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The Final State

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The Final State

The fate of relativistic compact objects after merger

Ka Wa Tsang

A thesis presented for the degree of
Doctor of Philosophy

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The Final State

The fate of relativistic compact objects after merger

PhD thesis

to obtain the degree of PhD at the
University of Groningen
on the authority of the
Rector Magnificus Prof. T. N. Wijmenga
and in accordance with
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諸天述說神的榮耀；穹蒼傳揚他的手段。
這日到那日發出言語；這夜到那夜傳出知識。
無言無語，也無聲音可聽。
它的量帶通遍天下，它的言語傳到地極。
(詩篇19：1-4上)

The heavens are telling the glory of God,
and the firmament proclaims the work of his hands.
Every day they pour forth speech,
and every night they tell knowledge.
There is no speech and there are no words;
their sound is inaudible.
Yet in all the world their line goes out,
and their words to the end of the world.
(Psalm 19:1-4a)

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CHAPTER 1

INTRODUCTION

Gravity is a daily natural phenomenon by which all massive objects tend to attract towards each other. In 1687, Isaac Newton (1642-1727) proposed Newton's law of universal gravitation in his book, *Philosophiæ Naturalis Principia Mathematica*, to quantify gravity. It states that every point mass exhibits an attractive gravitational force on other point mass along the line intersecting both points with a magnitude directly proportional to their masses and inversely proportional to the square of the distance apart,

$$\vec{\mathbf{F}}_1 = -\frac{m_1 m_2}{r^2} \hat{\mathbf{r}}_{21}, \quad (1.1)$$

where $\vec{\mathbf{F}}_1$ is the force acting on the first object due to the second object, m_1 and m_2 are the masses of the first and second object respectively, r is the distance separated between the two objects, $\hat{\mathbf{r}}_{21}$ is the unit vector pointing from the second object to the first object, and the negative sign represents that gravity is attractive. Together with the Newton's law of motions,

$$\vec{\mathbf{F}}_1 = m_1 \vec{\mathbf{a}}_1, \quad (1.2)$$

where $\vec{\mathbf{a}}_1$ is the acceleration of the first object, it successfully explains a set of famous empirical laws, namely Kepler's laws of planetary motion, which are

1. 'The orbit of a planet is an ellipse with one of whose foci being occupied by the Sun',
2. 'The vector drawn from the Sun to the planet's position sweeps equal areas in equal times', and
3. 'The squares of the periods are directly proportional to the cubes of the major axes of the ellipses'.

Despite the success of Newtonian gravity, it has serious drawbacks both theoretically and experimentally. For example, theoretically it assumes infinite speed of gravity. In other words, one object can gravitationally influence another object instantaneously. It violates special relativity, in which no signals can travel faster than the speed of light to avoid violation of causality. Experimentally it fails to explain the advance of perihelion of the Mercury first observed in 1859.

In 1915, Albert Einstein (1879-1955) proposed an alternative theory of gravity, that is the famous General Relativity (GR) in which gravity is no longer a result of force but spacetime curvature. GR satisfies the correspondence principle, meaning that it reduces to Newtonian

Figure 1.1: Orbital decay of the Hulse–Taylor binary consistent to the energy loss carried away by GWs [2].

gravity in the weak gravitational field and non-relativistic limits. John Wheeler, an American theoretical physicist, summarised GR in one sentence, *i.e.* “matter tells spacetime how to curve and curved spacetime tells matter how to move”. In fact, compared to Newtonian gravity, not only matter can curve spacetime but also radiation and dark energy.

GR not only explains what Newtonian gravity cannot explain but also makes new predictions. All classical tests are consistent with GR suggesting the adoption of GR over Newtonian gravity. They are

1. The advance of perihelion of the Mercury,
2. The bending of light around a massive object, and
3. The gravitational redshift of light.

In GR, the orbit of the Mercury is indeed not an ellipse as predicted by Newtonian gravity but its perihelion precesses with an amount consistent with the observed shift. In addition, GR predicts that the starlight bends around a massive object twice as much as what Newtonian gravity predicts and it was confirmed later on by Arthur Eddington (1882-1944) during the total solar eclipse on 29 May 1919. Finally, the gravitational redshift of light was verified in the Pound–Rebka experiment in 1959 [1].

More importantly, GR predicts the existence of black holes and gravitational waves. A black hole (BH) is a highly compact astrophysical object that significantly deforms its surrounding

Figure 1.2: The GW event GW150914 observed by the LIGO Hanford (H1, Left) and Livingston (L1, Right) detectors [3].

spacetime to an extent that a boundary is created, namely the event horizon, inside which neither matter nor radiation can escape from it. Gravitational waves (GWs) are a new kind of radiation that constitutes a disturbance in the spacetime curvature from accelerating mass and propagates outward at the speed of light. However, even with highly compact objects, the effect of GWs is predicted to be as tiny as a strain $\sim \mathcal{O}(10^{-21})$ which corresponds to a measurement of the size of an atom $\sim \mathcal{O}(10^{-10})$ m relative to the distance between the Sun and the Earth $\sim \mathcal{O}(10^{11})$ m and thus poses an observational challenge. The first indirect evidence of GWs comes from a Nobel Prize winning discovery of a binary neutron star (BNS) system known as Hulse–Taylor binary in 1974 [2]. Subsequent analysis showed that the orbital decay of this system is consistent with the loss of energy carried away by GWs [4], Fig. 1.1. With the advance of technology, on 14 September 2015, almost exactly 100 years after the publication of GR, the two detectors of the Laser Interferometer Gravitational-Wave Observatory (LIGO) in the United States, one in Hanford, Washington (LHO) and the other one in Livingston, Louisiana (LLO), simultaneously observed a transient GW signal which has a frequency range from 35 to 250 Hz with a peak strain of 10^{-21} . The signal is consistent with a waveform of binary black hole (BBH) merger as predicted by GR. It is the famous GW150914 event, Fig. 1.2, in human history which provides direct evidence of

1. The existence of GWs,
2. The existence of BHs,
3. The existence of BHs in a binary system, and

Figure 1.3: Aerial views of the LIGO Hanford Observatory, Washington (left), the LIGO Livingston Observatory, Louisiana (centre) and the Virgo detector, Italy (right).

4. BBH which merges within a Hubble time, *i.e.* 14.4 billion years.

In the old days, we observed the universe by looking at the starlight. With the help of telescope and computer, we could make precise measurement of stars. Now the discovery of GWs has opened a new alternative window for us to study the universe. We can now not just look at but also ‘listen’¹ to the universe. In 2017, the Nobel Prize in Physics was awarded to three physicists: Rainer Weiss, Kip Thorne and Barry Barish for their role in the direct detection of GWs.

After the participation of the Advanced Virgo detector in Italy in August 2017, the detector network becomes more sensitive to GWs and is able to see deeper into the universe. Aerial views of the LIGO and Virgo detectors are shown in Fig. 1.3. On 14 August 2017 the first three-detectors GW event from a BBH coalescence was observed, namely GW170814 [5]. After three days, on 17 August 2017, they observed GWs from a BNS coalescence, namely GW170817 [6]. Strikingly, after 1.7 s of the GW merger signal, a short-duration (~ 2 s) gamma ray burst (GRB) from a consistent sky location was also observed by the Fermi Gamma-ray Space Telescope (FGST) and INTErnational Gamma-Ray Astrophysics Laboratory (INTEGRAL) spacecraft, namely GRB170817A. The optical emission was also observed 11 hours after the GW merger signal, namely AT2017gfo. These electromagnetic (EM) observations are consistent with a kilonova which occurs when two NSs merge to give a neutron-rich environment where many heavy r-process nuclei are produced and subsequently decay to emit short GRB and EM radiations. The nucleosynthesis from BNS mergers can account for the origin of many elements heavier than iron such as gold and platinum [7]. GW170817 marked the start of multi-messenger astronomy, *i.e.* observing the same astronomical event via four messengers (EM radiation, GWs, neutrinos and cosmic rays). For neutrinos and cosmic rays, all GW sources observed so far are too far away from the Earth to be observed. During the first (O1), from 12 September 2015 to 19 January 2016, and second (O2), from 30 November 2016 to 25 August 2017, observing runs in total there are 10 BBH and 1 BNS confident detections summarised in the GW Transient Catalog (GWTC-1) of compact binary merger [8] (see Table 1.1).

Moreover, on 10 April 2019, the Event Horizon Telescope (EHT) has successfully taken the first BH image (Fig. 1.4) which is a supermassive black hole in M87 and gives further direct evidence of the existence of BHs [9]. In fact, the 2020 Nobel Prize in Physics was awarded one half to Roger Penrose ‘for the discovery that black hole formation is a robust prediction of the

¹The analogy is often made because the GW frequency detected by ground-based detectors falls within the human audible range.

Event	$m_1/M_\odot$	$m_2/M_\odot$	$\mathcal{M}/M_\odot$	χ_{eff}	$M_f/M_\odot$	a_f	$E_{\text{rad}}/(M_\odot c^2)$	$\ell_{\text{peak}}/(\text{erg s}^{-1})$	d_L/Mpc	z	$\Delta\Omega/\text{deg}^2$
GW150914	$35.6^{+4.7}_{-3.1}$	$30.6^{+3.0}_{-4.4}$	$28.6^{+1.7}_{-1.5}$	$-0.01^{+0.12}_{-0.13}$	$63.1^{+3.4}_{-3.0}$	$0.69^{+0.05}_{-0.04}$	$3.1^{+0.4}_{-0.4}$	$3.6^{+0.4}_{-0.4} \times 10^{56}$	440^{+150}_{-170}	$0.09^{+0.03}_{-0.03}$	182
GW151012	$23.2^{+14.9}_{-5.5}$	$13.6^{+4.1}_{-4.8}$	$15.2^{+2.1}_{-1.2}$	$0.05^{+0.31}_{-0.20}$	$35.6^{+10.8}_{-3.8}$	$0.67^{+0.13}_{-0.11}$	$1.6^{+0.6}_{-0.5}$	$3.2^{+0.8}_{-1.7} \times 10^{56}$	1080^{+550}_{-490}	$0.21^{+0.09}_{-0.09}$	1523
GW151226	$13.7^{+8.8}_{-3.2}$	$7.7^{+2.2}_{-2.5}$	$8.9^{+0.3}_{-0.3}$	$0.18^{+0.20}_{-0.12}$	$20.5^{+6.4}_{-1.5}$	$0.74^{+0.07}_{-0.05}$	$1.0^{+0.1}_{-0.2}$	$3.4^{+0.7}_{-1.7} \times 10^{56}$	450^{+180}_{-190}	$0.09^{+0.04}_{-0.04}$	1033
GW170104	$30.8^{+7.3}_{-5.6}$	$20.0^{+4.9}_{-4.6}$	$21.4^{+2.2}_{-1.8}$	$-0.04^{+0.17}_{-0.21}$	$48.9^{+5.1}_{-4.0}$	$0.66^{+0.08}_{-0.11}$	$2.2^{+0.5}_{-0.5}$	$3.3^{+0.6}_{-1.0} \times 10^{56}$	990^{+440}_{-430}	$0.20^{+0.08}_{-0.08}$	921
GW170608	$11.0^{+5.5}_{-1.7}$	$7.6^{+1.4}_{-2.2}$	$7.9^{+0.2}_{-0.2}$	$0.03^{+0.19}_{-0.07}$	$17.8^{+3.4}_{-0.7}$	$0.69^{+0.04}_{-0.04}$	$0.9^{+0.0}_{-0.1}$	$3.5^{+0.4}_{-1.3} \times 10^{56}$	320^{+120}_{-110}	$0.07^{+0.02}_{-0.02}$	392
GW170729	$50.2^{+16.2}_{-10.2}$	$34.0^{+9.1}_{-10.1}$	$35.4^{+6.5}_{-4.8}$	$0.37^{+0.21}_{-0.25}$	$79.5^{+14.7}_{-10.2}$	$0.81^{+0.07}_{-0.13}$	$4.8^{+1.7}_{-1.7}$	$4.2^{+0.9}_{-1.5} \times 10^{56}$	2840^{+1400}_{-1360}	$0.49^{+0.19}_{-0.21}$	1041
GW170809	$35.0^{+8.3}_{-5.9}$	$23.8^{+5.1}_{-5.2}$	$24.9^{+2.1}_{-1.7}$	$0.08^{+0.17}_{-0.17}$	$56.3^{+5.2}_{-3.8}$	$0.70^{+0.08}_{-0.09}$	$2.7^{+0.6}_{-0.6}$	$3.5^{+0.6}_{-0.9} \times 10^{56}$	1030^{+320}_{-390}	$0.20^{+0.05}_{-0.07}$	308
GW170814	$30.6^{+5.6}_{-3.0}$	$25.2^{+2.8}_{-4.0}$	$24.1^{+1.4}_{-1.1}$	$0.07^{+0.12}_{-0.12}$	$53.2^{+3.2}_{-2.4}$	$0.72^{+0.07}_{-0.05}$	$2.7^{+0.4}_{-0.3}$	$3.7^{+0.4}_{-0.5} \times 10^{56}$	600^{+150}_{-220}	$0.12^{+0.03}_{-0.04}$	87
GW170817	$1.46^{+0.12}_{-0.10}$	$1.27^{+0.09}_{-0.09}$	$1.186^{+0.001}_{-0.001}$	$0.00^{+0.02}_{-0.01}$	≤ 2.8	≤ 0.89	≥ 0.04	$\geq 0.1 \times 10^{56}$	40^{+7}_{-15}	$0.01^{+0.00}_{-0.00}$	16
GW170818	$35.4^{+7.5}_{-4.7}$	$26.7^{+4.3}_{-5.2}$	$26.5^{+2.1}_{-1.7}$	$-0.09^{+0.18}_{-0.21}$	$59.4^{+4.9}_{-3.8}$	$0.67^{+0.07}_{-0.08}$	$2.7^{+0.5}_{-0.5}$	$3.4^{+0.5}_{-0.7} \times 10^{56}$	1060^{+420}_{-380}	$0.21^{+0.07}_{-0.07}$	39
GW170823	$39.5^{+11.2}_{-6.7}$	$29.0^{+6.7}_{-7.8}$	$29.2^{+4.6}_{-3.6}$	$0.09^{+0.22}_{-0.26}$	$65.4^{+10.1}_{-7.4}$	$0.72^{+0.09}_{-0.12}$	$3.3^{+1.0}_{-0.9}$	$3.6^{+0.7}_{-1.1} \times 10^{56}$	1940^{+970}_{-900}	$0.35^{+0.15}_{-0.15}$	1666

Table 1.1: Selected source parameters of the 11 confident detections during O1 and O2 [8].

Figure 1.4: The first EHT image of the supermassive BH of M87 reconstructed by four independent teams with different techniques [9].

general theory of relativity’ and the other half to Reinhard Genzel and Andrea Ghez ‘for the discovery of a supermassive compact object at the centre of our galaxy.’.

After further upgrades of the Advanced LIGO and Advanced Virgo detectors, the third observing run (O3) started on 1 April 2019 and ended on 27 March 2020 ². On 12 April 2019, the detector network observed a GW event from a BBH, called GW190412 [10], which is different from observations during O1 and O2 because of its asymmetric masses. The median value of the mass ratio is measured to be $0.28^{+0.12}_{-0.07}$. It is an interesting event as it shows strong evidence of GW radiation beyond quadrupolar order (*i.e.* the higher multipoles) which indicates consistency with GR. In fact, the discovery of GWs offers a new way to test GR in the strong-field regime. Many tests of GR have been carried out. For the source dynamics, it includes parametrized test [11], inspiral-merger-ringdown consistency test [12], test of no-hair theorem [13–15], test of BH area increase law [16] *etc.* For properties of GWs, it includes test of Lorentz invariance violation [17], upper limit on the graviton mass [18], the speed of gravity bound [19] *etc.* Up to now, all observational evidence is consistent with what GR predicts and no deviations from GR have been found.

GW observations also give hints for us to study the formation of BHs. The mass of a black hole varies over a large range, for example stellar BH $< 10^2 M_\odot$, intermediate mass BH (IMBH)

²The original end date was 30 April 2020 but the observing run was suspended on 27 March 2020 because of the worldwide COVID-19 pandemic.

$[10^2 - 10^5] M_\odot$ and supermassive BH (SMBH) $> 10^5 M_\odot$. For stellar BHs, there are theoretical and observational arguments suggesting that stellar evolution may not produce BHs with mass less than $5 M_\odot$ [20–22]. In addition, the possible maximum mass of a NS is expected to be $\sim [2 - 3] M_\odot$ [23–27]. The range of $\sim [2 - 5] M_\odot$ is the so-called lower mass gap [28] in which no BHs are observed so far. On the other hand, there is the so-called upper mass gap in the range of $\sim [64 - 135] M_\odot$ in which no BH can form via core collapse of a single star due to the pair-instability supernova (PISN) [29–33]. Observational evidence for IMBHs has long been sought as IMBHs connect the gap between stellar BHs and SMBHs. The discovery of IMBHs will be a key to the formation of SMBHs which is still unknown. Surprisingly, on 25 April 2019, a single-detector event GW190425 was found in the LLO data [34]. It was of interest because the range of component masses, from 1.12 to 2.52 $M_\odot$, is consistent with the individual binary components being neutron stars, though the possibility that one or both binary components are BHs cannot be ruled out. If it was a BNS, it will have been the second BNS event detected via GWs and very different from the known populations of Galactic BNS because of its high total mass ($3.4^{+0.3}_{-0.1} M_\odot$) and chirp mass ($1.44^{+0.02}_{-0.02} M_\odot$). If it was a BBH, it will have contained the first BH observed in the lower mass gap. In any case, there are interesting implications for the formation of this system. In addition, on 21 May 2019, a GW event from a BBH was detected, called GW190521 [35]. It is the heaviest BBH system we have ever observed via GWs, which has a total mass $150^{+29}_{-17} M_\odot$ suggesting the mass of the resulting BH remnant to be $142^{+28}_{-16} M_\odot$. It means that a direct observation of the formation of an IMBH was made. Furthermore, the heavier component mass $85^{+21}_{-14} M_\odot$ sits in the upper mass gap. Many possible hypotheses are suggested to explain its formation, *e.g.* hierarchical mergers of lower-mass BHs [36].

GWs are also new tools for us to study the cosmology. For example, it allows a new measurement of the Hubble constant by combining the distance to the source inferred from GWs with the recession velocity inferred from the redshift using EM data [37, 38]. It does not require any form of cosmic distance ladder [39].

More importantly, we can study the nature of compact objects via GWs. How certain is the GR description of the observed BHs? Are there new types of compact objects other than BHs and NSs, *e.g.* wormholes, in the universe? In this thesis, we focus on the nature of the final state of compact binary merger. In Chapter 2 the theoretical foundation of GWs including their derivation from GR, their generation, and their interactions with matters are summarised. In Chapter 3 the Bayesian data analysis techniques used in the GW community are discussed. In Chapter 4 we analyze the nature of an excited BH resulting from a BBH from its well-known characteristic GWs, *i.e.* ringdown. Any abnormality captured by the additional deviation parameters signals the existence of exotic compact objects. In Chapter 5 we develop a morphology-independent algorithm to detect and characterize exotic compact objects resulting from a binary merger. This algorithm is important because there might be exotic compact objects that have not yet been envisaged. In Chapter 6 we discuss the general morphology of GWs emitted by the final NS after the BNS merger, establish the quasi-universal relations that connect the inspiral and postmerger properties, and build a simple GW waveform model of BNS postmerger from numerical simulations for the purpose of data analysis. In Chapter 7 we summarise our researches.

CHAPTER 2

GRAVITATIONAL WAVES

In this chapter, we are going to study the theoretical foundation of GWs. We will first define all notations and physical quantities, and review the basic GR. Afterwards, we will prove the existence of GWs by linearizing the Einstein field equations. We will then study how GWs interact with matter and are generated by matter.

2.1 Conventions and notations

2.1.1 Units and constants

Unless explicitly stated, geometric units are adopted such that the speed of light in vacuum c and the Newton's constant G are both set to unity,

$$c = G = 1. \quad (2.1)$$

Geometric units lead to the same dimension for time, length and mass,

$$[\text{Time}] = [\text{Length}] = [\text{Mass}]. \quad (2.2)$$

This is to be compared with another common unit convention used in particle physics, in which $\hbar = c = 1$ and the dimensions of length and time are not equal to the dimension of the mass but the inverse of it.

The solar mass is denoted by $M_\odot$ and equal to 1.989×10^{30} kg, 1.477 km or 4.927×10^{-6} s.

2.1.2 Einstein summation convention

Lower-case Greek letters (*e.g.* $\mu, \nu, \dots$) denote spacetime indices and take the values of 0, 1, 2 and 3, while lower-case Latin letters (*e.g.* $i, j, \dots$) denote space indices and take the values of 1, 2 and 3. A spacetime event x^μ is given by

$$x^\mu \equiv (t, \mathbf{x}) = (t, x^i) \quad (2.3)$$

where t represents the time coordinate and x^i represent the space coordinates. Depending on the context, x (or y) can denote either the x (or y)-coordinate in the Minkowski space or a spacetime event.

When an index variable appears twice in a single term, one subscript and one superscript, a summation of that term over all possible values of the index is implied. For example, the trace of the identity matrix δ^i_j (or the Kronecker delta) is equal to the number of dimensions,

$$\delta^i_i = 3. \quad (2.4)$$

2.1.3 Metric tensor

A position vector $\mathbf{s}$ at point P is considered and its infinitesimal displacement vector is

$$d\mathbf{s} = \mathbf{e}_\mu dx^\mu, \quad (2.5)$$

where $\mathbf{e}_\mu \equiv \frac{\partial \mathbf{s}}{\partial x^\mu}$ are the basis vectors. The metric tensor $g_{\mu\nu}$ is defined to be $g_{\mu\nu} \equiv \mathbf{e}_\mu \cdot \mathbf{e}_\nu$ such that the invariant line element is

$$ds^2 \equiv (d\mathbf{s}) \cdot (d\mathbf{s}) = (\mathbf{e}_\mu dx^\mu) \cdot (\mathbf{e}_\nu dx^\nu) = g_{\mu\nu} dx^\mu dx^\nu. \quad (2.6)$$

Physically $g_{\mu\nu}$ gives a rule to measure distance in a general space. In GR, $g_{\mu\nu}$ is a symmetric, positive-definite tensor and its determinant g does not vanish. Although it has $4^2 = 16$ components, only 10 of those are independent because it is symmetric. The metric signature is chosen to be $(-, +, +, +)$. The components of the metric tensor depend on the choice of the coordinate system. For example, in the flat Minkowski space, the metric tensor becomes the Minkowski metric $\eta_{\mu\nu} = \text{diag}\{-1, 1, 1, 1\}$. If the spatial part is in spherical coordinate system (r, θ, ϕ) , the invariant line element becomes

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (2.7)$$

It is being used to raise or lower the indices on tensors,

$$V^\mu = g^{\mu\nu} V_\nu, \quad (2.8)$$

$$V_\mu = g_{\mu\nu} V^\nu. \quad (2.9)$$

Combining Eqs. (2.8) and (2.9), we have

$$\delta^\mu_\nu = g^{\mu\lambda} g_{\lambda\nu}. \quad (2.10)$$

So $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. In GR, the metric compatibility $\nabla_\rho g_{\mu\nu} = 0$ is guaranteed where ∇_ρ is the covariant derivative and will be introduced shortly. Therefore, the covariant derivative commutes with raising or lowering of indices,

$$g_{\mu\nu} \nabla_\rho V^\nu = \nabla_\rho (g_{\mu\nu} V^\nu) = \nabla_\rho V_\mu. \quad (2.11)$$

2.1.4 Covariant derivatives

Given a coordinate system, every vector field $\mathbf{V}$ can be expanded in terms of basis vectors,

$$\mathbf{V} = V^\mu \mathbf{e}_\mu, \quad (2.12)$$

where V^μ are the components of $\mathbf{V}$ along the direction of the basis vectors $\mathbf{e}_\mu$. Because of the curved spacetime, the calculation of derivative, namely the covariant derivative ∇_μ , involves the change of the components and the change of the basis vectors. The former is captured by the normal partial derivative ∂_μ while the latter is captured by the Christoffel symbol $\Gamma^\lambda_{\mu\nu}$ (or the connection coefficients),

$$\partial_\mu \mathbf{e}_\nu \equiv \Gamma^\lambda_{\mu\nu} \mathbf{e}_\lambda. \quad (2.13)$$

With the help of linearity and Leibniz (product) rule, we have

$$\begin{aligned} \nabla_\mu \mathbf{V} &= \partial_\mu (V^\nu \mathbf{e}_\nu) \\ &= (\partial_\mu V^\nu) \mathbf{e}_\nu + V^\nu (\partial_\mu \mathbf{e}_\nu) \\ &= (\partial_\mu V^\nu) \mathbf{e}_\nu + V^\nu \Gamma^\lambda_{\mu\nu} \mathbf{e}_\lambda \\ &= (\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda) \mathbf{e}_\nu. \end{aligned} \quad (2.14)$$

By taking the ν component on both sides, we have

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda, \quad (2.15)$$

where the Christoffel symbol can be calculated in terms of $g_{\mu\nu}$,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}). \quad (2.16)$$

In the Minkowski space, the Christoffel symbol vanishes but in a curvilinear coordinate system it does not. In GR, the Christoffel symbol is torsion-free and thus symmetric with respect to the lower indices.

2.1.5 Riemann tensor

Definition

Because of the curved spacetime again, the covariant derivatives do not commute and its difference is quantified by the Riemann tensor $R^\lambda_{\mu\nu\kappa}$,

$$R^\lambda_{\mu\nu\kappa} V_\lambda \equiv (\nabla_\kappa \nabla_\nu - \nabla_\nu \nabla_\kappa) V_\mu \quad (2.17)$$

A space is flat if and only if the Riemann tensor vanishes. Physically Riemann tensor quantifies the tidal gravitational force which will be discussed more shortly. Substituting Eq. (2.15), the Riemann tensor can be expressed in terms of the Christoffel symbol,

$$R^\lambda_{\mu\nu\kappa} V_\lambda = (\partial_\nu \Gamma^\lambda_{\mu\kappa} - \partial_\kappa \Gamma^\lambda_{\mu\nu} + \Gamma^\alpha_{\mu\kappa} \Gamma^\lambda_{\alpha\nu} - \Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\alpha\kappa}) V_\lambda. \quad (2.18)$$

Substituting Eq. (2.16), the Riemann tensor can be expressed solely in terms of the first and second derivatives of $g_{\mu\nu}$,

$$\begin{aligned} R_{\lambda\mu\nu\kappa} = & \frac{1}{2} (\partial_\mu \partial_\nu g_{\lambda\kappa} + \partial_\lambda \partial_\kappa g_{\mu\nu} - \partial_\mu \partial_\kappa g_{\lambda\nu} - \partial_\lambda \partial_\nu g_{\mu\kappa}) \\ & + g_{\alpha\beta} (\Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\lambda\kappa} - \Gamma^\alpha_{\mu\kappa} \Gamma^\beta_{\lambda\nu}). \end{aligned} \quad (2.19)$$

Properties

From Eq. (2.19), several properties of the Riemann tensor can be obtained including the interchange symmetry

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu}, \quad (2.20)$$

the skew symmetry,

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu}, \quad (2.21)$$

the first Bianchi identity

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\nu\kappa\mu} + R_{\lambda\kappa\mu\nu} = 0, \quad (2.22)$$

and the second Bianchi identity

$$\nabla_\eta R_{\lambda\mu\nu\kappa} + \nabla_\nu R_{\lambda\mu\kappa\eta} + \nabla_\kappa R_{\lambda\mu\eta\nu} = 0. \quad (2.23)$$

Degrees of freedom

$R_{\lambda\mu\nu\kappa}$ has d^4 components where d is the number of spacetime dimensions. The interchange symmetry, Eq. (2.20), reduces the number of free parameters from d^4 to $\frac{d^2(d^2+1)}{2}$. The skew symmetry, Eq. (2.21), reduces the number of free parameters of the first two indices and also the last two indices from d^2 to $\frac{(d)(d-1)}{2}$. So there are $\left(\frac{(d)(d-1)}{2}\right)\left(\frac{(d)(d-1)}{2} + 1\right)/2$ free parameters left. If any two indices in the first Bianchi identity, Eq. (2.22), are equal, this identity gives no new information in addition to the above two symmetries. It gives additional constraints only if the indices are all different. There are $d(d-1)(d-2)(d-3)$ equations in which the indices are all different but only dC_4 distinct equations. Therefore, the free parameters reduce by this amount and equal to $\frac{1}{12}(d^2)(d^2-1)$. Because $d = 4$ in GR, only 20 out of $4^4 = 256$ are independent.

Derived physics quantities

Contracting the Riemann tensor yields the Ricci tensor

$$R_{\mu\kappa} \equiv g^{\lambda\nu} R_{\lambda\mu\nu\kappa}. \quad (2.24)$$

Further contraction of the Ricci tensor yields the Ricci scalar

$$R \equiv g^{\mu\kappa} R_{\mu\kappa} = R^\mu{}_\mu. \quad (2.25)$$

The typical variation of the metric is quantified by the curvature radius $\mathfrak{R}$, which is defined to be $\mathfrak{R} \equiv R^{-1/2}$. From the Ricci tensor and the Ricci scalar, one can construct the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \quad (2.26)$$

2.2 General relativity

“Matter tells spacetime how to curve and curved spacetime tells matter how to move”. GR is a field theory of gravitation and involves two parts:

1. the generation of the field, and
2. the law of motions in the field.

The former is governed by the Einstein field equations in which the metric tensor is obtained given a energy-momentum tensor while the latter is governed by the geodesic equations in which the particle trajectory is obtained given a metric tensor. Moreover, in the curved spacetime, a pair of parallel lines approaches each other due to the presence of tidal gravitational force. Their relative acceleration between two neighboring geodesics are related to the Riemann tensor in the geodesic deviation equations.

2.2.1 Einstein field equations

The Newton’s law of gravitation, Eq. (1.1), can be written in differential form, namely the Poisson equation,

$$\nabla^2\Phi = 4\pi\rho, \quad (2.27)$$

where ∇^2 is the Laplacian, Φ is the Newtonian gravitational potential such that $a^i = -\partial^i\Phi$ and ρ is the mass density. This equation suggests that the second derivative of the potential is directly proportional to the mass density. In 1912 Einstein proposed a relativistic version of the above equation by replacing the second derivative of the potential with the Ricci tensor $R_{\mu\nu}$ which is the second derivative of the metric tensor and the mass density with the energy-momentum tensor $T_{\mu\nu}$ which describes the density and flux of energy-momentum,

$$R_{\mu\nu} = 4\pi T_{\mu\nu}. \quad (2.28)$$

It turns out to be wrong but close. By applying ∇_μ on both sides, $\nabla_\mu T^{\mu\nu}$ is obtained on the right-hand side and vanishes because of the conservation of energy-momentum but $\nabla_\mu R^{\mu\nu}$ does not necessarily vanish. It gives a hint to what left-hand side should be. By using the second Bianchi identity, Eq. (2.23), one can show that $\nabla_\mu G^{\mu\nu}$ is identically zero. In 1915 Einstein published the correct Einstein field equations (EFEs),

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \quad (2.29)$$

which is a set of ten coupled second-order non-linear partial differential equations. By default, energy-momentum are covariantly conserved quantities,

$$\nabla_\mu T^{\mu\nu} = 0. \quad (2.30)$$

Due to the non-linearity of the EFEs, not many solutions can be obtained analytically by assuming certain symmetries including the flat space Minkowski metric and the Schwarzschild metric which describes the spacetime around a non-spinning chargeless BH.

2.2.2 Geodesic equation

The Newton's first law states that a particle moves in a straight line in the absence of external force. In other words, the momentum change of the particle is zero. In GR, the notion of straight line is replaced by the geodesic and the Newtonian momentum is replaced by the four-momentum $p^\mu \equiv m \frac{dx^\mu}{d\tau}$ where m is the rest mass of the particle and τ is the proper time given by $-d\tau^2 = ds^2$. As a result, the law of motions is simply the change of four-momentum vanishes,

$$dp^\mu = 0. \quad (2.31)$$

By following the steps similar to Eq. (2.14), the above equation can be written as

$$dp^\mu + \Gamma^\mu_{\nu\rho} p^\nu dx^\rho = 0. \quad (2.32)$$

Dividing both sides by $d\tau$, one can obtain the geodesic equation

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} p^\nu \frac{dx^\rho}{d\tau} = 0. \quad (2.33)$$

By dividing the rest mass on both sides, we have

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0. \quad (2.34)$$

In the case of a massless particle, τ should be replaced by some affine parameter on the geodesic. The $\mu = 0$ component of the geodesic equation relates the proper time and the coordinate time while the $\mu = i$ components of that can be understood if both sides are multiplied by a mass. Then the first term is simply the Newtonian momentum and the second term quantifies the gravitational force. If the space is flat, then the Christoffel symbol and thus the second term vanish, so it reduces to Newton's first law.

2.2.3 Geodesic deviation equation

Now consider two nearby time-like geodesics. Each geodesic is parametrized by its own proper time τ and the deviation vector $\xi^\mu(\tau)$ connects points with the same value of τ on the two geodesics. Both geodesics satisfy the geodesic equation, one is Eq. (2.34) and the other one is

$$\frac{d^2(x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x + \xi) \frac{d(x^\nu + \xi^\nu)}{d\tau} \frac{d(x^\rho + \xi^\rho)}{d\tau} = 0. \quad (2.35)$$

If $|\xi^\mu|$ is much smaller than the curvature radius, we can obtain an equation of ξ^μ by taking the difference between Eqs. (2.35) and (2.34) and keeping only up to the first-order term in ξ^μ ,

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{d\xi^\rho}{d\tau} + \xi^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \quad (2.36)$$

which is the geodesic deviation equation. It can be written in a more elegant way by introducing the directional covariant derivative of a vector field $V^\mu(x)$ along the curve $x(\tau)$. With the help of Eq. (2.15), we have

$$\frac{DV^\mu}{D\tau} \equiv \frac{dx^\nu}{d\tau} \nabla_\nu V^\mu = \frac{dV^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} V^\nu \frac{dx^\rho}{d\tau}. \quad (2.37)$$

Therefore, Eq. (2.36) can be written in terms of the Riemann tensor,

$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu_{\nu\rho\sigma} \xi^\rho \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau}. \quad (2.38)$$

This is an equation showing two time-like geodesics which experience a tidal gravitational force which is determined by the Riemann tensor.

2.3 Linearized theory

2.3.1 Weak-field metric

To study the properties of GWs, it is instructive to study the EFEs assuming the gravitational fields are weak. Every metric tensor can be written as a sum of the Minkowski metric and a correction term $h_{\mu\nu}$. In the so-called weak-field approximation, all the components of $h_{\mu\nu}$ are forced to be small, $|h_{\mu\nu}| \ll 1$, and all higher-order terms in the perturbation are ignored,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(|h_{\mu\nu}|^2). \quad (2.39)$$

Therefore, linearized theory of GR is a theory of the symmetric tensor field $h_{\mu\nu}$ propagating on a flat background spacetime. From Eq. (2.39) we immediately obtain

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad (2.40)$$

where $h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$. It means that $\eta^{\mu\nu}$ (or $\eta_{\mu\nu}$) can raise (or lower) indices. Our goal is to write the EFEs in terms of $h_{\mu\nu}$. We start with the Christoffel symbol. Eq. (2.16) becomes

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{\lambda\sigma} (\partial_\nu h_{\sigma\mu} + \partial_\mu h_{\nu\sigma} - \partial_\sigma h_{\mu\nu}). \quad (2.41)$$

So the Christoffel symbol is a first-order quantity and thus the Riemann tensor will come from the derivatives of the Γ 's but not Γ^2 terms. Eq. (2.19) becomes

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2} (\partial_\mu \partial_\nu h_{\lambda\kappa} + \partial_\lambda \partial_\kappa h_{\mu\nu} - \partial_\mu \partial_\kappa h_{\lambda\nu} - \partial_\lambda \partial_\nu h_{\mu\kappa}). \quad (2.42)$$

One important property is that the Riemann tensor in the linearized theory is gauge-invariant rather than just covariant in the full GR. From the contraction of the first and third indices of the Riemann tensor, the Ricci tensor is found to be

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \square h_{\mu\nu}), \quad (2.43)$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu} = h^\mu_\mu$ and $\square \equiv \eta_{\mu\nu} \partial^\mu \partial^\nu = \partial_\mu \partial^\mu = -\partial_t^2 + \nabla^2$ is the d'Alembertian operator. Contracting the Ricci tensor yields the Ricci scalar

$$R = \partial_\mu \partial_\nu h^{\mu\nu} - \square h. \quad (2.44)$$

All together, the Einstein tensor becomes

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R \\ &= \frac{1}{2}(\partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \square h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \square h), \end{aligned} \quad (2.45)$$

and the EFEs still have the form of

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \quad (2.46)$$

but $G_{\mu\nu}$ is linearized in the first order of $h_{\mu\nu}$ and computed via Eq. (2.45).

2.3.2 Gauge transformations

Physics is absolute but coordinate systems are arbitrary for describing the physics. In GR, when a new coordinate system x'^μ is chosen,

$$x^\mu \rightarrow x'^\mu(x), \quad (2.47)$$

all tensor quantities transform covariantly to retain the form of the laws of physics such that the latter are invariant under arbitrary differentiable coordinate transformations. Observers in any frames of reference can apply the same physics laws to obtain the same physics results although numerical values are different. No frames of reference are preferred. This is called the principle of covariance. In particular, the metric tensor transforms as

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x). \quad (2.48)$$

By choosing the background metric to be $\eta_{\mu\nu}$ and limiting $|h_{\mu\nu}|$ to be sufficiently small, the invariance of GR under coordinate transformation breaks down. It means that the choices of frame are limited to reference frames where Eq. (2.39) holds. However, the frame of reference is still not unique and there remains a residual gauge symmetry. A transformation of coordinates,

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x), \quad (2.49)$$

is considered where ξ^μ is an arbitrary function. Substituting Eq. (2.49) to Eq. (2.48), we obtain the gauge transformation

$$h_{\mu\nu}(x) \rightarrow h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu). \quad (2.50)$$

To preserve the condition $|h_{\mu\nu}| \ll 1$, it is necessary to limit $|\partial_\mu \xi_\nu|$ to be at most of the same order of smallness as $|h_{\mu\nu}|$,

$$|\partial_\mu \xi_\nu| \leq |h_{\mu\nu}|. \quad (2.51)$$

2.3.3 Lorentz transformations

If we consider a finite and global (*i.e.* x -independent) Lorentz transformation,

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu, \quad (2.52)$$

where the matrix Λ^μ_ν , by definition, satisfies

$$\Lambda^\rho_\mu \Lambda^\sigma_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}. \quad (2.53)$$

With the help of Eqs. (2.39) and (2.53), Eq. (2.48) becomes

$$\begin{aligned} g_{\mu\nu}(x) &\rightarrow g'_{\mu\nu}(x') = \Lambda^\rho_\mu \Lambda^\sigma_\nu g_{\rho\sigma}(x) \\ &= \Lambda^\rho_\mu \Lambda^\sigma_\nu [\eta_{\rho\sigma} + h_{\rho\sigma}(x)] \\ &= \eta_{\mu\nu} + \Lambda^\rho_\mu \Lambda^\sigma_\nu h_{\rho\sigma}(x). \end{aligned} \quad (2.54)$$

Because of $g'_{\mu\nu}(x') = \eta_{\mu\nu} + h'_{\mu\nu}(x')$, we have

$$h_{\mu\nu}(x) \rightarrow h'_{\mu\nu}(x') = \Lambda^\rho_\mu \Lambda^\sigma_\nu h_{\rho\sigma}(x). \quad (2.55)$$

It shows that $h_{\mu\nu}$ transforms like a tensor under Lorentz transformation. Rotations never spoil the condition $|h_{\mu\nu}| \ll 1$ but not boosts which have to be limited for not spoiling the condition. In addition, $h_{\mu\nu}$ is invariant under a constant translation c^μ ,

$$x^\mu \rightarrow x'^\mu = x^\mu + c^\mu. \quad (2.56)$$

Therefore linearized theory is invariant under Poincaré transformations (*i.e.* translations plus Lorentz transformations). In contrast, the full GR has general covariance rather than limiting to Poincaré symmetry and the infinitesimal transformation given by Eq. (2.50).

2.3.4 Transverse-traceless gauge

Lorenz gauge

To write the linearized EFEs, Eq. (2.46), in a more compact form, we define the trace-reversed weak-field metric

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad (2.57)$$

and by definition, the trace of $\bar{h}_{\mu\nu}$ is reversed,

$$\bar{h} \equiv \eta^{\mu\nu}h_{\mu\nu} = h - 2h = -h. \quad (2.58)$$

Therefore, Eq. (2.57) can be inverted to give

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}, \quad (2.59)$$

and the linearized EFEs, Eq. (2.46), become

$$\square \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} - \partial^\rho \partial_\nu \bar{h}_{\mu\rho} - \partial^\rho \partial_\mu \bar{h}_{\nu\rho} = -16\pi T_{\mu\nu}. \quad (2.60)$$

To simplify the above expression, we can make use of the gauge freedom, Eq. (2.50), and choose the Lorenz gauge ³,

$$\partial^\nu \bar{h}_{\mu\nu} = 0. \quad (2.61)$$

³Also known as the Hilbert gauge, or the harmonic gauge, or the De Donder gauge.

To prove that, we express Eq. (2.50) in terms of $\bar{h}_{\mu\nu}$,

$$\bar{h}_{\mu\nu} \rightarrow \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial_\rho \xi^\rho), \quad (2.62)$$

and so

$$\partial^\nu \bar{h}_{\mu\nu} \rightarrow (\partial^\nu \bar{h}_{\mu\nu})' = \partial^\nu \bar{h}_{\mu\nu} - \square \xi_\mu, \quad (2.63)$$

Because the d'Alembertian operator is invertible, it is always possible to find a ξ_μ such that $(\partial^\nu \bar{h}_{\mu\nu})' = 0$, *i.e.*

$$\xi_\mu(x) = \int d^4y G(x-y) \partial^\nu \bar{h}_{\mu\nu}(y), \quad (2.64)$$

where G is the Green's function of the d'Alembertian operator and $\square_x G(x-y) = \delta^4(x-y)$. In other words, it is always possible to choose a frame where Lorenz gauge holds. In Lorenz gauge, Eq. (2.60) reduces to an inhomogeneous wave equation,

$$\square \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \quad (2.65)$$

and the conservation of energy-momentum, Eq. (2.30), becomes

$$\partial^\nu T_{\mu\nu} = 0. \quad (2.66)$$

It will be seen shortly that physically Lorenz gauge ensures the propagation direction of GWs is orthogonal to its polarization ⁴. Therefore, gravitational wave is a transverse wave.

Residual gauge freedom

There is residual gauge freedom even the Lorenz gauge is chosen because, according to Eq. (2.63), a further coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu + \zeta^\mu$ does not spoil the Lorenz gauge with

$$\square \zeta_\mu = 0. \quad (2.67)$$

$h_{\mu\nu}$ has $4^2 = 16$ components. Because of its symmetricity, only 10 out of 16 are independent. Because of the Lorenz gauge, Eq. (2.61), which has four constraint equations, only $10 - 4 = 6$ degree of freedom left. Because of Eq. (2.67), which again has four functions of ζ being free to choose, so only $6 - 4 = 2$ degrees of freedom are left. Therefore, GWs have two polarization states, namely the plus and cross polarizations. In particular, outside the source, the residual gauge freedom is fixed by demanding

$$u^\mu h_{\mu\nu} = h_{0\nu} = 0, \quad (2.68)$$

where $u^\mu = (1, \mathbf{0})$ is the constant four-velocity of a time-like observer ⁵, and the traceless property

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} = 0. \quad (2.69)$$

Eq. (2.68) implies that the propagation direction of GWs is orthogonal to the worldline of the observer. All together, Eqs. (2.61), (2.68) and (2.69) define the transverse-traceless (TT) gauge. Eq. (2.68) gives only 3 additional constraints because $u^\mu \partial^\nu h_{\mu\nu} = 0$ is repeated. In summary, there are only $10 - 4 - (4 - 1) - 1 = 2$ independent components of $h_{\mu\nu}$ which are physically important and give two polarization states.

⁴It is similar to the Gauss's law in the Maxwell's equations which ensures the electromagnetic wave is transverse.

⁵In general, one can pick any u^μ with $u^\mu k_\mu \neq 0$ where k^μ is the four-wave vector of GWs.

2.3.5 Plane-wave solution

In order to understand the propagation of GWs, meaning that it is outside the source, $T_{\mu\nu}$ is set to zero and Eq. (2.65) reduces to a homogeneous wave equation,

$$\square \bar{h}_{\mu\nu} = 0, \quad (2.70)$$

which has a simple plane-wave solution,

$$\bar{h}_{\mu\nu}(x) = \Re\left(A_{\mu\nu}e^{ik_\lambda x^\lambda}\right), \quad (2.71)$$

where $\Re$ denotes the real part of a quantity, $A_{\mu\nu}$ is the constant (complex) amplitude tensor, $k^\mu = (\omega, \mathbf{k})$ is the four-wave vector with the angular frequency $\omega = |\mathbf{k}|$ and $k^\mu A_{\mu\nu} = 0$. $k^\mu k_\mu = 0$ implies that GWs travel in the speed of light in the flat background metric. A more general solution can be obtained by the superposition of plane-wave solutions with different propagation directions and frequencies,

$$\bar{h}_{\mu\nu}(x) = \Re\left(\int d^4k \mathcal{A}_{\mu\nu}(\mathbf{k})e^{ik_\lambda x^\lambda}\right), \quad (2.72)$$

where $\mathcal{A}_{\mu\nu}$ is the fourier component and the Lorenz gauge, Eq. (2.61), gives

$$k^\mu \mathcal{A}_{\mu\nu} = k^i \mathcal{A}_{ij} = 0, \quad (2.73)$$

which proves a previous statement that the propagation direction of GWs is orthogonal to its polarizations. Without the loss of generality, z -direction is chosen to be the propagation direction. By imposing the other two conditions, Eqs. (2.68) and (2.69), in the TT gauge, Eq. (2.71) can be written as

$$h_{\mu\nu}^{\text{TT}}(x) = \bar{h}_{\mu\nu}^{\text{TT}}(x) \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = h_+ e^+ + h_\times e^\times, \quad (2.74)$$

where $e^+ = \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_2$ and $e^\times = \hat{\mathbf{e}}_1 \otimes \hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_2 \otimes \hat{\mathbf{e}}_1$ are polarization tensors of h_+ and $h_\times$ respectively. They are the two remaining degrees of freedom in the linearized theory of GR, the plus and cross polarizations defined in the plane transverse to the propagation direction, given by

$$\begin{aligned} h_+(x) &= A_+ \cos(\omega(t - z) + \varphi_{+0}) \\ h_\times(x) &= A_\times \cos(\omega(t - z) + \varphi_{\times 0}), \end{aligned} \quad (2.75)$$

with φ_{+0} and $A_+ > 0$ are the constant phase shift and the constant amplitude for h_+ respectively, and similar definitions for the cross polarization.

2.3.6 Interaction with matter

Transverse-traceless frame

We can understand the effect of GWs by the geodesic equation. We consider a test mass at rest at $\tau = 0$ in the TT frame, *i.e.* $\frac{dx^i}{d\tau} = 0$, then $\mu = i$ components of Eq. (2.34) are

$$\left[\frac{d^2 x^i}{d\tau^2}\right]_{\tau=0} = -\left[\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2\right]_{\tau=0}, \quad (2.76)$$

where $\Gamma^i_{00} = \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00}) = 0$ by Eq. (2.41). So we have

$$\left[\frac{d^2 x^i}{d\tau^2} \right]_{\tau=0} = 0, \quad (2.77)$$

that is zero initial acceleration. In other words, in the TT frame, a test mass initially at rest before the arrival of GWs remains at rest even after the arrival of GWs. The coordinates of the TT frame stretch themselves, in response to the arrival of GWs, in such a way that the position of free test masses initially at rest do not change. This can be seen from the geodesic deviation equations as well. The $\mu = i$ components of Eq. (2.36) give

$$\left[\frac{d^2 \xi^i}{d\tau^2} \right]_{\tau=0} = - \left[2\Gamma^i_{0\rho} \frac{d\xi^\rho}{d\tau} + \xi^\sigma \partial_\sigma \Gamma^i_{00} \right]_{\tau=0}. \quad (2.78)$$

Because $\Gamma^i_{0\rho}$ is non-vanishing only if ρ is spatial by Eq. (2.41) but $\frac{d\xi^i}{d\tau} = 0$ at $\tau = 0$ while the second term is zero, we have $\left[\frac{d^2 \xi^i}{d\tau^2} \right]_{\tau=0} = 0$ and thus the coordinate separation ξ^i remains constant at all times.

This bizarre-looking behaviour can be interpreted as we used the gauge freedom in choosing coordinate system in which positions of free test masses mark the coordinates in the TT frame. In fact, because GR is invariant under coordinate transformations, the physical effects in GR are not reflected on what happens to the coordinates (or coordinates separation) but the proper distance. If we consider two events at $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$ with GWs propagating along the z -axis, then from Eq. (2.39), the proper distance in terms of the linear order in $h_+ = A_+ \cos \omega t$ is

$$\begin{aligned} s &= \int ds \\ &= \int_{x_1}^{x_2} dx \sqrt{1 + A_+ \cos \omega t} \\ &\approx (x_2 - x_1) \left(1 + \frac{1}{2} A_+ \cos \omega t \right), \end{aligned} \quad (2.79)$$

where $(x_2 - x_1)$ is the coordinate separation and thus remains constant. Therefore, the proper distance and hence the time taken by light to make a round trip changes periodically. More generally, if the spatial separation between the two events is given by a vector $\mathbf{L}$, then the proper distance is $s = \sqrt{L^2 + h_{ij}^{\text{TT}} L^i L^j}$.

Local Lorentz frame

In reality, coordinates are defined by a rigid ruler and test masses are free to be displaced by GWs. In GR, it is always possible to make a coordinate transformation at a spacetime event P to a local Lorentz frame (LLF) as a manifestation of equivalence principle where in general, for all α, β, μ, ν ,

$$\begin{aligned} g_{\mu\nu}(P) &= \eta_{\mu\nu} + \mathcal{O}\left(\frac{|\mathbf{x}|^2}{\mathfrak{R}^2}\right), \\ \partial_\alpha g_{\mu\nu}(P) &= 0, \end{aligned} \quad (2.80)$$

but at least for some α, β, μ, ν ,

$$\partial_\alpha \partial_\beta g_{\mu\nu}(P) \neq 0, \quad (2.81)$$

	g_{11}	Δx	$\Delta s = \sqrt{g_{11}(\Delta x)^2}$
TT	$1 + h_+$	L	$L(1 + \frac{1}{2}h_+)$
LLF	1	$L(1 + \frac{1}{2}h_+)$	

Table 2.1: A table showing how to calculate the proper distance in the transverse-traceless frame (TT) and the local Lorentz frame (LLF) if we consider two events on the x -axis. L stands for the spatial separation between two events before the arrival of GWs.

So the Christoffel symbol vanishes at P in the LLF. As long as the size of the region of interest is much smaller than the curvature radius, even in the presence of GWs, locally we live in a flat spacetime approximately where Newtonian physics is valid. The proper distance in LLF is the same as in TT because it is a scalar. Given the same proper distance and Minkowski metric, in LLF the coordinate distance is simply the proper distance. The calculations of proper distance in both TT frame and LLF are summarised in Table 2.1.

In the non-relativistic limit, *i.e.* $\frac{dx^0}{d\tau} \gg \frac{dx^i}{d\tau}$, the $\mu = i$ components of the geodesic deviation equation, Eq. (2.36), become

$$\frac{d^2 \xi^i}{d\tau^2} = -\xi^\sigma \partial_\sigma \Gamma^i_{00} \left(\frac{dx^0}{d\tau} \right)^2. \quad (2.82)$$

The $\mu = 0$ component of the geodesic equation, Eq. (2.34), shows that $\frac{dt}{d\tau}$ is constant, so we can divide both sides by $\left(\frac{dx^0}{d\tau} \right)^2$ to convert the independent variable from the proper time τ to the coordinate time t . Because the correction to the metric is quadratic in spatial distance, non-negligible contributions of the derivatives of the metric come only through two spatial derivatives and we can safely ignore all time derivatives of the Christoffel symbol. In particular, we have $\partial_0 \Gamma^i_{00} = \partial_0 \Gamma^i_{0j} = 0$ and thus the Riemann tensor, Eq. (2.18), gives

$$R^i_{0j0} = \partial_j \Gamma^i_{00} - \partial_0 \Gamma^i_{0j} = \partial_j \Gamma^i_{00}. \quad (2.83)$$

All together, Eq. (2.82) becomes

$$\ddot{\xi}^i = -R^i_{0j0} \xi^j, \quad (2.84)$$

where the overhead dot represents derivative with respect to coordinate time. The Riemann tensor is gauge-invariant and it can be computed easily in the TT frame. From Eq. (2.42), we have

$$R^i_{0j0} = R_{i0j0} = -\frac{1}{2} \ddot{h}^{\text{TT}}_{ij}. \quad (2.85)$$

Therefore, the geodesic deviation equation in the LLF is simply

$$\ddot{\xi}^i = \frac{1}{2} \ddot{h}^{\text{TT}}_{ij} \xi^j. \quad (2.86)$$

Remarkably, it shows that the effect of GWs on a point particle with mass m can be described in terms of a Newtonian force

$$F_i = \frac{m}{2} \ddot{h}^{\text{TT}}_{ij} \xi^j. \quad (2.87)$$

For a better visualization of the effect of GWs in LLF, we restrict our attention on a ring of test masses in the (x, y) plane because GWs are transverse. The locations of the test masses are

Figure 2.1: The deformation of a ring of test masses by two different polarizations of GWs, h_+ (first row) and $h_\times$ (second row), at different instants of time in the LLF. The dashed line represents the unperturbed position while the solid line represents the perturbed position due to the GWs.

denoted by $(x_0 + \delta x(t), y_0 + \delta y(t))$ where (x_0, y_0) and $(\delta x(t), \delta y(t)) \sim \mathcal{O}(h)$ are initial positions and perturbations due to GWs respectively. Eq. (2.86) becomes

$$\begin{aligned} \begin{pmatrix} \ddot{\delta x} \\ \ddot{\delta y} \end{pmatrix} &= \frac{1}{2} (\ddot{h}_+ e^+ + \ddot{h}_\times e^\times) \begin{pmatrix} x_0 + \delta x \\ y_0 + \delta y \end{pmatrix} \\ &\approx \frac{1}{2} (\ddot{h}_+ e^+ + \ddot{h}_\times e^\times) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \\ \text{so } \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} &= \frac{1}{2} (h_+(t) e^+ + h_\times(t) e^\times) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \end{aligned} \quad (2.88)$$

For the plus polarization of a plane-wave $h_+(t) = A_+ \cos \omega t$, Eq. (2.88) becomes

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}_+ = \frac{1}{2} A_+ \begin{pmatrix} x_0 \\ -y_0 \end{pmatrix} \cos \omega t. \quad (2.89)$$

Similarly, for the cross polarization $h_\times(t) = A_\times \cos \omega t$, we have

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}_\times = \frac{1}{2} A_\times \begin{pmatrix} y_0 \\ x_0 \end{pmatrix} \cos \omega t. \quad (2.90)$$

The visualizations of these two polarizations are shown in Fig. 2.1. Obviously the “plus” and “cross” label the direction of the deformation. The plus (or cross) polarization is animated in the bottom-right (or bottom-left) corner of this book.

To show the quadrupolar nature of the polarizations, we define coordinates, denoted by a prime, in the same TT frame but rotated by an angle ψ around the z -axis, then from Eq. (2.55) the new polarizations are given by

$$\begin{pmatrix} h'_+ \\ h'_\times \end{pmatrix} = \begin{pmatrix} \cos 2\psi & -\sin 2\psi \\ \sin 2\psi & \cos 2\psi \end{pmatrix} \begin{pmatrix} h_+ \\ h_\times \end{pmatrix}, \quad (2.91)$$

which shows that a rotation of $\frac{\pi}{4}$ changes one polarization into the other, *i.e.* the helicity is 2.

2.3.7 Generation

General solution

In order to study the generation of GWs, we need to solve the inhomogeneous wave equation, Eq. (2.65). Similar to Eq. (2.64), because of the invertibility of d'Alembertian operator, the general solution of Eq. (2.65) is given by

$$\bar{h}_{\mu\nu}(x) = -16\pi \int d^4y G(x-y) T_{\mu\nu}(y), \quad (2.92)$$

where x (or y) is the field (or source) spacetime event. By imposing the boundary condition of “no incoming radiation” from past null infinity, the retarded Green's function is selected,

$$G(x-y) = \frac{-1}{4\pi|\mathbf{x}-\mathbf{y}|} \delta(x_{\text{ret}}^0 - y^0), \quad (2.93)$$

where $x_{\text{ret}}^0 = t - |\mathbf{x} - \mathbf{y}|$ is the retarded time. The general solution becomes

$$\bar{h}_{ij}(x) = 4 \int d^3y \frac{1}{|\mathbf{x}-\mathbf{y}|} T_{ij}(t - |\mathbf{x}-\mathbf{y}|, \mathbf{y}). \quad (2.94)$$

Here all timelike components are dropped because T^{00} and T^{0i} can be recovered from T^{ij} by the conservation of energy-momentum, Eq. (2.66). Equivalently $\bar{h}^{00}$ and $\bar{h}^{0i}$ can be recovered from $\bar{h}^{ij}$ by the harmonic gauge, Eq. (2.61). In fact, we will express T^{ij} in terms of T^{00} and T^{0i} shortly in the general solution.

Far-field limit

If d denotes the typical size of the source, at $|\mathbf{x}| \gg d$, we have

$$|\mathbf{x}-\mathbf{y}| = |\mathbf{x}| - y^i \partial_i |\mathbf{x}| + \mathcal{O}\left(\frac{d^2}{|\mathbf{x}|}\right), \quad (2.95)$$

where $\partial^i |\mathbf{x}| = \hat{\mathbf{x}}^i$. The far-field approximation consists of retaining the zeroth-order term in the amplitude and the first-order term in the energy-momentum tensor,

$$\bar{h}_{ij}(x) = \frac{4}{|\mathbf{x}|} \int d^3y T_{ij}(t - |\mathbf{x}| + \mathbf{y} \cdot \hat{\mathbf{x}}, \mathbf{y}) + \mathcal{O}\left(\frac{d}{|\mathbf{x}|^2}\right). \quad (2.96)$$

In practice, the GW source of interest is millions of light-years away from the Earth and so the far-field limit is valid.

Multipole expansion

If the typical velocity v of the source is much smaller than the speed of light, $v \ll 1$, the wavelength of GWs will be much greater than d and the energy-momentum tensor can be expanded by using Taylor series,

$$T_{ij}(t - |\mathbf{x}| + \mathbf{y} \cdot \hat{\mathbf{x}}, \mathbf{y}) = \sum_{k=0}^{\infty} \frac{(\mathbf{y} \cdot \hat{\mathbf{x}} \partial_0)^k}{k!} T_{ij}(t - |\mathbf{x}|, \mathbf{y}). \quad (2.97)$$

We define the moments of the energy-momentum tensor T^{ij} ,

$$\begin{aligned} S^{ij}(t) &\equiv \int d^3y T^{ij}(t, \mathbf{y}), \\ S^{ij,k}(t) &\equiv \int d^3y T^{ij}(t, \mathbf{y}) y^k, \\ S^{ij,kl}(t) &\equiv \int d^3y T^{ij}(t, \mathbf{y}) y^k y^l, \end{aligned} \quad (2.98)$$

and so on, where S is symmetric with respect to (i, j) because T^{ij} is symmetric. Therefore, Eq. (2.97) becomes

$$\bar{h}_{ij}(x) = \frac{4}{|\mathbf{x}|} \left[S_{ij} + \hat{\mathbf{x}}_k \dot{S}_{ij}{}^{,k} + \frac{\hat{\mathbf{x}}_k \hat{\mathbf{x}}_l}{2!} \ddot{S}_{ij}{}^{,kl} + \dots \right]_{y^0=t-|\mathbf{x}|}. \quad (2.99)$$

In fact, this series is a low-velocity expansion in orders of v because each of the additional y in the multipole moments brings a factor of $\mathcal{O}(d)$ and thus together with the time derivatives it brings factor of $\mathcal{O}(v)$. The physical meaning of the various terms in Eq. (2.99) can be understood if all S^{ij} are replaced by linear combinations of the moments of the energy density T^{00} and of the (linear) momentum density T^{0i} , which, similar to Eq. (2.98), are defined to be

$$\begin{aligned} M(t) &\equiv \int d^3y T^{00}(t, \mathbf{y}), & P^i(t) &\equiv \int d^3y T^{0i}(t, \mathbf{y}), \\ M^k(t) &\equiv \int d^3y T^{00}(t, \mathbf{y}) y^k, & P^{i,k}(t) &\equiv \int d^3y T^{0i}(t, \mathbf{y}) y^k, \\ M^{kl}(t) &\equiv \int d^3y T^{00}(t, \mathbf{y}) y^k y^l, & P^{i,kl}(t) &\equiv \int d^3y T^{0i}(t, \mathbf{y}) y^k y^l, \end{aligned} \quad (2.100)$$

and so on, respectively. With the help of the conservation of energy-momentum, Eq. (2.66), and Stokes' theorem, we have the conservation of mass⁶,

$$\dot{M} = \int_V d^3y \partial_0 T^{00} = - \int_V d^3y \partial_i T^{0i} = - \int_{\partial V} d\hat{\mathbf{n}}_i T^{0i} = 0, \quad (2.101)$$

where V is taken to be large enough to contain the source entirely such that $T^{\mu\nu}$ vanishes at the boundary ∂V and $\hat{\mathbf{n}}$ is the unit normal vector to ∂V . Similarly, we have a bunch of identities

$$\begin{aligned} \dot{M} &= 0, & \dot{P}^i &= 0, \\ \dot{M}^i &= P^i, & \dot{P}^{i,j} &= S^{ij}, \\ \dot{M}^{ij} &= P^{i,j} + P^{j,i}, & \dot{P}^{i,jk} &= S^{ij,k} + S^{ik,j}, \\ \dot{M}^{ijk} &= P^{i,jk} + P^{j,ki} + P^{k,ij}, \end{aligned} \quad (2.102)$$

where $\dot{P}^i = 0$ is the conservation of (linear) momentum and $\dot{P}^{i,j} - \dot{P}^{j,i} = S^{ij} - S^{ji} = 0$ is the conservation of angular momentum. All together, we have

$$\begin{aligned} S^{ij} &= \frac{1}{2} \ddot{M}^{ij}, \\ \dot{S}^{ij,k} &= \frac{1}{6} \ddot{M}^{ijk} + \frac{1}{3} \left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij} \right), \end{aligned} \quad (2.103)$$

and similarly to other higher-order terms. To the leading order, Eq. (2.99) becomes

$$[\bar{h}_{ij}(x)]_{\text{quad}} = \frac{4}{|\mathbf{x}|} S_{ij} = \frac{2}{|\mathbf{x}|} \ddot{M}_{ij}(t - |\mathbf{x}|), \quad (2.104)$$

which shows that the leading order of GW radiation is the mass quadrupole radiation.

⁶A physical system radiating GWs does lose mass but in linearized theory the backreaction to the source dynamics due to the energy carried away by GWs is neglected.

Projection to TT gauge

Outside the source, this solution can be put into the TT gauge. We define the Lambda tensor

$$\Lambda_{ij}{}^{kl}(\hat{\mathbf{n}}) \equiv \mathcal{P}_i{}^k \mathcal{P}_j{}^l - \frac{1}{2} \mathcal{P}_{ij} \mathcal{P}^{kl}, \quad (2.105)$$

with the (spatial) projection tensor $\mathcal{P}_{ij}(\hat{\mathbf{n}}) \equiv \delta_{ij} - n_i n_j$ which projects tensor components into a surface orthogonal to the unit vector $\hat{\mathbf{n}}$. $\mathcal{P}_{ij}$ has several convenient properties including

1. symmetry: $\mathcal{P}_{ij} = \mathcal{P}_{ji}$,
2. transitivity: $\mathcal{P}_i{}^k \mathcal{P}_{kj} = \mathcal{P}_{ij}$,
3. transversality: $n^i \mathcal{P}_{ij}(\hat{\mathbf{n}}) = 0$, and
4. its trace: $\mathcal{P}_i{}^i = 3 - 1 = 2$.

From $\mathcal{P}_{ij}$ the Lambda tensor inherits several interesting properties including

1. symmetry: $\Lambda_{ij}{}^{kl} = \Lambda^{kl}{}_{ij}$,
2. transitivity: $\Lambda_{ij}{}^{kl} \Lambda_{kl}{}^{mn} = \Lambda_{ij}{}^{mn}$,
3. transversality on all indices: $n^i \Lambda_{ij}{}^{kl} = n^j \Lambda_{ij}{}^{kl} = \dots = 0$, and
4. tracelessness: $\Lambda_i{}^i{}^{kl} = \Lambda_{ij}{}^k{}_k = 0$.

Therefore, the Lambda tensor acts like a TT projection operator and by construction extracts the transverse-traceless part of any symmetric tensor. Eq. (2.104) becomes

$$[h_{ij}^{\text{TT}}(x)]_{\text{quad}} = \Lambda_{ij}{}^{kl} \bar{h}_{kl}(x) = \frac{2}{|\mathbf{x}|} \ddot{M}_{ij}^{\text{TT}}(t - |\mathbf{x}|). \quad (2.106)$$

Without loss of generality, the z -direction is chosen to be the propagation direction of GWs and so $\mathcal{P}_{ij} = \text{diag}\{1, 1, 0\}$. We have

$$[h_{ij}^{\text{TT}}(x)]_{\text{quad}} = \frac{2}{|\mathbf{x}|} \begin{bmatrix} (\ddot{M}_{11} - \ddot{M}_{22})/2 & \ddot{M}_{12} & 0 \\ \ddot{M}_{21} & -(\ddot{M}_{11} - \ddot{M}_{22})/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{t=t-|\mathbf{x}|}. \quad (2.107)$$

By comparing to Eq. (2.74), we obtain the two GW polarizations

$$\begin{aligned} h_+(x) &= \frac{1}{|\mathbf{x}|} [\ddot{M}_{11} - \ddot{M}_{22}]_{t=t-|\mathbf{x}|}, \\ h_\times(x) &= \frac{2}{|\mathbf{x}|} [\ddot{M}_{12}]_{t=t-|\mathbf{x}|}. \end{aligned} \quad (2.108)$$

In the case of arbitrary propagation direction, one can always perform rotations to obtain the h_+ and $h_\times$ in the other frame. We suppose in the unprimed frame (x, y, z) the propagation direction is a generic direction $\hat{\mathbf{n}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, while in the primed frame (x', y', z') the propagation direction is along z' -axis, *i.e.* $\hat{\mathbf{n}}' = (0, 0, 1)$. Then $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ are related by a rotation matrix $\mathcal{R}_j^i$,

$$n'^i = \mathcal{R}_j^i n^j \quad (2.109)$$

Figure 2.2: A compact binary system of two point masses, m_1 and m_2 , separated by distance $\mathcal{R}$ circularly orbiting around their centre of mass with an inclination angle ι between the orbital angular frequency vector $\boldsymbol{\omega}_{\text{orb}} = \omega_{\text{orb}}(0, \sin \iota, \cos \iota)$.

with

$$\mathcal{R}_j^i = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.110)$$

An additional rotation of polarization angle ψ along z' -axis can also be performed to mix h_+ and $h_\times$ similar to Eq. (2.91) but it is skipped here. Then the mass quadrupole transforms as

$$M'^{ij} = \mathcal{R}_k^i \mathcal{R}_l^j M^{kl} = (\mathcal{R} M \mathcal{R}^T)_{ij}. \quad (2.111)$$

So we obtain

$$\begin{aligned} h_+(x; \theta, \phi) &= \frac{1}{|\mathbf{x}|} \left[\ddot{M}'_{11} - \ddot{M}'_{22} \right]_{t=t-|\mathbf{x}|} \\ &= \frac{1}{|\mathbf{x}|} \left[\ddot{M}_{11} (\cos^2 \theta \cos^2 \phi - \sin^2 \phi) + \ddot{M}_{22} (\cos^2 \theta \sin^2 \phi - \cos^2 \phi) + \ddot{M}_{33} \sin^2 \theta \right. \\ &\quad \left. + \ddot{M}_{12} (1 + \cos^2 \theta) \sin 2\phi - \ddot{M}_{13} \sin 2\theta \cos \phi - \ddot{M}_{23} \sin 2\theta \sin \phi \right]_{t=t-|\mathbf{x}|}, \\ h_\times(x; \theta, \phi) &= \frac{2}{|\mathbf{x}|} \left[\ddot{M}'_{12} \right]_{t=t-|\mathbf{x}|} \\ &= \frac{1}{|\mathbf{x}|} \left[\left(\ddot{M}_{22} - \ddot{M}_{11} \right) \cos \theta \sin 2\phi + 2\ddot{M}_{12} \cos \theta \cos 2\phi \right. \\ &\quad \left. + 2\ddot{M}_{13} \sin \theta \sin \phi - 2\ddot{M}_{23} \sin \theta \cos \phi \right]_{t=t-|\mathbf{x}|}. \end{aligned} \quad (2.112)$$

These equations allow us to calculate the angular distribution of the quadrupole radiation given the M_{ij} .

Mass quadrupole radiation from compact binary systems

We consider a binary system of two point masses, m_1 and m_2 , separated by distance $\mathcal{R}$ circularly orbiting around their centre of mass with an inclination angle ι between the orbital angular

frequency vector $\boldsymbol{\omega}_{\text{orb}} = \omega_{\text{orb}}(0, \sin \iota, \cos \iota)$ and the line of sight on z -axis as shown in Fig. 2.2. We define

$$\begin{aligned}\mathbf{x}_1(t) &= \frac{m_2}{M} \mathcal{R} \hat{\mathbf{e}}(t), \\ \mathbf{x}_2(t) &= -\frac{m_1}{M} \mathcal{R} \hat{\mathbf{e}}(t),\end{aligned}\tag{2.113}$$

where $\hat{\mathbf{e}}(t) = (\cos(\omega_{\text{orb}} t), \cos \iota \sin(\omega_{\text{orb}} t), -\sin \iota \sin(\omega_{\text{orb}} t))$ and $M = m_1 + m_2$ is the total mass. By construction, the centre of mass is placed at the origin in the source frame. The energy density is

$$T^{00} = m_1 \delta(\mathbf{x} - \mathbf{x}_1) + m_2 \delta(\mathbf{x} - \mathbf{x}_2).\tag{2.114}$$

Substituting Eqs. (2.113) and (2.114) to Eq. (2.100), we have the mass quadrupole

$$M^{ij}(t) = \eta M R^2 \begin{pmatrix} \cos^2(\omega_{\text{orb}} t) & \cos \iota \cos(\omega_{\text{orb}} t) \sin(\omega_{\text{orb}} t) \\ \cos \iota \cos(\omega_{\text{orb}} t) \sin(\omega_{\text{orb}} t) & \cos^2 \iota \sin^2(\omega_{\text{orb}} t) \end{pmatrix},\tag{2.115}$$

where $\eta \equiv \frac{m_1 m_2}{M^2}$ is the symmetric mass ratio which ranges from 0 to $\frac{1}{4}$. So the two gravitational polarizations are given by Eq. (2.108)

$$\begin{aligned}h_+(x) &= -\frac{4\eta M \mathcal{R}^2 \omega_{\text{orb}}^2}{|\mathbf{x}|} \frac{1 + \cos^2 \iota}{2} \cos(2\omega_{\text{orb}}(t - |\mathbf{x}|)), \\ h_\times(x) &= -\frac{4\eta M \mathcal{R}^2 \omega_{\text{orb}}^2}{|\mathbf{x}|} \cos \iota \sin(2\omega_{\text{orb}}(t - |\mathbf{x}|)).\end{aligned}\tag{2.116}$$

If the two masses are sufficiently far apart, $\mathcal{R}$ can be expressed in terms of M and ω_{orb} using the Kepler's third law

$$\omega_{\text{orb}}^2 = \frac{M}{\mathcal{R}^3},\tag{2.117}$$

and Eq. (2.116) is further simplified to

$$\begin{aligned}h_+(x) &= -\frac{4M_c^{5/3} \omega_{\text{orb}}^{2/3}}{|\mathbf{x}|} \frac{1 + \cos^2 \iota}{2} \cos(2\omega_{\text{orb}}(t - |\mathbf{x}|)), \\ h_\times(x) &= -\frac{4M_c^{5/3} \omega_{\text{orb}}^{2/3}}{|\mathbf{x}|} \cos \iota \sin(2\omega_{\text{orb}}(t - |\mathbf{x}|)).\end{aligned}\tag{2.118}$$

where $M_c \equiv M \eta^{3/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$ is the chirp mass. These are solutions of monochromatic GWs propagating along the z -direction with h_+ leading $h_\times$ by $\frac{\pi}{2}$ and amplitudes which depend only on the chirp mass and the orbital angular frequency, and fall off inversely proportional to the distance to the observer. In the dominant mass quadrupole radiation mode, the GW frequency is twice as much as the orbital frequency,

$$\omega_{\text{GW}} = 2\omega_{\text{orb}}.\tag{2.119}$$

When the source is face-on, *i.e.* $\iota = 0$ and π , the two polarizations have equal amplitude. When the source is face-off (or edge-on), *i.e.* $\iota = \frac{\pi}{2}$, $h_\times$ vanishes and the GWs become linearly polarized. In higher-order corrections, more source properties will be taken into account, *e.g.* mass ratio, individual spins, spin-spin interaction and eccentricity. In this derivation, we have implicitly assumed that the source dynamics is unaffected by the emitted GWs, *i.e.* the orbit is perfectly circular. This assumption will be relaxed in the following section.

Figure 2.3: The angular dependence of a GW radiation, *i.e.* Eq. (2.122).

2.4 Compact binary coalescence

2.4.1 General features of an inspiral

In reality, as GWs carry energy, linear momentum and angular momentum away, the back reaction of GWs to the source dynamics of the binary should also be considered. The power of quadrupole radiation emitted per unit solid angle Ω is given by

$$\frac{d\mathcal{F}}{d\Omega} = \frac{|\mathbf{x}|^2}{16\pi} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle_t, \quad (2.120)$$

where $\langle \rangle_t$ means the time average [40]. Inserting Eq. (2.118), we have

$$\frac{d\mathcal{F}}{d\Omega} = \frac{2}{\pi} (\pi M_c f_{\text{GW}})^{10/3} g(\iota), \quad (2.121)$$

where $f_{\text{GW}} = \frac{\omega_{\text{GW}}}{2\pi}$ and the angular dependence function

$$g(\iota) = \left(\frac{1 + \cos^2 \iota}{2} \right)^2 + \cos^2 \iota, \quad (2.122)$$

which is shown in Fig. 2.3. The radiation attains its maximum at $\iota = 0$ and π , *i.e.* in the direction normal to the plane of the orbit. $g(\iota)$ is non-vanishing, meaning that the quadrupole radiation goes to all directions. No matter where the observer is, there is always a GW component which the observer can detect. By integrating over the solid angle, the total radiated power (or the GW luminosity) is

$$\mathcal{F} = \frac{32}{5} (\pi M_c f_{\text{GW}})^{10/3}. \quad (2.123)$$

The total energy of the source is the sum of the kinetic and potential energies of the orbit,

$$E = E_{\text{kin}} + E_{\text{pot}} = -\frac{m_1 m_2}{2\mathcal{R}}. \quad (2.124)$$

In order to compensate the energy loss in emitting GWs, the separation between two masses has to shrink for a more negative energy. According to Kepler's third law, Eq. (2.117), smaller separation leads to a higher frequency and thus the power of GWs which in turn leads to a further decrease in the separation. The frequency and amplitude increase as a function of time, the so-called chirp, and it becomes a runaway process that ends with the two masses merging to form a single object. In fact, from Eq. (2.117) again, we have

$$\dot{\mathcal{R}} = -\frac{2}{3}(\mathcal{R}\omega_{\text{orb}})\frac{\dot{\omega}_{\text{orb}}}{\omega_{\text{orb}}^2}. \quad (2.125)$$

As long as $\dot{\omega}_{\text{orb}} \ll \omega_{\text{orb}}^2$, the binary is in a so-called quasi-circular (or adiabatic) regime. We are safe to approximate that the binary is in a circular motion with a slowly varying radius. However, in the Schwarzschild geometry, there exists a radius of innermost stable circular orbit (ISCO) $\mathcal{R}_{\text{ISCO}} = 6M \approx 8.86 \text{ km} \left(\frac{M}{M_{\odot}}\right)$, which by Eq. (2.117) corresponds to a GW frequency

$$f_{\text{GW,ISCO}} = \frac{1}{6^{3/2}\pi M} \approx 2.2 \text{ kHz} \left(\frac{M_{\odot}}{M}\right), \quad (2.126)$$

beyond which no stable circular orbits are allowed and Eq. (2.117) is no longer valid. Moreover, if the two masses are not BHs but some stars such as NSs, we need to start worrying about the validity of the point-mass approximation when the masses are close. In other words, our derivation is valid at most up to the ISCO roughly.

By equating the GW luminosity and the rate of change of the energy loss of the binary system,

$$\mathcal{F} = -\frac{dE}{dt}, \quad (2.127)$$

we have

$$\dot{f}_{\text{GW}} = \frac{96}{5}\pi^{8/3}M_c^{5/3}f_{\text{GW}}^{11/3}, \quad (2.128)$$

which by integration gives the GW frequency evolution

$$\begin{aligned} f_{\text{GW}}(\tau) &= \frac{1}{\pi} \left(\frac{5}{256} \frac{1}{\tau} \right)^{3/8} M_c^{-5/8} \\ &\approx 134 \text{ Hz} \left(\frac{1.21 M_{\odot}}{M_c} \right)^{5/8} \left(\frac{1 \text{ s}}{\tau} \right)^{3/8}, \end{aligned} \quad (2.129)$$

where $\tau = t_{\text{coal}} - t$ is the time to coalescence at which f_{GW} formally diverges to infinity and $1.21 M_{\odot}$ is the chirp mass of a binary with $m_1 = m_2 = 1.4 M_{\odot}$ which is a typical mass of a NS. This equation clearly shows that the GW frequency is monotonically increasing with the time before the merger, *i.e.* so-called chirp signal. We can rewrite this equation to find the duration of a typical compact binary coalescence (CBC) signal between some frequency band $[f_{\text{min}}, f_{\text{max}}]$,

$$\Delta\tau \approx 2.19 \text{ s} \left(\frac{1.21 M_{\odot}}{M_c} \right)^{5/3} \left[\left(\frac{100 \text{ Hz}}{f_{\text{min}}} \right)^{8/3} - \left(\frac{100 \text{ Hz}}{f_{\text{max}}} \right)^{8/3} \right]. \quad (2.130)$$

It shows that in general a heavier system has a shorter signal in band. For $f_{\text{min}} = 10 \text{ Hz}$, which is a typical lowest frequency accessible to ground-based interferometers, and $f_{\text{max}} = f_{\text{GW,ISCO}} = 1566 \text{ Hz}$, $\Delta\tau$ is around 160 s. At 1 kHz the binary is only a few milliseconds to coalescence.

From Eq. (2.117), it corresponds to a separation of 33 km between two masses. Such a small separation without the objects touching each other, and such high masses can only be achieved by compact objects such as BHs and NSs but not larger stars such as white dwarfs⁷. A typical NS with $1.4 M_\odot$ mass has a radius of 10 km. Although the point-mass approximation breaks down at this stage, Eq. (2.130) is still a good approximation of the duration of a CBC signal.

Now we can combine everything to form a reasonable inspiral waveform. The GW phase can be obtained by integrating the GW angular frequency

$$\begin{aligned}\Phi_{\text{GW}}(t) &= \int_{t_0}^t dt' \omega_{\text{GW}}(t') \\ &= -2 \left(\frac{\tau}{5M_c} \right)^{5/8} + \Phi_0,\end{aligned}\tag{2.131}$$

where Φ_0 is an integration constant. Updating Eq. (2.118) by this equation and Eq. (2.129), we get

$$\begin{aligned}h_+(t) &= \frac{1}{|\mathbf{x}|} M_c^{5/4} \left(\frac{5}{\tau} \right)^{1/4} \frac{1 + \cos^2 \iota}{2} \cos \Phi_{\text{GW}}(\tau), \\ h_\times(t) &= \frac{1}{|\mathbf{x}|} M_c^{5/4} \left(\frac{5}{\tau} \right)^{1/4} \cos \iota \sin \Phi_{\text{GW}}(\tau),\end{aligned}\tag{2.132}$$

which are shown in Fig. 2.4. It in turn changes the power emitted by GWs and thus the binary orbit, but qualitatively the inspiral waveform looks the same.

2.4.2 Waveform approximants

A complete waveform consists of three parts: inspiral, merger and postmerger (IMP). As a rule of thumb, the inspiral phase is roughly determined by ISCO in which the point-mass approximation is valid and the leading-order term in the low-velocity expansion describes GWs accurately. Due to the energy carried away by GWs, the separation between two masses decreases to conserve the energy. Higher-order corrections in $\mathcal{O}(v)$ need to be taken into account. After ISCO, the binary enters the merger regime in which the point-mass approximation is no longer valid, the orbit is not quasi-circular anymore. If it involves NS, tidal effects need to be also taken into account. The two masses will then coalesce shortly and enter the postmerger regime. Depending on the type of progenitors, binary ends up to with different possible remnants. If it is a BBH, the postmerger phase is also called ringdown in which the two BHs form a single highly-excited Kerr BH which rapidly returns to a stationary Kerr BH by emitting GWs in the form of superposition of damped sinusoids. This process can be modelled by perturbation theory on a Kerr metric leading to quasi-normal mode (QNM) frequencies ω_{lmn} and lifetime τ_{lmn} (see also Chapter 4). If it is a binary of a neutron star and a black hole (NSBH), the final object will be a BH with or without undergoing ringdown depending on the mass ratio of the binary. If it is a BNS, the situation is more complicated. Depending on total mass and the equation of state (EOS), the remnant can be either a prompt-collapse BH or a massive neutron star. The former undergoes ringdown while the latter can be subclassified into a supermassive neutron star and a hypermassive neutron star, will emit GWs with a characteristic frequency f_2 (see also Chapter 6). Fig. 2.5 shows examples of an inspiral-merger-ringdown (IMR) waveform for BBH and an IMP waveform for BNS which results into a massive neutron star.

In GW data analysis, there are essentially 4 approaches to create a waveform approximant:

⁷White dwarf binaries will form a background for space-based interferometers [41] which have sensitivity at much lower frequencies than ground-based detectors.

Figure 2.4: The GW strains of both polarizations h_+ and $h_\times$, *i.e.* Eq. (2.132), in the inspiral stage of a face-on binary with $M_c = 30.4 M_\odot$ (*i.e.* a GW150914-like system) in a distance of 1 Mpc. The dashed line shows the time at where the system passes its ISCO before which Eq. (2.132) are valid. The plot clearly shows that h_+ leads $h_\times$ by $\pi/2$.

- Numerical relativity (NR)
- Post-Newtonian (PN) waveform
- Effective-one-body (EOB) waveform
- Phenomenological waveform

We are going to give a qualitative overview of each approach.

Numerical relativity

Up to now, no analytical solutions to a compact-objects binary are available and NR is to solve the full set of EFEs (coupled to the matter fields in the case of BNS) by numerical integration on supercomputers. The dynamical simulation is done by discretizing the spacetime geometry into grid, calculating derivatives using finite difference methods and evolving the dynamics and spacetime based on 3+1-form EFEs given initial conditions in time. In a NR simulation, many physics need to be taken into account, *e.g.* magnetohydrodynamics, finite-temperature EOS, thermal effect, neutrino emission, *etc.* The first successful NR simulation for an equal-mass non-spinning BBH was done by Pretorius in 2005 [43]. Nowadays, there are a lot of successful NR codes including BAM [44, 45], THC [46], LazEv [47], the Spectral Einstein Code (SpEC) [48],

Figure 2.5: A typical IMR waveform for BBH (first row) and IMP waveform for BNS (second row). The BNS signal is obtained from BAM:0035 on Computational Relativity catalog [42] with the equation-of-state of H4 and $m_1 = m_2 = 1.35 M_\odot$.

whisky [49] and sacra [50]. Some of NR waveforms are released and publicly available for BBH, NSBH and BNS, *e.g.* the Simulating eXtreme Spacetimes (SXS) catalog [51, 52], Georgia Tech Catalog [53], Rochester Institute of Technology (RIT) catalog [54, 55], Numerical INjection Analysis (NINJA) catalog [56, 57], NCSA Gravity Group [58], Computational Relativity (CoRe) catalog [42] and SACRA Gravitational Waveform Data Bank [59].

NR simulation gives accurate GW waveform without being truncated in any finite orders. All state-of-the-art waveform approximants are compared and calibrated to the NR waveform. Moreover, by simulating two binary systems, one with matter effects and one without, physicists can extract the tidal corrections based on NR waveforms which can be added to turn any existing BBH waveform to an accurate BNS waveform [60, 61]. In addition, in a NR simulation, one can freely choose the theory of gravity and systems of interest and obtain an accurate GW waveforms. Not only BBH and BNS simulations, recently physicists also start simulating system involving other kinds of compact stars such as axion star (AS) which is a potential dark matter candidate, *e.g.* a boson stars binary [62], a binary of a boson star and a NS [63], head-on collision between a AS and a NS or BH [64, 65].

Not only accurate waveforms, NR simulation allows to study the electromagnetic (EM) counterpart after the BNS merger. During the merger, a neutron-rich accretion disk around the remnant is created which is a paradise for production of elements heavier than iron such as gold and platinum through r-process. These radioactive elements undergo beta decay afterwards and emit a burst of EM radiation, which is known as kilonova (or macronova or even

mergernova). It is an astronomical source of EM radiation about 1 to 1000 times brighter than a regular supernova and last for hours to days in ultra-violet, optical, and infrared bands. In the first detection of BNS merger, GW170817, the associated kilonova has also been observed: GRB170817A, a gamma ray burst detected by Fermi-GBM 1.7 seconds after the merger, and AT2017gfo, a followup optical/infrared observations. NR allows us to simulate how these EM radiations are generated and faded out with time leading to a deeper understanding of NS.

However, NR simulation is computationally costly even on supercomputers. For the last few cycles of inspiral and postmerger, a typical waveform takes $\mathcal{O}(\text{months})$ to generate. In terms of parameter estimation in which millions of waveform need to be computed, this is not a good choice to use but it serves as a benchmark for making surrogates which can generate waveform fast, *e.g.* NRSur7dq2 [66].

Post-Newtonian waveform

Post-Newtonian (PN) formalism is a technique to calculate the inspiral phasing of a binary in a series expansion of the characteristic velocity of the binary $v \equiv \mathcal{R}\omega_{\text{orb}}$ while the amplitude is assumed to be Newtonian, *i.e.* $A = \frac{4M_c^{5/3}\omega_{\text{orb}}^{2/3}}{|\mathbf{x}|}$ in Eq. (2.118). In this formalism, the orbit is assumed to be quasi-circular. So we expect that PN waveforms are valid for small v . By rewriting the Kepler's third law, Eq. (2.117), and differentiating the energy balance equation, Eq. (2.127), with respect to v , we can obtain the orbital phase by a pair of differential equations,

$$\frac{d\Phi}{dt} = \frac{v^3}{M}, \quad (2.133)$$

$$\frac{dv}{dt} = -\frac{F(v)}{E'(v)}. \quad (2.134)$$

From Eq. (2.133), we can show that $v = (\pi M f_{\text{GW}})^{1/3}$. Equivalently they can be expressed in a pair of parametric equations,

$$\Phi(v) = \Phi(v_0) + \int_{v_0}^v dv v^3 \frac{E'(v)}{M\mathcal{F}(v)}, \quad (2.135)$$

$$t(v) = t(v_0) + \int_{v_0}^v dv \frac{E'(v)}{\mathcal{F}(v)}, \quad (2.136)$$

where v_0 is an arbitrary reference velocity. The total energy of the binary E is known up to 3PN order ⁸

$$E_3(v) = -\frac{1}{2}\mu v^2 \left\{ 1 + \left(-\frac{3}{4} - \frac{1}{12}\eta \right) v^2 + \left(-\frac{27}{8} + \frac{19}{8}\eta - \frac{1}{24}\eta^2 \right) v^4 \right. \\ \left. + \left[-\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2 \right) \eta - \frac{155}{96}\eta^2 - \frac{35}{5184}\eta^3 \right] v^6 + \mathcal{O}(v^8) \right\}, \quad (2.137)$$

⁸n PN order corresponds to $\mathcal{O}(v^{2n})$.

with $\mu = \frac{m_1 m_2}{M}$ to be the reduced mass while the GW luminosity is known up to 3.5PN order

$$\begin{aligned} \mathcal{F}_{3.5}(v) = \frac{32}{5} \eta^2 v^{10} & \left\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12} \right) v^2 + 4\pi v^3 \right. \\ & + \left(-\frac{44711}{9072} + \frac{9271}{504} \eta + \frac{65}{18} \eta^2 \right) v^4 + \left(-\frac{8191}{672} - \frac{583}{24} \eta \right) \pi v^5 \\ & + \left[\frac{6643739519}{69854400} + \frac{16}{3} \pi^2 - \frac{1712}{105} \gamma_E + \left(\frac{41}{48} \pi^2 - \frac{134543}{7776} \right) \eta - \frac{94403}{3024} \eta^2 \right. \\ & \quad \left. - \frac{775}{324} \eta^3 - \frac{856}{105} \ln(16v^2) \right] v^6 \\ & \left. + \left(-\frac{16285}{504} + \frac{214745}{1728} \eta + \frac{193385}{3024} \eta^2 \right) \pi v^7 + \mathcal{O}(v^8) \right\}, \end{aligned} \quad (2.138)$$

where $\gamma_E = 0.577216 \dots$ is the Euler-Mascheroni constant⁹. Different ways of dealing with the fraction $\frac{\mathcal{F}(v)}{E'(v)}$ leads to inequivalent PN phase evolutions, namely the *Taylor* family. Details of all approximants in the *Taylor* family can be found in [67]. We are now giving some examples of non-spinning approximants. The *TaylorT1* approximant is computed by numerically integrating Eqs. (2.133) and (2.134) where both $\mathcal{F}$ and E are left as what they are, *i.e.* Eqs. (2.137) and (2.138). The *TaylorT4* approximant is similar to the *TaylorT1* approximant but the fraction $\frac{\mathcal{F}(v)}{E'(v)}$ is first expanded as a Taylor series, and truncated at a consistent PN order. The *TaylorT2* approximant is computed by first expanding the fraction $\frac{E'(v)}{\mathcal{F}(v)}$ at a consistent PN order, then analytically integrating Eqs. (2.135) and (2.136) to obtain $\Phi(v)$ and $t(v)$, and finally numerically solving for $\Phi(t)$. Among all models in the *Taylor* family, the *TaylorT2* approximant is computationally the most expensive because the phase evolution $\Phi(t)$ involves solving a pair of transcendental equations

$$\begin{aligned} \Phi_{3.5}^{(T2)}(v) &= \Phi_{3.5}^{(T2)}(v_0) - \frac{1}{32\eta v^5} \left\{ 1 + \left(\frac{3715}{1008} + \frac{55}{12} \eta \right) v^2 - 10\pi v^3 \right. \\ &+ \left(\frac{15293365}{1016064} + \frac{27145}{1008} \eta + \frac{3085}{144} \eta^2 \right) v^4 + \left(\frac{38645}{672} - \frac{65}{8} \eta \right) \ln\left(\frac{v}{v_{\text{ISCO}}}\right) \pi v^5 \\ &+ \left[\frac{12348611926451}{18776862720} - \frac{160}{3} \pi^2 - \frac{1712}{21} \gamma_E + \left(\frac{2255}{48} \pi^2 - \frac{15737765635}{12192768} \right) \eta \right. \\ &\quad \left. + \frac{76055}{6912} \eta^2 - \frac{127825}{5184} \eta^3 - \frac{856}{21} \ln(16v^2) \right] v^6 \\ &\left. + \left(\frac{77096675}{2032128} + \frac{378515}{12096} \eta - \frac{74045}{6048} \eta^2 \right) \pi v^7 \right\}, \\ t_{3.5}^{(T2)}(v) &= t_{3.5}^{(T2)}(v_0) - \frac{5M}{256\eta v^8} \left\{ 1 + \left(\frac{743}{252} + \frac{11}{3} \eta \right) v^2 - \frac{32}{5} \pi v^3 \right. \\ &+ \left(\frac{3058673}{508032} + \frac{5429}{504} \eta + \frac{617}{72} \eta^2 \right) v^4 - \left(\frac{7729}{252} - \frac{13}{3} \eta \right) \pi v^5 \\ &+ \left[-\frac{10052469856691}{23471078400} + \frac{128}{3} \pi^2 + \frac{6848}{105} \gamma_E + \left(\frac{3147553127}{3048192} - \frac{451}{12} \pi^2 \right) \eta \right. \\ &\quad \left. - \frac{15211}{1728} \eta^2 + \frac{25565}{1296} \eta^3 + \frac{3424}{105} \ln(16v^2) \right] v^6 \\ &\left. + \left(-\frac{15419335}{127008} - \frac{75703}{756} \eta + \frac{14809}{378} \eta^2 \right) \pi v^7 \right\}, \end{aligned} \quad (2.139)$$

⁹The Euler-Mascheroni constant is defined to be $\gamma_E = \lim_{n \rightarrow \infty} (-\ln n + \sum_{k=1}^n 1/k)$.

where the velocity at ISCO v_{ISCO} is computed to be $1/\sqrt{6}$.

In the case of parameter estimation, we prefer a frequency domain approximant because the likelihood calculation is performed in the frequency domain, see Eq. (3.34) in the coming Sec. 3.3. The *TaylorF2* approximant is a waveform approximant in frequency domain obtained by analytically Fourier transforming the *TaylorT2* approximant using the stationary phase approximation (SPA). We assume that the observed GW signal can be written in a generic time domain form,

$$h(t) = A(t) \cos 2\Phi(t). \quad (2.140)$$

which is justified by Eq. (3.12) in the coming Sec. 3.1. The Fourier transform of $h(t)$ is then given by

$$\begin{aligned} \tilde{h}(f) &= \int_{-\infty}^{\infty} dt A(t) \cos 2\Phi(t) e^{2\pi i f t}, \\ &= \frac{1}{2} \int_{-\infty}^{\infty} dt A(t) [e^{i(2\pi f t + 2\Phi(t))} + e^{i(2\pi f t - 2\Phi(t))}]. \end{aligned} \quad (2.141)$$

The idea of the SPA is to evaluate the above integral only around the point where we expect to have the largest contribution. It means that the first term can be neglected because it oscillates rapidly and averages to zero while the second term also approximately averages to zero except around the saddle point t_* where

$$\begin{aligned} \left. \frac{d}{dt} [2\pi f t - 2\Phi(t)] \right|_{t=t_*} &= 0, \\ \dot{\Phi}(t_*) &= \pi f. \end{aligned} \quad (2.142)$$

Because $\dot{\Phi}(t) = 2\pi f_{\text{orb}} = \pi f_{\text{GW}}$, the saddle point happens when the Fourier variable f becomes equal to the GW frequency f_{GW} . Therefore, by expanding the phase at the second term up to $(t - t_*)^2$ and assuming the amplitude varies slowly around t_* , we have

$$\tilde{h}^{\text{SPA}}(f) = \frac{1}{2} A(t_*) e^{i[2\pi f t_* - 2\Phi(t_*)]} \int_{-\infty}^{\infty} dt e^{-i\ddot{\Phi}(t_*)(t-t_*)^2}. \quad (2.143)$$

With the fact that

$$\int_{-\infty}^{\infty} dx e^{-ix^2} = \sqrt{\pi} e^{-i\frac{\pi}{4}}, \quad (2.144)$$

Eq. (2.143) becomes

$$\tilde{h}^{\text{SPA}}(f) = \frac{1}{2} A(t_*) \sqrt{\frac{\pi}{\ddot{\Phi}(t_*)}} e^{i[2\pi f t_* - \frac{\pi}{4} - 2\Phi(t_*)]}. \quad (2.145)$$

Hence, the *TaylorF2* approximant is computed by substituting the parametric equations, $\Phi(v)$ and $t(v)$ in Eq. (2.139), obtained in the *TaylorT2* approximant. To change the variable from t to v , we make use of the Kepler's third law, Eq. (2.133), again to find out where the saddle point is and Eq. (2.142) becomes

$$v_* = (\pi M f)^{1/3}. \quad (2.146)$$

The *TaylorF2* approximant is obtained by substituting Eq. (2.139) into Eq. (2.145) and takes the form

$$\tilde{h}^{(\text{F2})}(f) = A^{(\text{F2})}(f) e^{-i\Phi^{(\text{F2})}(f)}. \quad (2.147)$$

The (mass quadrupole) amplitude reads

$$A^{(\text{F2})}(f) = \frac{\sqrt{F_+^2(1 + \cos^2 \iota)^2 + F_\times^2 4 \cos^2 \iota}}{2|\mathbf{x}|} \sqrt{\frac{5\pi}{96}} M_c^{5/6} (\pi f)^{-7/6}, \quad (2.148)$$

with the beam pattern functions F_+ , $F_\times$ which characterize the detector response to the GW polarizations and will be discussed in the coming Sec. 3.1. The phase is known up to 3.5 PN and reads

$$\Phi_{3.5}^{(\text{F2})}(f) = 2\pi f t_c - \Phi_c + \frac{3}{128\eta v^5} \sum_{n=0}^7 [\varphi_n + \varphi_n^l \ln(v)] v^n, \quad (2.149)$$

with v given by Eq. (2.146), an arbitrary time t_c , an arbitrary phase Φ_c and PN coefficients

$$\begin{aligned} \varphi_0 &= 1, \\ \varphi_1 &= 0, \\ \varphi_2 &= \frac{20}{9} \left(\frac{743}{336} + \frac{11}{4} \eta \right), \\ \varphi_3 &= -16\pi, \\ \varphi_4 &= 10 \left(\frac{3058673}{1016064} + \frac{5429}{1008} \eta + \frac{617}{144} \eta^2 \right), \\ \varphi_5 &= \pi \left(\frac{38645}{756} - \frac{65}{9} \eta \right) [1 - 3 \ln(v_{\text{ISCO}})], \\ \varphi_5^l &= \pi \left(\frac{38645}{252} - \frac{65}{3} \eta \right), \\ \varphi_6 &= \frac{11583231236531}{4694215680} - \frac{640}{3} \pi^2 - \frac{6848}{21} (\gamma_E - \ln 4) \\ &\quad + \left(-\frac{15737765635}{3048192} + \frac{2255\pi^2}{12} \right) \eta + \frac{76055}{1728} \eta^2 - \frac{127825}{1296} \eta^3, \\ \varphi_6^l &= -\frac{6848}{21}, \\ \varphi_7 &= \pi \left(\frac{77096675}{254016} + \frac{378515}{1512} \eta - \frac{74045}{756} \eta^2 \right). \end{aligned} \quad (2.150)$$

The PN coefficients can be extended to include spins effects. For BNS system, tidal effects can also be added to the coefficients. The *TaylorF2* approximant is good for parameter estimation in a sense that it is quick to evaluate because it has an analytical form in the frequency domain and does not require to numerically solve any equations. However, it is an inspiral-only waveform and starts to lose its accuracy when the binary is approaching ISCO. Because the GW luminosity is the brightest at the end of the inspiral and merger stage, it is important to develop usable approximant, which is accurate for the whole coalescence, to maximize the detection potential. Especially for BBH binaries, this becomes even more important as only a few cycles before the merger fall in the sensitivity band of the LIGO-Virgo detector network. *TaylorF2* will be one of the components to construct a more generic IMR waveform which will be discussed in the coming Sec. 2.4.2.

Effective-one-body waveform

In Newtonian mechanics, the complete two-body dynamics is solvable by re-formulating it as one-body problem: the centre of mass motion and the relative displacement vector motion.

The former motion is trivial while the latter motion is a motion of a particle with a reduced mass μ in an effective potential. Analogously, the effective-one-body (EOB) formalism maps a two-body problem in GR into a particle with a reduced mass μ moving under an effective EOB metric,

$$ds_{\text{eff}}^2 = -A(r)dt^2 + \frac{D(r)}{A(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \quad (2.151)$$

with polar coordinates (r, ϕ) and the conjugate momenta (p_r, p_ϕ) . It corresponds to the EOB Hamiltonian

$$H^{\text{real}}(r, p_r, p_\phi) = M\sqrt{1 + 2\eta\left(\frac{H^{\text{eff}}}{\mu} - 1\right)}, \quad (2.152)$$

with the effective Hamiltonian

$$H^{\text{eff}}(r, p_r, p_\phi) = \mu\sqrt{A(r)\left[1 + \frac{A(r)}{D(r)}p_r^2 + \frac{p_\phi^2}{r^2} + 2(4 - 3\eta)\eta\frac{p_r^4}{r^2}\right]}. \quad (2.153)$$

The Taylor-approximants to the coefficients $A(r)$ and $D(r)$ are written as

$$\begin{aligned} A_k(r) &= \sum_{i=0}^{k+1} \frac{a_i(\eta)}{r^i}, \\ D_k(r) &= \sum_{i=0}^k \frac{d_i(\eta)}{r^i}. \end{aligned} \quad (2.154)$$

These coefficients are currently known up to 3PN order (*i.e.* $k=4$) and can be found on [68]. The EOB waveforms can achieve a better agreement with NR waveforms by including a so-called pseudo-4PN coefficient $a_5(\eta) = 60\eta$ [68]. These coefficients can be extended to have spins and tidal effects for BNS system. The EOB Hamilton equations read

$$\begin{aligned} \frac{dr}{dt} &= \frac{\partial H^{\text{real}}}{\partial p_r}, \\ \frac{d\phi}{dt} &= \frac{\partial H^{\text{real}}}{\partial p_\phi}, \\ \frac{dp_r}{dt} &= -\frac{\partial H^{\text{real}}}{\partial r}, \\ \frac{dp_\phi}{dt} &= \mathcal{F}_\phi(r, p_r, p_\phi), \end{aligned} \quad (2.155)$$

where the radiation reaction force $\mathcal{F}_\phi$ can be related to Eq. (2.138).

The time domain EOB waveforms can be computed by solving the system differential equations in Eq. (2.155) with proper initial conditions. It is still slow to generate on a computer, *i.e.* $\mathcal{O}(\text{mins})$. In order to achieve a faster speed of generation, frequency domain reduced order models are needed to be built for the purpose of parameter estimation [69–71]. Based on the above framework, EOB waveforms are again inspiral-only but accurate up to merger. For BBH system, a full EOB IMR waveform can be constructed by adding the ringdown waveform and ensuring a smooth continuation at the matching point. Over the past years, many different EOB waveforms are developed and belong to the *SEOB*NR family, where *S* denotes spins.

Phenomenological waveform

The phenomenological waveform is an analytical IMR approximant in the frequency domain, namely the *IMRPhenom* family, which is constructed by adding phenomenological representations of the late inspiral and merger-ringdown regimes to the *TaylorF2* approximant described in the above Sec. 2.4.2. The phenomenological coefficients are obtained by fitting against many NR waveforms. The resulting waveforms have not only a similar quality to EOB, but also fast to generate on a computer thanks to its analytical form. In fact, this is waveform being used for the parametrized test, one of the data analysis pipelines for testing GR [11]. In particular, we will give a brief overview on *IMRPhenomD* waveform which describe an aligned-spin (non-precessing) BBH systems and details can be found on [72, 73]. Since we are dealing with an aligned-spin binary, the only concerned spin parameters are the dimensionless spin parameters defined as

$$\chi_i = \frac{\mathbf{S}_i \cdot \hat{\mathbf{L}}}{m_i^2}, \quad (2.156)$$

with the individual BH spin angular momenta $\mathbf{S}_i$, the orbital angular momentum $\hat{\mathbf{L}}$ and $\chi_i \in [-1, 1]$. The spin parameter being used in the model is

$$\chi_{\text{PN}} = \frac{m_1\chi_1 + m_2\chi_2}{M} - \frac{38\eta}{113}(\chi_1 + \chi_2). \quad (2.157)$$

In *IMRPhenomD*, the modelling of the complete BBH coalescence is divided into three regimes: inspiral, intermediate and ringdown. We first discuss about the phase and then the amplitude. The inspiral phase is modelled in the frequency range $Mf \in [0.0035, 0.018]$ and reads

$$\Phi_{\text{Ins}} = \Phi_{\text{TaylorF2}} + \frac{1}{\eta} \left(\sigma_0 + \sigma_1 f + \frac{3}{4} \sigma_2 f^{4/3} + \frac{3}{5} \sigma_3 f^{5/3} + \frac{1}{2} \sigma_4 f^2 \right), \quad (2.158)$$

where η is the symmetric mass ratio, the *TaylorF2* phase Φ_{TaylorF2} is similar to Eq. (2.150) but spins contributions are included, and σ_i are extra fitting parameters in $Mf \in [0.0035, 0.019]$ to correct the phase evolution of *TaylorF2* after ISCO. The intermediate phase is modelled in $Mf \in [0.018, 0.5Mf_{\text{RD}}]$ with the dominant ringdown QNM frequency f_{RD} and reads

$$\Phi_{\text{Int}} = \frac{1}{\eta} \left(\beta_0 + \beta_1 f + \beta_2 \ln f - \frac{\beta_3}{3} f^{-3} \right), \quad (2.159)$$

where β_i are fitting parameters in $Mf \in [0.017, 0.75Mf_{\text{RD}}]$. The ringdown phase is modelled in $f \in [0.045, 1.15]f_{\text{RD}}$ and reads

$$\Phi_{\text{MR}} = \frac{1}{\eta} \left[\alpha_0 + \alpha_1 f - \alpha_2 f^{-1} + \frac{4}{3} \alpha_3 f^{3/4} + \alpha_4 \arctan \left(\frac{f - \alpha_5 f_{\text{RD}}}{f_{\text{damp}}} \right) \right], \quad (2.160)$$

where f_{damp} is the ringdown damping frequency and α_i are fitting parameters. Between regimes, one constant and one linear coefficients are introduced to ensure $C(1)$ continuity at the matching point. Therefore the full IMR phase model is obtained by joining the piecewise regions with step functions

$$\theta_{f_0}^{\pm} = \frac{1}{2} [1 \pm \theta(f - f_0)], \quad (2.161)$$

with

$$\theta(f - f_0) = \begin{cases} -1 & \text{if } f < f_0 \\ 1 & \text{if } f \geq f_0 \end{cases}, \quad (2.162)$$

Waveform regime	Parameter	f dependence
Early-inspiral	φ_0	$f^{-5/3}$
	φ_1	$f^{-4/3}$
	φ_2	f^{-1}
	φ_3	$f^{-2/3}$
	φ_4	$f^{-1/3}$
	φ_{5l}	$\ln f$
	φ_6	$f^{1/3}$
	φ_{6l}	$f^{1/3} \ln f$
Late-inspiral	φ_7	$f^{2/3}$
	σ_2	$f^{4/3}$
	σ_3	$f^{5/3}$
Intermediate	σ_4	f^2
	β_2	$\ln f$
Merger-ringdown	β_3	f^{-3}
	α_2	f^{-1}
	α_3	$f^{3/4}$
	α_4	$\arctan(af + b)$

Table 2.2: Overview of the corresponding f -dependence for each phasing coefficients that appears in the *IMRPhenomD* approximant. a and b depend on f_{RD} and f_{damp} . For other not listed coefficients, they have either no or linear f dependence which are degenerate to the $C(1)$ continuity coefficients, and the arbitrary overall time and phase shift.

and is given by

$$\Phi_{\text{IMR}}(f) = \Phi_{\text{Ins}}(f)\theta_{f_1^\Phi}^- + \theta_{f_1^\Phi}^+ \Phi_{\text{Int}}(f)\theta_{f_2^\Phi}^- + \theta_{f_2^\Phi}^+ \Phi_{\text{MR}}(f), \quad (2.163)$$

with $Mf_1^\Phi = 0.018$ and $f_2^\Phi = 0.5f_{\text{RD}}$. Table 2.2 shows the corresponding f dependence for each phasing coefficients.

The amplitude model is also divided into these three regimes but defined by different frequency ranges. The inspiral amplitude A_{Ins} is defined with an upper frequency bound of $Mf = 0.014$ and based on a re-expanded PN amplitude

$$A_{\text{PN}}(f) = A_0(f) \sum_{i=0}^6 \mathcal{A}_i(\pi f)^{i/3}, \quad (2.164)$$

where A_0 has a leading order $f^{-7/6}$ similar in Eq. (2.148) and $\mathcal{A}_i$ is the re-expanded coefficients, plus the next three natural terms in the PN expansion for calibration, *i.e.*

$$A_{\text{Ins}} = A_{\text{PN}} + A_0 \sum_{i=7}^9 \rho_i f^{i/3}, \quad (2.165)$$

where ρ_i are the fitting parameters. The ringdown amplitude A_{MR} is modelled by a Lorentzian with an exponential decay in the frequency range $Mf \in [1/1.15, 1.2]f_{\text{RD}}$ and reads

$$\frac{A_{\text{MR}}}{A_0} = \gamma_1 \frac{\gamma_3 f_{\text{damp}}}{(f - f_{\text{RD}})^2 + (\gamma_3 f_{\text{damp}})^2} e^{-\frac{\gamma_2(f - f_{\text{RD}})}{\gamma_3 f_{\text{damp}}}}, \quad (2.166)$$

where γ_i are fitting parameters. The Lorentzian part has an amplitude peak at f_{RD} but the additional exponential factor shifts the peak to

$$f_{\text{peak}} = f_{\text{RD}} + \frac{f_{\text{damp}}\gamma_3\left(\sqrt{1-\gamma_2^2}-1\right)}{\gamma_2}. \quad (2.167)$$

The intermediate amplitude A_{Int} is modelled in the frequency range $Mf \in [0.014, f_{\text{peak}}]$ and reads

$$\frac{A_{\text{Int}}}{A_0} = \delta_0 + \delta_1 f + \delta_2 f^2 + \delta_3 f^3 + \delta_f f^4, \quad (2.168)$$

where δ_i are not fitting parameters but the solutions of the system of equations,

$$\begin{aligned} A_{\text{Int}}(f_1^A) &= A_{\text{Ins}}(f_1^A), \\ A_{\text{Int}}(f_2^A) &= A_{\text{NR}}(f_2^A), \\ A_{\text{Int}}(f_3^A) &= A_{\text{MR}}(f_3^A), \\ A'_{\text{Int}}(f_1^A) &= A'_{\text{Ins}}(f_1^A), \\ A'_{\text{Int}}(f_3^A) &= A'_{\text{MR}}(f_3^A), \end{aligned} \quad (2.169)$$

where $Mf_1^A = 0.014$, $f_3^A = f_{\text{peak}}$, $f_2^A = (f_1^A + f_3^A)/2$, $A_{\text{NR}}(f_2^A)$ is the amplitude of the NR waveform at f_2^A and can be modelled across the parameter space. Therefore, by construction, the amplitude is $C(1)$ continuous. The full IMR amplitude model is obtained by joining the piecewise regions with step functions and is given by

$$A_{\text{IMR}}(f) = A_{\text{Ins}}(f)\theta_{f_1^A}^- + \theta_{f_1^A}^+ A_{\text{Int}}(f)\theta_{f_3^A}^- + \theta_{f_3^A}^+ A_{\text{MR}}(f), \quad (2.170)$$

with $Mf_1^A = 0.014$ and $f_3^A = f_{\text{peak}}$.

All phenomenological fitting parameters

$$\Lambda^j = \underbrace{\left\{ \{\rho_i\}, \{A_{\text{NR}}(f_2^A)\} \right\}}_{\text{Amplitude Parameters}}, \underbrace{\left\{ \{\gamma_i\}, \{\sigma_i\}, \{\beta_i\}, \{\alpha_i\} \right\}}_{\text{Phase Parameters}} \quad (2.171)$$

are fit in terms of polynomials up to second order in η and third order in χ_{PN} ,

$$\Lambda^j = \sum_{m=0}^3 \sum_{n=0}^2 (\chi_{\text{PN}} - 1)^m (\lambda_{nm}^j \eta^n), \quad (2.172)$$

where λ_{nm}^j are the fitting coefficients which can be found on [73].

A precessing waveform approximant can be obtained by situating the non-precessing model in a comoving frame and transforming it by some Euler rotations [74],

$$h^{\text{prec}} = \mathbf{R} h^{\text{aligned}}, \quad (2.173)$$

where $\mathbf{R}$ is a rotation matrix built with Euler rotations. *IMRPhenomPv2* is a precessing waveform approximant built on *IMRPhenomD*.

CHAPTER 3

DATA ANALYSIS

In this chapter, we discuss how GWs are detected by ground-based laser interferometers, what noises affect the measurement and how to extract the GW signals deeply immersed in the noise.

3.1 Beam pattern functions

In a laser interferometer, the laser beam generated in the laser source is split by a beam splitter into two directions, denoted by two unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$. Each of these half rays bounce back and forth inside an optical resonator formed between the input and end mirrors to increase the effective optical path. The beam splitter combines the reflected half rays to form an interference pattern at the photodetector which converts the optical signal to the signal output strain $h(t)$. When the GWs travel through space for millions of light-years and reach our detectors, because it stretches the spacetime, changes the arm length, the resultant optical path difference and so the interference pattern therefore we can detect GWs. A schematic representation of a L-shaped GW interferometer is shown in Fig. 3.1.

We assume a L-shaped detector is placed at the origin in the (x, y) plane with end mirrors at $(L, 0)$ and $(0, L)$. A typical ground-based detector is sensitive to GWs with wavelength $\sim \mathcal{O}(10^3 \text{ km})$ which is a few orders of magnitude larger than the size of the detector $\sim \mathcal{O}(\text{km})$. Therefore, we can safely assume the detector is in local Lorentz frame (LLF)¹⁰. In a presence of GWs propagating along the z -axis, from Eq. (2.88), the two mirrors on the x and y -axis oscillate longitudinally with displacements

$$\begin{aligned} \delta x &= \frac{1}{2} h_+(t) L, \\ \text{and } \delta y &= -\frac{1}{2} h_+(t) L, \end{aligned} \tag{3.1}$$

respectively. Correspondingly, the oscillation of the relative difference in arm lengths is

$$h(t) \equiv \frac{\delta L}{L} = \frac{\delta x - \delta y}{L} = h_+(t). \tag{3.2}$$

The current GW interferometers in operation, *i.e.* the Advanced LIGO-Virgo detectors, are all L-shaped.

For a general interferometer where the arms are not perpendicular to each other, because the mirrors are free to move only along the longitudinal direction, the change of arm length is

¹⁰In case of space-based detector, the size of the detector $\sim \mathcal{O}(10^6 \text{ km})$ is comparable to the wavelength of the GWs of interest and so we do not have LLF approximation.

Figure 3.1: An oversimplified schematic representation of a L-shaped GW interferometer where the arms have length of L along the $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ directions.

sensitive to the projection of the oscillation direction, *i.e.* $u_i \delta u^i$ for the arm in $\hat{\mathbf{u}}$ direction and similarly $v_i \delta v^i$ for the other arm. Therefore, Eq. (3.2) can be generalized to

$$h(t) = D^{ij}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) h_{ij}^{\text{TT}}(t), \quad (3.3)$$

where $D^{ij}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) = \frac{1}{2}(u^i u^j - v^i v^j)$ is called the detector tensor. Einstein Telescope, one of the proposed third-generation ground-based GW observatory, is planned to be not L-shaped but composed of several V-shaped interferometers arranged in an equilaterally triangular shape [75], see Fig. 3.2.

It is straightforward to generalize the result also for GWs coming from a sky location (θ, ϕ) . In the detector frame (x, y, z) the propagation direction of GWs is

$$\hat{\mathbf{n}} = -(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (3.4)$$

which corresponds to the z' direction in the source frame (x', y', z') as shown in Fig. (3.3). We can perform three successive rotations by the three Euler's angles (θ, ϕ, ψ) to bring the source frame to the detector frame, so Eq. (3.3) becomes

$$h(t) = D^{ij}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) h_{ij}^{\text{Detector}}(t) = D^{ij}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) R_i^k R_j^l h_{kl}^{\text{TT}}(t), \quad (3.5)$$

with

$$R_j^i = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\pi - \theta) & 0 & -\sin(\pi - \theta) \\ 0 & 1 & 0 \\ \sin(\pi - \theta) & 0 & \cos(\pi - \theta) \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.6)$$

Figure 3.2: Design of the Einstein Telescope, one of the proposed third-generation ground-based GW observatory, composed of several V-shaped interferometers arranged in an equilaterally triangular shape [76].

Figure 3.3: a figure showing the three euler's angles relating the source frame (x', y', z') and the detector frame (x, y, z) . the polar angle θ and the azimuthal angle ϕ specify the sky location where the incident gw is. the polarization angle ψ mixes h_+ and $h_\times$.

In general, Eq. (3.5) can be written as a linear combination of h_+ and $h_\times$,

$$h(t; \theta, \phi, \psi) = F_+(\theta, \phi, \psi)h_+(t) + F_\times(\theta, \phi, \psi)h_\times(t), \quad (3.7)$$

where the angular dependence is contained in the beam (or antenna) pattern functions, F_+ and $F_\times$. In the case of a L-shaped interferometer with arms in $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions, they are

$$\begin{aligned} F_+^{(90^\circ)}(\theta, \phi, \psi) &= \frac{1}{2}(1 + \cos^2 \theta) \cos 2\phi \cos 2\psi - \cos \theta \sin 2\phi \sin 2\psi, \\ F_\times^{(90^\circ)}(\theta, \phi, \psi) &= \frac{1}{2}(1 + \cos^2 \theta) \cos 2\phi \sin 2\psi + \cos \theta \sin 2\phi \cos 2\psi, \end{aligned} \quad (3.8)$$

which are shown in Fig. (3.4) So the L-shaped interferometer is most sensitive if the GW comes from overhead. However, there exist directions (θ, ϕ) such that the strain $h(t)$ will be

Figure 3.4: Absolute value of beam pattern functions $|F_+(\theta, \phi, 0)|$ (left) and $|F_\times(\theta, \phi, 0)|$ (right) for an interferometer with arms in $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions. The middle plot is the combined of two beam pattern functions.

zero irrespective of what ψ is. It happens when

$$\begin{aligned} \cos 2\phi = 0 \quad \text{and} \quad \cos \theta = 0 \\ \phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \quad \text{and} \quad \theta = \frac{\pi}{2}. \end{aligned} \quad (3.9)$$

In these directions, the GWs do change the length of the detector arms, but the stretching and elongating effects are the same for both arms. This does not lead to a relative difference in arm length, so $h(t) = 0$. Although the blind spots of the detector reduce the observed signal-to-noise ratio (see also section 3.2), these do not harm the detection of GWs much because in reality we have multiple detectors. More detectors also lead to more precise sky localization [5] and moreover enable studies of GW polarizations.

In the case of a detector with detector arms at a 60° opening angle, the beam pattern functions are

$$\begin{aligned} F_+^{(60^\circ)}(\theta, \phi, \psi) &= \frac{\sqrt{3}}{2} \left[\frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \cos 2\psi - \cos \theta \sin 2\phi \sin 2\psi \right], \\ F_\times^{(60^\circ)}(\theta, \phi, \psi) &= \frac{\sqrt{3}}{2} \left[\frac{1}{2} (1 + \cos^2 \theta) \cos 2\phi \sin 2\psi + \cos \theta \sin 2\phi \cos 2\psi \right]. \end{aligned} \quad (3.10)$$

Compared to Eq. (3.8), we clearly see that a detector with detector arms at a 60° opening angle is less sensitive to GWs than a L-shaped detector assuming they are identical except the opening angle.

On the other hand, suppose the GW strains of the plus and cross polarizations can be written as

$$\begin{aligned} h_+(t) &= A(t) \frac{1 + \cos^2 \iota}{2} \cos \Phi(t), \\ h_\times(t) &= A(t) \cos \iota \sin \Phi(t), \end{aligned} \quad (3.11)$$

similar to Eq. (2.118). With the help of the subsidiary angle formula, Eq. (3.7) can be written as

$$h(t) = A'(t) \cos(\Phi(t) - \Phi'), \quad (3.12)$$

with

$$\begin{aligned} A'(t) &= A(t) \sqrt{F_+^2 \left(\frac{1 + \cos^2 \iota}{2} \right)^2 + F_\times^2 \cos^2 \iota}, \\ \tan \Phi' &= \frac{2F_\times \cos \iota}{F_+(1 + \cos^2 \iota)}. \end{aligned} \quad (3.13)$$

Effectively the beam pattern functions modify the GW amplitude and the phase shift by the above amounts.

3.2 Signal extraction

3.2.1 Noise sources

In reality, the detector's output $d(t)$ is not simply the signal $h(t)$ but contaminated with noises $n(t)$,

$$d(t) = h(t) + n(t). \quad (3.14)$$

The typical length change of the interferometer's arms is as tiny as $\mathcal{O}(10^{-18})$ m so it is critical the mirrors are free from any possible noise sources, *e.g.*

- Seismic noise: the motion of mirrors caused by ground vibrations, earthquakes, winds, ocean waves and human activities,
- Thermal noise: the microscopic fluctuation of the individual atoms in the mirrors and their suspensions,
- Quantum noise: displacement noise by exerting a fluctuating radiation pressure force that physically moves the mirrors, and the statistical uncertainty from the “photon counting” that is performed by the photodetector,
- Gas noise: displacement noise due to the collision of residual gas particles on the mirrors in the vacuum chamber.

Details and the corresponding minimization methods of each noise can be found on [77].

3.2.2 Characterization of noise

$n(t)$ can be modelled by stationary Gaussian noise. Stationarity implies the noise properties are not changing in time and thus the different Fourier components are uncorrelated. Gaussianity implies

$$\begin{aligned} \langle n(t) \rangle &= 0, \\ \langle \tilde{n}^*(f) \tilde{n}(f') \rangle &= \frac{1}{2} \delta(f - f') S_n(f), \end{aligned} \quad (3.15)$$

where $\langle \rangle$ is the ensemble average, tilde denotes the Fourier transform, star denotes the complex conjugate and $S_n(f)$ is the power spectral density (PSD) and has a dimension of Hz^{-1} . In reality a finite period of time T is used to measure $\tilde{n}(f)$, so we replace the ensemble average by a time average with an implicit assumption that the system is ergodic,

$$\frac{1}{2} S_n(f) = \langle |\tilde{n}(f)|^2 \rangle_t \Delta f, \quad (3.16)$$

Figure 3.5: The ASDs of LIGO-Hanford (red), LIGO-Livingston (blue) and Virgo (violet) detectors during O2 [8]. The spectra reveal a lot of peaks due to instrumental artifacts.

where $\Delta f = \frac{1}{T}$ is the resolution in frequency. The physical meaning of $\frac{1}{2}S_n(f)$ is clear from this equation. It gives the variance of the noise in a certain frequency bin. Fig. 3.5 shows the amplitude spectral density (ASD), which is the square root of the PSD, for the LIGO-Virgo detectors during O2. Integrating Eq. (3.15), we obtain the variance of the noise in time,

$$\begin{aligned}
 \langle n^2(t) \rangle &= \langle n^2(t=0) \rangle \\
 &= \int_{-\infty}^{\infty} df df' \langle n^*(f) n(f') \rangle \\
 &= \frac{1}{2} \int_{-\infty}^{\infty} df S_n(f) \\
 &= \int_0^{\infty} df S_n(f).
 \end{aligned} \tag{3.17}$$

3.2.3 Matched filtering

Suppose that we know $h(t)$, which is what we are searching for and can be a CBC signal or some burst signals. To extract the signal, we can compute the time average of the product of $d(t)$ and $h(t)$,

$$\frac{1}{T} \int_0^T dt d(t) h(t) = \frac{1}{T} \int_0^T dt h^2(t) + \frac{1}{T} \int_0^T dt n(t) h(t). \tag{3.18}$$

The first term gives the variance of the signal and the integral grows with T while the second term is like a random walk of $h(t)$ and the integral grows with $\sqrt{T}$.

$$\frac{1}{T} \int_0^T dt d(t)h(t) \sim h_0^2 + \sqrt{\frac{\tau_0}{T}} n_0 h_0, \quad (3.19)$$

where h_0 and n_0 give the characteristic amplitude of the signal and noise respectively while τ_0 gives a characteristic period of the signal. From this equation, the sufficient condition for extracting the signal is not $h_0 > n_0$, but

$$h_0 > \sqrt{\frac{\tau_0}{T}} n_0. \quad (3.20)$$

A typical CBC signal has a characteristic frequency of 100 Hz, Eq. (2.129), which is equivalent to a characteristic period of $\tau_0 = 0.01$ s, and a duration $T = 100$ s, Eq. (2.130). Therefore, even if h_0 is 100 times smaller than n_0 , it can in principle still be detected since $\sqrt{\frac{\tau_0}{T}} = 100$.

This kind of technique is called matched filtering. We can optimize the searching by applying a filter $K(t)$ to the detector output. We define the signal-to-noise ratio (SNR) in amplitude by S/N where S is the expected value of $s(t)$ when the signal is present and N is the root-mean-squared value of $s(t)$ when the signal is absent. After some calculations, we have

$$\frac{S}{N} = \frac{\int_{-\infty}^{\infty} df \tilde{h}(f) \tilde{K}^*(f)}{\sqrt{\int_{-\infty}^{\infty} df \frac{1}{2} S_n(f) |\tilde{K}(f)|^2}}. \quad (3.21)$$

We define the inner product between functions A and B ,

$$\begin{aligned} (A|B) &= \Re \int_{-\infty}^{\infty} df \frac{\tilde{A}^*(f) \tilde{B}(f)}{\frac{1}{2} S_n(f)} \\ &= 4 \Re \int_0^{\infty} df \frac{\tilde{A}^*(f) \tilde{B}(f)}{S_n(f)}. \end{aligned} \quad (3.22)$$

Eq. (3.21) can then be written in a compact way,

$$\frac{S}{N} = \frac{(u|h)}{\sqrt{(u|u)}}, \quad (3.23)$$

where $\tilde{u}(f) = \frac{1}{2} S_n(f) \tilde{K}_n(f)$. To maximize the SNR, u has to ‘point’ to the same direction as h . In this way, we obtain the Wiener filter,

$$\tilde{K}(f) \propto \frac{\tilde{h}(f)}{S_n(f)}, \quad (3.24)$$

which shows that the best filter is a combination of the knowledge of the signal and properties of the noise. In particular, if the noise is white, *i.e.* $S_n(f)$ is a constant, then intuitively the best filter will be the signal itself. If not, we should weight more in less noisy frequency region and vice versa. The optimal SNR is then obtained by inserting Eq. (3.24) into Eq. (3.23),

$$\frac{S}{N} = \sqrt{(h|h)} = 4 \int_0^{\infty} df \frac{|\tilde{h}(f)|^2}{S_n(f)}. \quad (3.25)$$

In practice, one of the GW sources of interest are CBC signals which can be characterized by 8 intrinsic parameters for BBH (or 10 for BNS) plus 7 extrinsic parameters, which are summarised in Table 3.1. LIGO-Virgo makes use of two separate pipelines, GstLAL [78] and PyCBC [79], to perform matched filtering using a collection of GW waveforms, called the template bank, on the detector’s data in near real time. The template bank of GstLAL in O2 is shown in Fig. 3.6 which contains 667,000 waveforms including BBH, NSBH and BNS. Up to now, all observed GW signals are discovered by these two pipelines.

Symbols	Intrinsic parameters	Symbols	Extrinsic parameters
m_1, m_2	Component masses	θ, ϕ	Sky location
$\vec{S}_1, \vec{S}_2$	Spins	ι, ψ	Orientation
Λ_1, Λ_2	Tidal deformabilities (for BNS)	r	Distance
		t_c	Time of coalescence
		ϕ_c	Coalescence phase

Table 3.1: Parameters of a CBC signal.

Figure 3.6: The template bank of GstLAL in O2 [78]. Each point represents a waveform characterized by masses m_1, m_2 . The green points denote BNS template. The blue points denote BBH template. The red points denote NSBH template. The magenta and black points represent additional templates that were added to aid in the background estimation for the scarcely populated, high-mass region of the template bank.

3.3 Bayesian inference

In this section, we talk about how to extract useful information from a signal once it has been detected. The result of parameter estimation will usually be a probability density function. The discrete probability is denoted by P and probability density function (PDF) is denoted by p . The probability for a continuous variable x to be in a certain interval $[x^{\min}, x^{\max}]$ is then

given by

$$P(x^{\min} \leq x \leq x^{\max}) = \int_{x^{\min}}^{x^{\max}} dx p(x). \quad (3.26)$$

3.3.1 Frequentist vs Bayesian

When we talk about probability, there are two kinds of meaning. Let us take flipping a coin to be an example. Suppose the probability of getting a head is $P(\text{head}) = \mu$, then $\mu = 0$ means the coin is double-tailed, $\mu = 0.5$ means the coin is fair and $\mu = 1$ means the coin is double-headed. In the frequentist approach, probability counts the proportion of outcome. If we flip a coin N times, there will be $N\mu$ times that the head faces up in average. There is a fixed true value of μ . Repeated experiments give a more accurate (with respect to the true fixed value) and precise (less systematics) estimate on μ . In the Bayesian approach, μ is a random variable associated to a probability distribution $p(\mu)$ rather than a fixed value. In addition, the probability measures the degree of belief and the standard deviation of $p(\mu)$ measures the uncertainty. Before we flip the coin, we have no clue what to say about the coin, so $p(\mu)$ is a uniform distribution which reflects our ignorance. When we flip the coin once, we know that the coin is either not double-tailed or not double-headed. When we flip the coin more and more times, $p(\mu)$ gets updated and peaks toward some value. In fact, we always use the Bayesian approach in our daily life, *e.g.* “it will rain tomorrow”. The frequentist approach is good for experiments which can be performed many times such as particle-colliding experiments, while the Bayesian approach is good for observation which can only be seen once such as a gravitational-wave signal on a particular day. We are going to use the Bayesian approach for GW data analysis.

3.3.2 Bayes’s theorem

The probability theory starts with two rules, the sum rule and the product rule,

$$\begin{aligned} 1 &= P(A|I) + P(\bar{A}|I), \\ P(A, B|I) &= P(A|B, I)P(B|I), \end{aligned} \quad (3.27)$$

where A, B are some propositions, $\bar{A}$ is the complement of A , and P ranges from 0 for impossibility to 1 for certainty. $P(A|B)$ is conditional probability and reads as ‘the probability that A is true given B ’. All probabilities are conditioned on I which stands for any relevant background information at hand because there is no such thing as absolute probability. From these two rules, we can construct Bayes’s theorem and the marginalization equation,

$$P(A|B, I) = \frac{P(A|I)P(B|A, I)}{P(B|I)}, \quad (3.28)$$

and

$$p(X|I) = \sum_k p(X, Y_k|I), \quad (3.29)$$

where $\{Y_k\}$ forms an exhaustive and mutually exclusive set. We can also write Eq. (3.29) in a continuous limit,

$$p(x|I) = \int_{y^{\min}}^{y^{\max}} dy p(x, y|I). \quad (3.30)$$

From the perspective of data analysis, the importance of Bayes's theorem becomes apparent if we let A and B to be the hypothesis $\mathcal{H}$ and data d respectively,

$$P(\mathcal{H}|d, I) = \frac{p(d|\mathcal{H}, I)P(\mathcal{H}|I)}{p(d|I)}. \quad (3.31)$$

On the right-hand side, the prior probability $P(\mathcal{H}|I)$ represents our initial belief of $\mathcal{H}$ before observing d . It is then updated by the likelihood $p(d|\mathcal{H}, I)$ (also known as evidence), which represents how well $\mathcal{H}$ can explain d , to give the posterior probability $P(\mathcal{H}|d, I)$, which represents our updated belief of $\mathcal{H}$ after observing d . $p(d|I)$ is simply an overall normalization constant. In other words, this theorem encapsulates the process of learning.

From the perspective of parameter estimation, if $\mathcal{H}$ depends on several parameters $\boldsymbol{\theta}$, then we can also write Bayes's theorem for $\boldsymbol{\theta}$,

$$p(\boldsymbol{\theta}|d, \mathcal{H}, I) = \frac{p(d|\boldsymbol{\theta}, \mathcal{H}, I)p(\boldsymbol{\theta}|\mathcal{H}, I)}{p(d|\mathcal{H}, I)}, \quad (3.32)$$

where $p(\boldsymbol{\theta}|\mathcal{H}, I)$ is the prior PDF of $\boldsymbol{\theta}$ representing our degree of belief of what $\boldsymbol{\theta}$ should be before observing d . This prior density is then updated by the likelihood function $p(d|\boldsymbol{\theta}, \mathcal{H}, I)$, which represents how likely it is to obtain the observed d given certain $\boldsymbol{\theta}$, to give the posterior PDF $p(\boldsymbol{\theta}|d, \mathcal{H}, I)$ representing our updated degree of belief in the light of d . The evidence $p(d|\mathcal{H}, I)$ can be calculated by integration over $\boldsymbol{\theta}$,

$$p(d|\mathcal{H}, I) = \int d\boldsymbol{\theta} p(d|\boldsymbol{\theta}, \mathcal{H}, I)p(\boldsymbol{\theta}|\mathcal{H}, I), \quad (3.33)$$

which is simply a normalization constant in this case. However, in model selection (see section 3.3.5), the evidence is an important quantity which tells us how well the data can be represented by $\mathcal{H}$.

In the case of GW data analysis, because noise is assumed to be Gaussian in the frequency domain with the PSD $S_n(f)$, the likelihood can be written as

$$p(d|\boldsymbol{\theta}, \mathcal{H}, I) \propto e^{-\frac{1}{2}(n|n)} \propto e^{-\frac{1}{2}(d-\tilde{h}(\boldsymbol{\theta};f)|d-\tilde{h}(\boldsymbol{\theta};f))}, \quad (3.34)$$

where the noise-weighted inner product $(a|b)$ is defined in Eq. (3.22).

3.3.3 Characterization of posterior

The methods of obtaining the posterior are explained in the coming Sec. 3.3.6 and 3.3.7. Once we obtain the posterior, it contains all information about the parameters $\boldsymbol{\theta}$ given the observed data. Because $\boldsymbol{\theta}$ can have multiple dimensions N , it is hard to visualize it by a plot. If we are only interested in a particular parameter θ_1 , we can eliminate other parameters by marginalization, Eq. (3.30),

$$p(\theta_1|d, \mathcal{H}, I) = \int_{\theta_2^{\min}}^{\theta_2^{\max}} \dots \int_{\theta_N^{\min}}^{\theta_N^{\max}} d\theta_2 \dots d\theta_N p(\boldsymbol{\theta}|d, \mathcal{H}, I). \quad (3.35)$$

One might even want to give a point estimate of θ from its 1D-posterior such as mean θ^{mean} and median θ^{median} ,

$$\begin{aligned} \theta^{\text{mean}} &\equiv \langle \theta \rangle = \int_{\theta^{\min}}^{\theta^{\max}} d\theta \theta p(\theta|d, \mathcal{H}, I), \\ \frac{1}{2} &= \int_{\theta^{\min}}^{\theta^{\text{median}}} d\theta p(\theta|d, \mathcal{H}, I). \end{aligned} \quad (3.36)$$

In particular, a Maximum a Posteriori (MAP) value is also a good point estimate of the posterior,

$$\theta^{\text{MAP}} = \arg \max_{\theta} p(\theta|d, \mathcal{H}, I), \quad (3.37)$$

which is equivalent to the maximum likelihood estimation (MLE) in the frequentist approach if the prior distribution is flat. We can also summarise the spread (or uncertainty) of the posterior by the standard deviation σ_{θ} and the credible interval $\theta^{\text{high}} - \theta^{\text{low}}$,

$$\begin{aligned} \sigma_{\theta}^2 &= \int_{\theta^{\text{min}}}^{\theta^{\text{max}}} d\theta (\theta - \theta^{\text{mean}})^2 p(\theta|d, \mathcal{H}, I), \\ \gamma &= \int_{\theta^{\text{low}}}^{\theta^{\text{high}}} d\theta p(\theta|d, \mathcal{H}, I), \end{aligned} \quad (3.38)$$

where the distance $\theta^{\text{high}} - \theta^{\text{low}}$ is minimized. γ is usually taken to be 0.68 or 0.95 which correspond to the probability of a Gaussian distribution in the interval of 1 or 2 standard deviations respectively. The larger the spread, the more uncertain about the point estimate we are.

To obtain the best signal reconstruction from model $h(\boldsymbol{\theta}, \mathcal{H}, I; t)$, we can plug in the best point estimate to the model, *i.e.* $h(\boldsymbol{\theta}_{\text{best}}, \mathcal{H}, I; t)$. Nevertheless, we can also embrace the probabilistic nature of parameters in Bayesian analysis rather than sticking to a point estimate. By averaging all possible parameter values weighted by their posterior probability, we have

$$\langle h(\boldsymbol{\theta}, \mathcal{H}, I; t) \rangle_{\boldsymbol{\theta}} = \int d\boldsymbol{\theta} h(\boldsymbol{\theta}, \mathcal{H}, I; t) p(\boldsymbol{\theta}|d, \mathcal{H}, I). \quad (3.39)$$

Moreover, we can also make prediction d^* at a future observed point $t = t^*$ with the posterior beliefs about the parameters,

$$p(d^*|d, \mathcal{H}, I) = \int d\boldsymbol{\theta} p(d^*|\boldsymbol{\theta}, \mathcal{H}, I) p(\boldsymbol{\theta}|d, \mathcal{H}, I), \quad (3.40)$$

where $p(d^*|d, \mathcal{H}, I)$ is the posterior predictive distribution and $p(d^*|\boldsymbol{\theta}, \mathcal{H}, I)$ is the likelihood of the future observation. It is instructive to write down also the prior predictive distribution,

$$p(d^*|\mathcal{H}, I) = \int d\boldsymbol{\theta} p(d^*|\boldsymbol{\theta}, \mathcal{H}, I) p(\boldsymbol{\theta}|\mathcal{H}, I), \quad (3.41)$$

which serves as making prediction with the prior beliefs about the parameters before observations. Compared to the evidence in Eq. (3.33), it has a similar form except d is replaced with d^* .

3.3.4 Combining posteriors

Suppose the parameters vector $\boldsymbol{\theta}$ is universal in all available data and we have N independent measurements of $\boldsymbol{\theta}$ from data $\{d_1, d_2, \dots, d_N\}$, we can combine all posteriors to obtain a tighter constraint on $\boldsymbol{\theta}$. According to the Bayes's theorem, the combined posterior can be written as

$$p(\boldsymbol{\theta}|\{d_1, d_2, \dots, d_N\}, \mathcal{H}, I) = \frac{p(\{d_1, d_2, \dots, d_N\}|\boldsymbol{\theta}, \mathcal{H}, I) p(\boldsymbol{\theta}|\mathcal{H}, I)}{p(\{d_1, d_2, \dots, d_N\}|\mathcal{H}, I)}. \quad (3.42)$$

Because all measurements are mutually independent, the joint probability equals the product of their probabilities,

$$\begin{aligned} p(\{d_1, d_2, \dots, d_N\}|\boldsymbol{\theta}, \mathcal{H}, I) &= \prod_{i=1}^N p(d_i|\boldsymbol{\theta}, \mathcal{H}, I), \\ p(\{d_1, d_2, \dots, d_N\}|\mathcal{H}, I) &= \prod_{i=1}^N p(d_i|\mathcal{H}, I). \end{aligned} \quad (3.43)$$

Therefore, the combined posterior becomes

$$\begin{aligned} p(\boldsymbol{\theta}|\{d_1, d_2, \dots, d_N\}, \mathcal{H}, I) &= p(\boldsymbol{\theta}|\mathcal{H}, I) \prod_{i=1}^N \frac{p(d_i|\boldsymbol{\theta}, \mathcal{H}, I)}{p(d_i|\mathcal{H}, I)} \\ &= p(\boldsymbol{\theta}|\mathcal{H}, I)^{1-N} \prod_{i=1}^N p(\boldsymbol{\theta}|d_i, \mathcal{H}, I). \end{aligned} \quad (3.44)$$

3.3.5 Model selection

To compare two hypotheses, X and Y , we can compute their relative posterior probabilities (known as the odds ratio or posterior odds),

$$O_Y^X \equiv \frac{P(X|d, I)}{P(Y|d, I)} = \frac{p(d|X, I)}{p(d|Y, I)} \frac{P(X|I)}{P(Y|I)}. \quad (3.45)$$

The prior odds $\frac{P(X|I)}{P(Y|I)}$ represents our relative initial beliefs of these two hypotheses. If there is no background information which favours either X or Y , the prior odds is equal to unity. The ratio of evidences (also known as Bayes factor) $\frac{p(d|X, I)}{p(d|Y, I)}$ compares the relative goodnesses of fit to the data by these two hypotheses. If $O_Y^X > 1$, the data tells us that X is more likely to be correct than Y and vice versa. If $O_Y^X = 1$, these two hypotheses are equally likely.

One may think that a hypothesis which has more degree of freedoms to fit the data is more favoured but it is not the case in model selection. Although extra free parameters allow a better fitting to data and thus increase the likelihood $p(d|\boldsymbol{\theta}, X, I)$, the bigger prior volume decreases the prior PDF $p(\boldsymbol{\theta}|X, I)$. Hence, from Eq. (3.33), the evidence $p(d|X, I)$ does not necessarily increase by having more free parameters. The best hypothesis is to explain the data with minimal parameters. It is known as Occam's razor, *i.e.* "it is vain to do with more what can be done with less", which is inbuilt in the Bayesian analysis.

3.3.6 Nested sampling

To obtain the evidence, we need to solve Eq. (3.33). We simplify the notation by letting $Z \equiv p(d|\mathcal{H}, I)$, $L(\boldsymbol{\theta}) \equiv p(d|\boldsymbol{\theta}, \mathcal{H}, I)$ and $\pi(\boldsymbol{\theta}) \equiv p(\boldsymbol{\theta}|\mathcal{H}, I)$, so Eq. (3.33) becomes

$$Z = \int d\boldsymbol{\theta} L(\boldsymbol{\theta}) \pi(\boldsymbol{\theta}), \quad (3.46)$$

Although the functional forms of $L(\boldsymbol{\theta})$ and $\pi(\boldsymbol{\theta})$ are known, it is difficult to solve it analytically because the integral is multi-dimensional (see Table 3.1) and the functional forms can be complicated. Instead we solve it numerically with nested sampling introduced by Skilling in 2006 [80]. We introduce the prior mass $X(L)$ which is defined to be the fraction of the prior volume with likelihood larger than L ,

$$X(L) = \int_{L(\boldsymbol{\theta}) \geq L} d\boldsymbol{\theta} \pi(\boldsymbol{\theta}). \quad (3.47)$$

Figure 3.7: A sketch of the likelihood as a function of prior mass and the corresponding likelihood contours in the parameter space [80].

So by definition, X is a monotonically decreasing function and is bounded, *i.e.* $0 \leq X(L) \leq 1$. The lower bound $X = 0$ corresponds to a surface within which no higher likelihood can be found, *i.e.* $L = L_{\max}$, while the upper bound $X = 1$ corresponds to a surface within which all points have higher likelihood, *i.e.* $L = L_{\min}$. The evidence reduces to a one-dimensional integral which can be approximated by a Riemann sum with a set of discrete and dense “dead points” (X_i^D, L_i^D) ,

$$Z = \int_0^1 dX L(X) \approx \sum_i L_i^D \Delta X_i^D, \quad (3.48)$$

where $L(X)$ is simply the inverse function of $X(L)$ in Eq. (3.47) and also monotonically decreasing as shown in Fig. 3.7. ΔX_i^D can be as simple as $X_i^D - X_{i+1}^D$, trapezoidal $(X_{i-1}^D - X_{i+1}^D)/2$ or more sophisticated method. Instead of solving Eq. (3.47) to get X_i which is in practice computationally unfeasible, X_i can be assigned statistically as a consequence of Probability Integral Transform (PIT).

Theorem 1 (Probability Integral Transform). *Suppose a random variable X has a continuous distribution f for which the CDF is F_X , then $Y = F_X(X)$ has a standard uniform distribution $U(0, 1)$. In other words, if $X \sim f$, then $F_X \sim U(0, 1)$.*

Proof. We consider the CDF of Y .

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(F_X(X) \leq y) \\ &= P(X \leq F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y)) \\ &= y, \end{aligned} \quad (3.49)$$

which is just the CDF of $U(0, 1)$. Therefore, $Y \sim U(0, 1)$. $\square$

Hence, according to PIT, if $L = 0$, $\theta \sim \pi$ is equivalent to $X \sim U(0, 1)$. For non-zero fixed L , we define the constrained prior

$$\pi_L(\theta) \equiv \begin{cases} \pi(\theta)/X(L) & \text{if } L \leq L(\theta) \\ 0 & \text{otherwise} \end{cases}, \quad (3.50)$$

and the constrained uniform distribution

$$U_X(x) \equiv \begin{cases} x/X & \text{if } 0 \leq x \leq X \\ 0 & \text{otherwise} \end{cases}. \quad (3.51)$$

By definition, $\pi_0 = \pi$ and $U_X = U(0, X)$. Then $\boldsymbol{\theta} \sim \pi_L$ is equivalent to $X \sim U_{X(L(\boldsymbol{\theta}))}$.

Imagine we draw a sample $\boldsymbol{\theta}_1 \sim \pi_0$ and a prior mass $X_1 \sim U(0, 1)$. $X(L(\boldsymbol{\theta}_1))$ does not necessarily lie close to X_1 . To make X_1 (arbitrarily) close to $X(L(\boldsymbol{\theta}_1))$, we introduce N live points, say 1000. Now we draw N samples $\boldsymbol{\theta}_i \sim \pi_0$ and N prior masses $X_i \sim U(0, 1)$. $L_i = L(\boldsymbol{\theta}_i)$ are sorted in ascending order while X_i are sorted in descending order.

$$\begin{aligned} 0 &< L_1 < L_2 < \dots < L_N, \\ 1 &> X_1 > X_2 > \dots > X_N > 0. \end{aligned} \quad (3.52)$$

Then the $X(L_1)$ will be much closer to X_1 than before and we can approximate $X(L_1)$ by X_1 . In this way, we obtain the first dead point $(X_1^D, L_1^D) = (X_1, L_1)$. The good news is that the remaining $N - 1$ live points can be recycled to obtain the next dead point. To keep the same number of live points, we draw an extra sample $\boldsymbol{\theta}_{\text{new}} \sim \pi_{L_1^D}$ and an extra prior mass $X_{\text{new}} \sim U(0, X_1^D)$. Now all live points automatically satisfy $X_i < X_1^D$ and $L_i > L_1^D$. After sorting of all live points, the live point with the highest prior mass becomes the next dead point $(X_2^D, L_2^D) = (X_1, L_1)$. This process continues until a termination condition, *e.g.*

$$L_{\text{max,current}} X_{\text{current}} < \alpha Z_{\text{current}}, \quad (3.53)$$

where $L_{\text{max,current}}$ is the largest likelihood so far observed by the sampling process, X_{current} is the current highest prior mass, Z_{current} is the current estimate of evidence already accumulated and α is a user-specified constant. Therefore the left hand side is an (over)estimate of the evidence that is still to be accumulated. By ‘climbing up’ the hill of likelihood from the bottom $(X_0^D, L_0^D) = (1, 0)$, *i.e.* progressively higher likelihood from 0 and smaller prior mass from 1, we can compute Z by a sum over all dead points using Eq. (3.48).

In fact, sorting of prior masses to look for the largest prior mass can be bypassed with the help of the PDF of the largest prior mass. The probability of having no prior mass larger than χ for N live points is given by

$$\begin{aligned} P(\{X_i\} \leq \chi) &= \prod_{i=1}^N P(X_i \leq \chi) \\ &= \prod_{i=1}^N \int_0^\chi dX_i \\ &= \chi^N, \end{aligned} \quad (3.54)$$

which implies that the largest prior mass χ follows a beta distribution $\beta(N, 1)$ ¹¹,

$$p(\chi) = \frac{\partial P(\{X_i\} \leq \chi)}{\partial \chi} = N \chi^{N-1}. \quad (3.55)$$

We can define a shrinkage ratio $t \in [0, 1]$ which follows the same distribution as χ ,

$$p(t) = N t^{N-1}. \quad (3.56)$$

¹¹The PDF for a general beta distribution is $\beta(a, b) \propto x^{a-1}(1-x)^{b-1}$ where $x \in [0, 1]$.

Then by drawing a sample $t_i \sim \beta(N, 1)$, the next prior mass is obtained by $X_i = t_i X_{i-1}$ with $X_0 = 1$. In this way, we form a sequence of X_i , *i.e.*

$$\{1, t_1, t_2 t_1, \dots\}, \quad (3.57)$$

Moreover, we can even choose X_i to be deterministic. From Eq. (3.56), we have

$$\begin{aligned} \langle t \rangle &= \frac{N}{N+1} \approx 1 - \frac{1}{N} \approx e^{-1/N} \\ \sigma_t^2 &= \frac{N}{(N+1)^2(N+2)} \approx \frac{1}{N^2}, \end{aligned} \quad (3.58)$$

Therefore, the prior mass is expected to shrink to $\ln X_i \approx -(i \pm \sqrt{i})/N$. We can pick $X_i = e^{-i/N}$ but it is better to acknowledge the uncertainties in X_i .

As a by-product, nested sampling provides also the posterior probability,

$$p(\theta_i | d, \mathcal{H}, I) \approx \frac{L_i \Delta X_i}{Z}. \quad (3.59)$$

3.3.7 Markov chain Monte Carlo

Suppose we are given a probability distribution p where the functional form is known but not the normalization constant. For example, p can be the posterior distribution defined in Eq. (3.32) where the functional form of the likelihood is defined in Eq. (3.34) and prior PDF can be taken to be uniform but the evidence is unknown. If we are not interested in the normalization constant but only posterior samples, direct sampling will be difficult because of the unknown normalization constant. In fact, we can still obtain posterior samples without knowing the normalization constant using Markov chain Monte Carlo (MCMC). Markov chain means a sequence of possible samples $\{\theta^{(1)}, \theta^{(2)}, \dots\}$ in which the probability of each sample $\theta^{(i)}$ depends solely on the previous sample $\theta^{(i-1)}$. In other words, Markov chain is memory-less (also known as Markov property). Monte Carlo are computational techniques in which random numbers are generated to obtain samples from a desired probability distribution. We are going to discuss one of the MCMC algorithms, *i.e.* the Metropolis-Hasting (MH) algorithm.

Metropolis-Hasting algorithm

Suppose we want to sample the posterior distribution p . We initialize an arbitrary point to be our first sample $\theta^{(0)}$, then a sample candidate θ' is generated from a chosen proposal distribution $Q(\theta' | \theta^{(i)})$. The transition from $\theta^{(i)}$ to θ' depends on the acceptance ratio

$$A = \frac{p(\theta')}{p(\theta^{(i)})} \frac{Q(\theta^{(i)} | \theta')}{Q(\theta' | \theta^{(i)})}. \quad (3.60)$$

Because A depends on the ratio of p , the normalization constant (*i.e.* the evidence) does not contribute to the computation of the ratio of p . Moreover, by construction, the easier-to-reach parameters (*i.e.* higher $Q(\theta' | \theta^{(i)})$) have a lower acceptance ratio and thus are harder to be accepted. If the proposal distribution is symmetric, *e.g.* Gaussian, then A reduces to

$$A = \frac{p(\theta')}{p(\theta^{(i)})}. \quad (3.61)$$

Now a random number $\alpha \sim U(0, 1)$ is drawn. If $A > \alpha$, then θ' is accepted to be the next sample by setting $\theta^{(i+1)} = \theta'$, otherwise, it is rejected by setting $\theta^{(i+1)} = \theta^{(i)}$. In other words,

θ' is accepted with an acceptance probability $a = \min\{1, A\}$. In this way, the Markov chain $\{\theta^{(i)}\}$, where $i = 1, 2, \dots, N$, will be equivalent to N samples being drawn from p . We can calculate different kinds of statistics using $\{\theta^{(i)}\}$, *e.g.* the mean of any arbitrary function $f(\theta)$

$$\langle f(\theta) \rangle = \int d\theta f(\theta) p(\theta) \approx \frac{1}{N} \sum_{i=1}^N f(\theta^{(i)}). \quad (3.62)$$

The disadvantage of the MH algorithm is the production of correlated samples. In order to obtain a set of independent samples, we have to throw away the majority of samples and keep only every n th sample (known as thinning). In addition, we need a burn-in stage which is to throw away the first $N_{\text{burn-in}} \sim \mathcal{O}(10^3 - 10^4)$ samples because our first-guessed sample $\theta^{(0)}$ is normally bad in the sense that it is in a low-probability region. Eq. (3.62) will be exact if $N \rightarrow \infty$ but in practice the Markov chain is truncated at finite N . If $\theta^{(i)}$ are in low-probability region, Eq. (3.62) gives a poor approximation of the true mean on the left. Therefore, the burn-in stage is to make sure that the Markov chain enters the high-probability region and so Eq. (3.62) gives better estimate of the true mean. Moreover, it reduces the dependence on the starting value $\theta^{(0)}$.

Thermodynamic integration

It is also possible to calculate the evidence using MCMC with thermodynamic integration (TI). We define the power posteriors

$$p_\beta(\theta|d, \mathcal{H}, I) \equiv \frac{q_\beta(\theta)}{Z_\beta} \equiv \frac{p(d|\theta, \mathcal{H}, I)^\beta p(\theta|\mathcal{H}, I)}{Z_\beta(d|\mathcal{H}, I)}, \quad (3.63)$$

where $q_\beta(\theta)$ is the unnormalized power posteriors, $Z_\beta(d|\mathcal{H}, I) = \int d\theta q_\beta(\theta)$ is the normalization constant and β is known as the inverse temperature parameter. By definition, p_0 is simply the prior distribution with $Z_0 = 1$ while p_1 is the posterior distribution and $Z_1 = Z$ is the evidence of interest. Varying $\beta \in [0, 1]$ gives a continuous path between these two distributions. We consider the derivative of $\ln Z_\beta$ with respect to β ,

$$\begin{aligned} \frac{\partial \ln Z_\beta}{\partial \beta} &= \frac{1}{Z_\beta} \frac{\partial Z_\beta}{\partial \beta} \\ &= \int d\theta \frac{1}{Z_\beta} \frac{\partial q_\beta(\theta)}{\partial \beta} \\ &= \int d\theta \frac{\partial \ln q_\beta(\theta)}{\partial \beta} p_\beta \\ &= \left\langle \frac{\partial \ln q_\beta(\theta)}{\partial \beta} \right\rangle. \end{aligned} \quad (3.64)$$

Therefore, we have

$$\ln Z_1 = \ln Z_1 - \ln Z_0 = \int_0^1 d\beta \left\langle \frac{\partial \ln q_\beta(\theta)}{\partial \beta} \right\rangle. \quad (3.65)$$

The main idea of TI is to approximate this one-dimensional integral by some quadrature rule over a discrete β ladder. For each β_i (known as the temperature rungs), we can perform a MCMC run to obtain the integrand.

Obviously, the sources of error are twofold: the discretization of β and MCMC error. They can be reduced by increasing the number of temperature rungs and the number of MCMC samples per temperature rung respectively. However, this also increases the computational workload.

3.3.8 Reversible-jump Markov chain Monte Carlo

Reversible-jump Markov chain Monte Carlo (RJMCMC) is an extension to the MCMC methodology that allows simulation of the posterior distributions with varying number of dimensions, *i.e.* “the number of unknowns is something we don’t know” [81]. Suppose we have K candidate models $\mathcal{M} = \{M_1, M_2, \dots, M_K\}$ and each model M_k has an n_k -dimensional vector of unknown parameters θ_k . For simplicity M_k and k will be used interchangeably. The joint posterior of (k, θ_k) is given by

$$p(k, \theta_k | d, I) = \frac{p(d|k, \theta_k, I)p(\theta_k|k, I)P(k|I)}{\sum_{k=1}^K \int d\theta_k p(d|k, \theta_k, I)p(\theta_k|k, I)P(k|I)}, \quad (3.66)$$

where $p(d|k, \theta_k, I)$ is the likelihood of M_k , $p(\theta_k|k, I)$ is the prior PDF on θ_k for M_k and $P(k|I)$ is the prior probability for M_k . If we do not favour a particular model, then $P(k|I) = 1/K$. $p(k, \theta_k | d, I)$ can be factorized as,

$$p(k, \theta_k | d, I) = P(k|d, I)p(\theta_k|k, d, I), \quad (3.67)$$

where $P(k|d, I)$ is the posterior probability for M_k and $p(\theta_k|k, d, I)$ is the posterior PDF on θ_k for M_k .

To obtain $p(k, \theta_k | d, I)$, we can make use of the MH algorithm again to create a Markov chain $\{(k^{(1)}, \theta^{(1)}), (k^{(2)}, \theta^{(2)}), \dots\}$. The acceptance ratio is simply

$$A = \frac{p(k', \theta')}{p(k^{(i)}, \theta^{(i)})} \frac{Q(k^{(i)}, \theta^{(i)} | k', \theta')}{Q(k', \theta' | k^{(i)}, \theta^{(i)})}, \quad (3.68)$$

where Q is the chosen proposal distribution and prime indicates the next candidate. Q can be factorized as,

$$Q(k', \theta' | k^{(i)}, \theta^{(i)}) = Q(k' | k^{(i)})Q(\theta' | k', k^{(i)}, \theta^{(i)}). \quad (3.69)$$

The first factor is the jumping probability from $k^{(i)}$ to k' while the second term denotes the PDF of θ' given $k', k^{(i)}$ and $\theta^{(i)}$. If $k' > k^{(i)}$, it is called a birth move. If $k' < k^{(i)}$, it is called a death move. If $k' = k^{(i)}$, it is called an update move. Because dimension of k' may be different from $k^{(i)}$, a random vector $\vec{u}^{(i)}$ with dimension $d_{\vec{u}^{(i)}}$ is drawn from $Q(\vec{u}^{(i)} | k', k^{(i)}, \theta^{(i)})$ such that

$$(\theta', \vec{u}') = g_{k^{(i)} \rightarrow k'}(\theta^{(i)}, \vec{u}^{(i)}) \quad (3.70)$$

where $g_{k^{(i)} \rightarrow k'}$ is a bijection between $(\theta^{(i)}, \vec{u}^{(i)})$ and $(\theta', \vec{u}')$ with

$$n_{k'} + d_{\vec{u}'} = n_{k^{(i)}} + d_{\vec{u}^{(i)}}, \quad (3.71)$$

which is known as the dimension matching. Since the change of variable of a density function is given by,

$$Q(\theta^{(i)} | k^{(i)}, k', \theta')Q(\vec{u}^{(i)} | k', k^{(i)}, \theta^{(i)}) = Q(\theta' | k', k^{(i)}, \theta^{(i)})Q(\vec{u}' | k^{(i)}, k', \theta') \left| \frac{\partial g_{k^{(i)} \rightarrow k'}(\theta^{(i)}, \vec{u}^{(i)})}{\partial(\theta^{(i)}, \vec{u}^{(i)})} \right|, \quad (3.72)$$

where the last term is the absolute value of the determinant of the Jacobian matrix, the acceptance ratio becomes

$$A = \frac{p(k', \theta')}{p(k^{(i)}, \theta^{(i)})} \frac{Q(k^{(i)} | k')}{Q(k' | k^{(i)})} \frac{Q(\vec{u}' | k^{(i)}, k', \theta')}{Q(\vec{u}^{(i)} | k', k^{(i)}, \theta^{(i)})} \left| \frac{\partial g_{k^{(i)} \rightarrow k'}(\theta^{(i)}, \vec{u}^{(i)})}{\partial(\theta^{(i)}, \vec{u}^{(i)})} \right|. \quad (3.73)$$

The use of acceptance ratio is similar to the standard MH algorithm.

With $p(k, \theta_k | d, I)$, we have two approaches to make use of the result. The first one is model averaging which is to treat the whole variable dimension model as a whole and estimate any quantities of interest. The second approach is to choose the best submodel with the largest $P(k|d, I)$ which is simply the ratio of the iterations spent within each model.

CHAPTER 4

RINGDOWN

After the inspiral stage, the binary enters a strongly non-linear merger regime and afterwards forms a single remnant. An excited Kerr BH is a common final state of CBC including BBH, NSBH and BNS (if total mass M is high enough; see Chapter 6). It can be studied by BH perturbation theory assuming that the final BH is simply a perturbed state of a stationary Kerr BH. Ringdown corresponds to the final transient state in which characteristic GW radiation is emitted from this newly-formed highly-excited Kerr BH in the form of a linear superposition of damped sinusoids [82], namely the quasi-normal modes (QNMs),

$$h(t) = h_+ - ih_\times = \frac{M_f}{r} \sum_{lmn} \mathcal{A}_{lmn} {}_{-2}S_{lmn} e^{i\tilde{\omega}_{lmn}t}, \quad (4.1)$$

where M_f is the remnant final mass, r is the distance from source to detector, $n = 0, 1, \dots$ is the overtone index, $l = 0, 1, \dots, n-1$ is the polar eigenvalue, $m = -l, -l+1, \dots, l$ is the azimuthal eigenvalue, $\mathcal{A}_{lmn}$ is the complex mode amplitude, ${}_{-2}S_{lmn}$ is the spin-weighted spheroidal harmonics (SWSHs) and $\tilde{\omega}_{lmn} = \omega_{lmn} + i/\tau_{lmn}$ is the angular complex QNM frequency whose real part is the QNM frequency ω_{lmn} and imaginary part is the inverse of damping time. The dominant QNM excitation is $l = m = 2, n = 0$. After ringdown, a stationary Kerr BH is formed and no GWs will be emitted anymore which indicates the end of a CBC. ${}_{-2}S_{lmn} = {}_{-2}S_{lm}(a_f \tilde{\omega}_{lmn}, \iota, \psi)$ satisfies the equation

$$\left[\frac{d}{du} \left((1-u^2) \frac{d}{du} \right) + (a_f \tilde{\omega}_{lmn} u)^2 - 2sa_f \tilde{\omega}_{lmn} u + s + {}_sA_{lmn} - \frac{(m+su)^2}{1-u^2} \right] {}_sS_{lm}(a_f \tilde{\omega}_{lmn}, \iota, \psi) = 0, \quad (4.2)$$

where $u \equiv \cos \iota$, s is the spin weight which is -2 in case of gravitational perturbation, a_f is the Kerr rotation parameter, and ${}_sA_{lmn}$ is the complex separation constant (or the eigenvalue). In the Schwarzschild limit of $a_f \rightarrow 0$, the SWSHs reduce to the spin-weighted spherical harmonics ${}_sY_{lm}$ and ${}_sA_{lmn} = l(l+1) - s(s+1)$. For non-zero a_f , the SWSHs are superpositions of ${}_sY_{lm}$ with the same value of m but different values of $l' \neq l$. Leaver found a series solution to the SWSHs [83]

$${}_sS_{lm}(a_f \tilde{\omega}_{lmn}, u, \psi) = e^{im\psi} e^{a_f \tilde{\omega} u} (1+u)^{k-} (1-u)^{k+} \sum_{p=0}^{\infty} a_p (1+u)^p, \quad (4.3)$$

where $k_{\pm} \equiv |m \pm s|/2$. The expansion coefficients a_p can be obtained from the three recursion relation

$$\begin{aligned}\alpha_0 a_1 + \beta_0 a_0 &= 0 \\ \alpha_p a_{p+1} + \beta_p a_p + \gamma_p a_{p-1} &= 0 \quad p = 1, 2, \dots,\end{aligned}\tag{4.4}$$

with

$$\begin{aligned}\alpha_p &= -2(p+1)(p+2k_-+1) \\ \beta_p &= p(p-1) + 2p(k_-+k_++1 - 2a_f \tilde{\omega}_{lmn}) \\ &\quad - [2a_f \tilde{\omega}_{lmn}(2k_-+s+1) - (k_-+k_+)(k_-+k_++1)] - [(a_f \tilde{\omega}_{lmn})^2 + s(s+1) + {}_s A_{lmn}] \\ \gamma_p &= 2a_f \tilde{\omega}_{lmn}(p+k_-+k_++s).\end{aligned}\tag{4.5}$$

The separation constant ${}_s A_{lm}$ can be obtained by solving numerically the continued fraction equation [83],

$$\beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 - \frac{\alpha_1 \gamma_2}{\beta_2 - \frac{\alpha_2 \gamma_3}{\beta_3 - \dots}}} = 0.\tag{4.6}$$

4.1 Testing the no-hair conjecture

In GR, no-hair conjecture states that a stationary isolated BH is determined uniquely by its mass M_f , dimensionless spin $j_f = a_f/M_f$ and electric charge Q which is expected to be zero for astrophysical objects. In fact, a stationary isolated Kerr BH does not emit GWs so it is not possible to perform any tests on it. However, as a manifestation of this conjecture, ω_{lmn} and τ_{lmn} are expected to depend only on M_f and j_f . It provides a ‘smoking gun’ to probe the true nature of BH known as testing no-hair conjecture or BH spectroscopy. Because there are only two independent parameters, if one can obtain three independent measurements of QNM parameters, *e.g.* ω_{220} , ω_{330} and τ_{220} , we can perform a consistency test in which M_f and j_f are inferred from the first and second measurements and the consistency is checked against the third measurement. Nevertheless, independent extractions of at least three QNM parameters are expected to be possible only if the SNRs in the ringdown stage is in an order of 100 [82,84,85] which is archivable for the Einstein Telescope [75] or the space interferometer LISA [86,87]. For GW150914, the SNR in the ringdown phase was ≈ 7 [88]. Instead of performing the above consistency test, in our work [14], we look for violations of the BH no-hair conjecture by introducing possible deviations in the QNM parameters,

$$\begin{aligned}\omega_{lmn}(M_f, j_f) &\rightarrow (1 + \delta\hat{\omega}_{lmn})\omega_{lmn}(M_f, j_f) \\ \tau_{lmn}(M_f, j_f) &\rightarrow (1 + \delta\hat{\tau}_{lmn})\tau_{lmn}(M_f, j_f),\end{aligned}\tag{4.7}$$

where $\delta\hat{\omega}_{lmn}$ and $\delta\hat{\tau}_{lmn}$ are relative deviations that we include as additional degrees of freedom in our inference. This parameterization has the advantages of being agnostic to specific families of violations and, most importantly, to be uniquely defined in GR,

$$\delta\hat{\omega}_{lmn} = \delta\hat{\tau}_{lmn} = 0 \quad \forall l, m, n.\tag{4.8}$$

By sampling over $\{M_f, j_f, \delta\hat{\omega}_{220}\}$ as well as other parameters (which will be discussed in Sec. 4.3), M_f and j_f give the predicted QNM frequencies and damping times while $\delta\hat{\omega}_{220}$ can capture any non-GR effect.

Figure 4.1: Dimensionless QNM frequency $M_f \omega_{lmn}$ (top) and damping time τ_{lmn}/M_f (bottom) as a function of j_f from [91]. The legend shows different QNMs labelled by lmn in descending order of the magnitude at $j_f = 0$.

In previous studies, it has been shown that Einstein Telescope, a third-generation GW observatory, is able to test the no-hair conjecture with an accuracy of a few percent by combining $\mathcal{O}(10)$ ringdown signals from BHs with masses in the range of $500 - 1000 M_\odot$ at distance up to 50 Gpc [13]. In our work, we provide a proof of principle that the *existing* Advanced LIGO and Virgo interferometric detectors, operating at design sensitivity [89, 90], will be capable of testing the no-hair conjecture with an accuracy of a few percent by combining $\mathcal{O}(5)$ ringdown signals from stellar-mass BHs at distance up to 1 Gpc.

4.2 Model

Our ringdown model is that of [92], where a robust method was developed to characterize QNMs up to $l = 5$, including overtones, by making use of NR waveforms. This model has the form of Eq. (4.1) and is calibrated by fitting against 68 NR waveforms of non-spinning BH binaries with mass ratio ranging from 1 to 15 starting from a time $t = 10M$ after the peak luminosity of the dominant mode. For the dependence of the complex QNM frequency on M_f and j_f , we can make use of the package on [91] which gives the fitting formulae of the QNMs as shown in Fig. 4.1. M_f and j_f can be predicted from the initial non-spinning BH progenitors by a polynomial of the symmetric mass ratio $\eta \equiv m_1 m_2 / (m_1 + m_2)^2$ [93] or a power series in $m_1 - m_2$ [94]. Here we use fits from NR that a boundary condition is imposed such that the dimensionless spin $j_f \rightarrow 0$ and $M_f/M \rightarrow 1$ as $\eta \rightarrow 0$,

$$j_f = 3.4339\eta - 3.7988\eta^2 - 5.7733\eta^3 + 6.3780\eta^4$$

$$M_f/M = 1 - 0.046297\eta + 0.71006\eta^2 + 1.5028\eta^3 - 4.0124\eta^4 + 0.28448\eta^5, \quad (4.9)$$

which are shown in Fig. 4.2. So from Eq. (4.9), given the same total mass, M_f/M is the smallest in the case of an equal-mass binary (*i.e.* $\eta = 1/4$) where around 5% of the initial total mass is carried away by GWs. In our analysis, M_f and j_f are sampled over rather than set by Eq. (4.9). The complex mode amplitudes do not depend on the properties of the final remnant but are set by how the remnant is formed. In CBC, they depend on the properties of the initial progenitors. In the case of non-spinning BH progenitors, they are well-captured by

Figure 4.2: Fits from NR of M_f (left) and j_f (right) of the BH remnant as a function of η assuming non-spinning BH progenitors [92].

series expansions in the symmetric mass ratio η ,

$$\begin{aligned}
\mathcal{A}_{220}(\eta) &= 0.9252\eta + 0.1323\eta^2 \\
\mathcal{A}_{221}(\eta) &= 0.1275e^{5.3106i}\eta + 1.1882e^{0.4873i}\eta^2 + 8.2709e^{3.3895i}\eta^3 + 26.2329e^{0.1372i}\eta^4 \\
\mathcal{A}_{210}(\eta) &= \sqrt{1-4\eta}(0.4795e^{3.5587i}\eta + 1.1736e^{1.5679i}\eta^2 + 1.2303e^{6.0496i}\eta^3) \\
\mathcal{A}_{330}(\eta) &= \sqrt{1-4\eta}(0.4247e^{5.4979i}\eta + 1.4742e^{3.6524i}\eta^2 + 4.3139e^{6.0787i}\eta^3 + 15.7264e^{3.2053i}\eta^4) \\
\mathcal{A}_{331}(\eta) &= \sqrt{1-4\eta}(0.1480e^{2.9908i}\eta + 1.4874e^{0.5635i}\eta^2 + 10.1637e^{4.2348i}\eta^3 + 29.4786e^{1.7619i}\eta^4) \\
\mathcal{A}_{320}(\eta) &= 0.1957e^{5.8008i}\eta + 1.5830e^{3.2194i}\eta^2 + 5.0338e^{0.6843i}\eta^3 + 3.7366e^{4.1217i}\eta^4 \\
\mathcal{A}_{440}(\eta) &= 0.2531e^{1.5961i}\eta + 2.4040e^{5.1851i}\eta^2 + 14.7273e^{1.9953i}\eta^3 + 67.3624e^{4.9143i}\eta^4 \\
&\quad + 126.5858e^{1.8502i}\eta^5 \\
\mathcal{A}_{430}(\eta) &= \sqrt{1-4\eta}(0.0938e^{3.2607i}\eta + 0.8273e^{0.7704i}\eta^2 + 3.3385e^{4.8264i}\eta^3 + 4.6639e^{2.7047i}\eta^4) \\
\mathcal{A}_{550}(\eta) &= \sqrt{1-4\eta}(0.1548e^{5.3772i}\eta + 1.5091e^{2.5764i}\eta^2 + 8.9333e^{5.5995i}\eta^3 + 42.3431e^{2.1269i}\eta^4 \\
&\quad + 89.1947e^{5.3348i}\eta^5).
\end{aligned} \tag{4.10}$$

In particular, the mode amplitudes are complex so as to take the relative phase shifts between modes into accounts which were neglected in the previous models [95] and subsequent Bayesian analyses [13, 96] that were based on them. Such an inclusion leads to a significant improvement in faithfulness against NR waveforms. This waveform model has also been extended to the case of initial BH progenitors with non-zero but aligned spins [97]; the use of such a ringdown model in data analysis is left for future study.

4.3 Simulations

Both to establish the effective ringdown start time and in subsequent simulations of no-hair conjecture tests, Bayesian parameter estimation is performed. The simulated signals are numerical IMR waveforms taken from the publicly available SXS catalog [51, 52], with mass ratio $q = m_1/m_2$ in the interval $[1, 3]$, and negligible initial spins as well as negligible residual eccentricity (SXS:BBH:0001, SXS:BBH:0030, SXS:BBH:0169, SXS:BBH:0198). These are coherently injected into synthetic, stationary, Gaussian noise for a network of Advanced LIGO and Advanced Virgo detectors at design sensitivity [89, 90]. The injected total mass M is uniformly distributed in the interval $[50, 90] M_\odot$, and the sky location (α, δ) as well as the

Parameters	Symbols	Prior ranges
Final mass	M_f	uniform in $[5, 200] M_\odot$
Final spin	j_f	uniform in $[-1, 1]$
Mass ratio	q	uniform in $[1, 15]$
Sky location	(α, δ)	uniform on the 2-sphere
Luminosity distance	r	constant number density in comoving volume; $[1, 1000]$ Mpc
Orientation	(ι, ψ)	uniform on the 2-sphere
Reference time	t_c	uniform in $[t_{\text{peak}} - 0.1 \text{ s}, t_{\text{peak}} + 0.1 \text{ s}]$
Reference phase	ϕ_c	uniform in $[0, 2\pi]$

Table 4.1: Prior ranges of different ringdown parameters.

orientation of the orbital plane (ι, ψ) at some reference time t_c and reference phase ϕ_c are uniformly distributed on the sphere. Luminosity distances r are chosen such that the total SNR in the IMR signal approximately equals 100, which is a plausible value for signals similar to GW150914 [3] assuming the Advanced LIGO-Virgo network at full sensitivity. This sets the average SNR contained just in the ringdown phase of our dataset to 15, if the start time is chosen to be $16M$ after the time at which the GW strain peaks (as will be demonstrated in Sec. 4.4, this is indeed a reasonable choice). By comparison, with the same choice of start time, the SNR in the ringdown of GW150914 with detectors at design sensitivity would have been $\text{SNR}_{\text{ringdown}} \approx 17$ [18].

The template waveforms used in our Bayesian analyses follow the aforementioned ringdown model, augmented with a windowing procedure for the start time, as explained in Sec. 4.4. Nested sampling (see Sec. 3.3.6) is done over 10 parameters:

$$\{M_f, j_f, q, \alpha, \delta, \iota, \psi, r, t_c, \phi_c\}. \quad (4.11)$$

Hence only parameters associated with the ringdown waveform are sampled over, with the mass ratio q determining the mode amplitudes. Bayesian inference is done using the LALInference library [98]. Priors are chosen to be uniform in $[5, 200] M_\odot$ for M_f , uniform in $[-1, 1]$ for j_f , uniform in $[1, 15]$ for q , and uniform in $[0, 2\pi]$ for ϕ_c . A constant number density in comoving volume sets the prior for the sky location and the distance, with a distance range of $[1, 1000]$ Mpc. The priors on (ι, ψ) are chosen to be uniform on the 2-sphere (with these angles being generated from the same distribution also for the simulation set). (For the SNRs considered, the impact of the specific shapes of the prior distributions has little impact on our results.) t_c is uniformly distributed within $[t_{\text{peak}} - 0.1 \text{ s}, t_{\text{peak}} + 0.1 \text{ s}]$ where t_{peak} is the peak time of the strain at which the signal is detected. These prior ranges are summarised in Table 4.1.

4.4 Start time

After the BBH merger, the system is still in a highly non-linear regime of gravity. The description of the GWs in the form of the superposition of QNMs is valid only if the system enters the linear ringdown regime where Eq. (4.1) is reliably to be used. However, the time at which the transition between the non-linear to the linear regime happens is not well-defined. For instance, in an early study of the ringdown, Fig. 4.3 shows how the inference of GW150914 with a single damped sinusoid on the QNM central frequency and characteristic time changes quite dramatically depending on the assumed time at which the transition occurs. Therefore it is critical to make a reasonable choice for the time at which the remnant black hole can be treated perturbatively and assess the effectiveness of such a choice.

Figure 4.3: The 90% credible regions in the joint posterior distributions for ω_{220} and τ_{220} using a single damped sinusoid assuming the start times $t_0 = t_M + 1, 3, 5, 6.5$ ms, where t_M is the merger time of the MAP waveform for GW150914. The black solid line shows the 90% credible region as derived from the posterior distributions of the remnant mass and spin parameters [18].

We choose the start time for the ringdown t_{start} from the analysis of numerical IMR waveforms added to stationary Gaussian noise with a power spectral density as expected for Advanced LIGO and Virgo at design sensitivity. To isolate the ringdown, we apply to the data a Planck window [99] whose starting time is varied in discrete steps over a range $[10, 20]$ M after the peak time of the strain in each detector. The choice of a specific windowing function has no significant impact on the analysis as long as the frequency range of interest is not altered. We indeed verified that different tapering functions give nearly identical results. The peak time t_{peak} itself can be estimated using analysis methods that can measure amplitudes and arrival times of a signal inside a detector, without relying on specific GR waveform models, such as *BayesWave* [100]. We choose a rise time for the Planck window of 1 ms, as we find that this choice gives a good compromise between the need to preserve the SNR and to avoid Gibbs phenomena. Consider a simulated signal with total mass M and mass ratio q , and a choice of window start time, e.g. $t_{\text{start}} = t_{\text{peak}} + \kappa M$ for some κ in $[10, 20]$. We then apply a similar window on the ringdown template waveforms, letting them start at $\kappa M'$ after the peak strain, where the mass M' is obtained from the sampled values M'_f and q' through Eq. (4.9). This leads to posterior PDFs for all parameters, and in particular for M_f and j_f . Our criterion to select the start time for the ringdown is built by minimizing the bias in the recovered parameters of the final object, while avoiding to select an arbitrarily large start time. Although large start times would ensure the validity of the linearized approximation employed in the waveform template, they would also drastically reduce the signal SNR, thus resulting in a poor estimation of the final parameters. The equilibrium point in this trade-off, arrived at as explained below, will ensure the analysis to take place in the linearized regime where our model is valid, while still allowing a precise estimation of the measured parameters. By looking at the covariance

between M_f and j_f and at the distance (induced by the covariance metric) between the true values M_f^I , j_f^I and the mean values $\bar{M}_f^I(\kappa)$ and $\bar{j}_f^I(\kappa)$, for each simulated signal I , we define the functions

$$\begin{aligned}\mathcal{B}(\kappa, I) &\equiv \sqrt{D_I^2(\kappa) + \det \mathbf{C}(\kappa)_I}, \\ D_I^2(\kappa) &\equiv \Delta \mathbf{x}(\kappa)_I^T \mathbf{C}^{-1}(\kappa)_I \Delta \mathbf{x}(\kappa)_I,\end{aligned}\quad (4.12)$$

where $\mathbf{C}(\kappa)_I$ is the two-dimensional covariance matrix between the posterior samples for M_f and j_f , $D_I(\kappa)$ is the (covariance induced) distance between the mean and the injected values and we defined the vector

$$\Delta \mathbf{x}(\kappa)_I^T = [(\bar{M}_f^I(\kappa) - M_f^I)/M_\odot, \bar{j}_f^I(\kappa) - j_f^I]. \quad (4.13)$$

The statistical uncertainty (larger for large start times), quantified by $\det \mathbf{C}(\kappa)_I$, is controlled by the SNR left in the ringdown part of the signal when the preceding stages are cut from the analysis. The distance $D_I(\kappa)$ quantifies the systematic uncertainty in the recovered parameters and is a proxy for the mismatch between the linear ringdown model and the non-linear signal. We thus let the effective ringdown start time be the one that minimizes $\mathcal{B}(\kappa)$ (thus minimizing the combination of statistical and systematic uncertainties), defined as the average of $\mathcal{B}(\kappa, I)$ over all simulated signals. The dataset consisted of 12 simulations at 11 different, equally spaced, values of $\kappa \in [10, 20]$, thus employing a total of 132 simulations. Fig. 4.4 illustrates the procedure by showing 90% credible regions for M_f and j_f , together with the value of the averaged $\mathcal{B}(\kappa)$ as a function of the ringdown start time, for a particular system which has an initial total mass $M = 72 M_\odot$ and a mass ratio $q = 1$. The value of κ minimizing $\mathcal{B}(\kappa, I)$ is $\kappa = 16$, which implies an effective ringdown start time of $t_{\text{start}} = t_{\text{peak}} + 16 M$ after the peak strain of the signal. This is consistent with the conclusion stated in an independent study by Bhagwat *et al.*, using a “Kerrness” measure on a single GW150914-like numerical signal [101]. The selection of the ringdown start time at $t_{\text{start}} = t_{\text{peak}} + 16 M$ uniquely determines the placement of the time domain Planck window. When dealing with real signals, the window is initially applied once to the data with $t_{\text{start}} = t_{\text{peak}} + 16 M_{\text{IMR}}$, with t_{peak} from a model-independent reconstruction, and M_{IMR} from a routine estimate (before performing our ringdown-only analysis) using an IMR model. While the posterior distribution for M (obtained from the sampled values of M_f and q through fitting formulae) is being explored, the window on the template model is instead recalculated and reapplied at each step, with its starting time set to the proposed value $16 M$ after the peak strain.

4.5 Results

We can test no-hair conjecture by introducing the relative deviation in the QNM parameters as in Eq. (7.1) and measuring at least three independent parameters characterizing the remnant geometry, which we chose to be M_f , j_f and $\delta\hat{\omega}_{220}$ (or $\delta\hat{\tau}_{220}$). In addition, the algorithm also samples all the other parameters in Table 4.1. The priors are unchanged except on M_f and j_f where we restrict to values inferred from M and q contained in the 90% credible intervals obtained from an earlier analysis including inspiral and merger using Eq. (4.9). The prior on the deviation parameters is chosen to be uniform in $[-1, 1]$. We consider GR signals with mass ratio $q \leq 3$. Figs. 4.5 shows the results of an analysis performed on a set of IMR signals added to stationary Gaussian noise. The PDFs are combined using Eq. (3.44). Upper bounds on the departures from the GR predictions for ω_{220} are smaller than $\sim 1.5\%$ at the 90% credible level already with six sources, while upper bounds on deviations from τ_{220} predictions are smaller than $\mathcal{O}(10\%)$. On the selected dataset higher mode deviations on both frequency and damping time are essentially unconstrained.

Figure 4.4: Estimated median values and 90% credible regions for final mass M_f (top panel) and spin j_f (central panel) as a function of the start time of the ringdown with respect to the strain peak time t_{peak} for a simulation with $M_f = 68.5 M_\odot$ and $j_f = 0.686$ (corresponding to $q = 1$). The bottom panel reports the value of the function $\mathcal{B}(\kappa)$, averaged over all simulations. The gray box highlights the value of $t_{\text{start}} - t_{\text{peak}} = 16 M$ for which $\mathcal{B}(\kappa)$ is at a minimum.

4.6 Conclusions

In this work [14], we demonstrated that observationally testing the black hole no-hair conjecture is possible within the next few years, once the LIGO-Virgo detector network reaches its design sensitivity. The ability to isolate the quasi-linear ringdown regime from the non-linear merger stage of the coalescence process enables estimating the parameters characterizing the ringdown. This also allows the identification of the time of the transition to be $16 M$ after the peak strain, M being the total mass of the merging system. Following our procedure, we showed that, with just $\mathcal{O}(5)$ plausible signals, violations from the no-hair conjecture, seen as changes in the dominant QNM frequency and damping time, can be constrained to be smaller than, respectively, $\sim 1.5\%$ and $\sim 10\%$ at 90% confidence. The results presented in this work can be extended to the recent spin-aligned ringdown model in [97].

Figure 4.5: Measurements of $\delta\hat{\omega}_{220}$ (Top panel) and $\delta\hat{\tau}_{220}$ (Bottom panel) characterizing a departure of the dominant QNM frequency and damping time from its GR value respectively, on a set of numerical simulations as described in the text. Left panel: evolution of medians and 90% credible intervals from the joint posterior distribution. Right panel: posterior PDFs for each individual signal (dotted lines), and the joint posterior distribution (solid line). With 5 detections the upper bound on $\delta\hat{\omega}_{220}$ is smaller than $\sim 1.5\%$ at 90% confidence while the upper bound on $\delta\hat{\tau}_{220}$ is smaller than $\mathcal{O}(10\%)$.

CHAPTER 5

ECHOES

How certain can we be that objects in CBC are the standard BHs of classical GR? In quantum gravity, Hawking’s information paradox has led to the suggestion of Planck-scale modifications of BH horizons (firewalls [102]) and other alterations of BH structure (fuzzballs [103]). In cosmology, dark matter particles have been proposed that congregate into star-like objects [104]. Yet another possibility concerns stars whose interior consists of self-repulsive, de Sitter spacetime, surrounded by a shell of ordinary matter (gravastars [105]). There is an idea of boson stars, macroscopic objects made up of scalar fields [106]. A variety of alternative objects, called BH mimickers, have been proposed (see [107] for an overview). When such objects are part of a binary system that undergoes coalescence, anomalous effects associated with them can leave an imprint upon the observed GW signal, including tidal effects [108, 109]; dynamical friction as well as resonant excitations due to dark matter clouds surrounding the objects [110]; violation of the no-hair conjecture [14, 111]; and finally through GW “echoes” that might follow after ringdown. They are formed if the final remnant has either no horizon or a non-totally absorbing surface above the would-be Schwarzschild radius. In the case of no horizon, any incoming GWs (*e.g.* resulting from the merger) will not be completely absorbed by the exotic compact object (ECO) but trapped in the potential well formed by the photon sphere $r = 3M$ ¹², with wave packets leaking out to infinity at regular time intervals. In the case of a non-totally absorbing surface above the would-be Schwarzschild radius, the potential well is instead formed by this inner barrier and the photon sphere.

We consider models in which the ECO has a radius r_0 that can be arbitrarily close to the would-be Schwarzschild radius [112],

$$r_0 = 2M + l, \quad (5.1)$$

with $l \ll M$. In the spherically symmetric case, the line element for these models is

$$ds^2 = -F(r) dt^2 + \frac{1}{B(r)} dr^2 + r^2 d\Omega^2, \quad (5.2)$$

where F and B are model-dependent. All matter is localized in the region $r \leq r_0$. In the region $r \geq r_0$, Birkhoff’s theorem guarantees that the metric is Schwarzschild metric (*i.e.* $F(r) = B(r) = 1 - 2M/r$). The time delay between two consecutive echoes is roughly estimated

¹²In GR and for Schwarzschild BH, $r = 3M$ is called light ring or photon sphere at which the gravity is strong enough for photon to form an unstable circular orbit. A little perturbation can cause the photon either being absorbed by the BH or sent off to infinity.

Figure 5.1: Qualitative features of the effective potential felt by the perturbations of a Schwarzschild BH compared to the one of wormholes and of star-like ECOs with a regular center. In the wormhole case, modes can be trapped between the photon spheres in the two “universes”. In the star-like case, modes are trapped between the photon sphere and the centrifugal barrier near the center of the star [112].

to be the time that light takes for a round trip between the potential barriers,

$$\Delta t \sim 2 \int_{r_{\min}}^{3M} \frac{dr}{\sqrt{FB}}, \quad (5.3)$$

where $r_{\min}$ is the location of the minimum of the potential shown in Fig. 5.1. For an ECO with mass M and a microscopic correction at the horizon scale of size l , the time between echoes tends to be constant, and is well-approximated by [112]

$$\Delta t \sim nM \log \frac{M}{l}, \quad (5.4)$$

with n a factor of order unity that is determined by the nature of the ECO (*e.g.* $n = 8$ for a wormhole, $n = 6$ for a gravastar, and $n = 4$ for an empty shell). As an example, taking M to be the detector frame mass of the remnant object resulting from the first GW detection GW150914 ($M \approx 65 M_{\odot}$) [88], setting $n = 4$, and identifying l with the Planck length, one has $\Delta t \approx 117$ ms. This is much longer than the duration of the ringdown of the remnant (~ 3 ms), but at the same time sufficiently short that it would be practical to search for echoes immediately following the main IMR signal.

To search for echoes, template-based methods were proposed using a heuristic expression for the echo waveforms in terms of Δt as well as a characteristic frequency, a damping factor, and a widening factor between successive echoes [113–115]. Though expressions exist for echo

waveforms from selected ECOs under various assumptions [112,116], concrete calculations have so far only been exploratory [117]. Moreover, there may well be other types of objects that also cause echoes but have not yet been envisaged. For this reason, it is desirable to have a generic search for echoes which can capture and characterize a wide variety of different waveform morphologies [118].

5.1 Morphology-independent search

A commonly used method to search for and reconstruct GW signals of a priori unknown form is through the **BayesWave** algorithm [100,119]. The output of a network of detectors, s , is written as

$$s = R * h + n_g + g, \quad (5.5)$$

where R is the response of the network to GWs, h is the signal, g denotes instrumental transients or glitches, and n_g is a stationary Gaussian noise component. Given a network of N_d detectors, the search is carried out by comparing Bayesian evidences of the following three hypotheses:

1. Signal hypothesis: signal model + noise model,
2. Glitch hypothesis: glitch model + noise model, and
3. Noise hypothesis: noise model.

The noise model consists of colored Gaussian noise whose power spectral density is computed using a combination of smooth spline curves and a collection of Lorentzians to fit sharp spectral features, while both signal and glitch models are characterized by a superposition of some basis functions, which will be explained in Sec. 5.2. Any signals present in the data will be coherent in the detector network in a sense that there exists a unique set of extrinsic parameters (sky location and polarization angle: see Sec. 3.1). Given a detector I the signal model in the frequency domain takes the form

$$(R * h)_I(f) = (F_+^I(\theta, \phi, \psi)h_+(f) + F_\times^I(\theta, \phi, \psi)h_\times(f))e^{2\pi i f \Delta t_I(\theta, \phi)}, \quad (5.6)$$

where $h_\times = \epsilon h_+ e^{i\pi/2}$ with the relative amplitude ϵ between h_+ and $h_\times$. The sky location (θ, ϕ) and the polarization angle ψ are consistent across detectors, whose beam pattern functions are denoted by F_+^I and $F_\times^I$. $\Delta t(\theta, \phi)$ is the time delay between the geocentric and detector arrival times. h_+ is decomposed into a sum of N basis functions. If each of the basis function has N_w parameters, typically a coherent signal can be described by $N_w N$ intrinsic parameters plus 4 extrinsic parameters (*i.e.* ϵ, θ, ϕ and ψ). In contrast, glitch model takes $N_w N$ parameters to describe the coherent signal in each detector and so in total $N_w N_d N$ parameters. Hence, when $N_d > 1$ and a signal is present, the signal hypothesis will be preferred over the glitch hypothesis because it enables a more parsimonious description. In other words, the signal hypothesis has a higher Bayesian evidence due to the penalty of Occam's razor acting on the glitch hypothesis.

For each of the three hypotheses, the corresponding parameter space is sampled over using RJMCMC algorithm, in which the number of basis functions N is free to vary (see Sec. 3.3.8). Bayesian evidences for the three hypotheses are then estimated by means of thermodynamic integration, giving the Bayes factors $B_{S/N}$ and $B_{S/G}$ for the signal versus noise and signal versus glitch hypotheses respectively. If an astrophysical signal present in the data, $B_{S/N}$ and $B_{S/G}$ are much larger than 1.

5.2 Basis functions

The choice of basis functions for signal and glitch models is not unique. Due to their simplicity, sine-Gaussians (or Morlet-Gabor wavelets) were originally employed and they have been shown to lead to efficient detection [120, 121] and reconstruction [122, 123] of a wide range of signal morphologies, though more options have been explored [124]. In the time domain, the sine-Gaussian is a function of 5 parameters (*i.e.* $N_w = 5$) and takes the form of

$$\Psi(A, f_0, t_0, \tau, \phi_0; t) = Ae^{-\left(\frac{t-t_0}{\tau}\right)^2} \cos(2\pi f_0(t - t_0) + \phi_0), \quad (5.7)$$

where A is the amplitude, f_0 and t_0 are the central frequency and time respectively, τ is the decay time and ϕ_0 is the phase shift. The Fourier transform of Eq. (5.7) is given by

$$\tilde{\Psi}(f) = \frac{AQ}{4\sqrt{\pi}f_0} e^{-\frac{Q^2}{4}\left(\frac{f}{f_0}-1\right)^2} \left(e^{i(2\pi f t_0 - \phi_0)} + e^{-Q^2 \frac{f}{f_0}} e^{i(2\pi f t_0 + \phi_0)} \right), \quad (5.8)$$

with the quality factor $Q = 2\pi f_0 \tau$. In our work and for the study of echoes we propose generalized wavelets to be the basis functions which are “combs” of sine-Gaussians, characterized by a time separation Δt between the individual sine-Gaussians as well as a fixed phase shift $\Delta\phi$ between them, an amplitude damping factor γ , and a widening factor w . The first sine-Gaussian in the generalized wavelet takes the same form as in Eq. (5.7). For subsequent n -th sine-Gaussian, it has the same central frequency but an amplitude $A\gamma^n$, a central time $t_0 + n\Delta t$, a decay time τw^n and phase shift $\phi_0 + n\Delta\phi$. So all together, the generalized wavelet is a function of 9 parameters (*i.e.* $N_w = 9$),

$$\begin{aligned} \Psi(A, f_0, t_0, \tau, \phi_0, \Delta t, \Delta\phi, \gamma, w; t) &= \sum_{n=0}^{N_G-1} \Psi_n(A\gamma^n, f_0, t_0 + n\Delta t, \tau w^n, \phi_0 + n\Delta\phi; t) \\ &= \sum_{n=0}^{N_G-1} \gamma^n A e^{-\left(\frac{t-(t_0+n\Delta t)}{w^n \tau}\right)^2} \cos(2\pi f_0(t - (t_0 + n\Delta t)) + \phi_0 + n\Delta\phi), \end{aligned} \quad (5.9)$$

where N_G is the number of sine-Gaussians in each generalized wavelet. In our work, we choose $N_G = 5$. Exponential damping at late times as well as widening is a feature of linearized calculations, and is also seen in NR simulations [125]. Even though actual echo signals are unlikely to resemble any single generalized wavelet and may not even have well-defined values for any of the aforementioned quantities, we do expect superpositions of generalized wavelets to be able to capture a wide variety of physical echo waveforms. Moreover, one can assume the distribution of samples over the generalized wavelet parameter space to yield basic information about the structure of the echoes signal, which should then be of help in identifying the nature of the object that is emitting them.

5.3 Simulations

In order to test the algorithm, we prepared a simulated stationary Gaussian noise for a network of two Advanced LIGO detectors at the predicted design sensitivity [90]. To this synthetic noise, we coherently add our simulated signal:

- (a) A single generalized wavelet as in Eq. (5.9) with $f_0 = 166.7$ Hz, $\tau = 0.0095$ s, $\phi_0 = 0$, $\Delta t = 0.04$ s, $\gamma = 0.7$ and $w = 1.2$, and

Figure 5.2: Two injections involving 10 echoes, generalized wavelet (left) with $f_0 = 166.7\text{Hz}$, $\tau = 0.0095\text{s}$, $\phi_0 = 0$, $\Delta t = 0.04\text{s}$, $\gamma = 0.7$ and $w = 1.2$, and a numerically-solved toy model involving the inspiral of a particle in a Kerr spacetime with mass ratio $q = 1000$ and Neumann reflective boundary conditions just outside the horizon (right).

- (b) A numerically-solved inspiral toy model involving an inspiral of a particle in a Kerr spacetime with mass ratio $q = 1000$ and Neumann reflective boundary conditions just above the horizon.

The signals are shown in Fig. 5.2. For both simulated signals, the amplitudes are chosen such that the combined SNR in all echoes is 8, 12, 18 and 25. The higher values correspond to the SNR in the ringdown signal of a GW detection like GW150914 [126] under the assumption that it would be seen in Advanced LIGO at final design sensitivity, whereas an SNR of 8 roughly equals the SNR that the ringdown actually had for GW150914 [18]. For both types of simulated signals, 10 echoes are injected (in reality one would expect infinitely many but only a finite number will be detectable). Case (a) has a well-defined damping factor γ and widening factor w , allowing us to establish that the method works as intended, by ascertaining that these parameters are recovered correctly. In case (b), γ and w may not have rigorous meaning, but the distributions on parameter space that are obtained should be indicative of the physics involved; moreover, the peaks of their distributions should correspond to what one estimates from a visual inspection of the signal. In the latter case, the stretch of data analyzed excludes the main IMR signal, as one would also do in reality.

5.3.1 Priors

In both cases the first echo is searched for in a window for t_0 that has a width of 0.5s. For the other parameters the prior distributions are flat in $f_0 \in [20, 1024]\text{ Hz}$ (respectively the detectors' lower cut-off frequency and half the sampling rate), $Q \in [2, 40]$ (so τ takes values roughly between $3 \times 10^{-4}\text{ s}$ and $2 \times 10^{-1}\text{ s}$), $\phi_0 \in [0, 2\pi]$, $\Delta t \in [0, 0.25]\text{ s}$, $\gamma \in [0, 1]$, $w \in [1, 2]$, and $\Delta\phi \in [0, 2\pi]$. The prior distribution of amplitude A is inferred from the SNR prior distribution which is different for signal and glitch models. For astrophysical GW signals we expect the sources to be distributed roughly uniform in volume. It implies a prior on the distance D which scales as $p(D) \sim D^2$. Since the SNR of a signal scales as $\text{SNR} \sim 1/D$, the corresponding prior on SNR scales as $p(\text{SNR}) \sim \text{SNR}^{-4}$. Therefore, for signal model, we choose

$$p_{\text{signal}}(\text{SNR}) = \frac{3\text{SNR}}{4\text{SNR}_*^2 \left(1 + \frac{\text{SNR}}{4\text{SNR}_*}\right)^5}, \quad (5.10)$$

Figure 5.3: The SNR distributions adopted for the signal and glitch models, which are Eqs. (5.10) and (5.11) respectively. SNR_* is chosen to be 5.

where SNR_* is the peak of the distribution. At large SNR, it scales as $p(\text{SNR}) \sim \text{SNR}^{-4}$. For the glitch model, the prior attempts to capture the fact that loud glitches are less common than quiet ones, but very quiet glitches are indistinguishable from noise. Empirically we choose

$$p_{\text{glitch}}(\text{SNR}) = \frac{\text{SNR}}{2\text{SNR}_*^2 \left(1 + \frac{\text{SNR}}{2\text{SNR}_*}\right)^3}. \quad (5.11)$$

In both distributions, SNR_* is chosen to be 5. Fig. 5.3 shows the plot of these distributions. By construction, both distributions smoothly go to zero for $\text{SNR} \rightarrow 0$ as well as $\text{SNR} \rightarrow \infty$. This avoids large numbers of low-amplitude wavelets to be included in the reconstruction. Moreover, the SNR distribution in the glitch model has a heavier tail than that in the signal model because if we observe something with $\text{SNR} > 50$, it is more likely to be a loud glitch than a real signal.

To infer the amplitude, we calculate the SNR of a single sine-Gaussian, Eq. (5.7), using Eq. (3.25) and obtain

$$\text{SNR}_{\text{sine-Gaussian}} = \frac{A\sqrt{Q}}{\sqrt{2\sqrt{2\pi}f_0 S_n(f_0)}}, \quad (5.12)$$

where $S_n(f)$ is the one-sided PSD of the noise. For the generalized wavelet, Eq. (5.9), the squared SNR is given by

$$\begin{aligned} \text{SNR}_{\text{generalized}}^2 &\equiv \left(\tilde{\Psi}|\tilde{\Psi}\right) = \sum_{i=0}^{N_G-1} \sum_{j=0}^{N_G-1} \left(\tilde{\Psi}_i|\tilde{\Psi}_j\right) \\ &= \sum_{i=0}^{N_G-1} \left(\tilde{\Psi}_i|\tilde{\Psi}_i\right) + 2 \sum_{i=0}^{N_G-1} \sum_{j=i+1}^{N_G-1} \left(\tilde{\Psi}_i|\tilde{\Psi}_j\right), \end{aligned} \quad (5.13)$$

Parameters	Priors
t_0	uniform in a window of 0.5s
f_0	uniform in [20, 1024] Hz
Q	uniform in [2, 40]
ϕ_0	uniform in $[0, 2\pi]$
A	inferred from SNR distributions in Eqs. (5.10) and (5.11)
Δt	uniform in $[0, 0.4]$ s
$\Delta\phi$	uniform in $[0, 2\pi]$
γ	uniform in $[0, 1]$
w	uniform in $[1, 2]$

Table 5.1: Priors of the morphology-independent search for echoes.

Figure 5.4: The distribution of samples for the case where the injected signal is a comb of sine-Gaussians, damping factor γ against the time Δt between echoes (left) and widening factor w against Δt (right). The colors indicate the number of samples per pixel, while the dashed lines show the injected values of parameters.

where the noise weighted inner product is defined in Eq. (3.22). The first term gives the contributions of the individual sine-Gaussians:

$$\left(\tilde{\Psi}_i|\tilde{\Psi}_i\right) = \left(\tilde{\Psi}_0|\tilde{\Psi}_0\right)(\gamma^2 w)^i, \quad (5.14)$$

where $\left(\tilde{\Psi}_0|\tilde{\Psi}_0\right)$ is simply the square of Eq. (5.12) while the second term gives the contributions of the overlap between sine-Gaussians,

$$\left(\tilde{\Psi}_i|\tilde{\Psi}_j\right) = \left(\tilde{\Psi}_0|\tilde{\Psi}_0\right) \frac{\sqrt{2}(\gamma w)^{i+j}}{\sqrt{w^{2i} + w^{2j}}} e^{\frac{-(i-j)^2 4\pi^2 f_0^2 \Delta t^2}{Q^2(w^{2i} + w^{2j})}} e^{i(\Delta\phi - 2\pi f_0 \Delta t)(i-j)}. \quad (5.15)$$

Table 5.1 gives us the summary of the above priors.

5.3.2 Parameter estimation

Fig. 5.4 shows the distribution of samples for case (a), for an SNR of 25 and injected echo-related parameters $\Delta t = 0.04$ s, $\gamma = 0.7$, and $w = 1.2$. These are measured correctly, with peak values and standard deviations $\Delta t = 0.040 \pm 0.007$ s, $\gamma = 0.69 \pm 0.05$, and $w = 1.16 \pm 0.09$. Fig. 5.5 shows the distribution of samples for case (b), again for an SNR of 25; visual inspection

Figure 5.5: Similar to Fig. 5.4 but the injected signal is the inspiral toy model.

Figure 5.6: Injected and recovered strain in one of the detectors for the inspiral toy model. The reconstructed signal (shaded orange band showing 90% credible interval) is indeed consistent with the signal that has been injected (solid maroon line). For completeness we also show the full detector data containing both signal and noise (gray background).

of the signal in Fig. 5.2 indicates similar values for Δt , γ , and w as for case (a), and these are indeed the values where sample distributions have their main peaks. The peak values and standard deviations are $\Delta t = 0.040 \pm 0.007s$, $\gamma = 0.71 \pm 0.11$, and $w = 1.12 \pm 0.12$. The distribution of $(w, \Delta t)$ samples also shows secondary peaks at $3\Delta t$ and $5\Delta t$. These correspond to secondary peaks with $\gamma \approx 0$ in $(\gamma, \Delta t)$ space, which are cases where essentially only one echo was found. However, the secondary modes are considerably weaker than the main one.

Fig. 5.6 shows that the recovered echoes signal is indeed consistent with what has been injected.

Figure 5.7: Background distributions for the (log) Bayes factors $B_{S/G}$ (left) and $B_{S/N}$ (right), containing 380 trials. The dashed lines show the values of these quantities for the injection of echoes from the inspiral toy model with SNRs of 8, 12, 18, and 25.

5.3.3 Background distribution

In existing GW data analysis implementations, the (log) Bayes factor can usually not be treated as a stand-alone idealized quantity. Both in unmodeled searches [120, 121], and in template-based inference [127], log Bayes factors tend to depend sensitively on the recovered SNR and can have relatively large values for SNRs that are below the detection threshold. For this reason, in our study, we do not use the Bayes factors computed from the data as the final products. Instead we factor in the “prior odds” to help normalize these Bayes factor values in order to confidently detect echoes. Following the common practice, we compare our foreground values of $\log B_{S/N}$ and $\log B_{S/G}$ to their respective background distributions in which these quantities are computed on stretches of detector noise, effectively introducing such “prior odds” [78, 128, 129]. These are shown in Fig. 5.7, together with the values obtained from the injection of echoes for the inspiral toy model. For all simulated signals considered here we find that, starting from $\text{SNR} = 12$, $\log B_{S/G}$ and $\log B_{S/N}$ are above their respective backgrounds; hence trains of echoes with this loudness would be detected with confidence. It is worth noting that very similar Bayes factors are obtained with the original **BayesWave** algorithm, which instead of the generalized wavelets of Eq. (5.9) uses the standard Morlet-Gabor wavelets of Eq. (5.7). Hence the use of generalized wavelets does not significantly improve detection. However, the generalized wavelets allow for the *characterization* of echoes, as in Figs. 5.4 and 5.5.

5.4 Search on GWTC-1

Here we apply this analysis method to all significant events in the first Gravitational Wave Transient Catalog (GWTC-1) [8], which comprises the signals from BBH and BNS coalescences found during the first and second observing runs of Advanced LIGO and Advanced Virgo.

5.4.1 Setup

In analyzing the stretches of data immediately preceding the events in GWTC-1 (for background calculation) or immediately after them (to search for echoes), we use the same priors as Table 5.1 but we take f_0 to be uniform in the interval $[30, 1024]$ Hz and Δt to be uniform in $[0, 0.7]$ s. For the prior of the central time of the first sine-Gaussian in a generalized wavelet, we want to start analyzing at a time that is safely beyond the plausible duration of the ringdown of

the remnant object. Let t_{event} be the arrival time for a given binary coalescence event as given in [8]; then we take t_0 to be uniform in $[t_{\text{event}} + 4\tau_{220}, t_{\text{event}} + 4\tau_{220} + 0.5]$ s. The value for τ_{220} is a conservatively long estimate for the decay time of the 220 mode in the ringdown, using the fitting formula $\tau_{220}(M_f, j_f, z)$ of [82],

$$\tau_{220} = 2M_f(1+z) \frac{0.7000 + 1.4187(1-j_f)^{-0.4990}}{1.5251 - 1.1568(1-j_f)^{0.1292}}, \quad (5.16)$$

where for the final mass M_f , the final spin j_f , and the redshift z we take values at the upper bounds of the 90% credible intervals listed in [8]; typically this comes to a few milliseconds. We note that our choices for parameter prior ranges, though pertaining to generalized wavelet decompositions rather than waveform templates, include the corresponding values for t_0 , γ , and A at which the template-based analysis of [113] claimed tentative evidence for echoes related to GW150914, GW151012, and GW151226.

To construct background distributions we use stretches of data preceding each coalescence event in GWTC-1 in the following way. In the interval between 1050 s and 250 s before the GPS time of a binary coalescence trigger as given in [8], we define 100 sub-intervals of 8 s each. (No signal in GWTC-1 will have been in the detectors' sensitive frequency band for more than 250 s, hence these intervals should effectively contain noise only.) For each of these intervals we compute $\log B_{S/N}$ and $\log B_{S/G}$ where the values for t_{event} are chosen to be at the start of each interval. The log Bayes factors from times preceding all the events that were seen in two detectors obtained in this way are put into histograms, and the same is done separately for log Bayes factors from times preceding all the 3-detector events. These histograms are normalized and smoothened using Gaussian kernel density estimates to obtain approximate probability distributions for $\log B_{S/N}$ and $\log B_{S/G}$ in the absence of echoes signals.

Finally, we calculate log Bayes factors for times following the coalescence events of GWTC-1, using the same priors, but now setting t_{event} to the arrival times for the events given in [8]. Considering $\log B_{S/N}$ or $\log B_{S/G}$ and the relevant number of detectors, the normalized, smoothened background distributions $\mathcal{P}(\log B)$ are used to compute p-values:

$$p = 1 - \int_{-\infty}^{\log B} \mathcal{P}(x) dx. \quad (5.17)$$

Combined p-values from all the events are obtained using Fisher's prescription [130]. Given individual p-values p_i , $i = 1, 2, \dots, N$, one defines

$$S = -2 \sum_{i=1}^N \log(p_i), \quad (5.18)$$

and the combined p-value is calculated as

$$p_{\text{comb}} = 1 - \int_0^S \chi_{2N}^2(x) dx, \quad (5.19)$$

where χ_{2N}^2 is the chi-squared distribution with $2N$ degrees of freedom.

5.4.2 Results and discussion

Figs. 5.8 and 5.9 show background and foreground for $\log B_{S/N}$ and $\log B_{S/G}$ in the case of, respectively, 2-detector and 3-detector signals, and in Tables 5.2 and 5.3 we list the specific log Bayes factors for these cases, as well as associated p-values. As expected from the prior on SNR, which peaks away from zero, the distributions of log Bayes factors tend to peak at

Event	$\log B_{S/N}$	$p_{S/N}$	$\log B_{S/G}$	$p_{S/G}$
GW150914	2.32	0.26	2.95	0.43
GW151012	-0.59	0.70	0.35	0.88
GW151226	-0.67	0.72	2.48	0.53
GW170104	1.09	0.44	3.80	0.28
GW170608	-0.90	0.75	0.90	0.82
GW170823	6.11	0.03	5.29	0.11
Combined	-	0.34	-	0.57

Table 5.2: Log Bayes factors for signal versus noise and signal versus glitch, and the corresponding p-values, for events seen in two detectors. The bottom row contains the combined p-values for all these events together.

Event	$\log B_{S/N}$	$p_{S/N}$	$\log B_{S/G}$	$p_{S/G}$
GW170729	4.24	0.67	5.64	0.62
GW170809	9.05	0.31	12.69	0.09
GW170814	8.75	0.33	8.54	0.34
GW170817	11.05	0.19	10.30	0.20
GW170817 + 1s	6.19	0.52	9.39	0.27
GW170818	10.39	0.23	9.36	0.27
Combined	-	0.47	-	0.22

Table 5.3: Similar to Table 5.2 but for events seen in three detectors. In the case of GW170817 we also include results for which the prior range for the time of the first echo was centered at 1.0 s after the event time, where [131] claimed tentative evidence for an echo. The combined p-values take the latter prior choice for this particular event.

Figure 5.8: Background distributions (orange histograms, smoothed in brown) and foreground (vertical dashed lines, shaded by their values and labelled left to right) for the log Bayes factors for signal versus noise $\log B_{S/N}$ (left) and signal versus glitch $\log B_{S/G}$ (right), for the 2-detector events of GWTC-1. The associated p-values can be found in Table 5.2.

positive values. Note also how both the $\log B_{S/N}$ and the $\log B_{S/G}$ background distributions have support for large and positive values, which is not surprising. As mentioned earlier, in practice log Bayes factors usually depend sensitively on the recovered SNR and can have relatively large values also for SNRs below the detection threshold. In particular, $\log B_{S/N}$ generically scales as $\log B_{S/N} \approx (1/2)\text{SNR}^2$ in unmodeled searches [120, 121] and in template-based inference [127]

Figure 5.9: Similar to Fig. 5.8 but for the 3-detector events of GWTC-1. The thin solid lines in each of the two panels are for an analysis of GW170817 in which the prior range for the time of the first echo was centered at 1s after the event time, where [35] claimed tentative evidence for an echo. The associated p-values can be found in Table 5.3.

alike. In the particular case of **BayesWave** and the assessment of signal versus glitches one has $\log B_{S/G} \approx N \log \text{SNR}$, where N is the number of wavelets used. As a consequence, even a recovered SNR below detection threshold (which is usually around 8 – 10) can result in large values of the Bayes factors. Also note how this effect is larger in the 3-detector case; this is because **BayesWave** is configured to expect SNR to increase with the number of detectors, which further increases the sensitivity of log Bayes factors to SNR, and the fact that the Virgo detector tends to be less quiet than the two LIGO detectors. (In this regard, see Fig. 1 of [132] and the surrounding discussion.)

All foreground results are in the support of the relevant background distributions. For signal versus noise, the smallest p-value is 3% (the case of GW170823), whereas for signal versus glitch the p-values do not go below 9% (see GW170809). In summary, we find no statistically significant evidence for echoes in GWTC-1. For the BBH observations in particular, this statement is in agreement with the template-based searches in [114, 133, 134]. (Note that a quantitative comparison of p-values is hard to make because very specific signal shapes are assumed in the latter analyses.)

Our results in Fig. 5.9 and Table 5.3 include the BNS inspiral GW170817, analyzed in the same manner as the BBH merger signals. In [135], an analysis using the original **BayesWave** algorithm yielded no evidence for a post-merger signal. With our generalized wavelets, we obtain $\log B_{S/N} = 11.05$ and $\log B_{S/G} = 10.30$, both consistent with background. Hence in particular we do not find evidence for an echoes-like postmerger signal either, at least not up to $\lesssim 0.5$ s after the event's GPS time. In [131], tentative evidence was claimed for echoes starting at $t_0 = t_{\text{event}} + 1.0$ s. Re-analyzing with the same priors as above but this time $t_0 \in [t_{\text{event}} + 0.75\text{s}, t_{\text{event}} + 1.25\text{s}]$, we find $\log B_{S/N} = 6.19$ and $\log B_{S/G} = 9.39$, both consistent with their respective background distributions. Hence also when the time of the first echo is in this time interval we find no significant evidence for echoes. That said, we explicitly note that in the case of a BH resulting from a BNS merger of total mass $\approx 2.7 M_{\odot}$ [135], we expect the dominant ringdown frequency and hence the central frequency f_0 of the echoes to be above 6000 Hz, *i.e.* above our prior upper bound, but also much beyond the detectors' frequency reach for plausible energies emitted [136]. Foreground and background analyses with a correspondingly high frequency range are left for future work.

Our non-observation of echoes can be used to put a crude upper limit on the reflectivity $\mathcal{R}$ of an ECO, under the assumption that the remnant objects of GWTC-1 were ECOs after

all. Following [137] and given that the GWTC-1 event with the highest SNR ratio in ringdown observation (namely GW150914) of echoes had $\text{SNR}_{\text{ringdown}} \approx 8.5$ [18], our non-observation of echoes leads to $\mathcal{R} \lesssim 0.998$ at $4 - \sigma$ confidence.

Finally, in Fig. 5.10 we show signal reconstructions (medians and 90% credible intervals) in terms of generalized wavelets for all the GWTC-1 events. For illustration purposes we also include the reconstruction of a simulated echoes waveform following the inspiral of a particle in a Schwarzschild spacetime with Neumann reflective boundary conditions just outside of where the horizon would have been, at mass ratio $q = 100$ [138, 139]. The simulated signal was embedded into detector noise at a SNR of 12, roughly corresponding to the SNR in the ringdown part of GW150914, had it been observed with Advanced LIGO sensitivity of the second observing run. In all cases the whitened raw data is shown along with the whitened signal reconstruction. We include these reconstructions for completeness and they are visually consistent with our core results which are the Bayes factor well inside the background, *i.e.* Figs 5.8 and 5.9.

Figure 5.10: Stretches of whitened data (gray) and signal reconstructions (red) for a simulated echoes signal as described in the main text (top left panel) and for data immediately after the events in GWTC-1. In the case of GW170817, the first echo is searched for in an interval centered at 1.0 s after the event time. In all cases the event GPS time corresponds to the left bound of the panel.

CHAPTER 6

POSTMERGER

6.1 Introduction

The extreme densities and conditions inside NSs cannot be reached in existing experiments. This makes NSs an unique laboratory to study the equation of state (EOS) governing cold-supranuclear dense material. Following the first detection of a GW signal originating from the coalescence of a BNS system, GW170817, by the Advanced LIGO [140] and Advanced Virgo detectors [89], it became possible to constrain the NS EOS by analyzing the measured GWs [6, 8, 135, 141, 142]. Because of the increasing sensitivity of GW interferometers, multiple detections of merging BNSs are expected in the near future [90]. This will make GW astronomy an inevitable tool within the nuclear physics community.

In general, there are two ways to extract information about the EOS governing the NS's interior from a GW detection. The first method relies on the modelling of the BNS inspiral [143–147] and on waveform approximants that include tidal effects, which represent accurately the system's properties, and are of sufficiently low computational cost that they can be used in parameter estimation pipelines, *e.g.* [144, 148, 149]. The zero-temperature EOS is then constrained by measuring a mass-weighted combination of the quadrupolar tidal deformability $\tilde{\Lambda}$ or similar parameters that characterize tidal interactions, *e.g.* [150, 151].

The second method relies on an accurate modelling of the postmerger GW spectrum, *e.g.* [152–156], and can deliver an independent estimate of the EOS at densities exceeding the ones present in single NSs [157]. Therefore, the postmerger modelling also allows to investigate interesting phenomena such as phase transitions happening inside the merger remnant at very high densities [158–160]. It is expected that all BNS merger remnants which do not undergo prompt collapse, *e.g.* [161, 162], will radiate a significant amount of energy in the form of GWs [163, 164]. This radiation has a characteristic GW spectrum composed of a few peaks at frequencies $f_{\text{GW}} \sim 2\text{--}4$ kHz. The main peak frequencies of the postmerger spectrum correlate to properties of a zero-temperature spherical equilibrium star as outlined in previous works, *e.g.* [123, 152–156, 165–172]. Fig. 6.1 shows an example of the GW mode from a BNS signal and the corresponding snapshot of the density profile.

To date, the advanced GW detectors have only been able to observe the inspiral of the two NSs [136, 174] and no postmerger signal has been observed. This non-observation is caused by the higher emission frequency at which current GW detectors are less sensitive. But, the increasing sensitivity of the 2nd generation of GW detectors (Advanced LIGO and Advanced Virgo) will not only increase the detection rate of BNS inspiral signals, there will also be the chance of observing the postmerger signal for a few ‘loud’ events. Dudi *et al.* [175] find that for sources similar to GW170817 but observed with Advanced LIGO and Advanced Virgo's design

Figure 6.1: NR simulation of a BNS merger showing the GW signal (top) and the matter evolution (bottom) [173]

sensitivities, the postmerger part of the BNS coalescence might have an SNR of ~ 2 -3, while the corresponding full inspiral signal will have an SNR of up to ~ 100 . The planned third generation of GW observatories, *e.g.* Einstein Telescope [176–178] or Cosmic Explorer [179], have the capability to detect the postmerger signal of upcoming BNS mergers with SNRs up to ~ 10 .

Unfortunately, the postmerger spectrum is influenced in a complicated way by thermal effects, magnetohydrodynamical instabilities, neutrino emissions, phase transitions, and dissipative processes, *e.g.* [158, 159, 180–185]. Currently, any postmerger study relies heavily on expensive NR simulations and there is to date no possibility to perform simulations incorporating all necessary microphysical processes. Therefore, our current theoretical understanding of this part of the BNS coalescence is overall limited. In addition, there has been no NR simulation yet which has been able to show convergence of the GW phase in the postmerger. While this observation can be generally explained by the presence of shocks, turbulence or discontinuities formed during the collision of the two stars, it also increases our uncertainty on any quantitative result.

Nevertheless, the community tried to construct postmerger approximants focusing on characteristic (robust) features present in NR simulations. The discovery of quasi-universal relations is a building block for most descriptions of the postmerger GW spectrum. Clark *et al.* [186] showed that a principle component analysis can be used to reduce the dimensionality of the spectrum for equal mass binaries once the different spectra are normalized and aligned such that the main emission frequencies coincide. Effort has also been put into modelling the plus polarization in time domain using a superposition of damped sinusoids incorporating quasi-universal relations [170]. Relying on an accurate f_2 estimate for rescaling of the waveforms Easter *et al.* [171] created a hierarchical model to estimate the postmerger spectra.

Here, we follow a similar path and try to describe the GW spectrum with a set of a three- and a six-parameter model function with a Lorentzian-like shape. Comparing our ansatz with a set of 54 NR simulations, we find average mismatches of 0.18 for the three-parameter and 0.15 for the six-parameter model; cf. Table A.1. Our approximants do not incorporate directly quasi-universal relations, but are constructed to describe generic postmerger waveforms. Thus, our analysis is flexible and allows to describe almost arbitrary configurations. Employing our model in standard parameter estimation pipelines [98, 187] of the LIGO and Virgo Collaborations, we find that we can extract the dominant emission frequency in the postmerger for a number of tests. To our knowledge, this is the first time a model-based (but configuration independent)

method is employed within a Bayesian analysis of the postmerger signal.

Once the individual parameters describing the postmerger spectra are extracted, we use fits for the peak frequency to connect the measured signal to the properties of the supranuclear EOS and the merging binary. This way, one can combine measurements from the inspiral and postmerger phase to provide a consistency test for our supranuclear matter description.

Although not used here, we want to mention an alternative approach, which employs the morphology-independent burst search algorithm called **BayesWave** [100, 122]. Chatziioannou *et al.* [123] showed that this approach is capable of reconstructing the postmerger signal and allows to extract properties from the measured GW signal. Even for a measured postmerger SNR of ~ 5 , the main emission frequency of the remnant could be determined within a few dozens of Hz. Compared to **BayesWave**, our simple model functions might have the advantage that without any modifications of the current code for statistical inferences, in particular the **LALInference** module [98] available in the LSC Algorithm Library (LAL) Suite, they can be added to existing frequency domain inspiral-merger waveforms describing the first part of the BNS coalescence, *e.g.* [61, 144, 148, 149, 188, 189], to construct a full inspiral-merger-postmerger (IMP) waveform directly employable for GW analysis. Such an IMP study can also be carried out within the **BayesWave** approach, but seems technically harder since one has to combine model-based and non-model-based algorithms.

Throughout the work, we employ the NR simulations published in the Computational Relativity (CoRe) database [42]. In addition, where explicitly mentioned, we increase our dataset by adding results published in [166, 168]. We refer the reader to Table A.1 in Appendix for further details about the individual numerical data.

6.2 Types of remnant

NR simulation is a fundamental tool to study the BNS dynamics especially at postmerger. Fig. 6.2 shows a schematic view of remnant characterization. After the coalescence of BNS, depending on total mass M and the EOS, the remnant can be classified into four categories,

1. Massive NS (MNS),
2. Supermassive NS (SMNS),
3. Hypermassive NS (HMNS), and
4. BH from prompt collapse.

If a BNS system has a total mass M smaller than the maximum Tolman–Oppenheimer–Volkoff mass M_{TOV} which is an upper bound of mass of a cold non-rotating NS, it forms MNS. Current estimates for M_{TOV} are based on the observation of J0740+6620 [190] with $M = 2.17^{+0.11}_{-0.10} M_{\odot}$ and the assumption that GW170817’s endstate was a BH [191–194] such that $M_{\text{TOV}} \lesssim 2.17\text{--}2.35 M_{\odot}$. If the remnant is self-rotating around its axis and together with the extra centrifugal force it becomes sufficient to support against its collapse due to the strong self-gravity, a SMNS is formed. In this way, the maximum allowed mass can be increased by 10 – 15% [195]. The remnant will collapse to a BH only if dissipation of its angular momentum is present. However, the merger does not always result in an uniformly rotating remnant but in a differentially rotating one in which its inner region is rotating faster than its outer region. This extra centrifugal force increases the maximum allowed mass even further for a given EOS and we called it HMNS. Once the angular momentum is redistributed significantly, HMNS will become unstable against the gravitational collapse and form a BH. Finally, if the remnant has a total

Figure 6.2: Schematic view of remnant characterization distinguishing between MNS, SMNS/HMNS, and prompt collapse. [173]

mass M greater than the mass threshold $M_{\text{thr}} = k_{\text{thr}} M_{\text{TOV}}$ with $1.3 \lesssim k_{\text{thr}} \lesssim 1.7$ [161, 162, 196]. the merger will undergo prompt collapse to form a BH within a few milliseconds. The above classification is based on cold-equilibrium configurations with zero temperature EOSs [197]. Thermal pressure from the finite temperature effect can increase the maximum allowed mass by $\sim 0.1 M_{\odot}$ [166].

6.3 Postmerger morphology

6.3.1 Time domain

In general, $h(t) = h_+ - ih_{\times}$ can be decomposed into modes using spin-weighted spherical harmonics ${}^{-s}Y_{lm}$ of weight -2 [199],

$$h(t) = h_+ - ih_{\times} = \frac{M}{r} \sum_{l=2}^{\infty} \sum_{m=-l}^l h_{lm}(t-r) {}^{-2}Y_{lm}(\iota, \phi), \quad (6.1)$$

where M is the total mass of the binary, r is the luminosity distance to source, ι is the inclination, ϕ is the polarization angle and $h_{lm}(t)$ is complex expansion coefficients. Modes with negative values of m can be obtained by

$$h_{l-m} = (-1)^l h_{lm}^*, \quad (6.2)$$

where $*$ denotes the complex conjugate. Details of the spin-weighted spherical harmonics is provided in Appendix B. As throughout the work, we restrict our considerations to the dominant 22-mode (as well as the 2-2-mode).

Figure 6.3: A typical time domain representation of a postmerger waveform (THC:0001 [42, 157, 198]); cf. Table A.1. As throughout the work, we restrict our considerations to the dominant 22-mode.

In contrast to BBH waveform, BNS systems are not mass-scale invariant since tidal interactions during the inspiral depend on the total mass and determine the merger outcome.

While the inspiral GW signal is characterized by a chirp, *i.e.* a monotonic increase of the GW amplitude and frequency, the postmerger emission shows a non-monotonic amplitude and frequency evolution. Fig. 6.3 presents one example of a possible postmerger waveform. In the following, we highlight some of the important features characterizing the signal.

Merger amplitude

By definition, the inspiral ends at the peak of the GW amplitude (merger) A_{mrg} marked with a red circle in Fig. 6.3. Breschi *et al.* [200] shows that A_{mrg} can be fit by rational functions

$$A_{\text{mrg}} = (3.4910 \times 10^{-1}) \frac{1 + (1.9272 \times 10^{-2})\xi_{5215.0} + (-4.3729 \times 10^{-6})\xi_{5215.0}^2}{1 + (2.8266 \times 10^{-2})\xi_{5215.0} + (9.3643 \times 10^{-6})\xi_{5215.0}^2}, \quad (6.3)$$

as a function of $\xi_{5215.0} = \kappa_2^T + (5215.0)(1 - 4\eta)$ with the symmetric mass ratio $\eta \equiv \frac{m_A m_B}{M^2}$ and the dimensionless quadrupole tidal coupling constant

$$\kappa_2^T = 3 \left[\frac{q^4}{(1+q)^5} \Lambda_A + \frac{q}{(1+q)^5} \Lambda_B \right], \quad (6.4)$$

assuming $q = m_A/m_B \geq 1$, where individual dimensionless tidal deformabilities $\Lambda = \frac{2}{3}k_2 \left(\frac{R}{M}\right)^5$ depend on the dimensionless $\ell = 2$ Love number and radius R of the isolated NSs. κ_2^T parametrizes, at the leading-order, the tidal interactions in the general relativistic 2-body Hamiltonian [144].

Figure 6.4: Dimensionless time between the merger and the first amplitude minimum within the postmerger as a function of the mass-weighted tidal deformability $\tilde{\Lambda}$. The color shows the mass ratio q . The error bar indicates the uncertainty obtained from simulations with the same physical parameters but different resolution as available in the CoRe database [42]. The cross marker indicates the data with error bars while the circle marker indicates those without. The shaded region indicates the 1-sigma (± 2.2913) uncertainty.

First postmerger minimum

After the merger, the amplitude decreases showing a clear minimum (red squared marker) shortly afterwards, see [45, 201] for further discussions. Around this intermediate and highly non-linear regime, different frequencies are excited for a few milliseconds, see *e.g.* [155, 169] for further details. We find that the time between merger and this amplitude minimum follows a quasi-universal relation. In Fig. 6.4 we show the time between the merger and the first amplitude minimum, $\Delta t_{\min}/M$, as a function of the mass-weighted tidal deformability

$$\tilde{\Lambda} = \frac{3}{13} \left[\frac{q^5 + 12q^4}{(1+q)^5} \Lambda_A + \frac{1 + 12q}{(1+q)^5} \Lambda_B \right], \quad (6.5)$$

with the color bar showing the mass ratio q . Based on the highest resolution R01 available in the CoRe database¹³, we find a clear correlation between $\Delta t_{\min}/M$ and the mass-weighted tidal deformability $\tilde{\Lambda}$. A good phenomenological representation is given by

$$\frac{\Delta t_{\min}}{M} = \alpha \frac{1 + \beta \tilde{\Lambda}}{1 + \gamma \tilde{\Lambda}}, \quad (6.6)$$

¹³All fits in this work are based on the resolution R01.

Figure 6.5: A scatter plot of the first peak in the postmerger spectra after merger versus mass ratio q with error bars from different resolutions. The cross marker indicates the data with error bar while the circle marker indicates those without. The color shows the mass-weighted tidal deformability $\tilde{\Lambda}$. The shaded region indicates the 1-sigma ($\pm 1.7915 \times 10^{-2}$) uncertainty.

with the parameters $\alpha = 2.4681 \times 10^1$, $\beta = 2.8477 \times 10^{-3}$, $\gamma = 6.6798 \times 10^{-4}$ obtained by a least-square fit for which the root-mean-square (RMS) error is 2.4608. Interestingly, the two highest mass ratio simulations do not follow Eq. (6.6). This is caused by the different post-merger evolution for these high-mass ratio setups. While the amplitude minimum is produced when the two NS cores approach each other and potentially get repelled, configurations with very high mass ratio show almost a disruption during the merger, *i.e.* the lower massive NS deforms significantly under the strong external gravitational field of its companion.

One possible application for this quasi-universal relation of $\Delta t_{\min}/M$ is the improvement of BNS waveform approximants, *i.e.* it might help to determine the amplitude evolution after the merger of the two NSs. In particular, incorporating an amplitude tapering after the merger with a width of $\Delta t_{\min}$ provides a natural ending condition for inspiral-only approximants, *e.g.* NRTidal [60, 61, 148] or tidal effective-one-body models [202–204]. Therefore, Eq. (6.6) might become a central criterion to connect inspiral and postmerger models.

First postmerger maximum

After the minimum of the GW amplitude, the amplitude grows and reaches a maximum, marked with a red diamond in Fig. 6.3. One finds that the main binary property determining the amplitude of this first postmerger GW amplitude maximum is the mass ratio of the binary q ,

Figure 6.6: Frequency domain gravitational waveform (for the dominant 22 mode) for the setup THC:0001. In blue we show the fast Fourier transformation of the time domain waveform shown in Fig. 6.3, while in green we show only the postmerger part obtained by discarding the inspiral signal in the time domain. The two marked characteristic frequencies are the merger frequency f_{mrg} and the main postmerger emission frequency f_2 .

cf. Fig. 6.5 with

$$\frac{|rh_{22}(t_{\text{max}_1})|}{M} = (2.8437 \times 10^{-1}) \frac{1 - (5.3149 \times 10^{-1})q}{1 - (2.3420 \times 10^{-1})q}. \quad (6.7)$$

The qualitative behavior is again related to the possible tidal disruption of the binary close to the merger for unequal mass systems. We note that even if the secondary star does not get disrupted, the maximum density in the remnant shows one peak rather than two independent cores [205, 206] which leads to a smaller first postmerger peak and overall on average a smaller GW amplitude.

6.3.2 Frequency domain

We obtain the frequency domain waveform by fast Fourier transformation of the time domain GW strain $h(t)$:

$$\tilde{h}(f) = \int_{-\infty}^{\infty} h(t) e^{-i2\pi ft} dt. \quad (6.8)$$

As before, we consider only the dominant 22-mode of the GW signal.

In Fig. 6.6 we show the frequency domain GW signal (blue solid line) of THC:0001. The merger frequency of this particular configuration is 1638 Hz and it is marked with a red circle.

Figure 6.7: Mf_2 as a function of ζ with error bars from different resolutions. The cross marker indicates the data with error bar while the circle marker indicates those without. The color shows the mass-weighted tidal deformability $\tilde{\lambda}$. The shaded region indicates the 1-sigma ($\pm 1.1025 \times 10^{-3}$) uncertainty. In addition to the CoRe-dataset employed to derive the previously shown quasi-universal relations, we include here the published results of [166, 168].

The main feature of the postmerger spectrum is the dominant peak characterizing the main emission frequency f_2 , which for the setup shown in the Figs. 6.3 and 6.6 is about 2354 Hz. For a better interpretation, we also present the frequency domain postmerger spectrum in green. Such postmerger-only waveforms are obtained by fast Fourier transformation after applying a Tukey window [207] with a shape parameter 0.05 at $t_{\min}$ (where the shape parameter represents the fraction of the window inside the cosine tapered region) and will be used for our injections to test our parameter estimation infrastructure.

Merger frequency

It was already known that the merger frequency can be expressed by a quasi-universal relation, *e.g.* [154, 208, 209]. One of the latest fits to the merger frequency f_{mrg} is obtained by Breschi *et al.* [200], *i.e.*

$$Mf_{\text{mrg}} = (3.3184 \times 10^{-2}) \frac{1 + (1.3067 \times 10^{-3})\xi_{3199.8}}{1 + (5.0064 \times 10^{-3})\xi_{3199.8}}, \quad (6.9)$$

with $\xi_{3199.8} = \kappa_2^T + (3199.8)(1 - 4\eta)$.

f_2 frequency

The dominant feature in the postmerger frequency spectrum is the dominant emission mode of the merger remnant at a frequency f_2 . As mentioned before, a number of works, *e.g.* [152–156] have discussed possible EOS-insensitive, quasi-universal relation for the f_2 frequency.

Building mostly on the work of Bernuzzi *et al.* [156], we derive a new relation for the f_2 frequency. First, we extend the dataset of 99 NR simulations employed in Bernuzzi *et al.* [156] and use a set of 121 data by incorporating additional setups published as a part of the CoRe database [42]; cf. Table A.1. Second, we have switched from

$$\kappa_2^T \equiv 2 \left[\frac{1}{q} \left(\frac{X_A}{C_A} \right)^5 k_2^A + q \left(\frac{X_B}{C_B} \right)^5 k_2^B \right], \quad (6.10)$$

to

$$\kappa_{\text{eff}}^T \equiv \frac{2}{13} \left[\left(1 + 12 \frac{X_B}{X_A} \right) \left(\frac{X_A}{C_A} \right)^5 k_2^A + (A \leftrightarrow B) \right] = \frac{3}{16} \tilde{\Lambda} \quad (6.11)$$

which yields a tiny improvement ($\sim 0.1\%$) in the RMS error against the NR data, but more notably relates directly to the mass-weighted tidal deformability $\tilde{\Lambda}$ measured most accurately from the inspiral part of the signal. In addition to the dependence of the mass-weighted tidal deformability $\tilde{\Lambda}$, the postmerger evolution depends also on the stability of the formed remnant and how close it is to the BH formation. This information is in part encoded in the ratio between the total mass M and the maximum allowed mass of a single non-rotating NS M_{TOV} . By incorporating an additional M/M_{TOV} dependence we are able to reduce the RMS error by $\approx 28\%$.

Therefore, we define a parameter ζ by a linear combination of κ_{eff}^T and $\frac{M}{M_{\text{TOV}}}$ (see also [200, 210] for a similar approach),

$$\zeta = \kappa_{\text{eff}}^T + a \frac{M}{M_{\text{TOV}}}. \quad (6.12)$$

The free parameter $a = -131.7010$ is determined by minimizing the RMS error. Finally, the dimensionless frequency Mf_2 is fitted against ζ using a Padé approximant:

$$Mf_2(\zeta) = \alpha \frac{1 + A\zeta}{1 + B\zeta} \quad (6.13)$$

with $\alpha = 3.4285 \times 10^{-2}$, $A = 2.0796 \times 10^{-3}$ and $B = 3.9588 \times 10^{-3}$. We present Eq. (6.13) together with our NR dataset and a one sigma uncertainty region (shaded area) in Fig. 6.7.

Frequency domain amplitude at f_2

Finally, we want to briefly discuss the dependence of the f_2 peak amplitude on the binary properties. While the f_2 frequency correlates clearly to ζ , we have not been able to find a similar tight relation between any combination of the binary parameters and the amplitude $|\tilde{h}_{22}(f_2)|$. The only noticeable imprint which we have been able to extract comes from the mass ratio q , where generally higher mass ratios lead to a smaller amplitude $|\tilde{r}\tilde{h}_{22}(f_2)/M|$ as shown in Fig. 6.8 with

$$\frac{|\tilde{r}\tilde{h}_{22}(f_2)|}{M} = (4.2319 \times 10^{-4}) \frac{1 - (5.4016 \times 10^{-1})q}{1 - (4.5927 \times 10^{-1})q}. \quad (6.14)$$

We note that because of the large uncertainty, we see Eq. (6.14) more as a qualitative rather than a quantitative statement about the postmerger spectrum. However, the overall amplitude

Figure 6.8: $|r\tilde{h}_{22}(f_2)/M|$ as a function of the mass ratio q with error bars from different resolutions. The cross marker indicates the data with error bar while the circle marker indicates those without. The shaded region indicates the 1-sigma ($\pm 7.4485 \times 10^{-5}$) uncertainty. The color shows the mass-weighted tidal deformability $\tilde{\Lambda}$.

decrease for an increasing mass ratio seems to be a robust feature and might help to interpret future GW observations. We suggest that this decrease is caused by the asymmetry of the density profile inside the merger remnant, since, in general, the density evolution of the high density regions in the remnant leads to the characteristic amplitude modulation of the signal visible in Fig. 6.3.

6.4 Model functions and simulation

6.4.1 Lorentzian approximants

Based on our previous discussion and the dominance of the characteristic f_2 frequency, we start our consideration with a simple damped sinusoidal time domain waveform to model the postmerger waveform. The Fourier transform of a damped sinusoidal function is a Lorentzian function, Eq. (6.15). In the simplest case which we consider, we use 3 unknown coefficients (c_0, c_1, c_2) corresponding to the amplitude, the dominant emission frequency and the inverse of the damping time, respectively, and write the frequency domain signal as:

$$\tilde{h}_{22}(f) = \frac{c_0 c_2}{\sqrt{(f - c_1)^2 + c_2^2}} e^{-i \arctan\left(\frac{f - c_1}{c_2}\right)}. \quad (6.15)$$

Figure 6.9: Mismatch between a subset of the NR data listed in Table A.1 and the three- and six-parameter models.

Eq. (6.15) suggests that the amplitude peak of the GW postmerger spectrum and also the main postmerger phase evolution are connected to the same frequency characterized by c_1 .

Minimizing over (c_0, c_1, c_2) , we compute the mismatches $\mathcal{M}$ between the used NR data from the CoRe database (Table A.1) and the model function, Eq. (6.15):

$$\mathcal{M} = 1 - \max_{\{t_c, \phi_c\}} \frac{\left(\tilde{h}_{\text{NR}} | \tilde{h}_{\text{Model}} e^{-i(\phi_c + 2\pi f t_c)} \right)}{\sqrt{\left(\tilde{h}_{\text{NR}} | \tilde{h}_{\text{NR}} \right) \left(\tilde{h}_{\text{Model}} | \tilde{h}_{\text{Model}} \right)}}, \quad (6.16)$$

where

$$(a|b) = 4\Re \int_{f_{\text{mrg}}}^{4096\text{Hz}} a^*(f) b(f) df, \quad (6.17)$$

t_c and ϕ_c are the reference time and phase respectively. Fig. 6.9 shows all mismatches, which on average are ~ 0.18 .

The mismatches can be further decreased by adding three additional coefficients:

$$\tilde{h}_{22}(f) = \frac{c_0 c_2}{\sqrt{(f - c_1)^2 + c_2^2}} e^{-ic_3 \arctan\left(\frac{f - c_1}{c_2}\right)}. \quad (6.18)$$

For Eq. (6.18) the amplitude and phase evolution are independent from each other and we obtain average mismatches of 0.15, *i.e.* about 17% better than for the three-parameter model. While one might argue that the additional introduced degrees of freedom hinder the extraction of individual parameters in a full Bayesian analysis, it might also be possible that the more

flexible 6-parameter model recovers signals with smaller SNRs. Thus, we continue our study with both model functions Eqs. (6.15) and (6.18).

Finally, one obtains the plus and cross polarizations from Eqs. (6.15) and (6.18) by incorporating the inclination ι dependence:

$$\tilde{h}_+ = \frac{1 + \cos^2 \iota}{2} \tilde{h}_{22}, \quad (6.19)$$

$$\tilde{h}_\times = -i \cos \iota \tilde{h}_{22}. \quad (6.20)$$

The inclination dependence is true as we are considering only the dominant 22-mode. From Eqs. (6.1), (6.2), (B.3) and (B.7), we have

$$\begin{aligned} h_+ &= \Re(h(t)) \\ &\propto \Re(h_{22}^{-2} Y_{22} + h_{2-2}^{-2} Y_{2-2}) \\ &\propto {}^{-2}Y_{22} + {}^{-2}Y_{2-2} \\ &\propto (1 + \cos \iota)^2 + (1 - \cos \iota)^2 \\ &\propto \frac{1 + \cos^2 \iota}{2}, \\ h_\times &= -\Im(h(t)) \\ &\propto -\Im(h_{22}^{-2} Y_{22} + h_{2-2}^{-2} Y_{2-2}) \\ &\propto -{}^{-2}Y_{22} + {}^{-2}Y_{2-2} \\ &\propto -(1 + \cos \iota)^2 + (1 - \cos \iota)^2 \\ &\propto -\cos \iota. \end{aligned} \quad (6.21)$$

The inclination dependence in the postmerger stage turns out to be the same as the mass quadrupole radiation emitted during the inspiral stage, *i.e.* Eq. (2.118).

$\tilde{h}_+$ and $\tilde{h}_\times$ can be employed directly to infer information from the postmerger part of a GW signal or to construct a full IMP-waveform for BNSs.

6.4.2 Validating the parameter estimation pipeline

In this section, we present for four selected cases the performance of the three- and six-parameter models. We inject the NR waveforms immersed in the same simulated Gaussian noise with total network SNRs ranging from SNR 0 to SNR 10 assuming that Advanced LIGO and Advanced Virgo detectors run at design sensitivity [90]¹⁴. Fig. 6.10 shows the injection of THC:0021 with SNR 8 in both the time (top panel) and frequency domain (bottom panel). For each injected waveform, a Tukey window with shape parameter 0.05 is applied at $t_{\min}$ to isolate the postmerger signal and avoid Gibbs phenomenon. All simulated signals are injected with zero inclination angle, zero polarization angle ψ and sky location (α, δ) to be $(0, 0)$.

We estimate parameters using Bayesian inference with the `LALInference` module [98] available in the `LALSuite` package. Sampling is done on 9 (12) parameters

$$\{c_i, \alpha, \delta, \iota, \psi, t_c, \phi_c\} \quad (6.22)$$

with nested sampling algorithm `lalinferencenest` [211], where i runs from 0 to 2 (5) for the three- (six-) parameter model. The priors are chosen to be uniform in $[0, 10^{-20}] \text{ s}^{-1}$ on c_0 , uniform in $[1500, 4096] \text{ Hz}$ on c_1 and c_5 , uniform in $[1, 400] \text{ Hz}$ on c_2 and c_4 , uniform in $[0, 6]$ on c_3 , uniform in $[0, 2\pi]$ on α, ψ and ϕ_c , uniform in $[-1, 1]$ on $\cos \iota$ and $\sin \delta$, and uniform in

Figure 6.10: Injection of THC:0021 with SNR 8, zero inclination angle, zero polarization angle and sky location (0,0). Left: time domain signal of the postmerger waveform highlighted within the detectors noise for H1–Hanford, L1–Livingston, and V1–Virgo. Right: frequency domain signal and design amplitude spectral density (ASD) for H1, L1, and V1.

Figure 6.11: Posteriors for the parameter c_1 in Eq. (6.15) (top panels) and Eq. (6.18) (bottom panels) for a variety of SNRs. c_1 can be directly related to the peak in the frequency domain spectrum and therefore relates to the f_2 frequency. The IDs of the four injected waveforms are # 18, 20, 32, 34, i.e., THC:0021, THC:0031, BAM:0048, BAM:0057 of [42]. The chosen set covers various EOSs, mass ratios, and masses and is therefore used as a testbed for our new algorithm.

[trigger time $- 0.05$ s, trigger time $+ 0.05$ s] on t_c where the trigger time is the signal arrival time in the geocentric frame.

Fig. 6.11 shows the posterior for c_1 , i.e. our best estimate of the f_2 frequency for SNRs up to 8 for our 4 examples, which we mark in Table A.1. We present the recovery with the 3- and the 6-parameter model in the top and bottom panels, respectively. The solid vertical line represents the injected f_2 frequency and the dashed line represents the estimate according to the quasi-universal relation Eq. (6.13) together with a one-sigma uncertainty (gray shaded region).

Let us summarise the main findings:

- (i) The three-parameter and six-parameter approximants perform similarly.
- (ii) Depending on the exact setting (*e.g.* intrinsic source properties, noise realization, sky location) one can recover the f_2 frequency with an SNR of ~ 4 for the best and ~ 8 for worst considered scenarios.
- (iii) Interestingly, one finds that also c_5 relates to a frequency which is close to the f_2 frequency, however, c_5 is significantly less constrained than c_1 (Fig. 6.12).
- (iv) Once 3rd generation detectors are available and so the postmerger SNRs of ~ 10 are obtained, the systematic uncertainties of the quasi-universal relations become larger than the statistical uncertainties; cf. dashed and solid, vertical black lines.

Figure 6.12: Posteriors for the parameter c_1 and c_5 in Eq. (6.18) for 2 SNRs. c_5 peaks at a frequency close to f_2 but it is significantly less constrained than c_1 .

6.4.3 Inspiral and postmerger consistency

Finally, we want to illustrate how a future detection of a postmerger GW signal will help to constrain the source properties and the internal composition of NSs. As shown before, the f_2 frequency can be extracted through a simple waveform model (or alternatively by using *BayesWave*, *e.g.* [123]). To connect the f_2 frequency with the source parameters, one needs to employ quasi-universal relations as presented in Sec. 6.3 and some information obtained from the analysis of the inspiral GW signal. In particular, the total mass M can be measured precisely using state-of-art BNS inspiral waveforms, *e.g.* [61, 69, 144, 148, 149, 188, 189, 202–204, 212, 213]. For GW170817 the uncertainty on M could be reduced to $\pm 0.04 M_\odot$ once EM information had been included [210, 214]. Thus, we will use an uncertainty of $\Delta M = \pm 0.04 M_\odot$ as a realistic estimate. In addition, we have to know the maximum TOV-mass M_{TOV} . As mentioned before, current estimates for M_{TOV} are based on the observation of J0740+6620 [190] with $M = 2.17^{+0.11}_{-0.10} M_\odot$ and the assumption that GW170817’s end state was a BH [191–194] such that $M_{\text{TOV}} \lesssim 2.17\text{--}2.35 M_\odot$. Due to the increasing number of BNS detections in the future, we expect that the uncertainty of M_{TOV} can be considerably reduced so that we will use an uncertainty of $\pm 0.04 M_\odot$. From this information, we can compute the ζ interval consistent with the observed inspiral signal from the $\tilde{\Lambda}$ posteriors, cf. the vertical red shaded region in

¹⁴The corresponding power spectral density (PSD) files, `LIGO-P1200087-v18-aLIGO_DESIGN.txt` and `LIGO-P1200087-v18-AdV_DESIGN.txt` are available in the `LALSimulation` directory of the `LALSuite` package.

Figure 6.13: Schematic plot showing how one can constrain ζ from f_2 measurements where the spread on f_2 is shown as one standard deviation for different SNRs. Top panel is for three-parameter model and the bottom panel is for six-parameter model. The vertical red shaded region corresponds to the ζ -interval consistent with a hypothetical inspiral signal assuming an uncertainty of $\pm 0.04 M_\odot$ for M and M_{TOV} , and ± 30 for κ_{eff}^T . The exact ζ and Mf_2 values are marked with vertical red-dashed line and horizontal black-dashed line respectively. The quasi-universal relation Eq. (6.13) is plotted as a black solid line.

Fig. 6.13. We then connect the ζ estimate obtained from the inspiral with Eq. (6.13) and the f_2 measurement of the postmerger signal. This consistency analysis is somewhat connected to the inspiral-merger-ringdown consistency test for BBH [12], but not only one has to assume the correctness of GR but also that our understanding of supranuclear matter and the EOS-insensitive quasi-universal relations are valid.

Fig. 6.13 summarises our main results. Generally the GW measurements can be considered consistent between the inspiral and postmerger observations as well as with the quasi-universal relation relating the main postmerger frequency with the binary properties. We find that for all cases, the quasi-universal relation and its 1-sigma uncertainty region lie within the intersection of the red shaded region (inspiral) and the blue/orange/green horizontal regions (postmerger). Thus, (as expected) all simulations are consistent with (i) GR and (ii) the nuclear physics descriptions used as a basis for NR simulations to derive the quasi-universal relations¹⁵. For future events this approach will allow us to probe our understanding of physical processes under extreme conditions, and in cases where the quasi-universal relation seems violated to even derive new relations based on GW measurements (under the assumption that GR is correct). We note that even in the case where either GR or the quasi-universal relations would be violated, we might not be able to determine reliably the violation based on one individual event, but stronger constraints can be obtained by combining multiple BNS events, see *e.g.* [11, 147] for similar approaches.

¹⁵One possible example of such a deviation would be the presence of a phase transition within the remnant, *e.g.* [158, 159].

6.5 Summary

In this work, we discussed the general morphology of a BNS postmerger in both the time and frequency domains. We presented quasi-universal relations for the time at which the first postmerger amplitude minimum happens and the strength of the first postmerger amplitude maximum. In general, the time between the merger and the amplitude minimum increases with an increasing $\tilde{\Lambda}$, while the amplitude of the first postmerger maximum decreases with an increasing mass ratio. In the frequency domain, we improved the existing quasi-universal relations of Mf_2 by extending the employed NR dataset (121 simulations in total) and adding an extra dependence of M/M_{TOV} . The extra term M/M_{TOV} characterizes how close the setup is to the BH formation.

We found that a three- (six-) parameter Lorentzian can model the postmerger waveform with average mismatch of 0.18 (0.15). To test these model functions, we performed an injection study, in which we simulated the detector strains with four different BNS configurations immersed in the same simulated Gaussian noise assuming Advanced LIGO and Advanced Virgo at design sensitivities. We found that in the best cases the Lorentzian models could measure the dominant emission frequency f_2 once the signal has an SNR of 4 or above; however, for most scenarios higher SNRs ~ 8 were required. Although there are rooms for improvement, the Lorentzian models serve as an intermediate step to construct a more accurate postmerger model and hence an inspiral-merger-postmerger approximant.

Employing the new quasi-universal relation for Mf_2 described in this work, we could present consistency tests between the inspiral and the postmerger signal; cf. Fig. 6.13.

CHAPTER 7

CONCLUSIONS

In 1915, Albert Einstein (1879-1955) proposed his alternative theory of gravity, that is the general relativity (GR), which predicts the existence of gravitational waves (GWs). The first detection of GWs from a binary black hole coalescence in September 2015, namely GW150914, by the two Advanced LIGO interferometers marked the beginning of a new era in GW astronomy. It has opened a new alternative window for us to study the universe. In this thesis, I presented our work on the nature of compact objects, especially the remnant resulting from the merger. The final state of compact binary coalescence completely depends on the nature and properties of the progenitors: for binary black hole (BBH) and neutron star black hole (NSBH) system it will be an excited Kerr black hole (BH) while for binary neutron star (BNS) it may be a black hole or massive neutron star (MNS) depending on the total mass and equation of state of the neutron star.

Ringdown from BBH

In GR, after the merger of two BHs, the remnant will be an excited Kerr BH which emits a characteristic GW radiation in the form of a linear superposition of quasi-normal modes (QNMs). No-hair conjecture states that a stationary isolated BH is determined uniquely by its mass M_f , dimensionless spin j_f and electric charge Q which is expected to be zero for astrophysical objects. As a manifestation of this conjecture, we expect that the QNM frequency ω_{lmn} and QNM damping time τ_{lmn} can be determined by the final mass and dimensionless spin completely. To probe the nature of the final BH, we look for violations of the BH no-hair conjecture by introducing possible deviations in the QNM parameters,

$$\begin{aligned}\omega_{lmn}(M_f, j_f) &\rightarrow (1 + \delta\hat{\omega}_{lmn})\omega_{lmn}(M_f, j_f) \\ \tau_{lmn}(M_f, j_f) &\rightarrow (1 + \delta\hat{\tau}_{lmn})\tau_{lmn}(M_f, j_f),\end{aligned}\tag{7.1}$$

where $\delta\hat{\omega}_{lmn}$ and $\delta\hat{\tau}_{lmn}$ are relative deviations that we include as additional degrees of freedom in our inference. If the final BH is a BH described by GR, any additional deviations parameters will be consistent to zero. In our work, we show that the BH no-hair conjecture can be tested observationally with just $\mathcal{O}(5)$ plausible signals detected by the LIGO-Virgo detector network operating at its design sensitivity. The use of the ringdown waveform to the case of the initial aligned-spins BH progenitors in data analysis is left for future study.

GW echoes from exotic stars

There are compact objects beyond GR which may exist in the universe. A variety of alternative objects, called BH mimickers, have been proposed, *e.g.* wormholes, boson stars, and gravastars,

which give different GW signatures, *e.g.* anomalous tidel effects, violation of no-hair conjecture, GW echoes *etc.* GW echoes are formed if the final remnant has either no horizon or a non-totally absorbing surface above the would-be Schwarzschild radius. In the case of no horizon, any incoming GWs (*e.g.* resulting from the merger) will not be completely absorbed by the exotic compact object (ECO) but trapped in the potential well formed by the photon sphere $r = 3M$ with wave packets leaking out to infinity at regular time intervals. In the case of a non-totally absorbing surface above the would-be Schwarzschild radius, the potential well is instead formed by this inner barrier and the photon sphere.

Because different BH mimickers give different waveforms depending on the theory and its nature, we have no any prior knowledge about the waveform. Therefore, we make use of the morphology-independent algorithm, especially **BayesWave**, which can detect any waveform by comparing the Bayesian evidences among signal, glitch and noise hypotheses. In our work, we modified the basis wavelet used in the algorithm to detect and characterise GW echoes specifically. We applied the methods to the detected signals of the first gravitational wave transient catalog and did not find any evidence for echoes.

Postmerger from BNS

When two neutron stars (NSs) merge, if the total mass is not high enough to undergo prompt collapse to form a BH, the final remnant will be a massive NS which emits GW with a dominant frequency $f_2 \sim 2\text{--}4$ kHz. In this frequency range, the existing interferometers are not sensitive enough to detect it. In the future, we expect to see BNS postmerger with third generation of GW interferometers. To extract the information about the EOS, we need to have accurate waveform approximants. Currently the waveforms are poorly known because there have been no comprehensive NR simulations incorporating all necessary microphysical processes. Nevertheless, the discovery of quasi-universal relations, especially for the f_2 frequency, serves as building blocks for modelling BNS postmerger. By constraining f_2 which contains rich information about the EOS at ultra-high density, we can in turn constrain the EOS. In our work, we discussed the general morphology of a BNS postmerger in both time and frequency domains. We presented the newly discovered quasi-universal relations, such as the time at which the first postmerger amplitude minimum happens and the strength of the first postmerger amplitude maximum, and improved existing Mf_2 quasi-universal relations by using more NR dataset and adding an extra dependence of M/M_{TOV} in the fitting. Because of the characteristic f_2 frequency, we modelled the postmerger by a three- (six-) parameter Lorentzian which has an average mismatch of 0.18 (0.15). With this waveform model, we demonstrated the possibility of performing inspiral-merger-postmerger (IMP) consistency tests. Until now, all BNS waveform approximants ignore the postmerger part. These works serve as an intermediate step to build an IMP waveform model of the BNS coalescence.

APPENDIX A

NUMERICAL RELATIVITY CONFIGURATIONS

Throughout the BNS postmerger work described in Chapter 6, we employ the NR simulations published in the Computational Relativity (CoRe) database [42]. In addition, where explicitly mentioned, we increase our dataset by adding results published in [166, 168]. These data are used for deriving and improving quasi-universal relations, and for the injection study.

ID	Code/CoRe-ID	EOS	M_{TOV} [$M_{\odot}$]	M [$M_{\odot}$]	q	$\kappa_{\text{eff}}^{\text{T}}$	$\frac{\Delta t_{\text{min}}}{M}$	$\frac{ rh_{22}(t_{\text{max}_1}) }{M}$ [10^{-2}]	f_2 (Hz)	$\frac{ r\tilde{h}_{22}(f_2) }{M}$ [10^{-4}]
#1	THC:0002	BHBlp	2.10	2.60	1.00	196	60.00	19.20	2458	4.29
#2	THC:0003	BHBlp	2.10	2.70	1.00	159	49.78	19.65	2726	3.97
#3	THC:0004	BHBlp	2.10	2.62	1.09	190	56.90	18.62	2602	3.58
#4	THC:0005	BHBlp	2.10	2.60	1.17	198	55.38	16.24	2478	3.60
#5	THC:0006	BHBlp	2.10	2.80	1.00	129	51.43	18.57	2912	3.01
#6	THC:0007	BHBlp	2.10	2.83	1.04	121	47.49	14.32	2767	3.19
#7	THC:0010	DD2	2.42	2.40	1.00	302	65.00	19.15	2231	3.11
#8	THC:0011	DD2	2.42	2.50	1.00	242	60.48	16.75	2354	3.02
#9	THC:0012	DD2	2.42	2.60	1.00	196	60.00	18.13	2478	2.52
#10	THC:0013	DD2	2.42	2.70	1.00	159	52.15	16.45	2664	2.75
#11	THC:0014	DD2	2.42	2.62	1.09	190	57.82	16.97	2437	2.68
#12	THC:0015	DD2	2.42	2.60	1.17	198	57.23	15.65	2478	2.91
#13	THC:0016	DD2	2.42	2.80	1.00	129	50.29	14.37	2571	3.39
#14	THC:0017	DD2	2.42	3.00	1.00	86	44.80	15.01	2767	4.17
#15	THC:0018	LS220	2.04	2.40	1.00	269	60.00	18.96	2520	4.52
#16	THC:0019	LS220	2.04	2.70	1.00	128	45.33	19.88	3015	4.53
#17	THC:0020	LS220	2.04	2.62	1.09	159	48.64	15.82	2850	4.80
#18	THC:0021	LS220	2.04	2.60	1.17	167	50.77	16.04	2726	4.14
#19	THC:0029	MS1b	2.76	2.70	1.00	287	60.80	16.53	2014	2.11
#20	THC:0031	SFHo	2.06	2.62	1.09	96	44.05	17.88	3222	3.87

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ID	Code/CoRe-ID	EOS	M_{TOV} [$M_{\odot}$]	M [$M_{\odot}$]	q	$\kappa_{\text{eff}}^{\text{T}}$	$\frac{\Delta t_{\text{min}}}{M}$	$\frac{ rh_{22}(t_{\text{max}_1}) }{M}$ [10^{-2}]	f_2 (Hz)	$\frac{ r\tilde{h}_{22}(f_2) }{M}$ [10^{-4}]
#21	THC:0032	SFHo	2.06	2.60	1.17	100	43.38	16.73	3056	3.86
#22	THC:0036	SLy	2.06	2.70	1.00	73	41.24	15.58	3459	3.05
#23	BAM:0002	2H	2.83	2.70	1.00	436	69.25	14.09	1871	4.99
#24	BAM:0003	ALF2	1.99	2.70	1.00	137	53.94	17.20	2720	2.96
#25	BAM:0004	ALF2	1.99	2.70	1.00	136	48.45	19.38	2791	4.36
#26	BAM:0009	ALF2	1.99	2.50	1.27	215	61.75	11.74	2391	3.47
#27	BAM:0010	ALF2	1.99	2.70	1.16	138	51.85	17.83	2633	3.27
#28	BAM:0022	ENG	2.25	2.70	1.00	89	44.59	18.48	2933	2.51
#29	BAM:0035	H4	2.03	2.70	1.00	208	52.08	13.72	2406	4.66
#30	BAM:0036	H4	2.03	2.70	1.00	207	56.85	15.20	2526	5.40
#31	BAM:0046	H4	2.03	2.70	1.16	210	59.62	13.42	2344	3.52
#32	BAM:0048	H4	2.03	2.75	1.25	191	54.30	13.08	2416	3.46
#33	BAM:0053	H4	2.03	2.75	1.50	205	58.18	7.98	2471	3.14
#34	BAM:0057	H4	2.03	2.75	1.75	223	105.69	2.19	2490	0.90
#35	BAM:0058	MPA1	2.47	2.70	1.00	114	49.16	19.34	2720	2.23
#36	BAM:0059	MS1	2.77	2.70	1.16	328	67.56	15.08	2065	3.32
#37	BAM:0061	MS1	2.77	2.70	1.00	323	67.84	14.13	2013	2.47
#38	BAM:0065	MS1b	2.76	2.70	1.00	287	62.25	18.77	2043	3.10
#39	BAM:0070	MS1b	2.76	2.75	1.00	260	61.82	14.85	2120	3.52
#40	BAM:0080	MS1b	2.76	2.50	1.27	439	76.67	11.83	2084	3.61
#41	BAM:0089	MS1b	2.76	2.75	1.25	266	65.45	12.67	2067	2.97
#42	BAM:0090	MS1b	2.76	3.20	1.00	112	48.33	17.35	2306	4.26
#43	BAM:0091	MS1b	2.76	2.75	1.50	278	59.63	8.37	1956	2.14
#44	BAM:0092	MS1b	2.76	3.40	1.00	79	41.57	19.48	2433	4.34
#45	BAM:0093	MS1b	2.76	2.75	1.75	293	73.70	5.84	1970	1.55
#46	BAM:0098	SLy	2.06	2.70	1.00	73	41.88	17.33	3340	3.29
#47	BAM:0107	SLy	2.06	2.46	1.22	135	49.78	13.86	2784	3.03
#48	BAM:0121	SLy	2.06	2.50	1.27	124	49.16	15.10	2787	3.89
#49	BAM:0122	SLy	2.06	2.60	1.17	95	45.95	13.03	3050	3.87
#50	BAM:0123	SLy	2.06	2.70	1.16	74	41.61	17.79	3362	3.31
#51	BAM:0124	SLy	2.06	2.50	1.50	134	50.27	8.96	2951	2.95
#52	BAM:0126	SLy	2.06	2.75	1.25	68	41.66	12.50	3460	1.89
#53	BAM:0128	SLy	2.06	2.75	1.50	76	45.73	6.60	3339	1.45
#54	whisky	Gamma2	1.82	2.90	1.00	277	-	-	2127	-
#55	whisky	Gamma2	1.82	2.85	1.00	324	-	-	2183	-
#56	whisky	Gamma2	1.82	2.80	1.00	379	-	-	2061	-
#57	whisky	Gamma2	1.82	2.75	1.00	442	-	-	2004	-
#58	whisky	Gamma2	1.82	2.70	1.00	516	-	-	1930	-
#59	whisky	GNH3	1.98	2.50	1.00	345	-	-	2272	-
#60	whisky	GNH3	1.98	2.55	1.00	306	-	-	2302	-
#61	whisky	GNH3	1.98	2.60	1.00	271	-	-	2425	-
#62	whisky	GNH3	1.98	2.65	1.00	240	-	-	2479	-

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ID	Code/CoRe-ID	EOS	M_{TOV} [$M_{\odot}$]	M [$M_{\odot}$]	q	$\kappa_{\text{eff}}^{\text{T}}$	$\frac{\Delta t_{\text{min}}}{M}$	$\frac{ rh_{22}(t_{\text{max}_1}) }{M}$ [10^{-2}]	f_2 (Hz)	$\frac{ r\tilde{h}_{22}(f_2) }{M}$ [10^{-4}]
#63	whisky	GNH3	1.98	2.70	1.00	213	-	-	2595	-
#64	whisky	ALF2	1.99	2.45	1.00	236	-	-	2443	-
#65	whisky	ALF2	1.99	2.50	1.00	212	-	-	2493	-
#66	whisky	ALF2	1.99	2.55	1.00	190	-	-	2574	-
#67	whisky	ALF2	1.99	2.60	1.00	170	-	-	2655	-
#68	whisky	ALF2	1.99	2.65	1.00	153	-	-	2693	-
#69	whisky	H4	2.03	2.50	1.00	327	-	-	2247	-
#70	whisky	H4	2.03	2.55	1.00	292	-	-	2377	-
#71	whisky	H4	2.03	2.60	1.00	260	-	-	2356	-
#72	whisky	H4	2.03	2.65	1.00	232	-	-	2449	-
#73	whisky	H4	2.03	2.70	1.00	208	-	-	2501	-
#74	whisky	SLy	2.06	2.50	1.00	118	-	-	3154	-
#75	whisky	SLy	2.06	2.55	1.00	105	-	-	3235	-
#76	whisky	SLy	2.06	2.60	1.00	93	-	-	3229	-
#77	whisky	SLy	2.06	2.65	1.00	82	-	-	3282	-
#78	whisky	SLy	2.06	2.70	1.00	73	-	-	3338	-
#79	whisky	SLy	2.06	2.60	1.08	93	-	-	3212	-
#80	whisky	APR4	2.20	2.55	1.00	85	-	-	3229	-
#81	whisky	APR4	2.20	2.60	1.00	75	-	-	3279	-
#82	whisky	APR4	2.20	2.65	1.00	67	-	-	3373	-
#83	whisky	APR4	2.20	2.70	1.00	60	-	-	3462	-
#84	sacra	APR4	2.20	2.70	1.00	60	-	-	3450	-
#85	sacra	APR4	2.20	2.70	1.00	60	-	-	3255	-
#86	sacra	APR4	2.20	2.70	1.00	60	-	-	3330	-
#87	sacra	APR4	2.20	2.60	1.00	76	-	-	3210	-
#88	sacra	SLy	2.06	2.70	1.25	76	-	-	3340	-
#89	sacra	SLy	2.06	2.70	1.16	74	-	-	3320	-
#90	sacra	SLy	2.06	2.70	1.08	73	-	-	3390	-
#91	sacra	SLy	2.06	2.70	1.00	73	-	-	3480	-
#92	sacra	SLy	2.06	2.60	1.00	93	-	-	3160	-
#93	sacra	ALF2	1.99	2.80	1.00	110	-	-	2920	-
#94	sacra	ALF2	1.99	2.70	1.25	139	-	-	2820	-
#95	sacra	ALF2	1.99	2.70	1.16	138	-	-	2650	-
#96	sacra	ALF2	1.99	2.70	1.08	137	-	-	2770	-
#97	sacra	ALF2	1.99	2.70	1.00	137	-	-	2770	-
#98	sacra	ALF2	1.99	2.60	1.00	170	-	-	2630	-
#99	sacra	H4	2.03	2.90	1.00	133	-	-	2930	-
#100	sacra	H4	2.03	2.80	1.15	168	-	-	2505	-
#101	sacra	H4	2.03	2.80	1.00	166	-	-	2780	-
#102	sacra	H4	2.03	2.70	1.25	214	-	-	2320	-
#103	sacra	H4	2.03	2.70	1.25	214	-	-	2340	-
#104	sacra	H4	2.03	2.70	1.25	214	-	-	2300	-

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ID	Code/CoRe-ID	EOS	M_{TOV} [$M_{\odot}$]	M [$M_{\odot}$]	q	$\kappa_{\text{eff}}^{\text{T}}$	$\frac{\Delta t_{\text{min}}}{M}$	$\frac{ rh_{22}(t_{\text{max}_1}) }{M}$ [10^{-2}]	f_2 (Hz)	$\frac{ r\tilde{h}_{22}(f_2) }{M}$ [10^{-4}]
#105	sacra	H4	2.03	2.70	1.16	210	-	-	2440	-
#106	sacra	H4	2.03	2.70	1.08	208	-	-	2475	-
#107	sacra	H4	2.03	2.70	1.00	208	-	-	2590	-
#108	sacra	H4	2.03	2.70	1.00	208	-	-	2530	-
#109	sacra	H4	2.03	2.70	1.00	208	-	-	2490	-
#110	sacra	H4	2.03	2.60	1.17	264	-	-	2370	-
#111	sacra	H4	2.03	2.60	1.08	261	-	-	2260	-
#112	sacra	H4	2.03	2.60	1.00	260	-	-	2310	-
#113	sacra	MS1	2.77	2.90	1.23	224	-	-	2120	-
#114	sacra	MS1	2.77	2.90	1.00	219	-	-	2110	-
#115	sacra	MS1	2.77	2.80	1.00	266	-	-	2045	-
#116	sacra	MS1	2.77	2.70	1.25	332	-	-	2110	-
#117	sacra	MS1	2.77	2.70	1.16	328	-	-	2050	-
#118	sacra	MS1	2.77	2.70	1.08	326	-	-	2050	-
#119	sacra	MS1	2.77	2.70	1.00	325	-	-	2020	-
#120	sacra	MS1	2.77	2.60	1.00	398	-	-	1960	-

Table A.1: All NR configurations employed to derive the quasi-universal relations. It includes simulations of the CoRe database [42] labeled as ‘THC’ or ‘BAM’, and additionally results published in [168] labeled as ‘whisky’ and [166] labeled as ‘sacra’. We highlight the simulations which have been used for the injection study discussed in Chapter 6. The individual columns refer to: the number of simulation, the employed code, the EOS, the maximum TOV mass for the employed EOS, the total mass of the system, the mass ratio, the effective tidal coupling constant, the time at which the first postmerger minimum appears, the amplitude of the first postmerger maximum, the f_2 frequency, and the amplitude of the f_2 frequency peak.

APPENDIX B

SPIN-WEIGHTED SPHERICAL HARMONICS

The explicit expression for the spin-weighted spherical harmonics can be written in terms of Wigner d-functions,

$$^{-s}Y_{lm} = (-1)^{-s} \sqrt{\frac{2l+1}{4\pi}} d_{m,s}^l(\iota) e^{im\phi}, \quad (\text{B.1})$$

where

$$d_{m,s}^l(\iota) = \sum_{k=k_1}^{k_2} \frac{(-1)^k \sqrt{(l+m)!(l-m)!(l+s)!(l-s)!}}{(l+m-k)!(l-s-k)!k!(k+s-m)!} \left(\cos\left(\frac{\iota}{2}\right)\right)^{2l+m-s-2k} \left(\sin\left(\frac{\iota}{2}\right)\right)^{2k+s-m}, \quad (\text{B.2})$$

with $k_1 = \max(0, m-s)$ and $k_2 = \min(l+m, l-s)$. For example,

$$^{-2}Y_{22} = \sqrt{\frac{5}{64\pi}} (1 + \cos \iota)^2 e^{2i\phi}, \quad (\text{B.3})$$

$$^{-2}Y_{21} = \sqrt{\frac{5}{16\pi}} \sin \iota (1 + \cos \iota) e^{i\phi}, \quad (\text{B.4})$$

$$^{-2}Y_{20} = \sqrt{\frac{15}{32\pi}} \sin^2 \iota, \quad (\text{B.5})$$

$$^{-2}Y_{2-1} = \sqrt{\frac{5}{16\pi}} \sin \iota (1 - \cos \iota) e^{-i\phi}, \quad (\text{B.6})$$

$$^{-2}Y_{2-2} = \sqrt{\frac{5}{64\pi}} (1 - \cos \iota)^2 e^{-2i\phi}. \quad (\text{B.7})$$

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LIST OF ACRONYMS

AS	Axion star
ASD	Amplitude spectral density
BBH	Binary black hole
BH	Black hole
BNS	Binary neutron star
CBC	Compact binary coalescence
CoRe	Computational Relativity [42]
ECO	Exotics compact object
EFEs	Einstein field equations
EM	Electromagnetic
EOS	Equation of state
GR	Gravitational relativity
GRB	Gamma ray burst
GW	Gravitational wave
GWTC-1	Gravitational wave Transient Catalog
HMNS	Hypermassive neutron star
IMBH	Intermediate black hole
IMP	Inspiral merger postmerger
IMR	Inspiral merger ringdown
ISCO	Innermost stable circular orbit
LIGO	Laser Interferometer Gravitational-Wave Observatory
LLF	Local Lorentz frame

MAP	Maximum a Posteriori
MCMC	Markov chain Monte Carlo
MH	Metropolis-Hasting
MLE	Maximum likelihood estimation
MNS	Massive neutron star
NR	Numerical relativity
NS	Neutron star
NSBH	Neutron star black hole
O1	Observing run 1
O2	Observing run 2
PDF	Probability density function
PIT	Probability integral transform
PSD	Power spectral density
QNM	Quasi-normal mode
RMS	Root-mean-square
RJMCMC	Reversible-jump Markov chain Monte Carlo
SMBH	Supermassive black hole
SMNS	Supermassive neutron star
SNR	Signal-to-noise ratio
SPA	Stationary phase approximation
SWSHs	Spin-weighted spheroidal harmonics
TI	Thermodynamic integration
TOV	Tolman–Oppenheimer–Volkoff
TT	Transverse-traceless

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PUBLIC SUMMARY

Whenever we look up at the stars in space, we are absorbed by the magnificence of the universe. By connecting a group of visible stars, we have constellations. We wonder what these stars are, how these stars move, where these stars come from and go eventually. By studying the starlight, we obtain rich information about the stars including their position, velocity and constituents. In fact, apart from light, there are many other cosmic messengers such as cosmic rays and neutrinos carrying information about the universe, travelling for billions light-years and reaching our Earth. In 14 September 2015, we found gravitational waves, a new member of the family of cosmic messengers.

Newtonian physics

The story begins with Isaac Newton (1642-1727) who wrote a book in 1687, *Philosophiæ Naturalis Principia Mathematica*, showing that our world can be accurately described by mathematical expressions called physics laws. One of the most profound results is the proof of elliptical orbits of the planets around the Sun as a consequence of a gravitational force satisfying the inverse square law. This force can travel through space and act on other objects. In addition, the force is universal in a sense that the nature of force between the Sun and the Earth is the same as that between us and the Earth and the only difference is the size of the forces. Despite the success of Newtonian gravity, it has serious drawbacks in its formalism. Theoretically, it assumes an infinite speed of gravity. This means that if the Sun is displaced by a tiny amount by some invisible hands, we on the Earth can feel it immediately despite the distance between the Sun and the Earth being huge. Such an infinite speed of transmission contradicts our understanding of causality. Moreover, empirically it fails to explain why the Mercury is not in a perfect elliptical orbit but the point of the closest approach advances a little bit every cycle.

General relativity

In 1915, Albert Einstein (1879-1955) proposed an alternative theory of gravity, that is the famous General Relativity (GR). Thanks to this theory, today with our mobile phone we have a global positioning system (GPS) to locate our positions. In his formalism, Einstein gave up the notion that gravity is a force, replacing it by spacetime curvature. Mathematically it can reduce to Newtonian gravity under weak-gravity assumptions showing that why the ‘wrong’ Newtonian gravity works so well. In other words, Newtonian gravity is a simplified version of GR. In this full version, the advance of the point of closest approach in the Mercury’s orbit can be explained accurately and is consistent with the observational measurement. The speed of gravity is no longer infinite but the same as the speed of light. GR provides not only

explanations for previously unsolved problems, but also gives new predictions including the bending of starlight around the Sun twice as much as what Newtonian gravity predicts, the existence of black holes and gravitational waves. The prediction about the bending of starlight was confirmed by the English astronomer Arthur Eddington (1882-1944) during a total solar eclipse on 29 May 1919.

A black hole is an astrophysical object having very high density to an extent that even light is not able to escape from its boundary, called the event horizon, because of its strong gravity. Until recently, there was indirect evidence of the existence of black holes from the X-ray measurements, but not direct evidence.

Gravitational waves

Gravitational waves (GWs) are ripples of the spacetime generated by accelerating masses. When GWs are incident on a human, this person will become taller and thinner at the first moment, and then shorter and fatter at the later moment, and the changes persist periodically until the GWs pass ¹⁶. But why do not we feel the existence of GWs? It is because the effect of GWs is predicted to be as tiny as a strain $\sim \mathcal{O}(10^{-21})$, which means that GWs stretch only the size of an atom $\sim \mathcal{O}(10^{-10})$ m relative to the distance between the Sun and the Earth $\sim \mathcal{O}(10^{11})$ m. Direct measurement of GWs looks impossible. Nevertheless, we have indirect evidence that GWs do exist. In 1974, a binary pulsar was discovered by Russell Alan Hulse and Joseph Hooton Taylor. It was then named Hulse-Taylor binary. The discovery of their binary pulsar earned Hulse and Taylor the Nobel Prize in Physics in 1993. Binary means that two stars are orbiting around each other. A pulsar is a rotating neutron star that emits beams of radial waves out of its magnetic poles. The emission can be observed only when the beam is pointing toward the Earth, and hence the pulsar acts like a lighthouse in space. By studying the radial waves emitted by this Hulse-Taylor binary, we can study the dynamics of the binary and check whether it is consistent with GR. According to GR, GWs carry energy away, shrink the separation between two stars and hence decrease the period. Indeed the orbital decay of the Hulse-Taylor binary is found to be consistent with GR.

GW150914 - first GW event from a binary black hole

In the early 1960s, physicists first suggested to use interferometers for the direct detection of GWs. Interferometer is a sensitive machine making use of laser beams to perform accurate and precise displacement measurements. With the advance of technology, in 2015, Advanced Laser Interferometer Gravitational-Wave Observatory (LIGO) in the United States was the first interferometer sensitive enough to begin GW observations. There are two Advanced LIGO detectors, one in Livingston and the other one in Hanford, separated by ~ 1000 km. More detectors are essential to confidentially detect GWs. If one interferometer detects the target signal but the other does not, we rather believe that it results from noises around the interferometer. However, if both interferometers detect the target signal simultaneously, we believe that its origin is astronomical. Moreover, each interferometer has some blind spots to certain directions, so that more interferometers ensure a full coverage of the entire sky. There are a lot of different GW sources but physicists focused on the collisions of binary black holes and binary neutron stars which have been modelled extensively and accurately. Dramatically, one day the two Advanced LIGO interferometers seemed to simultaneously detect GWs. Further analyses appeared to show that they came from a binary black hole collision. After all analyses were done, the signal was announced to be hardware-injected. It means that it was not astronomical but one deliberately used computer programs to vibrate the mirrors to mimic GWs. The

¹⁶This is the plus polarization, one of the two GW polarizations. Its animation is shown at the bottom-right corner of this book.

purpose of the hardware injection was to ensure all parts in the interferometer were working as what people expected without any human bias. We called it a blinded experiment. Fortunately, on 14 September 2015, they detected again GWs resulting from a pair of black holes of around 36 and 29 solar masses¹⁷ colliding each other. At that time, physicists thought probably it was another hardware injection. Instead, it was a real GW detection, the first in human history! It was named GW150914 (from “**G**ravitational **W**aves” and the date of observation **2015-09-14**). This detection, exactly 100 years after the publication of GR, provided direct evidence of the existence of GWs and black holes. In 2017, the Nobel Prize in Physics was awarded to three physicists: Rainer Weiss, Kip Thorne and Barry Barish, for their role in the direct detection of GWs.

GW170817 - first GW event from a binary neutron star

After the participation of the Advanced Virgo detector in Italy in August 2017, the detector network became more sensitive to GWs and was able to see deeper into the universe. On 14 August 2017 the first three-detectors GW event from a binary black hole was observed, namely GW170814. After three days, on 17 August 2017, they observed GWs from the collision of a binary neutron star, namely GW170817. A neutron star is a neutron-rich astrophysical object having very high density but slightly less dense than a black hole. In other words, if a neutron star were slightly heavier, it would collapse to form a black hole. When two neutron stars merge, it generates a highly neutron-rich environment where many heavy radioactive elements such as gold and platinum are produced and their subsequent radioactive decay emits very strong light known as the kilonova. After 1.7s of the GW merger signal, a short-duration (~ 2 s) gamma ray burst from a consistent sky location was also observed by the Fermi Gamma-ray Space Telescope (FGST) and INTErnational Gamma-Ray Astrophysics Laboratory (INTEGRAL) spacecraft. GW170817 marked the start of multi-messenger astronomy, *i.e.* observing the same astronomical event via various cosmic messengers such as light, GWs, neutrinos and cosmic rays. For all GWs seen so far, no accompanying neutrinos and cosmic rays have been observed because the GW sources were too far away from the Earth.

Study of black hole formation via GWs

The discovery of GWs offers a new way for us to study the mechanism of black hole formation. As we all know, the energy of a star is powered by nuclear fusion. When a star runs out of its fuel, it loses pressure to support against its gravity. If the star is massive enough $\sim \mathcal{O}(10) M_{\odot}$, it will explode known as the supernova and the core will collapse to form a black hole. The mass of a black hole varies over a large range, for example stellar black hole $< 10^2 M_{\odot}$, intermediate mass black hole $[10^2 - 10^5] M_{\odot}$ and supermassive black hole $> 10^5 M_{\odot}$. Stellar and supermassive black holes have long been found by X-ray observations. but not intermediate mass black holes. So intermediate mass black holes connect the gap between the stellar and supermassive black holes and their discovery will be a key to the formation of supermassive black holes, which is still unknown. For stellar black holes, there are theoretical and observational arguments suggesting that stellar evolution may not produce black holes with masses less than $5 M_{\odot}$. In addition, the possible maximum mass of a neutron star is predicted to be $\sim [2 - 3] M_{\odot}$. The range of $\sim [2 - 5] M_{\odot}$ is the so-called lower mass gap in which no black holes are observed so far. Physicists are curious about the reality of this lower mass gap. On the other hand, there is also theoretical arguments suggesting that no black holes with masses $\sim [64 - 135] M_{\odot}$, *i.e.* called the upper mass gap, can form via core collapse of a single star. Therefore, observations

¹⁷Solar mass $M_{\odot}$ means the mass of the Sun. It is a standard unit of mass in astronomy that is equal to around 2×10^{30} kg.

of black holes in the mass gaps or above the upper mass gap will give rich scientific insights to the mechanism of black hole formation.

Surprisingly, on 25 April 2019, a GW event GW190425 was found. It was of interest because the component masses range from 1.12 to 2.52 $M_{\odot}$, consistent with the individual binary components being neutron stars, or one or both binary components being black holes. If it was a binary neutron star, because of its high total mass 3.4 $M_{\odot}$, it will have been very different from the known population of binary neutron star. If it was a binary black hole, it will have contained the first black hole observed in the lower mass gap. In any case, there are interesting implications for the formation of this system. Moreover, on 21 May 2019, a binary black hole GW event GW190521 was detected which has a total mass of 150 $M_{\odot}$ and is the heaviest system we have ever observed via GWs. The collision of the two black holes resulted in a black hole with mass 142 $M_{\odot}$ which belongs to the class of intermediate mass black holes. In other words, we directly observed the formation of an intermediate mass black hole! Furthermore, the heavier component mass (85 $M_{\odot}$) sits in the upper mass gap. This means it was not formed via core collapse of a single star but something else. Probably it was a second generation black hole, *i.e.* resulting from two lighter black holes which are formed via core collapses, known as hierarchical mergers.

Study of the nature of a black hole via GWs

We can also make use of GWs to study the nature of very dense astrophysical objects including black holes. Do they really behave as what GR describes? Theoretically, there can be strange black hole mimickers such as wormholes which emit similar GWs as black hole. My research is to distinguish these black hole mimickers from black holes. In particular, I focus on the final state of the binary merger.

In the case of a binary black hole merger, the resulting remnant will again be a black hole which is excited and emits GWs with characteristic damping frequencies and damping times, called ringdown. According to the no-hair conjecture in GR, a black hole has no internal structure (*i.e.* no hairs) and is determined uniquely by its mass, rotation and charge which is expected to be zero for astrophysical objects. It sounds bizarre because a black hole has no memory about where it is formed and information seems to be lost. Therefore we want to test whether a black hole has hairs (*i.e.* whether it is a GR black hole or something else) via GWs. However, a stationary black hole does not emit GWs and so we need an excited black hole which can be found after a binary black hole merger. As a manifestation of the conjecture, the damping frequencies and the damping times are expected to depend on only its mass and rotation. To perform a consistency test, we include possible deviation parameters to the GR ringdown model and observe whether they are consistent with zero according to the data. If it is a GR black hole, they should be consistent with zero. If not, it signals either that GR is wrong, or that a new, very dense astrophysical object has been discovered. We provided a proof of principle that the existing Advanced LIGO-Virgo detectors operating at design sensitivity will be capable of testing the no-hair conjecture with an accuracy of a few percent. Details can be found in Chapter 4.

On the other hand, strange stars can be so strange that they have not yet been envisaged and the ringdown model with additional deviation parameters might not be sufficient to distinguish them from black holes. So we need a model-independent way to detect and characterize these strange stars. One of the expected signatures of these stars are gravitational wave echoes that are repeated burst signals at regular time intervals following the ringdown stage. Echoes exist if any incoming GWs can escape from the strange star (*i.e.* either no event horizon exists or some reflective boundary above the event horizon exists). Theoretically, this is strictly prohibited for GR black holes but not for some particular black hole mimickers. We created an algorithm to

look for gravitational wave echoes in the data but we do not find any. Details can be found in Chapter 5.

Study of the nature of a neutron star via GWs

As we have mentioned earlier, a neutron star is a very dense astrophysical object to an extent that it almost collapses to form a black hole because of its high mass. Physicists are interested in the equation of state, *i.e.* how matter behaves under such an extreme gravity condition. In particular, we study the equation of state using the remnant resulting from a binary neutron star merger. The remnant can be a black hole or an even more massive neutron star depending on the total component masses and the equation of state. In the latter case, the remnant will emit a very characteristic dominant frequency $f_2 \sim 2 - 4$ kHz which contains rich information about the equation of state. Although the existing interferometers are not sensitive enough to detect it, the future third generations of ground-based observatory can see it. To prepare for the future data analysis, we need to have a good model which is currently poorly known. We presented a simple model which on average has 80% match to the computer simulations. The method of using such a model was also demonstrated. Details can be found in Chapter 6.

Future of gravitational wave astronomy

The universe is always amazing and surprising. The discovery of GWs gives much excitement in the field of physics. Physicists around the world cannot wait to translate the messages carried by this new cosmic messenger. A lot of ground-based GW interferometers will be built around the world including India, Japan and China. Moreover, space-based interferometers are also planned. The future of GW astronomy is very promising. Many things are out there waiting for us to discover. Personally, I would like to ask this cosmic messenger how this universe formed. I believe that it will honestly tell me.

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¹⁸It stands for ‘Good Game Well-Played’ that is a jargon people say at the end of a game.