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Synchronization of coupled second-order Kuramoto oscillators

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DOI:
[10.33612/diss.155871911](https://doi.org/10.33612/diss.155871911)

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Document Version
Publisher's PDF, also known as Version of record

Publication date:
2021

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):
Gao, J. (2021). *Synchronization of coupled second-order Kuramoto oscillators*. [Thesis fully internal (DIV), University of Groningen]. University of Groningen. <https://doi.org/10.33612/diss.155871911>

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Synchronization of coupled second-order Kuramoto oscillators

Jian Gao

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The research for this doctoral dissertation has been carried out at the Faculty of Science and Engineering, University of Groningen, The Netherlands, within the Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence. This work was financially supported by the China Scholarship Council.

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PhD thesis, University of Groningen, the Netherlands

Printed by: IPSKAMP

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Synchronization of coupled second-order Kuramoto oscillators

PhD thesis

to obtain the degree of PhD at the
University of Groningen
on the authority of the
Rector Magnificus Prof. C. Wijmenga
and in accordance with
the decision by the College of Deans.

This thesis will be defended in public on

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Chapter 1

Introduction

1.1 Synchronization

From the Greek σύν =together and χρόνος=time, synchronization is a process of a population of dynamically interacting units, adjusting some properties of their trajectories, and making the system operate collectively and coherently. Synchronization is a ubiquitous natural phenomenon, playing an essential role in biology, sociology, ecology, and technology (Strogatz, 2004; Pikovsky et al., 2003). The scientific research of synchronization can be traced back to work by C. Huygens in 1665, discovering "an odd kind of sympathy" between coupled pendulum clocks (Huygens, 1899; Willms et al., 2017). He observed that two identical pendulum clocks, hanging on a beam, had identical periods but opposite displacements. The reason for this anti-phase synchronization is the weak coupling of the clocks through the beam. For the modern studies of the Huygens synchronization, we refer to (Pogromsky et al., 2010; Blekhman et al., 1997; Pena Ramirez, 2013; Oliveira and Melo, 2015).

From work by Huygens, the synchronization phenomenon gets into the scientists' vision. Much work is devoted to analyzing two coupled units, both from the physics and mathematics point of view (Acebrón et al., 2005). However, large populations' synchronization is another challenge, requiring many hypotheses. It did not come into scientific study scope until 1967 when Winfree introduced the first significant mathematical model of coupled limit-cycle oscillators with different intrinsic frequencies (Winfree, 1967). With this model, Winfree found that a population of non-identical oscillators can synchronize with aligned frequencies and coherent phases. This work starts the modern study of synchronization of coupled units by modeling the biological oscillators as phase oscillators and neglecting their amplitude. Based on the Winfree model, much work has been done to explore the synchronization phenomenon, as recently in (Giannuzzi et al., 2007; Ariaratnam and Strogatz, 2001; Basnarkov and Urumov, 2009; Gallego et al., 2017). However, though the amplitude of biological oscillators is neglected in the Winfree model,

it is still too general to be analytically tractable. Hence, following the Winfree model, a further simplified and more analytically tractable model has been introduced: the Kuramoto model. It becomes another milestone in synchronization and the most popular model with a lot of variations and applications.

1.2 Kuramoto oscillators

In 1975, the Kuramoto model was introduced by Toshiaki Kuramoto to study the spontaneous synchronization in collections of interacting systems with limit cycles (Kuramoto, 1975). In the later years, the Kuramoto model, both in this original form and its variations, becomes one of the most well-studied models for spontaneous synchronization in physics, biology, and social systems (Rodrigues et al., 2016). The Kuramoto model describes many systems such as the electrical power distribution networks (Filatrella et al., 2008), Josephson junction arrays (Wiesenfeld et al., 1998), cardiac pacemaker cells (Taylor et al., 2010), synchronization flashing in fireflies (Bechhoefer, 2005), vehicle coordination (Paley et al., 2007), rhythmic applause (Néda et al., 2000), and more. As a result, the model and its variations have attracted much attention from the dynamical systems, control theory, neuroscience, and other communities.

The Kuramoto model consists of coupled N phase oscillators whose dynamics read

$$\frac{d\theta_i}{dt} = \Omega_i - \frac{K}{N} \sum_{j=1}^N \sin(\theta_i - \theta_j), \quad i = 1, 2, \dots, N, \quad (1.1)$$

where N is the number of oscillators, K the coupling strength. Each oscillator is described by its phase θ_i and characterized by a natural frequency Ω_i . Each pair of oscillators is coupled through the periodic function $K \sin(\theta_i - \theta_j)$. If there is no coupling as $K = 0$, each oscillator follows $\dot{\theta} = \Omega_i$, which is the most straightforward limit cycle. On the other hand, if the coupling is strong enough, all oscillators will freeze to synchrony with $\dot{\theta}_i = \dot{\theta}_j$ for all i and j .

To show the basic phenomenon of coupled oscillators' synchronization, we consider the Kuramoto model's simplest case, two coupled oscillators $N = 2$. The dynamics read

$$\frac{d\theta_i}{dt} = \Omega_i - \frac{K}{2} \sum_{j=1}^2 \sin(\theta_i - \theta_j), \quad i = 1, 2. \quad (1.2)$$

We define the sum and difference of these two oscillators' phases as $\Theta = \theta_1 + \theta_2$ and $\varphi = \theta_1 - \theta_2$. Without loss of generality, we assume $\Omega_1 > \Omega_2 > 0$. Then the equation Eq. (1.2) can be rewritten as

$$\frac{d\Theta}{dt} = \Omega_1 + \Omega_2, \quad \frac{d\varphi}{dt} = \Omega_1 - \Omega_2 - K \sin \varphi. \quad (1.3)$$

The solution of Θ can be obtained straightforwardly as $\Theta(t) = (\Omega_1 + \Omega_2)t + \Theta(0)$ where $\Theta(0) = \theta_1(0) + \theta_2(0)$ is the initial state. The sum of phases $\Theta(t)$

Figure 1.1: Angular velocity of oscillators with different coupling strength K . The blue lines are for the oscillator one with $\Omega_1 = 1$, and the red lines for the oscillator two with $\Omega_2 = -1$. The blue and red dash-lines are the mean frequencies $\langle \dot{\theta}_{1,2} \rangle$ in the long time limit.

is independent of K , while the phase difference $\varphi(t)$ depends on the coupling strength K . If $K = 0$, we have $\varphi(t) = (\Omega_1 - \Omega_2)t + \varphi(0)$ with $\varphi(0) = \theta_1(0) - \theta_2(0)$. In this case the dynamics of these two oscillators read $\theta_1(t) = \Omega_1 t + \theta_1(0)$ and $\theta_2(t) = \Omega_2 t + \theta_2(0)$, following their natural frequencies, as shown in Fig. 1.1(a). This state is called the incoherence state.

With the increase of K , these two oscillators begin to affect each other. The frequency of the phase difference $\dot{\varphi}$ changes periodically. The mean value of $\dot{\varphi}$ reads

$$\langle \dot{\varphi} \rangle \equiv \frac{2\pi}{T} = 2\pi \left[\int_0^{2\pi} \frac{1}{\Omega_1 - \Omega_2 - K \sin \varphi} d\varphi \right]^{-1} = \sqrt{(\Omega_1 - \Omega_2)^2 - K^2}, \quad (1.4)$$

which works for $|\Omega_1 - \Omega_2| < K$. From $\varphi(t)$ and $\Theta(t)$ we have mean frequencies of oscillators as

$$\langle \dot{\theta}_{1,2} \rangle = \frac{1}{2} \langle \dot{\Theta} \pm \dot{\varphi} \rangle = \frac{1}{2} (\Omega_1 + \Omega_2) \pm \sqrt{(\Omega_1 - \Omega_2)^2 - K^2}, \quad (1.5)$$

The difference of these oscillators' mean frequencies becomes smaller with a larger coupling strength, and finally gets $\langle \dot{\theta}_1 \rangle = \langle \dot{\theta}_2 \rangle = \frac{1}{2}(\Omega_1 + \Omega_2)$ at $K = K_c$, as shown in Fig. 1.1(b-f).

Figure 1.2: The state transition (a) and bifurcation (b) of coupled two oscillators. The red and blue lines in (a) are the mean frequencies of the oscillators one and two with $\Omega_1 = 1$ and $\Omega_2 = -1$. The line and dash-line in (b) are for the stable and unstable solutions.

The change point happens at $K_c = |\Omega_1 - \Omega_2|$, with a saddle-node bifurcation for $\varphi(t)$ in Eq. (1.3). When $K \geq K_c$, setting $\dot{\varphi} = 0$ in Eq. (1.3), we have

$$\Omega_1 - \Omega_2 - K \sin \varphi = 0, \quad (1.6)$$

which has two equilibrium solutions as

$$\varphi_1^* = \arcsin \frac{\Omega_1 - \Omega_2}{K}, \quad \varphi_2^* = \pi - \arcsin \frac{\Omega_1 - \Omega_2}{K}. \quad (1.7)$$

The stability of these two solutions can be determined through the linear stability analysis. Substituting the small perturbation $\varphi = \varphi^* + \delta\varphi$ into the dynamics Eq. (1.3), we have the dynamics in the linear terms as

$$\dot{\delta\varphi} \approx -K \cos(\varphi^*) \delta\varphi. \quad (1.8)$$

The solution is linear stable if and only if $\cos(\varphi^*) > 0$. Hence, it is straightforward to check that the solution φ_1 is stable, and φ_2 unstable.

Hence when $K \geq K_c = |\Omega_1 - \Omega_2|$, we have the stable equilibrium of the phase difference as $\varphi(t) = \varphi_1^*$. The frequencies of these two oscillators are the same as $\dot{\theta}_1 = \dot{\theta}_2 = (\Omega_1 + \Omega_2)/2$. These two oscillators are synchronized, and we call this state the synchronization state.

From $K = 0$ to $K > K_c$, the system changes from the incoherence state to the synchronization state. It is a phase transition process in the macroscopic view.

Besides, we also observe the bifurcation of their phase difference for these two oscillators at $K = K_c$. The macroscopic transition process of the whole system is related to the microscopic bifurcation of specific oscillators, as shown in Fig. 1.2. This relation is more complicated for a more extensive system, but it is always the key feature of coupled oscillators' synchronization processes. Most of the studies follow the approach: first, find the phase transitions in the macroscopic view, and then try to explain it from the microscopic view.

Several analytical methods are developed to analyze the coupled oscillators. The first one is the self-consistent method proposed by Kuramoto (Kuramoto, 1975; Kuramoto and Nishikawa, 1987). In this method, one uses the order parameter to describe the states of the system, which is defined as

$$re^{i\Theta} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}. \quad (1.9)$$

With the assumption that amplitude r and the frequency of the angle Θ , $\Omega^r = \dot{\Theta}$, are constants in the infinite size limit $N \rightarrow \infty$, Kuramoto shows the first analytical method to obtain the phase transitions of coupled oscillators when their natural frequencies are symmetric and unimodal. Then this method is generalized to systems with non-symmetric distributions. With different distributions of natural frequencies, one gets first-order, second-order, and hybrid transitions (Strogatz, 2000).

Though the self-consistent method is intuitive and can describe the synchronization process, it is not rigorous in mathematics. Complementing the self-consistent method, a rigorous analysis based on the continuity equations has been developed (Strogatz, 2000). In this method, with the continuous assumption $N \rightarrow \infty$, one defines the probability density of phases as $\rho(\theta, t, \Omega)$ and derives its evolution as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \theta}(\rho v) = 0, \quad (1.10)$$

where $v = \Omega - \Omega^r - Kr \sin \theta$ is the angular velocity of a given oscillator with phase θ and natural frequency Ω . With this approach, the self-consistent method can be derived rigorously. Besides, from this continuity equation Eq. (1.10), many techniques based on partial differential equations can be applied to the study of synchronization.

Two primary techniques are the stability analysis (Crawford, 1994) and the Ott-Antonsen ansatz (Ott and Antonsen, 2008). The stability analysis is based on expanding ρ around the transition point, with the amplitude expansion method (Crawford, 1994), or the spectrum analysis (Chiba, 2015). With these techniques, one gets the details of bifurcations from Eq. (1.10). As for the Ott-Antonsen ansatz, it is a reduction method. With a particular ansatz, the density ρ can be written as

$$\rho(\theta, t, \Omega) = \frac{g(\Omega)}{2\pi} \left[1 + \left(\sum_{n=1}^{\infty} \alpha(\Omega, t)^n e^{in\theta} + c.c. \right) \right], \quad (1.11)$$

with which Eq. (1.10) can be rewritten as an ordinary differential equation of $\alpha(\Omega, t)$. With this method, a more detailed analysis is possible. Much work has been done based on this method, such as the study of bimodal distributions (Martens et al., 2009), oscillators in star networks (Xu et al., 2015), and chimera states (Laing, 2009).

Recently, the study of the Kuramoto model is focused on the synchronization of oscillators in complex networks. The dynamics can be written as

$$\frac{d\theta_i}{dt} + K \sum_{j=1}^N A_{ij} \sin(\theta_i - \theta_j) = \Omega_i, \quad i = 1, 2, \dots, N. \quad (1.12)$$

The connection matrix $A = \{A_{ij}\}$ is symmetric and non-negative. If there is a link between oscillators i and j , we have $A_{ij} = 1$ and otherwise zero. This connection matrix can describe various network structures between oscillators. Compared with the original Kuramoto model, where all the oscillators are connected, there are less analytical methods for the oscillators in complex networks. Many results are based on numerical simulations. For more information, we refer to the surveys (Arenas et al., 2008; Boccaletti et al., 2016; Dörfler and Bullo, 2014; Rodrigues et al., 2016).

After Kuramoto's work (Kuramoto, 1975), the Kuramoto model has many variations, such as the model with noise, phase shifts, time delay, or higher-order coupling functions (Rodrigues et al., 2016). Among these models, the Kuramoto model with inertias has recently attracted much attention for its applications in power grids and relation to statistical mechanics. These Kuramoto oscillators with inertias are called the second-order Kuramoto oscillators.

1.3 Second-order Kuramoto oscillators

In 1991, B. Ermentrout generalized the Kuramoto oscillator model by adding adaptations (inertias) to describe the perfect synchronization among three tropical Asian species of fireflies and got the second-order oscillator model (Ermentrout, 1991). Different from the Kuramoto oscillators, with inertias, the second-order oscillators can approach complete phase synchronization (Ermentrout, 1991). In 1997, Kuramoto's self-consistent analysis (Kuramoto, 1975) was extended to second-order oscillators by Tanaka *et al* (Tanaka et al., 1997a,b). Then, the effect of inertias in the synchronization of oscillators comes to researchers' attention. The coupled second-order oscillators show various properties different from the Kuramoto oscillators, such as hysteresis and abrupt transitions (Tanaka et al., 1997a), oscillatory states (Olmi et al., 2014), cluster explosive synchronization (Ji et al., 2014a), and non-equilibrium steady states (Gupta et al., 2014).

Compared with the Kuramoto model, the second-order oscillator is the one

with inertia and second-order derivative of its phase as

$$M_i \frac{d^2\theta_i}{dt^2} + D_i \frac{d\theta_i}{dt} + K \sum_{j=1}^N A_{ij} \sin(\theta_i - \theta_j) = \Omega_i, \quad i = 1, 2, \dots, N. \quad (1.13)$$

For each oscillator i , θ_i , $M_i > 0$, $D_i > 0$, Ω_i are respectively the phase, inertia, dissipate coefficient and natural frequency. The coupling K and connection matrix A_{ij} is the same as the Kuramoto model. Usually, we assume that all dissipate coefficients and inertias are the same for all the oscillators, as $D_i = D$, $M_i = M$ for all i .

The second-order oscillator model dynamics have been studied with hundreds of works in many aspects, from the dynamics (Manik et al., 2014; Dorfler and Bullo, 2012; Choi et al., 2011, 2013, 2014, 2015) to mean-field analysis (Tanaka et al., 1997a,b), from all-connected networks (Gupta et al., 2014; Hong et al., 1999a,b; Hong and Choi, 2000; Hong et al., 2002; Komarov et al., 2014; Bonilla, 2000; Acebrón et al., 2000; Olmi et al., 2014; Tanaka et al., 1997a,b; Acebrón and Spigler, 1998; Choi and Ha, 2012) to complex networks (Sasaki et al., 2015; Peron et al., 2015; Olmi et al., 2015; Jaros et al., 2015; Kachhvhah and Sen, 2014; Ji et al., 2013, 2014a; Ji and Kurths, 2014; Ji et al., 2014b; Belykh et al., 2016; Carareto et al., 2013; Olmi, 2015). The coupled oscillators show various behaviors, including oscillatory states (Olmi et al., 2014), cluster explosive synchronization (Ji et al., 2014a), and non-equilibrium steady states (Gupta et al., 2014).

All the parameters are somehow crucial to determine the properties of the system. Focusing on different parameters, people can explore different aspects of synchronization. Varying M is the way to understand the inertias' effect, where $M = 0$ corresponds to the Kuramoto model (Rodrigues et al., 2016). Changing the dissipate coefficient D builds the path to the Hamiltonian mean-field model with $D = 0$, which gives the idea of non-equilibrium steady states (Gupta et al., 2014). The increasing and decreasing of the coupling strength K serve as transition processes to synchronization and incoherence. Various states are found, as the weakly coupled state (Tanaka et al., 1997b) and oscillatory state (Tanaka et al., 1997a). Studies of different network structures $\{A_{ij}\}$ and natural frequency distributions $\{\Omega_i\}$ are essential to understand the relationship between topology and dynamics (Ji et al., 2014a).

Except for the interest of exploring the synchronization of coupled units, the second-order oscillator has various applications. These second-order oscillators can also be obtained with minor changes and applied to various systems, such as the Josephson junction arrays (Levi et al., 1978; Watanabe and Strogatz, 1994; Trees et al., 2005), goods markets (Ikeda et al., 2012), dendritic neurons (Sakyte and Ragulskis, 2011), and power generators (Filatrella et al., 2008; Rohden et al., 2012, 2014; Lozano et al., 2012; Witthaut and Timme, 2012; Menck et al., 2013; Bergen and Hill, 1981; Hill and Chen, 2006).

Significantly, the second-order oscillator model serves as an ideal model for power grids, for its simplicity and mathematical tractability. The system has two

primary states for coupled oscillators: incoherence and synchronization, which denotes the malfunctioning and functioning state of power grids. For the functioning state of power grids, one gets the existence and stability of it (Dörfler, 2013; Grzybowski et al., 2016). The effects of different operations or designs of a power grid on its stability can be estimated, as the decentralization process (Rohden et al., 2012, 2014), removing or adding edges (Lozano et al., 2012; Witthaut and Timme, 2012), and the seasonal changing of renewable sources (Maïzi et al., 2016). The optimization process of the stability of power grids can also be analyzed as in (Pinto and Saa, 2016). The oscillator model is also a dynamical model that can provide some dynamical information about the system, like the basin stability (Menck et al., 2013; Kim et al., 2015), survivability (Hellmann et al., 2016), resilience to perturbations or noise (Fortuna et al., 2011; Gambuzza et al., 2017), and the rerouting process and cascade failure of power grids (Rohden et al., 2016).

Unlike the complex network analysis, many significant results are about the synchronization state's stability, other than the network itself. Motter *et al* derive a stability condition for the synchronization state of a power grid (Motter et al., 2013). Lozano *et al* derive the minimal coupling strength required to ensure global frequency synchronization (Lozano et al., 2012). Menck *et al* find that the dead ends of power grids diminish the system's stability (Menck et al., 2013). Rohden *et al.* find that the decentralized grids become more sensitive to dynamical perturbations and more robust to topological failure (Rohden et al., 2012). Moreover, in (Witthaut and Timme, 2012), the Braess's paradox is found in power grids, where adding new links may not only promote but also destroy synchrony.

1.4 Outline and main results

Although the second-order oscillator model has been studied in many works, especially the stability analysis in the applications of power grids analysis, some basic properties of this model's synchronization are still unknown. What is the effect of inertias on the synchronization? What are the same and different points between the Kuramoto model and the second-order model?

Our primary goal is to explore the dynamics of the coupled second-order oscillators in synchronization in this project. Specifically, we try to answer how oscillators' dynamical properties affect the collective synchronization phenomenon through their coupling. The effect of inertias is the central topic of this project. Other aspects are also considered, such as the phase shifts between oscillators and the system's topology (networks).

Based on the second-order model, there are two primary analytical methods to explore oscillators' synchronization processes. One is the self-consistent method, generalized by Tanaka in (Tanaka et al., 1997a,b), and widely applied in studying the second-order oscillators' basic properties, such as the hysteresis (Tanaka et al., 1997a), oscillatory states (Olm et al., 2014), and cluster explosive synchronization (Ji et al., 2014a). The other is the stability analysis around the critical point. In

(Barre and Métivier, 2016), the change of transitions due to the inertias' effect is described for some systems using the stability analysis. Other than these two methods, some methods focus on the synchronization state's existence and stability conditions (Dörfler and Bullo, 2014).

We take the self-consistent method from this project's goal to explore the collective synchronization phenomenon from oscillators' dynamics. The self-consistent method has the advantage that it is simple and straightforward, especially considering the various oscillators' dynamics with different variations. Nevertheless, the limitation is also quite explicit. The self-consistent method can only be applied to incoherence, synchronization, and partial synchronization state where there is only zero or one synchronized cluster. It is also based on the mean-field assumption, only working for the infinitely many oscillators in all-connected and annealed random complex networks. As our main aim in this project is to explore the collective synchronization phenomenon from oscillators' dynamics, the self-consistent method is the proper one. But we will try to generalize it firstly for more kinds of states and systems.

In this thesis, the first two chapters are the basic ones, focusing on the self-consistent method itself, for the Kuramoto model and the second-order model. In chapter 2, we recall the self-consistent method for the Kuramoto oscillators. This method is widely applied in the study of partial synchronization states of infinitely many coupled oscillators and is rigorous in these cases. However, if we consider approximations, whether is it possible to apply the self-consistent method on a broader scope? In this chapter, we answer this question through the averaging method. Consequently, we find that the self-consistent method can be applied to states with several clusters and systems with only finite number of oscillators. This new scope depends on a new approach to understand and use the results from the self-consistent method. We do not take the self-consistent method to obtain the order parameter, but instead to determine the width of synchronized clusters. As to our knowledge, this new approach is the first analytical method to explore systems which have multi-cluster states or consist of finite number of oscillators, without specific settings of distributions or states.

In chapter 3, we go to the self-consistent method for the second-order oscillators generalized by Tanaka in (Tanaka et al., 1997a). Though this method has been applied widely in the studies of second-order oscillators, we find that there is a small but subtle error in the final expressions in (Tanaka et al., 1997a). This error results in the divergence of the critical coupling strength K_c , making the self-consistent method cannot be applied to obtain this most fundamental property of the synchronization transitions. This chapter follows and improves the methods in (Tanaka et al., 1997a), correcting the error, and obtaining the right self-consistent equations for the second-order oscillators. These new equations coincide with the numerical simulations and stability analysis. Besides, these equations converge to the Kuramoto self-consistent equations in the limit of zero inertias, building the bridge from the Kuramoto model to the second-order model. We analyze the abrupt synchronization transitions and hysteresis of oscillators with unimodal and

symmetric distributions of natural frequencies with these new equations.

In chapter 4 and 5, we focus on another phenomenon of the second-order oscillators, the oscillatory states. It is found in systems when the inertias are sufficiently large. It appears that the partial synchronization state becomes unstable in some parameter regions, and the system evolves to the one with multiple synchronization clusters. In chapter 4, we first do many numerical simulations in various systems with different natural frequencies distributions to explore such a phenomenon's mechanism. We find that this oscillatory state is not limited to the unimodal systems, but is a general phenomenon for second-order oscillators. The additional small clusters appear beside the major clusters when the inertias are sufficiently large. To explore the additional clusters' appearance, we consider another aspect of the self-consistent method as the fixed point of a map based on circle map lifts. The synchronized clusters of the system are the resonance stairs of the corresponding circle map lift. From this point, we show that additional clusters are closely related to the bi-stability of the second-order oscillators and are a general phenomenon for second-order oscillators. With the study of a few oscillators as the simplified system to show this phenomenon, we further show that these additional clusters are from the weakening of the major cluster's effect on the other oscillators.

In chapter 5, we continue the discussion of the oscillatory state. If it can be understood in the framework of circle map lifts, can it be analyzed by the self-consistent method? At first glance, the answer is no. We have applied the self-consistent method in chapter 4 and only obtain the partial synchronization states. In this chapter, we go to the formation process of oscillatory states in the time scale. Surprisingly, we find that the additional clusters always form after the major one, which inspires us that there is always a missing point in the self-consistent method. Once some oscillators are synchronized and have the same frequency, the system is changed. New states may appear from the changed system. From this point of view, we generalize the self-consistent method in a new approach with several rounds. In principle, the new states may appear one round after another, like a cascade. We can obtain the additional clusters and the oscillatory state analytically, coinciding with numerical simulations. Our solution is quite straightforward, but why do people not find the same phenomenon in Kuramoto oscillators? With the new approach of the self-consistent method, we find that the particular synchronization condition between oscillators covers the possibility of the cascade in Kuramoto models, making it stop at the first round. Due to the inertias' effect, the synchronization condition is changed, and we finally observe the cascade. This finding improves both the self-consistent method and our understanding of coupled oscillators.

From chapter 6, we start to consider the cross-effect of inertias and other factors. In this chapter, we consider the cross-effect of inertias and phase shifts. Following the same method in chapter 3, we first explore the dynamics of a single oscillator. We find that for the running oscillator (not synchronized), part of the effect of inertias can be expressed as an effective phase shift, mixing with the

original phase shift. The mixture's direct result is the destruction of the phase shift structure among oscillators, changing the phase shift dominated non-trivial synchronization transitions to the normal ones. Except for the direct mixture, another phenomenon is found for relatively large phases shifts and inertias. The oscillatory states that due to inertias are strengthened and protected by the phase shifts, resulting in the oscillating synchronization process. The oscillators stay in the oscillatory state with the increase of coupling strength and do not converge to the synchronization state. This non-synchronized oscillating process is a general phenomenon for second-order oscillators with phase shifts. We find it in various all-connected oscillators and oscillators in annealed complex networks if the network's mean-degree is large enough.

In chapter 7 and chapter 8, we consider the oscillators in complex networks. The self-consistent method is generalized to the complex networks with annealed network assumption. In chapter 7, the Kuramoto model in scale-free networks are considered with corresponding explosive synchronization and vanishing onset of the critical point. We find that in some cases, different systems of oscillators have the same self-consistent equations. From this self-consistent equation symmetry, we can transform the oscillators in annealed complex networks to the ones in all-connected networks. Both the topology and other factors in the dynamics are combined and transformed into the distributions of natural frequencies of all-connected oscillators. This perspective gives us a simple and intuitive way to understand Kuramoto oscillators' various synchronization transitions with different settings, including the complex network structure, the relation between natural frequencies and degrees, and some other manipulations such as the absolute frequency coupling. From the self-consistent equation symmetry, one can develop a classification system of different Kuramoto model variations.

In chapter 8, we apply the method in chapter 7 to second-order oscillators. The second-order oscillators have a cluster-explosive synchronization transition in the scale-free networks, other than the explosive synchronization of Kuramoto oscillators. We transform the oscillators in scale-free networks to the all-connected ones to explore and explain such a phenomenon. After the transformation, the topology introduces the same natural frequencies as the Kuramoto model. However, the inertias of oscillators are also changed in the transformation. The oscillators with higher degrees have higher inertias after the transformation. Consequently, such oscillators with high inertias are hard to synchronize, resulting in the cluster synchronization. In contrast, the other oscillators dominated by the topology introduced natural frequencies, resulting in explosive synchronization as the Kuramoto model. We can weaken oscillators' inertias in the scale-free networks to overcome the inertias' topological effect from the analysis. Consequently, we retrieve the explosive synchronization of second-order oscillators in scale-free networks.

In summary, in this thesis, we explore the synchronization processes of the second-order Kuramoto model, including the hysteresis in chapter 3, oscillatory state in chapter 4, cascading formation of clusters in chapter 5, oscillating synchronization processes with phase shifts in chapter 6, cluster explosive synchronization

in scale-free networks in chapter 8. During this study, we have to develop some new approaches such as the cluster-based self-consistent methods in chapter 2 and virtual frequency analysis in chapter 7. These methods are not limited to the second-order oscillators, but can be applied to various Kuramoto-like models.

Chapter 2

Self-consistent method and synchronization scenario

Synchronization of coupled Kuramoto oscillators and other generalized Kuramoto-like oscillators has attracted much attention in the last decade. The Kuramoto self-consistent method is an essential method in this field, which is analytically tractable and provides intuitive understandings. In this chapter, we propose a new approach to obtain and use the self-consistent equations. Besides the steady states' assumption in the continuous limit, our approach is based on multi-cluster states, with some approximations through the perturbation method. With this approach, we explore oscillators' synchronization processes in systems with bimodal distributions, finite number of oscillators, and absolute frequency weighted couplings. This new approach coincides well with numerical simulations. From the single-cluster to multi-cluster states, from finite networks of oscillators to large systems in the continuous limit, we find that the synchronization process follows a simple scenario. Clusters appear from the local concentration with the overtaking of the effect of the others. When several clusters overlap, they will merge and form a big cluster.

2.1 Introduction

The self-consistent method is the first analytical method for the Kuramoto model and proposed by Kuramoto (Kuramoto and Nishikawa, 1987; Strogatz, 2000). It is analytically tractable and provides intuitive understandings of synchronization processes. Based on this method, Kuramoto found phase transitions to synchronization with the increase of the coupling strength. When the oscillators have a unimodal and symmetric distribution of natural frequencies, from the critical coupling strength K_c , a group of oscillators is synchronized and forms a synchronization cluster. With the increase of the coupling strength, such synchronization cluster grows and contains more oscillators (Strogatz, 2000). The whole syn-

chronization process is similar to the second-order phase transition in statistical mechanics.

After proposed, the self-consistent method is widely used with various generalizations. The self-consistent method's validity has been checked through numerical simulations and other analysis methods, such as the stability analysis (Crawford, 1994) and the Ott-Antonsen method (Ott and Antonsen, 2008). Compared with other methods, the self-consistent method is more intuitive and not limited to particular states (Crawford, 1994) or system settings (Ott and Antonsen, 2008). However, it also has its crucial limitations. The self-consistent method is for steady states and can not be applied to the more interesting non-steady states, such as the standing wave states, chimera states (Abrams and Strogatz, 2004), Bellerophon states (Bi et al., 2016), and the finite systems with several oscillators. As one of the essential methods, this limitation of the self-consistent method limits our understanding of Kuramoto models and synchronizations.

In this chapter, we will generalize the usage of the self-consistent method. The basis of the self-consistent method is the resonance of oscillators with a driven signal. Given the driven signal, one can solve each oscillator's state analytically and get the collective field from all these states, which should be the same as the supposed driven signal. The equivalence of the driven signal and collective field is the systems' self-consistent condition. The classic driven signal corresponds to the steady states. This chapter considers more general driven signals and uses the perturbation method to obtain oscillators' approximate states.

We give a short outline of this chapter. In Sec. 2.2, the Kuramoto model is introduced, and a generalized self-consistent method is proposed. This method is applied in Sec. 2.3 to unimodal and bimodal distributions, infinite and finite cases, and Kuramoto-like models with various generalizations. The theoretical results coincide well with the numerical simulations. We conclude this chapter in Sec. 2.4.

2.2 Kuramoto model and self-consistent method

Kuramoto model describes the dynamics of coupled limit-cycle oscillators as

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad (2.1)$$

where N is the number of oscillators, K the coupling strength. Each oscillator is described by its phase θ_i and characterized by a natural frequency ω_i . If there is no coupling as $K = 0$, each oscillator follows $\dot{\theta} = \omega_i$, which is the most straightforward limit cycle. Each pair of oscillators is coupled through the periodic function $\sin(\theta_i - \theta_j)$.

With the increase of K , the oscillators can become synchronized, which means that their phases are correlated. To describe the level of synchronization, one can

define the order parameter Z as

$$Z(t) = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}. \quad (2.2)$$

The amplitude of Z describes the synchronization level, and its angle as the mean value of oscillators' phases. In the synchronization state, oscillators' phases are highly correlated. One gets $|Z| \approx 1$. On the other hand, if there is a weak correlation between oscillators, their phases are randomly distributed along the cycle. In this case, we have $|Z| \approx 0$, called incoherence state. With the increase of the coupling strength K , oscillators will change from the incoherence state to synchronization. One or several synchronization clusters appear, in which oscillators have the same mean-frequency.

In 1987, Kuramoto found that if there is only one cluster for a sufficiently large system $N \gg 1$, the order parameter follows a simple dynamics

$$\dot{Z}(t) = r e^{i(\Omega^r t + \Psi_n)} + \eta(t), \quad (2.3)$$

where the amplitude $r = |Z|$ and rotation frequency Ω^r are all constant. The noise term $\eta(t)$ converges to zeros in the limit $N \rightarrow \infty$ through $O(\eta) = 1/\sqrt{N}$ (Rodrigues et al., 2016). With the change of coupling strength K , one gets different $r(K)$ and $\Omega^r(K)$ as two functions of K . The former is the synchronization level, and the later describes the mean frequency of oscillators in the synchronization cluster. With the introduction of Eq. (2.3), getting these functions $r(K)$ and $\Omega^r(K)$ becomes the central question of synchronization.

To obtain $r(K)$ and $\Omega^r(K)$, Kuramoto rewrote the dynamics of oscillators Eq. (2.1) through substitution of Eq. (2.2) as

$$\dot{\theta}_i = \omega_i + K \text{Im}(Z(t) e^{-i\theta_i}) = \omega_i + K r \sin(\Omega^r t + \Psi_j - \theta_i). \quad (2.4)$$

The oscillators are decoupled with each other, only dependent on $Z(t)$. Without loss of generality, we can assume $\Psi_j \equiv 0$. If $Kr \geq |\omega_i|$, the i -th oscillator is locked with the field $Z(t)$. With the rotation frame with Ω^r and the transformation $\theta_i \mapsto \theta_i - \Omega^r t$, the dynamics of θ_i converge to the stable fixed point as $\theta_i = \arcsin((\omega_i - \Omega^r)/Kr)$. The mean-frequencies of these oscillators are the same as Ω^r . They form a synchronization cluster. The other oscillators with $Kr < |\omega_i|$ keep running with either positive and negative frequencies. We call them running oscillators. With the assumption $N \rightarrow \infty$, the distribution density of the running oscillators can be estimated through their phases' distributions $\rho(\theta_i) \propto |\dot{\theta}_i|^{-1}$.

The function $r(K)$ and $\Omega^r(K)$ can be determined by substituting all the oscillators' states, either locked or running, into the order parameter Eq (2.2). With the assumption $N \rightarrow \infty$, one gets the equation from Eq (2.2) as

$$r = \int_{-Kr}^{Kr} e^{i \arcsin((\omega - \Omega^r)/Kr)} g(\omega) d\omega + \left(\int_{-\infty}^{-Kr} + \int_{Kr}^{\infty} \right) e^{i\theta} \rho(\theta|\omega) g(\omega) d\omega, \quad (2.5)$$

where $g(\omega)$ is the distribution of natural frequency ω . We denote Eq. (2.5) as the self-consistent equation for it is the self-consistent (or necessary) condition of the steady states Eq. (2.3). From Eq. (2.5), one can obtain the functions $r(K)$ and $\Omega^r(K)$, which represent the phase transition of the oscillators.

The Kuramoto self-consistent method is simple and analytically tractable. It obtains a great success and has been used to study a variety of systems (Rodrigues et al., 2016). The basis of the self-consistent method is the assumption of $Z(t)$ as Eq. (2.3) and the infinite size approximation $N \rightarrow \infty$. From these two conditions, one can solve the dynamics of each oscillator Eq. (2.4). However, when we consider a more complicated field $Z(t)$ or finite-size system, the dynamics Eq. (2.4) are unsolvable.

In this chapter, we try to generalize the self-consistent method through its limitation. Specifically, we use the perturbation method to include both the noise term from finite-size effect in Eq. (2.4) and a more complicated $Z(t)$ for systems with several synchronization clusters. The order parameter $Z(t)$ is assumed as

$$Z(t) = \sum_{j=1}^n r_n e^{i(\Omega_n^r t + \Psi_n)} + \eta_z(t), \quad (2.6)$$

where n is the number of synchronized clusters. Each cluster has its frequency Ω_n^r , amplitude r_n , and phase constant Ψ_n . The noise term $\eta_z(t)$ satisfies $\langle \eta_z(t) \rangle = 0$, where $\langle \cdot \rangle$ means the time average process. Similar to the Fourier transformations, we can obtain the amplitude r_n of each group from $Z(t)$ as

$$r_n = \left\langle \frac{1}{N} \sum_{j=1}^N e^{i\theta_j - \Omega_n^r t - \Psi_n} \right\rangle. \quad (2.7)$$

Following the same approach of the self-consistent method from Eq. (2.1) to Eq. (2.4), we rewrite each oscillator's dynamics with the assumption Eq. (2.6), as

$$\dot{\theta}_i = \omega_i + K \text{Im}(Z(t) e^{-i\theta_i}) = \omega_i + \sum_{j=1}^n K r_j \sin(\Omega_j^r t + \Psi_j - \theta_i) + K(\eta(t) \sin(-\theta_i)). \quad (2.8)$$

In this case, each oscillator is driven by several periodic signals and noise. We know that each periodic signal needs to be supported by a synchronization cluster from the self-consistent method. Without the loss of generality, choosing one cluster denoted by $(r_m, \Omega_m^r, \Psi_m)$, we can rewrite the dynamics in this cluster's rotating frame $\theta_i^m \equiv \theta_i - \Omega_m^r t + \Psi_m$ as

$$\begin{aligned} \dot{\theta}_i^m = & \omega_i - \Omega_m^r - K r_m \sin(\theta_i^m) + \sum_{j \neq m} K r_j \sin((\Omega_j^r - \Omega_m^r)t + (\Psi_j - \Psi_m) - \theta_i^m) \\ & + K \eta(t) \sin(\Omega_m^r t + \Psi_m - \theta_i^m). \end{aligned} \quad (2.9)$$

If the clusters are all sufficiently far from each other, the frequency difference between clusters is much larger than the one within each group, as $|\Omega_j^r - \Omega_m^r| \gg |\omega_i - \Omega_m^r|$. Then take the average over the fast time scale, we have the approximate dynamics in the slow time scale, which is the one in each group as

$$\dot{\theta}_i^m = \omega_i - \Omega_m^r - Kr_m \sin(\theta_i^m). \quad (2.10)$$

Not surprisingly, the dynamics in each group Eq. (2.10) is the same as Eq. (2.4). The approximations above are equivalent to the independent assumption of synchronization clusters. If $|\omega_i - \Omega_m^r| \leq Kr_m$, the oscillator is locked and belongs to the synchronization cluster rotating with Ω_m^r . The phase of the locked oscillator reads

$$\sin \theta_i^m = b_i, \quad \cos \theta_i^m = \sqrt{1 - b_i^2}. \quad (2.11)$$

where $b_i \equiv (\omega_i - \Omega_m^r)/Kr_m$. On the other hand, around the cluster, running oscillators with $|\omega_i - \Omega_m^r| > Kr_m$ have the density distribution determined by the r_m and Ω_m^r as

$$\rho_i(\theta^m) = \frac{1}{2\pi} \frac{\sqrt{b_i^2 - 1}}{|b_i - \sin \theta^m|}. \quad (2.12)$$

For these running oscillators, the average of $\sin \theta_i^m$ and $\cos \theta_i^m$ reads

$$\begin{aligned} \langle \sin \theta_i^m \rangle &= \int_0^{2\pi} \rho_i(\theta^m) \sin \theta^m d\theta^m = b_i \left(1 - \sqrt{1 - \frac{1}{b_i^2}} \right), \\ \langle \cos \theta_i^m \rangle &= \int_0^{2\pi} \rho_i(\theta^m) \cos \theta^m d\theta^m = 0. \end{aligned} \quad (2.13)$$

The other oscillators around the cluster are all running oscillators for this m -th cluster. We approximate their contribution to the m -th cluster through Eq. (2.13). Some of these oscillators are synchronized, but from the assumption that frequency differences between clusters are much larger than those within each group, we further ignore the difference between their mean-frequencies and natural frequencies. As a result, the self-consistent equation for the m -th cluster reads

$$r_m = \left\langle \frac{1}{N} \sum_{j=1}^N e^{i\theta_j - \Omega_m^r - \Psi_m} \right\rangle = \left\langle \frac{1}{N} \sum_{j=1}^N e^{i\theta_j^m} \right\rangle = \frac{1}{N} \sum_{i=1}^N (\langle \cos \theta_i^m \rangle + i \langle \sin \theta_i^m \rangle). \quad (2.14)$$

The imaginary and real parts of the equation Eq. (2.14) are two conditions for Ω_m^r and r_m . We define a combined coupling strength $q_m = Kr_m$, and a combined frequency $\xi_i = q_m b_i$ as the new variables to solve the functions $r_m(K)$ and $\Omega_m^r(K)$. For the imaginary part of the self-consistent condition, we have

$$0 = \bar{\xi} \equiv q_m \frac{1}{N} \sum_{j=1}^N \langle \sin \theta_j^m \rangle = \sum_{j=1}^N \xi_j \left[\mathbf{1}_{\xi_j}(q_m) + (1 - \mathbf{1}_{\xi_j}(q_m)) \left(1 - \sqrt{1 - \frac{q_m^2}{\xi_j^2}} \right) \right] \quad (2.15)$$

where $\mathbf{1}_{\xi_j}(q_m) \equiv 1$ if $|\xi_j| \leq q_m$ and otherwise zero. The weighted mean-frequency $\bar{\xi}$ from all the oscillators should be zero in the rotation frame of Ω_m^r as the self-consistent condition.

From Eq. (2.15), one can get $\Omega_m^r(q_m)$. Substitution $\Omega_m^r(q_m)$ into the real part of Eq. (2.14) yields the equation for $r_m(q_m)$ as

$$r_m = \frac{1}{N} \sum_{j=1}^N \langle \cos \varphi_i^m \rangle = \frac{1}{N} \sum_{j=1}^N \sqrt{1 - \frac{\xi_j^2}{q_m^2}} \mathbf{1}_{\xi_j}(q_m). \quad (2.16)$$

From $\Omega_m^r(q_m)$ and $r_m(q_m)$ and the equation $K = q_m/r_m$, we obtain the function $\Omega_m^r(K)$ and $r_m(K)$ as the solution of Eq. (2.16). Hence, at a given coupling strength K , we have the m -th the synchronization cluster as

$$\omega_i \in \sigma_m(K) \equiv [\Omega_m^r(K) - q_m(K), \Omega_m^r(K) + q_m(K)]. \quad (2.17)$$

Note that in the synchronization region $|\xi| \leq q_m$ the average of ξ_i is defined as usual. In contrast, for the running oscillators with $|\xi| > q_m$, it is weighted by the factor $1 - \sqrt{1 - q_m^2/\xi^2}$. This fact further reduces the approximation error introduced by our assumptions about oscillators far from the cluster.

For the other clusters, it is straightforwardly to show that they follow the same equation as Eq. (2.14). The multi-solutions of the equation Eq. (2.14) describe the synchronization clusters. These solutions give the existence condition of synchronization clusters as a collection of $\sigma_m(K)$ as Eq. (2.17).

In Kuramoto's self-consistent method, the additional phase constant Ψ can be removed from proper initial time t_0 . Hence the amplitude r and frequency Ω^r contain all the information about the synchronization cluster. However, for multi-cluster states with the assumption Eq. (2.6), the phase differences between clusters are critical values to determine the order parameter $Z(t)$. Though we can obtain the approximate value of r_m and Ω_m^r , the phases of clusters Ψ_m are unknown. It is also an open question to obtain Ψ_m . We refer to (Gottwald, 2015) for further information about the phases and the collective-variable method to solve it.

In this chapter, we focus on synchronization clusters' formation through σ_m with r_m and Ω_m^r , not trying to obtain $Z(t)$.

2.3 Synchronization processes and formation of clusters

The assumptions and methods we proposed above are a new approach to understanding and using the Kuramoto self-consistent method. It includes a more general form of mean-field $Z(t)$ as Eq. (2.6) with some approximations. To show the validity of this new approach, we apply it to cases where the traditional self-consistent method can not be used directly, as the bimodal distributions and finite networks of oscillators.

Figure 2.1: Synchronization clusters of oscillators with double-Gaussian distribution with (a, b) $\omega_1^0 = \omega_2^0 = 0$, (c, d) $-\omega_1^0 = \omega_2^0 = 1.2$, (e, f) $-\omega_1^0 = \omega_2^0 = 1.4$, (g, h) $-\omega_1^0 = \omega_2^0 = 2$. Synchronization trees in the left column and the lines in the right column are solutions from the self-consistent method. The dashed lines in the right column are the frequency of synchronized groups. The numerical simulations are obtained from $N = 10000$ oscillators with an increase of coupling strength $dK = 0.01$ from $K = 0$. The initial states at $K + dK$ is the same as the final state at K . The synchronized oscillators are shown with yellow dots and the others with blue ones.

2.3.1 From unimodal to bimodal distributions

The first advantage of our approach is that it can be used to study the multi-cluster states. In this section, we show the change of synchronization processes of oscillators from unimodal distributions to bimodal ones in a unified framework. Specifically, consider Kuramoto oscillators with natural frequencies chosen randomly from the distribution $g_\omega(\omega)$

$$g_\omega(\omega) = C_1 \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(\omega-\omega_1^0)^2}{2\sigma_1^2}} + (1-C_1) \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(\omega-\omega_2^0)^2}{2\sigma_2^2}}. \quad (2.18)$$

The distribution is the sum of two Gaussian distributions with $\omega_1^0, \sigma_1, \omega_2^0, \sigma_2$ as the mean and variance of the first or second distribution. The number of oscillators is assumed to be infinite $N \rightarrow \infty$ in this section.

The factor C_1 controls the fraction of these two distributions. In the following, we set $C_1 = 0.5$ and $\sigma_1 = \sigma_2 = 1$. With the change of ω_1^0, ω_2^0 , we can obtain the transition from the unimodal case $\omega_1^0 = \omega_2^0$ to the bimodal case $\omega_2^0 - \omega_1^0 = O(1)$. We assume $\omega_2^0 \geq \omega_1^0$ without the loss of generality,

In the continuous limit $N \rightarrow \infty$, the self-consistent equation Eq. (2.15) and Eq. (2.16) can be rewritten straightforwardly as

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \xi \left[\mathbf{1}_\xi(q) + (1 - \mathbf{1}_{\xi_j}(q)) \left(1 - \sqrt{1 - \frac{q^2}{\xi_j^2}}\right) \right] \rho_\xi(\xi|\Omega) d\xi, \\ r &= \int_{\mathbb{R}} \sqrt{1 - \frac{\xi^2}{q^2}} \mathbf{1}_{\xi_j}(q) \rho_\xi(\xi|\Omega) d\xi, \end{aligned} \quad (2.19)$$

where $q \equiv Kr$ and Ω are for the synchronization level and frequency. The distribution of $\xi \equiv \omega - \Omega$ can be obtained from $g_\omega(\omega)$ as $\rho_\xi(\xi|\Omega) = \rho_\omega(\xi + \Omega)$. Given any distribution $g_\omega(\omega)$, the synchronization clusters with $r(K)$ and $\Omega^r(k)$ can be obtained implicitly from Eq. (2.19), which reads

$$\omega_i \in \sigma(K) \equiv [\Omega^r(K) - Kr(K), \Omega^r(K) + Kr(K)]. \quad (2.20)$$

To test and verify the theoretical results Eq. (2.20), we do numerical simulations of $N = 10000$ oscillators with an increase of coupling strength from $K = 0$ with $dK = 0.01$. The initial state at $K + dK$ is the same as the final state at K . After a sufficient transient time, the state of each oscillator is checked by its mean-frequency. For any two oscillators, if their mean-frequencies' difference is smaller than 1% of the difference of their natural frequencies, these two oscillators are both denoted as synchronized. The synchronized oscillators are shown with yellow dots and the others with blue ones in Fig. 2.1.

The synchronization process depends on the shape of the distribution $g_\omega(\omega)$. First of all, when $\omega_2^0 = \omega_1^0 = 0$, we get the unimodal case. It has been analyzed in detail by Kuramoto, as we have discussed above. For this system, there is only one

synchronization cluster in the middle of the oscillators. It grows from the center frequency $\omega = 0$ with the increase of coupling strength, shown in Fig. 2.1(a,b). The number of oscillators in the cluster also increases continuously from zero. One observes a continuous phase transition.

Secondly, with a slight separation of ω_2^0 and ω_1^0 , one gets a distribution with two peaks. Although, in general, it looks similar to the unimodal one, the second derivative $g_\omega(\omega)''$ changes from positive to negative around the center $\omega = 0$. In this case, we have two synchronization clusters. One is similar to the cluster of unimodal cases; the other is a back cluster, as shown in Fig. 2.1(c,d). If one follows the back cluster with an increase of K , the number of oscillators in this cluster decreases and finally to zero. However, for any pair of oscillators, an increase of K should strengthen their synchronization. The back cluster is unphysical from this view. As a matter of fact, in some particular systems, people have shown that it is unstable (Strogatz, 2000). It also has been shown that it works as a saddle in the dynamics of $Z(t)$, from a saddle-node bifurcation, where the synchronization state is the stable node.

In our approach, we stress that this back cluster is the folding back part of the normal synchronization cluster, which is quite common for finite cases, as we see in the next section. It appears as the result of the folding. Also, from the folding, one gets the flat bottom of the synchronization cluster, as shown in Fig. 2.1(e,f). There is a minimum requirement to form a cluster. The synchronization cluster cannot grow continuously from the center, but instead, jump to a relatively large cluster. At the same time, this back cluster describes the group of oscillators that are affected by the running oscillators greatly and cannot form a synchronization cluster spontaneously at a given level of K . As a result, the jump to synchronization happens at the critical coupling strength where the back cluster disappears.

Thirdly, a further separation of ω_2^0 and ω_1^0 is considered. As shown in Fig. 2.1(g,h), with the separation of the back cluster, two smaller clusters appear. They correspond to the two separated peaks of the distribution. These two clusters work as two triggers of synchronization. The oscillators can form these two smaller clusters, even though they are prevented from forming the one around $\Omega^r = 0$ by the back cluster.

Note that different from the giant cluster, these two smaller clusters exist in only a short region of K . They overlap and disappear with an increase of K , which means they are synchronized. Oscillators in these two clusters will merge and form a larger one. Besides, in Fig. 2.1(f), we see these two smaller clusters' integration also overcomes the giant cluster's minimum requirement. One gets the jump from a two-cluster state to the one with the single center cluster.

Finally, we have a more explicit bimodal distribution in Fig. 2.1(g,h) with a large separation of ω_2^0 and ω_1^0 . The two smaller synchronization clusters become relatively large in this case. They appear before the back cluster. With the coupling strength K increase, oscillators form a two-cluster state, called standing wave states (Martens et al., 2009) (or recently Bellerophon states (Li et al., 2019)).

The growth of each cluster is similar to the one of unimodal distributions in Fig. 2.1(a,b). With the increase of K , the theoretical region of these two clusters $\sigma_{1,2}$ overlaps. The oscillators around this overlap become unstable, attracting by two competitive clusters, shown as the blurred region in Fig. 2.1(h). With the increase of K further, these two clusters synchronize and form a center cluster.

The numerical simulations in Fig. 2.1 coincide with our approach and assumptions. The cluster boundaries σ depict the position and size of clusters. The independent assumption also works well. The oscillators strongly affected by two clusters only exist in a small overlap region of them. Besides, with the increase of K , smaller clusters' overlap will result in the formation of a large one, which is also captured in our approach.

At last, it is also mandatory to mention that, in the two-cluster state, there are smaller clusters as the blurred lines in Fig. 2.1(h). These are the higher-order resonance clusters, not captured in our approach from the assumptions. With the knowledge of the two major clusters, one can get these clusters approximately from the resonance analysis of oscillators. It is beyond the scope of this thesis. We refer to (Engelbrecht and Mirollo, 2012) for more information.

2.3.2 Finite networks of oscillators and synchronization tree

The second advantage of our method is the capacity to study finite networks of oscillators. The dynamic of finite networks of oscillators Eq. (2.1) is equivalent to the infinitely many oscillators system, which has a natural frequency distribution $g_\Omega(\Omega)$ as the sum of several δ functions. Hence our approach can be extended to finite networks of oscillators naturally. On the other hand, obtaining the synchronization tree of finite networks of oscillators is not easy work. The dynamic states of N oscillators are quite complicated, defined in a space of dimension $N - 1$. With the proper assumption, our approach only considers the synchronization part of such oscillators. This approach contains some approximations, but also actually gives the synchronization tree of all the oscillators.

This section is divided into two parts. First, we check the validity of our approach. The coupled $N = 2, 3, 5$ oscillators are considered with randomly picked natural frequencies. The result is shown in Fig. 2.2. For two oscillators, the dynamics Eq. (2.1) can be written as

$$\dot{\theta}_1 = \omega_1 + \frac{K}{2} \sin(\theta_2 - \theta_1), \quad \dot{\theta}_2 = \omega_2 + \frac{K}{2} \sin(\theta_1 - \theta_2), \quad (2.21)$$

where ω_1 and ω_2 are natural frequencies of these two oscillators. Without loss of generality, we assume $\omega_1 > \omega_2$. Besides, from the symmetry of different rotation frames, we further take $\omega_1 + \omega_2 = 0$.

The state of Eq. (2.21) can be obtained analytically. With the definition $\varphi \equiv \theta_1 - \theta_2$, one gets

$$\dot{\varphi} = \omega_1 - \omega_2 - K \sin(\varphi). \quad (2.22)$$

Figure 2.2: Synchronization trees of oscillators with two oscillators (a-b), three oscillators (c-d), and five oscillators (e-f). The blue tones are the synchronization tones around one oscillator. The red (black) tones are the stable (unstable) synchronization tones of at least two oscillators. In the right column, the unstable tones and the tones completely covered by the complete synchronization tone are neglected.

This dynamical equation has two fixed points when $K \geq |\omega_1 - \omega_2|$, one stable as $\varphi = \arcsin(|\omega_1 - \omega_2|/K)$ and one saddle $\varphi = \pi - \arcsin(|\omega_1 - \omega_2|/K)$. The stable fixed point is the synchronization state of these two oscillators. The saddle corresponds to the back cluster, as we will see later. In these states the mean-frequency of these two oscillators can be determined by the sum of the two equations of Eq. (2.21), as $\Omega^r = \dot{\theta}_1 = \dot{\theta}_2 = (\omega_1 + \omega_2)/2$.

Then, we apply our approach of the self-consistent method to these two oscillators. The equations Eq. (2.15) can be written as

$$0 = \sum_{j=1}^2 (\omega_j - \Omega^r) (\mathbf{1}_{\omega_j - \Omega^r}(q) + (1 - \mathbf{1}_{\omega_j - \Omega^r}(q)) (1 - \sqrt{1 - \frac{q^2}{(\omega_j - \Omega^r)^2}})) \quad (2.23)$$

where $q = Kr$ is the rescaled coupling strength. The function $\mathbf{1}_{\omega_j - \Omega^r}(q) = 1$ if $|\omega_j - \Omega^r| \leq q$ and otherwise zero. From Eq. (2.23), it is easy to obtain that any solution Ω^r should satisfy $\omega_1 < \Omega^r < \omega_2$. All the possible synchronized mean-frequencies are in the region $\Omega^r \in (\omega_1, \omega_2)$. There are three solutions of Eq. (2.23) as

$$\Omega_0^r = \frac{\omega_1 + \omega_2}{2}, \quad \Omega_1^r = -\frac{\omega_1 - \sqrt{4\omega_1^2 - 3q^2}}{3}, \quad \Omega_2^r = -\frac{\omega_2 + \sqrt{4\omega_2^2 - sq^2}}{3}. \quad (2.24)$$

In the limit $q \rightarrow 0$, we see that Ω_1^r and Ω_2^r are the frequencies of clusters grow from the oscillator θ_1 and θ_2 . In these two clusters, there is only one oscillator. As for Ω_0^r , it is the center cluster which includes both of these two oscillators. From the synchronization condition $|\omega_j - \Omega^r| \leq q$, we have that the center cluster only exist for $K \geq K_c = |\omega_1 - \omega_2|$.

The region of each cluster can be determined from Eq. (2.16). Specifically, for ω_1^r and ω_2 there is only one solution r_1 and r_2 . For ω_0 one gets two solutions $r_0^{(1)}$ and $r_0^{(2)}$ with $r_0^{(1)} \geq \frac{\sqrt{2}}{2} \geq r_0^{(2)}$. Corresponds to the bimodal cases, the solution $r_0^{(1)}$ describes the center cluster, and $r_0^{(2)}$ for the back as shown in Fig. 2.2(b). The value r does not contain as much information as the continuous limit with $N \rightarrow \infty$, for the only two oscillators in this system. In the continuous limit, r describes the real size of the cluster. In contrast, it only describes a kind of affective region of such oscillators or a group of oscillators for finite cases. We do not see a cluster's growth with r but instead the replacement of clusters from smaller to larger.

We can further prove that for three oscillators, our approach of the self-consistent method also gives the same result as the dynamical analysis, as shown in Fig. 2.2(c,d). One can find three single-oscillator clusters, with the colour blue in Fig. 2.2(d). Two oscillators form a two-oscillator cluster first, similar to the one described above. This two-oscillator cluster overlaps the single-oscillator one, and finally, we obtain a synchronization state with three oscillators. The formation of this three-oscillator cluster is similar to the one of the two-oscillator state. The only difference is the replacement of the single-oscillator cluster by the two-oscillator one.

Figure 2.3: Synchronization trees of oscillators with $N = 10$ oscillators with natural frequencies uniformly chosen from a Gaussian distribution (left column) and a uniform chosen distribution (right column). The red tones are the stable synchronization tones of at least two oscillators. The dashed lines are the tones in the continuous limit.

With the increase of the number of oscillators, the dynamical analysis becomes quite complicated for $N \geq 4$, if not practically impossible. We consider a system of five oscillators. The results are shown in Fig. 2.2(e,f). The formation of the synchronization tree in Fig. 2.2(e) is described well by the formation and replacement of synchronization clusters in Fig. 2.2(f). One can find the formation of two-oscillators clusters and, finally, the synchronization state with five oscillators. There are also two three-oscillator clusters, corresponding to the state where the center oscillator is synchronized with one of the two two-oscillator clusters. These two three-oscillator clusters largely overlap and are close to each other and also the five-oscillator one. As a result, we find the jump to the five-oscillator cluster directly.

As shown in Fig. 2.2, our approach shows the synchronization tree of finite networks of oscillators quite clear. At the quantitative level, though our approach does not give exact critical coupling strengths for systems with $N \geq 4$, the approximation is quite reasonable. In the second part of the section, we use our approach to study a special kind of system containing finite networks of oscillators, with natural frequencies chosen uniformly from a distribution $g_\omega(\omega)$. In this way, we explore the synchronization transitions at a microscopic level, either continuous or abrupt.

Two distributions are considered, as the Gaussian distribution and uniform

distribution. The natural frequencies of oscillators ω_j are determined from $g_\omega(\omega)$ as

$$\frac{j}{N} = G(\omega_j) = \int_{-\infty}^{\omega_j} g_\omega(\omega) d\omega, j = 1 \dots N \quad (2.25)$$

In this way, systems with only a few oscillators can show similar transition properties in the continuous limit. In Fig. 2.3 the results for the finite ω_j and continuous $g_\omega(\omega)$ are both shown and compared with numerical simulations.

For the symmetry of these two systems, the synchronization clusters share the same mean-frequency $\Omega^r = 0$. From the center, synchronization cluster form and replace one by one. The emergence of all the clusters agrees with the one in the continuous limit. Note that for Gaussian distribution, the sequence of clusters is quite stretched. In contrast, one gets an decompressed sequence for a uniform distribution. Clusters with different sizes form in a small region of the coupling strength. As a result, the synchronization process is also decompressed for the uniform distribution and show an abrupt one. The requirement of synchronized clusters with different size is almost the same. A random trigger can result in the complete synchronization of all the oscillators.

In Fig. 2.3, the calculation of the smallest two-oscillator cluster has the largest error compared with the numerical simulation. However, such calculation coincides with numerical results if there are only two oscillators, as we have proven before. With the other oscillators' effect, the two-oscillator cluster does not form at the same coupling strength. As a matter of fact, given N oscillators from a symmetric unimodal distribution $g_\omega(\omega)$, the natural frequency of the central pair of oscillators can be approximated as $\omega_2 = -\omega_1 \approx 1/(2Ng(0))$. In the limit $q \rightarrow 0$, ignoring all the other oscillators' contributions, the synchronization condition of this two oscillators can be obtained straightforwardly as

$$K > K_c^{(2)} = \frac{1}{2g(0)}. \quad (2.26)$$

At the start of the synchronization group, the synchronization condition $K_c^{(2)}$ depends negatively on the distribution's peak value $g_\omega(0)$ and independent of N . However considering other oscillators around the center, the synchronization condition is reduced to $K_c^{(\infty)} = 2/(\pi g(0))$ in the continuous limit (Strogatz, 2000). It is interesting to note that $K_c^{(2)} = (\pi/4)K_c^{(\infty)}$. The independent critical coupling strength $K_c^{(2)}$ is slightly smaller than $K_c^{(\infty)}$. Hence in the region $K_c^{(2)} \leq K \leq K_c^{(\infty)}$, if one cancels all the other oscillators' effect from the center two, they will become synchronized. They are the trigger of the synchronization process from this point of view.

2.3.3 Kuramoto-like models

In the above two sections, we have shown our approach to the Kuramoto model's self-consistent method. The self-consistent method is not limited to the Kuramoto

Figure 2.4: Distributions ρ_ω , transformed distributions of natural frequencies ρ_ω^t and synchronization trees of $N = 10000$ oscillators (right column) of the out-coupling model Eq. (2.28). The natural frequencies distribution ρ_ω is the standard Gaussian distribution.

model. It is widely applied in a verity of Kuramoto-like models, which are generalizations of the original one. In general, we can write some of them as

$$\dot{\theta}_i = \omega_i + f_i \frac{K}{N} \sum_{j=1}^N g_j \sin(\theta_j - \theta_i), \quad (2.27)$$

where f_i and g_j are two additional factors. In this way, the distribution of natural frequencies and topologies, coupling functions and other factors are considered (Ichinomiya, 2004; Oh et al., 2007; Xu et al., 2018; Peron and Rodrigues, 2012; Coutinho et al., 2013; Daido, 1987; Hong and Strogatz, 2011; Yuan et al., 2014; Hu et al., 2014; Skardal et al., 2013; Pinto and Saa, 2015; Dan and Jun-Zhong, 2014; Liu et al., 2013; Wang and Li, 2011). Much work has been done to focus either on the effect of specific factors or the cross-effect of them. A variety of synchronization processes have been studied well, especially through Kuramoto self-consistent method and the newly developed Ott-Antonsen ansatz (Ott and Antonsen, 2008).

It is interesting to note that even though the models are quite different from others, the synchronization transitions are quite similar for a large fraction of systems. The transitions are either continuous or discontinuous, containing either partial synchronization states or the multiple clusters, just as the original Kuramoto oscillators. Some of them follow exactly the same self-consistent equations, and correspondingly the same steady states (Gao and Efstathiou, 2020).

For all the Kuramoto like models Eq. (2.27), our approach can be applied straightforwardly. To give an example, in this section, we consider the model with

out-coupling as

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N |\omega_j| \sin(\theta_j - \theta_i). \quad (2.28)$$

This model has been studied in detail in (Xu et al., 2018, 2019). The authors find that this model has discontinuous transitions and Bellerophon states (or standing wave states). In (Gao and Efstathiou, 2018), we have shown that the self-consistent equations of Eq. (2.28) are the same as an original Kuramoto model with a bimodal distribution in the continuous limit as $g_\omega^t(\omega) = \langle |\omega| \rangle^{-1} |\omega| g_\omega(\omega)$ (see Chapter 7 for details).

Following our approach in this chapter, we firstly redefine the order parameter $Z(t)$ as

$$Z(t) = \frac{1}{N \langle |\omega| \rangle} \sum_{j=1}^N |\omega_j| e^{i\theta_j}. \quad (2.29)$$

Substitution of Eq. (2.29) into Eq. (2.28) yields the same equations as the Eq. (2.8) with our assumption Eq. (2.6). Following the same progress, we can obtain the self-consistent equations as

$$r = \frac{1}{N \langle |\omega| \rangle} \sum_{i=1}^N |\omega_i| (\langle \cos \theta_i \rangle + i \langle \sin \theta_i \rangle). \quad (2.30)$$

It is slightly different from the one Eq. (2.14) for the original Kuramoto model. The contribution of oscillators to r is weighted by its absolute frequency $|\omega|$. The synchronization clusters obtained from Eq. (2.30) are shown in Fig. 2.4. We obtain the similar synchronization processes as the one of Kuramomot oscillators with a bimodal distribution, coinciding with the transformed distribution.

2.4 Conclusions

In this Chapter, we propose a new approach to understand and use the Kuramoto self-consistent method. Other than the steady-state assumption, we use a more general assumption of the order parameter. With the approximation by perturbation methods, we obtain the same equations as the Kuramoto self-consistent method. Unlike the original self-consistent method, in our approach, we focus on the synchronization clusters σ other than the order parameter Z .

Three applications with numerical simulations are shown, from unimodal to bimodal distributions, from infinite to finite systems, and also the general Kuramoto-like models. It is shown that our approach works well with numerical results. The self-consistent equations can be applied in a much wilder way than expected. With this new approach, we explore the synchronization processes in a unified framework. From our point of view, the synchronization of oscillators can be understood

as two processes, formation of clusters and interaction between them. Clusters appear from the local concentration with the overtaking of the effect of the others. When several clusters overlap, they form a giant one with the increase of coupling strength. This simple scenario works from small to large systems, from the concentration of natural frequencies to other factors.

The limitation of our approach is also quite clear. It is based on the self-consistent method, which means that it can not contain any system's dynamical information, such as the stability and basins of attraction. In addition, we focus on the major synchronization clusters' formation with r_j and Ω_j^r . The inter-cluster interactions for Ψ_j and higher-order resonance clusters are neglected. After analyzing major clusters' formation, these two effects can be considered, using collective variables (Gottwald, 2015) and resonance analysis (Engelbrecht and Mirollo, 2012). In this way, one can expect a more complementary method.

The analysis is based on the mean-field form of oscillators, which can not be generalized to oscillators in complex networks directly. We believe the basic synchronization processes are the same. But it is an open question up to our knowledge. As for the finite cases, the number of resonance clusters increases steeply with the number of oscillators. Robust estimation of the synchronization process for finite networks of oscillators is a good topic.

Chapter 3

Steady States of Second-order Oscillators

As shown in Chapter 2, the self-consistent method, first introduced by Kuramoto, is a powerful tool for the analysis of the steady states of coupled oscillator networks. For second-order oscillator networks complications to the application of the self-consistent method arise because of the *bistable* behavior due to the co-existence of a stable fixed point and a stable limit cycle, and the resulting complicated boundary between the corresponding basins of attraction. In this chapter, we report on a self-consistent analysis of second-order oscillators which is simpler compared to previous approaches while giving more accurate results in the small inertia regime and close to incoherence. Though the second-order oscillators have a variety of dynamical states, in this chapter we only focus on the steady states of the second-order oscillators. Specifically, we apply the self-consistent method to analyze the steady states of coupled second-order oscillators and we introduce the concepts of *margin region* and *scaled inertia*. The improved accuracy of the self-consistent method close to incoherence leads to an accurate estimate of the critical coupling corresponding to transitions from incoherence.

3.1 Introduction

The Kuramoto model is not only simple and amenable to analytical considerations, but it is also easy to generalize in different directions. By adding frequency adaptations (inertias), the second-order oscillators model has been proposed and developed to describe the dynamics of several systems: tropical Asian species of fireflies (Ermentrout, 1991); Josephson junction arrays (Levi et al., 1978; Watanabe and Strogatz, 1994; Trees et al., 2005); goods markets (Ikeda et al., 2012); dendritic neurons (Sakye and Ragulskis, 2011); and power grids (Filatrella et al., 2008). Many important conclusions about the stability of power grids have been obtained through analysis of this model (Rohden et al., 2012, 2014; Lozano et al.,

2012; Witthaut and Timme, 2012; Menck et al., 2013; Hellmann et al., 2016; Kim et al., 2015; Gambuzza et al., 2017; Fortuna et al., 2011; Dörfler et al., 2013; Grzybowski et al., 2016; Maïzi et al., 2016; Manik et al., 2017a; Pinto and Saa, 2016; Rohden et al., 2017; Witthaut et al., 2016).

Kuramoto's self-consistent analysis (Kuramoto, 1975) has been extended to second-order oscillators by Tanaka *et al* (Tanaka et al., 1997a,b). In this chapter, we are revisiting the self-consistent method for the steady states of second-order oscillators. The benefits are twofold. First, we considerably simplify the derivation of the estimates of the limit cycles of the system that play a role in the self-consistent analysis. Second, the obtained estimates are much more accurate compared to earlier estimates, especially, for small inertias. Therefore, the method can be applied to the general case of the second-order oscillators, with large or small inertias, for both incoherence or synchronized states. The improved limit cycle estimates also lead to self-consistent equations that coincide well with numerical simulations. Moreover, the more accurate self-consistent method allows us to obtain the critical coupling strength K_c where steady state solutions bifurcate from the incoherence state. The results agree with the stability analysis of the incoherence state in (Barre and Métivier, 2016) obtained through an unstable manifold expansion of the associated continuity equation.

We give a short outline of the chapter. In Sec. 3.2, the model and the general framework of the self-consistent method are introduced. The dynamics of a single second-order oscillator is discussed in Sec. 3.3. Based on this, the self-consistent equation is obtained in Sec. 3.4 and several properties of steady states for arbitrary natural frequency distributions are discussed. In Sec. 3.5 the steady states of oscillators with symmetric and unimodal natural frequency distribution are discussed and the theoretical results are compared to numerical simulations. We conclude the chapter in Sec. 3.6.

3.2 Model and self-consistent method

The model for coupled second-order Kuramoto-type oscillators reads

$$m_i \ddot{\varphi}_i + D_i \dot{\varphi}_i = \Omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\varphi_j - \varphi_i), \quad (3.1)$$

for $i = 1, \dots, N$, where m_i , D_i , and Ω_i are respectively the inertia, damping coefficient, and natural frequency of the i -th oscillator. The dynamics of each oscillator is described by its phase φ and corresponding velocity $\dot{\varphi}$, with $(\varphi, \dot{\varphi}) \in \mathbb{S} \times \mathbb{R}$, where $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ (a circle of length 2π). Moreover, N is the number of oscillators and K is the (uniform) coupling strength.

To describe the collective behavior of the oscillators, one defines the order

parameter as

$$re^{i\phi} = \frac{1}{N} \sum_{j=1}^N e^{i\varphi_j}. \quad (3.2)$$

Here $r \in [0, 1]$ represents the *coherence* of the oscillators, since $r = 1$ if and only if all the oscillators are synchronized with $\varphi_i(t) = \varphi(t)$ for all $1 \leq i \leq N$, and in general (not rigorously) $r \cong 0$ corresponds to an almost uniform distribution of the phase terms $e^{i\varphi_i}$ over the unit circle. The rate of change of the collective phase ϕ is related to the mean frequency of the oscillators and describes the global rotation.

Using the amplitude $r(t)$ and phase $\phi(t)$ of the order parameter, the model (3.1) can be rewritten in a mean-field form as

$$m_i \ddot{\varphi}_i + D_i \dot{\varphi}_i = \Omega_i(t) + Kr(t) \sin(\phi(t) - \varphi_i), \quad (3.3)$$

with $i = 1, \dots, N$. In this chapter, we consider steady states given by

$$r(t) = r, \quad \phi(t) = \Omega^r t + \Psi, \quad (3.4)$$

where r , Ω^r , and Ψ are all constant. Passing to a frame rotating by $\phi(t) = \Omega^r t + \Psi$, and defining the phase difference between each oscillator and the frame as

$$\theta_i = \varphi_i - \phi(t), \quad (3.5)$$

one finds that the dynamics for the oscillators in the rotating frame is given by

$$m_i \ddot{\theta}_i + D_i \dot{\theta}_i = (\Omega_i - D_i \Omega^r) - Kr \sin \theta_i, \quad (3.6)$$

for $i = 1, \dots, N$. Dropping the index i from Eq. (3.6) and assuming $Krm \neq 0$, the dynamics of a single oscillator in the rotating frame can be transformed to the standard form

$$\ddot{\theta} + a\dot{\theta} = b - \sin \theta, \quad (3.7)$$

with only two effective parameters

$$a = \frac{D}{\sqrt{Krm}}, \quad b = \frac{\Omega - D\Omega^r}{Kr}, \quad (3.8)$$

and rescaled time $\tau = t\sqrt{Kr/m}$.

In this chapter we consider only the case where all the oscillators have the same inertia m and damping coefficient D even though our approach generalizes to the case of different inertias and damping coefficients. To pass to the continuum limit we replace by a density function $g(\Omega, \theta_0, \dot{\theta}_0)$ the collection of discrete oscillators characterized by natural frequency Ω_i and initial state $(\theta_i(0), \dot{\theta}_i(0))$. Note that, differently from the case of first-order Kuramoto oscillators, the initial state $(\theta_0, \dot{\theta}_0)$, is important for the dynamics of our case because of the bistable mechanism we discuss in Sec. 3.3.

Figure 3.1: Phase diagram for a single second order oscillator, Eq. (3.7), in the (a, b) parameter plane. The thick black curve represents $b_S(a)$, and the dashed curve represents $b_L(a) = 1$. Note that for $a \geq 1.193$ the two curves coincide. The two curves separate the parameter plane into the three regions shown in the diagram.

Then, in the continuum limit the self-consistent equation for r and Ω^r reads

$$r = \int_{\mathbb{S}} \int_{\mathbb{R}} \int_{\mathbb{R}} g(\Omega, \theta_0, \dot{\theta}_0) e^{i\theta(t)} d\theta_0 d\dot{\theta}_0 d\Omega \quad (3.9)$$

where $\theta(t)$ is obtained from the dynamics of a single oscillator (3.6) as

$$\theta(t) = \tilde{f}(t; a, b, \theta_0, \dot{\theta}_0). \quad (3.10)$$

In what follows, our goal is to understand the self-consistent equation (3.9) and use it to explore the properties of steady states of second-order oscillators.

3.3 Dynamics of a single oscillator and bistable region

In this section we recall facts about the dynamics of a single second-order oscillator described by Eq. (3.6) and then we compute an approximation to the limit cycle that plays a central role in what follows.

Fixed points and limit cycle

Depending on the values of the parameters a and b , Eq. (3.7) can have either a fixed point, or a globally attracting stable limit cycle, or a bistable region where

the fixed point and the limit cycle coexist (Levi et al., 1978; Tanaka et al., 1997b; Strogatz, 2014). A thorough qualitative study of the fixed points and limit cycle in this system can be found in (Levi et al., 1978) where it has been shown that for $b > 1$ the system has no fixed points and it has a globally attracting stable limit cycle. For $b < 1$ the system has exactly two fixed points, one stable and one unstable. Then for each fixed value $b < 1$ there is a value $a_*(b)$ of a for which if $0 < a < a_*(b)$ the system also has a stable limit cycle; a so-called *bistable* state. For $a > a_*(b)$ the limit cycle does not exist anymore. The transition at $a = a_*(b)$ occurs through a homoclinic tangency bifurcation. For $b = 1$ the situation concerning the limit cycle is similar, with corresponding $a_*(1) \cong 1.193$. However, for $b = 1$ the two fixed points merge and the system undergoes a saddle-node bifurcation so that for $b > 1$ there are no more fixed points. These results are summarized in the phase diagram shown in Fig. 3.1. The dynamics for three qualitatively different cases are shown in Fig. 3.2.

Remark 3.3.1. The limit cycle is a *running* or *rotating* limit cycle. That is, following the dynamics on the limit cycle, in one period the phase θ increases by 2π . Alternatively stated, the phase space of the system is a cylinder $(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ and the limit cycle is a non-homotopically-trivial circle on the cylinder, see Fig. 3.2(b) and Fig. 3.2(c).

For our purposes we use a different description of the phase diagram. We define two functions: the (constant) function $b_L(a) = 1$, and the function

$$b_S(a) = \begin{cases} (a_*)^{-1}(a), & 0 \leq a \leq a_*(1) \cong 1.193, \\ 1, & a \geq a_*(1), \end{cases} \quad (3.11)$$

where $(a_*)^{-1}$ is the inverse of $a_* : [0, 1] \rightarrow [0, a_*(1)]$. Clearly, adapting the discussion above, for $b > b_L(a) = 1$ there exists a globally attracting limit cycle and no fixed points. For $0 < b < b_S(a)$ the system has two fixed points and no limit cycle. Finally, the bistable state exists for $b_S(a) < b < b_L(a)$.

The condition $b < b_L(a) = 1$ for the existence of fixed points is easily obtained, since the fixed points correspond to solutions of $(\dot{\theta}, \ddot{\theta}) = (0, 0)$, giving the equation $b = \sin \theta$. Further computing the stability of the fixed points we obtain that for $b < b_L(a) = 1$ (and $a > 0$) the system has the fixed points

$$\begin{aligned} (\theta_0, \dot{\theta}_0) &= (\arcsin(b), 0), & (\text{stable}), \\ (\theta_0, \dot{\theta}_0) &= (\pi - \arcsin(b), 0), & (\text{saddle point}). \end{aligned} \quad (3.12)$$

As a result, we have

$$\exp(i\theta_0) = \sqrt{1 - b^2} + ib \quad (3.13)$$

for the calculation of order parameters Eq. (3.2).

Determining the existence region of the limit cycle, that is, the function $b_S(a)$, is more complicated. The limit cycle is always stable and it appears for $0 <$

Figure 3.2: Phase portraits for a single second order oscillator, Eq. (3.7). In (a), for $a = 0.5, b = 0.2$, there is a stable and an unstable fixed point and no limit cycles. In (b), for $a = 0.5, b = 0.7$, we have a bistable system where the stable and unstable fixed points co-exist with a limit cycle. In (c), for $a = 0.5, b = 1.5$, there exists only a limit cycle. The solid thick curve in figures (b) and (c) represents the limit cycle. The dashed curves in figures (a) and (b) represent stable and unstable asymptotic curves to the saddle fixed point. Note that the left and right sides of the picture must be identified because of the 2π -periodicity of θ .

$a < a_*(1)$ through a homoclinic bifurcation and for $a > a_*(1)$ through an infinite period bifurcation (Levi et al., 1978; Strogatz, 2014). For small values of a an application of Melnikov's method (Guckenheimer, 2013) gives

$$b_S(a) \approx 4\pi^{-1}a \approx 1.2732a, \quad (3.14)$$

cf. Fig. 3.1. The same result can also be obtained by Lyapunov's direct method (Risken, 1996) which gives a clear intuitive picture of this bifurcation (see details in (Risken, 1996)). Using numerical simulations a higher order approximation of this bifurcation line can be obtained as

$$b_S(a) \approx \begin{cases} 1.2732a - 0.3056a^3, & 0 \leq a \leq a_*(1) \approx 1.193, \\ 1, & a \geq a_*(1). \end{cases} \quad (3.15)$$

Approximation of the limit cycle

The analysis of the self-consistent equation for the second-order oscillators requires an analytic expression for the limit cycle. In general, the solution of the limit cycle cannot be obtained analytically. An approximate expression has been computed in (Tanaka et al., 1997b) through the use of the Poincaré-Lindstedt method at the

underdamped limit $a \ll 1$. Translating the result of (Tanaka et al., 1997b) to our notation we have

$$\theta(\tau) = \nu\tau + \frac{a^2}{b^2} \sin(\nu\tau) + \frac{a^4}{b^3} (\cos(\nu\tau) - 1) + \dots, \quad (3.16a)$$

where

$$\nu = \frac{b}{a} - \frac{a^3}{2b^3} + \dots. \quad (3.16b)$$

Here we derive a different approximation to the limit cycle which is valid for a larger range of parameter values and which at the underdamped limit $a \ll 1$ coincides with Tanaka's approximation, Eq. (3.16).

We start by expressing $\dot{\theta}$ as a function of θ for points on the limit cycle using a Fourier series. Keeping only the first harmonics we write

$$\dot{\theta}(\theta) = A_0 + A_1 \cos \theta + B_1 \sin \theta.$$

Substituting the last expression in Eq. (3.7) and computing the Fourier coefficients so that the first harmonics vanish we obtain

$$\begin{aligned} \dot{\theta}(\theta) &= \frac{b}{a} + \frac{ab}{a^4 + b^2} \cos \theta - \frac{a^3}{a^4 + b^2} \sin \theta \\ &= \nu_0 + \varepsilon \cos(\theta + \theta_*), \end{aligned} \quad (3.17)$$

where

$$\nu_0 = \frac{b}{a}, \quad \frac{1}{\varepsilon} e^{i\theta_*} = \nu_0 + ia.$$

The time-average of $e^{i\theta}$ on the limit cycle is given by

$$\begin{aligned} \langle e^{i\theta} \rangle &= \frac{1}{T} \int_0^T \exp(i\theta(\tau)) d\tau \\ &= \int_0^{2\pi} \frac{\exp(i\theta)}{\dot{\theta}(\theta)} d\theta \Big/ \int_0^{2\pi} \frac{1}{\dot{\theta}(\theta)} d\theta. \end{aligned} \quad (3.18)$$

These integrals can be exactly computed for $\dot{\theta}(\theta)$ given by Eq. (3.17). Computing the period integral we obtain

$$\nu = \frac{2\pi}{T} = \sqrt{\nu_0^2 - \varepsilon^2} = \sqrt{\nu_0^2 - (\nu_0^2 + a^2)^{-1}} \leq \nu_0. \quad (3.19)$$

The computation of $\langle e^{i\theta} \rangle$ gives

$$\begin{aligned} \langle e^{i\theta} \rangle &= e^{-i\theta_*} \varepsilon^{-1} \left[\sqrt{\nu_0^2 - \varepsilon^2} - \nu_0 \right] \\ &= -\nu_0(\nu_0 - \nu) + ia(\nu_0 - \nu). \end{aligned} \quad (3.20)$$

A Taylor series expansion for $\varepsilon \ll 1$ gives the expression

$$\begin{aligned}\langle e^{i\theta} \rangle &= \frac{1}{2} \left(-1 + \frac{ia}{\nu_0} \right) \varepsilon^2 + O(\varepsilon^4) \\ &= \frac{1}{2} \left(-1 + \frac{ia^2}{b} \right) \frac{a^2}{a^4 + b^2} + O(\varepsilon^4).\end{aligned}\tag{3.21}$$

which is valid for $a^2 + \nu_0^2 \gg 1$, that is, for $a \ll b$ or $a \gg 1$. With the same order of approximation, one can replace θ by $\nu_0 \tau$ in Eq. (3.17). Then integration with respect to τ gives

$$\theta(\tau) = \nu_0 \tau + \frac{a^4}{b(a^4 + b^2)} [\cos(\nu_0 \tau) - 1] + \frac{a^2}{a^4 + b^2} \sin(\nu_0 \tau),\tag{3.22}$$

with the constant of integration chosen so that $\theta(0) = 0$. Observe that for $a \ll 1$, Eq. (3.22) gives the approximation in Eq. (3.16). Moreover, this expression Eq. (3.22) can also be obtained through the Poincaré-Lindstedt method and valid when $a^2 \gg b \gg 1$, which corresponds to overdamping.

The previous results for Eq. (3.16) is calculated with condition $a \ll 1 \ll b$, where the value of the time-average of $\cos \theta$ on the limit cycle is approximated using the approach in (Tanaka et al., 1997b) by

$$\langle \cos \theta \rangle = -\frac{a^2}{2b^2},\tag{3.23}$$

see also ¹. This expression agrees with the lowest order term in the Taylor series of the real part of Eq. (3.21) with respect to $a \ll 1$.

As a result, when $a \ll 1 \ll b$ all the three approximate solutions, as Eq. (3.19), Eq. (3.22) and the previous one Eq. (3.16) gives the same estimation of the limit cycle. Both the solution Eq. (3.19) and Eq. (3.22) can be seen as a generalization of previous solutions. For the self-consistent method depending on the estimation of $\langle e^{i\theta} \rangle$, using numerical simulations (see Fig. 3.3), we found that both the computation in Eq. (3.20) or the one with Taylor expansion in Eq. (3.21) are significantly better estimates of $\langle \cos \theta \rangle$ on the limit cycle compared to the approximation obtained previously as Eq. (3.23). On the other hand, it is hard to distinguish the better one from Eq. (3.19) and Eq. (3.22) from the numerical simulations (see Fig. 3.3). Moreover, in the limit of large or small inertias, with $a \ll b$ or $a \gg 1$, Eq. (3.20) and Eq. (3.21) is the same in the order of $O(\varepsilon^2)$. Hence we consider both the expressions are proper expressions for self-consistent method.

In the computations in subsequent sections we will be using the expression Eq. (3.21) for the sake of simplicity. It is straightforwardly to show that both the quantitative (such as K_c) and qualitative (such as the margin regions) results we obtained following can also be obtained with another expression Eq. (3.20).

¹In Eq. (A.3) of (Tanaka et al., 1997b) the values of $\cos C$ and $\sin C$ have been interchanged leading to an incorrect estimation of the time-average of $\cos \theta$ along the limit cycle. In particular, the expression $\frac{1}{2} \tilde{r} \Delta^3$ in Eq. (34) of (Tanaka et al., 1997b) should have been $\frac{1}{2} \tilde{r} \Delta^2$ which, in our notation, corresponds to Eq. (3.23) in the present chapter.

Figure 3.3: Approximation errors of $\langle \cos \theta \rangle$ corresponding from left to right to Eq. (3.23), Eq. (3.21), and Eq. (3.20). The error is defined as the maximum of the absolute value of the difference between the numerical calculation of $\langle \cos \theta \rangle$ along the limit cycle (which exists only for $b > b_S(a)$) and the corresponding analytical estimation. Note that in panel (a) all errors above 0.5 are represented by the same color.

3.4 Self-consistent equation for two processes

Because of the complexity of the basins of attraction, it is difficult to study the problem of synchronization in its full generality. Instead, following Tanaka *et al.*'s approach in (Tanaka et al., 1997a,b), we consider the synchronization during the so-called *forward* and *backward* processes as a special collection of steady states.

In the forward process $[F]$ the system starts at the incoherence state with coupling $K = 0$ and then K progressively increases. Small coupling $K \ll 1$ and incoherence state $r \approx 0$ corresponds to large values of $a^2 = D^2/Krm$ and $b = (\Omega - D\Omega^r)/Kr$. In particular, it can be ensured that all oscillators are in the parameter region $b > b_L(a) = 1$ where there exists only a stable limit cycle (no stable fixed point), see Sec. 3.3. Similarly, in the backward process $[B]$ the system starts at a coherent state with a large value of K and then the coupling progressively decreases. Large coupling $K \gg 1$ and coherent state $r \approx 1$ corresponds to small values of $a^2 = D^2/Krm$ and $b = (\Omega - D\Omega^r)/Kr$. Here it can be ensured that all oscillators are in the parameter region $0 < b < b_S(a)$ where there exists only a stable fixed point (no stable limit cycle).

Thus, in these two processes the initial states of all the oscillators lie entirely in the basin of one stable state, the fixed point for $[B]$ and the limit cycle for $[F]$, and only leave it when this stable state disappears as K changes. This happens in the $[B]$ process when oscillators cross the boundary $b_P = b_L$ and in the $[F]$ process when they cross the boundary $b_P = b_S$. Note that because of the different values of the natural frequency Ω for each oscillator, the oscillators will move from one stable state to another one at different values of K .

The previous discussion implies that in the forward and backward processes the

role of the initial state $(\theta_0, \dot{\theta}_0)$ for each oscillator can be neglected and therefore we can consider the density $g(\Omega)$ obtained by integrating $g(\Omega, \theta_0, \dot{\theta}_0)$, that is,

$$g(\Omega) = \int_{\mathbb{S}} \int_{\mathbb{R}} g(\Omega, \theta_0, \dot{\theta}_0) d\theta_0 d\dot{\theta}_0.$$

Moreover, for fixed values of r and Ω^r the density $g(\Omega)$ is transformed to a density $G(b)$ as

$$G(b) = Kr g(Krb + D\Omega^r).$$

During the forward and backward processes, we can write the order parameter r as the sum of the coherence of two populations of oscillators: oscillators at the stable fixed point, which we call *locked*, and oscillators at the stable limit cycle, which we call *running*. We have

$$r = z_l + z_r,$$

where z_l and z_r represent the coherence of the locked and running oscillators, calculated in Eq. (3.21) and Eq. (3.13) respectively. In the forward and backward processes, the locked and running oscillators are separated by the boundary of the bistable region of a single oscillator, i.e., by b_S for $[F]$ and b_L for $[B]$.

With substitution of the solution of a single oscillator into the self-consistent equation for the locked oscillators, z_l reads

$$\begin{aligned} z_l &= \int_{|b| < b_P(a)} G(b) \left[\sqrt{1 - b^2} + ib \right] db \\ &= \int_{\mathbb{R}} G(b) \mathbf{1}_l \left[\sqrt{1 - b^2} + ib \right] db, \end{aligned}$$

where the indicator function $\mathbf{1}_l$ takes the value 1 if $|b| < b_P(a)$, corresponding to the condition for locked oscillators, and 0 otherwise. In the equation above we have $b_P = b_L$ for the backward process and $b_P = b_S$ for the forward process.

For the running oscillators the coherence z_r reads

$$\begin{aligned} z_r &= \int_{\mathbb{R}} G(b) \mathbf{1}_r \langle e^{i\theta} \rangle db \\ &= \int_{\mathbb{R}} G(b) \mathbf{1}_r \left[\frac{1}{2} \left(-1 + \frac{ia^2}{b} \right) \frac{a^2}{a^4 + b^2} \right] db, \end{aligned}$$

where the function $\mathbf{1}_r$ takes the value 1 if $|b| > b_P(a)$, corresponding to the condition for running oscillators, and 0 otherwise. Note that

$$\mathbf{1}_l + \mathbf{1}_r = 0.$$

Combining z_l and z_r , and separating the real and imaginary parts, we obtain the self-consistent equations for the second-order oscillators as

$$r = \int_{\mathbb{R}} G(b) \left[\mathbf{1}_l \sqrt{1 - b^2} - \mathbf{1}_r \frac{a^2}{2(b^2 + a^4)} \right] db, \quad (3.24a)$$

$$0 = \int_{\mathbb{R}} G(b) \left[\mathbf{1}_l b + \mathbf{1}_r \frac{a^4}{2b(b^2 + a^4)} \right] db. \quad (3.24b)$$

One checks that Eq. (3.24) always has the trivial solution $r = 0$. For $r > 0$ with the definition $q = Kr$, we have $a = D/\sqrt{qm}$ and Eq. (3.24) can be divided by $q > 0$ to obtain the equations

$$\frac{1}{K} = F_1(q, \Omega^r) \equiv \int_{\mathbb{R}} g(qb + D\Omega^r) \left[\mathbf{1}_l \sqrt{1 - b^2} - \mathbf{1}_r \frac{a^2}{2(b^2 + a^4)} \right] db, \quad (3.25a)$$

$$0 = F_2(q, \Omega^r) \equiv \int_{\mathbb{R}} g(qb + D\Omega^r) \left[\mathbf{1}_l b + \mathbf{1}_r \frac{a^4}{2b(b^2 + a^4)} \right] db. \quad (3.25b)$$

For values (q, Ω^r) satisfying $0 = F_2(q, \Omega^r)$, one can obtain the corresponding value of $K = [F_1(q, \Omega^r)]^{-1}$ provided that $F_1(q, \Omega^r) > 0$. Since $q = Kr$ we conclude that the triplet (K, q, Ω^r) can be transformed to the solutions of the self-consistent equation Eq. (3.25) as

$$\begin{aligned} (K, r, \Omega^r) &= (K, qK^{-1}, \Omega^r) \\ &= ([F_1(q, \Omega^r)]^{-1}, qF_1(q, \Omega^r), \Omega^r). \end{aligned}$$

These solutions can be (locally) parameterized in terms of K as families $(r(K), \Omega^r(K))$, except at points of bifurcation, i.e., at values of K where the number of families changes.

As an example of this approach we consider a system with the bimodal density function

$$g(\Omega) = \frac{3}{10} \sqrt{\frac{2}{\pi}} \exp[-2(\Omega + 1)^2] + \frac{7}{10} \sqrt{\frac{2}{\pi}} \exp[-2(\Omega - 1)^2], \quad (3.26)$$

see Fig. 3.4(a), and fix the parameter values to $D = 1$ and $m = 2$. The solution set of $F_2(q, \Omega^r) = 0$ is shown in Fig. 3.4(b) and the corresponding solutions are depicted in Fig. 3.4(c) in the (K, r, Ω^r) -space and projected onto the (K, r) plane in Fig. 3.4(d). Note the existence of more than one K -parameterized families $(r(K), \Omega^r(K))$. Moreover, note in Fig. 3.4(b) that some solutions of $F_2(q, \Omega^r) = 0$ lie in the region where $F_1(q, \Omega^r) \leq 0$, represented by the region between the dashed curves and the axis $q = 0$. Such solutions are rejected and do not appear in subsequent panels (c) and (d) in Fig. 3.4.

In the rest of this section we explore in more detail the properties of steady states obtained as solutions of the self-consistent equation, Eq. (3.25), for arbitrary distributions $g(\Omega)$. In Sec. 3.5 we focus in the case of unimodal symmetric distributions.

Figure 3.4: Steady state solutions of the self-consistent equations for a system with bimodal density $g(\Omega)$, Eq. (3.26), and $m = 2$, $D = 1$, $b_P = b_S$ which actually applies to the backward process [B]. Panel (a) depicts $g(\Omega)$. Panel (b) shows the zero-sets of $F_2(q, \Omega^r)$, Eq. (3.25b), with solid black curves, and of $F_1(q, \Omega^r)$, Eq. (3.25a), with dashed gray curves. We note the existence of three solution branches $\Omega^r(q)$ of Eq. (3.25b) for small q ; for larger q only one branch remains. Panel (c) shows the families $(K(q), r(q), \Omega^r(q))$ obtained through Eq. (3.25a). Panel (d) shows the projection of the families from panel (c) to the (K, r) -plane.

3.4.1 Multiple solution branches

Solutions to the self-consistent equation, Eq. (3.25), for second-order oscillators can have multiple branches. This feature is a natural consequence of the nonlinear nature of the self-consistent equations. First, for a given value of q , there may be multiple solution branches $\Omega^r(q)$ of the equation $F_2(q, \Omega^r) = 0$, Eq. (3.25b). The number of these branches is always odd (counting multiplicity), as a consequence of the continuity of $F_2(q, \Omega^r)$, and the fact that there is $M > 0$ such that $F_2(q, \Omega^r) < 0$ for $\Omega^r > M$ and $F_2(q, \Omega^r) > 0$ for $\Omega^r < -M$. Second, for each solution branch $\Omega_r(q)$, we have corresponding one-parameter families of steady states $(K(q), r(q), \Omega^r(q))$ through Eq. (3.25a). For each such family we can solve to obtain r as a function of K . However, for each family there may be more than one such branches $r(K)$. This is depicted in Fig. 3.4(d) for the bimodal distribution, Eq. (3.26), and in Fig. 3.5 for a Gaussian distribution, Eq. (3.30), with $\sigma = 1$.

In addition, since $F_1(q, \Omega^r)$ is bounded we conclude that K cannot take values smaller than some $K_{\min} > 0$ for solutions of the self-consistent equation, Eq. (3.25), or, equivalently, for the non-trivial solutions of Eq. (3.24). This implies that the only solution that is possible for $K < K_{\min}$ is the trivial solution $r = 0$ and other branches are the result of bifurcations that occur at values of K larger than $K_{\min}$.

Remark 3.4.1. The first-order Kuramoto model also exhibits multiple branches of steady state solutions. The effect of inertias in the second-order system is to introduce more branches to the system while the branches become more complicated.

3.4.2 Transition points

The trivial solution $r = 0$ represents the incoherence state. We are interested at the *transition points* where non-trivial solutions of the self-consistent equation either merge with or detach from the incoherence state for $r > 0$. Such transition points are important since often they coincide with the loss of stability of the incoherence state and the occurrence of a phase transition in the forward process. This is the reason why it is also usually denoted as the critical point (forward). Using Eq. (3.25), we can determine the transition points taking the limit $q \rightarrow 0^+$ corresponding to $r \rightarrow 0^+$.

When $q \rightarrow 0^+$, we have $a \rightarrow \infty$ and hence $b_P(a) = 1$ for both forward and backward processes. The transition points (Ω_c^r, K_c) are determined through non-trivial solutions of the self-consistent equation, Eq. (3.25), which we rewrite as

$$\begin{aligned}
\frac{1}{K_c} &= \lim_{q \rightarrow 0^+} \int_{-1}^1 g(qb + D\Omega_c^r) \sqrt{1 - b^2} db \\
&\quad - \lim_{q \rightarrow 0^+} \left[\int_1^\infty + \int_{-\infty}^{-1} \right] g(qb + D\Omega_c^r) \frac{a^2}{2(b^2 + a^4)} db, \\
0 &= \lim_{q \rightarrow 0^+} \int_{-1}^1 g(qb + D\Omega_c^r) b db \\
&\quad + \lim_{q \rightarrow 0^+} \left[\int_1^\infty + \int_{-\infty}^{-1} \right] g(qb + D\Omega_c^r) \frac{a^4}{2b(b^2 + a^4)} db.
\end{aligned}$$

The change of variables $x = qb$, the introduction of the *reduced mass* $\mu = m/D^2$, and subsequent calculations bring the previous equations to the form

$$\frac{1}{K_c} = \frac{\pi}{2} g(D\Omega_c^r) - \int_{\mathbb{R}} g(x + D\Omega_c^r) \frac{\mu}{2(1 + \mu^2 x^2)} dx, \quad (3.28a)$$

$$0 = \lim_{q \rightarrow 0^+} \int_q^\infty \frac{g(x + D\Omega_c^r) - g(-x + D\Omega_c^r)}{2x} \frac{1}{1 + \mu^2 x^2} dx. \quad (3.28b)$$

If the steady state branch that bifurcates at $K = K_c$ from the incoherence state is unstable then the transition between the incoherence state and corresponding coherent state is discontinuous. Otherwise, the transition is continuous.

Remark 3.4.2. Fig. 3.4(b) shows that it is possible that $\lim_{q \rightarrow 0^+} F_1(q, \Omega^r(q)) < 0$ and thus $K_c < 0$. We reject such solutions since for $q > 0$ (as we consider here) they give the non-physical $r < 0$. Consider the situation depicted in Fig. 3.4(b) where a curve $\mathcal{C}$ of solutions of $F_2(q, \Omega^r) = 0$ enters the region $F_1(q, \Omega^r) \leq 0$ by crossing the zero-set of $F_1(q, \Omega^r)$ (dashed curve in Fig. 3.4(b)) at a point (q_0, Ω_0^r) . Then, clearly, only the part $\mathcal{C}_+$ of $\mathcal{C}$ where $F_1(q, \Omega^r) > 0$ can be considered. Consider a point (q, Ω^r) on $\mathcal{C}_+$ that approaches (q_0, Ω_0^r) . Then the value of $F_1(q, \Omega^r)$ approaches 0 (while positive), and thus $K = [F_1(q, \Omega^r)]^{-1}$ approaches ∞ . This implies that in the (K, r) plane we obtain a family $(K(q), r(q))$ with $K(q) \rightarrow \infty$ and $r(q) \rightarrow 0$ as $q \rightarrow q_0$ in such a way so that $K(q)r(q) \rightarrow q_0$ as $q \rightarrow q_0$. In other words, for large enough K the corresponding curve $r(K)$ becomes asymptotic to the hyperbola $Kr = q_0$.

Remark 3.4.3. Compared with the Kuramoto model where $K_c = 2/(\pi g(D\Omega_c^r))$, the effect of inertias in Eq. (3.28a) is always to decrease the value of $1/K_c$ since the integral in this equation is non-negative. Hence with the same Ω_c^r , the critical coupling strength K_c for second-order oscillators is always larger than the one for Kuramoto oscillators.

3.4.3 Frequency scaling and scaled inertia

Consider a frequency distribution $g_s(\Omega)$ that depends on a *scale parameter* $s > 0$ so that

$$g_s(\Omega) = \frac{1}{s} g_1\left(\frac{\Omega}{s}\right).$$

Typical examples are the Gaussian distribution, Eq. (3.30), where $s = \sigma$, and the Lorentz distribution, Eq. (3.31), where $s = \gamma$.

Suppose that for inertia μ_1 and distribution g_1 one finds a steady state solution of the self-consistent equation, characterized by the parameters (q_1, w_1, K_1, r_1) . Here we introduce the parameter $w = D\Omega^r$ since Ω^r appears in the self-consistent equation only through w . Then a straightforward computation shows that for given inertia $\mu_s = s^{-1}\mu_1$ and for given distribution g_s there is a steady state solution characterized by parameters (q_s, w_s, K_s, r_s) with

$$q_s = sq_1, \quad w_s = sw_1, \quad K_s = sK_1, \quad r_s = r_1.$$

This property of steady-state solutions allows the translation of results from $s = 1$ to any value of $s > 0$. In particular, this allows the straightforward translation of the numerical results in Sec. 3.5, which have been obtained for a Gaussian distribution with $\sigma = 1$, to the case of arbitrary $\sigma > 0$.

Moreover, this observation suggests that we should introduce a more natural notion of inertia, the *scaled inertia*

$$\nu = s\mu,$$

so that $\nu_s = s\mu_s = \mu_1 = \nu_1$ is invariant under the scaling transformation. In what follows we do not directly use ν since we are interested in distributions that do not necessarily depend on a scale parameter.

3.4.4 Beyond the forward and backward processes – Dependence of steady states on initial states

For second-order oscillators, a crucial complication is the existence of the bistable state and the corresponding complicated basins of attraction. Considering the forward and backward processes, leading to Eq. (3.24), this complication is avoided since then the locked and running oscillators are separated by the boundaries of the bistable region.

The stead states in the forward and backward processes is only a special collection. In general, beyond these two processes with arbitrary choice of initial states, it is hard to find analytically the boundary between locked and running oscillators and consequently obtain the steady state. However, with different initial states, the oscillators will always separate into two groups. The corresponding fractions can be defined as $C_l(b; a)$ and $C_r(b; a)$ for locked and running groups respectively,

with normalization condition $C_l(b; a) + C_r(b; a) = 1$. In the special case of the forward process we have $C_l(b; a) = \mathbf{1}_{l, b_S}$ where $\mathbf{1}_{l, b_P}$ takes the value 1 if $|b| < b_P(a)$ and 0 otherwise. In the backward process we similarly have $C_l(b; a) = \mathbf{1}_{l, b_L}$. In terms of the fractions $C_l(b; a)$ and $C_r(b; a)$ the self-consistent equation reads

$$r = \int_{\mathbb{R}} G(b) \left[C_l(b; a) \sqrt{1 - b^2} - C_r(b; a) \frac{a^2}{2(b^2 + a^4)} \right] db, \quad (3.29a)$$

$$0 = \int_{\mathbb{R}} G(b) \left[C_l(b; a) b + C_r(b; a) \frac{a^4}{2b(b^2 + a^4)} \right] db. \quad (3.29b)$$

Even though we cannot easily determine $C_l(b; a)$, we note that

$$\mathbf{1}_{r, b_S} \leq C_l(b; a) \leq \mathbf{1}_{r, b_L}.$$

Therefore, different possibilities can be viewed as intermediate between the two considered processes. In particular, we can consider a boundary function $b_P(a)$ given as convex combination of the boundaries for the two processes, that is,

$$b_P(a) = c b_S(a) + (1 - c) b_L(a).$$

The previous discussion implies that, for fixed parameters (m, D, K) and fixed frequency distribution $g(\Omega)$, different initial states may reach different steady states. This is further discussed in Sec. 3.5.3 and demonstrated in Fig. 3.9 for a symmetric unimodal distribution.

3.4.5 Frequency shift and global rotation

For a single oscillator with natural frequency Ω , we can describe its dynamics in a frame rotating with frequency Ω' as having a new natural frequency $\Omega - D\Omega'$.

Consider now the case of coupled oscillators with a distribution of natural frequencies $g(\Omega)$. In a frame rotating with constant frequency Ω' the corresponding distribution of natural frequencies becomes $g(\Omega - D\Omega')$. Hence all the systems with a translation of $g(\Omega)$ by arbitrary $D\Omega'$ have the same dynamics. With substitution of $g(\Omega - D\Omega')$ into the self-consistent equations Eq. (3.24), the steady states for the translated system read $r(K)$, $\Omega^r(K) - \Omega'$.

Remark 3.4.4. Because of this behavior of solutions under frequency shifts we can assume, without loss of generality, that the mean of $g(\Omega)$ is 0 (when defined). This property also holds for first-order Kuramoto oscillators.

A natural rotating frame is the *center-of-mass frame* (Choi et al., 2011) with a rotating frequency $\Omega_{\text{cm}} = \int_{\mathbb{R}} \Omega g(\Omega) d\Omega$. It has been shown that the dynamics of oscillators in this rotating frame is exactly the same as the ones with a natural frequency with zero mean (Choi et al., 2011), considering with our remark above.

3.5 Symmetric unimodal natural frequency distribution

In this section, we consider the system with symmetric unimodal density function $g(\Omega)$. Because of the frequency shift property, see Sec. 3.4.5, we can assume that the median (and, when defined, also the mean) of $g(\Omega)$ is zero. We have $g(\Omega) = g(-\Omega)$, and $g(\Omega_1) \leq g(\Omega_2)$ if $\Omega_1 \geq \Omega_2 \geq 0$ from the unimodal property. Two typical symmetric unimodal distributions are the Gaussian distribution

$$g(\Omega) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{\Omega^2}{2\sigma^2}\right), \quad (3.30)$$

and the Lorentz (or Cauchy) distribution

$$g(\Omega) = \frac{1}{\pi} \frac{\gamma}{\Omega^2 + \gamma^2}. \quad (3.31)$$

3.5.1 Self-consistent equations

One crucial characteristic of the coupled oscillators with symmetric unimodal $g(\Omega)$ is that there is only one solution $\Omega^r(q) = 0$ of the equation $F_2(q, \Omega^r) = 0$. To show this, we rewrite Eq. (3.25b) as

$$\begin{aligned} 0 &= F_2(q, \Omega^r) \\ &= \int_0^\infty [g(qb + D\Omega^r) - g(-qb + D\Omega^r)] bW(|b|) db, \end{aligned} \quad (3.32)$$

where $W(|b|)$ is the even positive function given by

$$W(|b|) = \mathbf{1}_l + \mathbf{1}_r \frac{a^4}{2b^2(b^2 + a^4)} > 0.$$

Since the distribution $g(\Omega)$ is symmetric and unimodal, we have for $qb \neq 0$ that $g(qb + D\Omega^r) - g(-qb + D\Omega^r) = 0$ if and only if $\Omega^r = 0$. Hence the only solution of Eq. (3.32), or equivalently Eq. (3.25b), is $\Omega^r = 0$. In this case, the ensemble-mean frequency, see Sec. 3.4.5, reads $\omega_{\text{cm}} = 0 = \Omega^r$. Thus, with a symmetric unimodal $g(\Omega)$, the system has the simplest global rotating behavior.

Remark 3.5.1. Note that for a symmetric (not necessarily unimodal) $g(\Omega)$ the function $F_1(q, \Omega^r)$ is even in Ω^r while $F_2(q, \Omega^r)$ is odd in Ω^r . The latter property implies that $F_2(q, 0) = 0$ for all q , while the former property then implies that the corresponding value $F_1(q, 0)$ is a local maximum or minimum value of $F_1(q, \Omega^r)$ for fixed q .

With the substitution $\Omega^r = 0$, Eq. (3.25a) reads

$$\frac{1}{K} = \int_{\mathbb{R}} g(qb) \left[\mathbf{1}_l \sqrt{1 - b^2} - \mathbf{1}_r \frac{a^2}{2(b^2 + a^4)} \right] db, \quad (3.33)$$

Figure 3.5: The steady state solutions obtained by numerical integration of Eq. (3.33) for oscillators with Gaussian $g(\Omega)$, $\sigma = 1$, depicted together with the incoherence state $r = 0$. The damping coefficient is $D = 1$. From top left to bottom right: (a) $m = 0.5$; (b) $m = 1$; (c) $m = 3$; (d) $m = 6$. Solid lines (except $r = 0$) represent the backward process with $b_P = b_L$, and dashed lines the forward process with $b_P = b_S$. The gray-shaded area represents the margin region $\mathcal{M}$ discussed in Sec. 3.5.3. Transition points where steady state solutions merge with or detach from the incoherence state $r = 0$ are marked in the picture.

where we remind that $b_P = b_S$ for the forward process and $b_P = b_L = 1$ for the backward process. The solutions to the self-consistent equation, Eq. (3.33), for different values of $\mu = m/D^2$ and a Gaussian distribution with $\sigma = 1$ are shown in Fig. 3.5.

Even though for symmetric unimodal distributions $g(\Omega)$ the self-consistent equation $F_2(q, \Omega^r) = 0$ has only one solution $\Omega^r(q) = 0$, we can still obtain multiple solution branches $r(K)$ from the self-consistent equation $F_1(q, \Omega^r(q)) = 1/K$ as Eq. (3.33). Similar as Eq. (3.24a) with $\Omega^r = 0$, we can rewrite Eq. (3.33) as

$$r = L(r, K), \quad (3.34)$$

Figure 3.6: Numerical simulations of the backward and forward processes for $N = 5000$ oscillators with (a) $m = 0.2$ and (b) $m = 2$ where $D = 1$, and $g(\Omega)$ is the Gaussian distribution, Eq. (3.30), with $\sigma = 1$. Solid lines represent the backward process with $b_P = b_L$, and dashed lines the forward process with $b_P = b_S$. The red points represents the numerical results for the backward process and the blue points for the forward process. In panel (a) the points for the two processes largely overlap.

where $L(r, K)$ is the function with the original parameters as

$$L(r, K) = \int_{|\Omega| < Kr b_P(1/\sqrt{Kr\mu})} g(\Omega) \sqrt{1 - \frac{\Omega^2}{K^2 r^2}} d\Omega - \int_{|\Omega| > Kr b_P(1/\sqrt{Kr\mu})} g(\Omega) \frac{Kr\mu}{2(1 + \mu^2 \Omega^2)} d\Omega.$$

We note that Eq. (3.34) has the trivial solution $r = 0$, that is, $L(0, K) = 0$. Moreover, we have

$$\begin{aligned} L(1, K) &< \int_{|\Omega| < K b_P(1/\sqrt{K\mu})} g(\Omega) \sqrt{1 - \frac{\Omega^2}{K^2}} d\Omega \\ &< \int_{|\Omega| < K b_P(1/\sqrt{K\mu})} g(\Omega) d\Omega < 1. \end{aligned}$$

Hence there is at least one solution of Eq. (3.34) within $0 \leq r \leq 1$.

To check the analysis of steady states, we have performed several numerical simulations. We have numerically calculated the dynamics of a network with $N = 5000$ oscillators, following Eq. (3.1), using the fourth order Runge-Kutta method with fixed-size integration time-step $dt = 10^{-3}$. The natural frequency Ω_i for each oscillator is chosen randomly from a Gaussian distribution $g(\Omega)$ with $\sigma = 1$. At a given coupling strength K , after a transient period $t_0 = 40$, we calculate the order parameter r according to the definition in Eq. (3.2) as the

average value over a measurement period $\Delta t = 4$. In the forward and backward processes, we take $dK = 10^{-2}$ and $dK = -10^{-2}$ respectively as the increasing and decreasing coupling strength steps. In each step, the initial states of all the oscillators are the last states in the previous step. In the backward and forward processes, the initial states of the first step are chosen randomly from $\theta(0) \in [0, 2\pi]$, $\dot{\theta}(0) \in [0, 1]$ and $\theta(0) \in [0, 0.02\pi]$, $\dot{\theta}(0) \in [0, 1]$ respectively.

The phase transitions in the forward and backward processes are shown in Fig. 3.6. for oscillators with inertias $m = 0.2$ in panel (a) and $m = 2$ in panel (b). In all simulations $D = 1$. The figures show that our analytical results coincide with the numerical ones quite well and much better than the analytical results given in (Tanaka et al., 1997a,b). The error in the location of the transition point is due to the finite number ($N = 5000$) of oscillators used in the numerical simulations, whereas the self-consistent analysis is based on the limit $N \rightarrow \infty$; see (Olmi et al., 2014) for a more detailed discussion of this phenomenon.

3.5.2 Transition points

Near the incoherence state, that is for $r \cong 0$, we have $b_S = b_L = 1$, and there is no bistable behavior. Substituting $\Omega^r = 0$ into Eq. (3.28a) we obtain the critical value $K_c(\mu)$ of K as a function of the reduced inertia $\mu = m/D^2$. This is given by

$$K_c(\mu) = \frac{2}{\pi g(0) - A(\mu)}, \quad (3.35)$$

where

$$A(\mu) = \int_{\mathbb{R}} \frac{\mu}{1 + \mu^2 x^2} g(x) dx = \int_{\mathbb{R}} \frac{1}{1 + y^2} g\left(\frac{y}{\mu}\right) dy. \quad (3.36)$$

This critical coupling strength $K_c(\mu)$ coincides with the value of coupling strength where the incoherence state becomes unstable, see (Barre and Mtivier, 2016).

Since g is unimodal we have that

$$\frac{dA}{d\mu} = \int_{\mathbb{R}} \frac{1}{1 + y^2} \left[-g'\left(\frac{y}{\mu}\right) \frac{y}{\mu^2} \right] dy > 0, \quad (3.37)$$

and moreover, $\lim_{\mu \rightarrow 0} A(\mu) = 0$, and $\lim_{\mu \rightarrow \infty} A(\mu) = \pi g(0)$. Hence we have $0 < A(\mu) < \pi g(0)$ and

$$K_c > \frac{2}{\pi g(0)}$$

for $\mu > 0$. In particular, K_c increases with μ , see Fig. 3.5. In the limit $\mu \rightarrow 0$ (corresponding either to small inertia or to large damping coefficient), one obtains the critical coupling strength of Kuramoto oscillators, $K_c(0) = 2/(\pi g(0))$.

For Gaussian and for Lorentz distributions the value $K_c(\mu)$ can be explicitly computed. For the Gaussian distribution, Eq. (3.30), we find

$$K_c(\mu) = \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{\sigma}{1 - \exp\left(\frac{1}{2\mu^2\sigma^2}\right) \left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{2}\mu\sigma}\right)\right)},$$

which for small $\mu > 0$ gives

$$K_c(\mu) = \frac{2\sqrt{2}}{\sqrt{\pi}}\sigma + \frac{4}{\pi}\sigma^2\mu + O(\mu^2),$$

while for large μ it gives

$$K_c(\mu) = 2\sigma^2\mu + \sqrt{\frac{\pi}{2}}\sigma + O(\mu^{-1}).$$

For the Lorentz distribution, Eq. (3.31), we find

$$K_c(\mu) = 2\gamma + 2\gamma^2\mu.$$

3.5.3 Margin region

Recall that for symmetric unimodal distributions $g(\Omega)$ we have $F_2(q, 0) = 0$ for all $q > 0$ and any boundary function $b_P(a)$. This implies that steady states are parameterized by $q \geq 0$ through the relations $K(q) = F_1(q, 0)^{-1}$, $r(q) = qF_1(q, 0)$. Fix a value $q \geq 0$ and let $(K_L(q), r_L(q))$ and $(K_S(q), r_S(q))$ satisfy the self-consistent equations with $b_P = b_L$ and $b_P = b_S$ respectively, assuming that $r_L(q)K_L(q) = r_S(q)K_S(q) = q$, see Fig. 3.7. Then we find

$$\frac{1}{K_L(q)} = F_1(q, 0; b_L) \geq F_1(q, 0; b_S) = \frac{1}{K_S(q)}, \quad (3.38)$$

and subsequently $r_L(q) \geq r_S(q)$.

For these two points to be distinct it is necessary that $b_S(a) < b_L(a) = 1$, implying that $a < a_*(1) \approx 1.193$ or, equivalently, that

$$q > q_* \equiv \frac{1}{a_*(1)^2\mu} \approx \frac{1}{\sqrt{2}\mu}, \quad (3.39)$$

see Fig. 3.7. Therefore, we can consider the set $\mathcal{M}$ of steady states characterized by (r, K) with $K \in (K_L(q), K_S(q))$ and $r = q/K$ for $q > q_*$. We call $\mathcal{M}$ the *margin region*. Steady states in the margin region can be realized as solutions of the self-consistent equation by considering a boundary function $b_P(a)$ with $b_S(a) < b_P(a) < b_L(a)$, cf. Sec. 3.4.4.

Other than the forward and backward processes only, with different choices of initial states, the system may attain a different steady state in the margin region.

Figure 3.7: Representation of the margin region, cf. Fig. 3.5. In this picture $m = 2$, $D = 1$, and $g(\Omega)$ is Gaussian with $\sigma = 1$. The family of light gray curves represent sets of constant $q = Kr$. The margin region appears for $q > q_* \approx 0.35$ and is represented by the gray area. The set $q = q_*$ is represented by the dotted curve passing through the transitional point Q where the curves for the backward and forward process start differentiating. The two steady states marked by L and S correspond to the parameters (K_L, r_L) and (K_S, r_S) respectively, described in Sec. 3.5.3, for a value of q (here, $q = 1.75$).

Figure 3.8: The boundaries of the margin region correspond to steady state solutions of the self-consistent equation with $b_P = b_L$ and $b_P = b_S$. Points in the margin region can be realized as steady state solutions with $b_P = c b_S + (1 - c) b_L$, $0 < c < 1$. In this picture the corresponding curves for $c \in \{0.2, 0.4, 0.6, 0.8\}$ are represented by white curves inside the margin region. The parameters are $m = 3$, $D = 1$, and $g(\Omega)$ is Gaussian with $\sigma = 1$.

This is one crucial difference of second-order oscillators compared to first-order Kuramoto oscillators. When one only considers the forward and backward processes, this feature results in the well-known hysteresis of second-order oscillators, see (Tanaka et al., 1997b,a) and Fig. 3.6(b).

In Fig. 3.9, the initial states dependence of the steady states and the corresponding margin regions are shown for oscillators with Gaussian $g(\Omega)$ with $\sigma = 1$

Figure 3.9: The initial states dependence of steady states and margin region are shown with initial order parameter r_0 and order parameter r (mean: $\bullet$; maximum: ∇ ; minimum: $\triangle$) with (a) $K = 2.5$ and (b) $K = 4$. $N = 10000$ oscillators are used with $m = 2$ and $D = 1$. The boundaries of the margin region are calculated by setting $b_P = b_L$ and $b_P = b_S$ in Eq. (3.33).

for two given coupling strengths $K = 2.5$ (a) and $K = 4$ (b). The dynamics of $N = 10000$ oscillators with $m = 2$, and $D = 1$ has been calculated. The initial phases have been chosen randomly and uniformly in a connected subset of $[0, 2\pi]$ so that the corresponding initial order parameter is r_0 . The initial frequency is chosen randomly and uniformly from $[-0.5, 0.5]$ for all the cases. After a transient time period $t_0 = 200$, the order parameter r is measured as the time average over a period $\Delta t = 10$. The maximum and minimum of r is also recorded to show the variation of r . The boundary of regions of stable steady states is calculated with $b_P = b_L$ and $b_P = b_S$ in Eq. (3.33). We observe in Fig. 3.9 that states with different order parameters r_0 reach steady states with different r which either correspond the incoherence state or can be found inside the margin region.

3.6 Conclusions

In this chapter, we have considered the self-consistent method for second-order oscillators. Based on our analysis, and on the obtained self-consistent equations, we have discussed several properties of steady states. There are several important and novel points in this analysis.

First, instead of using the original parameters (m, D, K, r, Ω^r) we have introduced the rescaled parameters a and b in Eq. (3.3), thus simplifying the analysis of single oscillators but also of the network.

Second, we have given a significantly improved estimate of the limit cycle of the second-order oscillators, where the estimation proposed in (Tanaka et al., 1997b) is obtained as the lowest order of the Taylor series. Using numerical simulations,

we find that our estimation is more accurate for a much wider range of parameters compared to previously obtained estimations. Therefore, the new estimation results to more accurate self-consistent equations for second-order oscillators.

Third, using the more accurate self-consistent equations, we have performed a detailed analysis of the properties of the steady state solutions, such as the existence of multiple branches, and their dependence on the initial state. The critical transition point $K = K_c$ has also been calculated, coinciding with the stability analysis in (Barre and Métivier, 2016), obtained for symmetric and unimodal distribution $g(\Omega)$ through an unstable manifold expansion.

Finally, to better understand the dynamics and the steady states, we have introduced new concepts such as the *margin region*, Sec. 3.5.3, and the *scaled inertia* $\nu = s\mu$, Sec. 3.4.3.

The approach to self-consistent equations for second-order oscillators used in this chapter provides a framework that can be easily generalized to explore properties of steady states for more general systems, for example, with non-constant inertias and damping coefficients or with phase shifts. Moreover, combined with the development of generalized order parameters as in (Schröder et al., 2017), our approach can also pave the way to the analysis of second-order oscillators in complex networks, such as the power grids. The analysis in this chapter is from these points of view a basic building block in this research direction, as shown in the following chapters.

In addition, in this chapter we put our effort in the multi-stability and hysteresis of the oscillators introduced by inertia terms. However, there are several other ways to introduce multi-stability to Kuramoto oscillators, such as phase shift (Omel'chenko and Wolfrum, 2013) and topologies (Manik et al., 2017b). It is interesting and an open question to explore the similarity, difference and cross effect of such several factor.

Chapter 4

Oscillatory states, standing waves, and additional clusters

Other than the steady states analysed in Chapter 3, the second-order oscillators have a variety of dynamical states. The most important and interesting dynamical state is the oscillatory state. In this chapter, we try to explore the mechanism of the oscillatory state and its relationship with the effect of inertias. We discuss the appearance of oscillatory and standing wave states in second-order oscillator networks showing that it is a special case of a more general mechanism involving secondary synchronized clusters induced by inertia. Using a time-periodic mean-field ansatz, we find a bistable mechanism involving a stable fixed point and an invariant curve of an appropriate Poincaré map. The bistability and the devil's staircase associated to the rotation number on the invariant curve provide an explanation for the appearance of the secondary synchronized clusters. The effect of inertias in the self-organization process is analyzed through a simplified model. This shows that the effect of giant synchronized clusters on the other oscillators is weakened by inertias, thus leading to secondary synchronized clusters during the transition process to synchronization.

4.1 Introduction

In this chapter we consider a model of coupled second-order oscillators, where the dynamics is given by

$$m\ddot{\theta}_i + D\dot{\theta}_i = \Omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad i = 1, \dots, N. \quad (4.1)$$

Here m is the inertia and D the damping coefficient for all oscillators, N is the number of oscillators and K is the coupling strength. The natural frequencies Ω_i are randomly chosen from a distribution $g(\Omega)$. The state of the i -th oscillator is

described by its phase $\theta_i \in \mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. The collective state of the oscillators is described by the *order parameter*

$$re^{i\phi} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j},$$

where r measures the phase coherence, and ϕ represents a collective phase. If all the oscillators move in a single tight cluster we have $r \approx 1$. On the contrary, if the oscillators move incoherently, scattered around the circle, we have $r \approx 0$.

When $m = 0$, the second-order oscillators become Kuramoto oscillators. Kuramoto oscillators with a symmetric unimodal distribution $g(\Omega)$ have a continuous synchronization transition with the increase of K from $r \approx 0$ to $r \approx 1$ Kuramoto and Nishikawa (1987). In the presence of inertias, the dynamics of oscillators becomes much more complicated. In particular, with the increase of m , several new features manifest in second-order oscillators, such as hysteresis Olmi et al. (2014), change of the type of phase transitions (Tanaka et al., 1997a,b), and finally oscillatory states with periodic oscillations of the order parameter Olmi et al. (2014). Such oscillatory states are not only found in systems with unimodal distributions of Ω Tanaka et al. (1997b), but also with bimodal distributions Olmi and Torcini (2016) and in complex networks Olmi et al. (2014). With the help of the self-consistent method, the dynamics of hysteresis and discontinuous transitions have been recently analyzed in Gao and Efstathiou (2018). For oscillatory states, Olmi *et al* Olmi et al. (2014) have related the oscillation of the order parameter to the appearance of secondary synchronized clusters using numerical simulations. However, the dynamics of this oscillatory state and the appearance of additional synchronized clusters is still not well understood.

We give a short outline of the chapter. In Sec. 4.2, through numerical simulations, we show that the additional synchronized clusters as a general phenomenon of second-order oscillators. To explore the additional synchronized clusters' dynamics, in Sec. 4.3 we study oscillators with time-periodic mean-field. In Sec. 4.4 using a simplified model of oscillatory states, we provide approximate estimations of the inertias' effect to the self-organization process. We conclude the chapter in Sec. 4.5.

4.2 Additional synchronized clusters

To provide a more refined description of collective states (compared to the global description provided by the order parameter) we use the mean frequency $\langle \theta_j \rangle$ of each oscillator. Two oscillators are *synchronized* (or *frequency locked*) if they have the same mean frequency. A group of oscillators with the same value of mean frequency forms a *synchronized cluster*. States without any synchronized cluster are steady states with $r = 0$, called *incoherence states*. States with only one synchronized cluster are (*partial*) *synchronization states*. Their order parameters

have a constant modulus, $r(t) = r$ and a uniformly rotating phase $\phi = \Omega^r t + \Psi$. When all the oscillators have the same phase, we have the *complete synchronization state* with $r = 1$. If there are more than one synchronized clusters, the order parameter may have a time-dependent modulus, such as in *standing waves* and *oscillating states*.

To explore oscillatory states, we have numerically calculated the dynamics of a network with $N = 10000$ oscillators, following Eq. (4.1). The integration was done using the fourth order Runge-Kutta method with fixed-size time-step $dt = 10^{-3}$. The natural frequency Ω_i for each oscillator is chosen randomly from a distribution that is either Gaussian or a double Gaussian. To describe the upper and lower branches of hysteresis loops, we consider forward and backward processes. In the forward process the initial states of the oscillators are randomly chosen as $\theta(0) \in [0, 2\pi]$, $\dot{\theta}(0) \in [0, 1]$ (incoherence state) and then the coupling strength is gradually increased with step $dK = 0.01$. At each step, the initial states of all the oscillators are the final states in the previous step. After a transient period $t_0 = 100$, we calculate the order parameter r and mean frequencies $\langle \dot{\theta}_i \rangle$ over a measurement period $\Delta t = 10$ and then move to the next step increasing K by dK . In the backward process, the initial states of the oscillators are randomly chosen as $\theta(0) \in [0, 0.02\pi]$, $\dot{\theta}(0) \in [0, 1]$ (synchronized state) and the previously described procedure is followed with the value of K decreasing at each step by $dK = -0.01$.

In these calculations, we consider two effects: the effect of inertia, m , and the effect of the natural frequencies distribution $g(\Omega)$. In the first calculation oscillators with a Gaussian distribution

$$g_1(\Omega) = G(\Omega; 0, 1), \quad (4.2)$$

where

$$G(\Omega; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\Omega - \mu)^2}{2\sigma^2}\right),$$

are numerically explored with $m = 2$ and $m = 5$. When $m = 2$, the transition is discontinuous with a clear hysteresis, see Fig. 4.1(a). All the states are steady states, coinciding with the analytical calculation in Gao and Efstathiou (2018). For the larger $m = 5$, depicted in Fig. 4.1(b), the oscillators do not always reach a steady state and we have the appearance of oscillatory states comprising more than one synchronized clusters, as shown in Fig. 4.1(d,f). Note that the system still supports a steady state as predicted in Gao and Efstathiou (2018) but instead the oscillatory state is numerically observed. We thus conjecture that the steady state becomes unstable leading to the appearance of the oscillatory state. For both $m = 2$ and $m = 5$ there is a large synchronized cluster with average frequency 0, see Fig. 4.1(e,f). However, for $m = 5$ we observe the appearance in Fig. 4.1(f) of two additional synchronized clusters with non-zero average frequency at both sides of the main cluster. These two clusters lead to the oscillations of the modulus of the order parameter, shown in the inset in Fig. 4.1(f). In previous studies, this state

Figure 4.1: Backward and forward processes for $N = 10000$ oscillators with (left column) $m = 2$ and (right column) $m = 5$ for oscillators with Gaussian distributed natural frequencies, Eq. (4.2). The black circle (gray diamond) in panels (a,b) are the numerical results in the forward (backward) process showing the evolution of r with decreasing (increasing) K . The error bars show the minimum and maximum values of r for each K ; oscillatory states correspond to large error bars. The evolution of the mean frequency $\langle \dot{\theta} \rangle$ for the forward process (increasing K) is shown in panels (c,d). The dependence of $\langle \dot{\theta} \rangle$ on Ω is shown panels (e,f) for two typical states with different inertias and same coupling strength $K = 6$.

Figure 4.2: Backward and forward processes for $N = 1000$ oscillators with (left column) $m = 2$ and (right column) $m = 10$ for oscillators with bimodal natural frequency distribution $g_3(\Omega)$, Eq. (4.3). In panels (e,f) two typical states are shown with the same coupling strength $K = 5$. Other details are the same as in Fig. 4.1.

Figure 4.3: Backward and forward processes for $N = 1000$ oscillators with (left column) $m = 2$ and (right column) $m = 10$ for oscillators with bimodal natural frequency distribution $g_2(\Omega)$, Eq. (4.4). In panels (e,f) two typical states are shown with the same coupling strength $K = 9$. Other details are the same as in Fig. 4.1.

has been called *secondary synchronization* Tanaka et al. (1997b) or *oscillatory state* Olmi et al. (2014); Olmi and Torcini (2016). Fig. 4.1(c,d) show how oscillators abruptly join the main synchronized cluster for $m = 2$ when K increases above the transition value, but they form additional synchronized clusters for $m = 5$ even after K goes above the transition value.

In the second calculation we consider oscillators with $m = 2$ or $m = 10$ and with a bimodal Gaussian distribution of natural frequencies

$$g_3(\Omega) = \frac{1}{2} [G(\Omega; 1.5, 0.5) + G(\Omega; -1.5, 0.5)], \quad (4.3)$$

where the two modes have a small overlap, see Fig. 4.2. All the oscillators can be divided into two sub-groups with either positive or negative natural frequencies. For $m = 2$ we observe the appearance of two sub-populations, whereas the $\langle \dot{\theta} \rangle = 0$ synchronized cluster is completely missing in contrast to the unimodal case where it was the most prominent one. The collective behavior of the system resembles two oscillators rotating in opposite directions where the order parameter oscillates along a constant or slowly varying direction. This is the *standing wave* which is also observed for Kuramoto oscillators ($m = 0$) with a bimodal natural frequency distribution Martens et al. (2009); Bonilla et al. (1998). For $m = 10$, the oscillators are separated to two subgroups, with either negative or positive natural frequencies. For each of these subgroups, similar to the unimodal case, we observe the appearance in Fig. 4.2(f) of two additional synchronized clusters at both sides of its main cluster. In addition, the $\langle \dot{\theta} \rangle = 0$ synchronized cluster is also observed in contrast to the case with $m = 2$.

Finally, in the third calculation we consider oscillators with $m = 2$ or $m = 10$ and with a bimodal Gaussian distribution of natural frequencies

$$g_2(\Omega) = \frac{1}{2} [G(\Omega; 1, 0.7) + G(\Omega; -1, 0.7)], \quad (4.4)$$

where the two modes strongly overlap, see Fig. 4.3. For $m = 2$, Fig. 4.3(a,c,e), the system behaves very similarly to the unimodal case, which can be expected from the study of Kuramoto oscillators with $m = 0$. For $m = 10$, however, we observe states similar to a *standing wave* Fig. 4.3(b,d), with several clusters besides this main structure. With the effect of inertias, the intrinsic two sub-groups structure is activated, forming a pair of synchronized clusters that rotate like two giant oscillators.

These numerical results indicate that the appearance of additional synchronized clusters is a general phenomenon of second-order oscillators. In summary, the calculations indicate that increasing the inertia results in the appearance of additional synchronized clusters leading to oscillatory or standing wave states. In addition, the phenomenon of additional synchronized clusters always appears in the lower branch of hysteresis loops (along the forward process), while the states in the upper branches (along the backward process) are not affected.

4.3 Time-periodic mean-field

To understand the intrinsic synchronized clusters of the system, naturally the first step is to answer the question how synchronized clusters manifest under a given oscillatory mean-field. For Kuramoto oscillators this question has only been recently addressed in Engelbrecht and Mirollo (2012). For the second-order oscillators we consider here, the question becomes more complicated even though the idea behind our approach is similar to Engelbrecht and Mirollo (2012).

To analyze the oscillatory states, we first write the dynamics, Eq. (4.1), in mean-field form as

$$m\ddot{\theta}_i + D\dot{\theta}_i = \Omega_i + Kr(t)\sin(\Omega^r(t)t + \phi_0 - \theta_i), \quad (4.5)$$

for $i = 1, \dots, N$. In Eq. (4.5) the oscillators interact with the mean-field through the order parameter. In the particular case of oscillatory states we assume a time-dependent periodic mean-field modulus $r(t) = r_0(1 + \varepsilon f(t))$ and constant Ω^r . Here $f(t)$ is a T -periodic function with zero average and $\varepsilon \geq 0$ measures the relative size of the time-dependent term. With this assumption, the dynamics of oscillators can be written in mean-field form in a frame rotating as $\Omega^r t + \phi_0$ as

$$m\ddot{\theta} + D\dot{\theta} = \Omega - D\Omega^r - Kr_0(1 + \varepsilon f(t))\sin\theta. \quad (4.6)$$

Further defining $\omega = \dot{\theta}$ we obtain on $M = \mathbb{S} \times \mathbb{R}$ the system of first-order differential equations

$$\begin{aligned} \dot{\theta} &= \omega, \\ m\dot{\omega} &= -D\omega + (\Omega - D\Omega^r) - Kr_0(1 + \varepsilon f(t))\sin\theta. \end{aligned} \quad (4.7)$$

For a given initial state $(\theta(0), \omega(0))$, one can define the time- T Poincaré map induced by Eq. (4.7) as

$$F : M \rightarrow M : (\theta(0), \omega(0)) \mapsto (\theta(T), \omega(T)).$$

The case $\varepsilon = 0$ corresponds to a steady state with $r(t) = r_0$. In this case, Eq. (4.1) has two possible stable states Levi et al. (1978); Strogatz (2014); Guckenheimer (2013); Gao and Efstathiou (2018). Introducing the parameters $a = D/(Kr_0m)^{1/2}$ and $b = (\Omega - D\Omega^r)/(Kr_0)$, it is known that for $b \geq b_L := 1$ the only stable state is a limit cycle L where the motion has frequency $\Omega_L = \Omega/D - \Omega^r$. For $b \leq b_S(a)$ the only stable state is a fixed point $(\theta_0, 0)$. The bifurcation curve $b_S(a)$ is given by

$$b_S(a) \cong \begin{cases} 1.2732a - 0.3056a^3, & 0 \leq a \leq 1.193, \\ 1, & a \geq 1.193. \end{cases}$$

When $b_S(a) \leq b \leq b_L := 1$ the system is *bistable*—the stable fixed point and stable limit cycle co-exist. Several properties of second-order oscillators, such

Figure 4.4: Mean-frequency $\langle \dot{\theta} \rangle$ with respect to natural frequency Ω with $m = 1$ (a) and $m = 5$ (b) where $r(t) = 0.4 + 0.1 \sin(t)$, and $m = 1$ (c) and $m = 5$ (d) where $r(t) = 0.6 \sin(t)$. The other parameters read $D = 1, K = 4.5$. The gray circles are typical states independent of different inertias (steady states for $r(t) = 0.4 + 0.1 \sin(t)$ and standing wave for $r(t) = 0.6 \sin(t)$), and the black dots are the states from the bi-stability of oscillators with inertia effect. The bistable regions are colored with gray.

as the discontinuous phase transitions to synchronization and the corresponding hysteresis of steady states, are closely related to this bistability.

Consider the extended phase space $\widehat{M} = \mathbb{S} \times \mathbb{R} \times \mathbb{S}_T$ with coordinates (θ, ω, t) where t is viewed as a periodic variable in $\mathbb{S}_T := \mathbb{R}/T\mathbb{Z}$. The stable fixed point $(\theta_0, 0)$ becomes in $\widehat{M}$ a stable limit cycle $(\theta_0, 0, t)$, or equivalently a fixed point $(\theta_0, 0)$ of the Poincaré map F . Similarly, the stable limit cycle L becomes in $\widehat{M}$ a stable limit torus $L \times \mathbb{S}_T$ carrying quasi-periodic motions with frequencies $\omega_1 = \Omega_L$ and $\omega_2 = 2\pi/T$ and manifests on the Poincaré section as an invariant curve L_0 of F carrying a quasi-periodic circle map with rotation number

$$\rho_0 = \frac{\omega_1}{\omega_2} = \frac{\Omega_L T}{2\pi}.$$

Recall that the rotation number for an orbit of the Poincaré map F with initial condition (θ_0, ω_0) is

$$\text{rot}(F)(\theta_0, \omega_0) := \lim_{n \rightarrow \infty} \frac{\theta_n - \theta_0}{2\pi n},$$

where $(\theta_n, \omega_n) = F^n(\theta_0, \omega_0)$.

Both the stable limit cycle and the stable limit torus are compact normally hyperbolic invariant manifolds and thus by Fenichel's theory Fenichel (1971); Hirsch et al. (1970) we expect that for sufficiently small $\varepsilon > 0$ these structures will persist. In particular, the fixed point $(\theta_0, 0)$ of F persists as the fixed point $(\theta_\varepsilon, \omega_\varepsilon)$ while the invariant curve L_0 persists as the invariant curve L_ε . Extending terminology from the case $\varepsilon = 0$ we will refer to oscillators converging to the fixed point as *locked* and those converging to the invariant curve as *running*, cf. Gao and Efstathiou (2018).

The restriction of the Poincaré map F on the invariant curve L_ε gives rise to a circle map with a rotation number ρ_ε independent of the initial condition on the invariant curve Devaney (2003). Consider now an ensemble of oscillators characterized by different Ω while the other parameters determining the dynamics, that is, m , D , K , r_0 , Ω' , ε and the T -periodic function $f(t)$ are the same. Then the value of Ω determines whether the Poincaré map F for Eq. (4.7) has a fixed point, an invariant curve, or both. If there is only a (stable) fixed point $(\theta_\varepsilon, \omega_\varepsilon)$ then all orbits will eventually converge to it and their rotation number will be $\text{rot}(F)(\theta_0, \omega_0) = 0$. Similarly, if there is only a stable invariant curve L_ε then all orbits will have rotation number $\text{rot}(F)(\theta_0, \omega_0) = \rho_\varepsilon$. In the bistable case, where both a fixed point and an invariant curve co-exist, we will find some oscillators with rotation number 0 and some oscillators with rotation number ρ_ε depending on their initial condition, which determines if they converge to the fixed point or the invariant curve, respectively. Therefore, a plot of $\text{rot}(F)$ vs Ω for each oscillator will consist of three regions: one where all oscillators are running and have rotation number ρ_ε (which however depends on Ω and exhibits the typical devil's staircase structure), one where all oscillators are locked with rotation number 0, and the bistable region where some oscillators have rotation number 0 and some have rotation number ρ_ε .

With a larger bistable region, we have more and larger plateaus in the graph of ρ_ε vs Ω (corresponding to synchronized clusters) as shown in Fig. 4.4(a) and (b). With a larger bistable region and corresponding larger plateaus, the time-periodic mean-field can excite a larger oscillation of the order parameter of the oscillators. Taking the time-periodic mean-field as an oscillating perturbation around steady states, a sufficient large excited oscillation of the order parameter means the instability of such steady state and the formation of an oscillatory state.

The size of the bistable region depends on the value of a which in turn depends, for fixed Kr_0 , on the *reduced mass* $\mu = m/D^2$. In particular, for small μ we expect that there is no bistable region, while for sufficiently large μ , there is a bistable region whose size increases with μ . This observation explains why oscillatory states do not appear for small values of μ .

In addition, in the backward process, all the oscillators in the bistable region have rotation number 0 and they are not located on the plateaus where the rotation number is $\rho_\varepsilon \neq 0$. Therefore, in this case the appearance of bistable regions due to inertia does not contribute to the appearance of oscillatory states. This is the reason why backward processes are always similar with either large or small values of μ and do not support oscillatory states.

The inertia effect is not limited to the steady states. A special case of the previous analysis is when $r_0 = 0$ or very small and ε is large. The first condition, $r_0 = 0$ or small, implies that the order parameter oscillates, corresponding to the standing waves. The mean-field equation becomes

$$m\ddot{\theta} + D\dot{\theta} = (\Omega - D\Omega^r) - K\varepsilon f(t) \sin \theta.$$

For $\varepsilon = 0$, the system has a stable limit cycle L with frequency $\Omega_L = \Omega/D - \Omega^r$. As ε starts increasing the limit cycle persists as an invariant curve L_ε . However, for larger values of ε the dynamics on L_ε will give rise to fixed points when μ is large enough, see Fig. 4.4(c) and (d). With the increase of inertia, locked oscillators can also appear in the purely oscillating mean-field (standing wave) together with the running oscillators that also exist for small inertias. Note that this process is the opposite of the process that occurs in the case of oscillatory mean-field, see Fig. 4.4(a) and (b), where for small inertia we have only locked oscillators and with the increase of inertia we also observe the appearance of running oscillators.

These results, in particular the bi-stability of oscillators and devil's staircase structure of ρ_ε , explain why a periodic mean-field leads to the appearance of secondary synchronized clusters besides the cluster of locked oscillators. With the same time-periodic mean-field, the appearance of secondary synchronized clusters depends on the value of μ and the direction (backward and forward) of synchronization processes. Such analysis links the states of oscillators with the mean-field of the system and shows the inertia effect in these correlations. However, the mean-field of coupled oscillators is formed from the self-organization of all the oscillators. In the next section, to study the self-organization processes, we will focus on a more detailed model.

4.4 Self-organization processes

As a complement to the analysis based on the mean-field, this section is devoted to the dynamics of a few oscillators and their self-organization toward synchronization. Since all the oscillators are connected with each other, there are no topological differences and the only factor that affects their synchronization process is their natural frequencies' distribution $g(\Omega)$. To mimic the effect of $g(\Omega)$, we introduce a weighted model of finitely many oscillators where the dynamics is given for $i = 1, \dots, N$ by

$$m\ddot{\theta}_i + D\dot{\theta}_i = \Omega_i + \frac{K}{N} \sum_{j=1}^N a_j \sin(\theta_j - \theta_i). \quad (4.8)$$

Figure 4.5: (a) Weights for oscillators with different natural frequency Ω as $a_1 = 10, a_2 = 0.8, a_3 = 0.5$. (b) The factor C and $F_1/F_2 \equiv F(a_{12})/F(a_{23})$ with different rescaled inertia μ . (c) and (d) are the synchronization processes with increasing coupling K with $m = 0.2$ (c) and $m = 2.5$ (d). The damping coefficient is $D = 1$.

Here a_i is the weight, describing the fraction of oscillators with natural frequency Ω_i , and is used to mimic the distribution of oscillators.

To obtain more theoretical insights about the inertia effect, we begin by considering three coupled oscillators with $a_1 \gg a_2 > a_3$ and $\Omega_3 > \Omega_2 \gg \Omega_1$. Oscillator 1 is assigned the largest weight, describing the giant synchronized group, and oscillators 2 and 3 are assigned smaller weights but closer natural frequency, describing two small groups away from the giant group, see Fig. 4.5(a). Introducing the phase differences $\varphi_1 = \theta_1 - \theta_2$, $\varphi_2 = \theta_2 - \theta_3$, we rewrite the dynamics as

$$\begin{aligned}
 m\ddot{\varphi}_1 + D\dot{\varphi}_1 &= (\Omega_1 - \Omega_2) - \frac{K}{2} \left[(a_2 + a_1) \sin(\varphi_1) \right. \\
 &\quad \left. - a_3 \sin(\varphi_2) + a_3 \sin(\varphi_1 + \varphi_2) \right], \\
 m\ddot{\varphi}_2 + D\dot{\varphi}_2 &= (\Omega_2 - \Omega_3) - \frac{K}{2} \left[(a_3 + a_2) \sin(\varphi_2) \right. \\
 &\quad \left. - a_1 \sin(\varphi_1) + a_1 \sin(\varphi_1 + \varphi_2) \right].
 \end{aligned} \tag{4.9a}$$

Even though the system Eq. (4.9a) has various dynamical properties, we are only interested in the synchronization of each pair of oscillators, which means $\dot{\varphi}_1 \approx 0$ or $\dot{\varphi}_2 \approx 0$. The effect of the other oscillator is approximated as a periodic forcing by $\varphi_1 \approx \omega_1 t \equiv (\Omega_1 - \Omega_2)/Dt$ when $\dot{\varphi}_2 \approx 0$ and $\varphi_1 \approx \omega_2 t \equiv (\Omega_2 - \Omega_3)/Dt$ when $\dot{\varphi}_1 \approx 0$.

For the synchronization of the oscillators θ_1 and θ_2 , we have

$$m\ddot{\varphi}_1 + D\dot{\varphi}_1 = (\Omega_1 - \Omega_2) - \frac{K}{2} \left[(a_2 + a_1) \sin(\varphi_1) - a_3 \sin(\omega_2 t) + a_3 \sin(\varphi_1 + \omega_2 t) \right],$$

From the fact that a_1 is the largest weight and $(a_2 + a_1) \gg a_3$, we can ignore the periodic perturbations from φ_2 and obtain

$$m\ddot{\varphi}_1 + D\dot{\varphi}_1 = (\Omega_1 - \Omega_2) - \frac{K}{2} (a_2 + a_1) \sin(\varphi_1). \quad (4.10)$$

As for the synchronization of the oscillators θ_2 and θ_3 , we have

$$m\ddot{\varphi}_2 + D\dot{\varphi}_2 = (\Omega_2 - \Omega_3) - \frac{K}{2} \left[(a_3 + a_2) \sin(\varphi_2) - a_1 \sin(\omega_1 t) + a_1 \sin(\varphi_2 + \omega_1 t) \right].$$

From the fact that $\omega_2 \gg \omega_1$, we can average the fast periodic perturbation from φ_1 over time and obtain the dynamics of φ_2 as

$$m\ddot{\varphi}_2 + D\dot{\varphi}_2 = (\Omega_2 - \Omega_3) - \frac{K}{2} (a_3 + a_2) \sin(\varphi_2). \quad (4.11)$$

Both the dynamics Eq. (4.10) and Eq. (4.11) are the same as the dynamics of a single second-order oscillator Eq. (4.6) with $\varepsilon = 0$, studied in detail in Gao and Efstathiou (2018). Hence the synchronization of each pair of oscillators, $\dot{\varphi}_1 = 0$ and $\dot{\varphi}_2 = 0$, can be obtained respectively as,

$$\frac{2(\Omega_1 - \Omega_2)}{K(a_1 + a_2) b\left(\frac{\sqrt{2}}{\sqrt{K\mu(a_1 + a_2)}}\right)} \equiv \frac{\Delta\Omega_{12}}{F(a_{12})} \leq 1, \quad (4.12)$$

$$\frac{2(\Omega_2 - \Omega_3)}{K(a_3 + a_2) b\left(\frac{\sqrt{2}}{\sqrt{K\mu(a_3 + a_2)}}\right)} \equiv \frac{\Delta\Omega_{23}}{F(a_{23})} \leq 1, \quad (4.13)$$

where $F(a) = \frac{1}{2}Ka b\left(\frac{\sqrt{2}}{\sqrt{K\mu a}}\right)$ is a function of the weight a . The boundary function $b(x)$ equals either $b_S(x)$ or $b_L(x) \equiv 1$ in the forward and backward processes respectively. The frequency differences and sums of weights read $\Delta\Omega_{12} = \Omega_1 - \Omega_2$, $a_{12} = a_1 + a_2$ and $\Delta\Omega_{23} = \Omega_2 - \Omega_3$, $a_{23} = a_2 + a_3$. Since in the backward process the function $b_L(x) \equiv 1$ is constant, the transition process is independent of μ . However, in the forward process, with nonlinear boundary function $b_S(x)$, the synchronization processes depends on the value of μ crucially.

To compare these two synchronization conditions, we define the factor

$$C = \frac{\Delta\Omega_{12}}{\Delta\Omega_{23}} \frac{F(a_{23})}{F(a_{12})}. \quad (4.14)$$

If $C < 1$ the dominant synchronization process is the growth of the giant group from oscillator 1, while if $C > 1$ the additional synchronized cluster will form between oscillators 2 and oscillator 3.

The value of C and hence the synchronization processes depends on the value of oscillators' rescaled inertia μ , see Fig. 4.5(b). When μ is small, we have that $F(a) = Ka/2$, and hence $F(a_{23})/F(a_{12}) = a_{23}/a_{12}$. On the other hand, when μ is sufficient large the function F can be approximated as $F(a) \approx \sqrt{8K/\mu\pi^2}\sqrt{a}$ and correspondingly $F(a_{23})/F(a_{12}) \approx \sqrt{a_{23}/a_{12}}$. Even though the ratio $\Delta\Omega_{12}/\Delta\Omega_{23}$ is fixed, the ratio $F(a_{23})/F(a_{12})$ increases monotonically with the increase of μ , from a_{23}/a_{12} to $\sqrt{a_{23}/a_{12}}$, as shown in Fig. 4.5(b). The effect of weights is weakened by the increase of inertias along the lower boundary b_S . Therefore, for larger inertias the oscillators are more likely to synchronize among ones with closer natural frequencies than with the giant group with larger weights. Consequently, we observe the appearance of additional synchronized clusters in the forward processes with sufficient large inertias as shown in Fig. 4.5(d). As a generalization of these three-coupled oscillators, one can also consider three groups of oscillators as a limiting case of multimodal frequency distributions. This is beyond the scope of this chapter and we refer to Acebrón and Spigler (1998).

Following the analysis above for three oscillators, we further consider a larger system, where the synchronization process is a more complicated self-organization progress. Considering coupled $2N$ oscillators, with the dynamics Eq. (4.8) and uniformly spaced frequency Ω_i and different weights a_i , we could find different synchronization processes and states, see Fig. 4.6.

Firstly, for a system with a symmetric and unimodal natural frequency distribution, the weights of the central pair of oscillators θ_N and θ_{N+1} have the largest weights. From the fact that all nearest pairs of oscillators have the same natural frequencies difference $\Delta\Omega$, the central pair with the largest weights will synchronize first in the process of increasing coupling strength K from Eq. (4.14). Once they are synchronized, they form a synchronized cluster or equivalently an effective oscillator θ_0 with the frequency $\Omega_0 \equiv (\Omega_N + \Omega_{N+1})/2$ with weights $a_0 \equiv (a_N + a_{N+1})$.

With the further increase of coupling strength, for all the oscillators with θ_i with $i > N$ the next synchronization phenomenon will happen either between oscillators θ_{N+2} with θ_{N+3} or between the oscillator θ_{N+2} with the synchronized group θ_0 , determined by the condition Eq. (4.14) as

$$C = \frac{3}{2} \frac{F(a_{N+2} + a_{N+3})}{F(a_N + a_{N+1} + a_{N+2})}, \quad (4.15)$$

where $3/2$ is the ratio of their natural frequency differences.

Figure 4.6: Mean-frequency of $N = 12$ oscillators with increasing coupling strength K for unimodal distributed weights (a) ($\mu = 1$) and (b) ($\mu = 6$), large overlapped bimodal distributed weights (c) ($\mu = 0.2$) and (d) ($\mu = 6$), and small overlapped bimodal distributed weights (e) ($\mu = 0.2$) and (f) ($\mu = 6$). The natural frequencies are uniformly chosen as $\Omega_i = -5.5, -4.5, \dots, 5.5$ with weights $a_i = 0.2, 0.3, 0.5, 0.9, 1.2, 1.9, 1.9, 1.2, 0.9, 0.5, 0.3, 0.2$ in (a) and (b), $a_i = 0.3, 0.4, 0.5, 1.1, 1.4, 0.9, 0.9, 1.4, 1.1, 0.5, 0.4, 0.3$ in (c) and (d), $a_i = 0.5, 0.6, 0.9, 1.4, 0.9, 0.2, 0.2, 0.9, 1.4, 0.9, 0.6, 0.5$ in (e) and (f).

When μ is sufficient small, we have that $F(a)/F(b) = a/b$. From the unimodal property, we have $(a_{N+2} + a_{N+3})/(a_N + a_{N+1} + a_{N+2}) < 2/3$ hence $C > 1$ in Eq. (4.15). The synchronized group θ_0 grows bigger and includes θ_{N+2} . The same process also takes place for the oscillators θ_i with $i < N$ due to the symmetry. After such step of synchronization, the central synchronized group includes four oscillators θ_i with $i = N - 1, \dots, N + 2$. With the increase of K further, the next synchronization condition factor for the oscillators θ_{N+3} reads

$$C = \frac{5}{2} \frac{F(a_{N+3} + a_{N+4})}{F(a_{N-1} + a_N + a_{N+1} + a_{N+2} + a_{N+3})}, \quad (4.16)$$

With similar analysis and the unimodal property we assumed, we find that the oscillator θ_{N+3} will synchronize with the group θ_0 when μ is sufficient small. The central synchronized group gets larger. Following this process, with the increase of coupling strength, it is straightforward to show that all the oscillators will be included in this group one by one. This follows directly from the unimodal property and also the linear dependence of F on the weight, as shown in Fig. 4.6(a).

On the contrary, when inertia sufficiently increases, such linear dependence of F will be weakened to square root and the self-organization chain will be broken at some points θ_{N+n} . Instead of contributing to the growth of the central synchronized group θ_0 , a new cluster will form from the synchronization of θ_{N+n} and θ_{N+n+1} . Then the new cluster will grow larger with the increase of K to another point where the chain is broken again due to the non-linearity of $F(a)$ and a third synchronized group forms. Continuing this argument we obtain the multi-cluster devil's staircase structure, see Fig. 4.6(b). In this way, additional synchronized clusters are formed apart from the central one, forming the oscillatory state as in Fig. 4.1(f).

Secondly, in the bimodal case with sufficient large distance between the two peaks, the system can be approximated as two independent unimodal systems when the coupling strength is small enough. In this case two synchronized clusters will form and grow initially from oscillators with the largest weights, corresponding to the two peaks of the bimodal distribution. The self-organizing takes place independently for each cluster. For small inertias we have continuous growth from the two peaks of the distribution forming a standing wave, see Fig. 4.6(e). For large inertias we have the appearance of several small clusters, see Fig. 4.6(f). When the coupling strength K is large enough, these two branches of synchronization processes will merge to one, by creating one large central cluster.

Thirdly, for the case where the two peaks of the bimodal distributions have a large overlap, the synchronization process is more complicated. The two oscillators at each peak are close to each other and the oscillators between them also have relatively large weights. In this case, we need to consider about higher order terms in the synchronization processes, i.e. the synchronization condition of several oscillators.

Considering the case where θ_{N-1} and θ_{N+2} are the two peaks oscillators with the maximum weights, and the other two oscillators θ_N, θ_{N+1} between them have

slightly smaller weights. The condition of four-oscillator synchronization group $\theta_{N-1}, \dots, \theta_{N+2}$ can be estimated by the synchronization condition of the oscillator θ_{N+2} with the assumed synchronized cluster of $\theta_{N-1}, \theta_N, \theta_{N+1}$. Then the condition factor comparing the appearance of four-oscillator group and the synchronized central group between θ_{N+2} and θ_{N+3} reads

$$C = \frac{3}{2} \frac{F(a_{N+2} + a_{N+3})}{F(a_{N-1} + a_N + a_{N+1} + a_{N+2})}. \quad (4.17)$$

From the fact that $a_{N-1} = a_{N+2} > a_{N+3}$ as the maximum weight, if the two central oscillators have relatively large weights $a_N + a_{N+1} > a_{N+3}$, we have $C < 1$ and observe the abrupt appearance of four-oscillator synchronization group when μ is small, as shown in Fig. 4.6(c). On the contrary, if such weight is weakened by the inertia effect, the synchronization process will start from the appearance of two clusters of $\theta_{N-2}, \theta_{N-1}$ and $\theta_{N+2}, \theta_{N+3}$, forming standing wave states as the one for bimodal cases with smaller overlaps, see Fig. 4.6(d).

4.5 Conclusion

Based on the theoretical analysis in Sec. 4.3 and the simplified models in Sec. 4.4 we conclude that the main effect of inertias is the weakening of the synchronization influence of giant synchronized clusters on the other oscillators, when the system is in the lower branch of hysteresis loops. As a result, additional synchronized clusters appear besides the giant clusters when μ is sufficiently large, thus leading to the appearance of oscillatory states or standing waves.

Chapter 5

Cascading self-organizing synchronization

In Chapter 4, we analyse the oscillatory states, or namely the additional clusters, both from the macro view of mean-field and the micro view of coupled several oscillators. This chapter goes a step further, focusing on the clusters' formation in the time scale. We find the chain of clusters' formation. After the synchronization of some oscillators, they form a cluster with the same mean frequencies. The effect of such synchronized oscillators on the other ones depends on their synchronized frequencies not their natural frequencies. Hence, we have a new distribution of frequencies to determine the dynamics of the other oscillators. The system changed by the formation of clusters and new clusters could form from this the changed systems. Then the system changes again, and new clusters may also appear again. We call it the cascading formation of clusters. This phenomenon is natural but has not been found and analysed before. In this chapter, to analyze such process the self-consistent method developed in Chapter 2 and Chapter 3 are combined and generalized to include the changes of frequencies, which coincides well with numerical simulations consequently.

5.1 Introduction

The second-order oscillator is the Kuramoto oscillator with the consideration of inertias. The effect of inertias can be measured by $\mu = m/D^2$ (Gao and Efstathiou, 2018), where m is the inertia of oscillators and D the damping coefficient of the oscillators' rotation. When $\mu \rightarrow 0$ the dynamics of second-order oscillators converge to the one of Kuramoto model. On the contrary, when μ is large enough, several phenomena are found different from the Kuramoto model, such as the abrupt transitions and hysteresis (Tanaka et al., 1997a; Gao and Efstathiou, 2018), explosive cluster synchronization (Ji et al., 2013), and the oscillatory state and additional clusters (Olm et al., 2014; Olmi and Torcini, 2016).

In the Chapter 4, we have analyzed the oscillatory states and additional clusters from the view of circle maps and show that it is closely related to the bi-stability of oscillators. But we do not find the way to obtain such clusters. The self-consistent method only describe the formation of the central cluster. However, in Chapter 2 we have shown that the self-consistent method for the Kuramoto model can also been applied to multi-cluster states. The formation of different clusters is found approximately independently before overlapping. Hence a natural question is that why the method we developed for multi-clusters can not capture the clusters in the oscillatory state.

The key point is that the clusters we considered in Chapter 2 are all independent clusters, while the ones we find in Chapter 4 is not. From numerical simulations, we find that the partial synchronization state with only the major cluster is metastable. After a long-lasting time, two smaller synchronized clusters form and result in the oscillatory states. The formation of clusters follows a cascading process. Only after the major one, other clusters can form.

In this chapter, we generalize the self-consistent method to explore this cascading formation of clusters. We show that even in the Kuramoto model, such cascading formation also exists but is hidden by Kuramoto oscillators' special synchronization condition. With the effect of inertia, the synchronization condition changes dramatically, and consequently, we encounter this new phenomenon. For the oscillatory states with two steps, our theory coincides nicely with the numerical simulation.

We give a short outline of the chapter. In Sec.5.2, we show the cascading formation of synchronized clusters with numerical simulations. In Sec.5.3, the self-consistent method is developed for this process. With this new approach, we analyze the cascading formation and compare the theoretical result with numerical simulations. The relation of this cascading formation with the effect of inertias is discussed in Sec.5.4.

5.2 Chains of state transitions

The second-order oscillators are the Kuramoto oscillators with inertia, whose dynamics read

$$m\ddot{\theta} + D\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i), \quad (5.1)$$

where N is the number of oscillators, K the coupling strength, ω_i the natural frequency. The inertia m and damping coefficient D are assumed the same for all the oscillators. Each oscillator is described by its phase θ_i . Through the coupling function $\sin(\theta_j - \theta_i)$ all the oscillators are coupled with each other.

To describe the synchronization level of oscillators, one can define the complex

order parameter $Z(t)$ as

$$r(t)e^{i\Psi(t)} = Z(t) \equiv \frac{1}{N} \sum_{j=1}^N e^{i\theta_j}, \quad (5.2)$$

where $r(t)$ is the amplitude and $\Psi(t)$ is the global phase. If all the oscillators share the same phase, one gets $r(t) \equiv 1$, called the complete synchronization state. On the contrary, when the phases are scattered uniformly around the circle, we get $r(t) \equiv 0$, called the incoherence state. Between these two states, one gets various states depending on the pattern of $r(t)$ and $\Psi(t)$. In general, with larger amplitude r , one gets a higher level of synchronization of oscillators.

In this chapter, we assume the natural frequencies of oscillators follow unimodal and symmetric distributions. Without loss of generality, we set $D = 1$. Consider $N = 10000$ oscillators with randomly chosen natural frequencies ω from a Gaussian distribution

$$g(\omega) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}}. \quad (5.3)$$

Here, we take the standard Gaussian distribution $\omega_0 = 0$ and $\sigma = 1$.

According to (Gao and Efstathiou, 2018), the transition to and from the incoherence state is abrupt and with hysteresis in general. There is a margin region filled with possible steady states of the system. In the forward or backward process with increasing or decreasing the coupling strength K progressively, one gets the process following the lower or upper boundary of this margin region. In other words, given the same K , the system converges to different steady states in the margin region from different initial states. The appearance of additional clusters, or oscillatory states, is only observed in the forward processes around the lower boundary of the margin region when m is large enough.

In this chapter, we focus on the formation of the oscillatory states. Hence, we take $m = 5$ for all the oscillators. The initial states of all the oscillators are chosen randomly from $\theta_i(0) \in [0, 2\pi]$ and $\dot{\theta}_i(0) \in [-1, 1]$ to ensure that the system will be captured by the lower branch of the margin region of steady states. We take the coupling strength $K = 6$ and use the variable step-size Runge-Kutta method with the maximum step $dt = 1$ for numerical simulation.

The numerical results are shown in Fig. 5.1. The system starts from the initial incoherence state with $r(0) = 0$. After a short transient period, the system converges rapidly to the partial synchronization state around $t = 200$. The oscillators stay in the partial synchronization state with $r(t) \approx 0.69$ for about 400 time units, before the order parameter beginning to oscillate. After $t = 600$, the oscillation of $r(t)$ grows bigger and finally to the maximum around $t = 1500$. This final state is the oscillatory state we expected. The order parameter $r(t)$ shows a periodic oscillation, with three major synchronization clusters, one in the middle as the largest and two smaller ones on each side. These three clusters separate the oscillators into five groups depending on their natural frequencies as $\omega \in [-\Omega_0, \Omega_0]$ for

the central one, $\omega \in (\Omega_0, \Omega_1]$ and $\omega \in (-\Omega_1, -\Omega_0]$ for the two smaller ones, and finally $\omega \in (\Omega_1, \infty)$ and $\omega \in (-\infty, -\Omega_1)$ for the other oscillators.

To show more details of the formation of this oscillatory state, we generalize the order parameter definition to a group of oscillators $i \in I$ as

$$r_I(t)e^{i\Psi_I(t)} = Z_I(t) \equiv \frac{1}{n} \sum_{j=1}^N e^{i\theta_j} \Theta_I(j) \quad (5.4)$$

where n is the size of I , $\Theta_I(j) = 1$ if $j \in I$ and otherwise zero. Five groups of oscillators are defined as the separation at $t = 1500$. Correspondingly, five order parameters are calculated for each group, respectively. As shown in Fig. 5.1, in the process from $t = 0$ to $t = 2000$, the central group forms from $t = 100$ to $t = 200$, while the other two smaller groups begin to form from $t = 600$. As for the other oscillators, the synchronization among them stays at a low level at $r_I \approx 0.1$.

Combining these five groups, it is clear that for the global state, one gets the incoherence state with $r \approx 0$ before $t = 100$. After $t = 100$, the central cluster begins to form, and finally, at $t = 200$ one gets the synchronization state with an approximate constant r . The system stays in this states for about 400 units. And then the other two smaller clusters begin to form. This formation takes around 500 time units. In the period from $t = 0$ to $t = 100$, and $t = 200$ to $t = 600$, the system is in the incoherence state and partial synchronization state. They are both metastable states. With 1000 trails of the random initial states, one gets a different length of such period, which follows a distribution, as shown in Fig. 5.2, following a power law as $P(\tau) \sim \tau^{-\alpha}$. In all the realizations, these two metastable states always appear in order.

From the numerical simulation, we see that the transition from the initial incoherence to the oscillatory state is a combination of two steps of the synchronization clusters' formation, the central one as the first step, the others as the second step. The two smaller clusters only appear after the formation of the central one. The only approach from initial incoherence states to oscillating states is through the metastable partial synchronization state. The formation of the clusters follows a chain or a cascading rule in this system.

5.3 Two-step self-consistent method

Different from the multi-cluster Kuramoto oscillators, where the clusters form simultaneously, the three clusters form in a order, as shown in Fig. 5.1(b). Correspondingly, the classical approach of the self-consistent method does not give any clues of these clusters. However, it is proposed to be a good method to analyze the multi-cluster states of Kuramoto oscillators.

In this section, we generalize the self-consistent method to a two-step method to analyse this process. First of all, following the classic self-consistent method,

Figure 5.1: Transition process and final state of $N = 10000$ oscillators. The global order parameter $r(t)$ Eq. (5.2) is shown in (a), and five group order parameters $r_I(t)$ Eq. (5.4) are shown in (b), denoted as $r_1, r_2, \dots, r_5$. These five groups are separated by the oscillators' natural frequencies as $\Omega \in (-1.16, 1.16)$ for r_1 , $\Omega \in (1.16, 1.55)$ and $\Omega \in (-1.55, -1.16)$ for r_2 and r_3 , and $\Omega > 1.55$ and $\Omega < -1.55$ for r_4 and r_5 . The final state at $t = 2000$ is shown in (c) with mean frequencies $\bar{\theta}_i$ of all the oscillators. Three stairs are for the three synchronization clusters.

substitution of Eq. (5.2) into the dynamics Eq. (5.1) yields

$$m\ddot{\theta} + D\dot{\theta}_i = \omega_i + \frac{K}{2i}(Z(t)e^{-i\theta_i} - Z^*(t)e^{i\theta_i}), \quad (5.5)$$

Based on the new frame of self-consistent method proposed recently, for any independently formed clusters, we can assume $Z(t)$ with a constant amplitude $r(t) = r$ and a constant rotation frequency $\Psi(t) = \bar{\omega}t + \psi(0)$. From the previous studies, one can use the self-consistent method to study the steady state of the system. The oscillators can be separated into two groups, namely locked oscillators and drifting ones. The locked oscillators are synchronized with such mean-field $Z(t)$ with $D\dot{\theta}_i = \bar{\omega}$ while the drifting ones not. In the rotating frame with the mean-field $Z(t)$, the dynamics of the locked and drifting oscillators are solved analytically as one fixed point and one limit cycle. Combining the these two solutions, and substitute it into the definition of mean-field, we have the self-consistent equations as

$$r = \int_{\mathbb{R}} g(\omega) \left[\rho_l \sqrt{1 - b^2} - \rho_r \frac{a^2}{2(a^4 + b^2)} \right] d\omega, \quad (5.6a)$$

$$0 = \int_{\mathbb{R}} g(\omega) \left[\rho_l b + \rho_r \frac{a^4}{2b(a^4 + b^2)} \right] d\omega. \quad (5.6b)$$

where $a = D/\sqrt{Krm}$, $b = (\omega - D\Omega_0)/Kr$, and $\rho_l(\omega, \Omega)$ and $\rho_r(\omega, \Omega)$ describe the fraction of locked and running oscillators respectively, with the normalization condition $\rho_l + \rho_r = 1$. The fractions have a clear boundary

$$\mathbf{1}_S(\omega, \Omega) \leq \rho_l(\omega, \Omega) \leq \mathbf{1}_L(\omega, \Omega). \quad (5.7)$$

Figure 5.2: Distribution of lasting time for incoherence state (a) and partial synchronization state (b) for 1000 realizations of random initial states chosen from $\theta_i \in [0, 2\pi]$ and $\dot{\theta}_i(0) \in [-1, 1]$. The global order parameter $r(t)$ used to determine the incoherence state with $0 < r(t) < 0.1$ and partial synchronization state $0.67 < r(t) < 0.72$. The lasting time is the time period in which the system stays in one state. The inset figures are shown the distributions with double logarithmic coordinates with fitting lines for part of the tails. The slope of these fitting lines read -3.79 (a) and -3.95 (b).

The indicator functions $\mathbf{1}_{S,L}$ defines the limit cases, taking the value 1 if $|b| < b_S(a)$ or $|b| < b_L(a)$, corresponding to the condition for locked oscillators, and 0 otherwise. From the dynamics of second-order oscillators, we have $b_L \equiv 1$ and

$$b \geq b_S = \begin{cases} \frac{4}{\pi}a - 0.305a^3, & a \leq 1.193, \\ 1, & a > 1.193. \end{cases} \quad (5.8)$$

All the steady states with (r, Ω^r) are the possible states of the system if they satisfy the self-consistent equations. From Eq. (5.6), the incoherence $r = 0$ keeps stable in a region $K < K_c$. Other than this solution, it has been proved that the only solution of Ω^r is $\Omega^r = 0$, which is symmetric and around the center $\omega = 0$. A more detailed analysis shows two steady states in the region of $K \in (k_0, K_c)$. One is the stable partial synchronization state. The other is unstable, working as the separation of the stable partial synchronization state and metastable incoherence state. This result corresponds with numerical simulations we observed, as the transition from incoherence state to the partial synchronization state.

Note that before the second transition to the oscillating state, the numerical simulation coincides nicely with the self-consistent method. One can obtain the global order parameter $r = 0.69$ from the self-consistent method. The self-consistent method does not give any hints about the existence of another two smaller clusters.

As a matter of fact, after the formation of the synchronization cluster, the locked oscillators follow the solution $\theta_j(t) \approx \Omega_0 t + f(\omega_j)$. The order parameter $Z_I(t)$ for these oscillators can be obtained from Eq. (5.4). Substituting of $\theta_j(t)$ of the locked oscillators and the corresponding order parameter $Z_I(t) \equiv r_I \exp(i\Omega_0 t + \Psi_0(0))$ into the dynamics of the other oscillators, we have

$$m\ddot{\theta} + D\dot{\theta}_i = \omega_i + K \left[\frac{n}{N} r_I \sin(\Omega_0 t + \Psi_I(0) - \theta_i) + \frac{N-n}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i)(1 - \Theta_I(j)) \right], \quad (5.9)$$

where n is the number of locked oscillators. Without any approximation, Eq. (5.9) can be rewritten as

$$m\ddot{\theta} + D\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N [\sin(\theta_j - \theta_i)(1 - \Theta_I(j)) + r_I \sin(\Omega_0 t - \theta_i + \Psi_I(0))\Theta_I(j)]. \quad (5.10)$$

Hence we can assume that all of them contribute equally to all the other oscillators with the additional weight r_I . In addition, from the fact that once the synchronized cluster is formed, the locked oscillators in this cluster are affected little by the formation of the other clusters. We can further approximate all the locked oscillators as the identical ones with the same natural frequency as $\omega_j = \Omega_0$. In this way, Eq. (5.10) can be approximated as

$$m\ddot{\theta} + D\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N \eta_j \sin(\theta_j - \theta_i). \quad (5.11)$$

where the additional weight η_j reads $\eta_j = r_I$ for $j \in I$ and otherwise $\eta_j = 1$. Hence, with a group of oscillators synchronized, through Eq. (5.10) we redefine a system with a new distribution of natural frequencies $g(\omega)'$,

$$g(\omega)' = C [g(\omega)(1 - \Theta_I(\omega)) + r_I \delta(\omega - \Omega_0)] \quad (5.12)$$

where C is the normalization constant.

Replace $g(\omega)$ by $g(\omega)'$. The synchronization clusters can be calculated with the self-consistent method as the second-round. Note that in the second round, we only focus on the oscillators that are not synchronized in the first round. Recall that in the first round with $g(\omega)$, we obtain two solutions. One is stable corresponding to the central cluster, and the other is a saddle to separation the formation of the cluster with incoherence. In the second round, after taking the transformation to $g(\omega)'$, we obtain two more stable solutions together with two saddle points, as shown in Fig. 5.3. These two stable solutions correspond to two more clusters, and the related saddle points also work as the separation of their formation. The theoretical result of these three clusters are shown in Fig. 5.3 and compared with numerical simulations. The theoretical result coincide well with the numerical ones.

Figure 5.3: Stable and unstable fixed solutions of the self-consistent method (a). The red and blue points are from the first and second round of the self-consistent method. The mean frequency of oscillators at $t = 2000$ versus their natural frequencies is shown in (b). The analytical results are shown with vertical lines and colored blocks. The red is for the cluster one and the blue is for the cluster two and three in (a).

5.4 Effect of inertias

The formation of synchronization clusters and the transformation of $g(\omega)$ exist not only for second-order oscillators but also for the Kuramoto ones. It is interesting to ask why there is no additional cluster in the Kuramoto model. After the transformation, the distribution $g'(\omega)$ is trimodal. From this point of view, the three clusters corresponding to these three peaks are straightforward to be understood. This section will show that the potential additional cluster always exists, but it only comes to reality when the inertia effect is considered for oscillators.

After the formation of the central synchronization cluster, the distribution $g(\omega)$ is transformed to the one

$$g'(\omega) = C[g(\omega)(1 - \Theta_I(\omega)) + r_I\delta(\omega - \Omega_0)]. \quad (5.13)$$

The function $\Theta_I(\omega)$ defines the synchronized cluster around Ω_0 . For the symmetric Gaussian distribution, we have $\Omega_0 = 0$ as the unique solution. We can approximate the order parameter as $r_I \approx 1$ for the synchronized cluster, from which we have

$$g'(\omega) = \begin{cases} \delta(\omega) \int_{-\omega^*}^{\omega^*} g(\omega) d\omega, & |\omega| \leq \omega^* \\ g(\omega), & |\omega| > \omega^*. \end{cases} \quad (5.14)$$

The cluster boundary ω^* ($-\omega^*$) is the largest (smallest) oscillator's natural frequency in the central cluster. The value of ω^* can be obtained from the self-consistent method, depending on the type of oscillators.

Substituting ω^* and the distribution $g'(\omega)$ into the self-consistent equations Eq. (5.6), in the limit $q \rightarrow 0^+$, one gets the solution of Ω_c^r as

$$0 = \lim_{q \rightarrow 0^+} \int_q^\infty \frac{g'(x + \Omega_c^r) - g'(-x + \Omega_c^r)}{2x} \frac{1}{1 + m^2 x^2} dx. \quad (5.15)$$

From the symmetry of $g'(\omega)$, the solutions always come in pairs, with one negative and one positive. Without loss of generality, we focus on the positive one and assume it as $\Omega_c^r = \omega^* + \delta$. If $\delta < 0$, it is easy to show that all the oscillators are running in the limit $q \rightarrow 0^+$. As a result, for the formation of clusters, we have $\delta > 0$. Substitution of $\Omega_c^r = \omega^* + \delta$ into Eq. (5.15) yields

$$\begin{aligned} F(\omega^*) = & \lim_{q \rightarrow 0^+} \int_q^\delta \frac{g(x + \omega^* + \delta) - g(-x + \omega^* + \delta)}{2x} \frac{1}{1 + m^2 x^2} dx \\ & + \int_\delta^{2\omega^* + \delta} \frac{g(x + \omega^* + \delta)}{2x} \frac{1}{1 + m^2 x^2} dx \\ & - \frac{1}{2(\omega^* + \delta)(1 + m^2(\omega^* + \delta)^2)} \\ & + \int_{2\omega^* + \delta}^\infty \frac{g(x + \omega^* + \delta) - g(-x + \omega^* + \delta)}{2x} \frac{1}{1 + m^2 x^2} dx. \end{aligned} \quad (5.16)$$

In the limit $\delta \rightarrow 0^+$, from the second part of the integral, we have $F(\omega^*) \rightarrow \infty > 0$. On the contrary, in the limit $\delta \rightarrow \infty$, we have $F(\omega^*) < 0$ from the condition $g(x + \omega^* + \delta) - g(-x + \omega^* + \delta) < 0$. From the continuity of $F(\omega^*)$, there is at least one solution Ω_c^r in the region (ω^*, ∞) in the limit $q \rightarrow 0$. The critical coupling strength K_c of such solution reads

$$\frac{1}{K_c} = \frac{\pi}{2} g(\Omega_c^r) - \int_{\mathbb{R}} g(x + \Omega_c^r) \frac{m}{2(1 + m^2 x^2)} dx. \quad (5.17)$$

In the limit $m \rightarrow 0$, from the expansion of r around K_c , one gets the continuous transitions from the convex condition. Hence, the minimum coupling strength for the cluster is exactly the coupling strength K_c as

$$K_c = \frac{2}{\pi g(\Omega_c^r)} > \frac{2}{\pi g(\omega^*)}. \quad (5.18)$$

As for the central cluster, from the self-consistent equation Eq. (5.6) in the limit $m \rightarrow 0$ we have

$$\frac{1}{K_0} = \frac{1}{q} \int_{-q}^q g(\omega) \sqrt{1 - \frac{\omega^2}{q^2}} d\omega = \int_{-1}^1 g(q\eta) \sqrt{1 - \eta^2} d\eta, \quad (5.19)$$

From the condition that $g(q\eta) \geq g(q)$ for $-1 \leq \eta \leq 1$ and $t = \pm 1$, we have

$$\frac{1}{K_0} \geq g(q) \int_{-1}^1 \sqrt{1 - \eta^2} d\eta = \frac{\pi}{2} g(q). \quad (5.20)$$

With the synchronization condition $q = \omega^*$ for Kuramoto oscillators, the minimum coupling strength for the additional clusters is always larger than the coupling strength of the central cluster for the case with the limit $m \rightarrow 0$,

$$K_c > \frac{2}{\pi g(\omega^*)} > K_0. \quad (5.21)$$

As a result, with the increase of coupling strength $K = K_0$ and the growth of the central cluster, the boundary condition ω^* becomes larger and larger. But no matter what the value ω^* is, the minimum coupling strength needed for the formation of other clusters always larger than K , as $K_c > K_0 = K$. As a result, there are no additional clusters in the zero inertia limit (Kuramoto oscillators) for symmetric unimodal distributions.

Figure 5.4: Critical coupling strengths K_c, K_0^L, K_0^S versus synchronization level of the central cluster ω^* with inertias $m = 5$ (a), $m = 2$ (b) and $m = 0.1$ (c).

With the effect of inertia, the minimum coupling strength K_c for the additional clusters changes slightly. But for the central cluster, there are two boundaries of the margin region of possible states b_L and b_S . In the upper boundary b_L , the synchronization condition of oscillators is the same as Kuramoto ones as $q = b_L \omega^* \equiv \omega^*$. Hence the stable part of the coupling strength K_0^L is approximately independent of inertias, same as the Kuramoto oscillators. On the contrary, in the lower boundary, the synchronization condition $q = b_S \omega^* < \omega^*$ decreases largely with an increase of m . Hence, for a given ω^* , the coupling strength K_0^S in the forward process increases largely. When m is sufficiently large, one gets the cross of k_c and K_0^S and a region where

$$K_0^S > K_c > K_0^L. \quad (5.22)$$

In this region, we can find the additional clusters' appearance round the lower boundary b_S , as in the forward process where the coupling strength is increased progressively.

The additional clusters are from the overlooked fact that forming a synchronization cluster changes the state of the system from which it forms. For the Kuramoto oscillators, this changed state does not support any more synchronized cluster. But for the second-order oscillators, with the dramatically changed synchronization condition, new clusters appear from the changed state. As a matter of fact, for the Kuramoto oscillator or the upper bound of the second-order oscillator, the synchronization condition is from a saddle-node bifurcation, while for the lower bound of the second-order oscillators, it is from a homo-clinic bifurcation. Different bifurcations result in different conditions and also different phenomenon for the changed systems.

5.5 Conclusions

In this chapter, we show that there are two transitions from the incoherence to the oscillatory state. The first one is from the incoherence to the partial synchronization state and the second from the partial synchronization state to the oscillatory state. Both the incoherence and the partial synchronization state are metastable for a finite system. To analyze this two-step process, we generalize the self-consistent method to include the change of system from the formation of synchronization clusters. The three synchronized clusters can be obtained analytically. Other than the stable fixed point for the central cluster, two more stable fixed points appear after the system's change. The numerical simulations show that our theory works well for second-order oscillators.

Even though many efforts have been put into the study of the synchronization of coupled oscillators, most of the research is focused on the final states, steady or non-steady. As for the self-organization processes in time scale, there are few works and almost no analytical methods (Ghosh and Gupta, 2013; Park and Kahng, 2018; Pluchino and Rapisarda, 2006). Even though the method we generalized in this chapter is not a real time-scale method, we hope it may inspire more works in this direction.

Chapter 6

Second-order Kuramoto-Sakaguchi models

In this chapter we study the synchronization of oscillators with inertias and phase shifts, namely the second-order Kuramoto-Sakaguchi model. Using the self-consistent method, we find that the effect of inertia is the introduction of effective phase shifts. The discontinuous synchronization transition of the Kuramoto-Sakaguchi model changes to a continuous one when the value of inertia is small. In addition, we find a new synchronization process, in which with increasing coupling strength the system reaches an oscillating state instead of complete synchronization due to the cross-effect of phase shifts and inertias. Through numerical simulations, the same type of synchronization process is also found for oscillators in complex networks, including scale-free, small-world and random networks.

6.1 Introduction

Synchronization of oscillators, especially synchronization in complex networks Arenas et al. (2008), has been recognized as one of the important phenomena in nature. Among the different models of oscillator dynamics, the Kuramoto model Kuramoto and Nishikawa (1987), and its various generalizations Rodrigues et al. (2016), are some of the most popular models. Within this class of generalized Kuramoto models, second-order oscillator models, that is, oscillators with inertias, have been used for describing the dynamics of fireflies Ermentrout (1991), Josephson junction arrays Levi et al. (1978); Watanabe and Strogatz (1994); Trees et al. (2005), goods markets Ikeda et al. (2012), dendritic neurons Sakyte and Ragulskis (2011), and power grids Filatrella et al. (2008). Due to the effect of inertias, phenomena such as hysteresis, bi-stability and abrupt transitions are found for these second-order oscillators Tanaka et al. (1997a,b); Gao and Efstathiou (2018). In Tanaka et al. (1997a); Gao and Efstathiou (2018) the changes from continuous to abrupt phase transition for second-order oscillators have been studied in detail

using the self-consistent method.

As a natural generalization, oscillators with both inertias and phase shifts, namely the second-order Kuramoto-Sakaguchi model, are considered in Barre and Métivier (2016). It has been found that due to the effect of inertias the synchronization transition of oscillators can be changed from continuous to abrupt and vice versa. In this chapter, we generalize the self-consistent method presented in Gao and Efstathiou (2018) to the second-order Kuramoto-Sakaguchi model. We find that the inertias introduce effective phase shifts and that the type of synchronization transition is affected by the mixture of these inertia-induced phase shifts and the ones built into the model.

Moreover, we find a new type of synchronization process with increasing coupling strength. In this process, oscillators converge to an oscillating state by forming several synchronization clusters, which cannot be further synchronized by increasing the coupling strength. This process is quite different from the common belief that with sufficient large coupling strengths the coupled Kuramoto-like oscillators are typically synchronized to a highly coherent steady state, except for some specific choice of parameters, such as with phase shift $\pm\pi/2$. Through the self-consistent method and dynamical analysis of the synchronized clusters, we show that this process is due to the cross-effect of inertias and phase shifts, and is not limited to the case of all-connected oscillators. Through numerical simulations, this new type of synchronization process is also found in oscillators connected in complex networks.

This chapter is organized as follows. In Section 6.2, we generalize the self-consistent method to oscillators with inertias and phase shifts. The mixture of effective (inertia-induced) and intrinsic phase shifts is associated to the change of properties of the synchronization transition. Using the self-consistent method, in Section 6.3, we find the new synchronization process to oscillating states and study it through the self-consistent method and dynamical analysis. Using numerical simulations, this process is also observed for oscillators on complex networks. We conclude this chapter in Section 6.4.

6.2 Effective phase shifts

To focus on the effect of phase shifts, we assume that all the oscillators have the same inertia m and damping constant D . The dynamics of the second-order Kuramoto-Sakaguchi model reads

$$m\ddot{\varphi}_i + D\dot{\varphi}_i = \Omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\varphi_j - \varphi_i - \alpha), \quad i = 1, 2, \dots, N \quad (6.1)$$

where N is the number of oscillators and K is the uniform coupling strength. Each oscillator is described by its phase $\varphi_i \in \mathbb{S}$ with Ω_i as its natural frequency. The intrinsic phase shift α is added in the coupling term $\sin(\varphi_j - \varphi_i - \alpha)$. The standard second-order model corresponds to $\alpha = 0$.

Following the work by Kuramoto Kuramoto and Nishikawa (1987) we define the order parameter

$$re^{i\phi} = \frac{1}{N} \sum_{j=1}^N e^{i\varphi_j}, \quad (6.2)$$

where r and ϕ represent the coherence and mean-phase of the oscillators. If all the oscillators run independently, their phases will almost uniformly distribute along the unit circle. As a result, we have $r \cong 0$, and the state is called incoherence. On the other hand, if all of the oscillators are synchronized and have the same phase $\theta_i(t) \equiv \theta(t)$, we have $r = 1$. This is called the complete synchronization state of the system.

Using r and ϕ , the model (6.1) can also be rewritten in a mean-field form as

$$m\ddot{\phi} + D\dot{\phi} = \Omega + Kr(t) \sin(\phi(t) - \varphi - \alpha), \quad (6.3)$$

where the subscripts have been dropped. In Eq. (6.3) each oscillator interacts with other oscillators only through the mean-field terms r and ϕ . Therefore, the dynamics of the system can be obtained through the analysis of each single oscillator with a presupposed mean-field.

For simplicity, in this chapter we assume an infinite number of oscillators $N \rightarrow \infty$, and that the distribution of natural frequencies of oscillators $g_\Omega(\Omega)$ is symmetric, $g_\Omega(\Omega) = g_\Omega(-\Omega)$, and unimodal. The essential states of the system are the steady states defined as

$$r(t) = r, \quad \phi(t) = \Omega^r t + \Psi, \quad (6.4)$$

where the order parameter $r(t)$ is independent of time, and the phase $\phi(t)$ has a constant rotation velocity. Without loss of generality, we set $\Psi \equiv 0$. Define the phases θ of each oscillator in a rotating coordinate frame through the transformation $\theta = \varphi - \phi(t)$. Substitution of Eq. (6.4) into Eq. (6.3) yields

$$m\ddot{\theta} + D\dot{\theta} = (\Omega - D\Omega^r) - Kr \sin(\theta + \alpha). \quad (6.5)$$

For $\alpha = 0$, Eq. (6.5) is exactly the same as the one for a single second-order oscillator without intrinsic phase shift Gao and Efstathiou (2018). Following Gao and Efstathiou (2018), Eq. (6.5) can be rewritten in the standard form as

$$\ddot{\theta} + a\dot{\theta} = b - \sin(\theta + \alpha), \quad (6.6)$$

with rescaled time $\tau = t/\sqrt{m/Kr}$ and

$$a = \frac{D}{\sqrt{Krm}}, \quad b = \frac{\Omega - D\Omega^r}{Kr}. \quad (6.7)$$

Because of its dependence on Ω , the parameter b follows the distribution $g_b(b) = Krg_\Omega(Krb + D\Omega^r)$.

It is known from the earlier studies Tanaka et al. (1997a,b); Gao and Efstathiou (2018) that the system Eq. (6.6) has two stable states, one fixed point and one limit cycle Strogatz (2014); Tanaka et al. (1997b). The rotation frequency of oscillators is defined as $\omega = \dot{\theta}$. Taking $a > 0$, the stable fixed point reads

$$\theta_0 = \arcsin(b) - \alpha, \quad \omega_0 = 0, \quad (6.8)$$

with the existence condition $b \leq b_L(a) = 1$. On the other hand, for the limit cycle, using the same estimation method as in Gao and Efstathiou (2018) we have the approximate expression $\dot{\theta}(\tau)$ for the limit cycle, given by

$$\dot{\theta}(\tau) = \frac{b}{a} - \frac{1}{a}\sigma \sin(\theta(\tau) + \Delta + \alpha), \quad (6.9)$$

where the coupling factor σ and phase shift term Δ are

$$\sigma = \frac{a^2}{\sqrt{b^2 + a^4}}, \quad \Delta = \arcsin\left(\frac{-b}{\sqrt{b^2 + a^4}}\right). \quad (6.10)$$

The existence condition of the limit cycle can be calculated through Melnikov's method Guckenheimer (2013) or Lyapunov's direct method Riskin (1996) and numerical simulations Gao and Efstathiou (2018) as

$$b \geq b_S = \begin{cases} (4/\pi)a - 0.305a^3, & a \leq 1.193, \\ 1, & a > 1.193. \end{cases} \quad (6.11)$$

Eq. (6.9) shows that running oscillators have the same dynamics as Kuramoto-Sakaguchi oscillators with coupling factor σ and *effective phase shift* $\alpha + \Delta$ as the combination of intrinsic phase shift α and inertia-induced phase shift $\Delta \in (-\pi/2, \pi/2)$.

Even a small inertia value can introduce the mixture effect of phase shifts α and Δ . As a result, several non-trivial transitions of Kuramoto-Sakaguchi oscillators that depend on the specific choice of phase shifts will be undermined by inertias. These include the non-universal transition processes in Omel'chenko and Wolfrum (2012, 2013), shown in Fig. 6.1(a-b), and the discontinuous transition Fig. 6.1(c-d). Note that with the introduction of inertias, the transition processes are not always changed from continuous to abrupt. The opposite also happens when there are phase shifts, as pointed out in Barre and Mtivier (2016) using the stability analysis around the critical point.

When the inertia is not zero, there is a bistable parameter region, where the system has both a fixed point and a limit cycle, given by $b_L \geq b \geq b_S$. Each oscillator is either *locked* at the fixed point or *running* along the limit cycle. Taking $N \rightarrow \infty$, the order parameter defined in Eq. (6.2) can be rewritten as

$$r = \int_{\mathbb{R}} \int_{\mathbb{S}} \int_{\mathbb{R}} e^{i\theta(t)} p(\Omega, \theta_0, \omega_0) d\omega_0 d\theta_0 d\Omega, \quad (6.12)$$

Figure 6.1: Synchronization for oscillators with a double Gaussian distribution $g_{\Omega}(\Omega) = 0.6 \times \frac{1}{\sqrt{2\pi}} e^{-\Omega^2/2} + 0.4 \times \frac{1}{\sqrt{2\pi \times 0.1}} e^{-\Omega^2/(2 \times 0.1^2)}$ with $\alpha = 1.07$ and different inertias $m = 0$ (a), $m = 0.1$ (b); or a double Lorentz distribution $g_{\Omega}(\Omega) = 0.8 \times \frac{1}{\pi} \frac{1}{\Omega^2+1} + 0.2 \times \frac{1}{\pi} \frac{0.075}{\Omega^2+0.075^2}$ with $\alpha = 0.8$ and different inertias, $m = 0$ (c), $m = 0.1$ (d). The solid (dashed) lines are solutions of self-consistent equations, and correspond to stable (unstable) steady states. Circles are from the numerical simulations of 10000 oscillators with proper initial states.

where $p(\Omega, \theta_0, \omega_0)$ represents the distribution of initial conditions and natural frequencies, and the dynamics $\theta(t)$ for each oscillator depends on its initial conditions and Ω . Note that $\int_{\mathbb{S}} \int_{\mathbb{R}} p(\Omega, \theta_0, \omega_0) d\theta_0 d\omega_0 = g_{\Omega}(\Omega)$. If we know the ratio of locked and running oscillators in the system then the last expression can be simplified. Substituting the solution of locked and running oscillators, Eq. (6.8) and Eq. (6.9), into Eq. (6.12), together with their existence conditions, we have the self-consistent

equations

$$r = \int_{\mathbb{R}} g_{\Omega}(\Omega) \rho_l(a, b) \left(\sqrt{1 - b^2} \cos \alpha - b \sin \alpha \right) - g_{\Omega}(\Omega) \rho_r(a, b) \left(\frac{b}{\sigma} + \sqrt{\frac{b^2}{\sigma^2} - 1} \right) \sin(\Delta + \alpha) d\Omega, \quad (6.13a)$$

$$0 = \int_{\mathbb{R}} g_{\Omega}(\Omega) \rho_l(a, b) \left(b \cos \alpha - \sqrt{1 - b^2} \sin \alpha \right) + g_{\Omega}(\Omega) \rho_r(a, b) \left(\frac{b}{\sigma} - \sqrt{\frac{b^2}{\sigma^2} - 1} \right) \cos(\Delta + \alpha) d\Omega. \quad (6.13b)$$

The fraction functions $\rho_l(a, b)$ and $\rho_r(a, b)$ are the fraction of locked and running oscillators respectively, satisfying the normalization condition $\rho_l(a, b) + \rho_r(a, b) = 1$, and the boundaries $\mathbf{1}_S(a, b) \leq \rho_l(a, b) \leq \mathbf{1}_L(a, b)$. The indicator functions $\mathbf{1}_{S,L}$ take the value 1 if $|b| < b_S(a)$ or $|b| < b_L(a)$ and 0 otherwise, corresponding to cases of running or locked oscillators. The most commonly used fraction functions are the two indicator functions $\rho_l(a, b) = \mathbf{1}_S(a, b)$ and $\rho_l(a, b) = \mathbf{1}_L(a, b)$ where all the oscillators are in the limit cycle state, or the fixed point state, as long as it is possible. These two functions correspond to the so-called forward and backward processes. In the forward process, the initial state for small coupling strength is the incoherence state, and the coupling strength is then progressively increased. In the backward process, the initial state for large coupling strength is the synchronization state, and the coupling strength is then progressively decreased. For second-order oscillators, these two processes in general do not coincide with each other, a phenomenon known as *hysteresis* Tanaka et al. (1997a); Gao and Efstathiou (2018).

Compared with the previous results, it is easy to verify that when $\alpha = 0$, from Eq. (6.13) one regains the self-consistent equations for second-order oscillators without phase shifts in Gao and Efstathiou (2018) using the approximation $b/\sigma - \sqrt{b^2/\sigma^2 - 1} \approx \sigma/(2b)$ which is valid for small σ . On the other hand, in the limit $m \rightarrow 0$, we have $b_{S,L}(a) \rightarrow 1$, $\sigma \rightarrow 1$, $\Delta \rightarrow 0$. The self-consistent equations (6.13) in this case are the same as the ones obtained for Kuramoto-Sakaguchi models in Omel'chenko and Wolfrum (2012, 2013),

Following Omel'chenko and Wolfrum (2013); Gao and Efstathiou (2018), by defining $q = Kr$ and correspondingly $a = D/\sqrt{qm}$ and $g_b(b) = qg_{\Omega}(qb + D\Omega^r)$,

the self-consistent equations (6.13) can be rewritten as

$$\begin{aligned} \frac{\cos \alpha}{K} = F_1(q, \Omega^r) &\equiv \int_{-\infty}^{\infty} g_{\Omega}(qb + D\Omega^r) \\ &\left[\rho_l \sqrt{1 - b^2} + \rho_r \left(\frac{b}{\sigma} - \sqrt{\frac{b^2}{\sigma^2} - 1} \right) \sin \Delta \right] db, \end{aligned} \quad (6.14a)$$

$$\begin{aligned} \frac{\sin \alpha}{K} = F_2(q, \Omega^r) &\equiv \int_{-\infty}^{\infty} g_{\Omega}(qb + D\Omega^r) \\ &\left[\rho_l b + \rho_r \left(\frac{b}{\sigma} - \sqrt{\frac{b^2}{\sigma^2} - 1} \right) \cos \Delta \right] db, \end{aligned} \quad (6.14b)$$

Eq. (6.14) defines a map from (q, Ω^r) to (α, K) . The solutions of the self-consistent equations can be denoted as the quad (q, Ω^r, K, α) corresponding to points on the graph of this map. From the quad (q, Ω^r, K, α) , it is straightforward to obtain the solutions for the order parameter as the triplets (K, α, r) and (K, α, Ω^r) . These are depicted in Fig. 6.1.

The results of the numerical simulation that demonstrates the mixture effect of intrinsic and inertia-induced phase shifts are shown in Fig. 6.1. Here, we consider $N = 10000$ oscillators with either no or small inertias $m = 0.1$. The natural frequencies are chosen from a double Gaussian or a double Lorenz distribution considered in Omel'chenko and Wolfrum (2012, 2013). The coupling strength is increased from $K = 0$ to $K = 4$ with increment $dK = 0.1$. To obtain the stable states at each coupling strength K , two initial states of oscillators are considered. One is the incoherence state, and the other is the synchronization state. From these two initial states, after sufficient long transient time $t = 100$, we obtain the stable states at each coupling strength K , shown as circles in Fig. 6.1. The theoretical results are obtained from the self-consistent method in Omel'chenko and Wolfrum (2012, 2013) for $m = 0$ and the equations Eq. (6.14) for $m = 0.1$. Due to the fact that the inertia $m = 0.1$ is quite small, the difference between b_S and b_L is negligible. The synchronization transitions can be obtained directly from the stable states. If there is only one stable state for each K , the transition is continuous. On the contrary, if there are multiple and discontinuous branches of stable states, the transitions are abrupt.

Comparing the numerical simulations and theoretical results, we firstly find that the theoretical predictions of the self-consistent method coincide well with the results of the numerical simulations. Secondly, even with a small value of inertias, such as $m = 0.1$, the stable states of oscillators change dramatically, resulting in corresponding changes in the synchronization transitions. This phenomenon is found in Barre and Mtivier (2016) through the stability analysis around the critical point. In this chapter, through Eq. (6.9) and the self-consistent method, we show that the physical mechanism of these transitions is the inertia-induced phase shift Δ and its direct mixture with the intrinsic phase shift α . This mixture results

Figure 6.2: (a) Synchronization transitions for oscillators with a Gaussian distribution $g_{\Omega}(\Omega) = \frac{1}{\sqrt{2\pi}}e^{-\Omega^2/2}$ with $m = 2, \alpha = 0.5$ in shown. The dotted and dash-dotted lines are the solutions of self-consistent equations in the backward, forward processes. Squares and circles with error bar are from the numerical simulations of 10000 oscillators in the forward and backward processes, where the error bar is the standard deviation of $r(t)$. At $k = 16$ in the forward process, the oscillating state is in (b) with the order parameter $r(t)$, (c) the mean frequencies of oscillators versus their natural frequencies with the distribution of the mean frequencies in the inner figure. (d) The mean frequencies of two largest synchronized clusters in the forward process.

in the cancellation of the effect of the phase shift and consequently leads to the continuous synchronization transitions for oscillators with unimodal distributions. Interestingly, this analysis can also be applied to the second-order oscillators with $\alpha = 0$, where the phase shift Δ in general introduces abrupt transitions Gao and Efstathiou (2018).

6.3 Oscillating synchronization process

In the previous section we saw how the cross-effect of phase shifts and inertias leads to changes in the synchronization transitions from abrupt to continuous

Figure 6.3: (a) Phase diagram of 1000 oscillators in the forward processes up to $k = 40$ with inertias m and phase shifts α . The natural frequencies of oscillators are chosen randomly from a Gaussian distribution $g_{\Omega}(\Omega) = \frac{1}{\sqrt{2\pi}}e^{-\Omega^2/2}$. (b) The basin of attraction of oscillating states at $K = 20$ with $m = 2, \alpha = 0.5$. The oscillators are sorted separated into two groups by their natural frequencies, with fractions n_1 and n_2 where $n_1 + n_2 = 1$. The oscillators initial frequencies are chosen randomly from $[\Omega_1 - \delta\omega, \Omega_1 + \delta\omega]$ and $[\Omega_2 - \delta\omega, \Omega_2 + \delta\omega]$ respectively. The initial phases of all oscillators are chosen randomly from $[0, 2\pi]$. We set $\Omega_1 = -1$ and $\delta\omega = 0.1$. The separation of oscillating states and synchronization states is determined by the standard deviation $\sigma_r = 0.1$ of $r(t)$.

or vice versa through the direct mixture of α and Δ . Here, we show that the same cross-effect to a different synchronization transition where oscillators do not reach a steady state with increasing coupling strength but instead they reach an oscillating state for arbitrarily large coupling strength. This phenomenon is due to the formation of several synchronized clusters and appears in the parameter region of relatively large inertias and phase shifts.

First, we check the existence of complete synchronization state in the limit $K \rightarrow \infty$. From the self-consistent equations, when the coupling strength is sufficiently large if the system converges to the complete synchronization as $r \rightarrow 1$ and all the oscillators are locked, we have $q \approx K \gg 1$. Then Eq. (6.14b) reads

$$r \sin \alpha \approx \int_{-qb_{S,L}(a)+D\Omega^r}^{qb_{S,L}(a)+D\Omega^r} g_{\Omega}(\Omega) \frac{\Omega - D\Omega^r}{q} d\Omega \quad (6.15)$$

where $qb_{S,L}(a) + D\Omega^r \gg 0$ and $-qb_{S,L}(a) + D\Omega^r \ll 0$ are the self-consistent conditions for the complete synchronization. From the property that $g_{\Omega}(\Omega)$ is a normalized distribution, we have the solution

$$D\Omega_r \approx \bar{\Omega} - K \sin \alpha \approx \bar{\Omega} - q \sin \alpha, \quad (6.16)$$

where $\bar{\Omega}$ is the mean-frequency of natural frequencies Ω . From the symmetry of

$g_\Omega(\Omega)$ as we assumed, we have $\bar{\Omega} = 0$. Note that the collective frequency Ω_r depends on the coupling strength.

Such complete synchronization state only exists if the self-consistent conditions

$$qb_{S,L}(a) + D\Omega^r \gg 0, \quad -qb_{S,L}(a) + D\Omega^r \ll 0, \quad (6.17)$$

are satisfied. When q is sufficiently large, we have $qb_L = q$ and $qb_S \approx C\sqrt{q}$ with the constant $C = 4D/(\pi\sqrt{m})$. From the solution $D\Omega_r = -q\sin\alpha$, we deduce that only in the backward process with $qb_L = q$ the self-consistent conditions Eq. (6.17) are satisfied if $\alpha \neq \pm\pi/2$ and hence the complete synchronization states exist. On the contrary, in the forward process, both expressions in Eq. (6.17) are either positive or negative, depending on the value of α . In both cases, the self-consistent conditions are not satisfied and oscillators cannot converge to the complete synchronization states.

The critical coupling strength K_n for this new synchronization process can be estimated by $qb_{S,L}(a) = |D\Omega^r|$, which gives

$$K_n = \frac{16D^2}{\pi^2 m \sin^2 \alpha}. \quad (6.18)$$

When $\alpha \rightarrow 0$ or $m \rightarrow 0$, we have $K_n \rightarrow \infty$. In this case, all the oscillators are already synchronized with each other and therefore this new synchronization process does not manifest. The new synchronization process only appears in the forward process when K_n is smaller than the critical point of the appearance of complete synchronization states. Hence one gets the usual synchronization processes with either small inertias or small phase shifts.

In addition, when both α and m are large enough, another effect of inertias should also be included, namely the appearance of additional synchronized clusters. As discussed in Olmi et al. (2014); Gao and Efstathiou (2018), for second-order oscillators with large enough inertias, the steady states with only one cluster are not stable and several additional clusters can form besides the central cluster. In this case, the amplitude of the order parameter $r(t)$ exhibits a periodic oscillation. This kind of state is called *oscillating state*, and is the direct result of inertias Gao and Efstathiou (2018). Hence, as shown in Fig. 6.2, numerical simulations reveal that the synchronization process converges to oscillating states and not to the steady states calculated with the self-consistent method. In this case, the oscillators form two major synchronized clusters. We name this synchronization process the oscillating synchronization process to distinguish it from the classic synchronization process that leads to the complete synchronization state.

The numerical results for $N = 10000$ oscillators are shown in Fig. 6.2. The natural frequencies of oscillators are chosen randomly from a Gaussian distribution. The inertia and phase shift of the oscillators are $m = 2$, $\alpha = 0.5$. Both the forward and backward processes are considered in the region $K \in [0, 20]$ with $dK = 0.1$. Comparing with the theoretical results from the self-consistent equations Eq. (6.13), the numerical result in the backward process coincides well with

the result from Eq. (6.13), as shown in Fig. 6.2(a). However, in the forward process, the order parameter exhibits a large oscillation. For a specific state in the forward process at $K = 16$, we show the order parameter $r(t)$, and mean-frequency $\bar{\omega}$ in Fig. 6.2(b,c). We see the periodic oscillation of $r(t)$ and correspondingly the multi-synchronization clusters shown as the stairs in Fig. 6.2(c). To check the properties of the oscillating state, we show the mean-frequency of the largest two clusters in the forward process. As shown in Fig. 6.2(d), the two mean-frequencies of these clusters depend linearly on the coupling strength K . Though the results in Fig. 6.2 are shown up to $K = 20$, we have checked that these non-synchronized oscillating states are still stable up to $K = 500$.

Recall that when there is no phase shift, these synchronized clusters will merge into a single one for sufficiently large coupling strength Gao and Efstathiou (2018). However, due to the phase shift α the separation of such clusters is strengthened. The frequency of these two clusters depends on the coupling strength K approximately linearly with a slope proportional to the fraction of oscillators in it, as shown in Fig. 6.2(d). As a matter of fact, these two clusters cannot be synchronized by increasing the coupling strength. As a simple model exhibiting the same behaviour, consider a special system with only two values of natural frequencies, i.e. N_1 oscillators with Ω_1 and N_2 oscillators with Ω_2 , following

$$m\ddot{\theta}_i + D\dot{\theta}_i = \Omega_1 + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i - \alpha), \quad (6.19)$$

when $i = 1, \dots, N_1$, and

$$m\ddot{\theta}_i + D\dot{\theta}_i = \Omega_2 + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j - \theta_i - \alpha), \quad (6.20)$$

when $i = N_1 + 1, \dots, N = N_1 + N_2$. The oscillators are naturally divided into two groups and synchronized within each group. The dimension of the system can be reduced and one finds

$$m\ddot{\theta}_1 + D\dot{\theta}_1 = \Omega_1 - Kn_1 \sin \alpha + Kn_2 \sin(\theta_2 - \theta_1 - \alpha), \quad (6.21a)$$

$$m\ddot{\theta}_2 + D\dot{\theta}_1 = \Omega_2 - Kn_2 \sin \alpha + Kn_1 \sin(\theta_1 - \theta_2 - \alpha), \quad (6.21b)$$

where θ_1 and θ_2 are the common phases of the oscillators in the first and second group respectively, and $n_1 = N_1/N, n_2 = N_2/N$ with $n_1 + n_2 = 1$. With the definition of phase difference $\varphi = \theta_1 - \theta_2$ we have

$$m\ddot{\varphi} + D\dot{\varphi} = \Omega_1 - \Omega_2 - K(n_1 - n_2) \sin \alpha - K[n_2 \sin(\varphi + \alpha) + n_1 \sin(\varphi - \alpha)]. \quad (6.22)$$

Without loss of generality, taking $n_1 > n_2$, Eq. (6.22) can be rewritten as

$$m\ddot{\varphi} + D\dot{\varphi} = \Delta\Omega - \bar{K} \sin(\varphi + \bar{\alpha}), \quad (6.23)$$

where

$$\Delta\Omega = \Omega_1 - \Omega_2 - K(n_1 - n_2) \sin \alpha, \quad (6.24a)$$

$$\bar{K} = K \sqrt{\cos^2 \alpha + (n_1 - n_2)^2 \sin^2 \alpha} \equiv Kq(\alpha), \quad (6.24b)$$

$$\bar{\alpha} = \arcsin \left(\frac{(n_1 - n_2) \sin \alpha}{\sqrt{\cos^2 \alpha + (n_1 - n_2)^2 \sin^2 \alpha}} \right). \quad (6.24c)$$

The simplified dynamics in Eq. (6.23) is the same as the dynamics for second-order oscillators in the mean field Eq. (6.6). Hence the synchronization condition for the two clusters is determined by the two parameters

$$a = \frac{D}{\sqrt{Kq(\alpha)m}}, \quad b = \frac{\Omega_1 - \Omega_2}{Kq(\alpha)} - \frac{(n_1 - n_2) \sin \alpha}{q(\alpha)}. \quad (6.25)$$

As a result, with $K \rightarrow \infty$, we have $a \rightarrow 0$ and $|b| \rightarrow (n_1 - n_2) \sin \alpha / q(\alpha) > 0$. From the fact that $b_S(a) \rightarrow 0$ in the limit $a \rightarrow 0$, the synchronization condition $|b| < b_S(a)$ can not be satisfied with increasing K . In this case, we have the non-synchronized process, where the two clusters cannot be synchronized.

It is clear that the non-synchronized process is due to the cross-effect of inertia and phase shift. If $\alpha = 0$, we have $q(\alpha) = 1$. Substitution of α and $q(\alpha)$ into Eq. (6.25) yields

$$a = \frac{D}{\sqrt{Km}}, \quad b = \frac{\Omega_1 - \Omega_2}{K}. \quad (6.26)$$

In the limit $K \rightarrow \infty$, we have $b_S \approx 4D/\sqrt{Km\pi}$. Hence no matter how large $|\Omega_1 - \Omega_2|$ we always have $K_c = \pi^2(\Omega_1 - \Omega_2)^2 m / 16D^2$ where the two clusters will become synchronized with $K > K_c$ with increasing K . On the other hand, if $m = 0$ one gets $b_S = 1$. From the fact that $(n_1 - n_2) \sin \alpha / q(\alpha) < 1$ we have $|b| < 1 = b_S$ in the limit $K \rightarrow \infty$. As a result, these two clusters will be synchronized when K is large enough.

In addition, similar to the analysis for $m = 0$, in the backward processes with $b_L \equiv 1$, the synchronization states are not affected by the inertias and phase shifts. As a result, from the quite different properties of b_L and $b_S(a)$, we have the non-trivial bi-stability of complete synchronization and oscillating states.

To test the conclusion above, we calculate the phase diagram of non-synchronized oscillating states. $N = 10000$ oscillators are considered with a Gaussian distribution of their natural frequencies. With different inertias $m \in [0.1, 4]$ and phase shifts $\alpha \in [0.01, 1]$, we follow the oscillators in the forward process to a sufficient large coupling $K = 40$. The boundary between oscillating states and partial synchronization states is determined by ± 0.1 standard deviations of the order parameter r from its mean value. The result is shown in Fig. 6.3(a). The oscillating states exist when both the inertia and phase shift are relatively large. The fitting curve for the boundary lines read $\alpha = (\pi/2)/(1 + 2.96k)$. In Fig. 6.3(b), we

Figure 6.4: Synchronization transitions for 1000 oscillators with a Gaussian distribution $g_{\Omega}(\Omega) = \frac{1}{\sqrt{2\pi}}e^{-\Omega^2/2}$ with $m = 2, \alpha = 0.5$ in backward process on (a) Erdos-Rényi random networks Erdos (1960) with $p = 0.3$ the probability for edge creation, (b) Watts-Strogatz small-world networks Watts and Strogatz (1998) with $k = 100$ the nearest connection in a ring and $p = 0.3$ the probability for edge creation, (c) Barabási-Albert scale-free networks Barabási and Albert (1999) with the minimum degree $k_0 = 50$, (d) Barabási-Albert scale-free networks with the minimum degree $k_0 = 3$.

check the basin of attraction of the oscillating state at $K = 20$. The oscillators sorted and separated into two groups according to their natural frequencies. The fraction of the two groups is defined by n_1 and n_2 with $n_1 + n_2 = 1$. The initial frequencies of the oscillators are chosen randomly from a small region around Ω_1 and Ω_2 , and their initial phases are chosen randomly from $[0, 2\pi]$. Without loss of generality, we take $\Omega_1 = -1$. From the numerical simulations, we see that there is clear large basin of attraction of the oscillating state as shown in Fig. 6.3(b). As we can see from the expression for the parameters a, b in Eq. (6.25), the basin of attraction of oscillating states is closely related to the frequency and fraction separation $|\Omega_1 - \Omega_2|$ and $|n_1 - n_2|$ of two groups.

To check the generality of the oscillating synchronization process, we considered various systems of the second-order Kuramoto-Sakaguchi oscillators. For all-

connected oscillators, this new oscillating synchronization process is found in all the cases we considered, including uniform, Lorentz and double-Gaussian/Lorentz distributions of the natural frequencies. For oscillators in complex networks, we consider scale-free, ER random, and small-world networks. The non-synchronization processes are found in all the systems as shown in Fig. 6.4. The oscillating states appear in these processes when the mean-degree of such networks is large. On the other hand, with a smaller mean-degree, the second synchronization cluster is suppressed by the topology of the network. We find the non-synchronized steady states converge to $r = 0$ in the limit $K \rightarrow \infty$ as shown in Fig. 6.4(d). The oscillating synchronization process depends on the mean-degree of networks, not their densities. This fact is closely related to the conditions determining whether the mean-field assumption works for random networks. The suppression of the oscillating states is beyond the scope of this chapter, and will be considered in a forthcoming work.

6.4 Conclusions

In this chapter we analyse the second-order oscillators with phase shifts, namely second-order Kuramoto-Sakaguchi model. The self-consistent method is generalized and used to study the steady states of oscillators. With the inertia introduced phase shifts, the non-universal transitions of Kuramoto-Sakaguchi oscillators Omel'chenko and Wolfrum (2012) are canceled out by a small value of m . The changing of abrupt to continuous transitions with the effect of inertias proposed in Barre and Métivier (2016) is also shown and studied by the self-consistent method.

In addition, the cross-effect of inertia and phase shifts also results in the oscillating synchronization forward processes. Instead of synchronization states, the system will stay in the oscillating state and can not be synchronized with increasing coupling strength. This interesting phenomenon is due to the combination of additional synchronized clusters as an effect of inertias and the dependence of Ω^r on K as an effect of phase shifts. Using numerical simulations, such non-synchronized processes are also found in different distributions of natural frequencies and topologies of the networks.

Chapter 7

Reduction of oscillator dynamics on complex networks

In Chapter 6, we have encountered the system in complex systems. In this chapter, we focus on the synchronization of oscillators in complex networks and try to understand the interplay between the oscillator dynamics and the network topology. Through a remarkable variable transformation and the introduction of virtual frequencies we show that Kuramoto oscillators on annealed networks, with or without frequency-degree correlation, can be transformed to Kuramoto oscillators on complete graphs with a re-arranged, virtual frequency distribution, which encodes both the natural frequency distribution (dynamics) and the degree distribution (topology). We exploit this observation to give alternative, straightforward, explanations to a variety of phenomena that have been observed in complex networks, such as explosive synchronization and vanishing synchronization onset. We further extend this method to the study of the frequency-weighted coupling model which also exhibits explosive synchronization.

7.1 Introduction

Recently, *explosive synchronization* has been found in scale-free networks where each oscillator's natural frequency is linearly correlated with its degree (Gómez-Gardenes et al., 2011). Such transition process is first-order-like, discontinuous and irreversible, and is closely related to explosive percolation and cascading failures (Boccaletti et al., 2016). Explosive synchronization has also been found in oscillators on a complete graph with frequency-weighted coupling (Hu et al., 2014). At the same time, oscillators on scale-free networks without frequency-degree correlation exhibit the opposite phenomenon, that is, a continuous transition with vanishing onset (Ichinomiya, 2004).

In both cases the scaling exponent γ of the scale-free networks is a critical parameter. In particular, vanishing onset and explosive synchronization are only

observed when $2 < \gamma < 3$ (Coutinho et al., 2013; Ichinomiya, 2004). Even though these phenomena have been extensively studied (Zou et al., 2014; Zhang et al., 2015; Xu et al., 2015), their mechanism is still unclear.

In this chapter, with an annealed network approximation (Ichinomiya, 2004; Coutinho et al., 2013), we use the self-consistent method as a tool to revisit the synchronization phenomenon with a mean-field in Sec. 7.2. We introduce transformations that incorporate the effect of the network topology by modifying the distribution of natural frequencies. Therefore, the different cases are reduced to the classical setting of Kuramoto oscillators on complete graphs with various distributions of natural frequencies. These cases include the explosive synchronization and vanishing onset in scale-free networks with or without frequency-degree correlation in Sec. 7.3 and Sec. 7.4, and also the explosive synchronization in frequency-weighted coupling models in Sec. 7.5. Therefore, an intuitive understanding of these models' properties can be more easily obtained. The method can also be applied to a variety of Kuramoto like oscillators, as shown in Sec. 7.6. A short conclusion is given in Sec. 7.7.

7.2 Self-consistent method in complex networks

The basic idea of the Kuramoto model (Kuramoto and Nishikawa, 1987) to explore synchronization is to consider a group of coupled oscillators with different natural frequencies as

$$\dot{\theta}_i = \omega_i - \frac{\lambda}{N} \sum_{j=1}^N \sin(\theta_i - \theta_j), \quad 1 \leq i \leq N, \quad (7.1)$$

where θ_i is the oscillator's phase, and ω_i is its natural frequency. The coupling strength is given by λ , and N is the size of the system. To describe the coherent state of oscillators, the order parameter is introduced as $re^{i\phi} = \sum_{j=1}^N \exp(i\theta_j)/N$. Using the order parameter, the dynamics in Eq. (7.1) can be rewritten in mean-field form as

$$\dot{\theta}_i = \omega_i - \lambda r \sin(\theta_i - \phi), \quad 1 \leq i \leq N. \quad (7.2)$$

In this work, we are only interested in the steady states where $r(t) = r > 0$ is constant and $\phi = \Omega t + \phi_0$. In this case, analytical results on the onset of synchronization can be obtained from the analysis of each single oscillator through the self-consistent method (Kuramoto and Nishikawa, 1987; Strogatz, 2000). The dynamics in Eq. (7.2) can be further rewritten in the frame rotating as $\Omega t + \phi_0$ and a rescaled time $\tau = qt$, where $q = \lambda r$, as

$$\dot{\theta} = b - \sin \theta, \quad (7.3)$$

where $b = (\omega - \Omega)/q$, and we have suppressed the indices of oscillators. When $|b| \leq 1$ the oscillator synchronizes with the mean-field (locked), while if $|b| > 1$ it keeps running. In the continuous limit $N \rightarrow \infty$, combining steady states of these

two kinds of oscillators, we obtain the self-consistent equations for the parameters r and Ω . The self-consistent equations can be written as

$$\begin{aligned}\frac{1}{\lambda} &= \frac{1}{q} \int_{\mathbb{R}} g_{\omega}(\omega) \mathbf{1}_b \sqrt{1 - b^2} d\omega, \\ 0 &= \int_{\mathbb{R}} g_{\omega}(\omega) \left[\mathbf{1}_b b + (1 - \mathbf{1}_b) b \left(1 - \sqrt{1 - \frac{1}{b^2}} \right) \right] d\omega, \\ b &= (\omega - \Omega)/q,\end{aligned}\tag{7.4}$$

where $g_{\omega}(\omega)$ is the density function of natural frequencies ω . The indicator function $\mathbf{1}_b$ takes the value 1 if $|b| \leq 1$ corresponding to locked oscillators, and 0 otherwise. Further details on the self-consistent method can be found in (Kuramoto and Nishikawa, 1987; Strogatz, 2000; Acebrón et al., 2005; Rodrigues et al., 2016).

In complex networks, the model for coupled oscillators reads

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^N A_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \dots, N.\tag{7.5}$$

The adjacency matrix A_{ij} describes the connection of oscillators. If there is a link between the oscillator i and j , we have $A_{ij} = 1$, and $A_{ij} = 0$ otherwise. For uncorrelated networks with randomly picked links and large order N (annealed networks), the adjacency matrix can be approximated with the mean-field assumption $A_{ij} = k_i k_j / N \langle k \rangle$, where k_i is the degree of the i -th node (oscillator) and $\langle k \rangle$ is the mean degree (Ichinomiya, 2004; Boccaletti et al., 2016). The model now reads

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^N \frac{k_i k_j}{N \langle k \rangle} \sin(\theta_j - \theta_i), \quad i = 1, \dots, N.\tag{7.6}$$

A generalized order parameter (mean-field) can be defined as

$$re^{i\phi} = \sum_{j=1}^N \frac{k_j}{N \langle k \rangle} \exp(i\theta_j).\tag{7.7}$$

Substituting the order parameter into Eq. (7.6), we obtain

$$\dot{\theta}_i = \omega_i - \lambda r k_i \sin(\theta_i - \phi), \quad 1 \leq i \leq N,$$

which then reduces to the same mean-field form as Eq. (7.3) with parameter $b = (\omega - \Omega)/kq$ depending on both natural frequency ω and degree k .

7.3 Explosive synchronization

We first consider the case of explosive synchronization, in a network where each oscillator's natural frequency ω is correlated to its degree as $\omega = Ak$. The corre-

Figure 7.1: Transition processes (r vs λ) and re-arranged distributions $G(\nu)$ of oscillators with the frequency-degree correlation $\omega = Ak$ in annealed complex networks: (a-c) scale-free networks $g(k) \sim k^{-\gamma}$ with (a) $\gamma = 2.6$, (b) $\gamma = 3$, and (c) $\gamma = 3.4$; (d) random network with exponential degree distribution $g(k) \sim e^{-k}$. The minimum degree is assumed $k_0 = 50$. Numerical results are shown for $N = 10000$ oscillators in the forward process (blue $\triangle$ with increasing λ) and backward process (red ∇ with decreasing λ).

sponding self-consistent equations read

$$\begin{aligned} \frac{1}{\lambda} &= \frac{1}{q\langle k \rangle} \int_{\mathbb{R}} k g_k(k) \mathbf{1}_b \sqrt{1 - b^2} dk, \\ 0 &= \int_{\mathbb{R}} k g_k(k) \left[\mathbf{1}_b b + (1 - \mathbf{1}_b) b \left(1 - \sqrt{1 - \frac{1}{b^2}} \right) \right] dk, \\ b &= (Ak - \Omega)/kq, \end{aligned} \quad (7.8)$$

where $g_k(k)$ is the degree density, see also (Coutinho et al., 2013).

We then define new parameters (Λ, Q, W) and a *virtual frequency* ν by

$$\nu = \frac{1}{k}, \quad W = \frac{A}{\Omega}, \quad Q = \frac{q}{\Omega}, \quad \Lambda = \frac{\lambda}{\Omega}. \quad (7.9)$$

Then the self-consistent equations Eq. (7.8) can be rewritten as

$$\begin{aligned}\frac{1}{\Lambda} &= \frac{1}{Q} \int_{\mathbb{R}} G(\nu) \mathbf{1}_b \sqrt{1 - b^2} d\nu, \\ 0 &= \int_{\mathbb{R}} G(\nu) \left[\mathbf{1}_b b + (1 - \mathbf{1}_b) b \left(1 - \sqrt{1 - \frac{1}{b^2}} \right) \right] dk, \\ b &= (\nu - W)/Q,\end{aligned}\tag{7.10}$$

where

$$G(\nu) = \frac{1}{\langle k \rangle} \frac{1}{\nu^3} g_k \left(\frac{1}{\nu} \right).\tag{7.11}$$

This is the same form as Eq. (7.4) which holds for complete graphs, where $(\lambda, q, \Omega, \omega)$ are replaced by (Λ, Q, W, ν) and the natural frequency density $g_\omega(\omega)$ is replaced by the *virtual frequency density* $G(\nu)$. Therefore, the self-consistent equations for the systems we consider here become the self-consistent equations for Kuramoto oscillators on complete graphs and (virtual) frequency density $G(\nu)$.

The re-arranged distribution $G(\nu)$ is determined by the degree density $g_k(k)$. Depending on the divergence of quadratic mean degree $\langle k^2 \rangle$ of $g_k(k)$, there is a clear distinction between two types of $G(\nu)$. Consider, for example, scale-free networks with $g_k(k) \sim k^{-\gamma}$. Then we have

$$G(\nu) = C\nu^{\gamma-3},\tag{7.12}$$

with $\nu \in (0, 1/k_0]$, where k_0 is the minimum degree of the network, and C is the normalization factor. For $2 < \gamma < 3$ (divergent $\langle k^2 \rangle$), $G(\nu)$ is monotonically decreasing with ν and divergent at $\nu = 0$, while for $\gamma > 3$ (convergent $\langle k^2 \rangle$), it is monotonically increasing and stays finite in the region $\nu \in (0, 1/k_0]$. Between these two cases when $\gamma = 3$ the distribution $G(\nu)$ is uniform. The weight of oscillators with large degrees is enlarged dramatically when $2 < \gamma < 3$.

From well-known results of Kuramoto oscillators on complete graphs (Strogatz, 2000), the uniform distribution of natural frequencies (corresponding to $\gamma = 3$) has a hybrid synchronization transition which is abrupt and without hysteresis, see also Fig. 7.1(b). On each side of such hybrid synchronization transition, we find the discontinuous transitions with hysteresis if $2 < \gamma < 3$ (Fig. 7.1(a)) and the continuous transition if $\gamma > 3$ (Fig. 7.1(c)), see (Coutinho et al., 2013). It follows from Eq. (7.11), that other distributions $g_k(k)$ that fall for large k faster than k^{-3} and thus have finite $\langle k^2 \rangle$, are also monotonically increasing in the vicinity of $\nu = 0$ from the fact that $G(\nu) \rightarrow 0$ as $\nu \rightarrow 0$. Consequently, such distributions result in monotonically increasing or unimodal distributions $G(\nu)$ and consequently give continuous transitions similarly to scale-free networks with $\gamma > 3$. The example of the exponential distribution is shown in Fig. 7.1(d). Numerical results shown in the figures are for networks generated by the static model (Goh et al., 2001) with size $N = 10000$.

Figure 7.2: Transition processes (r vs λ) and re-arranged distributions $G(\nu)$ of oscillators without frequency-degree correlation in annealed complex networks: scale-free networks $g(k) \sim k^{-\gamma}$ with (a, c) $\gamma = 2.6$ and (b, d) $\gamma = 3.4$. The minimum degree is assumed $k_0 = 50$. The maximum degree is $k_m = \infty$ (a, b) and $k_m = 500$ (c, d). The minimum degree is $k_0 = 50$. In all cases the distribution of natural frequencies is Gaussian, $g_\omega(\omega) = (1/\sqrt{2\pi\sigma^2}) \exp(-\omega^2/2\sigma^2)$ with $\sigma = \langle k \rangle$. In (c, d) the numerical results are for $N = 10000$ oscillators and shown as Fig. 7.1.

We remark that the degree distribution affects the transition to synchronization. With the increase of the scaling exponent γ , the critical frequency Ω_c with $q \rightarrow 0$ increases. Since the oscillators that synchronize have $\nu \approx \Omega_c$ and $k = 1/\nu$ we conclude that the increase of γ implies that the transition to synchronization is increasingly driven by low-degree nodes. At the same time, due to the concavity of the re-arranged density $G(\nu)$ (Strogatz, 2000) for $\gamma > 3$, the synchronized state bifurcating from the incoherence state becomes stable when low-degree nodes, which form the majority of the network, are involved. Since the low-degree nodes play the most important role for the transition to synchronization, even if the distribution $g_k(k) \sim k^{-\gamma}$, $2 < \gamma < 3$, is truncated at a maximum degree k_0 , giving a broad-scale network, the network will still exhibit explosive synchronization.

7.4 Vanishing onset

Another interesting phenomenon for oscillators in scale-free networks is the vanishing onset of synchronization when the distributions of ω and k are independent (Ichinomiya, 2004). From the dynamical equations Eq. (7.6), assuming the distribution $g_\omega(\omega)$ is unimodal and symmetric, we have $\Omega = 0$ and the self-consistent equation

$$\frac{1}{\lambda} = \frac{1}{q} \int_{\mathbb{R}} \frac{k}{\langle k \rangle} g_k(k) \left(\int_{\mathbb{R}} g_\omega(\omega) \mathbf{1}_b \sqrt{1 - b^2} d\omega \right) dk. \quad (7.13)$$

In this case, given that $b = \omega/kq$, we define the virtual frequency as $\nu = \omega/k$ and its distribution as

$$G(\nu) = \int_{\mathbb{R}} \frac{k^2}{\langle k \rangle} g_k(k) g_\omega(k\nu) dk. \quad (7.14)$$

With these choices, the self-consistent equation Eq. (7.13) becomes

$$\frac{1}{\lambda} = \frac{1}{q} \int_{\mathbb{R}} G(\nu) \mathbf{1}_b \sqrt{1 - b^2} d\nu,$$

where $b = \nu/q$, that is, it takes the same form as the self-consistent equation for Kuramoto oscillators on complete graphs with unimodal and symmetric (virtual) frequency density $G(\nu)$. Therefore, the interplay of the natural frequency density $g_\omega(\omega)$ (dynamics) and the degree density $g_k(k)$ (topology) is expressed through the re-arranged virtual frequency density $G(\nu)$.

As an example, consider the uniform distribution $g_\omega(\omega) = 1/2$ with $\omega \in [-1, 1]$. For scale-free networks $g_k(k) \sim k^{-\gamma}$, with $\gamma \neq 3$, we obtain the symmetric and unimodal distribution density

$$G(\nu) = C(|\nu|^{\gamma-3} - k_0^{3-\gamma}), \quad (7.15)$$

where $\nu \in [-1/k_0, 1/k_0]$, and C is the normalization constant (negative for $\gamma > 3$ or positive for $2 < \gamma < 3$). When $2 < \gamma < 3$, $G(\nu)$ diverges at $\nu = 0$, while for $\gamma > 3$, the distribution density remains finite. In addition, for $\gamma = 3$ one finds $G(\nu) = C \ln(|\nu|/k_0)$ for $\nu \in [-1/k_0, 1/k_0]$. The transition onset of Kuramoto oscillators with unimodal and symmetric (virtual) frequency density $G(\nu)$ is determined by $\lambda_c = 2/\pi G(0)$. Therefore, the divergence of $G(\nu)$ at $\nu = 0$ results to vanishing $\lambda_c = 0$.

Depending on the divergence of the quadratic mean degree $\langle k^2 \rangle$, networks can also be divided into two categories. If and only if $\langle k^2 \rangle$ is convergent (e.g., for exponential distributions), the re-arranged distributions remain finite, similarly to the case $\gamma > 3$, and thus we do not have vanishing onset. The same result was previously uncovered in (Ichinomiya, 2004). With Gaussian distributions of natural frequency, the transition processes and re-arranged distributions are shown in Fig. 7.2(a-b).

Different from the explosive synchronization which results from the scaling exponent γ in the bulk and is largely insensitive to the tail, the vanishing onset is

Figure 7.3: Transition processes (r vs λ) and re-arranged distributions $G(\nu)$ of frequency-weighted-coupling oscillators: (a) out-coupling model and (b) in-coupling model. The natural frequencies density is Gaussian, $g_\omega(\omega) = (1/\sqrt{2\pi})\exp(-\omega^2/2)$. The numerical results are for $N = 10000$ oscillators and shown as Fig. 7.1.

sensitive to the tail of the distribution. Specifically, when we consider broad-scale networks with truncated distributions $g_k(k) \sim k^{-\gamma}$, the corresponding re-arranged distributions $G(\nu)$ are finite at $\nu = 0$, as shown in Fig. 7.2(c-d). On the contrary, fat-tailed distributions with divergent $\langle k^2 \rangle$ have vanishing synchronization onset. Note that any finite system has a natural bond for the maximum degree k_m of $g_k(k)$. Hence the vanishing onset can only be observed in the infinite limit $N \rightarrow \infty$.

7.5 Frequency-weighted coupling

Except for the model with frequency-degree correlation in scale-free networks, another model that exhibits explosive synchronization, is the Kuramoto model with absolute frequency-weighted coupling (Hu et al., 2014; Xu et al., 2016). It is defined on complete graphs as

$$\dot{\theta}_i = \omega_i - \frac{\lambda}{N} \sum_{j=1}^N F_{ij} \sin(\theta_i - \theta_j), \quad 1 \leq i \leq N, \quad (7.16)$$

where $F_{ij} = |\omega_i|$ (in-coupling model) or $F_{ij} = |\omega_j|/\langle |\omega| \rangle$ (out-coupling model), mimicking the frequency-degree correlation (Bi et al., 2016; Xu et al., 2018). The frequency-weighted coupling model typically shows explosive synchronization (and also oscillatory states, such as standing waves and Bellerophon states) (Xu et al., 2018; Bi et al., 2016).

For the out-coupling model, one can define an order parameter as

$$r = \sum_j |\omega_j| / (N \langle |\omega| \rangle) \exp(i\theta_j). \quad (7.17)$$

Encoding the frequency-coupling, the self-consistent equation can be rewritten, using the virtual frequencies $\nu = \omega$, in standard form with re-arranged distribution $G(\nu) = |\nu| g_\omega(\nu) / \langle |\omega| \rangle$. For any normalized distributions $g_\omega(\omega)$, $G(\nu) \rightarrow 0$ as either $\nu \rightarrow 0$ or $\nu \rightarrow \pm\infty$. Thus for any unimodal symmetric distribution $g_\omega(\omega)$ the re-arranged distribution $G(\nu)$ is bimodal and symmetric, see Fig. 7.3(a).

For the in-coupling model, the case becomes more complicated. For the steady-state solution with $\Omega = 0$, we have $b = \omega/|\omega|q$ and thus we define the virtual frequency $\nu = \text{sign}(\omega)$, which is naturally bimodal. For $\Omega \neq 0$, we define the virtual frequency through the transformation

$$\nu = \frac{1}{\omega}, W = \frac{1}{\Omega}, Q = \frac{q}{\Omega} \text{sign}(\omega), \Lambda = \frac{\lambda}{\Omega} \text{sign}(\omega), \quad (7.18)$$

with density

$$G(\nu) = \frac{1}{\nu^2} g_\omega \left(\frac{1}{\nu} \right). \quad (7.19)$$

The latter is bimodal and symmetric when $g_\omega(\omega)$ is unimodal and symmetric, see Fig. 7.3(b). Note, that in this case the coupling strength Λ can be either positive or negative, unlike the standard Kuramoto model.

For coupled oscillators, bimodal frequency distributions and the coexistence of the positive and negative coupling strength contribute to abrupt transitions and oscillatory states (standing wave, π state) (Hong and Strogatz, 2011; Martens et al., 2009). The frequency-weighted coupling model, especially the in-coupling one, includes these two factors and hence one can anticipate its explosive synchronization and the existence of oscillatory (Bellerophon) states (Bi et al., 2016).

7.6 Kuramoto-like models

With different generalizations, a variety of Kuramoto-like models have been studied. Such models have the general form

$$\dot{\theta}_i = \omega_i + \epsilon_i \frac{K}{N} \sum_{j=1}^N \eta_j \sin(\theta_j - \theta_i) \quad (7.20)$$

where $\omega_i, \epsilon_i, \eta_j$ are three parameters, which can be either independent or correlated with each other. The order parameter can be defined as $re^{i\Phi} = \frac{1}{N} \sum_{j=1}^N \eta_j e^{i\theta_j}$ and then one gets the same dynamics with the *virtual frequency* $\nu_i = (\omega_i - \Omega)/\epsilon_i$. The translated frequencies ν_i for the Kuramoto model are a simple example with

$\epsilon_i \equiv 1$. The self-consistent equation for the dynamics Eq. (7.20) reads

$$r = \frac{1}{N} \sum_{j=1}^N \eta_i (\langle \cos \theta_i \rangle + i \langle \sin \theta_i \rangle) \quad (7.21)$$

$$= \int_{-\infty}^{\infty} \rho_{\omega}(\omega) \eta (\langle \cos \theta \rangle + i \langle \sin \theta \rangle) d\omega \quad (7.22)$$

$$\equiv \int_{-\infty}^{\infty} \rho_{\nu}(\nu|\Omega) (\langle \cos \theta \rangle + i \langle \sin \theta \rangle) d\nu, \quad (7.23)$$

where $\rho_{\nu}(\nu|\Omega)$ is the distribution of *virtual frequencies* ν , and η is the *additional weight*. For the Kuramoto model, one gets $\rho_{\nu}(\nu|\Omega) = \rho_{\omega}(\nu + \Omega)$, while for various Kuramoto-like models $\rho_{\nu}(\nu|\Omega)$ combines all the information from all the three variables $\omega_i, \epsilon_i, \eta_j$. A summary of different Kuramoto-like models (Bi et al., 2016), (Bonilla et al., 1992), (Coutinho et al., 2013), (Daido, 1987), (Hong and Strogatz, 2011), (Liu et al., 2013), (Wang and Li, 2011), (Xu et al., 2018), (Yuan et al., 2014) and the corresponding virtual frequency distributions is given in Table 1.

Model	Virtual frequency
$\dot{\theta}_i = \omega_i + \frac{J}{k_i^{1-\eta}} \sum_j \frac{k_i k_j}{N \langle k \rangle} \sin(\theta_j - \theta_i)$	$\nu_i = (\omega_i - \Omega) / k_i^{\eta}$
$\dot{\theta}_i = k_i + \lambda \sum_j \frac{k_i k_j}{N \langle k \rangle} \sin(\theta_j - \theta_i)$	$\nu_i = (k_i - \Omega) / k_i$
$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_j \omega_j \sin(\theta_j - \theta_i)$	$\nu_i = (\omega_i - \Omega) / \langle \omega \rangle$
$\dot{\theta}_i = \omega_i + K \sum_j s_i s_j \sin(\theta_j - \theta_i)$	$\nu_i = (\omega_i - \Omega) / (\langle s \rangle s_i)$
$\dot{\theta}_i = \omega_i + \frac{\eta_i K}{N} \sum_j \sin(\theta_j - \theta_i)$	$\nu_i = (\omega_i - \Omega) / \eta_i$
$\dot{\theta}_i = \omega_i + \frac{K \omega_i }{N} \sum_j \sin(\theta_j - \theta_i)$	$\nu_i = (\omega_i - \Omega) / \omega_i $
$\dot{\theta}_i = \omega_i + \lambda \frac{ \omega_i }{k_i} \sum_j \frac{k_i k_j}{N \langle k \rangle} \sin(\theta_j - \theta_i)$	$\nu_i = (\omega_i - \Omega) / \omega_i $
$\dot{\theta}_i = \eta_i \alpha k_i^{\beta} + K \sum_j \frac{k_i k_j}{N \langle k \rangle} \sin(\theta_j - \theta_i)$	$\nu_i = (\eta_i \alpha k_i^{\beta} - \Omega) / k_i$
$\dot{\theta}_i = \omega_i \eta_i + (1 - \eta_i) k_i + K \sum_j \frac{k_i k_j}{N \langle k \rangle} \sin(\theta_j - \theta_i)$	$\nu_i = (\omega_i \eta_i + (1 - \eta_i) k_i - \Omega) / k_i$
$\dot{\theta}_i = \frac{k_i^{\alpha}}{\beta} + K \sum_j \frac{k_i k_j}{N \langle k \rangle} \sin(\theta_j - \theta_i)$	$\nu_i = (k_i^{\alpha} / \beta - \Omega) / k_i$
$\dot{\theta}_i = k_i + \eta_i + K \sum_j \frac{k_i k_j}{N \langle k \rangle} \sin(\theta_j - \theta_i)$	$\nu_i = (k_i + \eta_i - \Omega) / k_i$
$\dot{\theta}_i = \omega_i + \frac{K \omega_i^{\beta}}{N} \sum_j \sin(\theta_j - \theta_i)$	$\nu_i = (\omega_i - \Omega) / \omega_i^{\beta}$

Table 7.1: Kuramoto-like models

As the combination of various generalizations, the virtual frequency distribution $\rho_{\nu}(\nu|\Omega)$ provides the overall information of the model. If $\rho_{\nu}(\nu|\Omega)$ can be rewritten in the form $\rho_{\nu}(\nu|\Omega) = \rho_{\omega}(\nu + \Omega)$, the synchronization process of such system can be understood and analyzed easily through its counterpart of Kuramoto model with $\rho_{\omega}(\omega)$ as its distribution of natural frequencies. Other than these special cases, in general $\rho_{\nu}(\nu|\Omega)$ shows more complicated patterns. Specifically, with a given Ω , the system can be analyzed as the corresponding Kuramoto model with the same $\rho_{\nu}(\nu|\Omega)$. With the change of Ω , the system can show different properties for the nonlinear change of $\rho_{\nu}(\nu|\Omega)$ on Ω .

7.7 Conclusions

In conclusion, we have shown that with appropriate transformations, several oscillator systems on complex networks are transformed to the standard Kuramoto model on complete graphs with a re-arranged virtual frequency distribution. Such re-arranged distributions combine the effect of topology, dynamics, and their correlation, leading to a deeper intuitive understanding of the onset of synchronization. The analysis in this chapter is based on complete graphs or annealed complex networks assumption that works for random complex networks with a large mean-degree $\langle k \rangle \gg 1$ (Ichinomiya, 2005; Sonnenschein and Schimansky-Geier, 2012; Peron and Rodrigues, 2012). Our method can be generalized to more complicated cases, such as the partial degree-frequency correlation (Pinto and Saa, 2015) and the degree correlations (Sendiña-Nadal et al., 2015; Restrepo and Ott, 2014). Including such systems, we can obtain a more general framework of Kuramoto-like synchronization, where the models studied in this chapter are the linear cases.

Chapter 8

Cluster explosive synchronization

As discussed in Chapter 7, the explosive synchronization of Kuramoto oscillators has attracted much attention, where oscillators are coupled through a scale-free network with degree-related natural frequencies. Different from the Kuramoto oscillators, in second-order oscillators with inertias, a more complicated cluster explosive synchronization process is found. In this chapter, we study this cluster explosive synchronization in detail. Using the self-consistent method in Chapter 3 and the virtual frequency method of oscillators on complex networks in Chapter 7, we find the topology introduced inertia-degree correlations. The topology strengthens the inertia effect of oscillators with large degrees. Hence, this fraction of oscillators follows the typical cluster synchronization for second-order oscillators. Together with the explosive synchronization of oscillators with small degrees, the system gets the cluster explosive synchronization. Inspired by this inertia-degree correlation, we further propose the frequency-inertia correlation to control the synchronization processes. The explosive synchronization of second-order oscillators can be restored in scale-free networks with a proper correlation. Besides, we show that in all-connected oscillators with different frequency-inertia correlations, one can get cluster explosive, explosive, and inverse cluster explosive synchronization processes.

8.1 Introduction

Synchronization has been recognized as one of the most important phenomena in nature and engineering. The study of synchronization ranges from mathematics, physics to biology and social science (Arenas et al., 2008). Among all the models used in various studies, Kuramoto-type oscillator is one of the most important models (Kuramoto and Nishikawa, 1987; Rodrigues et al., 2016). Many interesting phenomena have been found, such as the explosive synchronization for Kuramoto oscillators in scale-free networks. All the oscillators get synchronized or desynchronized at critical coupling strengths, closely related to explosive percolations

and cascading failures (Gambuzza et al., 2017; Boccaletti et al., 2016). When the oscillators have nonzero inertias (second-order oscillators), a more complicated phenomenon has been found. The explosive process of synchronization is divided into two parts. The oscillators with lower degrees keep the abrupt property, while those with larger degrees get synchronized hierarchically after the critical coupling strength (Ji et al., 2013, 2014a). Besides, the de-synchronization process is not affected by the inertias and keeps abrupt and explosive. This phenomenon is called the cluster explosive synchronization (Ji et al., 2013).

In this chapter, we will try to explore the mechanism of cluster explosive synchronizations. Two methods are used, the self-consistent method (Gao and Efstathiou, 2018) and the virtual frequency method of oscillators on complex networks (Gao and Efstathiou, 2020). We show that cluster explosive synchronizations of second-order oscillators are from the degree-inertia correlation introduced by networks' topology. The inertial effect of oscillators with large degrees are strengthened. They follow the typical cluster synchronization of second-order oscillators. On the other hand, the oscillators with small degrees are mainly affected by the topology and the degree-frequency correlation, resulting in explosive synchronization. The cluster explosive synchronization is a combined phenomenon of these two transitions. Inspired by this mechanism, we introduce an additional inertia-degree correlation to balance topology's effect and obtain the explosive synchronization of second-order oscillators. This mechanism can also be applied to all-connected oscillators, where cluster explosive synchronization can be introduced through the frequency-inertia correlations.

This chapter is organized as follows. In Sec. 8.2, the model and self-consistent method are introduced as the basis of the analysis. In Sec. 8.3 we analyze the difference between explosive de-synchronizations and cluster explosive synchronizations, showing the intrinsic hysteresis feature of second-order oscillators. In Sec. 8.4, we get the rearranged distributions of virtual frequencies and the inertia-frequency correlation from distributed degrees of the network. An additional degree-inertia correlation is designed to balance such a topological effect on inertias and, consequently, introduce explosive synchronizations. We conclude this chapter in Sec. 8.5.

8.2 Model and self-consistent method

In complex networks, the model of coupled second-order Kuramoto-type oscillators reads

$$m\ddot{\varphi}_i + D\dot{\varphi}_i = \Omega_i + \lambda \sum_{j=1}^N A_{ij} \sin(\varphi_j - \varphi_i), \quad i = 1, \dots, N, \quad (8.1)$$

where m and D are the (common) inertia and damping coefficient, and Ω_i is the natural frequency of the i -th oscillator. The adjacency matrix A_{ij} describes the connection of oscillators. If there is a link between the oscillator i and j , we

have $A_{ij} = 1$, and $A_{ij} = 0$ otherwise. For uncorrelated networks with randomly picked links, when the number of nodes N and its mean-degree $\bar{k}$ are large enough the adjacency matrix can be approximated with the mean-field assumption $A_{ij} = k_i k_j / N \bar{k}$, where k_i is the degree of the i -th node (oscillator) (Ichinomiya, 2004; Peron and Rodrigues, 2012). With such mean-field assumption, the model reads

$$m\ddot{\varphi}_i + D\dot{\varphi}_i = \Omega_i + \lambda \sum_{j=1}^N \frac{k_i k_j}{N \bar{k}} \sin(\varphi_j - \varphi_i), \quad i = 1, \dots, N. \quad (8.2)$$

A generalized order parameter can be defined in complex networks as

$$r e^{i\psi} = \frac{1}{\bar{k}} \sum_{j=1}^N k_j e^{i\varphi_j}, \quad (8.3)$$

The radius r represents the coherence of the oscillators. The phase ψ describes the mean phase of the oscillators. Substituting the order parameter into Eq. (8.2), we have the mean-field form as

$$m\ddot{\varphi} + D\dot{\varphi} = \Omega - \lambda k r \sin(\varphi - \psi), \quad (8.4)$$

where the index i is neglected.

In this chapter, we are interested in the cases where the oscillator's natural frequency is assumed to be linear related to its degree $\Omega = Ak$. For the steady states of the oscillators, we have $r(t) = r$ and $\psi = \Omega^* t + \psi_0$. Assuming $\lambda k r m \neq 0$ and taking the rotation frame with $\theta = \varphi - \psi$, one gets the dynamics of a single oscillator as,

$$\ddot{\theta} + a\dot{\theta} = b - \sin(\theta), \quad (8.5)$$

where the time is rescaled as $\tau = t / \sqrt{m/k\lambda r}$, and

$$a = \frac{D}{\sqrt{kqm}}, \quad b = \frac{Ak - D\Omega^*}{kq}. \quad (8.6)$$

The dynamics Eq. (8.5) is the same as the one for second-order oscillators in complete graphs. Hence, from previous studies, each oscillator in the mean-field following Eq. (8.5) has two stable states, one fixed point and one limit cycle (Strogatz, 2014; Tanaka et al., 1997b,a; Gao and Efstathiou, 2018).

Supposing $a > 0$, the stable fixed point reads

$$\theta_0 = \arcsin(b), \quad \omega_0 = 0, \quad (8.7)$$

with the condition

$$b \leq b_L(a) = 1. \quad (8.8)$$

The limit cycle can be approximated as

$$\theta(\tau) = \frac{b}{a}\tau + \frac{a^4}{b(a^4 + b^2)} \left[\cos\left(\frac{b}{a}\tau\right) - 1 \right] + \frac{a^2}{a^4 + b^2} \sin\left(\frac{b}{a}\tau\right), \quad (8.9)$$

with the condition

$$b \geq b_S = \begin{cases} \frac{4}{\pi}a - 0.305a^3, & a \leq 1.193, \\ 1, & a > 1.193. \end{cases} \quad (8.10)$$

With $b_L \geq b_S$, there is a bistable region $b_L \geq b \geq b_S$ for these two solutions when $a < 1.193$. Depending on different initial states, all the oscillators are either at the fixed point or along the limit cycle, called the locked and running respectively.

Supposing $N \rightarrow \infty$, at the continuum limit we have a distribution of degree k , denoted as $g(k)$. The order parameter defined in Eq. (8.3) reads now

$$re^{i\psi} = \int_{\mathbb{R}} ke^{i\varphi(k)}g(k)dk. \quad (8.11)$$

Substituting of the solutions of locked and running oscillators, Eq. (8.7) and Eq. (8.9), into Eq. (8.11), we have the self-consistent equations as

$$r = \frac{1}{\bar{k}} \int_{\mathbb{R}} kg(k) \left[\rho_l \sqrt{1-b^2} - \rho_r \frac{a^2}{2(a^4+b^2)} \right] dk, \quad (8.12a)$$

$$0 = \frac{1}{\bar{k}} \int_{\mathbb{R}} kg(k) \left[\rho_l b + \rho_r \frac{a^4}{2b(a^4+b^2)} \right] dk. \quad (8.12b)$$

The fraction functions $\rho_l(k, \Omega)$ and $\rho_r(k, \Omega)$ describe the fraction of locked and running oscillators respectively, satisfying the normalization condition

$$\rho_l(k, \Omega) + \rho_r(k, \Omega) = 1, \quad (8.13)$$

and the boundaries

$$\mathbf{1}_S(k, \Omega) \leq \rho_{lock}(k, \Omega) \leq \mathbf{1}_L(k, \Omega). \quad (8.14)$$

The indicator functions $\mathbf{1}_{S,L}$ takes the value 1 if $|b| < b_S(a)$ or $|b| < b_L(a)$, corresponding to the condition for locked oscillators, and 0 otherwise.

Without loss of generality, in the following, we only focus on the two boundary cases, as $\rho_{lock}(k, \Omega) = \mathbf{1}_P(k, \Omega)$ with $P = S$ or $P = L$. The steady states within these two boundaries are all solutions of the self-consistent equations with proper fraction functions. Such two boundaries are also called the forward (with $\mathbf{1}_S(k, \Omega)$) and backward processes (with $\mathbf{1}_L(k, \Omega)$) for they are the steady states if one prepares the system in incoherence or complete synchronization states and increase or decrease the coupling strength progressively (Tanaka et al., 1997a,b).

Other than the trivial solution $r = 0$, the self-consistent equations can be easily solved through the form

$$\frac{1}{\lambda} = \int_{\mathbb{R}} \frac{kg(k)}{q\bar{k}} \left[\mathbf{1}_P \sqrt{1-b^2} - \frac{(1-\mathbf{1}_P)a^2}{2(a^4+b^2)} \right] dk, \quad (8.15a)$$

$$0 = \int_{\mathbb{R}} kg(k) \left[\mathbf{1}_P b + \frac{(1-\mathbf{1}_P)a^4}{2b(a^4+b^2)} \right] dk, \quad (8.15b)$$

as two equations of $q = \lambda r$ and Ω^r . For values (q_0, Ω_0^r) satisfying Eq. (8.15b), we can obtain the solution $\lambda_0(q_0, \Omega_0^r)$ from Eq. (8.15a) provided that $\lambda_0 > 0$. Since $q = \lambda r$, the triplet solution $(\lambda_0, q_0, \Omega_0^r)$ can be transformed to $(\lambda_0, r_0, \Omega_0^r)$ and locally parameterized in terms of λ as families $(r_0(\lambda_0), \Omega_0^r(\lambda_0))$, which are the steady states in general, except at points of bifurcation where the number of families changes.

8.3 Cluster explosive synchronization

The self-consistent equation Eq. (8.15) keeps invariant under the transformation $(q, \Omega^r, \lambda, A) \mapsto (q, -\Omega^r, \lambda, -A)$. There is a one to one correspondence between the solution (q, Ω^r, λ) with positive correlation factor A and the one $(q, -\Omega^r, \lambda)$ with a negative factor $-A$ for the system. The systems with positive and negative degree-frequency correlations follow the same dynamics. Without loss of generality, we assume $A = \bar{k}^{-1} > 0$ in the following. Consequently, we have $\Omega^r > 0$ from Eq. (8.15b).

From the view of micro synchronization processes, given (q, Ω^r) in the backward process with $\mathbf{1}_L$, we have the synchronization condition as

$$k(A - q) \leq D\Omega^r \leq k(A + q). \quad (8.16)$$

With $A > 0$, and consequently $D\Omega^r > 0$, Eq. (8.16) can be rewritten as

$$\frac{D\Omega^r}{A + q} \leq k \leq \frac{D\Omega^r}{A - q}, \quad \text{if } q < A, \quad (8.17a)$$

$$\frac{D\Omega^r}{A + q} \leq k, \quad \text{if } q > A. \quad (8.17b)$$

It is worth to note that when $q > q_1 = A$, there is no upper bound of oscillator's degree for the synchronization group. Hence when the lowest degree of oscillators k_0 satisfies the lower bound, all the oscillators will be synchronized, which gives the condition $q \geq q_2 = D\Omega^r/k_0 - A$. For the complete synchronization state, it is easily to obtain $D\Omega^r = A\bar{k}$ from the sum of the dynamical equations of all the oscillators. As a result, from q_1, q_2 the condition for complete synchronization state reads

$$q \geq q_c = A \max\left(\left(\frac{\bar{k}}{k_0} - 1\right), 1\right). \quad (8.18)$$

For scale-free networks, we have $k_0 = \bar{k}(\gamma - 2)/(\gamma - 1)$. Hence if $\gamma > 3$ for power-law distributions one has the same threshold as $q_c = A$. It is worthy to note that this condition does not depend on the choice of inertias. It is directly from the linear correlation between degrees and frequencies.

As for the forward process with $\mathbf{1}_S$, the dynamics are quite different. There is no threshold for the complete frequency synchronization. Instead, the system shows a slow expansion of the synchronization cluster with respect to q . To see

Figure 8.1: Backward and forward processes for $N = 10000$ oscillators with (left column) $m = m_0$ and (right column) $m = m_0 \bar{k} k^{-1}$ for oscillators on scale free networks $g(k) \sim k^{-\gamma}$ with $\gamma = 2.8$ and minimum degree $k_0 = 50$. The blue $\triangle$ (orange ∇) in panels (a,b) are the numerical results in the forward (backward) process showing the evolution of r with decreasing (increasing) λ . The error bars show the minimum and maximum values of r for each K ; oscillatory states correspond to large error bars. The dashed (solid) lines are from the self-consistent equation in the forward (backward) processes. The evolution of mean frequencies of oscillators $\langle \omega_i \rangle$ for the forward process (increasing λ) is shown in panels (c,d) and (e, f) respectively.

Figure 8.2: Backward and forward processes for $N = 1000$ oscillators with (upper row) $m = m_0 A_2 \Omega^2$, (middle row) $m = m_0$ and (lower row) $m = m_0 A_{-2} \Omega^{-2}$ for all-connected oscillators with natural frequencies chosen randomly in $[0.5, 1.5]$. The blue $\triangle$ (orange ∇) in the left column are the numerical results in the forward (backward) process showing the evolution of r with decreasing (increasing) K . The dashed (solid) lines are from the self-consistent equation in the forward (backward) processes. The evolution of mean frequencies of oscillators $\langle \omega_i \rangle$ for the forward process (increasing K) is shown in right column. The constant $A_{-2} = 1$ and $A_2 = 2$ is chosen to make the inertia regions comparable.

this point, assuming $q \gg 1$, one gets the boundary $b_S(a)$ approximated as $b_S(a) \approx (4/\pi)a$. We have the synchronization condition with $\mathbf{1}_S$ as

$$kA - \frac{4}{\pi}D\sqrt{\frac{q}{m}}k \leq D\Omega^r \leq kA + \frac{4}{\pi}D\sqrt{\frac{q}{m}}k. \quad (8.19)$$

From the left inequality, after some calculations we have that there is always a maximum of synchronized oscillators' degree

$$k_m = (\sqrt{D\Omega^r + \frac{16D^2q}{\pi^2mA}} + \frac{2D}{\pi A}\sqrt{\frac{q}{m}})^2. \quad (8.20)$$

The oscillators with $k > k_m$ belong to the running group. The value of k_m increases almost linearly with the increase of q when q is sufficient large with $D\Omega^r \approx Ak$. The complete synchronization state can only be obtained at the limit $q \rightarrow \infty$, quite different from backward processes with a finite threshold q_c .

By the self-consistent equations, the steady states are calculated in the forward and backward processes, as shown in Fig. 8.1 (a). In the forward process, due to the nonzero contribution of running oscillators, one has a folding of the steady states around the incoherence. As a result, with the increase of λ , there is a jump from the state with $q = 0$ to a state $q = q_0 > 0$, where a relatively large synchronized cluster abruptly appears from the incoherence states with the condition

$$kA - kq_0b_S(\frac{D}{\sqrt{kq_0m}}) \leq D\Omega^r \leq kA + kq_0b_S(\frac{D}{\sqrt{kq_0m}}). \quad (8.21)$$

After this jump, from the nonlinear boundary b_S , the synchronized cluster slowly grows with increasing q . Combining these two processes, we have the cluster explosive synchronization, shown in Fig. 8.1 (e).

8.4 Virtual inertia-frequency correlations

The cluster synchronization in the forward processes is from the effect of inertias. In our recent work (Gao and Efstathiou, 2020), the explosive synchronization of Kuramoto oscillators is studied intuitively with the virtual frequency method. Using the same method, for the second-order oscillators, with the reduction of oscillator dynamics on the complex networks to the one on complete graphs, one gets the virtual frequencies and the virtual inertias.

For the second-order oscillators, the reduction transformation of parameters reads

$$\Omega_n = \frac{A}{D\Omega^r}, q_n = \frac{q}{D\Omega^r}, \lambda_n = \frac{\lambda}{D\Omega^r}. \quad (8.22)$$

The virtual frequency ω_n and inertias m_n are defined as

$$\omega_n = \frac{1}{k}, m_n = mkD\Omega^r, \quad (8.23)$$

with which the self-consistent equation Eq. 8.15 can be transformed to the one in complete graphs

$$\frac{1}{\lambda_n} = \int_{\mathbb{R}} \frac{g_n(\omega_n)}{q_n} \left[\mathbf{1}_P \sqrt{1-b^2} - \frac{(1-\mathbf{1}_P)a^2}{2(a^4+b^2)} \right] d\omega_n, \quad (8.24a)$$

$$0 = \int_{\mathbb{R}} g_n(\omega_n) \left[\mathbf{1}_P b + \frac{(1-\mathbf{1}_P)a^4}{2b(a^4+b^2)} \right] d\omega_n, \quad (8.24b)$$

with

$$a = \frac{D}{\sqrt{q_n m_n}}, \quad b = \frac{\Omega_n^r - \omega_n}{q_n}. \quad (8.25)$$

The new defined natural frequencies ω_n have a distribution

$$g_n(\omega_n) = \frac{1}{k} \frac{1}{\omega_n^3} g_k\left(\frac{1}{\omega_n}\right), \quad (8.26)$$

which is independent of inertia and is the same as the one for Kuramoto oscillators with $m = 0$, resulting in the explosive de-synchronization process.

However, with the transformation Eq. (8.23), one gets the new defined inertia m_n proportional to the oscillator's degree k . The topological effect improves the weight of oscillators with a large degree and improves their inertias. With the transformation to complete graphs, one has the inertia-frequency correlations

$$m_n = m D \Omega^r / \omega_n. \quad (8.27)$$

This is the reason for the appearance of cluster explosive synchronizations from the view of complete graphs.

From the independence of $b_L \equiv 1$ on m_n , in the upper branch, the de-synchronization transition of the second-order oscillators is mainly based on the natural frequencies' distribution $g_n(\omega_n)$ which is the same for Kuramoto oscillators. One has the similar explosive de-synchronization transitions as the one for Kuramoto oscillators. On the other hand, in the forward processes where the inertias m_n are crucial in the boundary b_S , the oscillators with larger inertias need a large coupling strength to synchronize with the others. In this case, one finds the cluster synchronization processes, quite different from the one of Kuramoto oscillators.

The cluster explosive synchronization results from the virtual frequency-degree correlation Eq. (8.27). As a matter of fact, with inertia-degree correlations we can either strengthen or weaken such virtual frequency-degree correlation. Specifically, Eq. (8.27) can be balanced by an inertia-degree correlation as $m_i = m/k_i$. The virtual inertias m_n reads

$$m_n = m D \Omega^r, \quad (8.28)$$

which is a constant for all the oscillators.

In this case, when $q \gg 1$ in the forward process we have the synchronization condition as

$$k(A - \frac{4}{\pi}D\sqrt{\frac{q}{m_0}}) \leq D\Omega^r \leq k(A + \frac{4}{\pi}D\sqrt{\frac{q}{m_0}}). \quad (8.29)$$

Similar as the analysis before, we have the threshold q'_c for the the the complete synchronization state in the forward process as

$$q > q'_c = \max((\frac{\bar{k}}{k_0} - 1)^2, 1) \cdot \frac{16A^2}{\pi^2} \frac{m_0}{D^2}. \quad (8.30)$$

In this case, both the backward and forward processes follow the similar dynamics as the Kuramoto oscillators.

Using the self-consistent equations, with inertia-degree correlations, one gets two similar transition processes with b_S and b_L in Fig. 8.1 (b). The explosive cluster synchronization process in the forward process is replaced by an explosive one, as shown in Fig. 8.1 (f).

While the backward processes are kept the same, the forward transition processes of second-order oscillators are controlled by the inertia-frequency correlations, see Fig. 8.1 (c-f). In principle, if we can change the negative correlation Eq. (8.27) to a positive one through a much stronger inertia-degree correlations, we can observe an inverse cluster explosive synchronizations. The oscillators with large degrees synchronized abruptly at a critical point while the other oscillators are synchronized hierarchically. However, the non-symmetry of $g_n(\omega_n)$ Eq. (8.26) makes this phenomenon hard to be observed.

To make the effect of inertia-frequency correlations clear, consider the oscillators in all-connected networks with uniform distributions of natural frequencies, with the inertia-frequency correlations $m_i = mA_\alpha\omega_i^\alpha$, as shown in Fig. 8.2. The constant factor A_α is added to make the systems comparable for different α . When $\alpha = 0$, one gets an explosive synchronization similar to Kuramoto oscillators, in Fig. 8.2(b). When $\alpha > 0$ the effective inertias m_i is positively correlated to ω_i and one gets the cluster explosive synchronization processes in Fig. 8.2(a). There is a lower bound of the synchronized cluster. On the contrary, when $\alpha < 0$, the inertia-frequency correlation is negative, one gets an upper bound of frequencies in the synchronized cluster. All the oscillators with larger frequencies than such bound get synchronized abruptly at a critical point while the other oscillators will be synchronized hierarchically after it in the process of increasing coupling strength. The inertia-frequency correlations determine the changing of the synchronization order of oscillators.

8.5 Conclusions

In this chapter, we study the cluster explosive synchronization processes of second-order oscillators in scale-free networks. Using the self-consistent method, we analyzed the synchronization conditions in both forward and backward processes. The

appearance of the cluster explosive synchronization is found from the cross-effect of topology and inertias. When the inertias' effect is weakened in the backward process, we have the explosive de-synchronization from the topology's effect and frequency-degree correlation.

To explore topology and inertias' cross-effect, we use the virtual frequency method of oscillators on complex networks, transforming the system to all-connected oscillators. Other than the convex distributions of natural frequencies, the topology introduced inertia-degree(frequency) correlation is found. The oscillators with large degrees (low virtual frequencies) also have large inertias, resulting in a large threshold of coupling strength to be synchronized.

Inspired by the topology introduced inertia-frequency correlations, we introduce additional inertia-frequency correlations to control the synchronization processes. First, we can balance such topology by introducing correlations and observing the explosive synchronizations of second-order oscillators in scale-free networks. Second, we reproduce the cluster explosive, explosive, and inverse cluster explosive synchronization in all-connected oscillators with different inertia-frequency correlations.

Chapter 9

Future Work

In this project, we've explored several aspects of the second-order oscillators' synchronization processes, and developed some methods to do that. Besides these results, we also have many questions and unfinished work, some of which are quite interesting and I would try to finish them in the future.

Kuramoto-like models

The first part of the future work should be Kuramoto-like models. We have introduced the generalized self-consistent method in Chapter 2 and used it in Chapter 5 to find the cascading formation. Besides, the virtual frequency analysis based on the self-consistent method in Chapter 7 is a powerful tool to explore the cross-effect of different parameters, as we show in Chapter 8. The models' differences can be explored directly through the self-consistent equations with these two methods. Hence, we start to consider some questions as:

1. Can we classify the Kuramoto-like models based on their self-consistent equations, corresponding to their synchronization processes?
2. Can we design a Kuramoto-like model to show a particular synchronization process?

To answer these questions, I will try to define the systems through the self-consistent equations, not the opposite way from the coupled oscillators. In this way, we have found that the classic Kuramoto models are the linear models, and there is quite an open area of the non-linear models need to be explored.

Effect of inertias

The second part of the future work are about the effect of inertias. We have analysed many aspects of such effect of inertias, such as the hysteresis, oscillatory

states, cascading synchronization, cluster-explosive synchronization, and oscillating synchronization processes. But there are still a lot of open questions, among which the most important ones are:

1. What is the effect of inertias of oscillators in complex networks?
2. How is the synchronization analysis we did related to the stability and control of such models?

To explore a single parameter comprehensively in a model is not an easy job, especially when such model has so many variations. As a matter of fact, even the most straightforward parameter, natural frequency, has not been studied completely clearly. So the key point here is not to study the open problems one by one, but to identity the important ones and find a systematic approach.

Self-organization process

The third-one is the more general topic, self-organization process. Although it takes our four years to obtain the results shown in this thesis, but the model we considered is simple in the domain of synchronization. One can study the complex networks as the links of oscillators, and also more complicated oscillators such as the chaotic ones. The more explicated models you considered, the richer dynamical behaviours you can find. However, in all kinds of these models, you can always find synchronization. So, after we explored the synchronization of Kuramoto-like models, it is naturally to ask the questions as:

1. Can we apply our methods to more complicated models, with either complicated networks or complicated dynamics of single oscillators?
2. Are there some simple but general rules under all these kinds of synchronization phenomena?

To answer these questions, I think we should combine the techniques of data science with the analytical methods we relied on in this thesis.

Specifically, let's take the numerical simulations of coupled oscillators as multi-dimensional time series. One can apply various methods based on the time series' analysis, such as clustering and classification methods for the clusters' forming and topological data analysis for the critical transitions to synchronization. This methods are independent of models. By combing the analysis from dynamical system theory, statistical mechanics which have developed for each specific models, I think we could build a set of methods to study various models of coupled units and find the general mechanism of synchronization. Analytical analysis of simple models give us the ideas and the model-free data-based methods provide the tools to check them in a general setting. Moreover, in this way, more details of synchronization will be explored, other than the several states people defined artificially.

Relation to statistical mechanics

We have mentioned the relationship between the second-order oscillator and Hamiltonian mean-field model in Chapter 1. One of our main research goals is relating the dynamical properties of single oscillators to the collective phenomena of synchronization, which is similar to the statistical mechanics' topics. From this point of view, we can ask the questions as:

1. Can we apply the methods from statistical mechanics to study the self-organization processes in the synchronization?
2. Can we apply the methods from the synchronization analysis to study some questions in statistical mechanics, relating the finite-size cases and infinite ones?

We didn't go far in this direction, but I think some of the recently developed statistical mechanics can be generalized and applied to our models, such as the non-extensive statistical mechanics for long-range coupling and small systems. On the other hand, the coupled Hamiltonian systems that statistical mechanics deals with are also complex systems with simple collective behaviours. Hence, our experience of exploring the coupled oscillators with their collective synchronization could also be generalized to the study such coupled Hamiltonian systems.

As a matter of fact, these complex systems are not too complicated to be understood, especially when they show quite simple collective behaviours. Is it possible that there are some intuitive and simple rules under such simple collective behaviours? Can we build a systematic approach to find it in various systems?

Summary

In this project, we focus on how the dynamics of single oscillators affect the synchronization process of the whole system. There are two frameworks that can be used to answer this question: dynamical systems theory and statistical mechanics. However, both of these have clear obstacles. On the one hand, from the point of view of dynamical systems theory, the system is defined as N coupled dynamical equations. When N is large, with $N = 10000$ being a realistic number for applications, it is impossible to analyze such a dynamical system directly. On the other hand, from the point of view of statistical mechanics, the system is defined as the evolution of a density distribution. Due to the long-range coupling and network structure, the system is not in statistical equilibrium, and the theory of statistical mechanics in these cases is also an open question. As a result, even though synchronization is an ubiquitous phenomenon in nature and engineering, it is still not an easy job to completely understand all the intricacies of the problem. In particular, there is no method that describes all aspects of synchronization. Instead, we have several different methods, each one being applicable to different parts of the problem, while several questions remain open.

In this project we face these challenges directly. Specifically, we extend the previous version of the self-consistent method to study coupled second-order oscillators and generalize it beyond the case of steady states of infinitely many oscillators, to multi-cluster states, networks of finitely many oscillators, and cascading synchronization. This generalized self-consistent method covers a large fraction of systems typically considered in the study of synchronization and serves as the basis of a unified framework. Moreover, we show how using certain symmetries of the systems under consideration, we can transform the self-consistent equations to ones for systems with different distributions of natural frequencies and inertias. This transformation can be further applied to systems with complicated coupling functions, correlations between parameters, and different network topologies. It provides a systematic approach to explore the cross-effect of various factors and intuitive and straightforward explanations for several surprising phenomena, such as explosive synchronization, cluster-explosive synchronization, and standing wave states.

Combined with the dynamical analysis, this new framework provides us a powerful tool to explore the relation between the microscopic oscillators' dynamics and

the macroscopic synchronization phenomena. Specifically, we explore the effect of the second-order oscillators' inertia on their synchronization and relate their dynamical behavior to specific synchronization phenomena, such as hysteresis due to bi-stability, appearance of different synchronization transitions and oscillatory states due to internal phase shifts, cascading formation of clusters due to homoclinic bifurcations, and explosive cluster synchronization due to virtual inertia-frequency correlations. Combining all these results, we obtain a comprehensive understanding of inertia's effect on the synchronization of coupled oscillators.

Moreover, in the study of second-order oscillators, we find two new synchronization-related phenomena. The first one is the cascading formation of synchronization clusters. Here, the synchronized oscillators form a cluster that affects the system's dynamics and leads to the formation of additional synchronized clusters. This process can lead to a cascade of new synchronized clusters. This phenomenon does not appear in the Kuramoto model due to its specific synchronization condition. However, the inertias in second-order oscillators modify the synchronization condition, leading to the cascading formation of synchronized clusters, which is analyzed through the generalized self-consistent method. The second phenomenon is the oscillating synchronization process. In the presence of phase-shifts, their cross-effect with inertias can affect the system dynamics so that it does not converge to the complete synchronization state as the coupling strength increases. Instead, the system divides into several synchronization clusters, whose frequency differences grow with the coupling strength. These results demonstrate that the intuitive belief that all oscillators will synchronize for a sufficient large coupling strength is erroneous since the appearance of synchronized clusters modifies the dynamics in such a way that results in more clusters.

Synchronization of phase oscillators, serving as one of the main themes in the study of the collective behavior of coupled units, has attracted a lot of attention from the dynamical systems, control theory, neuroscience, and other communities. We believe that the generalized self-consistent method that we introduced in this work will have broad applications in the study of the Kuramoto model and the coupled second-order oscillators model. A further generalization and combination of the self-consistent method with other methods, such as stability analysis, inter-cluster dynamics, and resonance analysis, appear to be promising research directions.

The synchronization of phase oscillators is a simple phenomenon which is highly non-trivial to comprehensively understand. Even though our generalized self-consistent method provides a unified framework to study synchronization in mean-field cases, new methods, such as geometry analysis, non-equilibrium statistical mechanics, and data-driven analysis, could be useful for more complicated systems. With additional effort, I believe that a complete set of methods to study synchronization will be developed and we will finally be able to understand this ubiquitous phenomenon in nature and engineering.

Samenvatting

In dit project ligt de focus op hoe de dynamica van een enkele oscillator het synchronisatie proces van het hele systeem beïnvloeden. Er zijn twee manieren om deze vergelijking te beantwoorden: de dynamische systeemtheorie, het systeem is gedefinieerd als N gekoppelde dynamische vergelijkingen. Als N groot is, normaal gesproken $N = 10000$ in de simulatie, is het onmogelijk om direct zo een dynamisch systeem te analyseren. Aan de andere kant, van statistische mechanica, het systeem is gedefinieerd als een evolutie van dichtheid distributie. Met lange afstand koppelen en gecompliceerde structuren, het systeem is niet in statistisch evenwicht, en de theorie van statistische mechanica in dit geval is ook een open vraag. Als een resultaat, ook al is de synchronisatie een Alomtegenwoordigheid in de natuur en ingenieurswerk., is het nog steeds niet een makkelijk taak om het probleem dat we hebben op te lossen. In de studie van synchronisatie van fase oscillators, hebben we niet een complete set van methodes om het te bestuderen. Alleen een klein gedeelte van de synchronisatie, behandeld apart door meerde methodes, is duidelijk voor ons.

In dit project, om de vraag hierboven te beantwoorden, ontwikkelen en gebruiken we een zelf-consistente methode in de studie van Kuramoto en tweede-order oscillators. Wij corrigeren de eerdere versie van de zelf-consistente methode om tweede-order oscillators te bestuderen en het te generaliseren voor multi-cluster staten, eindige hoeveelheid oscillators en trapsgewijze synchronisatie. Deze gegeneraliseerde zelf-consistente methode bedekt een groot gedeelte van systemen met dezelfde zelf-consistente vergelijkingen. Daarnaast, door de symmetrie van systemen met dezelfde zelf-consistente vergelijkingen, transformeren we het model zijn variaties naar systemen met andere verdelingen van natuurlijke frequenties en traagheid. Deze transformatie bedekt het model met ingewikkelde gekoppelde functies, links tussen parameters en netwerkstructuren. Verschillende verrassende fenomenen worden intuïtief en rechte doorzee met deze benadering, zoals de explosieve en cluster-explosieve synchronisatie en staande golfstaten, Met deze twee benaderingen, ontdekken we het effect van traagheid op de synchronisatie van tweede-order oscillatoren en relateren we de dynamische kenmerken aan het synchronisatie fenomeen., zoals hysteresis, verandering van overgang trillende staten, trapsgewijze ontstaan van clusters en explosieve cluster synchronisatie.

Bovendien, in de studie van tweede-order oscillatoren, vinden we twee nieuwe

fenomenen van synchronisatie. De eerste is het trapsgewijs ontstaan van synchronisatie clusters, dit is een gemist punt in eerdere onderzoeken naar fase oscillatoren. Wij vonden dat gesynchroniseerde oscillatoren de staat van het systeem veranderen en de nieuwe gesynchroniseerde clusters kunnen ontstaan uit het nieuwe systeem. Dit proces kan doorgaan als een waterval. Voor eerdere studies van het Kuramoto model zijn deze effecten beschreven door specifieke synchronisatie voorwaarden. Door het traagheid effect ontstaat een andere synchroniseer conditie en daardoor vinden we dit fenomeen en generaliseren de zelf-consistente methode om het te analyseren. Het tweede fenomeen is het trillende synchronisatie proces. Door het kruis-effect van traagheid en fase verschuivingen, kan het systeem niet convergeren naar de complete synchronisatie staat met de gekoppelde kracht toename. In plaats daarvan verdeelt het systeem zich in verschillende synchronisatie clusters., wiens frequentie verschil groeit met koppelings kracht. Hierdoor, met alleen de toename van koppeling kracht kan het systeem niet meet synchroniseren maar heeft grotere gaten van frequenties.

Synchronisatie van fase oscillatoren, functionerende als een van de grote onderwerpen in de studie van collectief gedrag van gekoppelde voorwerpen, heeft veel aandacht getrokken van de dynamische systemen, controle theorie, neurowetenschap en andere gemeenschappen. Voor toekomstige werken geloven wij dat de zelf-consistente methode die we gegeneraliseerd hebben in dit project brede toepassingen heeft in de studie van het kuramoto model en het tweede-order model. Een verdere generalisatie en combinatie van de zelf-consistente methode met andere methodes zijn veelbelovend, zoals de stabiliteit analyse, inter-cluster dynamiek en resonantie analyse van oscillatoren. In het studeren van synchronisatie moeten er veel misverstanden gecorrigeerd worden, zoals het intuïtieve geloof dat alle oscillatoren synchroniseren als de koppeling kracht groot genoeg is en het gemiste punt dat de gesynchroniseerde clusters kunnen veranderen en resulteren in meer clusters.

Als laatste wil ik er graag op wijzen dat de synchronisatie van fase-oscillatoren simpel zijn als fenomeen maar niet triviaal om te studeren. Het ontbreken van rigoureuze methodes is het grootste obstakel. Ik denk dat meer moderne analyse geïntroduceerd zou moeten worden in dit veld voor toekomstige onderzoeken, zoals geometrie analyse, niet-evenwichtige statistische mechanica en data-georiënteerde analyse. Deze methoden zullen ons helpen het synchronisatie proces beter te begrijpen.

List of Publications

The publications related to the PhD work are listed below.

- **(Chapter 2)** J. Gao, K. Efstathiou. "From infinite to finite, unimodal to multimodal: A unified framework of Kuramoto-like models" *to be submitted*
- **(Chapter 3)** J. Gao, K. Efstathiou. "Self-consistent method and steady states of second-order oscillators." *Physical Review E* 98.4 (2018): 042201.
- **(Chapter 4)** J. Gao, K. Efstathiou. " Oscillatory states, standing waves, and additional synchronized clusters in networks of second-order oscillators: uncovering the role of inertia" *Submitted*
- **(Chapter 5)** J. Gao, K. Efstathiou. "Cascading self-organizing synchronization of coupled Kuramoto oscillators with inertia." *to be submitted*
- **(Chapter 6)** J. Gao, K. Efstathiou. "Synchronization of coupled second-order Kuramoto-Sakaguchi oscillators." *submitted*
- **(Chapter 7)** J. Gao, K. Efstathiou. "Reduction of oscillator dynamics on complex networks to dynamics on complete graphs through virtual frequencies." *Physical Review E* 101.2 (2020): 022302.
- **(Chapter 8)** J. Gao, K. Efstathiou. "Topological effect and cluster explosive synchronization of second-order oscillators." *to be submitted*

Acknowledgements

Being a scientist is my dream when I was a child. At this moment, I am finally getting there, becoming a doctor, and starting my career as a scientist. During this whole journey, I feel so lucky and happy.

Konstantinos, I remember the first time I met you at the Beijing Airport in 2016. I was quite nervous and introduced to you my master project with several papers at a coffee shop. You asked me about my contribution to such articles and said it was a good job. From then on, you became my supervisor, and we worked together for more than four years. In the first month after I came to Groningen, you sent me a book about how to be a supervisor of Chinese students. I knew that I could rely on you for any problems in the project, just as I did in these four years. You told me how to make sure that the readers could go through our papers smoothly, be a referee, reply to emails from unknown researchers interested in our work, and do a presentation or prepare a poster. I learned quite a lot from you, Konstantinos. Thank you very much!

Henk, I feel lucky to have you as my supervisor and promoter. We had the first meeting in the sixth month after I came to Groningen. I was quite unconfident at that time about my mathematics skills, writing that I have limited mathematics skills in the first R&D interview form. You told me no one knows everything and I already did a good job. You said I could always come to your office and share some interesting points in my research. I appreciate it very much, though I lack the courage to do that. The first time I did a poster presentation at the Bernoulli Institute, you came to me and voted for my poster, smiling and saying "kind bias." In my office, I can always hear your hearty laughter. You made me feel that it is quite happy and lucky to be in this research group. Thank you, Henk!

I want to thank the assessment committee members: Prof. G. Vegter, Prof. M. Cao, and Prof. Z. Liu, for going through my thesis and your kind suggestions and comments.

Many thanks to all the staff and all my colleagues in the Bernoulli building. In particular, I would like to thank Dirk, Mahdi, Nikolay, Pan Li, Pariya, Reza, Vladimire, Benjamin, Chengtao, Eric, Jiajia Jia, Junjie Jiao, Lorenzo, Markus, Georg, Gariel for creating the pleasant, friendly, and stimulating atmosphere. Special thanks should be given to Yongjiao, Bohuan, and Shuaipeng. We discussed a lot about mathematics, researching, and many other things. From you guys,

I learned how a mathematician works and thinks, which are quite different from my physicist view. I realized that maybe I would never become a good mathematician like you, but I can learn to work with you and appreciate the beauty of mathematics. Besides the academic life in the Bernoulli building, I wish to thank Lulu and Xukai in Groningen and Can Xu in Xiamen for sharing your ideas and projects with me. You bring me excellent opportunities to learn something new.

There are more people to mention who gave me a wonderful experience in Groningen, my friends and neighbors, Shun Fang, Yafeng Song, Shuai Feng, Weiteng Guo, Mengfan Zhang, Qian Chen, Xiu Jia, Yuan Fang, Beibei Guo, Yi Yu, Jiajia Jia, Lulu Gong, Jianjun Liu, Chongnan Ye, Yuhao Luo, Haoqi Chen, Qian Wang, Xukai Zhang, Heng Zhang, Xiaotian Zhu, Ling Liu. With you guys, I never felt lonely in this city. Special thanks to Ids for translating the summary and abstract of this thesis.

Lastly, I would like to thank my parents and my girlfriend Rongjing for your love, support, and continuous encouragement. I don't know how to express my gratitude to you. I love you!

Jian Gao
Groningen
January, 2021

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