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Topics in Inhomogeneous Bernoulli Percolation

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A study of two models

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Onderwerpen in inhomogene Bernoulli-percolatie

Een studie van twee modellen

Tópicos em Percolação de Bernoulli não-homogênea

Um estudo de dois modelos

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Introduction

The object of study of this thesis. Percolation Theory is a well-established discipline, having its mathematical roots back in 1957, with the seminal work of Broadbent and Hammersley [6]. Initially proposed as a model for the transport of fluid through a porous medium, the theory has evolved up to the present days, reaching a wide range of topics, specially in mathematics and physics. From the physicist's standpoint, it has been applied to the study of disordered physical systems, such as random electrical networks, the modeling of epidemics and oil recovery. As for the mathematician's point of view, it has become the source of elegant and challenging problems in combinatorics, probability theory, graph theory and analysis.

This thesis is an investigation of some mathematical aspects of inhomogeneous Bernoulli bond percolation on two different graphs $G = (V, E)$; in each of them, we consider a decomposition $E' \cup E''$ of the relevant edge set E and, given $p, q \in [0, 1]$, we assign parameters p and q to the edges of E' and E'' , respectively. In such a formulation, one of the sets, say E'' , is regarded as the set of inhomogeneities. In our study, we analyze, in both models, some properties of the critical curve $q \mapsto p_c(q)$ (or $p \mapsto q_c(p)$), where $p_c(q)$ is the supremum of the values of p for which percolation with parameters p, q does not occur. In one of the models, we also prove the uniqueness of the infinite cluster in the supercritical phase.

The above-mentioned topics are of intrinsic interest. Perhaps one of the earliest works concerning the behavior of critical curves is due to Kesten, presented in [19]. Considering the square lattice $\mathbb{L}^2 = (\mathbb{Z}^2, E^2)$ and choosing E'' and E' to be respectively the sets of vertical and horizontal edges, he proves that $p_c(q) = 1 - q$. Later on, Zhang [25] also considers the square lattice, but with the edge set E'' being only the vertical edges within the y -axis and $E' = E^2 \setminus E''$. He proves that, for any $q < 1$, there is no percolation at $p = 1/2$, which implies that $p_c(q)$ is constant in the interval $[0, 1)$. In the context of long-range percolation, de Lima, Rolla and Valesin [9] consider an oriented, d -regular, rooted tree $T_{d,k}$, where besides the usual set of “short bonds” E' , there is a set E'' of “long edges” of length $k \in \mathbb{N}$, pointing from each vertex x to its d^k descendants at distance k . They show that $q \mapsto p_c(q)$ is continuous and strictly

decreasing in the region where it is positive. This conclusion is also achieved by Couto, de Lima and Sanchis [8], where the authors consider the slab of thickness k induced by the vertex set $\mathbb{Z}^2 \times \{0, \dots, k\}$, with $\mathbb{E}'$ and $\mathbb{E}''$ being respectively the sets of edges parallel and perpendicular to the xy -plane. As for the number of infinite clusters in a supercritical percolation configuration, this is one of the most basic questions studied in percolation theory. For invariant percolation on the d -dimensional lattice, major contributions in proving the uniqueness of the infinite cluster are those of Aizenman, Kesten and Newman [1] and Burton and Keane [7]. An extension of the latter's argument to more general graphs can be found in the book of Lyons and Peres [20].

Overview of this thesis. This thesis consists of three chapters, each of them containing one main result. In Chapter 1, we present an extension of the recent work of Szabó and Valesin [23], published in [10]. In [23], the authors have proved the continuity of the critical curve $q \mapsto p_c(q)$ on the interval $(0, 1)$, when $G = (V, E)$ is the graph induced by the cartesian product between an infinite and connected graph $G = (V, E)$ and the set of integers $\mathbb{Z}$, the set $\mathbb{E}''$ is obtained by selecting finite subsets $V' \subset V$, $E' \subset E$ and defining

$$\mathbb{E}'' = (\cup_{u \in V'} \{(u, n), (u, n+1)\} : n \in \mathbb{Z}\}) \cup (\cup_{\{u, v\} \in E'} \{(u, n), (v, n)\} : n \in \mathbb{Z}\}),$$

and $\mathbb{E}' = E \setminus \mathbb{E}''$. In Chapter 1 ([10]), we extend their result, in the sense that the continuity of $q \mapsto p_c(q)$ still holds if V' and E' are infinite sets, provided that the set $V' \cup (\cup_{e \in E'} e)$ do not possess arbitrarily large connected components in G , and the graph-theoretic distance between any two such components is bigger than 2. This is achieved through the construction of a coupling which allows us to understand how a small change in the parameters p and q of the model affects the percolation behavior and is largely based on the ideas of [9] and [23].

Chapters 2 and 3 are devoted to the study of inhomogeneous percolation on $\mathbb{Z}^d$, $d \geq 3$, with a sublattice of inhomogeneities. In this setting, the graph G is the d -dimensional lattice $\mathbb{L}^d = (\mathbb{Z}^d, E^d)$, the set $\mathbb{E}''$ is the set of edges within the subspace $\mathbb{Z}^s \times \{0\}^{d-s}$, $2 \leq s < d$, and $\mathbb{E}' = E^d \setminus \mathbb{E}''$. Some properties of this model have already been addressed by Iliev, Janse van Rensburg and Madras [18]. Besides several classical results that have been transferred from the homogeneous to the inhomogeneous percolation setting, the authors have proved that the critical function $p \mapsto q_c(p)$ is strictly decreasing in the interval $[0, p_c(d)]$, where $p_c(d) \in (0, 1)$ here denotes the critical point for percolation on $\mathbb{L}^d$ in the homogeneous case. This is particularly interesting since it shows the existence of parameters (p, q) , $p < p_c(d) < q < p_c(s)$, for which there is an infinite cluster almost surely. To complement [18], we provide the

following additional results:

In Chapter 2, we prove the uniqueness of the infinite cluster in the supercritical phase. As we shall discuss further, the lack of invariance of the percolation measure under a transitive group of automorphisms of $\mathbb{L}^d$ plays against a direct application of the existing techniques of [1, 7, 20]. We will then explore some other properties of our model, so that we can overcome this issue and conveniently adapt the known arguments to prove the uniqueness of the infinite cluster in our case.

In Chapter 3, we address the problem of whether for any $p \in [0, p_c(d))$, the critical point $(p, q_c(p)) \in [0, 1]^2$ can be approximated by the critical point of the restriction of the inhomogeneous process to a slab $\mathbb{Z}^2 \times \{-N, \dots, N\}^{d-2}$, for large $N \in \mathbb{N}$. Here, the classical work of Grimmett and Marstrand [13] serves as the standard reference for providing the building blocks that give an affirmative answer to this question. We shall see that, since we are dealing with a supercritical regime of parameters (p, q) , where $p < p_c(d) < q < p_c(s)$, the construction of a suitable renormalization process for our case possesses some particularities that contrast with the usual approach of [13]. As a consequence, the finite-size criterion used in the construction of long-range connections should be modified accordingly, which in turn introduces some technical obstacle in the renormalization procedure that must be properly dealt with.

Basic definitions and notations. The requirements for the reader to follow this study are the knowledge of elementary probability theory, real analysis and some concepts from graph theory and ergodic theory. Fundamental definitions that are common to every chapter of this text are introduced in the following. A detailed account on percolation theory can be found in the book of Grimmett [14].

We begin with some terminology from graph theory. We say that $G = (\mathbb{V}, \mathbb{E})$ is a **graph** with vertex set $\mathbb{V}$ and edge set $\mathbb{E}$ if $\mathbb{V}$ is a non-empty countable set and $\mathbb{E}$ is a subset of the family of subsets of $\mathbb{V}$ with two elements. For example, the d -dimensional lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ is the graph with vertex set $\mathbb{Z}^d$ and edge set $\mathbb{E}^d := \{\{x, y\} \subset \mathbb{Z}^d : \|x - y\|_1 = 1\}$, where $\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i|$. If $G' = (\mathbb{V}', \mathbb{E}')$ is a graph, $\mathbb{V}' \subset \mathbb{V}$ and $\mathbb{E}' \subset \mathbb{E}$, we say that G' is a **subgraph** of G . Given $x, y \in \mathbb{V}$, a **path** $\pi(x, y)$ from x to y on G is a set of distinct vertices $\pi(x, y) = \{x = v_0, v_1, \dots, v_m = y\} \subset \mathbb{V}$, such that $\{v_i, v_{i+1}\} \in \mathbb{E}$ for every $i = 1, \dots, m$. Denoting by $\mathcal{P}(x, y)$ the set of paths from x to y , the graph-theoretic **distance** between x and y on G is the number $\text{dist}_G(x, y) := \inf\{|\pi(x, y)| : \pi \in \mathcal{P}(x, y)\} - 1$. That is, $\text{dist}_G(x, y)$ is the number of edges within the shortest path from x to y . The **degree** of a vertex $x \in \mathbb{V}$ is the number $\deg(x) := |\{y \in \mathbb{V} : \{x, y\} \in \mathbb{E}\}|$. If $\deg(x) = 0$, we say that x is an **isolated vertex**. The graph $G = (\mathbb{V}, \mathbb{E})$ is said to have **bounded degree** if there is some $k \in \mathbb{N}$ such that $\deg(x) \leq k$ for every $x \in \mathbb{V}$.

Next, we introduce the relevant definitions for working with inhomogeneous Bernoulli bond percolation. Briefly, a percolation process on a graph $G = (V, E)$ is defined as a probability measure on the set of subgraphs of G . Among the many possible variants, we shall work with bond percolation models, in which every edge of E can be **open (retained)** or **closed (removed)**, states represented by 1 and 0, respectively. Thus, a typical percolation configuration is an element of $\Omega = \{0, 1\}^E$; this set can be regarded as the set of subgraphs of G induced by their open edges. That is, an element $\omega \in \Omega$ is associated with the subgraph $(V(\omega), E(\omega))$, where $E(\omega) = \{e \in E : \omega(e) = 1\}$ and $V(\omega) = \{x \in V : \exists e \in E(\omega) \text{ such that } x \in e\}$, and conversely, a subgraph $(V', E') \subset G$ with no isolated vertices induces the configuration $\omega \in \Omega$, given by $\omega(e) = 1$ if $e \in E'$ and $\omega(e) = 0$ otherwise. The underlying σ -algebra $\mathcal{F}$ of the process is the one generated by the finite-dimensional cylinder sets of Ω . As for the probability measure, let $b(\alpha)$ be the Bernoulli measure with parameter $\alpha \in [0, 1]$ and let $E' \cup E''$ be a decomposition of the edge set E . Given $p, q \in [0, 1]$, we define $P_{p,q} := \prod_{e \in E'} b(p) \times \prod_{e \in E''} b(q)$. That is, $P_{p,q}$ is the product measure on $(\Omega, \mathcal{F})$ with densities p and q on E' and E'' , respectively.

Given a configuration $\omega \in \Omega = \{0, 1\}^E$, an **open path** in $G = (V, E)$ is a set of vertices $\{v_0, v_1, \dots, v_m\} \subset V$, such that $\omega(\{v_i, v_{i+1}\}) = 1$ for every $i = 0, \dots, m-1$. For $x, y \in V$, we say that x is **connected** to y in ω if either $x = y$ or there is an open path from x to y , this event being denoted by $\{x \leftrightarrow y\}$. The **cluster** $C(x)$ of x in ω is the random set $C(x) := \{y \in V : x \leftrightarrow y\}$. If $|C(x)| = \infty$, we say that the vertex x **percolates** and write $\{x \leftrightarrow \infty\}$ for the event of such configurations.

We end this introductory section recalling an important result, extensively used in percolation theory, called the FKG Inequality, named after Fortuin, Kasteleyn and Ginibre [12]. This result is inserted in the more general context of correlation inequalities on partially ordered sets, but here we state a specific version, first proved by Harris [17], for the case of Bernoulli percolation. We refer the reader to [14] for a proof of the result.

Consider the following partial ordering of the elements of $\Omega = \{0, 1\}^E$: given $\omega, \omega' \in \Omega$, we say that $\omega \leq \omega'$ if and only if $\omega(e) \leq \omega'(e)$ for every $e \in E$. In this context, we say that an event $A \in \mathcal{F}$ is **increasing** if the following property holds: if $\omega \in A$, $\omega' \in \Omega$ and $\omega \leq \omega'$, then $\omega' \in A$. The event $\{x \leftrightarrow \infty\}$ is an example of an increasing event. An event $A \in \mathcal{F}$ is **decreasing** if A^c is increasing.

Theorem 0.1 (FKG Inequality). *If $A, B \in \mathcal{A}$ are both increasing or both decreasing events, then*

$$P_{p,q}(A \cap B) \geq P_{p,q}(A)P_{p,q}(B). \quad (1)$$

This result is fairly intuitive in the sense that, if A and B are positively correlated events, then the probability of A conditioned that B occurs must be at least the probability of A itself. For instance, if we condition on the event that there is an open path

joining two vertices $x, y \in \mathbb{V}$, then it is more likely to have an open path joining x to a third vertex $z \in \mathbb{V}$.