University of Groningen

Projective geometry for blueprints

López Peña , Javier; Lorscheid, Oliver

Published in:
Comptes Rendus Mathématique

DOI:
[10.1016/j.crma.2012.05.001](https://doi.org/10.1016/j.crma.2012.05.001)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2012

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):
López Peña , J., & Lorscheid, O. (2012). Projective geometry for blueprints. *Comptes Rendus Mathématique*, 350(9-10), 455-458. <https://doi.org/10.1016/j.crma.2012.05.001>

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Algebra/Algebraic Geometry

Projective geometry for blueprints

*Geometrie projective pour les canevas bleus*Javier López Peña^{a,1}, Oliver Lorscheid^b^a Department of Mathematics, University College London, 25 Gower Street, London WC1E 6BT, United Kingdom^b Department of Mathematics, University of Wuppertal, Gaußstr. 20, 42097 Wuppertal, Germany

ARTICLE INFO

Article history:

Received 8 March 2012

Accepted after revision 27 April 2012

Available online 17 May 2012

Presented by Christophe Soulé

ABSTRACT

In this Note, we generalize the Proj-construction from usual schemes to blue schemes. This yields the definition of projective space and projective varieties over a blueprint. In particular, it is possible to descend closed subvarieties of a projective space to a canonical $\mathbb{F}_1$ -model. We discuss this in case of the Grassmannian $\text{Gr}(2, 4)$.

© 2012 Published by Elsevier Masson SAS on behalf of Académie des sciences.

R É S U M É

Dans cette Note, nous généralisons la Proj-construction des schémas usuels aux schémas bleus. Cela entraîne la définition d'espace projectif et de variétés projectives sur un canevas bleu. En particulier, il est possible de descendre une sous-variété fermée d'un espace projectif en un $\mathbb{F}_1$ -modèle canonique. Nous discutons cela dans le cas de la Grassmannienne $\text{Gr}(2, 4)$.

© 2012 Published by Elsevier Masson SAS on behalf of Académie des sciences.

1. Introduction

Blueprints are a common generalization of commutative (semi)rings and monoids. The associated geometric objects, blue schemes, are therefore a common generalization of usual scheme theory and $\mathbb{F}_1$ -geometry (as considered by Kato [5], Deitmar [3] and Connes–Consani [2]). The possibility of forming semiring schemes allows us to talk about idempotent schemes and tropical schemes (cf. [11]). All this is worked out in [9].

It is known, though not covered in literature yet, that the Proj-construction from usual algebraic geometry has an analogue in $\mathbb{F}_1$ -geometry (after Kato, Deitmar and Connes–Consani). In this note we describe a generalization of this to blueprints. Privately, Koen Thas has announced a treatment of Proj for monoidal schemes (see [13]).

We follow the notations and conventions of [10]. Namely, all blueprints that appear in this note are proper and with a zero. We remark that the following constructions can be carried out for the more general notion of a blueprint as considered in [9]; the reason that we restrict to proper blueprints with a zero is that this allows us to adopt a notation that is common in $\mathbb{F}_1$ -geometry.

Namely, we denote by $\mathbb{A}_B^n$ the (blue) affine n -space $\text{Spec}(B[T_1, \dots, T_n])$ over a blueprint B . In case of a ring, this does not equal the usual affine n -space since $B[T_1, \dots, T_n]$ is not closed under addition. Therefore, we denote the usual affine

E-mail addresses: jlp@math.ucl.ac.uk (J. López Peña), lorscheid@math.uni-wuppertal.de (O. Lorscheid).

¹ J. López Peña's research was supported by MCIM grant MTM2010-20940-C02-01, research group FQM-266 (Junta de Andalucía) and Max-Planck Institute for Mathematics in Bonn.

n -space over a ring B by ${}^+\mathbb{A}_B^n = \text{Spec}(B[T_1, \dots, T_n]^+)$. Similarly, we use a superscript “+” for the usual projective space ${}^+\mathbb{P}_B^n$ and the usual Grassmannian $\text{Gr}(k, n)_B^+$ over a ring B .

2. Graded blueprints and Proj

Let B be a blueprint and M a subset of B . We say that M is *additively closed* in B if for all additive relations $b \equiv \sum a_i$ with $a_i \in M$ also b is an element of M . Note that, in particular, 0 is an element of M . A *graded blueprint* is a blueprint B together with additively closed subsets B_i for $i \in \mathbb{N}$ such that $1 \in B_0$, such that for all $i, j \in \mathbb{N}$ and $a \in B_i, b \in B_j$, the product ab is an element of B_{i+j} and such that for every $b \in B$, there are a unique finite subset I of $\mathbb{N}$ and unique non-zero elements $a_i \in B_i$ for every $i \in I$ such that $b = \sum a_i$. An element of $\bigcup_{i \geq 0} B_i$ is called *homogeneous*. If $a \in B_i$ is non-zero, then we say, more specifically, that a is *homogeneous of degree i* .

We collect some immediate facts for a graded blueprint B as above. The subset B_0 is multiplicatively closed, i.e. B_0 can be seen as a subblueprint of B . The subblueprint B_0 equals B if and only if for all $i > 0$, $B_i = \{0\}$. In this case we say that B is *trivially graded*. By the uniqueness of the decomposition into homogeneous elements, we have $B_i \cap B_j = \{0\}$ for $i \neq j$. This means that the union $\bigcup_{i \geq 0} B_i$ has the structure of a wedge product $\bigvee_{i \geq 0} B_i$. Since $\bigvee_{i \geq 0} B_i$ is multiplicatively closed, it can be seen as a subblueprint of B . We define $B_{\text{hom}} = \bigvee_{i \geq 0} B_i$ and call the subblueprint B_{hom} the *homogeneous part* of B .

Let S be a multiplicative subset of B . If b/s is an element of the localization $S^{-1}B$ where f is homogeneous of degree i and s is homogeneous of degree j , then we say that b/s is a homogeneous element of degree $i - j$. We define $S^{-1}B_0$ as the subset of homogeneous elements of degree 0. It is multiplicatively closed, and inherits thus a subblueprint structure from $S^{-1}B$. If S is the complement of a prime ideal $\mathfrak{p}$, then we write $B_{(\mathfrak{p})}$ for the subblueprint $(B_{\mathfrak{p}})_0$ of homogeneous elements of degree 0 in $B_{\mathfrak{p}}$.

An ideal I of a graded blueprint B is called *homogeneous* if it is generated by homogeneous elements, i.e. if for every $c \in I$, there are homogeneous elements $p_i, q_j \in I$ and elements $a_i, b_j \in B$ and an additive relation $\sum a_i p_i + c = \sum b_j q_j$ in B .

Let B be a graded blueprint. Then we define $\text{Proj } B$ as the set of all homogeneous prime ideals $\mathfrak{p}$ of B that do not contain $B_{\text{hom}}^+ = \bigvee_{i > 0} B_i$. The set $X = \text{Proj } B$ comes together with the topology that is defined by the basis

$$U_h = \{\mathfrak{p} \in X \mid h \notin \mathfrak{p}\}$$

where h ranges through B_{hom} and with a structure sheaf $\mathcal{O}_X$ that is the sheafification of the association $U_h \mapsto B[h^{-1}]_0$ where $B[h^{-1}]$ is the localization of B at $S = \{h^i\}_{i \geq 0}$.

Note that if B is a ring, the above definitions yield the usual construction of $\text{Proj } B$ for graded rings. In complete analogy to the case of graded rings, one proves the following theorem:

Theorem 1. *The space $X = \text{Proj } B$ together with $\mathcal{O}_X$ is a blue scheme. The stalk at a point $\mathfrak{p} \in \text{Proj } B$ is $\mathcal{O}_{X, \mathfrak{p}} = B_{(\mathfrak{p})}$. If $h \in B_{\text{hom}}^+$, then $U_h \simeq \text{Spec } B[h^{-1}]_0$. The inclusions $B_0 \hookrightarrow B[h^{-1}]_0$ yield morphisms $\text{Spec } B[h^{-1}]_0 \rightarrow \text{Spec } B_0$, which glue to a structural morphism $\text{Proj } B \rightarrow \text{Spec } B_0$. $\square$*

If B is a graded blueprint, then the associated semiring B^+ inherits a grading. Namely, let $B_{\text{hom}} = \bigvee_{i \geq 0} B_i$ the homogeneous part of B . Then we can define B_i^+ as the additive closure of B_i in B^+ , i.e. as the set of all $b \in B$ such that there is an additive relation of the form $b \equiv \sum a_k$ in B with $a_k \in B_i$. Then $\bigvee B_i^+$ defines a grading of B^+ . Similarly, the grading of B induces a grading on a tensor product $B \otimes_C D$ with respect to blueprint morphisms $C \rightarrow B$ and $C \rightarrow D$ under the assumption that the image of $C \rightarrow B$ is contained in B_0 . Consequently, a grading of B implies a grading of $B_{\text{inv}} = B \otimes_{\mathbb{F}_1} \mathbb{F}_{12}$ (see [9, Lemma 1.4] and [10, p. 11]) and of the ring $B_{\mathbb{Z}}^+ = B_{\text{inv}}^+$. Analogously, if both B and D are graded and the image of C lies in both B_0 and D_0 , then $B \otimes_C D$ inherits a grading from the gradings of B and D .

3. Projective space

The functor Proj allows the definition of the projective space $\mathbb{P}_B^n$ over a blueprint B . Namely, the free blueprint $C = B[T_0, \dots, T_n]$ over B comes together with a natural grading (cf. [9, Section 1.12] for the definition of free blueprints). Namely, C_i consists of all monomials $bT_0^{e_0} \cdots T_n^{e_n}$ such that $e_0 + \dots + e_n = i$ where $b \in B$. Note that $C_0 = B$ and $C_{\text{hom}} = C$. The projective space $\mathbb{P}_B^n$ is defined as $\text{Proj } B[T_0, \dots, T_n]$. It comes together with a structure morphism $\mathbb{P}_B^n \rightarrow \text{Spec } B$.

In case of $B = \mathbb{F}_1$, the projective space $\mathbb{P}_{\mathbb{F}_1}^n$ is the monoidal scheme that is known from $\mathbb{F}_1$ -geometry (see [4], [1, Section 3.1.4]) and [10, Ex. 1.6]). The topological space of $\mathbb{P}_{\mathbb{F}_1}^n$ is finite. Its points correspond to the homogeneous prime ideals $(S_i)_{i \in I}$ of $\mathbb{F}_1[S_0, \dots, S_n]$ where I ranges through all proper subsets of $\{0, \dots, n\}$.

In case of a ring B , the projective space $\mathbb{P}_B^n$ does not coincide with the usual projective space since the free blueprint $B[S_0, \dots, S_n]$ is not a ring, but merely the blueprint of all monomials of the form $bS_0^{e_0} \cdots S_n^{e_n}$ with $b \in B$. However, the associated scheme ${}^+\mathbb{P}_B^n = (\mathbb{P}_B^n)^+$ coincides with the usual projective space over B , which equals $\text{Proj } B[S_0, \dots, S_n]^+$.

4. Closed subschemes

Let $\mathcal{X}$ be a scheme of finite type. By an $\mathbb{F}_1$ -model of $\mathcal{X}$ we mean a blue scheme X of finite type such that $X_{\mathbb{Z}}^+$ is isomorphic to $\mathcal{X}$. Since a finitely generated $\mathbb{Z}$ -algebra is, by definition, generated by a finitely generated multiplicative

Fig. 1. Points of the Grassmannian $\text{Gr}(2, 4)_{\mathbb{F}_1}$. Generator x_{ij} belonging to an ideal is depicted as segment $i-j$ in $\begin{smallmatrix} 4 & \square & 3 \\ 1 & & 2 \end{smallmatrix}$.

subset as a $\mathbb{Z}$ -module, every scheme of finite type has an $\mathbb{F}_1$ -model. It is, on the contrary, true that a scheme of finite type possesses a large number of $\mathbb{F}_1$ -models.

Given a scheme $\mathcal{X}$ with an $\mathbb{F}_1$ -model X , we can associate to every closed subscheme $\mathcal{Y}$ of $\mathcal{X}$ the following closed subscheme Y of X , which is an $\mathbb{F}_1$ -model of $\mathcal{Y}$. In case that $X = \text{Spec } B$ is the spectrum of a blueprint $B = A//\mathcal{R}$, and thus $\mathcal{X} \simeq \text{Spec } B_{\mathbb{Z}}^+$ is an affine scheme, we can define Y as $\text{Spec } C$ for $C = A//\mathcal{R}(Y)$ where $\mathcal{R}(Y)$ is the pre-addition that contains $\sum a_i \equiv \sum b_j$ whenever $\sum a_i = \sum b_j$ holds in the coordinate ring $\Gamma \mathcal{Y}$ of $\mathcal{Y}$. This is a process that we used already in [10, Section 3].

Since localizations commute with additive closures, i.e. $(S^{-1}B)_{\mathbb{Z}}^+ = S^{-1}(B_{\mathbb{Z}}^+)$ where S is a multiplicative subset of B , the above process is compatible with the restriction to affine opens $U \subset X$. This means that given $U = \text{Spec}(S^{-1}B)$, which is an $\mathbb{F}_1$ -model for $\mathcal{X}' = U_{\mathbb{Z}}^+$, then the $\mathbb{F}_1$ -model Y' that is associated to the closed subscheme $\mathcal{Y}' = \mathcal{X}' \times_{\mathcal{X}} \mathcal{Y}$ of $\mathcal{X}'$ by the above process is the spectrum of the blueprint $S^{-1}C$. Consequently, we can associate with every closed subscheme $\mathcal{Y}$ of a scheme $\mathcal{X}$ with an $\mathbb{F}_1$ -model X a closed subscheme Y of X , which is an $\mathbb{F}_1$ -model of $\mathcal{Y}$; namely, we apply the above process to all affine open subschemes of $\mathcal{X}$ and glue them together, which is possible since additive closures commute with localizations.

In case of a projective variety, i.e. a closed subscheme $\mathcal{Y}$ of a projective space ${}^+\mathbb{P}_{\mathbb{Z}}^n$, we derive the following description of the associated $\mathbb{F}_1$ -model Y in $\mathbb{P}_{\mathbb{F}_1}^n$ by homogeneous coordinate rings. Let C be the homogeneous coordinate ring of $\mathcal{Y}$, which is a quotient of $\mathbb{Z}[S_0, \dots, S_n]^+$ by a homogeneous ideal I . Let $\mathcal{R}$ be the pre-addition on $\mathbb{F}_1[S_0, \dots, S_n]$ that consists of all relations $\sum a_i \equiv \sum b_j$ such that $\sum a_i = \sum b_j$ in C . Then $B = \mathbb{F}_1[S_0, \dots, S_n]//\mathcal{R}$ inherits a grading from $\mathbb{F}_1[S_0, \dots, S_n]$ by defining B_i as the image of $\mathbb{F}_1[S_0, \dots, S_n]_i$ in B . Note that $B \subset C$ and that the sets B_i equal the intersections $B_i = C_i \cap B$ for $i \geq 0$ where C_i is the homogeneous part of degree i of C . Then the $\mathbb{F}_1$ -model Y of $\mathcal{Y}$ equals $\text{Proj } B$.

5. $\mathbb{F}_1$ -models for Grassmannians

One of the simplest examples of projective varieties that is not toric is the Grassmannian $\text{Gr}(2, 4)$. The problem of finding $\mathbb{F}_1$ -models for Grassmannians was originally posed by Soulé in [12], and solved by the authors by obtaining a torification from the Schubert cell decomposition (cf. [8,7]). In this note, we present $\mathbb{F}_1$ -models for Grassmannians as projective varieties defined through (homogeneous) blueprints. The proposed construction fits within a more general framework for obtaining blueprints and totally positive blueprints from cluster data (cf. [6]).

Classically, the coordinate ring for $\text{Gr}(k, n)$ is obtained by quotienting out the homogeneous coordinate ring of the projective space $\mathbb{P}^{(k)-1}$ by the homogeneous ideal generated by the Plücker relations. A similar construction can be carried out using the framework of (graded) blueprints. We make that construction explicit for $\text{Gr}(2, 4)$.

Define the blueprint $\mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4)) = \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]//\mathcal{R}$ where the congruence $\mathcal{R}$ is generated by the Plücker relation $x_{12}x_{34} + x_{14}x_{23} \equiv x_{13}x_{24}$ (the signs have been picked to ensure that the totally positive part of the Grassmannian is preserved, cf. [6]). Since $\mathcal{R}$ is generated by a homogeneous relation, $\mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4))$ inherits a grading from the canonical morphism

$$\pi : \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}] \longrightarrow \mathbb{F}_1[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]//\mathcal{R}.$$

Let $\text{Gr}(2, 4)_{\mathbb{F}_1} := \text{Proj}(\mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4)))$. The base extension $\text{Gr}(2, 4)_{\mathbb{Z}}^+$ is the usual Grassmannian, and π defines a closed embedding of $\text{Gr}(2, 4)_{\mathbb{F}_1}$ into $\mathbb{P}_{\mathbb{F}_1}^5$, which extends to the classical Plücker embedding $\text{Gr}(2, 4)_{\mathbb{Z}}^+ \hookrightarrow \mathbb{P}_{\mathbb{Z}}^5$.

Homogeneous prime ideals in $\mathcal{O}_{\mathbb{F}_1}(\text{Gr}(2, 4))$ are described by their generators as the proper subsets $I \subsetneq \{x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}\}$ such that I is either contained in one of the sets $\{x_{12}, x_{34}\}$, $\{x_{14}, x_{23}\}$, $\{x_{13}, x_{24}\}$, or otherwise I has a non-empty intersection with all three of them. In other words, I cannot contain elements in two of the above sets without also containing an element of the third one. $\text{Gr}(2, 4)_{\mathbb{F}_1}$ is depicted in Fig. 1. It consists of $6 + 12 + 11 + 6 + 1 = 36$ prime ideals of ranks

0, 1, 2, 3 and 4, respectively (cf. [10, Def. 2.3] for the definition of rank), thus resulting in a model essentially different to the one presented in [8], which had 35 points corresponding to the coefficients of $N_{\text{Gr}(2,4)}(q) = 6 + 12(q-1) + 11(q-1)^2 + 5(q-1)^3 + 1(q-1)^4$. In spite of arising from different constructions, both $\mathbb{F}_1$ -models for $\text{Gr}(2,4)$ have $6 = \binom{4}{2}$ closed points, supporting the naive combinatorial interpretation of $\text{Gr}(2,4)_{\mathbb{F}_1}$. These six points correspond to the $\mathbb{F}_1$ -rational Tits points of $\text{Gr}(2,4)_{\mathbb{F}_1}$, which reflect the naive notion of $\mathbb{F}_1$ -rational points of an $\mathbb{F}_1$ -scheme (cf. [10, Section 2.2]).

As in the classical setting, the Grassmannian $\text{Gr}(2,4)_{\mathbb{F}_1}$ does admit a covering by six $\mathbb{F}_1$ -models of affine 4-space, which correspond to the open subsets of $\text{Gr}(2,4)_{\mathbb{F}_1}$ where one of the generators is non-zero. However, these $\mathbb{F}_1$ -models of affine 4-space are not the standard model $\mathbb{A}_{\mathbb{F}_1}^4 = \text{Spec}(\mathbb{F}_1[a, b, c, d])$, but the “ 2×2 -matrices” $M_{2,\mathbb{F}_1} = \text{Spec}(\mathbb{F}_1[a, b, c, d, D]//\langle ad \equiv bc + D \rangle)$ in case that one of x_{12} , x_{34} , x_{14} or x_{23} is non-zero, and the “twisted 2×2 -matrices” $M_{2,\mathbb{F}_1}^{\tau} = \text{Spec}(\mathbb{F}_1[a, b, c, d, D]//\langle ad + bc \equiv D \rangle)$ in case that one of x_{13} or x_{24} is non-zero.

References

- [1] C. Chu, O. Lorscheid, R. Santhanam, Sheaves and K -theory for $\mathbb{F}_1$ -schemes, *Adv. Math.* 229 (4) (2012) 2239–2286.
- [2] A. Connes, C. Consani, Characteristic 1, entropy and the absolute point, preprint, arXiv:0911.3537v1, 2009.
- [3] A. Deitmar, Schemes over $\mathbb{F}_1$, in: *Number Fields and Function Fields—Two Parallel Worlds*, in: *Progr. Math.*, vol. 239, Birkhäuser Boston, Boston, MA, 2005, pp. 87–100.
- [4] A. Deitmar, $\mathbb{F}_1$ -schemes and toric varieties, *Beiträge Algebra Geom.* 49 (2) (2008) 517–525.
- [5] K. Kato, Toric singularities, *Amer. J. Math.* 116 (5) (1994) 1073–1099.
- [6] J. López Peña, $\mathbb{F}_1$ -models for cluster algebras and total positivity, in preparation.
- [7] J. López Peña, O. Lorscheid, Mapping $\mathbb{F}_1$ -land an overview of geometries over the field with one element, in: *Noncommutative Geometry, Arithmetic and Related Topics*, Johns Hopkins University Press, 2011, pp. 241–265.
- [8] J. López Peña, O. Lorscheid, Torified varieties and their geometries over $\mathbb{F}_1$, *Math. Z.* 267 (3–4) (2011) 605–643.
- [9] O. Lorscheid, The geometry of blueprints. Part I: Algebraic background and scheme theory, *Adv. Math.* 229 (3) (2012) 1804–1846.
- [10] O. Lorscheid, The geometry of blueprints. Part II: Tits–Weyl models of algebraic groups, preprint, arXiv:1201.1324, 2012.
- [11] G. Mikhalkin, Tropical geometry, unpublished notes, 2010.
- [12] C. Soulé, Les variétés sur le corps à un élément, *Mosc. Math. J.* 4 (1) (2004) 217–244, 312.
- [13] K. Thas, Notes on $\mathbb{F}_1$, I. Combinatorics of $\mathcal{D}_0$ -schemes and $\mathbb{F}_1$ -geometry, in preparation.