}_L^* + r\mathbb{1}). \quad (5.4)$$

Proof. ($\Rightarrow$): The matrix Y_{LL} is an irreducible Z-matrix and $\mu > 0$, and so $g(\mu)$ is an irreducible Z-matrix by Lemma 2.5. We let s and $v > 0$ denote respectively the Perron root and Perron vector of $g(\mu)$. The matrix $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is an M-matrix, and therefore a Z-matrix. Lemma 4.2 states that $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is a Z-matrix if and only if $V_L^* > 0$, and so $\tilde{V}_L > 0$. Using the fact that (r, μ) is an eigenpair to $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ and substituting (5.3), we observe that

$$r\mu = -\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)\mu = g(\tilde{V}_L)\mu - [\mathcal{I}_L^*]\mu = g(\mu)\tilde{V}_L - [\mu]\mathcal{I}_L^*.$$

By rearranging terms it follows that

$$[\mu]\mathcal{I}_L^* + r\mu = g(\mu)\tilde{V}_L. \quad (5.5)$$

Multiplying (5.5) by $v^\top$ results in

$$v^\top([\mu]\mathcal{I}_L^* + r\mu) = v^\top g(\mu)\tilde{V}_L = sv^\top\tilde{V}_L. \quad (5.6)$$

Note that $\tilde{V}_L > 0$, $v > 0$, $\mu > 0$, $r \geq 0$ and $\mathcal{I}_L^* \geq 0$. It follows that the left hand side of (5.6) is positive. Since $v^\top\tilde{V}_L$ is also positive, we deduce that the Perron root s is positive. This means that $g(\mu)$ is a nonsingular M-matrix (i.e., $\mu \in \mathcal{M}$), and that (5.4) follows from (5.5).

($\Leftarrow$): If $\mu \in \mathcal{M}$, then $\mu > 0$ by Lemma B.1. The rest of the proof follows by reversing the steps of the " $\Rightarrow$ "-part. $\square$

Note that (5.4) is invariant under scaling of μ , and since $\mathcal{M}$ is a cone we may normalize μ . For this purpose we define

$$\mathcal{M}_1 := \mathcal{M} \cap \{ \mu \mid \|\mu\|_1 = 1 \} = \mathcal{M} \cap \{ \mu \mid \mathbb{1}^\top \mu = 1 \}.$$

Proposition 4.3 and Corollary 4.4 together with Lemma 5.4 give rise to an alternative parametrization of $\mathcal{D}$.

Theorem 5.5. *The set $\mathcal{D}$ of long-term voltage stable operating points, its closure $\text{cl}(\mathcal{D})$, and its boundary $\partial\mathcal{D}$ are parametrized by*

$$\begin{aligned} \mathcal{D} &= \{ g(\mu)^{-1}[\mu](\mathcal{I}_L^* + r\mathbb{1}) \mid \mu \in \mathcal{M}_1, r > 0 \}; \\ \text{cl}(\mathcal{D}) &= \{ g(\mu)^{-1}[\mu](\mathcal{I}_L^* + r\mathbb{1}) \mid \mu \in \mathcal{M}_1, r \geq 0 \}; \\ \partial\mathcal{D} &= \{ g(\mu)^{-1}[\mu]\mathcal{I}_L^* \mid \mu \in \mathcal{M}_1 \}. \end{aligned}$$

Furthermore, the map

$$(\mu, r) \mapsto g(\mu)^{-1}[\mu](\mathcal{I}_L^* + r\mathbb{1})$$

from $\mathcal{M}_1 \times \mathbb{R}_{\geq 0}$ to $\text{cl}(\mathcal{D})$ is a bicontinuous map.

The proof of Theorem 5.5 is analogous to the proof of Theorem 4.7, and is therefore omitted.

To simplify notation, we define for $\mu \in \mathcal{M}$ the map

$$\psi(\mu) := g(\mu)^{-1}[\mu]\mathcal{I}_L^*. \tag{5.7}$$

Note that Theorem 5.5 implies that $\psi(\mathcal{M}_1) = \partial\mathcal{D}$, which is a parametrization of the boundary of $\mathcal{D}$.

Figure 5.1 illustrates that $\text{cl}(\mathcal{D})$ is in fact a subset of $\mathcal{M}_1$, which is shown in Lemma B.2.

Theorem 4.7 and Theorem 5.5 present two different parametrizations of $\partial\mathcal{D}$. The next lemma relates these two parametrizations, and will be instrumental for identifying which $\lambda \in \Lambda$ satisfy $P_c(\varphi(\lambda)) \geq 0$ in Lemma 5.3.

Lemma 5.6. *Let $\tilde{V}_L \in \partial\mathcal{D}$, then there exist*

1. *a unique vector $\lambda \in \Lambda_1$ such that $\tilde{V}_L = \varphi(\lambda)$;*
2. *a unique vector $\mu \in \mathcal{M}_1$ such that $\tilde{V}_L = \psi(\mu)$;*
3. *a positive scalar c such that*

$$[\lambda]\tilde{V}_L = c\mu. \tag{5.8}$$

Consequently, μ may be expressed in terms of λ , and vice versa, by

$$\mu = (\lambda^\top \varphi(\lambda))^{-1} [\lambda] \varphi(\lambda) \in \mathcal{M}_1; \quad (5.9)$$

$$\lambda = (\mathbb{1}^\top [\psi(\mu)]^{-1} \mu)^{-1} [\psi(\mu)]^{-1} \mu \in \Lambda_1. \quad (5.10)$$

Proof. The existence and uniqueness of λ and μ follows respectively from Theorem 4.7 and Theorem 5.5. Since Y_{LL} is symmetric we have

$$\begin{aligned} -\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)[\tilde{V}_L] &= [\tilde{V}_L]Y_{LL}[\tilde{V}_L] + [\tilde{V}_L][Y_{LL}(\tilde{V}_L - V_L^*)] \\ &= -[\tilde{V}_L]\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^\top. \end{aligned} \quad (5.11)$$

Note that $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ and its transpose are singular M-matrices by Corollary 4.4, and are irreducible by Lemma 4.2 since $\tilde{V}_L > 0$. Lemma 2.13 states that the kernels of $-\frac{\partial P_c}{\partial V_L}$ and its transpose are spanned by any of their respective Perron vectors. Hence, if $\lambda \in \Lambda_1$ is such that $\tilde{V}_L = \varphi(\lambda)$, then λ is a Perron vector of $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^\top$ by Lemma 4.6. We deduce from (5.11) that

$$0 = -[\tilde{V}_L]\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)^\top \lambda = -\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)[\tilde{V}_L]\lambda.$$

It follows that $[\tilde{V}_L]\lambda$ spans in the kernel of $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$. Lemma 5.4 implies that (5.8) holds for some scalar c . Since $\tilde{V}_L > 0$, $\lambda > 0$ and $\mu > 0$ we have $c > 0$. Moreover, since $\mu^\top \mathbb{1} = 1$, multiplying (5.8) by $\mathbb{1}^\top$ yields $c = \lambda^\top \tilde{V}_L = \lambda^\top \varphi(\lambda)$. By taking c to the other side of (5.8) we obtain (5.9). Similarly, since $\lambda^\top \mathbb{1} = 1$, multiplying (5.8) by $\mathbb{1}^\top [\tilde{V}_L]^{-1}$ yields $1 = \mathbb{1}^\top [\tilde{V}_L]^{-1} \mu c = \mathbb{1}^\top [\psi(\mu)]^{-1} \mu c$. By multiplying (5.8) by $[\psi(\mu)]^{-1}$ we obtain (5.10). $\square$

Lemma 5.6, and in particular (5.8), establishes a duality between the two parametrizations of $\partial\mathcal{D}$. Note that (5.9) and (5.10) describe their correspondence.

5.3 Two parametrizations of the boundary of $\mathcal{F}$

We continue by studying parametrizations of the boundary of $\mathcal{F}$. Corollary 4.20 states that $\partial\mathcal{D}$ is in one-to-one correspondence with $\partial\mathcal{F}$. Since $\partial\mathcal{D}$ is parametrized both by $\varphi(\lambda)$ for $\lambda \in \Lambda_1$ (Theorem 4.7) and by $\psi(\mu)$ for $\mu \in \mathcal{M}_1$ (Theorem 5.5), it follows that $\partial\mathcal{F}$ can be parametrized as

$$\partial\mathcal{F} = \{ P_c(\varphi(\lambda)) \mid \lambda \in \Lambda_1 \} = \{ P_c(\psi(\mu)) \mid \mu \in \mathcal{M}_1 \}.$$

The next theorem gives an alternative formulation for both of these parametrizations.

Theorem 5.7. *Let $\tilde{P}_c \in \partial F$, then there exist unique vectors $\tilde{V}_L \in \partial D$ and $\mu \in \mathcal{M}_1$ such that $\tilde{P}_c = P_c(\tilde{V}_L)$ and $\tilde{V}_L = \psi(\mu)$. These vectors satisfy*

$$\tilde{P}_c = [\tilde{V}_L]^2[\mu]^{-1}Y_{LL}\mu. \quad (5.12)$$

This implies that the boundary of $\mathcal{F}$ is parametrized by

$$\partial\mathcal{F} = \{ [\psi(\mu)]^2[\mu]^{-1}Y_{LL}\mu \mid \mu \in \mathcal{M}_1 \}. \quad (5.13)$$

Proof. The existence and uniqueness of $\tilde{V}_L$ and μ follows respectively from Corollary 4.20 and Theorem 5.5. By (5.7), (5.2) and (3.10) we have

$$\begin{aligned} \psi(\mu) &= g(\mu)^{-1}[\mu]\mathcal{I}_L^* \\ &= ([\mu]Y_{LL} + [Y_{LL}\mu])^{-1}[\mu]Y_{LL}V_L^* \\ &= V_L^* - ([\mu]Y_{LL} + [Y_{LL}\mu])^{-1}[Y_{LL}\mu]V_L^*. \end{aligned} \quad (5.14)$$

We deduce that

$$[\mu]Y_{LL}(V_L^* - \psi(\mu)) = [\mu]Y_{LL}([\mu]Y_{LL} + [Y_{LL}\mu])^{-1}[Y_{LL}\mu]V_L^*. \quad (5.15)$$

Observe that for any two square matrices A, B such that $A + B$ is nonsingular we have the identity¹

$$A(A + B)^{-1}B = B(A + B)^{-1}A. \quad (5.16)$$

Using (5.16) with $A = [\mu]Y_{LL}$ and $B = [Y_{LL}\mu]$ in (5.15) yields

$$[\mu]Y_{LL}(V_L^* - \psi(\mu)) = [Y_{LL}\mu]([\mu]Y_{LL} + [Y_{LL}\mu])^{-1}[\mu]Y_{LL}V_L^* \quad (5.17)$$

$$= [Y_{LL}\mu]\psi(\mu) \quad (5.18)$$

$$= [\psi(\mu)]Y_{LL}\mu, \quad (5.19)$$

where we substituted (5.14). By (5.17) it follows that

$$\begin{aligned} P_c(\psi(\mu)) &= [\psi(\mu)]Y_{LL}(V_L^* - \psi(\mu)) \\ &= [\psi(\mu)][\mu]^{-1}[\psi(\mu)]Y_{LL}\mu = [\psi(\mu)]^2[\mu]^{-1}Y_{LL}\mu, \end{aligned}$$

which proves (5.13). Since $\tilde{P}_c = P_c(\psi(\mu))$ and $\tilde{V}_L = \psi(\mu)$ we have $\tilde{P}_c = [\tilde{V}_L]^2[\mu]^{-1}Y_{LL}\tilde{V}_L$, which proves (5.12). $\square$

The duality of Lemma 5.6 implies the following corollary.

Corollary 5.8. *Let $\tilde{P}_c \in \partial F$, then there exists unique vectors $\tilde{V}_L \in \partial D$ and $\lambda \in \Lambda_1$ such that $\tilde{P}_c = P_c(\tilde{V}_L)$ and $\tilde{V}_L = \varphi(\lambda)$. These vectors satisfy*

$$\tilde{P}_c = [\varphi(\lambda)][\lambda]^{-1}Y_{LL}[\lambda]\varphi(\lambda). \quad (5.20)$$

This implies that the boundary of $\mathcal{F}$ is parametrized by

$$\partial\mathcal{F} = \{ [\varphi(\lambda)][\lambda]^{-1}Y_{LL}[\lambda]\varphi(\lambda) \mid \lambda \in \Lambda_1 \}. \quad (5.21)$$

¹This identity may be verified by adding $A(A + B)^{-1}A$ to both sides of the equation and simplifying.

5.4 The boundary of $\mathcal{F}$ in the nonnegative orthant

Theorem 5.7 gives an explicit relation between the boundary of $\mathcal{F}$ and the vectors $\mu \in \mathcal{M}_1$. The next lemma characterizes all $\mu \in \mathcal{M}_1$ for which the corresponding power demand in $\partial\mathcal{F}$ lies in the nonnegative orthant.

Lemma 5.9. *Given $\tilde{P}_c \in \partial\mathcal{F}$, let $\tilde{V}_L \in \partial\mathcal{D}$ and $\mu \in \mathcal{M}_1$ be the unique vectors so that $\tilde{P}_c = P_c(\tilde{V}_L)$ and $\tilde{V}_L = \psi(\mu)$, then $\tilde{P}_c \in \mathcal{N}$ if and only if $Y_{LL}\mu \in \mathcal{N}$. Consequently, the boundary of $\mathcal{F}$ in the nonnegative orthant is parametrized by*

$$\partial\mathcal{F} \cap \mathcal{N} = \{ P_c(\psi(\mu)) \mid \mu \in \mathcal{M}_1, Y_{LL}\mu \in \mathcal{N} \}.$$

Proof. The existence and uniqueness of $\tilde{V}_L$ and μ follow respectively from Corollary 4.20 and Theorem 5.7. Note that $\tilde{V}_L > 0$, and $\mu > 0$ by Lemma B.1. Hence, it follows from (5.12) that $\tilde{P}_c \geq 0$ if and only if $Y_{LL}\mu \geq 0$. The parametrization follows directly from Theorem 5.7. $\square$

Lemma 5.9 shows that any power demand $\tilde{P}_c$ in $\partial\mathcal{F} \cap \mathcal{N}$ is uniquely associated to the vector $Y_{LL}\mu$ in $\mathcal{N}$. Conversely, we now show that any nonzero vector ν in $\mathcal{N}$ is, up to scaling of ν , uniquely associated to a power demand in $\partial\mathcal{F} \cap \mathcal{N}$. We require the following lemma.

Lemma 5.10. *For each nonzero vector $\nu \in \mathcal{N}$ we have $Y_{LL}^{-1}\nu \in \mathcal{M}$.*

Proof. It suffices to show that $g(Y_{LL}^{-1}\nu)$ is a nonsingular M-matrix. Note that

$$g(Y_{LL}^{-1}\nu) = [Y_{LL}^{-1}\nu]Y_{LL} + [\nu].$$

The matrix Y_{LL} is a nonsingular irreducible M-matrix, and its inverse is a positive matrix by Lemma 2.15. Since $\nu \geq 0$ it follows that $Y_{LL}^{-1}\nu > 0$. Hence $[Y_{LL}^{-1}\nu]Y_{LL}$ is an irreducible nonsingular M-matrix by Lemma 2.5 and Lemma 2.18. Since $\nu \geq 0$, Lemma 2.19 implies that $[Y_{LL}^{-1}\nu]Y_{LL} + [\nu]$ is a nonsingular M-matrix. $\square$

We normalize the nonzero vectors in $\mathcal{N}$ by

$$\begin{aligned} \mathcal{N}_1 &:= \mathcal{N} \cap \{ \nu \mid \|\nu\|_1 = 1 \} \\ &= \{ \nu \in \mathbb{R}^n \mid \nu \geq 0, \mathbf{1}^\top \nu = 1 \}. \end{aligned} \tag{5.22}$$

We remark that $\mathcal{N}_1$ is known as the standard $n - 1$ -simplex.

Lemma 5.9 and Lemma 5.10 suggest that each $\nu \in \mathcal{N}_1$ is uniquely associated to a vector $\mu \in \mathcal{M}_1$ for which the associated power demand $P_c(\psi(\mu))$ is nonnegative. Since there is a one-to-one correspondence between $\mathcal{M}_1$ and Λ_1 by Lemma 5.6, this would mean that there is a one-to-one correspondence between $\mathcal{N}_1$, and the vectors $\lambda \in \Lambda_1$ for

which the associated power demand $P_c(\varphi(\lambda))$ is nonnegative. To this end we define for nonzero $\nu \in \mathcal{N}$ the map

$$\begin{aligned}\chi(\nu) &:= [\psi(Y_{LL}^{-1}\nu)]^{-1} Y_{LL}^{-1}\nu \\ &= \left[[Y_{LL}^{-1}\nu]^{-1} g(Y_{LL}^{-1}\nu)^{-1} [Y_{LL}^{-1}\nu] \mathcal{I}_L^* \right]^{-1} \mathbf{1}.\end{aligned}\tag{5.23}$$

Since Y_{LL} is symmetric we have for all $\mu > 0$ that

$$[\mu]^{-1} g(\mu) [\mu] = (Y_{LL} + [\mu]^{-1} [Y_{LL}\mu]) [\mu] = g(\mu)^\top\tag{5.24}$$

by using (5.2), and hence $\chi(\nu)$ can also be written as

$$\chi(\nu) = \left[g(Y_{LL}^{-1}\nu)^{-\top} \mathcal{I}_L^* \right]^{-1} \mathbf{1}.\tag{5.25}$$

The following theorem establishes a one-to-one correspondence between the set $\mathcal{N}_1$ and the sets $\partial\mathcal{F}$, $\partial\mathcal{D}$, $\mathcal{M}_1$ and Λ_1 for which their associated power demands are nonnegative. In addition, we present a parametrization of the boundary of $\mathcal{F}$ restricted to the nonnegative orthant, in terms of $\mathcal{N}_1$.

Theorem 5.11. *There is a one-to-one correspondence between the following sets:*

- i) *The nonnegative feasible power demands $\tilde{P}_c$ on the boundary of $\mathcal{F}$ (i.e., $\tilde{P}_c \in \partial\mathcal{F} \cap \mathcal{N}$);*
- ii) *The operating points $\tilde{V}_L$ on the boundary of $\mathcal{D}$ such that $Y_{LL}\tilde{V}_L \leq \mathcal{I}_L^*$;*
- iii) *The vectors $\mu \in \mathcal{M}_1$ such that $Y_{LL}\mu \in \mathcal{N}$;*
- iv) *The vectors $\lambda \in \Lambda_1$ such that $Y_{LL}[\lambda]\varphi(\lambda) \in \mathcal{N}$;*
- v) *The vectors $\nu \in \mathcal{N}_1$.*

These correspondences satisfy the equations

$$\begin{aligned}\tilde{P}_c &= P_c(\tilde{V}_L); & \mu &\propto [\lambda]\varphi(\lambda) \propto Y_{LL}^{-1}\nu; \\ \tilde{V}_L &= \psi(\mu) = \psi(Y_{LL}^{-1}\nu) & \lambda &\propto [\psi(\mu)]^{-1}\mu \propto \chi(\nu); \\ &= \varphi(\lambda) = \varphi(\chi(\nu)); & \nu &\propto Y_{LL}[\lambda]\varphi(\lambda) \propto Y_{LL}\mu;\end{aligned}$$

where by $\propto$ we mean that equality holds up to a positive scaling factor. In particular, χ is a one-to-one correspondence between $\mathcal{N}_1$ and (iv), up to scaling. Moreover, the boundary of $\mathcal{F}$ in the nonnegative orthant is parametrized by

$$\partial\mathcal{F} \cap \mathcal{N} = \left\{ P_c(\psi(Y_{LL}^{-1}\nu)) \mid \nu \in \mathcal{N}_1 \right\},$$

and the corresponding operating points are parametrized by

$$\left\{ \tilde{V}_L \in \partial\mathcal{D} \mid P_c(\tilde{V}_L) \geq 0 \right\} = \left\{ \psi(Y_{LL}^{-1}\nu) \mid \nu \in \mathcal{N}_1 \right\}.$$

Proof. (i $\leftrightarrow$ ii): The map P_c from $\partial\mathcal{D}$ to $\partial\mathcal{F}$ is a one-to-one by Corollary 4.20. Lemma 5.1 therefore implies that the map P_c from i) and ii) is one-to-one.

(i $\leftrightarrow$ iii): The map ψ from $\mathcal{M}_1$ to $\partial\mathcal{D}$ is one-to-one by Theorem 5.5, and hence $P_c \circ \psi$ from $\mathcal{M}_1$ to $\partial\mathcal{F}$ is one-to-one. Lemma 5.9 therefore implies that the map $P_c \circ \psi$ from i) to iii) is one-to-one.

(iii $\leftrightarrow$ iv): Lemma 5.6 establishes that $\mathcal{M}_1$ and Λ_1 are in one-to-one correspondence, and that $\tilde{V}_L = \psi(\mu) = \varphi(\lambda)$. Note that (5.9) and (5.10) imply that $\mu \propto [\lambda]\varphi(\lambda)$ and $\lambda \propto [\psi(\mu)]^{-1}\mu$. Substituting (5.8) in iii) results in iv) and are therefore equivalent.

(v $\leftrightarrow$ iii): Lemma 5.10 shows that the map $v \mapsto (\mathbb{1}^\top Y_{LL}^{-1}v)^{-1}Y_{LL}^{-1}v$ is a map $\mathcal{N}_1$ to $\mathcal{M}_1$. This map is injective since Y_{LL}^{-1} is nonsingular, and is therefore one-to-one on its image, which is exactly the set iii). This shows that $\mu \propto Y_{LL}^{-1}v$ and $v \propto Y_{LL}\mu$.

Since $\mu \propto [\lambda]\varphi(\lambda)$ and $v \propto Y_{LL}\mu$, it follows that $v \propto Y_{LL}[\lambda]\varphi(\lambda)$. Due to (5.7) and (5.24) we have

$$[\mu]^{-1}\psi(\mu) = [\mu]^{-1}g(\mu)^{-1}[\mu]\mathcal{I}_L^* = g(\mu)^{-\top}\mathcal{I}_L^*.$$

Since $\lambda \propto [\psi(\mu)]^{-1}\mu$, it follows that $\lambda \propto [g(\mu)^{-\top}\mathcal{I}_L^*]^{-1}\mathbb{1}$. Because $\mu \propto Y_{LL}^{-1}v$, we deduce that $\lambda \propto \chi(v)$ by (5.25). Thus, the map χ from $\mathcal{N}_1$ to Λ is one-to-one, up to scaling.

Finally, the parametrizations follow directly from (i $\leftrightarrow$ iii) and (v $\leftrightarrow$ iii). $\square$

Remark 5.12. From a computation standpoint, the parametrization of $\partial\mathcal{F} \cap \mathcal{N}$ in Theorem 5.11 is cheaper to compute than the parametrizations of $\partial\mathcal{F}$ in Theorem 5.7 or Corollary 5.8. Indeed, to compute the set $\partial\mathcal{F}$ we require to identify either $\mathcal{M}_1$ or Λ_1 by Theorem 5.7 or Corollary 5.8, respectively, which both are sets that are (in essence) described by the eigenvalues of $n \times n$ matrices. In contrast, the parametrization of $\partial\mathcal{F} \cap \mathcal{N}$ in Theorem 5.11 is in terms of the set $\mathcal{N}_1$, which is merely an $n - 1$ -simplex and requires no additional computation.

5.5 Refined results for nonnegative power demands

We conclude this chapter by presenting a refinement of Theorem 4.18 and Theorem 4.22 for nonnegative power demands. This is obtained by applying Theorem 5.11 to Lemma 5.3.

Theorem 5.11 states that the map χ is a one-to-one correspondence between the set $v \in \mathcal{N}_1$ and vectors $\lambda \in \Lambda_1$ for which the associated power demand $P_c(\varphi(\lambda))$ is nonnegative. More specifically, we have

$$\{ \lambda \in \Lambda_1 \mid P_c(\varphi(\lambda)) \geq 0 \} = \{ (\mathbb{1}^\top \chi(v))^{-1}\chi(v) \mid v \in \mathcal{N}_1 \} \subseteq \Lambda_1. \quad (5.26)$$

By substituting this result in Lemma 5.3 we obtain a geometric characterization of $\mathcal{F}$ in terms of $\mathcal{N}_1$.

Corollary 5.13. *The set $\mathcal{F} \cap \mathcal{N}$ is the intersection over all $\nu \in \mathcal{N}_1$ of the half-spaces $H_{\chi(\nu)}$, i.e.,*

$$\mathcal{F} \cap \mathcal{N} = \mathcal{N} \cap \bigcap_{\nu \in \mathcal{N}_1} H_{\chi(\nu)}.$$

Proof. The statement follows from substituting (5.26) in Lemma 5.3, and by noting the half-spaces H_λ are invariant under scaling of λ . $\square$

We may now present a necessary and sufficient condition for a vector of nonnegative power demands to be feasible. This condition can be regarded as a refinement of Theorem 4.22 for nonnegative power demands, and is obtained from Corollary 5.13 by rewriting the half-spaces $H_{\chi(\nu)}$.

Theorem 5.14. *Let $\tilde{P}_c$ be a nonnegative power demand (i.e., $\tilde{P}_c \in \mathcal{N}$), then $\tilde{P}_c$ is feasible (i.e., $\tilde{P}_c \in \mathcal{F} \cap \mathcal{N}$) if and only if*

$$\chi(\nu)^\top \tilde{P}_c \leq \frac{1}{2} \nu^\top V_L^* \quad (5.27)$$

for all $\nu \in \mathcal{N}_1$, where $\chi(\nu)$ was defined in (5.23), where V_L^* are the open-circuit voltages (3.10), and where $\mathcal{N}_1$ is the standard $n - 1$ -simplex (5.22). More explicitly, (5.27) is equivalent to

$$\mathbb{1}^\top \left[([Y_{LL}^{-1} \nu] + Y_{LL}^{-1} [\nu])^{-1} V_L^* \right]^{-1} \tilde{P}_c \leq \frac{1}{2} \nu^\top V_L^*.$$

Similarly, $\tilde{P}_c$ is feasible under small perturbation (i.e., $\tilde{P}_c \in \text{int}(\mathcal{F}) \cap \mathcal{N}$) if and only if the inequality in (5.27) holds strictly for all $\nu \in \mathcal{N}_1$.

Proof. Corollary 5.13 implies that $\tilde{P}_c \in \mathcal{F} \cap \mathcal{N}$ if and only if $\tilde{P}_c \in \mathcal{N}$ and $\tilde{P}_c \in H_{\chi(\nu)}$ for all $\nu \in \mathcal{N}_1$. By definition of H_λ , the latter is equivalent to

$$\chi(\nu)^\top \tilde{P}_c \leq \|\varphi(\chi(\nu))\|_{h(\chi(\nu))}^2 \quad (5.28)$$

for all $\nu \in \mathcal{N}_1$. We continue by rewriting the right-hand side of (5.28). Note that

$$\begin{aligned} \|\varphi(\chi(\nu))\|_{h(\chi(\nu))}^2 &= \varphi(\chi(\nu))^\top h(\chi(\nu)) \varphi(\chi(\nu)) \\ &= \frac{1}{2} \varphi(\chi(\nu))^\top [\chi(\nu)] \mathcal{I}_L^*, \end{aligned} \quad (5.29)$$

where we substituted (4.15) and (4.12). By substituting (5.23) in (5.29) it follows that the right-hand side of (5.28) equals

$$\frac{1}{2} \varphi(\chi(\nu))^\top [\psi(Y_{LL}^{-1} \nu)]^{-1} [Y_{LL}^{-1} \nu] \mathcal{I}_L^*. \quad (5.30)$$

Theorem 5.11 states that $\varphi(\chi(\nu)) = \psi(Y_{LL}^{-1} \nu)$, and hence from (5.30) we deduce that the right-hand side of (5.28) equals

$$\frac{1}{2} \mathbb{1}^\top [Y_{LL}^{-1} \nu] \mathcal{I}_L^* = \frac{1}{2} \nu^\top Y_{LL}^{-1} \mathcal{I}_L^* = \frac{1}{2} \nu^\top V_L^*.$$

where we used (3.10). The left-hand side of (5.27) can be rewritten by observing in (5.25) that

$$\begin{aligned} g(Y_{LL}^{-1}\nu)^{-\top} \mathcal{I}_L^* &= (Y_{LL}[Y_{LL}^{-1}\nu] + [\nu])^{-1} \mathcal{I}_L^* \\ &= ([Y_{LL}^{-1}\nu] + Y_{LL}^{-1}[\nu])^{-1} Y_{LL}^{-1} \mathcal{I}_L^* \\ &= ([Y_{LL}^{-1}\nu] + Y_{LL}^{-1}[\nu])^{-1} V_L^*, \end{aligned}$$

where we used (5.2) and (3.10).

Lemma 4.9 states that we have equality in (5.28) if and only if $\tilde{P}_c = P_c(\varphi(\chi(\nu)))$. Theorem 5.11 implies that $\tilde{P}_c \in \partial\mathcal{F} \cap \mathcal{N}$ if and only if there exists $\nu \in \mathcal{N}_1$ such that $\tilde{P}_c = P_c(\varphi(\chi(\nu)))$. Hence, $\tilde{P}_c \notin \partial\mathcal{F} \cap \mathcal{N}$ if and only if equality in (5.27) does not hold for all $\nu \in \mathcal{N}_1$. Thus, $\tilde{P}_c \in \text{int}(\mathcal{F}) \cap \mathcal{N}$ if and only if the inequality in (5.27) holds strictly for all $\nu \in \mathcal{N}_1$. $\square$

Similar to Remark 5.12, we note that the necessary and sufficient condition for feasibility of nonnegative power demands in Theorem 5.14 is cheaper to compute than the LMI condition in Theorem 4.22. Indeed, Theorem 4.22 relies on the computation of the definiteness of an $n+1 \times n+1$ -matrix. Similar to the proof of Theorem 4.22, it can be shown by means of the Haynsworth inertia additivity formula that this computation is equivalent to identifying set Λ_1 and verifying that the inequality

$$\lambda^\top \tilde{P}_c \leq \frac{1}{4} (\mathcal{I}_L^*)^\top [\lambda] h(\lambda)^{-1} [\lambda] \mathcal{I}_L^* = \|\varphi(\lambda)\|_{h(\lambda)}^2 \quad (5.31)$$

holds for all $\lambda \in \Lambda_1$. In contrast, Theorem 5.14 proves that for a nonnegative power demand it is sufficient to consider only vectors λ in the subset of Λ_1 described by (5.26), which is parametrized by the simplex $\mathcal{N}_1$. Since $\chi(\nu) \in \Lambda_1$ for all $\nu \in \mathcal{N}_1$, it is not necessary to compute Λ_1 to decide the feasibility of nonnegative power demands. Alternatively we note that, since the set Λ_1 is convex by Lemma A.1, it is sufficient to verify (5.31) for all λ in the convex hull of the set (5.26) in order to decide if a nonnegative power demand is feasible.

Remark 5.15. Similar results for positive power demand are obtained by taking $\mathcal{N} = \{ \nu \in \mathbb{R}^n \mid \nu > \mathbb{0} \}$ throughout this chapter. In particular, analogous to Theorem 5.14, it can be shown that a vector of positive power demands $\tilde{P}_c > \mathbb{0}$ is feasible if and only if (5.27) holds for all $\nu > \mathbb{0}$, and similar for feasibility under small perturbation.

5.6 Conclusion

In this chapter we focused on nonnegative power demands, and gave a necessary and sufficient condition for the feasibility (under small perturbation) for nonnegative power

demands. This condition is cheaper to compute than the LMI condition proposed of Theorem 4.22. We developed a second parametrization of $\mathcal{D}$, and established a duality between this parametrization and the parametrization in Chapter 4. From these parametrizations we obtained two parametrizations of the boundary of $\mathcal{F}$, one of which allows for an elegant parametrization of the boundary of $\mathcal{F}$ in the positive orthant.

6 | Sufficient conditions for power flow feasibility

In the remainder of this thesis we return to the case where power demands are not restricted to the nonnegative orthant. In this chapter we prove two novel sufficient conditions for the feasibility of a vector of power demands, which generalize the sufficient conditions found in [7] and [52]. In addition we show how the conditions in [7] and [52] are recovered from the conditions proposed in this chapter.

The benefit of these sufficient conditions for power flow feasibility over a necessary and sufficient condition such as Theorem 4.22 is that they are cheaper to compute, and may therefore be more applicable in practical applications. However, since these sufficient condition are not necessary, they cannot guarantee that a power demand is not feasible.

This chapter is structured as follows. First we show that each feasible vector of power demands gives rise to a sufficient condition for power flow feasibility (Lemma 6.1), and derive a sufficient condition from Theorem 5.14 (Corollary 6.3). Next, we propose a weaker sufficient condition (Theorem 6.5), which generalizes the condition in [52] and identifies for which vectors the latter condition is tight. Finally we show that Theorem 6.5 generalizes the sufficient condition in [7], and argue why the latter condition is not tight in general (Lemma 6.9).

6.1 Sufficient conditions by element-wise domination

Lemma 6.1. *Let $\tilde{P}_c$ be a feasible power demand (i.e., $\tilde{P}_c \in \mathcal{F}$). If a power demand $\hat{P}_c$ satisfies $\hat{P}_c \preceq \tilde{P}_c$, then $\hat{P}_c$ is feasible under small perturbation (i.e., $\hat{P}_c \in \text{int}(\mathcal{F})$).*

Proof. Since $\tilde{P}_c \in \mathcal{F}$ we have by Theorem 4.18 that

$$\lambda^\top \tilde{P}_c \leq \|\varphi(\lambda)\|_{h(\lambda)}^2$$

for all $\lambda \in \Lambda$, where we used (4.24). Note that $\lambda > 0$ for $\lambda \in \Lambda$ by Lemma A.2. Since $\hat{P}_c \preceq \tilde{P}_c$ we have

$$\lambda^\top \hat{P}_c < \lambda^\top \tilde{P}_c \leq \|\varphi(\lambda)\|_{h(\lambda)}^2, \tag{6.1}$$

for all $\lambda \in \Lambda$. Hence, $\hat{P}_c \in \mathcal{F}$ by Theorem 4.18 and (4.24). Since the inequality in (6.1) is strict, Lemma 4.17 implies that $\hat{P}_c \neq P_c(\varphi(\lambda))$ for all $\lambda \in \Lambda$, and therefore $\hat{P}_c \notin \partial\mathcal{F}$ by Corollary 4.20 and Theorem 4.7. $\square$

Lemma 6.1 shows that any feasible power demand gives rise to a sufficient condition for power flow feasibility. In particular, note that the power demand $\mathbf{0} = P_c(V_L^*)$ is feasible under small perturbation. Lemma 6.1 therefore implies the following corollary.

Corollary 6.2. *Any nonpositive power demand is feasible under small perturbation.*

We remark that a vector of nonpositive power demands corresponds to a case in which none of the power loads drain power from the grid and therefore behave as sources. Intuitively it is clear that such a vector of power demands is feasible. Consequently, some of the sources may act as loads and drain the power that is not dissipated in the lines.

Recall that Theorem 5.14 gives a necessary and sufficient condition for the feasibility of a nonnegative power demand. Lemma 6.1 allows us to extend Theorem 5.14 to a sufficient condition for vectors of power demands which have negative entries. We define $\mathbf{max}(a, b) \in \mathbb{R}^n$ as the vector obtained by taking the element-wise maximum of $a, b \in \mathbb{R}^n$, i.e.,

$$\mathbf{max}(a, b)_i := \max(a_i, b_i).$$

Note for $\tilde{P}_c \in \mathbb{R}^n$ that $\mathbf{max}(\tilde{P}_c, \mathbf{0})$ is nonnegative, and that $\tilde{P}_c \leq \mathbf{max}(\tilde{P}_c, \mathbf{0})$. Hence, Theorem 5.14 and Lemma 6.1 directly imply the following sufficient condition for the feasibility of a vector of power demands.

Corollary 6.3. *A vector of power demands $\tilde{P}_c \in \mathbb{R}^n$ is feasible (i.e., $\tilde{P}_c \in \mathcal{F}$) if*

$$\chi(\nu)^\top \mathbf{max}(\tilde{P}_c, \mathbf{0}) \leq \frac{1}{2} \nu^\top V_L^*$$

for all $\nu \in \mathcal{N}_1$.

Note that Corollary 6.3 is necessary and sufficient for nonnegative power demands by Theorem 5.14.

6.2 A generalization of the sufficient condition of Simpson-Porco *et al.* (2016)

We proceed by studying known sufficient conditions in the literature and comparing them to Corollary 6.3. The paper [52] studies the decoupled reactive power flow equations for lossless AC power grids with constant power loads. The analysis and results in [52] translate naturally to DC power grids. In [52] a sufficient condition for the feasibility of a vector of constant power demands is proposed, which we state in the context of DC power grids.

Proposition 6.4 ([52, Supplementary Theorem 1]). *Let $\tilde{P}_c$ be a nonnegative vector of power demands (i.e., $\tilde{P}_c \in \mathcal{N}$), then $\tilde{P}_c$ is feasible under small perturbation (i.e., $\tilde{P}_c \in \text{int}(\mathcal{F})$) if*

$$\|(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\tilde{P}_c\|_\infty < 1. \quad (6.2)$$

This sufficient condition for feasibility is tight since we have

$$\|(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}P_{\max}\|_\infty = 1,$$

where $P_{\max} \in \partial\mathcal{F}$ is the maximizing power demand defined in Lemma 3.18, and lies on the boundary of $\mathcal{F}$.

Proposition 6.4 applies only to nonnegative power demands and is not necessary in general. It is therefore weaker than Theorem 5.14 and Corollary 6.3. The proof of Proposition 6.4 in [52] relies on a fixed point argument. The following result generalizes Proposition 6.4, and identifies all power demands for which the condition (6.2) is tight (i.e., the power demands on the boundary of $\mathcal{F}$ so that equality in (6.2) holds).

Figure 6.1: A depiction of the set $\mathcal{F}$ of feasible power demands for a power grid with two loads. The yellow area corresponds to the sufficient condition in Theorem 6.5. The green shaded area corresponds to the set described by the sufficient condition in [52] (see Proposition 6.4), and does not include the yellow boundary. The black points are the power demands for which the condition in [52] is tight, and correspond to the black operating points in Figure 5.1. The red area corresponds to the sufficient condition in [7] (see Proposition 6.8). The red point indicates a point of intersection of the boundary of the condition in [7] with either the boundary of the condition in [52], or the boundary of the condition in Theorem 6.5.

Theorem 6.5. A vector of power demands $\tilde{P}_c$ is feasible (i.e., $\tilde{P}_c \in \mathcal{F}$) if

$$(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\mathbf{max}(\tilde{P}_c, 0) \leq \mathbf{1}, \quad (6.3)$$

and feasible under small perturbation (i.e., $\tilde{P}_c \notin \text{int}(\mathcal{F})$) if $\tilde{P}_c$ is not of the form

$$\begin{aligned} (\tilde{P}_c)_{[\alpha]} &= \frac{1}{4}[(V_L^*)_{[\alpha]}]((Y_{LL}/(Y_{LL})_{[\alpha^c, \alpha^c]})(V_L^*)_{[\alpha]}; \\ (\tilde{P}_c)_{[\alpha^c]} &= 0 \end{aligned} \quad (6.4)$$

for all nonempty $\alpha \subseteq \mathbf{n}$.

Proof. For the sake of notation we follow Lemma 6.7 and define $\hat{Y}_{LL} := [V_L^*]Y_{LL}[V_L^*]$, which is a nonsingular irreducible M-matrix. It follows from Lemma 2.15 that the inverse of $\hat{Y}_{LL}$ is positive. Let S be the set of $\tilde{P}_c$ defined by

$$\tilde{P}_c \geq 0; \quad \hat{Y}_{LL}^{-1}\tilde{P}_c \leq \frac{1}{4}\mathbf{1}, \quad (6.5)$$

which corresponds to all $\tilde{P}_c \in \mathcal{N}$ so that (6.3) holds. The set S is convex and (6.5) describes the intersection of $2n$ closed half-spaces. The normals to these half-spaces are given by the canonical basis vectors $e_1, \dots, e_n$ and the rows of $\hat{Y}_{LL}^{-1}$. The set S is bounded since $\hat{Y}_{LL}^{-1}$ is positive. Weyl's Theorem [38, pp. 88] states that S is the convex hull of the points which lie on the boundary of n half-spaces in (6.5) so that their corresponding normals span $\mathbb{R}^n$. To this end we define $P_c^\emptyset := 0$, which lies on the boundary of the n half-spaces described by $\tilde{P}_c \geq 0$. Similarly, we let $\alpha \subseteq \mathbf{n}$ be nonempty and let $P_c^\alpha \in S$ be a point described by Weyl's Theorem for which $(\hat{Y}_{LL}^{-1}\tilde{P}_c)_{[\alpha]} = \frac{1}{4}\mathbf{1}$ and $(\hat{Y}_{LL}^{-1}\tilde{P}_c)_{[\alpha^c]} < \frac{1}{4}\mathbf{1}$. The corresponding normals are given by the rows of $\hat{Y}_{LL}^{-1}$ indexed by α . Since $\hat{Y}_{LL}$ is positive definite we know that $(\hat{Y}_{LL})_{[\alpha, \alpha]}$ is positive definite and therefore nonsingular. The only choice of normals of the half-spaces which complete the span of $\mathbb{R}^n$ are e_i for $i \in \alpha^c$, which implies $(P_c^\alpha)_{[\alpha^c]} = 0$. Since $(P_c^\alpha)_{[\alpha^c]} = 0$ we have

$$\frac{1}{4}\mathbf{1} = (\hat{Y}_{LL}^{-1}P_c^\alpha)_{[\alpha]} = (\hat{Y}_{LL}^{-1})_{[\alpha, \alpha]}(P_c^\alpha)_{[\alpha]},$$

and therefore $(P_c^\alpha)_{[\alpha]} = \frac{1}{4}(\hat{Y}_{LL}^{-1})_{[\alpha, \alpha]}^{-1}\mathbf{1}$. By the block matrix inverse formula [62, Eq. (0.8.1)] we observe that

$$(P_c^\alpha)_{[\alpha]} = \frac{1}{4}(\hat{Y}_{LL}/(\hat{Y}_{LL})_{[\alpha^c, \alpha^c]})\mathbf{1}. \quad (6.6)$$

The above exhaustively describes all points specified by Weyl's Theorem, and hence we have

$$S = \text{conv}(\{ P_c^\alpha \mid \alpha \subseteq \mathbf{n} \}).$$

Recall that $P_c^\emptyset = 0 = P_c(V_L^*) \in \text{int}(\mathcal{F})$. The points P_c^α for nonempty $\alpha \subseteq \mathbf{n}$ correspond to the power demands described in (6.4) through substitution of $\hat{Y}_{LL} = [V_L^*]Y_{LL}[V_L^*]$. We

show that these points lie on the boundary of $\mathcal{F}$. Note that for $\alpha = \mathbf{n}$ we have by (6.6), (3.10) and (3.21) that

$$P_c^n = \frac{1}{4}\widehat{Y}_{LL}\mathbb{1} = \frac{1}{4}[V_L^*]Y_{LL}[V_L^*]\mathbb{1} = \frac{1}{4}[V_L^*]\mathcal{I}_L^* = P_{\max},$$

which lies on the boundary of $\mathcal{F}$. Consider any feasible power demand $\tilde{P}_c \in \mathcal{F}$ such that $(\tilde{P}_c)_{[\alpha]} = \mathbb{0}$ with $\alpha \neq \emptyset, \mathbf{n}$. Let $\tilde{V}_L > \mathbb{0}$ be a so that $\tilde{P}_c = P_c(\tilde{V}_L)$. By (3.15) we have

$$\mathbb{0} = P_c(\tilde{V}_L)_{[\alpha^c]} = [(\tilde{V}_L)_{[\alpha^c]}](Y_{LL}(V_L^* - \tilde{V}_L))_{[\alpha^c]} \quad (6.7)$$

where we used (3.10). Since $(\tilde{V}_L^\alpha)_{[\alpha^c]} > \mathbb{0}$ it follows from (6.7) that $(Y_{LL}(V_L^* - \tilde{V}_L))_{[\alpha^c]} = \mathbb{0}$. Since $(Y_{LL})_{[\alpha^c, \alpha^c]}$ is nonsingular, we may solve for $(\tilde{V}_L)_{[\alpha^c]}$. Similar to [62, Eq. (0.7.4)], substitution of $(\tilde{V}_L)_{[\alpha^c]}$ in $P_c(\tilde{V}_L)_{[\alpha^c]}$ yields

$$P_c(\tilde{V}_L)_{[\alpha]} = [(\tilde{V}_L)_{[\alpha]}]Y_{LL}/(Y_{LL})_{[\alpha^c, \alpha^c]}((V_L^*)_{[\alpha]} - (\tilde{V}_L)_{[\alpha]}),$$

which corresponds to the power flow equations of a Kron-reduced power grid (see, e.g., [16, 60]). Analogous to Lemma 3.18, the maximizing feasible power demand for the Kron-reduced power grid is obtained by taking $(\tilde{V}_L)_{[\alpha]} = \frac{1}{2}(V_L^*)_{[\alpha]}$, which corresponds in the power demand P_c^α . Hence P_c^α lies on the boundary of $\mathcal{F}$. Since $\mathcal{F}$ is convex by Theorem 4.18, and $P_c^\alpha \in \mathcal{F}$ for all $\alpha \subseteq \mathbf{n}$, we have that $S \subseteq \mathcal{F} \cap \mathcal{N}$. Each supporting half-space of $\mathcal{F}$ has a unique point of support (Theorem 4.12), and so the boundary of $\mathcal{F}$ does not contain a line piece. Consequently, the all points in S other than the points P_c^α for $\alpha \neq \emptyset$ lie in the interior of $\mathcal{F}$. Lemma 6.1 implies (6.3) from (6.5). $\square$

Note that Theorem 6.5 is weaker than Corollary 6.3, but is cheaper to compute. Proposition 6.4 is recovered from Theorem 6.5 as follows.

Proof of Proposition 6.4. Let $\tilde{P}_c \in \mathcal{N}$ which implies that $\tilde{P}_c = \max(\tilde{P}_c, \mathbb{0})$. Let $\tilde{P}_c$ satisfy (6.2), which is therefore equivalent to

$$-\mathbb{1} < (\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\max(\tilde{P}_c, \mathbb{0}) < \mathbb{1}. \quad (6.8)$$

It follows from Theorem 6.5 that $\tilde{P}_c \in \mathcal{F}$. The latter inequality in (6.8) is strict, and therefore $\tilde{P}_c$ lies in the interior of the set described by (6.3), and hence $\tilde{P}_c \in \text{int}(\mathcal{F})$. $\square$

Theorem 6.5 states that the power demands described by (6.4) are the only power demands which satisfy (6.3) and lie on the boundary of $\mathcal{F}$. The condition in (6.2) is therefore tight for such power demands.

Remark 6.6. Note that if $\alpha = \mathbf{n}$ in (6.4) we obtain the maximizing power, since

$$\frac{1}{4}[V_L^*]Y_{LL}[V_L^*]\mathbb{1} = \frac{1}{4}[V_L^*]\mathcal{I}_L^* = P_{\max},$$

by (3.10) and (3.21). The proof of Theorem 6.5 shows that the power demands described by (6.4) correspond to the maximizing power demands of all power grids obtained by Kron-reduction (see, e.g., [16]). The power flow of such power grids is equivalent to power flow of the full power grid, with the additional restriction that the currents at the loads indexed by α vanish (i.e., $(\mathcal{I}_L)_{[\alpha]} = \mathbb{0}$).

6.3 On the sufficient condition of Bolognani & Zampieri (2015)

We conclude this chapter by showing that Theorem 6.5 also generalizes the sufficient condition in [7]. The paper [7] studies the power flow equation of an AC power grid with constant power loads and a single source node. The analysis and results in [7] translate naturally to DC power grids with a single source node. The next lemma shows that the results in [7] apply to DC power grids with multiple sources as well, which allows us to compare Theorem 6.5 and [7].

Lemma 6.7. *Let $\mathcal{P}$ denote a DC power grid with constant power loads with n loads and m sources. Let $\hat{\mathcal{P}}$ denote the DC power grid with n loads and a single source that is described by the Kirchhoff matrix*

$$\hat{Y} = \begin{pmatrix} \hat{Y}_{LL} & \hat{Y}_{LS} \\ \hat{Y}_{SL} & \hat{Y}_{SS} \end{pmatrix} = \begin{pmatrix} [V_L^*]Y_{LL}[V_L^*] & -[V_L^*]\mathcal{I}_L^* \\ -(\mathcal{I}_L^*)^\top[V_L^*] & (V_L^*)^\top\mathcal{I}_L^* \end{pmatrix}. \quad (6.9)$$

and where the source voltage equals $\hat{V}_S = 1$. The feasibility of the power flow equations of $\mathcal{P}$ and $\hat{\mathcal{P}}$ is equivalent.

Proof. We first verify that $\hat{Y}$ is indeed a Kirchhoff matrix. Note that $[V_L^*]Y_{LL}[V_L^*]\mathbf{1} = [V_L^*]\mathcal{I}_L^*$ by (3.10), and so $\hat{Y}\mathbf{1} = \mathbb{0}$. Also, since Y_{LL} is an irreducible Z-matrix, and $V_L^* > \mathbb{0}$ and $\mathcal{I}_L^* \not\geq \mathbb{0}$, $\hat{Y}$ is also an irreducible Z-matrix, and therefore a Kirchhoff matrix. The powers injected at the loads in power grid $\hat{\mathcal{P}}$ satisfy

$$\begin{aligned} \hat{P}_L(\hat{V}_L) &= [\hat{V}_L](\hat{Y}_{LL}\hat{V}_L + \hat{Y}_{LS}\hat{V}_S) \\ &= [\hat{V}_L]([V_L^*]Y_{LL}[V_L^*]\hat{V}_L - [V_L^*]\mathcal{I}_L^*) \\ &= [\hat{V}_L][V_L^*](Y_{LL}[V_L^*]\hat{V}_L - Y_{LL}V_L^*) = P_L([V_L^*]\hat{V}_L). \end{aligned}$$

where we used (3.10) and (3.4). We therefore have $\hat{P}_L(\hat{V}_L) = P_L(V_L)$ by taking $V_L = [V_L^*]\hat{V}_L$. Hence, given $P_c \in \mathbb{R}^n$, we have that $\hat{P}_L(\hat{V}_L) = P_c$ is feasible for some $\hat{V}_L > \mathbb{0}$ if and only if $P_L(V_L) = P_c$ is feasible for some $V_L > \mathbb{0}$. $\square$

We continue by formulating the sufficient condition in [7]. We follow [7] and define for $p \in [1, \infty]$ the matrix norm

$$\|A\|_p^* := \max_j \{\|A_{[j,n]}\|_p\}, \quad (6.10)$$

where $A_{[j,n]}$ denotes the j -th row of A . The sufficient condition for power flow feasibility of [7] in the context of DC power grids is given as follows.

Proposition 6.8 ([7, Thm. 1]). *Let $\tilde{P}_c \in \mathbb{R}^n$ be a vector of power demands, then $\tilde{P}_c$ is feasible under small perturbation (i.e., $\tilde{P}_c \in \text{int}(\mathcal{F})$) if for some $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\|(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\|_q^* \|\tilde{P}_c\|_p < 1. \quad (6.11)$$

The proof of Proposition 6.8 in [7] also relies on a fixed point argument. Proposition 6.8 is recovered from Lemma 6.1 and Proposition 6.4 as follows.

Proof. Let $\tilde{P}_c$ satisfy (6.11). Let $\hat{P}_c \in \mathcal{N}$ be such that $(\hat{P}_c)_i = |(\tilde{P}_c)_i|$. It follows that $\|\hat{P}_c\|_p = \|\tilde{P}_c\|_p$, and hence $\hat{P}_c$ satisfies (6.11). The matrix Y_{LL} is a nonsingular irreducible M-matrix, and hence $[V_L^*]Y_{LL}[V_L^*]$ is a nonsingular irreducible M-matrix by Lemma 2.5 and Lemma 2.18. Its inverse is a positive matrix by Lemma 2.15. Let $v^j \in \mathbb{R}^n$ be such that $(v^j)^\top$ is the j -th row of $(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}$. We have $v^j > 0$. By (6.10) it follows from (6.11) that for all j

$$\|v^j\|_q \|\hat{P}_c\|_p < 1.$$

By Hölder's inequality (see, e.g., [48, pp. 303]) we have

$$\|[v^j]\hat{P}_c\|_1 \leq \|v^j\|_q \|\hat{P}_c\|_p < 1.$$

Since $[v^j]P_c \geq 0$ we know that $\|[v^j]\hat{P}_c\|_1 = (v^j)^\top \hat{P}_c$. This implies that $(v^j)^\top \hat{P}_c < 1$ for all i , and hence

$$(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\hat{P}_c < \mathbb{1}.$$

Hence (6.2) holds for $\hat{P}_c$, and $\hat{P}_c \in \text{int}(\mathcal{F})$ by Proposition 6.4. Since $\tilde{P}_c \leq \hat{P}_c$, Lemma 6.1 implies that $\tilde{P}_c \in \text{int}(\mathcal{F})$. $\square$

Our proof of Proposition 6.8 shows that the sufficient condition in [7] for nonnegative power demands is more conservative in comparison to the sufficient condition in [52]. This also shows that Theorem 6.5 generalizes both results. The next lemma gives a more intuitive interpretation of the condition (6.11), by showing that (6.11) describes the largest open p -ball such that (6.2) holds for nonnegative power demands.

Lemma 6.9. *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The sufficient condition for power flow feasibility (6.11) in Proposition 6.8 describes the open ball centered at 0*

$$\mathcal{B} := \{ y \in \mathbb{R}^n \mid \|y\|_p < r \}$$

where the radius $r = (\|(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\|_q^*)^{-1} > 0$ is the largest scalar such that (6.2) holds for all $\tilde{P}_c \in \mathcal{B} \cap \mathcal{N}$.

Proof. We show that there exists a nonnegative vector of power demands on the boundary of $\mathcal{B}$ such that equality in (6.2) holds, which therefore defines the radius r . We continue our proof of Proposition 6.8. Let j be such that $\|v^j\|_q = \|(\frac{1}{4}[V_L^*]Y_{LL}[V_L^*])^{-1}\|_q^*$. If $p \neq 1$, then equality in Hölder's inequality $\|[v^j]\tilde{P}_c\|_1 \leq \|v^j\|_q \|\tilde{P}_c\|_p$ holds if $(\tilde{P}_c)_i = c((v^j)_i)^{q-1}$ for all i and for any $c \in \mathbb{R}^n$. Consider the positive vector of power demands $\hat{P}_c$ given by $(\hat{P}_c)_i = c((v^j)_i)^{q-1}$ and where $c^{-1} = \|v^j\|_q^q$. For this vector we have $\|[v^j]\hat{P}_c\|_1 = \|v^j\|_q \|\hat{P}_c\|_p = 1$. Hence, by following proof of Proposition 6.8, $\hat{P}_c$ satisfies equality in both (6.2) and (6.11). Thus, $\hat{P}_c \in \partial\mathcal{B} \cap \mathcal{N}$, and $\|\hat{P}_c\|_p = (\|v^j\|_q)^{-1} = r$. If $p = 1$, the same holds when we take $\hat{P}_c = e_i \|v^j\|_\infty^{-1}$, where i is a single index such that $(v^j)_i = \|v^j\|_\infty$, and $(\hat{P}_c)_i = 0$ otherwise. $\square$

Note that $\hat{P}_c$ constructed in the proof of Lemma 6.9 is not necessarily of the form (6.4). Since (6.2) is tight only for such points, this suggests that the condition (6.11) is not tight in general. This can be observed for $p = q = 2$ in Figure 6.1 by the red dot, which does not lie on the boundary of $\mathcal{F}$.

6.4 Conclusion and discussion

In this chapter we have presented two novel sufficient conditions for the feasibility of a power demand, which were shown to generalize known sufficient conditions in the literature. In addition we proved that any power demand dominated by a feasible power demand is also feasible.

We make a few remarks and suggestions for future research.

In the proof of Theorem 6.5 shows that (6.3) is equal to the vectors dominated by the convex hull of points on the boundary of $\mathcal{F}$. Similarly, other sufficient conditions of feasibility may be obtained by taking multiple points on the boundary of $\mathcal{F}$, and checking if a point lies in their convex hull. More generally, it can be shown that $\mathcal{F}$ is the convex hull of its own boundary. This indicates that by meshing $\partial\mathcal{F}$, we may obtain a sufficient condition which is “arbitrarily close to being necessary”.

In Section 6.3 we showed that the results in [7] when applied to DC power grids can be recovered from Theorem 6.5. However, the paper [7] considers the more general case of AC power grids with constant (complex) power loads. It is unclear how our approach can be extended so that we may also recover this more general case.

7 | A plug-and-play certificate for the power flow problem

A microgrid is a small localized power grid with sources and loads, which is interconnected to a larger power grid. One of the key features of a microgrid is their plug-and-play capability: a microgrid can disconnect from the main grid to operate independently in “island mode”, and reconnect whenever necessary. This allows for, for example, leveling out the fluctuations of power generation and consumption, and for a microgrid to disconnect and operate independently. The latter scenario is particularly useful in case the main grid suffers from a power outage.

Microgrids, and in particular DC microgrids, have gained a lot of attention over the last decades. DC microgrids are considered a promising alternative to the classical power grid paradigm of centralized AC power generation and distribution. Indeed, since a large portion of the day-to-day energy consumption is converted to DC, together with the rise of photo-voltaic cells and advances in battery storage, it is sensible to implement DC power grids on a larger scale.

There has been some research on the control of DC microgrids. A distributed control scheme was proposed in [63], and a plug-and-play control scheme was proposed in [56], although neither paper consider constant-power loads. To the best of our knowledge, no theoretical advances have been made towards power flow feasibility where new sources and/or constant-power loads are introduced. And so, the best available approach to ensure the feasibility after interconnection is to re-verify the feasibility of the power flow in the altered power grid.

To this end, the goal of this chapter is to formulate a *certificate* $\mathcal{C}$ and a *test* $\mathcal{T}$ for a power grid consisting of multiple DC microgrids with constant-power loads. The certificate $\mathcal{C}$ and test $\mathcal{T}$ are conditions on the power grid and a to-be-attached microgrid such that

- (i) the certificate $\mathcal{C}$ implies that a power grid is feasible; and
- (ii) the test $\mathcal{T}$ guarantees that the certificate $\mathcal{C}$ also holds for the interconnection of the power grid with the microgrid, which in turn guarantees that the power flow of the interconnection is feasible as well.

The main advantage of this approach is that, when new microgrids are attached to the power grid, re-verification of power flow feasibility for the interconnected power grid is not necessary, which is therefore computationally cheaper. We refer to this as the *plug-and-play property*.

The novelty of our approach to design such a certificate is that it uses the block Cholesky decomposition (BCD) as a theoretical tool to analyze the power flow equations. Commonly, whenever a new load is introduced to a power grid, conditions for the feasibility of the power grid should be re-evaluated. The BCD allows for a sequential verification of such a condition, so that a complete re-evaluation is not necessary.

Throughout this chapter we consider a DC power grid consisting of multiple interconnected DC microgrids with constant-power loads. Our aim is to study the related power flow problem, while respecting the topological structure of the microgrids. In particular, we study under which conditions the power flow of a power grid remains feasible after the interconnection of a new microgrid. To this end, we aim to derive a certificate $\mathcal{C}$ and a test $\mathcal{T}$, as described above. Both $\mathcal{C}$ and $\mathcal{T}$ are formulated in terms of the individual microgrids, their interconnections, and adhere to a hierarchical structure. This hierarchical structure is precisely the order in which the microgrids are connected to the power grid. To clarify: microgrids which are lower in the hierarchy were added before the ones that are higher in the hierarchy.

This chapter is organized as follows. In Section 7.1 we analyze the interconnection of multiple microgrids and we construct the block Cholesky decomposition. From this we derive a sufficient condition for power flow feasibility of multiple interconnected microgrids. In Section 7.2 we consider the process of interconnecting a microgrid to a power grid, and derive a certificate $\mathcal{C}$ and test $\mathcal{T}$.

7.1 Microgrids with a fixed interconnection topology

We consider a DC power grid with n loads and m sources, with Kirchhoff matrix

$$Y = \begin{pmatrix} Y_{LL} & Y_{LS} \\ Y_{SL} & Y_{SS} \end{pmatrix}.$$

We define $r := n + m$, and note that $Y \in \mathbb{R}^{r \times r}$. Our power grid is subdivided into k microgrids. We denote the i -th microgrid by $\mathcal{M}_i$, and let the numbering of the microgrids represents the order in which the microgrids were attached to the power grid. This induces a hierarchical structure on the microgrids.

We let Γ denote the graph corresponding to Y . Similarly, we let $\Gamma_{\mathcal{M}_i}$ denote the graph corresponding to the microgrid $\mathcal{M}_i$ in “island mode”. If we let $\alpha \subseteq r$ denote the nodes corresponding to $\mathcal{M}_i$, then the Kirchhoff matrix corresponding to $\Gamma_{\mathcal{M}_i}$ is given by $Y_{[\alpha, \alpha]} - [Y_{[\alpha, \alpha]} \mathbb{1}]$.

We do not specifically assume that Γ is connected, but instead assume the following.

Assumption 7.1. Each load node v in a microgrid $\mathcal{M}_i$ satisfies at least one the following conditions:

Figure 7.1: An example of the interconnection of two microgrids $\mathcal{M}_1$, $\mathcal{M}_2$ satisfying Assumption 7.1. Source nodes are represented by $\blacktriangle$ and load nodes are represented by $\circ$.

- i) The node $v \in \mathcal{M}_i$ is path-connected with respect to the graph $\Gamma_{\mathcal{M}_i}$ to a source node in $\mathcal{M}_i$;
- ii) For the node $v \in \mathcal{M}_i$ there exists a path $v = v_0, v_1, \dots, v_l$ in the graph Γ , meaning that $v_{k-1} \sim v_k$ for $k \in l$, such that $v_{k-1} \in \mathcal{M}_i$ for $k \in l$, and $v_k \in \mathcal{M}_j$ where $j < i$. Put differently, the node $v \in \mathcal{M}_i$ is path-connected with respect to the graph $\Gamma_{\mathcal{M}_i}$ to a node in $\mathcal{M}_i$ which has a neighbor that is part of a microgrid which is lower in the hierarchy.

These conditions can be interpreted as follows: Condition 7.1.i is equivalent to the condition that the node is path-connected to a source node in $\mathcal{M}_i$ when the microgrid operates in island mode. Condition 7.1.ii is equivalent to the condition that every load node is path-connected to a node lower in the hierarchy, and where this path does not ascend the hierarchy.

Conceptually speaking, a microgrid without loads cannot operate in island mode. Note that Assumption 7.1 only requires that the first microgrid is able to operate in island mode, while this is not necessarily the case for the other microgrids, meaning that they can consist purely of loads. In addition, note that we do not assume that the power grid is connected.

Assumption 7.1 prevents the case where a microgrid is connected to a power grid while some of its load nodes are not path-connected to a source node. More precisely, it states that for each load node there exists a path $(v_1, \dots, v_l)$ to a source node such that it descends the hierarchy of microgrids, meaning that, if $v_i \in \mathcal{M}_s$ and $v_j \in \mathcal{M}_t$, then $i < j$ implies $s \geq t$.

Example 7.2. Figure 7.1 represents an interconnection of two microgrids for which Assumption 7.1 holds: Nodes 1, 2, 4 and 5 satisfy 7.1.i, while nodes 3, 4 and 5 satisfy 7.1.ii.

Note that node 3 is not path-connected with respect to $\Gamma_{\mathcal{M}_2}$ to a source in $\mathcal{M}_2$. This implies that Assumption 7.1 would not hold for node 3 if the numbering of $\mathcal{M}_1$ and $\mathcal{M}_2$ would be interchanged. *I.e.*, $\mathcal{M}_2$ would not be able to operate in island mode.

Assumption 7.1 implies the following Lemma.

Lemma 7.3. *The matrix Y_{LL} is an invertible M-matrix.*

Proof. It follows from Assumption 7.1 that every load node is path-connected to a source node. This implies that each connected component $(Y_{LL})_{[\sigma,\sigma]}$ of Y_{LL} satisfies that $(Y_{LL})_{[\sigma,\sigma]}\mathbb{1} \not\geq \mathbb{0}$, and hence is a grounded Laplacian. Therefore Y_{LL} is a grounded Laplacian matrix, and is nonsingular by Lemma 2.25. $\square$

7.1.1 The block Cholesky decomposition of Y_{LL}

In this section we derive the block Cholesky decomposition (BCD) of the grounded Laplacian matrix $Y_{LL} \in \mathbb{R}^{n \times n}$.

We let α_i be nested subsets of $\mathbf{n}$ which correspond to the load nodes of the first i microgrids. More formally, we let $\alpha_1, \dots, \alpha_{k+1} \subseteq \mathbf{n}$ be nested subsets of $\mathbf{n}$ such that $\alpha_i \subseteq \alpha_{i+1}$ for $i \in \mathbf{k}$, and $\alpha_{k+1} = \mathbf{n}$.

We define $a_i := |\alpha_i|$, the number of entries in α_i . By relabeling the loads we assume without loss of generality that $\alpha_i = \{1, \dots, a_i\}$. This means that, for $i \in \mathbf{k}$, the matrices $(Y_{LL})_{[\alpha_i, \alpha_i]}$ are leading principal submatrices of Y .

Assumption 7.1 implies that each load node is path-connected to a source in the same microgrid as the load, or is path-connected to a source lower in the hierarchy (or both). This implies that each irreducible component $(Y_{LL})_{[\sigma,\sigma]}$ of the principal submatrices $(Y_{LL})_{[\alpha_i, \alpha_i]}$ of Y satisfies $(Y_{LL})_{[\sigma,\sigma]}\mathbb{1} \not\geq \mathbb{0}$. Hence, $(Y_{LL})_{[\alpha_i, \alpha_i]}$ are grounded Laplacian matrices, and nonsingular M-matrices by Lemma 2.25.

We let $\beta_i \subseteq \mathbf{n}$ denote the loads in to the microgrid $\mathcal{M}_i$. Put differently, we let $\beta_i := \alpha_{i+1} \setminus \alpha_i$ for $i \in \mathbf{k}$, which implies that $\alpha_i \cup \beta_i = \alpha_{i+1}$. In addition, we define $b_i := |\beta_i|$, the number of load nodes in $\mathcal{M}_i$.

Note that $(Y_{LL})_{[\alpha_i, \alpha_i]}$ is a principal submatrix of $(Y_{LL})_{[\alpha_{i+1}, \alpha_{i+1}]}$. We define

$$\begin{aligned} S_i &:= (Y_{LL})_{[\alpha_{i+1}, \alpha_{i+1}]} / (Y_{LL})_{[\alpha_i, \alpha_i]} \\ &= (Y_{LL})_{[\beta_i, \beta_i]} - (Y_{LL})_{[\beta_i, \alpha_i]} (Y_{LL})_{[\alpha_i, \alpha_i]}^{-1} (Y_{LL})_{[\alpha_i, \beta_i]} \in \mathbb{R}^{b_i \times b_i}, \end{aligned}$$

to be their Schur-complement. Recall from Section 2.2.2 that this Schur-complement corresponds to the Kron reduction of the circuit described by $(Y_{LL})_{[\alpha_{i+1}, \alpha_{i+1}]}$ with respect to the internal nodes $\alpha_i \subseteq \alpha_{i+1}$. This means that S_i is also a grounded Laplacian matrix. By Lemma 2.25, S_i is a positive definite nonsingular M-matrix.

In addition we define

$$T_i := (Y_{LL})_{[\beta_i, \alpha_i]} (Y_{LL})_{[\alpha_i, \alpha_i]}^{-1}. \quad (7.1)$$

A simple computation shows that

$$(Y_{LL})_{[\alpha_{i+1}, \alpha_{i+1}]} = \begin{pmatrix} I & 0 \\ T_i & I \end{pmatrix} \begin{pmatrix} (Y_{LL})_{[\alpha_i, \alpha_i]} & 0 \\ 0 & S_i \end{pmatrix} \begin{pmatrix} I & T_i^\top \\ 0 & I \end{pmatrix} \quad (7.2)$$

We shall now construct the block Cholesky decomposition of Y_{LL} . Let the matrices $C_i, D_i \in \mathbb{R}^{a_i \times a_i}$ be defined by $C_1 := I$ and $D_1 := (Y_{LL})_{[\alpha_1, \alpha_1]}$ for $i = 1$, and for $i \in \mathbf{k}, i \neq k$ be iteratively defined by

$$C_{i+1} := \begin{pmatrix} C_i & 0 \\ T_i C_i & I \end{pmatrix}; \quad (7.3)$$

$$D_{i+1} := \begin{pmatrix} D_i & 0 \\ 0 & S_i \end{pmatrix}. \quad (7.4)$$

We observe for $i \in \mathbf{k}, i \neq k$ that

$$C_{i+1} D_{i+1} C_{i+1}^\top = \begin{pmatrix} I & 0 \\ T_i & I \end{pmatrix} \begin{pmatrix} C_i D_i C_i^\top & 0 \\ 0 & S_i \end{pmatrix} \begin{pmatrix} I & T_i^\top \\ 0 & I \end{pmatrix}. \quad (7.5)$$

Since $C_1 D_1 C_1^\top = (Y_{LL})_{[\alpha_1, \alpha_1]}$, it follows from (7.5) and (7.2) by induction on i that for $i \in \mathbf{k}$ we have

$$C_i D_i C_i^\top = (Y_{LL})_{[\alpha_i, \alpha_i]}. \quad (7.6)$$

We define $C := C_k$ and $D := D_k$. This yields $CDC^\top = Y_{LL}$, which is the block Cholesky decomposition of Y_{LL} . We observe from (7.3) and (7.4) that C_i is a lower triangular matrix with ones on its diagonal, and that D_i is a block diagonal matrix.

We proceed by studying the inverse of C_i and D_i . Each diagonal block of D_i is a nonsingular M-matrix. Hence D_i is a nonsingular M-matrix. By studying its irreducible components, it follows by Lemma 2.15 that D_i^{-1} is a nonnegative matrix. The inverse of D_i is iteratively given by $D_1^{-1} = (Y_{LL})_{[\alpha_1, \alpha_1]}^{-1}$ and

$$D_{i+1}^{-1} := \begin{pmatrix} D_i^{-1} & 0 \\ 0 & S_i^{-1} \end{pmatrix}.$$

Similarly, the inverse of C_i satisfies $C_1^{-1} = I$ and

$$C_{i+1}^{-1} := \begin{pmatrix} C_i^{-1} & 0 \\ -T_i & I \end{pmatrix} = \begin{pmatrix} C_i^{-1} & 0 \\ -(Y_{LL})_{[\beta_i, \alpha_i]} C_i^{-\top} D_i^{-1} C_i^{-1} & I \end{pmatrix}, \quad (7.7)$$

where we substituted (7.1) and used (7.6). By the same substitution we have

$$C_{i+1} := \begin{pmatrix} C_i & 0 \\ T_i C_i & I \end{pmatrix} = \begin{pmatrix} C_i & 0 \\ (Y_{LL})_{[\beta_i, \alpha_i]} C_i^{-\top} D_i^{-1} & I \end{pmatrix}. \quad (7.8)$$

The matrix T_i is nonpositive, and it follows from (7.7) that C_i^{-1} is nonnegative. Observe from (7.7) and (7.8) that to compute the Cholesky decomposition of Y_{LL} , or its inverse $C^{-\top} D^{-1} C^{-1} = Y_{LL}^{-1}$, it is necessary and sufficient to compute the inverse of the diagonal blocks of the matrix D .

In the case that $k = n$ we recover the Cholesky decomposition of Y_{LL} . The BCD therefore generalizes the Cholesky decomposition.

7.1.2 Iterative feasibility condition using the BCD

A circuit-theoretic interpretation of the BCD of Y_{LL} is not straight-forward, but can be seen as follows. Each diagonal block of D describes the voltage-current relation of at the nodes in a microgrid when all load nodes of lower hierarchy draw no current, and all load nodes of higher hierarchy are grounded.

The matrix C describes all weighted paths between load nodes in distinct microgrids which ascend the hierarchy, and is therefore (block) lower triangular. The off-diagonal elements of C are nonpositive, while C has ones on its diagonal.

The matrix CD roughly represents the conductances between load nodes in a microgrid, under the assumption that load nodes which are lower in hierarchy draw no current, compensated by the conductances between load nodes in microgrids that are lower in hierarchy and adhere to the same assumption. This compensation is based on paths which descend the hierarchy, which gives CD a block lower triangular structure.

We proceed by stating a weaker version of Theorem 6.5 which involves the BCD of Y_{LL} . In order to state the theorem we need the following lemmas.

Lemma 7.4. *The load nodes k and l in $\mathcal{M}_i$ are path-connected with respect to the graph induced by the load nodes in $\mathcal{M}_1, \dots, \mathcal{M}_i$ if and only if $(S_i^{-1})_{kl} > 0$.*

Proof. The connected components of the graph corresponding to the interconnection of the nodes in $\mathcal{M}_1, \dots, \mathcal{M}_i$ are the irreducible components of $(Y_{LL})_{[\alpha_i, \alpha_i]}$. We can permute its rows and columns such that $(Y_{LL})_{[\alpha_i, \alpha_i]}$ is block-diagonal. The inverse of each block is irreducible, and positive by Lemma 2.15. It follows that the (k, l) -entry of $(Y_{LL})_{[\alpha_i, \alpha_i]}^{-1}$ is positive if and only if k and l are path-connected with respect to the graph. It can be shown that S_i^{-1} is the principal submatrix of $(Y_{LL})_{[\alpha_i, \alpha_i]}^{-1}$ which corresponds to load nodes of $\mathcal{M}_i$. $\square$

The next lemma is the the reason why we need Assumption 7.1.

Lemma 7.5. *The vector $C^\top V_L^*$ is positive.*

Proof. Recall from 3.10 that

$$V_L^* = Y_{LL}^{-1} \mathcal{I}_L^* = C^{-\top} D^{-1} C^{-1} \mathcal{I}_L^*,$$

and therefore $C^\top V_L^* = D^{-1} C^{-1} \mathcal{I}_L^*$. Note that $V_S > 0$ and $-Y_{LS} \geq 0$, and therefore $-Y_{LS} V_S \geq 0$. Since Y_{LL} is an invertible M-matrix, we have seen that $C^{-1} \geq 0$ and $D^{-1} \geq 0$. Since $\mathcal{I}_L^* \geq 0$, this implies that $D^{-1} C^{-1} \mathcal{I}_L^* \geq 0$.

We consider the rows of $C^\top V_L^*$ corresponding to the load nodes in a microgrid $\mathcal{M}_i$, which are given by

$$(C^\top V_L^*)_{[\beta_i]} = D_{[\beta_i, \beta_i]}^{-1} (C^{-1} \mathcal{I}_L^*)_{[\beta_i]} = S_i^{-1} \begin{pmatrix} -T_i^{-1} & I & 0 \end{pmatrix} \mathcal{I}_L^*. \quad (7.9)$$

Consider a load node k in $\mathcal{M}_i$.

If Assumption 7.1.i holds for k , then k is path-connect to a node l in $\mathcal{M}_i$ with respect to the graph $\Gamma_{\mathcal{M}_i}$ (or coincides with l) such that l shares an edge with a source node in $\mathcal{M}_i$. This implies that $(S_i^{-1})_{kl} (\mathcal{I}_L^*)_l > 0$ by Lemma 7.4, and since $(\mathcal{I}_L^*)_l = (-Y_{LS} V_S)_l$ is positive if there is an edge between l and a source node in the same microgrid.

If Assumption 7.1.ii holds for k , then k is path-connected to a node l in $\mathcal{M}_j$ with respect to Γ such that $j < i$, and l is the only node in the path not in $\mathcal{M}_i$.

If l is a source node, then again

$$S_{i k l'}^{-1} (\mathcal{I}_L^*)_{l'} > 0$$

where l' is the node in the path which shares an edge with l . If l is a load node, then, by applying induction on i in the above, Assumption 7.1 implies that l is path-connected to a load node m in $\mathcal{M}_{j'}$ such that $j' \leq j$, which shares an edge with a source node. Hence

$$(-S_i^{-1} (Y_{LL})_{[\beta_i, \alpha_i]})_{kl} ((Y_{LL})_{[\alpha_{i-1}, \alpha_{i-1}]}^{-1})_{lm} (\mathcal{I}_L^*)_m > 0$$

by Lemma 7.4. This implies that each row of (7.9) is positive. $\square$

The vector $C^\top V_L^*$ is a lower bound for the open circuit voltages V_L^* and roughly represents the effect of the potentials at the sources when there is no current flow at loads and where currents only flow over paths which descend the hierarchy.

The next theorem gives a sufficient condition for feasibility which is more conservative than Theorem 6.5, but incorporates the topological structure of the microgrids by considering its block Cholesky decomposition.

Theorem 7.6. *Consider a power grid which consists of interconnected DC microgrids with constant-power loads, and where Assumption 7.1 holds. Let $CDC^\top = Y_{LL}$ be the block Cholesky*

decomposition, as formulated in Section 7.1.1. If P_c is a vector of constant power demands which satisfies

$$D^{-1}C^{-1}[C^\top V_L^*]^{-1}\mathbf{max}(P_c, 0) \leq \frac{1}{4}C^\top V_L^*, \quad (7.10)$$

then P_c is feasible (i.e., $P_c \in \mathcal{F}$).

Proof. Let (7.10) be satisfied. From Lemma 7.5 it follows that $[C^\top V_L^*]^{-1}$ is well-defined and nonnegative. The matrix $C^{-\top}$ has ones on its diagonal, and since $C^{-\top}$ is nonnegative, it follows that $C^{-\top} - I_n \geq 0$. This means that $I_n \leq C^{-\top}$ and we therefore have

$$C^\top V_L^* \leq C^{-\top} C^\top V_L^* = V_L^*.$$

This further implies that

$$[V_L^*]^{-1}\mathbf{max}(P_c, 0) \leq [C^\top V_L^*]^{-1}\mathbf{max}(P_c, 0). \quad (7.11)$$

The matrix $D^{-1}C^{-1}$ is nonnegative and hence multiplying both sides of (7.11) by $D^{-1}C^{-1}$ preserves the inequality. This yields

$$D^{-1}C^{-1}[V_L^*]^{-1}\mathbf{max}(P_c, 0) \leq D^{-1}C^{-1}[C^\top V_L^*]^{-1}\mathbf{max}(P_c, 0). \quad (7.12)$$

It follows from (7.12) and our assumption (7.10) that

$$D^{-1}C^{-1}[V_L^*]^{-1}\mathbf{max}(P_c, 0) \leq \frac{1}{4}C^\top V_L^*.$$

Multiplying both sides by the nonnegative matrix $[V_L^*]^{-1}C^{-\top}$ preserves the inequality, and yields

$$[V_L^*]^{-1}C^{-\top}D^{-1}C^{-1}[V_L^*]^{-1}\mathbf{max}(P_c, 0) = [V_L^*]^{-1}Y_{LL}^{-1}[V_L^*]^{-1}\mathbf{max}(P_c, 0) < \frac{1}{4}\mathbf{1}.$$

Theorem 6.5 implies that $P_c \in \mathcal{F}$. □

The condition (7.10) in Theorem 7.6 is equivalent to

$$D^{-1}C^{-1}[D^{-1}C^{-1}\mathcal{I}_L^*]^{-1}P_L < \frac{1}{4}D^{-1}C^{-1}\mathcal{I}_L^*, \quad (7.13)$$

where we substituted (3.8) and used $CDC^\top = Y_{LL}$.

The block structure of (7.13) is schematically represented by Figure 7.2. The introduction of a new microgrid gives rise to a new block row of C^{-1} and D^{-1} . Figure 7.2 illustrates a plug-and-play certificate, in the sense that, if a power grid satisfies (7.10) and a microgrid satisfies a condition similar to the last block row in Figure 7.2, then their interconnection satisfies (7.10) as well. However, the problem in this approach is that adding a new microgrid will change the diagonal elements of Y_{LL} , which breaks down the iterative construction of the block Cholesky decomposition. This is investigated further in the next section.

$$\begin{bmatrix} D^{-1} & C^{-1} & I_L^* \end{bmatrix}^{-1} \cdot P_c < \frac{1}{4} \cdot \begin{bmatrix} D^{-1} & C^{-1} & I_L^* \end{bmatrix}$$

Figure 7.2: A schematic representation of inequality (7.13) for three microgrids. Due to the block lower triangular structure of C^{-1} and D^{-1} , the inequality concerning the first two block rows is independent of the third block row of matrices D^{-1} , C^{-1} , and the vectors I_L^* and P_c . This suggests that the introduction of a fourth block row to D^{-1} and C^{-1} will therefore not compromise the inequality of the first three block rows.

7.2 Microgrids with a changing interconnection topology

In this section we relate the feasibility of a power grid to an associated “virtual power grid”, which has a more conservative power flow. By studying this virtual power grid we may use Theorem 7.6 to derive a plug-and-play certificate $\mathcal{C}$, along with a test $\mathcal{T}$, to iteratively guarantee power flow feasibility. This forms the main result of this chapter. The following example illustrates why Theorem 7.6 cannot be applied directly, to obtain such a certificate $\mathcal{C}$.

Example 7.7. Consider the power grid in Figure 7.1 where we assume unit line conductances. We consider the microgrid $\mathcal{M}_1$ in island mode and let Y represent its Kirchhoff matrix. We have $Y_{LL} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We proceed by connecting $\mathcal{M}_2$ to $\mathcal{M}_1$ in the same fashion as in Figure 7.1, and let $\hat{Y}$ represent the Kirchhoff matrix of the interconnected microgrids. It follows that

$$\hat{Y}_{LL} = \left(\begin{array}{cc|ccc} 2 & 0 & -1 & 0 & 0 \\ 0 & 3 & 0 & -1 & -1 \\ \hline -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 4 & -1 \\ 0 & -1 & 0 & -1 & 3 \end{array} \right),$$

where the partitioning of the matrix correspond to the two microgrids. Note that the diagonal elements corresponding to node 1 and node 2 have increased. Hence, the matrices Y_{LL} and $\hat{Y}_{LL}$ do not share the relation (7.2), and the BCD of Y_{LL} cannot be used to compute the BCD of $\hat{Y}_{LL}$.

To formulate a plug-and-play certificate as suggested in Section 7.1.2, it is essential for Theorem 7.6 that the blocks of the BCD of Y_{LL} remain unaltered when new microgrids

are attached to the power grid. However, from Example 7.7 we see that, when new lines are introduced, the diagonal elements of the matrix $\hat{Y}_{LL}$ change compared to Y_{LL} .

7.2.1 Virtual shunts and virtual power grids

To address the issue above, we increase the diagonal elements of the Laplacian by temporarily introducing resistive shunts at load nodes. These resistive shunts are later replaced (and potentially removed) when more lines are interconnected to a load node. We refer to these shunts as *virtual shunts*. This approach ensures that the diagonal elements of the Laplacian matrix remain constant when new microgrids are attached.

Virtual shunts are not physically present in the power grid, and should be seen as placeholders for prospective lines. We refer to a power grid with virtual shunts as a *virtual power grid*. We refer to other power grids as *physical power grids*, to prevent ambiguity. It should be understood that a virtual power grid does not fully capture the behavior of the associated physical power grid. However, the feasibility of the two is related by the following lemma.

Lemma 7.8. *Let E be a nonnegative diagonal matrix. If*

$$(Y_{LL} + E)^{-1}[(Y_{LL} + E)^{-1}\mathcal{I}_L^*]^{-1}\mathbf{max}(P_c, 0) \leq \frac{1}{4}(Y_{LL} + E)^{-1}\mathcal{I}_L^*, \quad (7.14)$$

then also (6.3) holds, and P_c is feasible.

Proof. The nonsingular M-matrix Y_{LL}^{-1} is nonnegative, by Lemma 2.15, and the same holds for $(Y_{LL} + E)^{-1}$. The matrix $Y_{LL}^{-1}(Y_{LL} + E) = I + Y_{LL}^{-1}E \geq I$ is nonnegative as well. Hence, multiplying by $Y_{LL}^{-1}(A + E)$ preserves inequalities. By multiplying (7.14) with $Y_{LL}^{-1}(A + E)$, we obtain

$$Y_{LL}^{-1}[(Y_{LL} + E)^{-1}\mathcal{I}_L^*]^{-1}\mathbf{max}(P_c, 0) \leq \frac{1}{4}Y_{LL}^{-1}\mathcal{I}_L^*. \quad (7.15)$$

The matrix $Y_{LL}^{-1}E(Y_{LL} + E)^{-1}$ is nonnegative, and therefore

$$\begin{aligned} Y_{LL}^{-1}\mathcal{I}_L^* &= Y_{LL}^{-1}(A + E)(Y_{LL} + E)^{-1}\mathcal{I}_L^* \\ &= (Y_{LL} + E)^{-1}\mathcal{I}_L^* + Y_{LL}^{-1}E(Y_{LL} + E)^{-1}\mathcal{I}_L^* \\ &\geq (Y_{LL} + E)^{-1}\mathcal{I}_L^* > 0, \end{aligned}$$

which implies that

$$[Y_{LL}^{-1}\mathcal{I}_L^*]^{-1} \leq [(Y_{LL} + E)^{-1}\mathcal{I}_L^*]^{-1}. \quad (7.16)$$

Multiplying both sides of (7.16) by the nonnegative matrix $[Y_{LL}^{-1}\mathcal{I}_L^*]^{-1}Y_{LL}^{-1}$ from the left and the nonnegative vector $\mathbf{max}(P_c, 0)$ from the right preserves the inequality, and

yields

$$\begin{aligned} & [Y_{LL}^{-1}\mathcal{I}_L^*]^{-1}Y_{LL}^{-1}[Y_{LL}^{-1}\mathcal{I}_L^*]^{-1}\mathbf{max}(P_c, 0) \\ & \leq [Y_{LL}^{-1}\mathcal{I}_L^*]^{-1}Y_{LL}^{-1}[(Y_{LL} + E)^{-1}\mathcal{I}_L^*]^{-1}\mathbf{max}(P_c, 0). \end{aligned} \quad (7.17)$$

It follows from (7.17) and (7.15) that

$$[Y_{LL}^{-1}\mathcal{I}_L^*]^{-1}Y_{LL}^{-1}[Y_{LL}^{-1}\mathcal{I}_L^*]^{-1}\mathbf{max}(P_c, 0) < \frac{1}{4}\mathbb{1}.$$

and so (6.3) holds. $\square$

The diagonal matrix E in Lemma 7.8 represents the virtual shunts of a virtual power grid. Lemma 7.8 shows that if P_c satisfies the sufficient condition of Theorem 6.5 for a virtual power grid, then P_c also the same sufficient condition for the physical power grid. More generally, we can show that the introduction of shunts in a power grid makes the power flow more conservative—see Lemma C.3.

Note that virtual shunts are not needed for source nodes. This follows directly from the observation that Y_{SS} does not appear in the power flow equations (3.13). Hence, lines between sources can be altered freely, without compromising feasibility.

It is essential that the virtual shunts are chosen properly. Choosing E large leads to more conservative conditions, whereas choosing E small restricts the possible interconnections with potential microgrids. However, from a practical point of view, it is reasonable to assume that the conductances of prospective lines are known *a priori*. For example, some lines may already exist but are not in use. Also, there might be limitations on the number of lines a node can connect with, which would give an upper bound for the virtual shunt.

7.2.2 A plug-and-play certificate for power flow feasibility

In this section we present the main result of the chapter. The main line of this section is as follows.

We consider a power grid and define a virtual counterpart in the sense of Section 7.2.1. We define a BCD of the lines of the loads in the virtual power grid. We assume that the BCD satisfies the conditions for Theorem 7.6. This implies that both the power grid and its virtual counterpart are feasible, by Lemma 7.8.

We proceed by introducing a microgrid and again define a virtual counterpart. We interconnect both virtual grids, assuming that the physical interconnection satisfies Assumption 7.1 and does not exceed the virtual shunts. We extend the BCD of the virtual power grid to the virtual interconnection and use this to state the main theorem.

The main theorem gives a sufficient condition such that this interconnected virtual power grid is feasible. Moreover, the theorem states that the physical interconnection of the power grid and the microgrid is therefore also feasible, by Lemma 7.8.

Consider a DC power grid described by the Laplacian matrix $\begin{pmatrix} Y_{L_1 L_1} & Y_{L_1 S_1} \\ Y_{S_1 L_1} & Y_{S_1 S_1} \end{pmatrix}$, together with the virtual shunts E_{L_1} and load voltages V_{L_1} . Let P_{L_1} be the power demands at the load nodes and $V'_{L_1} := -(Y_{L_1 L_1} + E_{L_1})^{-1} Y_{L_1 S_1} V_{S_1}$ the virtual open-circuit voltages, where V_{S_1} are the source voltages. The vector V'_{L_1} is a lower bound for the open-circuit voltages of the physical power grid.

Let C, D be such that $CDC^\top = Y_{L_1 L_1} + E_{L_1}$ is the block Cholesky decomposition.

Assumption 7.9. The virtual power grid satisfies the certificate $\mathcal{C}$, which is given by

$$D^{-1}C^{-1}[C^\top V'_{L_1}]^{-1}P_{L_1} < \frac{1}{4}C^\top V'_{L_1}.$$

Assumption 7.9 implies that $(Y_{L_1 L_1} + E_{L_1}, Y_{L_1 S_1} V_{S_1}, P_{L_1})$ is feasible by Theorem 7.6. It follows by Lemma 7.8 that $(Y_{L_1 L_1}, Y_{L_1 S_1} V_{S_1}, P_{L_1})$, the physical counterpart, is feasible as well.

In addition, consider a microgrid described by the Laplacian matrix $\begin{pmatrix} Y_{L_2 L_2} & Y_{L_2 S_2} \\ Y_{S_2 L_2} & Y_{S_2 S_2} \end{pmatrix}$, together with the virtual shunts E_{L_2} and load voltages V_{L_2} . Let P_{L_2} be the power demand at the load nodes.

Let the physical interconnection of the power grid and the microgrid be given by the Laplacian matrix

$$\begin{pmatrix} Y_{L_1 L_1} + \widehat{E}_{L_1} & Y_{L_1 L_2} & Y_{L_1 S_1} & Y_{L_1 S_2} \\ Y_{L_2 L_1} & Y_{L_2 L_2} + \widehat{E}_{L_2} & Y_{L_2 S_1} & Y_{L_2 S_2} \\ Y_{S_1 L_1} & Y_{S_1 L_2} & Y_{S_1 S_1} + \widehat{E}_{S_1} & Y_{S_1 S_2} \\ Y_{S_2 L_1} & Y_{S_2 L_2} & Y_{S_2 S_1} & Y_{S_2 S_2} + \widehat{E}_{S_2} \end{pmatrix},$$

where $\widehat{E}_{L_1}, \widehat{E}_{L_2}, \widehat{E}_{S_1}, \widehat{E}_{S_2}$ are defined as follows:

$$\begin{aligned} \widehat{E}_{L_1} &:= -[Y_{L_1 L_2} \mathbb{1} + Y_{L_1 S_2} \mathbb{1}]; \widehat{E}_{S_1} := -[Y_{S_1 L_2} \mathbb{1} + Y_{S_1 S_2} \mathbb{1}]; \\ \widehat{E}_{L_2} &:= -[Y_{L_2 L_1} \mathbb{1} + Y_{L_2 S_1} \mathbb{1}]; \widehat{E}_{S_2} := -[Y_{S_2 L_1} \mathbb{1} + Y_{S_2 S_1} \mathbb{1}]. \end{aligned}$$

We make the following assumption on the interconnection.

Assumption 7.10. The interconnection of the power grid and the microgrid is such that $\widehat{E}_{L_1} \leq E_{L_1}, \widehat{E}_{L_2} \leq E_{L_2}$ and Assumption 7.1 holds.

The bounds on $\widehat{E}_{L_1}$ and $\widehat{E}_{L_2}$ imply that the virtual interconnection of both virtual grids does not exceed the virtual shunts. This will allow us to use Lemma 7.8 in Theorem 7.11.

Let $\tilde{C}, \tilde{D}$ such that

$$\tilde{C}\tilde{D}\tilde{C}^\top = \begin{pmatrix} Y_{L_1L_1} + E_{L_1} & Y_{L_1L_2} \\ Y_{L_2L_1} & Y_{L_2L_2} + E_{L_2} \end{pmatrix}$$

is a BCD. From Section 7.1.1 it follows that

$$\begin{aligned} \tilde{C}^{-1} &= \begin{pmatrix} C^{-1} & 0 \\ -Y_{L_2L_1}(CDC^\top)^{-1} & I_n \end{pmatrix}; \\ \tilde{D}^{-1} &= \begin{pmatrix} D^{-1} & 0 \\ 0 & R^{-1} \end{pmatrix}, \end{aligned}$$

where we define

$$R := Y_{L_2L_2} + E_{L_2} - Y_{L_2L_1}(CDC^\top)^{-1}Y_{L_1L_2}.$$

Define $V'_L := -(\tilde{C}\tilde{D}\tilde{C}^\top)^{-1} \begin{pmatrix} Y_{L_1S_1} & Y_{L_1S_2} \\ Y_{L_2S_1} & Y_{L_2S_2} \end{pmatrix} \begin{pmatrix} V_{S_1} \\ V_{S_2} \end{pmatrix}$, which is a lower bound for the open-circuit voltages of the interconnected power grid.

Finally, let $(\tilde{C}^\top V'_L)_{L_1}$ denote the entries of $\tilde{C}^\top V'_L$ corresponding to the load nodes of the original power grid, and similar for $(\tilde{C}^\top V'_L)_{L_2}$ and the microgrid.

Theorem 7.11. *Suppose that Assumptions 7.9 and 7.10 are satisfied. If the test $\mathcal{T}$, given by*

$$-R^{-1}Y_{L_2L_1}(CDC^\top)^{-1}[(\tilde{C}^\top V'_L)_{L_1}]^{-1}P_{L_1} + R^{-1}[(\tilde{C}^\top V'_L)_{L_2}]^{-1}P_{L_2} < \frac{1}{4}(\tilde{C}^\top V'_L)_{L_2}, \quad (7.18)$$

is satisfied, then the certificate $\mathcal{C}$ for the virtual interconnected microgrids, given by

$$\tilde{D}^{-1}\tilde{C}^{-1}[\tilde{C}^\top V'_L]^{-1} \begin{pmatrix} P_{L_1} \\ P_{L_2} \end{pmatrix} < \frac{1}{4}\tilde{C}^\top V'_L, \quad (7.19)$$

holds. Therefore the interconnection of the power grid and the microgrid is feasible.

Proof. Due to the block triangular structure of $\tilde{C}$, the rows of (7.19) corresponding to the virtual power grid are given by

$$D^{-1}C^{-1}[(\tilde{C}^\top V'_L)_{L_1}]^{-1}P_{L_1} < \frac{1}{4}(\tilde{C}^\top V'_L)_{L_1}. \quad (7.20)$$

For the same reason we have

$$\begin{aligned} (\tilde{C}^\top V'_L)_{L_1} &= -D^{-1}C^{-1}(Y_{L_1S_1}V_{S_1} + Y_{L_1S_2}V_{S_2}) \\ &\geq -D^{-1}C^{-1}Y_{L_1S_1}V_{S_1} = C^\top V'_{L_1}, \end{aligned}$$

which implies $[(\tilde{C}^\top V'_L)_{L_1}]^{-1} \leq [C^\top V'_{L_1}]^{-1}$. Assumption 7.9 implies that

$$D^{-1}C^{-1}[(\tilde{C}^\top V'_L)_{L_1}]^{-1}P_{L_1} \leq D^{-1}C^{-1}[C^\top V'_{L_1}]^{-1}P_{L_1} < \frac{1}{4}C^\top V'_{L_1} \leq \frac{1}{4}(\tilde{C}^\top V'_L)_{L_1}.$$

Therefore (7.20) is satisfied.

Note that (7.18) is equivalent to the rows of (7.19) corresponding to the virtual microgrid. Hence, if (7.18) holds, then so does (7.19).

If (7.19) is satisfied, then by Theorem 7.6 the virtual interconnection

$$\begin{pmatrix} Y_{L_1 L_1} + E_{L_1} & Y_{L_1 L_2} \\ Y_{L_2 L_1} & Y_{L_2 L_2} + E_{L_2} \end{pmatrix}, \begin{pmatrix} Y_{L_1 S_1} & Y_{L_1 S_2} \\ Y_{L_2 S_1} & Y_{L_2 S_2} \end{pmatrix} \begin{pmatrix} V_{S_1} \\ V_{S_2} \end{pmatrix}, \begin{pmatrix} P_{L_1} \\ P_{L_2} \end{pmatrix}$$

is feasible. Since $\widehat{E}_{L_1} \leq E_{L_1}$ and $\widehat{E}_{L_2} \leq E_{L_2}$, we may use Lemma 7.8 to conclude that the physical interconnection

$$\begin{pmatrix} Y_{L_1 L_1} + \widehat{E}_{L_1} & Y_{L_1 L_2} \\ Y_{L_2 L_1} & Y_{L_2 L_2} + \widehat{E}_{L_2} \end{pmatrix}, \begin{pmatrix} Y_{L_1 S_1} & Y_{L_1 S_2} \\ Y_{L_2 S_1} & Y_{L_2 S_2} \end{pmatrix} \begin{pmatrix} V_{S_1} \\ V_{S_2} \end{pmatrix}, \begin{pmatrix} P_{L_1} \\ P_{L_2} \end{pmatrix}$$

is feasible as well. □

Note that Assumption 7.9 and (7.19) are of the same form. Hence, microgrids can be attached to a power grid in an iterative fashion by using Theorem 7.11 to guarantee feasibility. This provides a method to determine if a power grid remains feasible after connecting a specific microgrid.

7.2.3 Example of the plug-and-play certificate

The following example illustrates the application of Theorem 7.11 to determine if a power grid remains feasible after interconnecting a specific microgrid.

Example 7.12. Consider again the power grid in Figure 7.1 with unit line conductance, where we connect $\mathcal{M}_2$ to the power grid $\mathcal{M}_1$. We continue in the notation of Section 7.2.2.

Let the voltage potentials at each source be 1 volt, and consider constant-power loads such that $P_{L_1} = \frac{1}{25} \begin{pmatrix} 2 & 2 \end{pmatrix}^\top$ and $P_{L_2} = \frac{1}{25} \begin{pmatrix} 1 & 9 & 7 \end{pmatrix}^\top$.

For microgrids $\mathcal{M}_1$ and $\mathcal{M}_2$ we have

$$Y_{L_1 L_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad Y_{L_2 L_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

We consider the virtual shunts

$$E_{L_1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}; \quad E_{L_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $D = Y_{L_1 L_1} + E_{L_1} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. The virtual open-circuit voltages for $\mathcal{M}_1$ are $V'_{L_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \end{pmatrix}^\top$. It follows that $C^\top V'_{L_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \end{pmatrix}^\top$ and

$$D^{-1}C^{-1}[C^\top V'_{L_1}]^{-1}P_L = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \left[\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \right]^{-1} \begin{pmatrix} \frac{2}{25} \\ \frac{2}{25} \end{pmatrix} = \begin{pmatrix} \frac{2}{25} \\ \frac{2}{25} \end{pmatrix} < \frac{1}{4} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} = \frac{1}{4}C^\top V'_{L_1}.$$

Hence Assumption 7.9 is satisfied and $\mathcal{M}_1$ is feasible in island mode.

We consider the interconnection in Figure 7.1, which satisfies $\hat{E}_{L_2} = E_{L_2}$ and $\hat{E}_{L_2} = E_{L_2}$. Since Assumption 7.1 holds for the interconnection, Assumption 7.10 is satisfied.

We find $V'_L = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^\top$ and $\tilde{C}^\top V'_L = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & 1 & 1 & 1 \end{pmatrix}^\top$. Note that

$$Y_{L_2 L_1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

We compute $R^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{11}{24} \end{pmatrix}$. Continuing from the left-hand side of (7.18), we

have

$$\begin{aligned} & - \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{11}{24} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \left[\begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \right]^{-1} \begin{pmatrix} \frac{2}{25} \\ \frac{2}{25} \end{pmatrix} \\ & + \begin{pmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{11}{24} \end{pmatrix} \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]^{-1} \begin{pmatrix} \frac{1}{25} \\ \frac{9}{25} \\ \frac{7}{25} \end{pmatrix} \approx \begin{pmatrix} 0.24 \\ 0.21 \\ 0.24 \end{pmatrix} < \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}(C^\top V'_L)_{L_2}. \end{aligned}$$

Hence, the test $\mathcal{T}$ is satisfied and hence the certificate $\mathcal{C}$ holds for the interconnected power grid. This implies that the interconnection of the two microgrids is feasible, by Theorem 7.11.

7.3 Conclusion

In this chapter we studied the interconnection of DC microgrids with constant-power loads. We have presented a certificate $\mathcal{C}$ for the feasibility of a DC power grid. In addition, we have presented a test $\mathcal{T}$ for the interconnection of a microgrid such that condition $\mathcal{C}$

also holds for their interconnection. This establishes a method to determine if a power grid remains feasible after connecting a specific microgrid.

The proposed certificate $\mathcal{C}$ makes use of the block Cholesky decomposition and is based on the sufficient condition of Theorem 6.5, which is a condition that is cheap to compute but conservative. In this chapter we have seen that the use of virtual shunts adds another degree of conservativeness to the power flow. To obtain a stronger certificate it may be interesting to again makes use of the block Cholesky decomposition and develop a certificate based on the necessary and sufficient LMI condition of Theorem 4.22.

8 | Variations in line conductances

This chapter studies how power flow feasibility is affected by changes in line conductances. Section 8.1 shows that an analogue of Braess' paradox can also occur in DC power grids with constant-power loads, and that this may occur for most practical power grids with multiple sources. The prevalent occurrence of this phenomenon motivates Section 8.2, where we study power grids where the exact line conductances are not known, and only bounds on line conductances are available. For such power grids we formulate the concept of strong feasibility, and give a sufficient condition which guarantees that the power flow in the power grid is strongly feasible.

Recall from Section 2.1.5 that if nodes i and j are connected by a line, we let $G_{ij} = G_{ji} > 0$ denote the conductance of this line. We now extend this definition by putting $G_{ij} = G_{ji} = 0$ if the nodes i and j are not connected by a line.

8.1 Braess' paradox in DC power grids with constant-power loads

Braess' paradox is a phenomenon in traffic flow networks, first observed in [8]. It corresponds with the counterintuitive observation that adding a road or improving the capacity of a road does not necessarily improve the traffic flow. It has been shown that this phenomenon is a prevalent property of such networks [53]. Similar phenomena have been observed for DC power grids. The paper [43] reports a physical example (inspired by [10]) of a resistive circuit with diodes and constant current loads which exhibits behavior analogous to Braess' paradox in traffic flow networks. The paper [34] studies the effect of power transportation limits in the lines of DC power grids, and reports an example for which the maximal total power demand is subject to Braess' paradox. Braess' paradox has also been observed in AC power grids. The paper [11] shows that Braess' paradox can occur in AC power grids. Also, in [54] it was observed for AC power grids that introducing a new line or increasing the conductance of a line can destabilize the power grid.

The aim of this section is to show that Braess' paradox also occurs for most DC power grids with constant-power loads, which in accordance with the observations for other flow networks in the previously mentioned literature.

8.1.1 Braess' paradox for power flow feasibility

The classical formulation of Braess' paradox is that, after increasing line conductances or adding a line, the quality of the flow in the power grid becomes measurably worse. In this thesis, however, we employ a stronger formulation of Braess' paradox, by studying how adding lines or increasing conductances can cause the power grid to lose stability, and in particular, lose power flow feasibility. This phenomenon is in line with observations in [34] and [54].

Definition 8.1. *Braess' paradox for power flow feasibility* is the phenomenon that adding a line or increasing a line conductance in a power grid destroys the feasibility of the power flow and destabilizes the power grid.

In the remainder of this chapter we refer to Definition 8.1 simply as the Braess' paradox, for the sake of brevity.

8.1.2 A sufficient condition for the occurrence Braess' paradox

It was suggested in [34] that the maximal total demand $\mathbb{1}^\top P_{\max}$ is subject to Braess' paradox. In this section we study $\mathbb{1}^\top P_{\max}$ and give a formal proof for the occurrence of the phenomenon. We start by showing that the maximum total power that can be transported by the power grid never increases when the conductance of a line between two load nodes increases.

Theorem 8.2. *The maximal total power demand $\mathbb{1}^\top P_{\max}$ is nonincreasing as a function of the conductances of lines between loads.*

Proof. Consider two distinct loads i and j . Increasing G_{ij} by a positive scalar c corresponds to adding the matrix $\Delta := c(e_i - e_j)(e_i - e_j)^\top$ to Y_{LL} . If $G_{ij} = 0$ this means that we are adding a new line, and if $G_{ij} > 0$ this corresponds to increasing the conductance of a line. By substituting (3.10) in (3.21) we observe that

$$\mathbb{1}^\top P_{\max} = -\frac{1}{4}(V_L^*)^\top Y_{LS} V_S = \frac{1}{4}V_S^\top Y_{SL} Y_{LL}^{-1} Y_{LS} V_S.$$

We let $\hat{P}_{\max}$ be the maximizing power demand corresponding the power grid after increasing G_{ij} by c . We have

$$\mathbb{1}^\top \hat{P}_{\max} = \frac{1}{4}V_S^\top Y_{SL} (Y_{LL} + \Delta)^{-1} Y_{LS} V_S.$$

The difference between the two total power demands is $d := \mathbb{1}^\top P_{\max} - \mathbb{1}^\top \hat{P}_{\max}$. By Lemma C.1 we have

$$\begin{aligned} d &= \frac{1}{4}c \frac{(V_S^\top Y_{SL} Y_{LL}^{-1} (e_i - e_j))^2}{1 + c(e_i - e_j)^\top Y_{LL}^{-1} (e_i - e_j)} \\ &= c \frac{(\frac{1}{2}(V_L^*)_i - \frac{1}{2}(V_L^*)_j)^2}{1 + c(e_i - e_j)^\top Y_{LL}^{-1} (e_i - e_j)}, \end{aligned} \tag{8.1}$$

where we substituted (3.10). Since Y_{LL} is positive definite and $c > 0$, this implies that $d \geq 0$, which means that $\mathbb{1}^\top \hat{P}_{\max} \leq \mathbb{1}^\top P_{\max}$. Hence, $\mathbb{1}^\top P_{\max}$ is nonincreasing as a function of the conductance G_{ij} where i and j correspond to loads. $\square$

Theorem 8.2 shows that an increase of the a line conductance can make the total power decrease. This means that the power flow in a power grid can become unfeasible after such an increase, when the total power demand $\mathbb{1}^\top P_c$ is close to its maximum $\mathbb{1}^\top P_{\max}$, or in particular when $P_c = P_{\max}$. Hence, in such a case Braess' paradox occurs (see Definition 8.1). This is formalized in the following theorem, which presents a sufficient condition for the existence of a vector of constant power demands for which Braess' paradox occurs.

Theorem 8.3. *Consider a power grids with Kirchhoff matrix Y and source voltages V_S . Let P_c be a vector of constant power demands such that the power flow is feasible. If there are load nodes i and j such that the open-circuit voltages (3.10) satisfy $(V_L^*)_i \neq (V_L^*)_j$, and if P_c satisfies*

$$\mathbb{1}^\top P_c > \mathbb{1}^\top P_{\max} - \frac{(\frac{1}{2}(V_L^*)_i - \frac{1}{2}(V_L^*)_j)^2}{(e_i - e_j)^\top Y_{LL}^{-1}(e_i - e_j)}, \quad (8.2)$$

then there exists a scalar $c > 0$ such that the power flow becomes unfeasible after increasing G_{ij} by c , either through increasing the conductance of the line between loads i and j , or through adding a new line between loads i to j . In particular, if we consider $P_c = P_{\max}$, then the power flow becomes unfeasible for any increase of G_{ij} .

Proof. We continue the proof of Theorem 8.2. Since $(V_L^*)_i \neq (V_L^*)_j$ we have $d > 0$, and therefore $\mathbb{1}^\top \hat{P}_{\max} < \mathbb{1}^\top P_{\max}$. Since $\hat{P}_{\max}$ is the maximizing power demand corresponding the power grid after increasing G_{ij} , we have by Proposition 3.20 that $\mathbb{1}^\top \hat{P}_c < \mathbb{1}^\top \hat{P}_{\max}$ for all power demands $\hat{P}_c \neq \hat{P}_{\max}$ for which the power flow of the *altered* power grid is feasible. Since $\mathbb{1}^\top \hat{P}_{\max} < \mathbb{1}^\top P_{\max}$, this implies that the power flow is no longer feasible after increasing G_{ij} when $P_c = P_{\max}$. By taking $c \rightarrow \infty$ in (8.1), (*i.e.*, by shorting the line between i and j), we observe that

$$d < \frac{(\frac{1}{2}(V_L^*)_i - \frac{1}{2}(V_L^*)_j)^2}{(e_i - e_j)^\top Y_{LL}^{-1}(e_i - e_j)}.$$

And so, for every P_c such that the power flow is feasible and such that (8.2) holds we can take c large enough such that $\mathbb{1}^\top \hat{P}_{\max} < \mathbb{1}^\top P_c$, which implies by Proposition 3.20 that the power flow becomes unfeasible. $\square$

We remark that the denominator of the fraction in (8.2) corresponds to the effective resistance between nodes i and j with respect to the grounded Laplacian Y_{LL} (*e.g.*, see [16]).

Figure 8.1: A schematic depiction of a DC power grid with two loads and two sources ($n = 2$, $m = 2$) for which Braess' paradox can occur.

Figure 8.2: Plots for the sets of constant power demands P_c for which the power flow of the power grid in Example 8.4 is feasible. The blue and green regions correspond to respectively $G_{12} = 0.3$ and $G_{12} = 1$. This corresponds to increasing the conductance G_{12} of the line between node 1 and node 2 by 0.7 (see Figure 8.1). The blue and green dashed lines are the points for which equality holds in (3.23) for the respective power grids. The decrease of the maximal total power demand is indicated by d . We observe that $P_{\max}$ is no longer feasible after the conductance is increased. The black dashed line corresponds the points for which equality in (8.2) holds. Theorem 8.3 shows that Braess' paradox may occur for all feasible vectors of constant power demands beyond the black dashed line, such as for $P_{\max}$.

Theorem 8.3 tells us that if the open-circuit voltages are not all equal (*i.e.*, $V_L^* \notin \text{span}\{\mathbf{1}\}$), then Braess' paradox can occur when the power demands are close to the maximizing power demands $P_{\max}$. This is illustrated by the following example.

Example 8.4. Consider the power grid with two loads and two sources, as depicted in Figure 8.1, where $V_S = \begin{pmatrix} 1 & 3 \end{pmatrix}^\top$, $G_{12} = 0.3$, $G_{14} = 1$, $G_{23} = 5$ and $G_{34} = 1$. The corresponding open-circuit voltages are $V_L^* = \frac{1}{6.8} \begin{pmatrix} 17.4 & 7.4 \end{pmatrix}^\top$, and are not a multiple of $\mathbb{1}$. The blue area in Figure 8.2 depicts the corresponding set $\mathcal{F}$ of P_c such that the power flow is feasible. Since $(V_L^*)_1 \neq (V_L^*)_2$, Theorem 8.3 states that we will observe Braess' paradox if we increase the line between load 1 and load 2. We increase the conductance of this line by 0.7. This results in the green area in Figure 8.2 corresponding to the set $\hat{\mathcal{F}}$ of P_c such that the power flow is feasible. It is observed that the blue area is not contained in the green area. Hence there are vectors of power demands for which the power flow has become unfeasible after increasing the conductance of the line. In particular we see that $P_{\max}$ is no longer feasible after the conductance is increased. We remark that, although the maximal total power demand decreased when the conductance was increase, the displayed area of feasibility increased by 4.79%.

Remark 8.5. For constant-power loads we have that the constant power demand $(P_c)_i$ at a load i vanishes if and only if the injected current $(\mathcal{I}_L)_i$ at the load vanishes, since $(V_L)_i > 0$ and

$$(P_c)_i = -(P_L)_i = -(V_L)_i(\mathcal{I}_L)_i.$$

When the injected current at a load node vanishes, the load can be (algebraically) eliminated from the network via Kron reduction (see Section 2.2.2). We observe in Figure 8.2 that if $P_{c,2} = 0$, then the maximal value of $P_{c,1}$ decreases from 2.1 to 2 when G_{12} is increased. This shows that Braess' paradox can also occur in a Kron reduced power grid. The opposite effect is observed for $P_{c,2}$ when $P_{c,1} = 0$.

We continue this section by stating a slightly sharper version of Theorem 8.3, which shows that Braess' paradox can also occur if we only consider increasing the conductance of an existing line, and do not add lines.

Corollary 8.6. Consider a power grid with Y and $V_S > \mathbb{0}$ given. If not all open-circuit voltages (3.10) are equal (i.e., $V_L^* \notin \text{span}\{\mathbb{1}\}$), then there exist a vectors of power demand P_c and a line between two load nodes such that the power flow is feasible and becomes unfeasible after increasing the conductance of the line.

Proof. Suppose that for all lines between two load nodes i and j we have $(V_L^*)_i = (V_L^*)_j$. The load nodes in the power grid form a connected graph by assumption, which means that $(V_L^*)_i = (V_L^*)_j$ for all i, j , and hence $V_L^* \in \text{span}\{\mathbb{1}\}$. This contradicts our assumption on V_L^* . Hence there exists loads i and j which are connected by a line and satisfy $(V_L^*)_i \neq (V_L^*)_j$. The proof of Theorem 8.3 shows that Braess' paradox can occur by increasing the conductance of the line between loads i and j . $\square$

We remark that Theorem 8.3 and Corollary 8.6 extend naturally to DC power grids with resistive shunts (see Section 2.2.3).

8.1.3 Prevalence of Braess' paradox for power flow feasibility

In [53] it was shown that Braess' paradox is a widespread phenomenon which can occur in most traffic flow networks. Similarly, [11] showed that Braess' paradox may occur in any coupled-oscillator network such as AC power grids. In this section we validate that Braess' paradox is also common for DC power flow feasibility, and can occur in most practical DC power grids with constant-power loads.

Corollary 8.6 states that Braess' paradox may occur in the case where not all open-circuit voltages are equal. Due to the relation (3.10) between the source voltages and the open-circuit voltages, we may restate the theorem in terms of the source voltages. As a result, we show that the occurrence of Braess' paradox does not depend on the topology of the power grid, but is due to the voltage sources.

Theorem 8.7. *Consider a power grid with Y and $V_S > \mathbb{0}$ given, and assume that Y_{LS} has full column rank. If the source voltages are not all equal (i.e., $V_S \notin \text{span}\{\mathbb{1}\}$), then there exist a vector of power demands P_c and a line between two load nodes such that the power flow is feasible and becomes unfeasible after increasing the conductance of the line.*

Proof. Recall from Section 2.1.5 that $Y\mathbb{1} = \mathbb{0}$, which implies that

$$Y_{LL}\mathbb{1} + Y_{LS}\mathbb{1} = \mathbb{0}.$$

Rewriting this expression yields $\mathbb{1} = -Y_{LL}^{-1}Y_{LS}\mathbb{1}$. The matrix $-Y_{LL}^{-1}Y_{LS}$ has full column rank, and so by (3.10) we have $V_L^* = c\mathbb{1}$ for some $c > \mathbb{0}$ if and only if $V_S = c\mathbb{1}$. Corollary 8.6 implies the result. $\square$

Theorem 8.7 shows that Braess' paradox may occur if Y_{LS} has full column rank and not all source voltages are equal. The condition that Y_{LS} has full column rank can be interpreted as the property that there are no redundant voltage sources, in the sense that the removal of one source cannot be compensated by other sources. A more precise formulation and proof for this interpretation is given in Lemma C.2. To illustrate, Y_{LS} has full column rank if each load shares a line with at most one source and each source shares a line with at least one load. Conversely, if three sources share lines with only two loads, then (at least) one source is redundant.

For all power grids with a single source node we have that Y_{LS} has full column rank. This shows that Theorem 8.3, Corollary 8.6 and Theorem 8.7 are not satisfied for power grids with a single source. However, since practical power grids commonly have

multiple sources¹, and all source voltages are not likely to be the same, we conclude that Braess' paradox for power flow feasibility may occur in most practical DC power grids.

A natural follow-up question is if Braess' paradox can also occur if all source voltages are equal, for example in the case of a single source. When all voltage potentials at the sources are equal, we may replace all sources by a single source, which results in an equivalent power flow feasibility problem. Hence, to prove that Braess' paradox cannot occur when all source voltages are equal, studying the case of a single source node would be sufficient. The following example suggests that Braess' paradox does not occur for such power grids. A formal proof of this claim is lacking, and thus this question remains open.

Example 8.8. Consider again the power grid with two loads and two sources, as depicted in Figure 8.1, where now $V_S = \begin{pmatrix} 2 & 2 \end{pmatrix}^\top$, and as in Example 8.4 we have $G_{12} = 0.3$, $G_{14} = 1$, $G_{23} = 5$ and $G_{34} = 1$. The corresponding open-circuit voltages are $V_L^* = \frac{2.24}{1.1} \begin{pmatrix} 1 & 1 \end{pmatrix}^\top$, which is a multiple of $\mathbf{1}$. The blue area in Figure 8.2 depicts the corresponding the set $\mathcal{F}$ of power demands P_c such that the power flow is feasible. We increase the conductance of this line by 0.7 and 1.7, which results in respectively the green and yellow areas in Figure 8.3, corresponding to the P_c such that the power flow is feasible. We observe that the blue area is contained in the green area, which in turn is contained in the yellow area. This suggests that power flow feasibility is not compromised when G_{ij} is increased. We also observe that $P_{\max}$ does not change when the conductance is increased, which follows from (8.1) in the proof of Theorem 8.3, since now $(V_L^*)_1 = (V_L^*)_2$. Note that taking $G_{12} \rightarrow \infty$ in the power grid in Figure 8.1 corresponds to shorting the line between nodes i and j . This corresponds to combining the two loads into a single constant power load, where the constant power demand of the new load is the sum of the demands of the two loads. Since $n = 2$, this sum is bounded from above by $\mathbf{1}^\top P_{\max}$ by Proposition 3.20, this implies that the set $\mathcal{F}$ converges to half-space described by (3.23), which has the dashed line as boundary, which suggests that Braess' paradox does not occur for this power grid when the source voltages are equal.

8.2 Bounded uncertainties in line conductances

Section 8.1 argued that Theorem 8.7 applies to most practical power grids, which shows that Braess' paradox is a fundamental problem for power flow feasibility when line conductances are not precisely known. Indeed, Example 8.4 illustrates that the

¹With the possible exception of power distribution grids, which are commonly modeled with a single source or slack bus.

Figure 8.3: Plots for the set of feasible power demands for the power grid in Example 8.8. The blue, green and yellow regions correspond to respectively $G_{12} = 0.3$, $G_{12} = 1$ and $G_{12} = 2$. We observe that $P_{\max}$ is the same for all G_{12} .

line conductances can influence the power flow feasibility, especially when a power grid operates close to its feasibility limits. The existence of Braess' paradox for power flow feasibility shows that exact measurements of line conductances are desirable for guaranteeing the safe operation of a power grid. Unfortunately, such measurements are often not available, and only rough estimates of line conductances may be known. This motivates us to study DC power grids with constant power loads such that power flow feasibility is resilient to inaccuracies or deviations of the line conductances.

In the remainder of this chapter we assume that no exact measurements of the line conductances are available. Instead, we assume that upper and lower bounds on the line conductances are known, which are represented by $\overline{G}_{ij} \geq 0$ and $\underline{G}_{ij} \geq 0$, respectively, and satisfy $\underline{G}_{ij} \leq \overline{G}_{ij}$. Note that the case where $\underline{G}_{ij} = 0$ and $\overline{G}_{ij} > 0$ corresponds to the possibility that nodes i and j are possibly not connected by a line.

Throughout this section we assume that $V_S > 0$ is known and fixed. We also assume that the conductance in least one line between a load and another node is not exactly known, which means $\underline{G}_{ij} \neq \overline{G}_{ij}$ some pair of nodes (i, j) where i or j is a load. We will write $\mathcal{F}_Y$ for the set of feasible power demands, to indicate its dependence on the Kirchhoff matrix Y . This notation is more formally defined as follows.

Definition 8.9. For a given Kirchhoff matrix Y we let $\mathcal{F}_Y$ denote the set of all constant power demands $P_c \in \mathbb{R}^n$ such that the power flow is feasible. *I.e.*,

$$\mathcal{F}_Y := \{ P_c \mid \exists V_L > 0 : P_c = -[V_L](Y_{LL}V_L + Y_{LS}V_S) \}.$$

Note that $\mathcal{F}_Y$ is independent of the principal submatrix Y_{SS} .

In addition, we introduce the following definitions.

Definition 8.10. We let $\mathcal{Y} \subseteq \mathbb{R}^{(n+m) \times (n+m)}$ denote the set of all Kirchhoff matrices Y given by (8.4) that satisfy $\underline{G}_{ij} \leq G_{ij} \leq \overline{G}_{ij}$.

Definition 8.11. Consider a power grid with line uncertainties described by $\mathcal{Y}$ and with constant power demands $P_c \in \mathbb{R}^n$. We say that the power grid is *strongly feasible* if the power flow is feasible for all Kirchhoff matrices $Y \in \mathcal{Y}$, or equivalently, if P_c is contained in the set

$$\mathcal{F}_\mathcal{Y} := \bigcap_{Y \in \mathcal{Y}} \mathcal{F}_Y. \quad (8.3)$$

We remark that $\mathcal{F}_\mathcal{Y}$ is closed convex, since it is the intersection of closed convex sets due to Theorem 4.18. The set $\mathcal{F}_Y$ can be constructed via its boundary—see Section 5.3. We may therefore numerically compute $\mathcal{F}_\mathcal{Y}$ for examples with few uncertainties, by explicitly computing the intersection of the sets $\mathcal{F}_Y$. This method is used to obtain Figure 8.4, which depicts the set of constant power demands for which the power grid described in Example 8.4 is strongly feasible. Note however that this explicit method is computationally intensive when many line conductances are uncertain. Hence a more tractable method to check is therefore desirable. An example with multiple uncertainties is found in Figure 8.5, where we consider the power grid in Example 8.4 with $G_{12} = 1$ and consider an uncertainty of 20% in the line conductances.

8.2.1 A sufficient condition for strong feasibility

We conclude this section by presenting a sufficient condition for the strong feasibility of a power grid with line uncertainties. Our methodology relies on constructing an auxiliary power grid that features resistive shunts at the load nodes (see Section 2.2.3). Adding resistive shunts to a power grid makes the power flow feasibility of a power grid more conservative, which we prove in Lemma C.3. By taking an appropriate choice of the line conductances and resistive shunts, we prove that power flow feasibility of the auxiliary power grid implies that the power flow of the power grid with line uncertainties is strongly feasible.

To this end we introduce the following auxiliary power grid, based on the bounds $\underline{G}_{ij}$ and $\overline{G}_{ij}$, with same number of nodes and the same source voltages, but different line conductances. In this power grid we take $\underline{G}_{ij}$ as the conductance of the line between nodes i and j . In addition we consider resistive shunts at the nodes, which are lines that connect a node to ground. The line conductance of each shunt equals

$$G_{\text{shunt},i} := \sum_k (\overline{G}_{ki} - \underline{G}_{ki}).$$

Figure 8.4: Plots of constant power demands for the power grid in Example 8.4. This means that Kirchhoff matrices $Y \in \mathcal{Y}$ are described by the bounds $\underline{G}_{12} = 0.3$, $\overline{G}_{12} = 1$, $\underline{G}_{14} = \overline{G}_{14} = 1$, $\underline{G}_{23} = \overline{G}_{23} = 5$ and $\underline{G}_{34} = \overline{G}_{34} = 1$. The gray regions depicts the power demands such that the power flow is feasible for different choices of line conductances. The blue region corresponds to the set $\mathcal{F}_Y$ of power demands such that the power flow is strongly feasible, and is the intersection of all gray regions. The red region corresponds to the sufficient condition for strong feasibility in Theorem 8.12, and is contained in the blue region.

Since $\underline{G}_{ij} \neq \overline{G}_{ij}$ for some load node i , we know that at $G_{\text{shunt},i} > 0$ for at least one load node i . The grounded Laplacian matrix $\hat{Y} \in \mathbb{R}^{(n+m) \times (n+m)}$ associated to this auxiliary power grid is given by

$$\hat{Y}_{ij} := \begin{cases} \sum_k \underline{G}_{ki} + G_{\text{shunt},i} = \sum_k \overline{G}_{ki} & \text{if } i = j \\ -\underline{G}_{ij} & \text{if } i \neq j \end{cases}. \quad (8.4)$$

Theorem 8.12. *If a vector $P_c \in \mathbb{R}^n$ of constant power demands is such that the auxiliary power grid with shunted Kirchhoff matrix $\hat{Y}$ has feasible power flow, then any power grid with Kirchhoff matrix $Y \in \mathcal{Y}$ also has feasible power flow, implying that the power grid is strongly feasible for this P_c . Put differently, we have $\mathcal{F}_{\hat{Y}} \subseteq \mathcal{F}_Y$. Moreover, P_c never lies on the boundary of $\mathcal{F}_Y$ for any $Y \in \mathcal{Y}$.*

Proof. Let $P_c \in \mathcal{F}_{\hat{Y}}$, which means that there exists $\tilde{V}_L > 0$ such that

$$P_c = -[\tilde{V}_L](\hat{Y}_{LL}\tilde{V}_L + \hat{Y}_{LS}V_S).$$

Let $Y \in \mathcal{Y}$. The matrix $\hat{Y}$ satisfies $\hat{Y} - Y \geq 0$ since $\underline{G}_{ij} \leq G_{ij} \leq \overline{G}_{ij}$. This means that for all $V_L > 0$ we have

$$(\hat{Y}_{LL} - Y_{LL})V_L + (\hat{Y}_{LS} - Y_{LS})V_S \geq 0. \quad (8.5)$$

Figure 8.5: Plots of constant power demands for the power grid in Figure 8.1 with $V_S = (1 \ 3)^\top$.

The green region corresponds to the power demands such that the power grid with line conductances $\tilde{G}_{12} = 1$, $\tilde{G}_{14} = 1$, $\tilde{G}_{23} = 5$ and $\tilde{G}_{34} = 1$ has feasible power flow, similar to Example 8.4. The gray region describes the union of same sets when the line conductances deviate by at most 20%. In this case we have $\underline{G}_{ij} = 0.8 \tilde{G}_{ij}$ and $\overline{G}_{ij} = 1.2 \tilde{G}_{ij}$ for all i, j . The blue region corresponds to the power demands such that the power grid is strongly feasible under the line uncertainties. The red region depicts the sufficient condition of Theorem 8.12, and is contained in the blue region.

Since $\underline{G}_{ij} \neq \overline{G}_{ij}$ for some load node i we have $\hat{Y}_{LL} \neq Y_{LL}$ or $\hat{Y}_{LS} \neq Y_{LS}$, and hence we never have equality in (8.5). Using (8.5) we observe that

$$P_c = -[\tilde{V}_L](\hat{Y}_{LL}\tilde{V}_L + \hat{Y}_{LS}V_S) \not\leq -[\tilde{V}_L](Y_{LL}\tilde{V}_L + Y_{LS}V_S) =: \tilde{P}_c.$$

By definition of $\mathcal{F}_Y$ we have that $\tilde{P}_c \in \mathcal{F}_Y$. Since $P_c \not\leq \tilde{P}_c$ it follows from Lemma 6.1 that $P_c \in \text{int}(\mathcal{F}_Y)$, the interior of $\mathcal{F}_Y$. Hence, $\mathcal{F}_{\hat{Y}} \subseteq \text{int}(\mathcal{F}_Y)$ for all $Y \in \mathcal{Y}$, and thus $\mathcal{F}_{\hat{Y}} \subseteq \mathcal{F}_Y$. $\square$

Theorem 8.12 presents a sufficient condition for the strong feasibility of a power grid, in terms of the inclusion $P_c \in \mathcal{F}_{\hat{Y}}$. There exists effective methods to verify this inclusion. It was argued in Section 4.8 the methods of Chapter 4 apply to a general grounded Laplacian such as $\hat{Y}$. The condition $P_c \in \mathcal{F}_{\hat{Y}}$ is therefore equivalent to solving a linear matrix inequality of Theorem 4.22. This condition may also be verified using the method proposed in [41]. Alternatively, if P_c is nonnegative then this condition can be verified using the scalar-valued inequality in Theorem 5.14.

The fact that any $P_c \in \mathcal{F}_{\hat{Y}}$ never lies on the boundary of $\mathcal{F}_Y$ indicates that this condition is not tight and therefore conservative. This is verified in Figures 8.4 and 8.5,

Figure 8.6: Plots of constant power demands for the power grid in Figure 8.1 with $V_S = (1 \ 3)^\top$ and with bounded line conductances. We have $\underline{G}_{12} = \overline{G}_{12} = 0.3$, $\underline{G}_{14} = 1$, $\overline{G}_{14} = 1.7$, $\underline{G}_{23} = 5$, $\overline{G}_{23} = 5.7$ and $\underline{G}_{34} = \overline{G}_{34} = 1$. The gray regions depicts the power demands such that the power flow is feasible for different choices of line conductances. The blue region corresponds to the set $\mathcal{F}_Y$ of power demands such that the power flow is strongly feasible, and is the intersection of all gray regions. The red region corresponds to the sufficient condition for strong feasibility in Theorem 8.12, and is contained in the blue region. We observe that the set $\mathcal{F}_{\hat{Y}}$ coincides with the set $\mathcal{F}_{\hat{Y}}$ in Figure 8.4. This follows since the matrices $\hat{Y}$ are the equal, with the exception of the Y_{SS} submatrix.

in which the set $\mathcal{F}_{\hat{Y}}$ does not touch the boundary of $\mathcal{F}_Y$. This conservativeness can be attributed to the resistive shunts in the auxiliary power grid, since resistive shunts make power flow feasibility more conservative by Lemma C.3, and since shunts are not present in the power grid described by $\mathcal{Y}$.

The auxiliary power grid is not unique to the power grid described by $\mathcal{Y}$. This is observed by comparing Figures 8.6 and 8.4, which consider two power grids with different uncertainties in the lines, but where sets $\mathcal{F}_{\hat{Y}}$ are equal. Note that the sets $\mathcal{F}_Y$ in these figures are not equal, which shows that, loosely speaking, the set $\mathcal{F}_{\hat{Y}}$ is not an accurate approximation of $\mathcal{F}_Y$.

8.3 Conclusion and discussion

In this chapter we have shown that an analogue of Braess' paradox may occur in the power flow feasibility of a DC power grid. The observed phenomenon states that an increase of a line conductance has the potential to destroy the feasibility of the power

flow in a power grid. We have shown that this phenomenon may occur in most practical power grids with multiple sources. This has motivated the study of power grids where the exact line conductances are unknown, and only upper and lower bounds are available. We introduced the concept of strong feasibility, which asks when power flow is feasible for all line conductances within these bounds. We concluded by presenting a sufficient condition for the strong feasibility of the power flow in such power grids.

Further directions of research are as follows. We have suggested to combat Braess' paradox by considering bounded line conductances and introducing the notion of strong feasibility. However, there may be alternative methods to avoid Braess' paradox that are worth studying, such as a probabilistic approach. In addition we wonder if and how our observations for DC power grids are applicable to other research areas, such as traffic networks. Furthermore, many question regarding strong feasibility can be studied. For example, is the intersection in (8.3) equivalent to a finite intersection? If so, we can verify strong feasibility by checking a finite number of LMI's, using Theorem 4.22. More ambitiously, can we formulate a necessary and sufficient condition for a power grid with bounded line conductances to have strongly feasible power flow?

9 | Conclusion

This thesis concerned the feasibility of power flow in DC power grids with constant-power loads. For such power grids it is known that the feasibility of power flow is a nontrivial and elusive problem, and essential to the safe long-term operation of the power grid. In this thesis we presented an in-depth study of this feasibility problem.

To facilitate the analysis in this thesis we compiled an overview of results in matrix theory, graph theory, and circuit theory. This overview was found in Chapter 2.

In Chapter 3 we stated the problem of power flow feasibility for DC power grid with constant-power loads. We illustrated the difficulties of this power flow problem by means of several academic examples. Moreover, we showed that multiple solutions to the power flow problem may exist, and discussed the problem of selecting the “correct” operating point in terms of three desirable properties.

Chapter 4 formed the keystone of our analysis of the power flow problem. In this chapter we gave a full characterization of the set of constant-power demands for which the power flow is feasible. Moreover, we showed that the three desirable properties of operating points belong to a single unique operating point, and that there is a one-to-one correspondence between a feasible vector of power demands and such an operating point. It was shown that the operating point associated to a vector of feasible power demands can be found by solving an initial value problem. Furthermore, we showed that the set of feasible constant-power load is closed and convex, and that the feasibility of a vector of constant-power demands can be verified by a necessary and sufficient LMI condition. Also, in our analysis we obtained a parametrization of the set of desirable operating points.

In Chapter 5 we continued the analysis of Chapter 4, and considered the case where the constant power demands are positive. This case corresponds to a more classical setup where loads cannot supply power to the grid. For such loads we derived an alternative necessary and sufficient condition, which is (to some extent) cheaper to compute than its counterpart in Chapter 4. In addition we derived a second parametrization of the set of desirable operating points, and established a duality between this parametrization and the one from Chapter 4. We also obtained two parametrizations of the boundary of the set of feasible power demands, a parametrization of the boundary of this set in the nonnegative orthant, and a geometric characterization of the nonnegative feasible power demands.

In Chapter 6 we returned to the case where power demands can also be negative, and derived multiple sufficient conditions for power flow feasibility. The advantages of such conditions are that they are cheaper to compute than, for example, the necessary and

sufficient LMI condition from Chapter 4. Furthermore we showed how our conditions generalize known conditions in the literature, and even showed how these conditions can be recovered from our condition.

Chapter 7 dealt with the feasibility of the interconnection of multiple DC microgrids with constant-power loads. Microgrids have the feature to connect and disconnect from the main power grid, and instead operate in “island mode”. We devised a “plug-and-play” certificate, along with a test, to guarantee that the power flow of a power grid remains feasible after a new microgrid is connected. This certificate relies on the block Cholesky decomposition, and to facilitate this approach we introduced the concept of a virtual power grid. If the power flow in this virtual power grid is feasible, then the power flow of the physical power grid is also feasible.

Finally, in Chapter 8 we considered uncertainty in conductances of the line. First, we showed that for DC power grids with constant-power loads, and at least two sources, there is a phenomenon akin to Braess’ paradox. In this context, the paradox states that, when a new line is added to a power grid or the conductance of a line in the power grid is increased, it can happen that the power flow is no longer feasible. We also proved that this phenomenon is not specific to any particular power grid topology, and that it can occur in any power grid with at least two sources. In the second half of this chapter we considered power grids where the exact line conductances are unknown and only upper and lower bounds are known, in an attempt to combat our analogue of Braess’ paradox. We gave a sufficient condition for the feasibility of the power flow in all power grids that satisfy the bounds on the line conductances.

9.1 Suggestions for future research

One of the more fundamental questions is how the results of this thesis can be extended to AC power grids. In such an extension one expects to recover the results within this thesis in the case of DC power grids. However, in the case of AC power grids, the model describing the system at steady state is complex-valued. For this reason the theory developed in this thesis cannot be applied, and it is not clear which steps should be taken to generalize our results successfully.

A second topic of interest is a closer study of the voltage domain. One may wonder how many solutions to the power flow equation exists, when it is feasible, and how we may systematically obtain (compute) these operating points. We have seen that the desired operating point can be obtained by solving an initial value problem. By studying this initial value problem one may obtain higher-order approximations of the desired operating point. It would be insightful to study the quality of these approximations, and which of the desired properties these operating points possess.

In this thesis we avoided any operational limits on the power grid, such as bounds on voltage potentials, source power injections, current limits in the lines, and more. This topic ties in with the previously mentioned study of the voltage domain, since many of these limits can be stated in terms of the selected operating point. These operational limits are commonly considered in the optimal power flow literature, where the power flow problem is addressed by optimization techniques. The necessary and sufficient LMI condition of Chapter 4 could be a powerful tool for this reason. However, at this time the exact implications of this LMI condition to the optimal power flow literature are not clear, and require more study.

Finally, it could be valuable to explore other branches of mathematics and other research fields, to find out how the results in this thesis can be applied to other problems. Prospective research subjects may include traffic networks, chemical networks, and imaging theory.

A | Properties of Λ_1 and its closure

Recall from (4.6) and (4.10) that Λ (Λ_1) is the set of vectors λ such that $h(\lambda) = \frac{1}{2}([\lambda]Y_{LL} + Y_{LL}[\lambda])$ is positive definite (and $\|\lambda\|_1 = 1$).

Lemma A.1. *The set Λ is an open convex cone.*

Proof. The convex combination of positive definite matrices is again positive definite, and the set of all positive definite matrices is open. The result follows since $h(\lambda)$ is linear in λ . $\square$

Lemma A.2. *The set Λ is contained in the positive orthant. I.e., $\lambda > \mathbb{0}$ for $\lambda \in \Lambda$.*

Proof. Let λ be such that $h(\lambda)$ is positive definite. Recall that the matrix Y_{LL} is positive definite. A matrix is positive definite only if its diagonal elements are positive. The diagonal elements of Y_{LL} and $h(\lambda)$ are respectively given by $(Y_{LL})_{ii}$ and $\lambda_i(Y_{LL})_{ii}$, and therefore $(Y_{LL})_{ii} > 0$ and $\lambda_i(Y_{LL})_{ii} > 0$. This implies that $\lambda_i > 0$ for all i . $\square$

Lemma A.3. *The set Λ_1 is a bounded convex set.*

Proof. Let $\lambda \in \Lambda$. Since $\lambda > \mathbb{0}$ by Lemma A.2, it follows that $\|\lambda\|_1 = \lambda^\top \mathbf{1}$. Hence $\Lambda_1 = \Lambda \cap \{ \lambda \mid \|\lambda\|_1 = 1 \} = \Lambda \cap \{ \lambda \mid \lambda^\top \mathbf{1} = 1 \}$. The latter expression is an intersection of convex sets (see Lemma A.1). Hence Λ_1 is convex. The set $\{ \lambda \mid \|\lambda\|_1 = 1 \}$ is bounded and thus Λ_1 is bounded. $\square$

Lemma A.4. *The closure of Λ_1 satisfies*

$$\text{cl}(\Lambda_1) = \{ \lambda \mid h(\lambda) \text{ is positive semi-definite, } \|\lambda\|_1 = 1 \}.$$

Proof. Since Λ_1 is nonempty, this follows directly from linearity of h , and the fact that the positive semi-definite matrices form the closure of the positive definite matrices. $\square$

Lemma A.5. *The set $\text{cl}(\Lambda_1)$ is contained in the positive orthant. Moreover, the matrix $h(\lambda)$ for $\lambda \in \text{cl}(\Lambda_1)$ is an irreducible M-matrix.*

Proof. The vectors in Λ_1 are positive, and so the vectors in $\text{cl}(\Lambda_1)$ are nonnegative. To show that $\text{cl}(\Lambda_1)$ lies in the positive orthant, it suffices to show that if a vector $\lambda \in \partial\Lambda_1$ contains zeros, then Y_{LL} is not irreducible, which is a contradiction.

Suppose $\lambda \in \partial\Lambda_1$ such that $\lambda_{[\alpha]} = \mathbb{0}$ and $\lambda_{[\alpha^c]} > \mathbb{0}$ for some nonempty set $\alpha \subseteq \mathbf{n}$. Let w be a vector such that $w_{[\alpha^c]} = \mathbb{0}$ and $w_{[\alpha]}$ is arbitrary. We therefore have $[\lambda]w = \mathbb{0}$. Since

$h(\lambda)$ is positive semi-definite, the following inequality holds for every vector v and scalar β :

$$\begin{aligned} 0 &\leq (v - \beta w)^\top h(\lambda)(v - \beta w) = (v - \beta w)^\top [\lambda]Y_{LL}(v - \beta w) \\ &= v^\top [\lambda]Y_{LL}(v - \beta w) = v^\top [\lambda]Y_{LL}v - \beta v^\top [\lambda]Y_{LL}w \quad (\text{A.1}) \end{aligned}$$

If v is such that $v^\top [\lambda]Y_{LL}w \neq 0$, then (A.1) is violated when we take β such that $\beta v^\top [\lambda]Y_{LL}w$ is sufficiently large. It follows that $v^\top [\lambda]Y_{LL}w = 0$ for all v . This implies that

$$[\lambda]Y_{LL}w = 0. \quad (\text{A.2})$$

The rows of (A.2) corresponding to $\alpha^c \subseteq \mathbf{n}$ satisfy

$$[\lambda_{\alpha^c}]((Y_{LL})_{[\alpha^c, \alpha]}w_{[\alpha]} + (Y_{LL})_{[\alpha^c, \alpha^c]}w_{[\alpha^c]}) = 0. \quad (\text{A.3})$$

Recall that $\lambda_{[\alpha^c]} > 0$ and $w_{[\alpha^c]} = 0$, and thus (A.3) implies

$$(Y_{LL})_{[\alpha^c, \alpha]}w_{[\alpha]} = 0. \quad (\text{A.4})$$

Since $w_{[\alpha]}$ is arbitrary, (A.4) should hold for all $w_{[\alpha]}$, and hence $(Y_{LL})_{[\alpha^c, \alpha]} = 0$. However, this contradicts the assumption that Y_{LL} is irreducible. We conclude that $\lambda > 0$.

Since Y_{LL} is an irreducible Z-matrix and $\lambda > 0$, Lemma 2.5 implies that $h(\lambda)$ is an irreducible Z-matrix. Since $h(\lambda)$ is positive semi-definite, it is an M-matrix by Lemma 2.17. $\square$

Lemma A.6. *Let $\lambda \in \partial\Lambda_1$. For every $v \geq 0$ in the kernel of $h(\lambda)$ we have $\lambda^\top \frac{\partial P_c}{\partial V_L}(x)v > 0$ for all $x \in \mathbb{R}^n$.*

Proof. Since $\lambda \in \partial\Lambda_1$, the matrix $h(\lambda)$ is singular. By Lemma A.5, it is a singular irreducible M-matrix, and its Perron root is zero. The kernel of $h(\lambda)$ is spanned by any Perron vector, by Lemma 2.13. This implies that v is a Perron vector and $v > 0$. Let x be any vector. By substituting (4.1), we note that $\lambda^\top \frac{\partial P_c}{\partial V_L}(x)v$ is equivalent to

$$\begin{aligned} \lambda^\top \frac{\partial P_c}{\partial V_L}(x)v &= \lambda^\top [Y_{LL}V_L^*]v - \lambda^\top ([x]Y_{LL} + [Y_{LL}x])v \\ &= \lambda^\top [\mathcal{I}_L^*]v - 2x^\top h(\lambda)v \\ &= \lambda^\top [\mathcal{I}_L^*]v, \end{aligned} \quad (\text{A.5})$$

where we used the fact that $h(\lambda)v = 0$. Since $v > 0$, $\lambda > 0$ and $\mathcal{I}_L^* \geq 0$, it follows from (A.5) that $\lambda^\top [\mathcal{I}_L^*]v > 0$. We conclude that $\lambda^\top \frac{\partial P_c}{\partial V_L}(x)v > 0$. $\square$

B | Properties of $\mathcal{M}$

Lemma B.1. *If $\mu \in \mathcal{M}$, then $\mu > \mathbb{0}$. Moreover, $\mathcal{M}$ is an open cone and is simply connected.*

Proof. If $\mu \in \mathcal{M}$, then $g(\mu) = [\mu]Y_{LL} + [Y_{LL}\mu]$ is a nonsingular M-matrix, and therefore a Z-matrix. Recall that Y_{LL} is an irreducible Z-matrix, which implies that $(Y_{LL})_{[i,i^c]} \not\leq 0$ for all i . If $\mu_i < 0$, then $g(\mu)_{[i,i^c]} = \mu_i(Y_{LL})_{[i,i^c]} \not\geq 0$, which contradicts the fact that $g(\mu)$ is a Z-matrix. Hence $\mu \geq \mathbb{0}$. We will show that a vector μ which contains zeros does not yield an M-matrix. Suppose $\mu \in \mathcal{M}$ such that $\mu_{[\alpha]} = \mathbb{0}$ and $\mu_{[\alpha^c]} > \mathbb{0}$ for some nonempty set $\alpha \subseteq \mathbf{n}$. Since $\mu_{[\alpha]} = \mathbb{0}$, the following submatrices of $[\mu]Y_{LL} + [Y_{LL}\mu]$ satisfy

$$([\mu]Y_{LL} + [Y_{LL}\mu])_{[\alpha,\alpha^c]} = [\mu_{[\alpha]}](Y_{LL})_{[\alpha,\alpha^c]} = 0 \quad (\text{B.1})$$

and

$$\begin{aligned} ([\mu]Y_{LL} + [Y_{LL}\mu])_{[\alpha,\alpha]} &= [\mu_{[\alpha]}](Y_{LL})_{[\alpha,\alpha]} + [(Y_{LL}\mu)_{[\alpha]}] \\ &= 0 + [(Y_{LL})_{[\alpha,\alpha]}\mu_{[\alpha]} + (Y_{LL})_{[\alpha,\alpha^c]}\mu_{[\alpha^c]}] \\ &= [(Y_{LL})_{[\alpha,\alpha^c]}\mu_{[\alpha^c]}]. \end{aligned} \quad (\text{B.2})$$

It follows from (B.1) that $[\mu]Y_{LL} + [Y_{LL}\mu]$ is block-triangular. Hence, the eigenvalues of each diagonal block are also eigenvalues of $[\mu]Y_{LL} + [Y_{LL}\mu]$. The principal submatrix given in (B.2) is such a diagonal block. Note from (B.2) that this block is diagonal, and so its eigenvalues are the elements of the vector $(Y_{LL})_{[\alpha,\alpha^c]}\mu_{[\alpha^c]}$. Since Y_{LL} is an irreducible Z-matrix we have $(Y_{LL})_{[\alpha,\alpha^c]} \not\leq 0$. Recall that $\mu_{[\alpha^c]} > \mathbb{0}$, which implies that

$$(Y_{LL})_{[\alpha,\alpha^c]}\mu_{[\alpha^c]} \not\leq \mathbb{0},$$

and so $[\mu]Y_{LL} + [Y_{LL}\mu]$ has nonpositive eigenvalues. However, since $[\mu]Y_{LL} + [Y_{LL}\mu]$ is an M-matrix, its Perron root is positive and is a lower bound for all other eigenvalues, which is a contradiction. We conclude that $\mu > \mathbb{0}$.

The matrix $g(\mu)$ is linear in μ . Hence, scaling of μ gives rise to a scaling of the eigenvalues of $g(\mu)$, and in particular of the Perron root of $g(\mu)$. Hence $\mathcal{M}$ is a cone. The set of nonsingular M-matrices is open, and so $\mathcal{M}$ is an open set.

The set $\partial\mathcal{D}$ is simply connected by Theorem 4.7. Theorem 5.5 shows that there exists a bicontinuous map between $\partial\mathcal{D}$ and $\mathcal{M}_1$. Topological properties are preserved by bicontinuous maps, and hence $\mathcal{M}_1$ is also simply connected. Its conic hull $\mathcal{M}$ is therefore also simply connected. $\square$

Lemma B.2. *The set of long-term voltage semi-stable operating points is contained in $\mathcal{M}$ (i.e., $\text{cl}(\mathcal{D}) \subseteq \mathcal{M}$).*

Proof. Recall from Corollary 4.4 that if $\tilde{V}_L \in \text{cl}(\mathcal{D})$, then $-\frac{\partial P_c}{\partial V_L}(\tilde{V}_L)$ is an M-matrix, which is irreducible by Lemma 4.2. This means that $g(\tilde{V}_L) - [\mathcal{I}_L^*]$ is an irreducible M-matrix by (5.3). By adding $[\mathcal{I}_L^*]$ to $g(\tilde{V}_L) - [\mathcal{I}_L^*]$, Lemma 2.19 implies that $g(\tilde{V}_L)$ is a nonsingular M-matrix since $\mathcal{I}_L^* \geq \mathbf{0}$. $\square$

C | Appendix to Chapter 8

Lemma C.1. Let $A = A^\top \in \mathbb{R}^{n \times n}$ be nonsingular, let $b, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$. If $1 + cb^\top A^{-1}b \neq 0$ then

$$v^\top (A + cbb^\top)^{-1}v = v^\top A^{-1}v - c \frac{(v^\top A^{-1}b)^2}{1 + cb^\top A^{-1}b}.$$

Proof. By the Sherman-Morrison formula [45, E9.1.1] we have that $(A + cbb^\top)^{-1}$ exists and satisfies

$$(A + cbb^\top)^{-1} = A^{-1} - c(1 + cb^\top A^{-1}b)^{-1}A^{-1}bb^\top A^{-1}.$$

The result is obtained through multiplication by $v^\top$ and v , and noting that $A^{-\top} = A^{-1}$. $\square$

Lemma C.2. The matrix $Y_{LS} \in \mathbb{R}^{n \times m}$ has full column rank if and only if for each $V_S > 0$ there does not exist a vector $\hat{V}_S \geq 0$ such that $Y_{LS}V_S = Y_{LS}\hat{V}_S$ and $(\hat{V}_S)_i = 0$ for some i (i.e., none of the sources are redundant).

Proof. ($\Rightarrow$): If $Y_{LS}V_S = Y_{LS}\hat{V}_S$ and $(\hat{V}_S)_i = 0 < (V_S)_i$, then $0 \neq V_S - \hat{V}_S \in \ker Y_{LS}$ and so Y_{LS} does not have full column rank.

($\Leftarrow$): Suppose $Y_{LS} \in \mathbb{R}^{n \times m}$ does not have full column rank. Let w be a nonzero vector in the kernel of Y_{LS} . Without loss of generality we assume that w has a negative entry. It follows that for some $\theta > 0$ we have $\hat{V}_S := V_S + \theta w \geq 0$ with equality for at least one entry. But this means that $Y_{LS}V_S = Y_{LS}\hat{V}_S$ and $(\hat{V}_S)_i = 0$ for some i , which contradicts our premise. Hence $Y_{LS} \in \mathbb{R}^{n \times m}$ has full column rank. $\square$

Lemma C.3. Consider a power grid with Kirchhoff matrix Y . Adding resistive shunts to the load nodes makes power flow feasibility more conservative. More explicitly, let D be a nonnegative diagonal matrix with D_{ii} positive for some load node i . Let P_c and $V_S > 0$ be given. If a power grid with Kirchhoff matrix $Y + D$ has feasible power flow (i.e., $P_c \in \mathcal{F}_{Y+D}$), then a power grid with Kirchhoff matrix Y has feasible power flow (i.e., $P_c \in \mathcal{F}_Y$) and P_c does not lie on the boundary of $\mathcal{F}_Y$.

Proof. Let $P_c \in \mathcal{F}_{Y+D}$, which means that there exists $\tilde{V}_L > 0$ such that

$$P_c = -[\tilde{V}_L]((Y + D)_{LL}\tilde{V}_L + (Y + D)_{LS}V_S).$$

For all $V_L > 0$ we have $D_{LL}V_L \geq 0$, where equality does not hold since D_{ii} positive for some load node i . Note also that $D_{LS} = 0$ since D is diagonal. We therefore have

$$P_c = -[\tilde{V}_L](\hat{Y}_{LL}\tilde{V}_L + \hat{Y}_{LS}V_S) \leq -[\tilde{V}_L](Y_{LL}\tilde{V}_L + Y_{LS}V_S) =: \tilde{P}_c.$$

By definition of $\mathcal{F}_Y$ we have that $\tilde{P}_c \in \mathcal{F}_Y$. Since $P_c \leq \tilde{P}_c$, Lemma 6.1 implies that P_c lies in the interior of $\mathcal{F}_Y$. $\square$

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Summary

Over the past century our demand for and dependence on electrical energy has grown exponentially. A society without electricity has simply become unthinkable. It is therefore of great importance to secure the reliability of our power grid. The past decades have seen an increase in new (renewable) energy sources, such as solar and wind energy, of which the power output is sometimes difficult to predict. These technologies therefore introduce a degree of uncertainty to the power grid, which asks for the development of novel control strategies. However, before we can consider such control strategies it is important to establish if the control problem in question is actually feasible. It is therefore important to develop a good understanding of the limits of our power grids.

Power grids traditionally make use of alternating current (AC). In the past decades there have also been ideas to apply direct current (DC) on a larger scale for the transportation and distribution of power. One advantage of DC networks is that these networks are easier to analyze in comparison with AC networks. One particular topic concerns the so-called DC microgrids, which are self-sufficient DC power grids which can operate independently from the main power grid, for example in the case of a blackout.

This thesis is centered around DC power grids. It considers power grids which consist of two types of nodes: loads and sources. A source is a node which injects power into the grid, such as a power plant or a solar farm, whereas a load drains power from the grid, such as a factory or a household. More specifically, the load that are considered are constant-power loads, which demand a fixed amount of power from the grid. Such loads are known to destabilize the grid, due to the simple fact that their demand cannot always be satisfied by the sources. A sustained mismatch of supply and demand can lead to a steep decline in the voltage potentials and to blackouts, which are to be avoided. This thesis presents a formal analysis of this problem, and derives a method to identify if (and when) the demands of the loads can be satisfied.

This thesis also deals with a number of follow-up questions on the different cases in which the constant-power demands are satisfied. For such a case it is desirable that the power dissipated in the lines is minimized, and that the voltage potentials at the loads are as high as possible. It is shown that there exists a unique case which satisfies both of these properties. In addition, a method is presented to construct this unique case.

Thereafter, (sufficient) conditions are studied and derived, which guarantee that the demand of the loads can be satisfied. Furthermore, a certificate is formulated to guarantee that the load in multiple DC microgrids can be satisfied after interconnecting the microgrids. Finally, observations are made on the effects of changes and uncertainties

in power lines, which leads to the formulation of a phenomenon analogous to Braess' paradox, and an approach is presented on how to deal with such uncertainties.

Samenvatting

Gedurende de afgelopen eeuw is onze vraag naar en afhankelijkheid van elektrische energie exponentieel gegroeid. Een maatschappij zonder elektriciteit is dan ook ondenkbaar geworden. Het is daarom van groot belang dat de betrouwbaarheid van ons elektriciteitsnetwerk gewaarborgd is. In de laatste decennia is het aandeel van nieuwe (hernieuwbare) energiebronnen zoals zonne-energie en windenergie toegenomen. Helaas valt de opbrengst van deze energiebronnen niet altijd goed te voorspellen. Dit brengt een mate van onzekerheid met zich mee, waardoor we genoodzaakt zijn om nieuwe regeltechnieken te ontwikkelen om hier mee om te gaan. Nog voordat we kunnen nadenken over zulke regeltechnieken moeten we weten of hetgeen wat we willen bereiken ook daadwerkelijk haalbaar is. Hiervoor is het belangrijk om een goed beeld te hebben van de grenzen van onze elektriciteitsnetwerken.

Ons elektriciteitsinfrastructuur maakt traditioneel gebruik van wisselstroom. In de afgelopen decennia wordt er ook nagedacht om op een grotere schaal gelijkstroom te gebruiken voor de transportatie en distributie van elektrische stroom. Één voordeel van gelijkstroomnetwerken is dat dit netwerktype zich gemakkelijker laat analyseren in vergelijking met wisselstroomnetwerken. In het bijzonder wordt er nagedacht over kleinschalige zelfvoorzienende gelijkstroomnetwerken die zelfstandig van het hoofdelektriciteitsnet kunnen functioneren, bijvoorbeeld in het geval van een grootschalige stroomuitval.

In dit proefschrift staan gelijkstroomnetwerken centraal. Er wordt gekeken naar netwerken die bestaan uit twee soorten knopen: ladingen en bronnen. Een bron-knoop representeert een energiebron, zoals een elektriciteitscentrale of een windmolenpark, en een lading-knoop staat gelijk aan een gebruiker van het netwerk, zoals een fabriek of een huishouden. Meer specifiek wordt er gekeken naar lading-knopen die een vaste hoeveelheid vermogen vragen. Het zijn dit soort ladingen die het netwerk kunnen doen destabiliseren, simpelweg doordat aan deze vraag niet altijd voldaan kan worden. Wanneer de mismatch in vraag en aanbod aanhoudt kan dit leiden tot een abrupte daling van de spanningen in het netwerk en tot stroomuitval, wat voorkomen dient te worden. Dit proefschrift presenteert een formele analyse van dit probleem, en leidt een methode af om te identificeren of (en wanneer) aan de vraag van de ladingen voldaan kan worden.

Dit proefschrift gaat ook in op een aantal vervolgvragen over de verschillende cases waarin aan de vraag van de ladingen voldaan is. Het is voor zo een casus wenselijk dat het vermogensverlies dat optreedt door de transport van elektriciteit geminimaliseerd is, en dat de spanningspotentialen bij de ladingen zo hoog mogelijk zijn. Er wordt aangetoond dat er een unieke casus bestaat wanneer deze beide eigenschappen optreden.

Tevens wordt er een methode gepresenteerd voor het construeren van deze casus.

Vervolgens worden (voldoende) voorwaarden bestudeerd en afgeleid die garanderen dat aan de vraag van de ladingen ook voldaan kan worden. Ook wordt er gekeken naar garanties voor het voldoen aan de vraag na het koppelen van meerdere kleinschalige zelfvoorzienende gelijkstroomnetwerken. Tot slot wordt er geobserveerd wat het effect is van veranderingen en onzekerheden van elektriciteitsleidingen, wat leidt tot formulering van een fenomeen dat verwant is aan de Braess-paradox, en wordt er nagedacht over hoe om te gaan met deze onzekerheden.