Branes and Wrapping Rules

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Abstract. We show that the branes of ten-dimensional IIA/IIB string theory must satisfy, upon toroidal compactification, specific wrapping rules in order to reproduce the number of supersymmetric branes that follows from a supergravity analysis. The realization of these wrapping rules suggests that IIA/IIB string theory contains a whole class of so-called non-standard Kaluza-Klein monopoles. Whereas branes couple to \( p \)-form potentials these non-standard Kaluza-Klein monopoles can be associated with mixed-symmetry potentials. We discuss a possible \( E_{11} \)-symmetry underlying these new structures.

1. Introduction
It is by now well-understood that branes form a crucial ingredient of string theory. For instance, they have been used to calculate the entropy of certain black holes \cite{1} and they are at the heart of the AdS/CFT correspondence \cite{2}. In general, branes are massive objects that divide spacetime into a number of worldvolume and transverse directions. For instance, a ten-dimensional string corresponds to 2 worldvolume and 8 transverse directions. The question we would like to address in this talk is: what can we learn about branes by using as input supergravity as a low-energy approximation to string theory? Often, the presence of a \( p \)-brane in string theory can be deduced from the presence of a rank \( (p+1) \)-form potential in the corresponding supergravity theory. At first sight the relation between the branes of string theory and the potentials of its supergravity approximation could have been investigated many years ago. The new twist we want to give to this old question is to make use of the relatively new insight that the potentials of a given supergravity theory are not only the ones that describe the physical degrees of freedom of the supermultiplet. It turns out that the supersymmetry algebra allows additional high-rank potentials that do not describe any degree of freedom but, nevertheless, play an important role in describing the coupling of branes to background fields.

One can divide branes into \textit{standard} branes, with \( T \geq 3 \) transverse directions, and \textit{non-standard} branes, with \( 0 \leq T \leq 2 \) transverse directions. The standard branes are asymptotically flat. The remaining set of non-standard branes are not asymptotically flat. The consistency of these non-standard branes requires to consider a given number of them, in combination with a so-called orientifold. In this talk we will not pursue this but, instead, consider single branes only and see whether they satisfy some necessary, but not necessarily sufficient, criteria. It is easy to see that the standard branes always couple to potentials that describe physical degrees of freedom. For instance, in \( D = 10 \) dimensions, the standard \( p \)-branes, with \( 0 \leq p \leq 6 \), couple to
physical \((p + 1)\)-form potentials, which include the dual potentials. The highest-rank potential is a 7-form potential which is dual to a vector. The non-standard branes with \(T = 2\) transverse directions are special in the sense that they couple to \((D - 2)\)-form potentials that are dual to the scalars of the supergravity non-linear sigma models. Due to the non-linearity of the scalars this duality is non-trivial and unusual in the sense that the number of physical scalars and dual potentials are not the same. For the exact relation between the numbers, we refer to [3] where branes with \(T = 2\) have been denominated “defect branes” since they include objects such as four-dimensional cosmic strings and ten-dimensional Dirichlet 7-branes. The non-standard branes with \(T = 1\) transverse directions are domain-walls and they couple to \((D - 1)\)-form potentials. One can view these potentials as being the duals of an integration constant such as the massive Romans parameter in IIA supergravity or any gauge coupling constant in gauged supergravity. Finally, the non-standard branes with zero transverse directions are called “space-filling” branes. They are special in the sense that they only allow a double dimensional reduction to a lower-dimensional space-filling brane. These space-filling branes play an important role in describing superstring theories with less than the maximum number of supercharges.

**Table 1.** This table lists the duality groups of all \(3 \leq D \leq 10\) maximal supergravity theories.

<table>
<thead>
<tr>
<th>(D)</th>
<th>U-duality</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIA</td>
<td>(\text{SL}(2, \mathbb{R}))</td>
</tr>
<tr>
<td>IIB</td>
<td>(\text{SL}(2, \mathbb{R}) \times \mathbb{R}^+)</td>
</tr>
<tr>
<td>9</td>
<td>(\text{SL}(3, \mathbb{R}) \times \text{SL}(2, \mathbb{R}))</td>
</tr>
<tr>
<td>8</td>
<td>(\text{SL}(5, \mathbb{R}))</td>
</tr>
<tr>
<td>7</td>
<td>(\text{SO}(5, 5))</td>
</tr>
<tr>
<td>6</td>
<td>(E_6(6))</td>
</tr>
<tr>
<td>5</td>
<td>(E_7(7))</td>
</tr>
<tr>
<td>4</td>
<td>(E_8(8))</td>
</tr>
</tbody>
</table>

All potentials, whether describing physical degrees of freedom or not, can be classified according to the allowed U-duality representations. The U-duality groups for \(3 \leq D \leq 10\) maximal supergravity are given in table 1. The U-duality representations of the physical potentials have been classified a long time ago and they follow from the representation theory of the supersymmetry algebra. The physical potentials of the different maximal supergravity theories are related to each other via toroidal reduction. The lower-dimensional ones all follow from the reduction of the ten-dimensional IIA or IIB potentials. Remarkably, the U-duality representations of the remaining higher-rank potentials that do not describe physical degrees of freedom have also been classified recently [4, 5, 6]. In principle these representations can be derived by the requirement that the supersymmetry algebra is realized on these fields. This has been explicitly verified in \(D = 10\) dimensions in which case the physical potentials of IIA and IIB supergravity can be extended with the potentials given in table 2 [7].

A distinguishing feature of the un-physical potentials is that, when considered in different dimensions, they are not related to each other by toroidal compactification. This is unlike the “physical” potentials, including the dual potentials, whose numbers are fixed by the representation theory of the supersymmetry algebra. Supergravity is therefore not complete in the sense that the lower-dimensional supergravity theories, including the un-physical potentials,
Table 2. This table lists the U-duality representations of all potentials, both physical and un-physical, that are consistent with the IIA or IIB supersymmetry algebra. The representations in the IIB case refer to the SL(2, \mathbb{R}) S-duality group.

<table>
<thead>
<tr>
<th>(D \setminus p)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIA</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2×1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IIB</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

do not follow from the reduction of the ten-dimensional supergravity theory. It is this incomplete nature of supergravity that will lead us to suggest later in this talk a class of non-standard Kaluza-Klein (KK) monopoles in string theory.

In this talk we will consider the supersymmetric branes of IIA/IIB string theory compactified on a torus, which couple to the fields of the corresponding maximal supergravities. As mentioned above these fields do not only include the physical potentials, i.e. the \(p\)-forms with \(0 \leq p \leq D-2\) but also the un-physical potentials, i.e. \((D-1)\)-forms (which are dual to constant parameters) and \(D\)-forms (that have no field strength). While standard branes are automatically classified because their number coincides with the dimension of the U-duality representation of the corresponding field we find that this is in general not true for the non-standard branes. In fact we find two new features for the non-standard branes:

1. Not every U-duality representation corresponds to supersymmetric branes
2. Not each component of a U-duality representation corresponds to a supersymmetric brane

For instance, of all potentials corresponding to the non-standard branes in \(D = 10\) dimensions, see table 2 for \(p = 7, 8\) and 9, only a subset, see table 3, corresponds to a supersymmetric brane.

Table 3. This table shows that the only supersymmetric non-standard branes in \(D = 10\) dimensions are the D7-brane and its S-dual (IIB), the D8-brane (IIA) and the D9-brane and its S-dual (IIB).

<table>
<thead>
<tr>
<th>(D \setminus p)</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIA</td>
<td>0 \subset 1</td>
<td>1</td>
<td>0 \subset 2×1</td>
</tr>
<tr>
<td>IIB</td>
<td>2 \subset 3</td>
<td>(2 \subset 4) \oplus (0 \subset 2)</td>
<td></td>
</tr>
</tbody>
</table>

To determine whether a given potential couples to a supersymmetric brane or not we first construct a gauge-invariant Wess-Zumino (WZ) term which is always possible at the cost of having to introduce a number of world-volume potentials. Next, we impose the following supersymmetric brane criterion [8, 9]:

**supersymmetric brane criterion**: a potential can be associated to a supersymmetric brane if the corresponding gauge-invariant WZ term requires the introduction of world-volume fields that fit within the bosonic sector of a suitable supermultiplet.

We will give two examples elucidating the supersymmetric brane criterion above. The first example concerns the M-theory branes. In M-theory we have a three-form potential \(C_3\) and a dual 6-form potential \(C_6\). Together, they form the following non-trivial “\(p\)-form algebra”: 

\[ C_3 \oplus C_6 \]
\[ \delta C_3 = d \Lambda_2 , \quad \delta C_6 = d \Lambda_5 + H_4 \Lambda_2 . \] 

(1)

The 3-form potential couples to the M2-brane while the 6-form potential couples to the M5-brane. Since the 3-form potential is inert under the gauge transformations of the 6-form its Wess-Zumino term is simply given by

\[ L_{\text{WZ}}(M_2) = C_3 , \]

(2)

where \( C_3 \) now stands for the pull-back of the target-space three-form. Obviously, this WZ term only contains the 8 embedding scalars which precisely fit into an 8 + 8 scalar multiplet on the worldvolume. The situation for the M5-brane is more subtle. Since the 6-form potential also transforms under the gauge transformations of the 3-form we need to introduce, on top of the 5 embedding scalars, a worldvolume 2-form potential \( c_2 \), describing 6 degrees of freedom, in the following way:

\[ L_{\text{WZ}}(M_5) = C_6 + H_3^+ \wedge C_3 , \quad H_3 = dc_2 + C_3 . \]

(3)

In itself the introduction of a world-volume 2-form potential \( c_2 \) is enough to construct a gauge-invariant WZ term. However, by just doing that we would have introduced \( 5+6=11 \) degrees of freedom on the M5-brane world-volume which is too much. We therefore impose a self-duality condition on the curvature \( H_3 \) which we have indicated with the superscript index + in eq. (3). After imposing this duality relation we end up with \( 5 + 3 = 6 \) degrees of freedom which fit into a self-dual tensor multiplet on the 6-dimensional world-volume. The lesson to be learnt from this example is that the construction of a gauge-invariant WZ term can go at the cost of having to introduce too many worldvolume degrees of freedom. One may solve this by imposing additional duality relations between the worldvolume potentials but sometimes this is not enough. This explains why there are in general more potentials than corresponding supersymmetric branes.

The second example concerns the 7-branes of IIB string theory. In this case the supersymmetry algebra closes on an \( \text{SL}(2, \mathbb{R}) \) triplet of 8-forms. This leads to a triplet of 9-form curvatures which satisfy a non-linear constraint such that the three dual potentials together describe just 2 dual scalar degrees of freedom. It should be stressed that the non-linear constraint is on the curvatures and not on the potentials themselves. All three potentials therefore can in principle be used to couple to a supersymmetric 7-brane. Using a real \( \text{SO}(2,1) \) notation the corresponding S-duality covariant WZ term contains terms of the form \( (i = 1, 2, 3) \)

\[ (\text{WZ term})_i \sim (\text{WV curvature}) \Gamma_i (\text{TS gauge field}) , \]

where \( \Gamma_i \) are the gamma-matrices of \( \text{SO}(2,1) \) and both the world-volume (WV) fields and the target space (TS) ones are \( \text{SO}(2,1) \) spinors, whose indices we have left understood. The issue is now that we have again introduced too many world-volume potentials. The world-volume curvature, being a two-component spinor of \( \text{SO}(2,1) \) contains both the Born-Infeld vector as well as its S-dual. The question is now, whether for a given value of \( i \) the Gamma matrix \( \Gamma_i \) projects out one of these two worldvolume vectors. Using lightcone notation \( i = (+, -, 3) \), it turns out that this is indeed the case for \( i = +, - \) but not for \( i = 3 \). Therefore, only two of the 8-form potentials can actually be associated to supersymmetric branes [10]: the D7-brane and its S-dual.
Before discussing the non-standard branes we will first discuss the standard ones in the next section and show how the counting of the supersymmetric branes leads to interesting so-called “wrapping rules”. Before we will discuss these “usual” branes we wish to make one more remark about the classification of branes. Since many different branes will pass by in this talk it is useful to classify them in different ways. We already discussed the distinction between standard branes, with \( T \geq 3 \) transverse directions, and the non standard ones, with \( 0 \leq T \leq 2 \) transverse directions. These are the defect branes (\( T = 2 \)), the domain walls (\( T = 1 \)) and the space-filling branes (\( T = 0 \)). Another useful way to classify the branes of string theory is according to the way that the string tension \( T \) scales with the string coupling constant \( g_s \). Introducing an integer number \( \alpha \leq 0 \) this scaling is given by

\[
T \sim (g_s)^{\alpha}.
\]

This leads us to fundamental branes (\( \alpha = 0 \)), Dirichlet branes (\( \alpha = -1 \)), solitonic branes (\( \alpha = -2 \)) etc. In this talk we will make use of both ways to classify branes. To determine the value of \( \alpha \) corresponding to a given potential it is easiest to decompose in each dimension \( D = 10 - d \) the U-duality representations in terms of T-duality representations as

\[
U - \text{duality} \supset SO(d, d) \times \mathbb{R}^+.
\]

The value of \( \alpha \) then follows from the \( \mathbb{R}^+ \)-weight of the corresponding potential.

One final remark concerns the set up of this talk. Our approach will be bottom-up in the sense that we will start with simple supergravity considerations to classify the supersymmetric branes of toroidally compactified IIA/IIB string theory. Once we have obtained the complete answer we will observe that there is an underlying symmetry structure related to the very extended Kac-Moody algebra \( E_{11} \). This we will discuss at the end of this talk. We wish to stress that the analysis we will perform in the next sections does not rely in any way on the assumption of an underlying \( E_{11} \) symmetry algebra. It will purely be a (surprising) outcome of the supergravity analysis we are going to describe.

2. The “Standard” branes

It is well-known that both IIA and IIB string theory have a single fundamental string that couples to the NS-NS 2-form potential. Since strings can wrap we have in \( D < 10 \) dimensions both strings and wrapped strings, i.e. 0-branes, which couple to 2-forms and 1-forms, respectively. Naively, one would expect one wrapped string for each compactified direction. Instead, we end up with two 0-branes for each compactified direction. This is due to the fact that IIA/IIB string theory also contains a pp-wave which, upon reduction, gives rise to an additional 0-brane. Effectively, we therefore end up with two 0-branes for each compactified direction. This is precisely what we need in order that the corresponding 1-forms \( B_{1,A} (A = 1, \cdots, 2d) \) organize themselves as a vector of the T-duality group \( SO(d, d) \).

It turns out that in each dimension \( D < 10 \) the T-duality singlet 2-form \( B_2 \) and the T-duality vector \( B_{1,A} \) transform under each other’s gauge transformation and together form a “p-form algebra”. Therefore, both are needed to construct a gauge-invariant WZ term, like we discussed for the M5-brane in the previous section, see eq. (3). To construct a gauge-invariant WZ term we need to introduce a T-duality vector \( b_{0,A} \) of additional worldvolume scalars:

\[
\mathcal{L}_{\text{WZ}}(D < 10) = B_2 + \eta^{AB} \mathcal{F}_{1,A} B_{1,B} , \quad \mathcal{F}_{1,A} = db_{0,A} + B_{1,A} .
\]

Together with the embedding scalars these “extra” scalars will not fit into a worldvolume scalar multiplet. To get the correct counting we need to impose a self-duality condition on the extra scalars like in doubled geometry [11].
The lower-dimensional fundamental branes (0-branes $F_0^A$ and string $F_1$) can be nicely understood as the result of the following simple “wrapping rule” \(^1\)

\[
\begin{align*}
\text{wrapped} & \rightarrow \text{doubled} \\
\text{unwrapped} & \rightarrow \text{undoubled},
\end{align*}
\]

when applied to the single fundamental IIA/IIB string, see table 4.

**Table 4.** Applying the fundamental wrapping rule (7) to the IIA/IIB fundamental string gives rise, in each dimension $3 \leq D \leq 9$, to a singlet fundamental string $F_1$ and a T-duality vector of 0-branes $F_0^A$.

<table>
<thead>
<tr>
<th>$F_p$-brane</th>
<th>IIA/IIB</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1/1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

We next continue our analysis with the Dirichlet branes, i.e. the branes with $\alpha = -1$. Unlike the fundamental string there are many D-branes in ten dimensions. The IIA theory contains all Dp-branes with $p$ even while the IIB theory contains the Dp-branes with $p$ odd. The Dp-branes couple to the RR $(p+1)$-form potentials which under gauge transformations transform into each other. Therefore, all RR potentials are needed to write down a gauge-invariant WZ term. In $D < 10$ dimensions these RR potentials organize themselves into spinor representations of the T-duality group of alternating chirality. To simplify the notation the $(p+1)$-form potentials are often written down using a notation where $\mathcal{C}$ stands for the formal sum of all $(p+1)$-form potentials $C_{p+1}$. Furthermore, we do not indicate the spinor indices explicitly.

Gauge-invariance can only be obtained at the cost of introducing an extra so-called Born-Infeld vector $b_1$ and the same extra scalars $b_{0,A}$ as we introduced in the fundamental case. This leads to the following WZ term [20]

\[
\mathcal{L}_{WZ}(D \leq 10) = e^{\mathcal{F}_2} e^{\mathcal{F}_1 A \Gamma^A} \mathcal{C}, \quad \mathcal{F}_2 = db_1 + B_2, \quad \mathcal{F}_{1,A} = db_{0,A} + B_{1,A},
\]

where it is understood that one first expands the exponentials and then picks out the relevant $(p+1)$-form potential out of $\mathcal{C}$. The $\Gamma^A$ are the gamma matrices of the T-duality group whose spinor indices act on the RR potentials. Note that this WZ term is a spinor under T-duality where each component represents the WZ term of a given Dp-brane. Like in the fundamental case the introduction of the extra scalars $b_{0,A}$ introduces too many degrees of freedom to fit into a worldvolume vector multiplet. However, in this case we cannot solve this issue by imposing a duality relation on the extra scalars since we are not dealing with a two-dimensional worldvolume now. Surprisingly, everything works out due to the fact that, although all $b_{0,A}$ scalars are present in the WZ term (8), if you consider just one component of this spinor equation the $\Gamma^A$ matrices project out precisely half of the extra scalars such that we end up with the correct counting.

\(^1\) Since there are two theories in $D = 10$ (IIA and IIB) it is understood that this wrapping rule and the ones given below are applied as follows when reducing from $D = 10$ to $D = 9$ dimensions: a nine-dimensional “undoubled” brane can be seen as coming from IIA and from IIB, and as a consequence the set of undoubled branes coming from either IIA or IIB is the same; a nine-dimensional “doubled” brane has only one origin in terms of ten-dimensional branes, which is a IIA or a IIB brane, and the set of doubled branes results from both IIA and IIB, treating each resulting brane as different.
Like in the previous case the different spinor representations \((Dp)_{\alpha} (\alpha = 1, \cdots, 2^{9-D})\) of the lower-dimensional Dp-branes can be obtained by applying the following simple Dp-brane wrapping rule

\[
\begin{align*}
\text{wrapped} & \rightarrow \text{undoubled} \\
\text{unwrapped} & \rightarrow \text{undoubled}
\end{align*}
\] (8)

on the different Dp-branes of IIA/IIB string theory, see table 5. Unlike the fundamental branes the D-branes are complete by themselves in the sense that the realization of the D-brane wrapping rule (8) does not require the input of any gravitational solutions.

**Table 5.** Applying the D-brane wrapping rule (8) to the D-branes of IIA/IIB string theory leads to the different lower-dimensional Dp-branes \((Dp)_{\alpha} (\alpha = 1, \cdots, 2^{9-D})\).

<table>
<thead>
<tr>
<th>Dp-brane</th>
<th>IIA/IIB</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
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<tr>
<td>1</td>
<td>0/1</td>
<td>1</td>
<td>2</td>
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<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
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<td>2</td>
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<td>8</td>
<td>1/0</td>
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<tr>
<td>9</td>
<td>0/1</td>
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</tbody>
</table>

The final set of standard branes we will consider are the standard solitons, i.e. branes with \(\alpha = -2\) and \(T \geq 3\) transverse directions. The IIA/IIB string theory has a single solitonic NS-NS 5-brane. Upon wrapped reduction it gives rise to a single D=9 solitonic 4-brane and upon unwrapped reduction it leads to a single D=9 solitonic 5-brane. It turns out that in this case the solitonic 5-brane is doubled due to the presence of a single Kaluza-Klein monopole in IIA/IIB string theory. This leads to the following dual or solitonic wrapping rule:

\[
\begin{align*}
\text{wrapped} & \rightarrow \text{undoubled} \\
\text{unwrapped} & \rightarrow \text{doubled}
\end{align*}
\] (9)

When applied to the solitonic NS-NS 5-brane of IIA/IIB string theory it gives rise to a singlet \(S(D-5)\)-brane soliton and a T-duality vector \(S(D-4)_{A}\) of brane-solitons, see table 6.

This finishes our discussion of the standard branes. The question is now what happens with the non-standard branes, i.e. the branes with \(T \leq 2\) transverse directions. We will discuss this in the next section.

3. The “non-standard” branes

Our discussion on the solitonic branes, started in the previous section, has not finished yet. A supergravity analysis, making use of the decomposition (5), shows that there are more “non-standard” solitonic branes, i.e. branes with \(0 \leq T \leq 2\) transverse directions. They occur as anti-symmetric tensor representations of the T-duality group, see table 7. This table should be read as follows. In each dimension \(D\) there are solitonic branes in antisymmetric representations of increasing rank, starting with rank 0 at the top row up to a maximum rank \(r_{\text{max}}\) given by
Table 6. Applying the solitonic wrapping rule (9) to the NS-NS solitonic 5-brane of IIA/IIB string theory leads to a lower-dimensional singlet $S(D-5)$-brane soliton and a vector $S(D - 4)_A$ of brane-solitons.

<table>
<thead>
<tr>
<th>Sp-brane</th>
<th>IIA/IIB</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1</td>
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<td>8</td>
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<tr>
<td>4</td>
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<td>1</td>
<td>4</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>1/1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$r_{\text{max}} = d$ if $D \geq 6$, which always corresponds to a solitonic 5-brane, and $r_{\text{max}} = 4$ if $D \leq 6$. The solitonic 5-brane with maximum rank representation decomposes into a self-dual and anti-self-dual representation of the SO($d, d$) T-duality group. One of these representations has a worldvolume vector-multiplet while the other has a worldvolume self-dual tensor multiplet.

Table 7. This table indicates all $D$-dimensional solitonic branes, standard as well as non-standard ones. For a given dimension $D$ the maximum rank under T-duality is given by $r_{\text{max}} = d$ if $D \geq 6$, corresponding to a solitonic 5-brane, and $r_{\text{max}} = 4$ if $D \leq 6$.

<table>
<thead>
<tr>
<th>$S(D - 5)$-brane</th>
<th>$[S(D - 4)$-brane$]_A$</th>
<th>$[S(D - 3)$-brane$]_{AB}$</th>
<th>$[S(D - 2)$-brane$]_{ABC}$</th>
<th>$[S(D - 1)$-brane$]_{ABCD}$</th>
</tr>
</thead>
</table>

As we already anticipated in the introduction the non-standard branes behave differently than the standard ones in the sense that not each component of the antisymmetric tensor representations occurring in table 7 corresponds to a supersymmetric solitonic brane. Imposing our supersymmetric brane criterion discussed in the introduction leads to the correct number of supersymmetric branes. The result of this analysis can be found in table 8 which contains the supersymmetric standard solitonic branes as well.

Surprisingly, we find that the numbers of supersymmetric solitons, given in table 8 are precisely the same as the ones one obtains by extending the solitonic wrapping rule (9) from supersymmetric standard solitons only to standard as well as non-standard supersymmetric solitons! Strictly speaking, without saying explicitly we also extended the wrapping rule for the fundamental branes and D-branes from standard to non-standard ones. The difference is that in that case all components of the scalar, vector and spinor T-duality representations involved correspond to supersymmetric branes. In the case of fundamental branes non-standard strings only happen for $D \leq 4$ dimensions while non-standard 0-branes only occur in $D = 3$ dimensions. Non-standard D-branes already occur in $D = 10$ dimensions.

We are now faced with the following question: where do the solitons that realize the solitonic
Table 8. This table indicates the number of supersymmetric solitonic branes, both the standard and the non-standard ones for dimensions $3 \leq D \leq 10$.

<table>
<thead>
<tr>
<th>Sp-brane</th>
<th>IIA/IIB</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>1</td>
<td>12</td>
<td>84</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td>10</td>
<td>60</td>
<td>280</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td>8</td>
<td>40</td>
<td>160</td>
<td>560</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1</td>
<td>6</td>
<td>24</td>
<td>80</td>
<td>240</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>1</td>
<td>4</td>
<td>12</td>
<td>32</td>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>1/1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

wrapping rule (9) come from? In the case of the standard solitons the answer to this question is that the solitonic wrapping rule can be realized due to the presence of the Kaluza-Klein (KK) monopole in $D = 10$ dimensions. The difference between the KK monopole and the other branes is that these monopoles divide spacetime into three inequivalent directions. Besides the worldvolume and transverse directions which we already encountered with the branes there is a third so-called “isometry” direction. We call the KK monopole from now on the “standard” KK monopole because it has three transverse directions. It turns out that in the same way that the standard KK monopole is needed to realize the solitonic wrapping rule (9) a new kind of so-called “non-standard” KK monopoles, with $T \leq 2$ transverse directions, are needed to realize the same wrapping rule to obtain the non-standard solitonic branes. Precisely this class of non-standard KK monopoles have been analyzed and classified some time ago in [12]. They are local solutions of the corresponding supergravity theory. Since they are non-standard, we expect that multiple monopole solutions and/or orientifolds are needed to turn them into finite-energy solutions, like it is the case for the non-standard branes.

We will indicate a general KK monopole with $T$ transverse, $p$ spatial and $I$ isometry directions in $D$ dimensions corresponding to a general value of $\alpha$ with $(T + p + I = D - 1)$

$$(T, p, I)_{\alpha},$$

where the special case $I = 0$ refers to branes. It turns out that in order to realize the solitonic wrapping rule (9) for the standard as well as non-standard solitons, one needs the following class of branes and KK monopoles [9]:

$$(4 - n, 5, n)_{-2},$$

$n = 0, 1, 2, 3, 4)$. (11)

Here $n = 0$ corresponds to the NS-NS 5-brane, $n = 1$ to the standard KK monopole with $T = 3$ transverse directions and $n = 2, 3, 4$ to the non-standard KK monopoles with $0 \leq T \leq 2$ transverse directions.

The only ten-dimensional supersymmetric brane which is left aside by this analysis $^2$ is the S-dual of the D7-brane of the IIB theory. The tension of this brane scales like $(g_S)^{-3}$ in the string frame. In any dimension below ten, one can deduce the T-duality representations of the $\alpha = -3$ fields by simply looking at the tables in ref. [9]. We find that the T-duality representations of the potentials corresponding to supersymmetric branes are in this case given by tensor-spinor representations, see table 9.

$^2$ We are not taking into account the ten-dimensional space-filling branes. These branes can only wrap.
Table 9. Forms with $\alpha = -3$ in any dimension that couple to supersymmetric branes. All representations are meant to be irreducible, and the $T$-duality vector indices $AB$ are meant to be antisymmetrised. The $a, \dot{a}$ denote chiral and anti-chiral $T$-duality spinor indices.

<table>
<thead>
<tr>
<th>$D$-form</th>
<th>$E_{D-2,\dot{a}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(D-2)$-form</td>
<td>$E_{D-2,\dot{a}}$</td>
</tr>
<tr>
<td>$(D-1)$-form</td>
<td>$E_{D-1,A\dot{a}}$</td>
</tr>
<tr>
<td>$D$-form</td>
<td>$E_{D,ABA}$</td>
</tr>
</tbody>
</table>

Applying our supersymmetric brane criterion we can find out which components of the potentials in table 9 correspond to supersymmetric branes. The number of supersymmetric branes with $\alpha = -3$ in $3 \leq D \leq 10$ that follows from this analysis can be found in table 10.

Table 10. This table indicates the number of $\alpha = -3$ supersymmetric branes in $3 \leq D \leq 10$ dimensions.

<table>
<thead>
<tr>
<th>$p$-brane</th>
<th>IIA/IIB</th>
<th>9</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>64</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>32</td>
<td>448</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>16</td>
<td>192</td>
<td>1344</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>8</td>
<td>80</td>
<td>480</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>4</td>
<td>32</td>
<td>160</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>2</td>
<td>12</td>
<td>48</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>1</td>
<td>4</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0/1</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Remarkably, the numbers of supersymmetric branes given in table 10 can also be obtained by introducing yet one more wrapping rule:

\[
\text{wrapped } \rightarrow \text{ doubled},
\]
\[
\text{unwrapped } \rightarrow \text{ doubled}. \tag{12}
\]

That is, one obtains the right counting if, going from $D + 1$ to $D$ dimensions, both wrapped and unwrapped branes get doubled. The initial condition is the S-dual of the D7-brane in IIB string theory. Clearly, the new wrapping rule (12) can only be realised by the help of yet more non-standard KK monopoles. Note that the same number of supersymmetric branes can be obtained starting from IIA string theory, in which case all the branes can be seen to arise from compactifications of non-standard KK monopoles as there is no $\alpha = -3$ brane in IIA string theory.

We find that all the branes in table 10, satisfying the wrapping rule (12), can be obtained from the ten-dimensional IIA/IIB branes/monopoles given in table 11. Note that we now also encounter non-standard KK monopoles with two in-equivalent isometry directions (giving different contributions to the mass formula) which we have indicated by

\[
(T, p, I_1, I_2). \tag{13}
\]
Table 11. This table indicates the ten-dimensional $\alpha = -3$ brane/mode origin of all $\alpha = -3$ branes in $D < 10$ dimensions. We have not indicated the sub-index $\alpha = -3$ in this table.

<table>
<thead>
<tr>
<th>$T$</th>
<th>IIA</th>
<th>IIB</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$(2,6,1)+(2,4,3)+(2,2,5)+(2,0,7)$</td>
<td>$(2,7)+(2,5,2)+(2,3,4)+(2,1,6)$</td>
</tr>
<tr>
<td>1</td>
<td>$(1,7,0,1)+(1,5,2,1)+(1,3,4,1)+(1,1,6,1)$</td>
<td>$(1,6,1,1)+(1,4,3,1)+(1,2,5,1)$</td>
</tr>
<tr>
<td>0</td>
<td>$(0,6,1,2)+(0,4,3,2)+(0,2,5,2)$</td>
<td>$(0,7,0,2)+(0,5,2,2)+(0,3,4,2)$</td>
</tr>
</tbody>
</table>

The only branes left at this stage are the non-standard ones that have no parent brane in either IIA or IIB string theory. They all arise from non-standard IIA/IIB KK monopoles. All these branes are highly non-perturbative in the sense that they have large negative values of $\alpha$ such that $\alpha \geq -11$ for $D \geq 3$. We will not discuss the different KK monopoles involved here any further.

This finishes our discussion of the higher-dimensional origin of all supersymmetric branes. In the next section we wish to discuss some general properties of the KK monopoles, in particular, their relation to mixed-symmetry tensors.

4. KK monopoles and mixed-symmetry tensors

It turns out that in the same way that one can associate a $(p+1)$-form potential to a $p$-brane we can associate a mixed-symmetry tensor to a KK monopole. We stress that, unlike the forms, this is a formal relationship. We do not know precisely whether and how these mixed-symmetry tensors have a Wess-Zumino coupling to the KK monopoles neither whether and in which sense they can be incorporated into a supergravity multiplet like the form fields. Nevertheless, there are some intriguing correspondences which we wish to discuss.

Mixed-symmetry potentials with a fixed value of $\alpha$ are associated to KK monopoles as follows: the symmetry of the potential $A_{m,n}$ is that of a Young tableau with two columns, one with $m$ rows and one with $n$ rows,\(^3\) and it corresponds to the KK monopole

$$A_{m,n} \leftrightarrow (D-m,m-n-1,n)_{\alpha}, \quad \text{or} \quad (T,p,I)_{\alpha} \leftrightarrow A_{D-T,I}.\quad (14)$$

This rule can be extended to include monopoles with two inequivalent isometry directions as follows

$$A_{m,n_1,n_2} \leftrightarrow (D-m,m-n_1-1,n_1-n_2,n_2)_{\alpha}, \quad \text{or} \quad (T,p,I_1,I_2)_{\alpha} \leftrightarrow A_{D-T,I_1+I_2,I_2}.\quad (15)$$

The generalization to more than two inequivalent isometry directions is obvious.

For simplicity, we will denote the correspondence between the mixed-symmetry fields and the solutions with an equality, i.e. $A_{m,n} = (D-m,m-n-1,n)_{\alpha}$. In this notation, the string-theory origin of the solitonic branes discussed in the previous section reads

$$D_{6+n,n} = (4-n,5,n)_{-2}, \quad n = 0, 1, 2, 3, 4.\quad (16)$$

The $D_6$ field ($n = 0$), the magnetic dual of the NS-NS 2-form $B_2$, couples to the NS-NS 5-brane while the $D_{7,1}$ field, the so-called dual graviton, is associated with the standard KK monopole. Although this dual graviton field $D_{7,1}$ can only be introduced consistently at the linearized level,

\(^3\) An anti-symmetric potential is denoted by $A_{m,0} = A_m$. 

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it can still be considered as a tool to determine all the lower-dimensional standard solitons that follow from it by dimensional reduction, see the discussion below. In the same way the string theory origin of the $\alpha = -3$ branes, given by the branes/monopoles in table 11 can be associated with the form/mixed-symmetry fields given in table 12.

**Table 12.** This table indicates the form and mixed-symmetry fields that can be associated with the $\alpha = -3$ non-standard brane/monopole solutions given in table 11.

<table>
<thead>
<tr>
<th>$T$</th>
<th>IIA</th>
<th>IIB</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$E_{8,1} + E_{8,3} + E_{8,5} + E_{8,7}$</td>
<td>$E_{8} + E_{8,2} + E_{8,4} + E_{8,6}$</td>
</tr>
<tr>
<td>1</td>
<td>$E_{9,1,1} + E_{9,3,1} + E_{9,5,1} + E_{9,7,1}$</td>
<td>$E_{9,2,1} + E_{9,4,1} + E_{9,6,1}$</td>
</tr>
<tr>
<td>0</td>
<td>$E_{10,3,2} + E_{10,5,2} + E_{10,7,2}$</td>
<td>$E_{10,2,2} + E_{10,4,2} + E_{10,6,2}$</td>
</tr>
</tbody>
</table>

The whole set of supersymmetric branes in any dimensions can be obtained from the ten-dimensional mixed-symmetry fields provided one imposes a restricted reduction rule which guarantees that one only obtains supersymmetric branes after reduction. This restricted reduction rule, for the general case of a mixed-symmetry field $A_{m,n_1,n_2}$ corresponding to a Young tableaux with 3 columns reads as follows:

**Restricted reduction rule:** for a mixed-symmetry field $A_{m,n_1,n_2}$ to yield, upon toroidal reduction, a potential corresponding to a supersymmetric brane, we require that the $n_2$ indices are internal and along directions parallel to $n_2$ of the $n_1$ indices and $n_2$ of the $m$ indices, and that the remaining $n_1 - n_2$ indices in the second set are also internal and along directions parallel to $n_1 - n_2$ of the $m$ indices.

When applied to the special case of a mixed-symmetry field $A_{m,n}$ corresponding to a Young tableaux with two columns the above rule states that that a supersymmetric brane is only obtained when the $n$ indices on the right of the comma in $A_{m,n}$ are internal and along directions that coincide with $n$ of the indices on the left of the comma.

As an example we show how the restricted reduction of the IIA $\alpha = -3$ mixed-symmetry fields given in table 12 gives rise to the seven-dimensional $\alpha = -3$ branes given in table 10. We have, from IIA,

$$
E_{8,1} \rightarrow E_{\bar{6}ijk,1} (3) \quad E_{\bar{6}ij,i} (6) \quad E_{\bar{7}i,i} (3), \\
E_{8,3} \rightarrow E_{\bar{6}ijk,ijk} (1), \\
E_{9,1,1} \rightarrow E_{\bar{6}ijk,i,i} (3) \quad E_{\bar{7}ij,i,i} (6), \\
E_{9,3,1} \rightarrow E_{\bar{6}ijk,i,jk,i} (3), \\
E_{10,3,2} \rightarrow E_{\bar{7}ijk,i,jk,i} (3).
$$

(17)

Note that not all mixed-symmetry fields in table 12 contribute since we have to reduce over all isometry directions to obtain a form field. We thus obtain four 4-branes, twelve 5-branes and twelve 6-branes, in accordance with table 10. One can show that the IIB compactification gives the same result. Similarly, one can show that the reduction to all the other dimensions reproduces the correct number of supersymmetric branes.

Summarizing, we conclude that one can associate to each KK monopole, standard and non-standard, a mixed-symmetry field which in many ways plays the same role as the form fields...
that can be associated to branes. It remains to be seen what the precise role is they are playing in the classification of the supersymmetric branes. It could vary from just a book keeping device to predict the number supersymmetric branes in lower dimensions to a complete democratic treatment of forms and mixed-symmetry fields where both forms and mixed-symmetry fields, together with appropriate duality relations, are part of the same supergravity multiplet.

5. An underlying $E_{11}$ structure?
At this point we have concluded that a number of forms and mixed-symmetry fields play a role in the classification of supersymmetric branes. One could wonder whether there is a symmetry algebra that predicts all these fields. Surprisingly, there is! The forms and mixed-symmetry fields we have encountered so far are precisely the ones that are predicted by the very extended $E_{11}$ Kac-Moody algebra. This algebra was introduced earlier in the context of supergravity in [13]. Its representation theory in $D$ dimensions produces an infinite number of $\text{GL}(D,\mathbb{R})$ representations which can be sliced by a so-called level decomposition [16, 17]. So far, this algebra has led to three important predictions about the fields and branes of maximal supergravity theories:

(1) $E_{11}$ predicts the number of physical fields of maximal supergravity [13, 14].
(2) $E_{11}$ predicts the U-duality representations of all supergravity forms that do not describe physical degrees of freedom [4, 5].
(3) $E_{11}$ predicts the number of ten-dimensional non-standard KK monopoles [18] and the number of supersymmetric branes [19] in $3 \leq D \leq 11$ dimensions.

Given these predictions it is clear that $E_{11}$ must play a role in the scheme of things. Its precise role depends crucially on the role of the mixed-symmetry fields. Any progress in their understanding will clarify the role of $E_{11}$.

6. Conclusions
In this talk we showed, by following simple supergravity considerations, that branes whose tension scales as $T \sim (g_s)^{\alpha}$ for $\alpha = 0, -1, -2, -3$ satisfy the following wrapping rules

\begin{align*}
\text{wrapped} & \rightarrow \text{doubled, undoubled, undoubled, doubled}, \\
\text{unwrapped} & \rightarrow \text{undoubled, undoubled, doubled, doubled},
\end{align*}

(18)

where the four terms at the right of the arrow correspond to $\alpha = 0, -1, -2$ and $-3$, respectively. For $\alpha = 0$ the doubling of branes is due to the reduction of pp-waves. Dirichlet branes, with $\alpha = -1$, have no doubling and are complete by themselves. For standard solitonic branes, with $\alpha = -2$, the doubling is due to the presence of the standard KK monopole. We showed that the doubling in the case of non-standard solitons is due to the presence of so-called non-standard KK monopoles [20]. Similar non-standard KK monopoles are also needed to realize the wrapping rule in the case of branes with $\alpha = -3$.

At present it is not clear what the precise status of the non-standard KK monopoles is. We are able to associate a set of mixed-symmetry fields to them with a restricted reduction rule such that all branes suggested by supergravity are generated upon reduction. The explicit solution for all non-standard KK monopoles have been given in [12]. What is not yet clear is whether a finite energy solution can be obtained, possibly by taking superpositions of such non-standard KK monopoles and by including non-standard orientifolds. In the introduction we stated that supergravity is incomplete in the sense that the maximal supergravity theories in different dimensions are not related to each other by toroidal reduction. In some sense the new structure we introduced, non-standard KK monopoles or mixed-symmetry fields, takes this
incomplete nature of supergravity away. Whether this is merely a book keeping trick or a true physical meaning can be given to these non-standard KK monopoles remains to be explored.

The role of the very extended Kac-Moody algebra $E_{11}$ [13] in this is intriguing. Not only does $E_{11}$ predict the number of physical and un-physical potentials of maximal supergravity, it also predicts the number of supersymmetric branes and mixed-symmetry tensors.

Ten-dimensional string theory does not contain branes with $\alpha < -4$. The IIB theory contains a space-filling brane with $\alpha = -4$, the S-dual of the D9-brane, but space-filling branes can only wrap and therefore no non-trivial wrapping rule can be associated with them. Indeed, for $\alpha = -4$ we do not find a general pattern like for the higher values of $\alpha$. Interestingly, lower-dimensional maximal supergravity suggests the existence of non-space-filling branes with $\alpha = -4$. For instance, in $D \leq 6$ dimensions there are domain walls with $\alpha = -4$ and in $D = 3, 4$ dimensions there are branes of co-dimension 2 with $\alpha = -4$. Clearly, such branes do not follow from the reduction of the ten-dimensional IIB space-filling brane and must be the result of reducing a non-standard KK monopole with $\alpha = -4$. Similarly, in $D \leq 6$ dimensions maximal supergravity suggests branes with $\alpha \leq -5$ and such branes too must be the result of non-standard KK monopoles with $\alpha \leq -5$.

Summarizing, we find that all branes of IIA and IIB string theory, excluding the space-filling branes which should be treated separately, satisfy the wrapping rule (18). The deeper meaning of why branes should satisfy such a simple wrapping rule is unclear to us. It would be interesting to see whether some geometrical interpretation could be given of this rule. In this respect it would be interesting to investigate the doubled wrapping rule we find for the S-dual of the D7-brane and to see whether this could be understood from an F-theory [21] point of view.

Finally, it would be interesting to see whether there is any relation between this work and work on generalized geometry and/or non-geometry. This also could lead to interesting applications to phenomenology, like, e.g., the construction of de Sitter solutions in string theory.

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References


