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Topology Identification of Heterogeneous Networks of Linear Systems

Henk J. van Waarde, Pietro Tesi, and M. Kanat Camlibel

Abstract—This paper deals with the problem of reconstructing the graph structure of a dynamical network using measured data. This problem is often referred to as topology identification and has received considerable attention in recent literature. Most existing literature focuses on topology identification for networks of single integrators or networks of single-input single-output systems. The aim of this paper is to study topology identification for a more general class of so-called heterogeneous networks, in which the dynamics of the nodes are modeled by general (possibly distinct) linear systems. Our two main contributions are the following. First, we state conditions for topological identifiability, i.e., conditions under which the network topology can be uniquely reconstructed from measured data. Secondly, we develop a topology identification scheme that is based on subspace methods and that is able to reconstruct the network topology for the general class of systems.

I. INTRODUCTION

The last years have seen an increasing interest in networks of dynamical systems. Such networks appear in a variety of contexts, including biological networks, power grids and robotics. The behavior of a dynamical network is greatly influenced by its underlying network topology. Indeed, it is well-known that a consensus network reaches consensus if and only if its network graph is connected. Network topologies are also known to affect the controllability of dynamical networks.

However, in many scenarios the topology of the network is unknown or not known exactly. This problem is particularly apparent in biology, for example in neural networks and genetic networks [1]. To deal with this problem, several topology identification methods have been studied [2]–[11]. Such methods aim at reconstructing the topology (and weights) of a dynamical network on the basis of measured data obtained from the network.

The paper [2] studies necessary and sufficient conditions for dynamical structure reconstruction, see also [5]. A node-knockout scheme for topology identification was introduced in [3] and further investigated in [7]. Moreover, the paper [4] studies topology identification using compressed sensing, while [6] considers network reconstruction using Wiener filtering. A distributed algorithm for network reconstruction has also been studied [8]. The authors of [9] study topology identification using power spectral analysis. In [11],

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the network topology was reconstructed by solving certain Lyapunov equations. We also note that a related identification problem for dynamical networks with *known* topology has been considered, see e.g. [12]–[16].

Most existing work on topology identification emphasizes the role of the network topology by considering relatively simple node dynamics. For example, networks of single integrators have been studied in [2], [3], [5], [8], [10], [11]. In addition, the papers [7] and [9] consider networks of single-input single-output systems. The current paper builds on this work by studying topology identification for a more general class of heterogeneous networks, in which the node dynamics are modeled by general, possibly distinct, linear systems. In particular, our main contributions are the following.

- 1) We provide conditions under which the network topology can be uniquely reconstructed from measured data, a notion referred to as *topological identifiability*.
- 2) We establish a topology identification scheme based on subspace methods and formally prove its correctness under the assumptions that an identifiability condition and a persistency of excitation condition are satisfied.

The outline of the paper is as follows. In Section II we formulate the problem. Sections III and IV contain our results on topological identifiability and topology identification, respectively. Finally, we give our conclusions in Section V.

Notation: The set of natural numbers is denoted by $\mathbb{N}$ and the set of real numbers is denoted by $\mathbb{R}$. We denote the set of real $n \times m$ matrices by $\mathbb{R}^{n \times m}$ and define $\mathbb{R}^n := \mathbb{R}^{n \times 1}$. The image and rank of a real matrix A are denoted by $\text{im } A$ and $\text{rank } A$. Finally, we denote the Kronecker product by $\otimes$.

II. PROBLEM FORMULATION

We follow a similar setup as in [17, Ch. 9] and consider networks composed of N discrete-time systems of the form

$$\begin{aligned} x_i(t+1) &= A_i x_i(t) + B_i v_i(t) \\ w_i(t) &= C_i x_i(t), \end{aligned} \quad (1)$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state of the i -th node system, $v_i(t) \in \mathbb{R}^{m_i}$ is its input and $w_i(t) \in \mathbb{R}^{p_i}$ is its output for $i = 1, 2, \dots, N$. The real matrices A_i , B_i and C_i are of appropriate dimensions. We use the shorthand notation (A_i, B_i, C_i) to denote (1). The coupling between nodes is induced by the inputs $v_i(t)$, which are specified as

$$v_i(t) = \sum_{j=1}^N Q_{ij} w_j(t) + R_i u(t),$$

where $u(t) \in \mathbb{R}^m$ is the external network input and Q_{ij} and R_i are real matrices of appropriate dimensions. Let S_i be a

real $p \times p_i$ matrix and consider the external network output $y(t) \in \mathbb{R}^p$, defined by

$$y(t) = \sum_{i=1}^N S_i w_i(t).$$

Then, by introducing the block diagonal matrices

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_N \end{pmatrix}, B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_N \end{pmatrix}$$

$$C = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & C_N \end{pmatrix}$$

and the matrices

$$Q = \begin{pmatrix} Q_{11} & \cdots & Q_{1N} \\ \vdots & \ddots & \vdots \\ Q_{N1} & \cdots & Q_{NN} \end{pmatrix}, R = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix}, S^\top = \begin{pmatrix} S_1^\top \\ \vdots \\ S_N^\top \end{pmatrix},$$

we can represent the network dynamics compactly as

$$\begin{aligned} x(t+1) &= (A + BQC)x(t) + BRu(t) \\ y(t) &= SCx(t). \end{aligned} \quad (2)$$

Here the vector $x(t) \in \mathbb{R}^n$ is the concatenation of the states of all network nodes and $n := \sum_{i=1}^N n_i$.

There are a few important special cases of node dynamics (1) and resulting network dynamics (2) which we will discuss next. If $A_i = A_0$, $B_i = B_0$ and $C_i = C_0$ for all $i = 1, 2, \dots, N$, the dynamics of all nodes in the network are the same and the resulting dynamical network is called *homogeneous*. The more general setting in which the node dynamics are not necessarily the same is referred to as a *heterogeneous* network. Another special case of node dynamics occurs when $m_i = p_i = 1$ for all $i = 1, 2, \dots, N$. In this case, the node systems are single-input single-output (SISO) systems, and the resulting dynamical network is referred to as a *SISO network*. Topology identification of homogeneous SISO networks has been studied in [7] and [9]. In addition, topology identification has been well-studied (see e.g. [2], [3], [10], [11]) for networks of so-called *single-integrators*, in which the node dynamics are described by $\dot{x}_i(t) = v_i(t)$. This type of node dynamics can be seen continuous-time counterpart of (1) where $A_i = 0$, $B_i = 1$ and $C_i = 1$ for $i = 1, 2, \dots, N$.

The purpose of this paper is to study topology identification for general, heterogeneous dynamical networks of the form (2). Although we focus on discrete-time systems, our results can be stated for continuous-time systems as well. In order to make the problem more precise, we first introduce the topology of (2). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a weighted directed graph with $\mathcal{V} = \{1, 2, \dots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ such that $(j, i) \in \mathcal{E}$ if and only if $Q_{ij} \neq 0$. Each edge $(j, i) \in \mathcal{E}$ is weighted by the nonzero matrix Q_{ij} . We refer

to $\mathcal{G}$ as the *topology* of the dynamical network (2). Then, the problem of *topology identification* concerns finding $\mathcal{G}$ (equivalently, finding Q) using measurements of the input $u(t)$ and output $y(t)$ of (2). We assume knowledge of the local node dynamics (i.e., the matrices A, B and C) as well as the external input/output matrices R and S ¹.

An important aspect of topology identification concerns the question of whether Q can be *uniquely* reconstructed from input/output data obtained from the network. We formalize this question by defining a notion of *topological identifiability*. To this end, we denote the transfer matrix from u to y of (2) by

$$F_Q(z) := SC(zI - A - BQC)^{-1}BR, \quad (3)$$

where the subscript Q emphasizes the dependence on the interconnection matrix Q .

Definition 1: The topology of system (2) is *identifiable* if the following implication holds:

$$F_Q(z) = F_{\bar{Q}}(z) \text{ for real } \bar{Q} \implies Q = \bar{Q}.$$

This paper has two main goals. First, we want to establish conditions under which the topology of (2) is identifiable. This is formalized in the following problem.

Problem 1: Find necessary and sufficient conditions on the node dynamics A, B, C , the external input/output matrices R, S and the interconnection matrix Q under which the topology of (2) is identifiable.

Our second goal is to identify Q from input/output data.

Problem 2: Develop a methodology to identify the interconnection matrix Q from measurements of the input $u(t)$ and output $y(t)$ of system (2).

III. RESULTS ON TOPOLOGICAL IDENTIFIABILITY

In this section, we focus on Problem 1. We start with the simple case of ‘single integrators’ in which $A_i = 0$, $B_i = 1$ and $C_i = 1$, before studying the general case in Section III-B.

A. Identifiability for $A_i = 0$, $B_i = 1$ and $C_i = 1$

In the case that $A_i = 0$, $B_i = 1$ and $C_i = 1$ for all $i \in \mathcal{V}$, $F_Q(z)$ simplifies to $F_Q(z) = S(zI - Q)^{-1}R$. The following lemma states a necessary condition for identifiability in this simplified scenario.

Lemma 3: Suppose that $A_i = 0$, $B_i = 1$ and $C_i = 1$ for all $i \in \mathcal{V}$. If the topology of (2) is identifiable then (Q, R) is controllable and (S, Q) is observable.

Proof: Suppose on the contrary that (S, Q) is unobservable. Let $v \in \mathbb{R}^N$ be a nonzero vector in the unobservable subspace of (S, Q) , i.e.,

$$SQ^k v = 0 \text{ for all } k = 0, 1, 2, \dots$$

This implies that $S(zI - Q)^{-1}R = S(zI - Q - vv^\top)^{-1}R$. However, since $v \neq 0$, the matrices Q and $\bar{Q} := Q + vv^\top$ are distinct. Hence, the topology of (2) is not identifiable. The

¹This is a standard assumption in the literature on topology identification (see, e.g., [7] and [9]). Without knowledge of the node dynamics, topology identification becomes a full system identification problem.

proof for necessity of controllability of (Q, R) is analogous and therefore omitted. ■

Next, we give another necessary condition for identifiability.

Lemma 4: Suppose that $A_i = 0$, $B_i = 1$ and $C_i = 1$ for all $i \in \mathcal{V}$. If the topology of (2) is identifiable then $\text{rank } S = N$ or $\text{rank } R = N$.

Proof: Suppose on the contrary that $\text{rank } R < N$ and $\text{rank } S < N$. There exist nonzero vectors $v_1, v_2 \in \mathbb{R}^N$ such that $Sv_1 = 0$ and $v_2^\top R = 0$. We can scale v_2 in such a way that $v_2^\top v_1 \neq -1$. Next, we define $T := I + v_1 v_2^\top$. By the Sherman-Morrison formula, T is invertible if and only if $1 + v_2^\top v_1 \neq 0$, equivalently $v_2^\top v_1 \neq -1$. By our assumption on v_2 , the matrix T is hence invertible, and its inverse is

$$T^{-1} = I - \frac{v_1 v_2^\top}{1 + v_2^\top v_1}.$$

Now, we define the matrix

$$\bar{Q} := T^{-1}QT = \left(I - \frac{v_1 v_2^\top}{1 + v_2^\top v_1}\right)Q(I + v_1 v_2^\top). \quad (4)$$

We distinguish two cases: $Q \neq \bar{Q}$ and $Q = \bar{Q}$. First suppose that $Q \neq \bar{Q}$. Since we have $\bar{Q} = T^{-1}QT$, $TR = R$ and $ST = S$, we see that

$$S(zI - Q)^{-1}R = S(zI - \bar{Q})^{-1}R.$$

Hence, the topology is not identifiable. Next, suppose that $Q = \bar{Q}$. It follows from (4) that

$$Qv_1 v_2^\top - \frac{v_1 v_2^\top}{1 + v_2^\top v_1}Q - \frac{v_1 v_2^\top}{1 + v_2^\top v_1}Qv_1 v_2^\top = 0,$$

equivalently,

$$(1 + v_2^\top v_1)Qv_1 v_2^\top - v_1 v_2^\top Q - v_1 v_2^\top Qv_1 v_2^\top = 0.$$

We multiply from right by v_2 and rearrange terms to obtain

$$(1 + v_2^\top v_1)v_2^\top Qv_1 = (v_2^\top Qv_2 + v_2^\top Qv_1 v_2^\top v_2)v_1.$$

This means that v_1 is an eigenvector of Q contained in the kernel of S . Therefore, (S, Q) is unobservable (cf. [18, Ch. 3]). By the previous lemma, this implies that the topology of (2) is not identifiable. ■

Finally, based on the previous results, we state necessary and sufficient conditions for topological identifiability.

Theorem 5: Suppose that $A_i = 0$, $B_i = 1$ and $C_i = 1$ for all $i \in \mathcal{V}$. Then the topology of (2) is identifiable if and only if either $\text{rank } S = N$ and (Q, R) is controllable or $\text{rank } R = N$ and (S, Q) is observable.

Proof: The ‘only if’ statement follows from the previous two lemmas. To prove the ‘if’ statement, suppose that $\text{rank } S = N$ and (Q, R) is controllable. Let $\bar{Q}$ be a matrix such that $S(zI - Q)^{-1}R = S(zI - \bar{Q})^{-1}R$. Since S has full column rank, (S, Q) is observable. In addition, since (Q, R) is controllable, there exists a nonsingular matrix $T \in \mathbb{R}^{N \times N}$ such that $TQT^{-1} = \bar{Q}$, $TR = R$ and $ST^{-1} = S$. Using the fact that $\text{rank } S = N$, we see that $T^{-1} = I$ and hence $T = I$. Therefore $Q = \bar{Q}$. We conclude that the topology of (2) is identifiable. The proof for the case that $\text{rank } R = N$ and (S, Q) is observable is analogous, and hence omitted. ■

Remark 1: Theorem 5 shows that either full state measurement or full state excitation is necessary for topological identification in the ‘single integrator’ case. Similar observations were made in [19, Thm. 1] in the scenario where both Q and R are to-be-identified, and in [11, Rem. 2] in the context of autonomous networked systems.

B. Identifiability for general dynamics

Next, we study identifiability for general network dynamics given by (2). For analysis purposes we first rewrite the network transfer matrix $F_Q(z)$. To this end, note that

$$zI - A = (zI - A - BQC) + BQC.$$

We premultiply by $(zI - A)^{-1}$ and postmultiply by $(zI - A - BQC)^{-1}$ to obtain

$$(zI - A - BQC)^{-1} = (zI - A)^{-1} + (zI - A)^{-1}BQC(zI - A - BQC)^{-1}.$$

Premultiplication with C and postmultiplication with the matrix B yields

$$C(zI - A - BQC)^{-1}B = G(z) + G(z)QC(zI - A - BQC)^{-1}B,$$

where $G(z) = C(zI - A)^{-1}B$ is a block diagonal matrix containing the transfer matrices of all node systems. By rearranging terms, we obtain

$$C(zI - A - BQC)^{-1}B = (I - G(z)Q)^{-1}G(z). \quad (5)$$

Note that the inverse of $I - G(z)Q$ exists as a rational matrix. Indeed, since $(zI - A)^{-1}$ is strictly proper we see that $\lim_{z \rightarrow \infty} (I - G(z)Q) = I$. Finally, we conclude from (5) that the transfer matrix $F_Q(z)$ equals

$$F_Q(z) = S(I - G(z)Q)^{-1}G(z)R. \quad (6)$$

We remark that (6) is an attractive representation of the network transfer matrix, since the known matrices A , B and C are grouped and contained in the transfer matrix $G(z)$.

Remark 2: Using the expression (6), we can represent the networked system (2) by the block diagram in Figure 2. Hence, the problem of topology identification can be viewed as the identification of the *static output feedback gain* Q , assuming knowledge of the system $G(z)$ and the external input/output matrices R and S .

Fig. 1. Block diagram of the networked system (2).

Next, we find conditions for topological identifiability in the general case. Let $G_i(z) := C_i(zI - A_i)^{-1}B_i$ denote the transfer matrix from v_i to w_i of node system $i \in \mathcal{V}$.

Theorem 6: If the topology of (2) is identifiable then for all $i, j \in \mathcal{V}$ the constant kernel of $G_i^\top(z) \otimes G_j(z)$ is trivial, that is,

$$(G_i^\top(z) \otimes G_j(z)) w = 0 \text{ for } w \in \mathbb{R}^{p_i m_j} \implies w = 0. \quad (7)$$

In addition, if $R = I$ and $S = I$ then the condition (7) is necessary and sufficient for topological identifiability of (2).

Proof: We first prove the second statement. That is, we assume that $R = I$ and $S = I$ and we want to show that the topology of (2) is identifiable if and only if (7) holds. Suppose that $F_Q(z) = F_{\bar{Q}}(z)$, where $\bar{Q}$ is real. Then, from (6) we have

$$(I - G(z)Q)^{-1} G(z) = (I - G(z)\bar{Q})^{-1} G(z). \quad (8)$$

We define $\Delta := Q - \bar{Q}$. Then, (8) is equivalent with each of the following statements:

$$(I - G(z)\bar{Q})(I - G(z)Q)^{-1} G(z) = G(z) \quad (9)$$

$$(I - G(z)(Q - \Delta))(I - G(z)Q)^{-1} G(z) = G(z) \quad (10)$$

$$G(z)\Delta(I - G(z)Q)^{-1} G(z) = 0 \quad (11)$$

$$G(z)\Delta G(z)(I - QG(z))^{-1} = 0 \quad (12)$$

$$G(z)\Delta G(z) = 0 \quad (13)$$

$$(G^\top(z) \otimes G(z)) \text{vec}(\Delta) = 0, \quad (14)$$

where $\text{vec}(\Delta)$ denotes the vectorization of Δ . By (14) it is clear that the topology of (2) is identifiable if and only if the constant kernel of $G^\top(z) \otimes G(z)$ is trivial. Now, by the block diagonal structure of $G(z)$, we conclude that the topology of (2) is identifiable if and only if the constant kernel of $G_i^\top(z) \otimes G_j(z)$ is trivial for all $i, j \in \mathcal{V}$.

Finally, it is not difficult to prove the necessity of (7) for topological identifiability with general matrices R and S . Indeed, suppose that (7) is not satisfied. By the previous discussion, (8) is satisfied for some matrix $\bar{Q} \neq Q$. By premultiplication with S and postmultiplication with R we conclude that $F_Q(z) = F_{\bar{Q}}(z)$. In other words, the topology of (2) is not identifiable. ■

Remark 3: The assumption $S = I$ and $R = I$ in the second part of Theorem 6 means that the *outputs* of all nodes in the network are measured, and the *inputs* $v_i(t)$ of all network nodes can be fully excited. It is worthwhile to note that full state measurement or full state excitation is generally *not* necessary for topological identifiability of (2). This is in contrast with the special case studied in Section III-A (see Rem. 1). The intuition is that the interconnection matrix Q is in general of lower dimension than the network state-space dimension. In fact, the seemingly restrictive conditions of Theorem 5 arise from the fact that Q is an $n \times n$ matrix whenever $A_i = 0$, $B_i = 1$ and $C_i = 1$.

A consequence of Theorem 6 is that identifiability of the topology of (2) implies that the constant kernels of $G_i^\top(z)$ and $G_i(z)$ are trivial for all $i \in \mathcal{V}$. Based on this fact, we relate topological identifiability and output controllability of the node systems. We first recall the definition of output controllability.

Definition 2: Consider the system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \quad (15)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$, and let $y_{u, x_0}(\cdot)$ denote the output trajectory of (15) for a given initial condition x_0 and input $u(\cdot)$. System (15) is called *output controllable* if for every $x_0 \in \mathbb{R}^n$ and $y_1 \in \mathbb{R}^p$ there exists an input $u(\cdot)$ and time instant $T \in \mathbb{N}$ such that $y_{u, x_0}(T) = y_1$.

Corollary 7: If the topology of (2) is identifiable then the systems (A_i, B_i, C_i) and $(A_i^\top, C_i^\top, B_i^\top)$ are output controllable for all $i \in \mathcal{V}$.

Proof: By Theorem 6, identifiability of the topology of (2) implies that the constant kernel of $G_i^\top(z)$ is trivial for all $i \in \mathcal{V}$. Now, for $w \in \mathbb{R}^{p_i}$ we have $w^\top G_i(z) = 0$ if and only if $w^\top C_i A_i^k B_i = 0$ for all $k = 0, 1, \dots$, equivalently, $w^\top C_i A_i^k B_i = 0$ for all $k = 0, 1, \dots, n_i - 1$. Hence, the implication $w^\top G_i(z) = 0 \implies w = 0$ is equivalent with the implication

$$w^\top (C_i B_i \quad C_i A_i B_i \quad \dots \quad C_i A_i^{n_i-1} B_i) = 0 \implies w = 0.$$

The latter implication holds if and only if the output controllability matrix of (A_i, B_i, C_i) has full row rank, equivalently (A_i, B_i, C_i) is output controllable [18, Ex. 3.22]. The proof for the necessity of output controllability of $(A_i^\top, C_i^\top, B_i^\top)$ is analogous and hence omitted. ■

Remark 4: Output controllability of (A_i, B_i, C_i) can be interpreted as an ‘excitability’ condition. Indeed, it guarantees that we have enough freedom in steering the output $w_i(t)$ of each node $i \in \mathcal{V}$.

IV. RESULTS ON TOPOLOGY IDENTIFICATION

In this section, we focus on the problem of topology identification, as formulated in Problem 2. We assume that $R = I$ and $S = I$. As we will see, the problem is still interesting under this simplifying assumption, although we expect to relax this condition in future work. Our approach to reconstruct Q will use ideas from subspace identification [20], [21]. Roughly speaking, subspace identification methods can identify a minimal (i.e., controllable and observable) state-space realization of the transfer matrix $F_Q(z)$ on the basis of input/output data obtained from (2). Therefore, under the assumption that (2) is minimal, a possible strategy for topology identification is the following. First, identify a minimal state-space realization (K, L, M) of $F_Q(z)$ using subspace identification. Then, find a similarity transformation T that brings K to the required form, i.e., solve the following equations in the unknowns² T and Q :

$$\begin{aligned} TKT^{-1} &= A + BQC, & TL &= B, \\ MT^{-1} &= C, & \text{rank } T &= n. \end{aligned}$$

However, this approach is rather indirect, and it would be more attractive to identify Q directly from input/output data.

²We recall that the node system dynamics described by A, B, C are assumed to be known. Therefore, the unknowns are T and Q .

Therefore, in what follows we establish a tailor-made scheme for topology identification.

For the sake of simplicity, we first assume that (2) is minimal. Later, we will also outline the approach for non-minimal systems, see Remark 6. We follow the narrative of [20] and define the block Hankel matrices

$$U_p(\ell) := \begin{pmatrix} u(0) & u(1) & \cdots & u(\ell-1) \\ u(1) & u(2) & \cdots & u(\ell) \\ \vdots & \vdots & & \vdots \\ u(k-1) & u(k) & \cdots & u(k+\ell-2) \end{pmatrix}$$

and

$$U_f(\ell) := \begin{pmatrix} u(k) & u(k+1) & \cdots & u(k+\ell-1) \\ u(k+1) & u(k+2) & \cdots & u(k+\ell) \\ \vdots & \vdots & & \vdots \\ u(2k-1) & u(2k) & \cdots & u(2k+\ell-2) \end{pmatrix},$$

where k and ℓ are sufficiently large integers such that $k \geq n$ and $\ell \geq 2mk + n$. The argument (ℓ) refers to the number of columns of U_p and U_f and is omitted if the dimensions of U_p and U_f are clear from the context. We define the matrices $Y_p(\ell)$ and $Y_f(\ell)$ in an analogous way, using the output vectors $y(0), y(1), \dots, y(2k+\ell-2)$. The matrices U_p and Y_p are often referred to as *past* (input/output) data, while U_f and Y_f are referred to as the *future* data. For ease of notation, we define $A_Q := A + BQC$. Then, by introducing the extended observability matrix Γ_k and the Toeplitz matrix of Markov parameters H_k as

$$\Gamma_k := \begin{pmatrix} C \\ CA_Q \\ \vdots \\ CA_Q^{k-1} \end{pmatrix} \text{ and } H_k := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ CA_Q^{k-2}B & \cdots & CB & 0 \end{pmatrix},$$

we obtain the equations

$$Y_p = \Gamma_k X_p + H_k U_p \quad (16)$$

$$Y_f = \Gamma_k X_f + H_k U_f, \quad (17)$$

where X_p and X_f are matrices of state vectors, i.e.,

$$X_p := \begin{pmatrix} x(0) & x(1) & \cdots & x(\ell-1) \\ x(k) & x(k+1) & \cdots & x(k+\ell-1) \end{pmatrix}.$$

We write $X_p(\ell)$ and $X_f(\ell)$ whenever the dimensions of X_p and X_f might lead to confusion. One of the central ideas of subspace identification is that the row space of the state sequence X_f can be obtained from measured input/output data assuming that a persistency of excitation condition holds. In particular, Willems *et al.* [22, Cor. 2(iii)] have shown that

$$\text{rank} \begin{pmatrix} U_p \\ U_f \\ X_p \end{pmatrix} = 2mk + n \quad (18)$$

if the input u is persistently exciting of sufficiently high order. By exploiting (16), (17) and the fact that (2) is

minimal, it can be verified that (18) implies that

$$\text{rank} \begin{pmatrix} U_p \\ Y_p \end{pmatrix} = \text{rank} \begin{pmatrix} U_f \\ Y_f \end{pmatrix} = mk + n, \text{ and} \quad (19)$$

$$\text{rank} \begin{pmatrix} U_p \\ Y_p \\ U_f \\ Y_f \end{pmatrix} = 2mk + n. \quad (20)$$

Then, following [20, Sec. 4.1], it can be shown that

$$\text{im } X_f^\top = \text{im} \begin{pmatrix} U_p \\ Y_p \end{pmatrix}^\top \cap \text{im} \begin{pmatrix} U_f \\ Y_f \end{pmatrix}^\top, \quad (21)$$

meaning that the row space of X_f coincides with the intersection of the row spaces of past and future data.

Remark 5: Several papers (e.g. [20], [21]) on subspace identification start from the basic assumption that the three rank conditions in (19) and (20) hold. However, we argue that it is more natural to start from the *single* condition (18), which is equivalent to (19), (20). Condition (18) can be easily related to a persistency of excitation condition on the input u via [22, Cor. 2(iii)]. This allows us to enforce the rank condition (18) by suitable choice of inputs.

Next, choose $\hat{X}_f \in \mathbb{R}^{\ell \times n}$ such that the columns of $\hat{X}_f^\top$ form a basis for the subspace in (21). This means that $T\hat{X}_f = X_f$ for some nonsingular $T \in \mathbb{R}^{n \times n}$. We define X_f^+ as the matrix obtained from X_f by removing its first column, and X_f^- as the matrix obtained from X_f by removing its last column. The matrices $\hat{X}_f^+$ and $\hat{X}_f^-$ are obtained from $\hat{X}_f$ in similar fashion. At this point, subspace algorithms typically use the data $\hat{X}_f^+$, $\hat{X}_f^-$, U_f and Y_f , and solve a system of linear equations to obtain a minimal state-space system compatible with the data, see e.g. [20, Sec. 4.2]. However, in our setup, the matrices A , B and C are known and we are looking for a state matrix $A + BQC$ of *specific structure*. In fact, we have the following result which gives conditions under which Q (and T) can be obtained as the unique solution of a set of linear equations in terms of the data matrices and local system matrices.

Theorem 8: Consider system (2) and assume that $R = I$, $S = I$ and (2) is minimal. Let $k \geq n$ and $\ell > 2mk + n$ and assume that the following two conditions hold:

- 1) The topology of (2) is identifiable.
- 2) The input u is persistently exciting of sufficiently high order [22] such that

$$\text{rank} \begin{pmatrix} U_p(\ell-1) \\ U_f(\ell-1) \\ X_p(\ell-1) \end{pmatrix} = 2mk + n. \quad (22)$$

Then (T, Q) is the unique solution to the following two linear equations in the unknown (T, Q) :

$$T\hat{X}_f^+ = AT\hat{X}_f^- + BQ\hat{Y}_f + B\hat{U}_f \quad (23)$$

$$\hat{Y}_f = CT\hat{X}_f^-. \quad (24)$$

Here $\hat{U}_f := (u(k) \cdots u(k+\ell-2))$ and $\hat{Y}_f$ is defined as $\hat{Y}_f := (y(k) \cdots y(k+\ell-2))$.

Proof: Since (22) holds for $U_p(\ell - 1)$, $U_f(\ell - 1)$ and $X_p(\ell - 1)$, it is also satisfied for $U_p(\ell)$, $U_f(\ell)$ and $X_p(\ell)$. By (21), the columns of $\hat{X}_f^\top$ form a basis for $\text{im } X_f^\top(\ell)$. Note that $X_f^+ = (A + BQC)X_f^- + B\hat{U}_f$ and $\hat{Y}_f = CX_f^-$. By substitution of $X_f^+ = T\hat{X}_f^+$ and $X_f^- = T\hat{X}_f^-$, it follows that (T, Q) is a solution to (23), (24). What remains to be shown is the uniqueness. Let $(\bar{T}, \bar{Q})$ satisfy (23), (24). As T is nonsingular by definition, we have that

$$\begin{pmatrix} \bar{T}T^{-1}A_QT & \bar{T}T^{-1}B \\ CT & 0 \end{pmatrix} \begin{pmatrix} \hat{X}_f^- \\ \hat{U}_f \end{pmatrix} = \begin{pmatrix} A_Q\bar{T} & B \\ C\bar{T} & 0 \end{pmatrix} \begin{pmatrix} \hat{X}_f^- \\ \hat{U}_f \end{pmatrix}.$$

Since X_f can be expressed as a linear combination of X_p and U_p (cf. [20, Sec. 4.1]), it can be shown that (22) implies

$$\text{rank} \begin{pmatrix} \hat{X}_f^- \\ \hat{U}_f \end{pmatrix} = n + m.$$

This means that $\bar{T}T^{-1}A_Q = A_Q\bar{T}T^{-1}$, $\bar{T}T^{-1}B = B$ and $C\bar{T}T^{-1} = C$. As a consequence, $\Gamma(Q) = \Gamma(\bar{Q})\bar{T}T^{-1}$, where $\Gamma(Q)$ denotes the observability matrix of the pair (C, A_Q) . By observability of (C, A_Q) , the matrix $\bar{T}$ has full rank. This shows that (A_Q, B, C) and $(A_{\bar{Q}}, B, C)$ are similar, hence $F_Q(z) = F_{\bar{Q}}(z)$. Since the topology of (2) is identifiable, we conclude that $Q = \bar{Q}$. Finally, from $\Gamma(Q) = \Gamma(\bar{Q})\bar{T}T^{-1}$ we conclude that $\Gamma(Q) = \Gamma(Q)\bar{T}T^{-1}$. By observability, $\bar{T}T^{-1} = I$, that is, $T = \bar{T}$. ■

To summarize, the proposed topology identification approach uses the data $U_p(\ell)$, $U_f(\ell)$, $Y_p(\ell)$ and $Y_f(\ell)$ to find a state sequence $\hat{X}_f$. Then, Q is obtained by solving the linear equations (23), (24). These equations can be solved, for instance, via Kronecker products. Note that equations of the form (23) have also been studied in [23, Sec. 7.2].

Remark 6: In Theorem 8 we have assumed that (2) is minimal. A similar theorem can be stated for non-minimal systems. To do so, let $(\hat{A}, \hat{B}, \hat{C})$ be a minimal realization of $G(z)$. It follows from the equivalence of (3) and (6) that $F_Q(z) = \hat{C}(zI - \hat{A} - \hat{B}Q\hat{C})^{-1}\hat{B}$. In fact, $(\hat{A} + \hat{B}Q\hat{C}, \hat{B}, \hat{C})$ is a minimal realization of $F_Q(z)$ since controllability and observability are invariant under output feedback (see e.g. [18, Ex. 3.9]). Using these facts, Theorem 8 can be adapted for the case of non-minimal systems by replacing (A, B, C) in (23) and (24) by $(\hat{A}, \hat{B}, \hat{C})$, and by replacing n by the dimension of $\hat{A}$. Note that this also reduces the dimension of the unknown T in (23), (24) to the dimension of $\hat{A}$.

V. CONCLUSIONS

In this paper, we have studied topology identification for heterogeneous networks of linear systems. First, we have established conditions for topological identifiability. Subsequently, we have provided a topology identification scheme based on subspace methods. The approach uses Hankel matrices of observed input/output data to compute a state sequence. Then, the interconnection matrix is obtained by solving linear equations involving the data matrices and the local system matrices. For future work, it is of interest to extend the approach to arbitrary external input/output matrices R and S , and to consider networks with disturbances.

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