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Brief paper

Descriptive vector, relative error matrix, and interaction analysis of multivariable plants[☆]Nima Monshizadeh-Naini^{*}, Alireza Fatehi, Ali Kahki-Sedigh

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ABSTRACT

In this paper, we introduce a vector which is able to describe the Niederlinski Index (NI), Relative Gain array (RGA), and the characteristic equation of the relative error matrix. The spectral radius and the structured singular value of the relative error matrix are investigated. The cases where the perfect result of the Relative Gain Array, equal to the identity matrix, coincides with the least interaction in a plant are pointed out. Then, the Jury Algorithm is adopted to get some insight into interaction analysis of multivariable plants. In particular, for interaction analysis of 3×3 plants, simple yet promising conditions in terms of the Relative Gain Array and the Niederlinski Index are derived. Several examples are also discussed to illustrate the main points.

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1. Introduction

Despite the significant advances of centralized controllers, decentralized controllers are still preferred in practice since they are fault tolerant, simple to tune, and easy to understand. A key issue in decentralized control of multivariable plants is the so-called *input–output pairing*. An appropriate pairing decision is the one which minimizes the interaction among loops.

In the last four decades, several interaction measures (IM) have been introduced to achieve the best control configuration for decentralized control of multivariable plants (see Khaki-Sedigh & Moaveni, 2009). Notions of spectral radius and structured singular value along with introducing the relative error matrix led to some of the best known interaction measures (Grosdidier & Morari, 1986). In fact, these measures provide critical information regarding both the stability and performance of the systems under decentralized control. In addition, they are further adopted to provide sufficient conditions for two key system properties which are Integral Controllability (IC) and Decentralized Integral Controllability (DIC) (Skogestad & Morari, 1992).

On the other hand, the well-known Relative Gain Array (RGA), proposed by Bristol (1966) is recognized as a basic tool for

input–output pairing of multivariable plants. The advantages of Relative Gain Array mainly lie in its simple calculation, scaling invariant property, and the fact that it has to be computed only once. However, this simplicity has cost some drawbacks.

The fact that the original RGA was defined at steady-state led to advent of numerous dynamic pairing methods (e.g. Mc Avoy, Arkun, Chen, Robinson, & Schnelle, 2003; Monshizadeh-Naini, Fatehi, & Khaki-Sedigh, 2009).

Hovd and Skogestad (1992) introduced a counterexample to the RGA conventional pairing rules. Moreover, an example is provided by Johnson and Shapiro (1986) which implies that strong interaction may exist in a plant enjoying a perfect RGA (i.e. RGA equals to the Identity matrix).

As a whole, the RGA itself cannot provide sufficient information regarding the actual interaction present in a plant. On the other hand, adopting the spectral radius and specially the structured singular value for interaction analysis implies high computational load.

The Niederlinski Index (NI), introduced by Niederlinski (1971), provides a necessary condition for stability of a system under decentralized control. The result with slight modification is reported by Grosdidier, Morari, and Holt (1985), and has been generalized for block decentralized controllers in Grosdidier and Morari (1986), Kariwala, Forbes, and Meadows (2003).

In spite of the fact that, the conventional interaction analysis tools such as the RGA, the NI, and the Relative Error Matrix (REM) have different viewpoints, and provide dissimilar descriptions of the interaction, they are not entirely independent. To investigate the relation, we introduce a vector, according to which, the mentioned three tools can be parametrically described. We call it the *descriptive vector* since it is capable of describing the

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determinant of the plant, RGA, NI, and characteristic equation of the REM, which gives the spectral radius and describes the structured singular value. The characteristic equation of the REM is further investigated to point out the cases where the interaction is significant and where it is negligible. Consequently, the cases where the perfect result of RGA refers to interaction-free plants are determined. It will be seen that these cases are identifiable by number of nonzero elements in the descriptive vector.

One may expect that computationally involved interaction measures, regarding the REM, may be simplified in the plants with limited number of inputs and outputs. The interaction analysis for 2×2 plants has been widely discussed in previous works (see Grosdidier & Morari, 1986; Skogestad & Morari, 1992). In this paper, interesting conditions for interaction analysis of 3×3 plants are derived. Those conditions only depend on the Relative Gain Array (RGA) and the Niederlinski Index (NI).

The paper is organized as follows: In Section 2, some basic concepts are firstly discussed. In Section 3, the descriptive vector is introduced and its applications are discussed. In Section 4, the characteristic equation of the REM is investigated to point out the cases where the interaction is significant and where it is negligible. The points made in this section, are also accompanied by several illustrative examples. The Jury Algorithm is adopted in Section 5 to derive some information and attractive conditions for interaction analysis. This section is mainly dedicated to 3×3 plants, and some examples are discussed to demonstrate the simplicity and effectiveness of the proposed conditions. Finally, Section 6 is allocated to conclusion and summaries.

2. Preliminaries

Let $G(s)$ denote a stable square $m \times m$ plant as follows:

$$G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) & \dots & g_{1m}(s) \\ g_{21}(s) & g_{22}(s) & \dots & g_{2m}(s) \\ \dots & \dots & \dots & \dots \\ g_{m1}(s) & g_{m2}(s) & \dots & g_{mm}(s) \end{bmatrix}. \quad (1)$$

Definition 1. The Relative Error Matrix (REM) is defined as (Grosdidier & Morari, 1986)

$$E(s) = (G(s) - G_d(s))G_d^{-1}(s) \quad (2)$$

where $G_d(s)$ denotes the diagonal part of $G(s)$. The REM provides a powerful tool for interaction analysis. In particular, computing the spectral radius and the structured singular value of the REM at steady-state provide the following valuable information (Skogestad & Morari, 1992):

- (i) Sufficient conditions for IC and DIC of a system:
 - The system is Integral Controllable if $\rho(E(0)) < 1$, where ρ denotes the spectral radius.
 - The plant is Decentralized Integral Controllable if $\mu(E(0)) < 1$, where μ denotes the structured singular value computed with respect to a diagonal structure.
- (ii) Design simplicity, i.e. smaller values of $\rho(E(0))$ and $\mu(E(0))$ account for less interactions and simpler design.

Definition 2. The Relative Gain Array of the plant $G(s)$ is defined as (Hovd & Skogestad, 1992):

$$\Lambda(s) = G(s) \cdot \times G^{-T}(s) \quad (3)$$

where $\cdot \times$ denotes the element by element product.

Definition 3. The Niederlinski Index of the plant $G(s)$ is defined as (Niederlinski, 1971):

$$NI = \frac{\det(G(0))}{\prod_{i=1}^m g_{ii}(0)}. \quad (4)$$

Note that, in stable plants, positive NI is a necessary condition for stability under decentralized control (Grosdidier et al., 1985).

3. Descriptive vector

As mentioned earlier, the RGA, the NI, the spectral radius, and the structured singular value of the REM provide different descriptions of the interaction present in plant. Here, through the definition of the descriptive vector, the underlying relation among those interaction analysis tools is investigated.

Definition 4. The descriptive vector for an $m \times m$ multivariable plant is defined as

$$v(s) = [v_1(s), v_2(s), \dots, v_n(s)]^T \quad (5)$$

where $v_i(s)$ is the product of the corresponding transfer function elements of the i th possible pairing selection, and n is equal to the number of possible pairing choices which is $m!$. For instance, the descriptive vector for a 3×3 plant, is given by

$$v = [g_{11}g_{22}g_{33}, g_{11}g_{23}g_{32}, g_{12}g_{21}g_{33}, g_{13}g_{22}g_{31}, g_{12}g_{23}g_{31}, g_{13}g_{21}g_{32}]^T. \quad (6)$$

Note that in (6) and what follows, the s variables with respect to the descriptive vector and transfer function elements are dropped for simplicity.

Each element of the descriptive vector refers to a certain pairing choice. That is, v_1 refers to the first possible pairing choice, v_2 indicates the second, and v_n refers to the n th possible pairing selection. There is no mandatory order for the descriptive vector elements; however, we assume that v_1 corresponds to the diagonal pairing.

Remark 1. Permuting a plant can only change the order of the descriptive vector elements since these elements refer to alternative pairing schemes.

Here, some definitions from the theory of determinants are discussed (Muir, 1960).

Definition 5. Number of inversion. With the r quantities $a_1, a_2, \dots, a_r$ we may form $r!$ permutations. The elements of a permutation are said to be arranged in their natural order when the suffixes are arranged in an ascending order. Any permutation in which an element precedes another with a less suffix is said to contain an inversion. The number of inversion is defined as the number of interchanges of consecutive elements necessary to arrange them in their natural order.

As a case in point, the array a_3, a_2, a_4, a_1 has 4 inversions which are

$$a_3 \rightarrow a_2, \quad a_3 \rightarrow a_1, \quad a_2 \rightarrow a_1, \quad \text{and} \quad a_4 \rightarrow a_1.$$

Assume that we represent a pairing choice with an array of elements where the suffixes referring to the rows are in an ascending order. Now, the permutations and the number of inversions in Definition 5 can be deduced solely based on the

suffixes referring to the columns. For instance, the pairing choice $\{(y_1, u_3), (y_2, u_2), (y_3, u_4), (y_4, u_1)\}$ in a 4×4 plant is represented as $\{g_{13}, g_{22}, g_{34}, g_{41}\}$ and contains 4 inversions as mentioned above. Bear in mind that the array corresponding to the diagonal pairing is in natural order and has no inversion.

As mentioned earlier, several interaction analysis tools can be described through the elements of the descriptive vector.

Let w be a vector defined as

$$w = [(-1)^{\gamma_1}, (-1)^{\gamma_2}, \dots, (-1)^{\gamma_n}]^T \quad (7)$$

where γ_i is the number of inversion corresponding to the i th pairing choice, identified by the i th element of the descriptive vector v_i . Obviously, the elements of w are independent of the values of plant elements, and they are either $+1$ or -1 . Now, the following items can be stated in terms of descriptive vector elements.

Determinant of the Plant. According to the theory of determinants (Muir, 1960), it is directly inferred that

$$\det(G) = w^T v. \quad (8)$$

The Niederlinski Index. Based on the definition of the descriptive vector, (4), and (8)

$$NI = \frac{w^T v}{v_1} \Big|_{s=0}. \quad (9)$$

The Relative Gain Array. According to (3), the RGA elements, denoted by λ_{ij} , are obtained as

$$\lambda_{ij} = (-1)^{i+j} \frac{g_{ij} \det(G^{ij})}{\det(G)} \quad (10)$$

where G^{ij} denotes the matrix obtained after removing the i th row and j th column. Considering the expansion of the determinant along the i th row, it can be inferred that the value of λ_{ij} indicates the share of the (i, j) th subsystem in the plant determinant. Hence, in terms of the descriptive vector, RGA elements can be expressed as

$$\lambda_{ij} = 1 - \frac{w^T v|_{g_{ij}=0}}{w^T v}. \quad (11)$$

The REM. Assuming that the plant is rearranged so that the selected pairs are along the diagonal, the characteristic equation of the REM is computed as follows:

$$\det(E - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & g_{12} & \dots & g_{1m} \\ g_{21} & -\lambda & \dots & g_{2m} \\ g_{11} & \dots & -\lambda & \dots \\ \dots & \dots & \dots & -\lambda \\ g_{m1} & g_{m2} & \dots & -\lambda \end{vmatrix} = 0. \quad (12)$$

Defining $z = -\lambda$, the characteristic polynomial of the REM is in the form:

$$H(z) = \frac{1}{g_{11}g_{22} \dots g_{mm}} \begin{vmatrix} g_{11}z & g_{12} & \dots & g_{1m} \\ g_{21} & g_{22}z & \dots & g_{2m} \\ \dots & \dots & \dots & \dots \\ g_{m1} & g_{m2} & \dots & g_{mm}z \end{vmatrix}. \quad (13)$$

Note that replacing λ by $z = -\lambda$ does not change the magnitude of the eigenvalues; hence, the derivation of the spectral radius is not affected by this replacement. Let $H_0(z)$ denote the determinant in the right-hand side of the above equation. Then $H_0(z)$ can be represented in a polynomial form as

$$H_0(z) = a_m z^m + a_{m-2} z^{m-2} + \dots + a_1 z + a_0. \quad (14)$$

Clearly, $a_m = \Pi$, where Π denotes the product of g_{ii} , $i = 1, 2, \dots, m$. In terms of the descriptive vector, it is straightforward to investigate that the polynomial $H_0(z)$ can be expressed as

$$H_0(z) = [z^{q_1}, z^{q_2}, \dots, z^{q_n}](w \cdot \times v) \quad (15)$$

where q_i denotes the number of diagonal elements in the i th pairing selection, corresponding to the i th element of the descriptive vector. Obviously, the largest root of the polynomial $H_0(z)$ or $H(z)$ is the spectral radius of the REM.

Using the same notations as above, the discussed results are summarized in the following proposition.

Proposition 1. Based on the definition of the descriptive vector, the following items can be deduced.

Niederlinski Index (given by (9)).

Relative Gain Array: (given by (11)).

The polynomial which gives $\rho(E)$ (given by (15)).

4. Further investigation of the REM

It is well known that the RGA for triangular plants is equal to the identity matrix (Hovd & Skogestad, 1992). The reverse result holds for 2×2 plants, and similarly for 3×3 plants providing that we also take into account the essentially triangular plants. Essentially triangular plants are those that can be rendered triangular by rows and columns permutations. Unfortunately, this cannot be generalized to $m \times m$ plants. In fact, in 4×4 plants or larger, $\Lambda(s) = I$ is not sufficient to conclude the overall stability based on the stability of the individual loops (see Example 2). An alternative condition in terms of input–output pairs is stated in following theorem.

Theorem 1. If any possible pairing selection, except the diagonal pairing, corresponds to at least one transfer function element with a zero value, then we have:

- (i) $\Lambda(s) = I$
- (ii) $\rho(E(s)) = \mu(E(s)) = 0$.

Proof. (i) If any possible pairing selection, except the diagonal pairing, corresponds to at least one transfer function element with a zero value, then the descriptive vector is in the form

$$v = [\Pi \quad 0 \quad \dots \quad 0]^T. \quad (16)$$

According to Proposition 1,

$$\lambda_{ij} = 1 - \frac{w^T v|_{g_{ij}=0}}{w^T v}. \quad (17)$$

Based on (16), $w^T v = \Pi$ which is independent of any off-diagonal element. Hence, the ratio in the right-hand side of (17) is equal to 1 with respect to any off-diagonal element, and $\Lambda(s) = I$.

(ii) Considering (15) and (16), $H_0(z) = \Pi z^m$ which yields $\rho(E(s)) = 0$.

The relation between the spectral radius and structured singular value is given by Skogestad and Postlethwaite (2005):

$$\mu_\Delta(E) = \max_{\Delta, \bar{\sigma}(\Delta) \leq 1} \rho(E\Delta). \quad (18)$$

Here, μ is computed with respect to a diagonal structure (i.e. $\Delta = \text{diag}\{\delta_1, \delta_2, \dots, \delta_m\}$). The diagonal entries of REM are zero. Hence, only off-diagonal elements of the plant are perturbed as $\hat{g}_{ij} = g_{ij}\delta_j$ in formation of $H(z)$ in (13). Now, since the values of δ_j appear as multiplicative terms, they cannot make any zero element of the descriptive vector nonzero and the descriptive vector remains in the same form as (16). Hence, $\mu(E(s)) = 0$. $\square$

Example 1. Consider the following plants:

$$G_1 = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & g_{23} \\ g_{31} & 0 & g_{33} \end{bmatrix}, \quad G_2 = \begin{bmatrix} g_{11} & g_{11} & 0 & g_{14} \\ 0 & g_{22} & 0 & g_{24} \\ g_{31} & g_{32} & g_{31} & g_{34} \\ 0 & 0 & 0 & g_{44} \end{bmatrix}.$$

It is obvious that, for both plants, any pairing decision other than the diagonal pairing implies to pair on at least one transfer function element with a zero value. So, according to [Theorem 1](#), the RGA is the Identity matrix and $\mu(E) = 0$.

Remark 2. As stated before, $\Lambda(s) = I$ does not generally imply the least interaction. Indeed, a plant with $\Lambda(s) = I$ may still suffer from strong interaction (see [Example 2](#)). [Theorem 1](#), points out the cases where the result of $\Lambda(s) = I$ is reliable. In fact, $v = [\pi \ 0 \ \dots \ 0]^T$ guarantees the least interaction, and implies that the overall stability is not jeopardized as long as the individual loops are stable.

Remark 3. The assumption of diagonal pairing is just for simplicity. In fact, we deal with an interaction-free plant if the descriptive vector contains exactly one nonzero element. By interaction-free, we mean that $\mu(E(s)) = 0$ for the permuted plant, and individual stability, therefore, results in overall stability.

Remark 4. It is easy to observe that the maximum number of nonzero elements in descriptive vector of a triangular or essentially triangular plant is 1. Actually, the statement is trivial for triangular plants, and based on [Remark 1](#), the result is also valid for essentially triangular plants. Consequently, the plant is neither triangular nor essentially triangular if the number of nonzero elements in the corresponding descriptive vector is more than 1.

Example 2. Consider the following matrix ([Skogestad & Postlethwaite, 2005](#))¹:

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & \alpha & 1 & 1 \\ 1 & 1 & \beta & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

where α and β are positive scalars. The RGA is the identity matrix in this case. So, there is no interaction from RGA viewpoint. However, there are multiple pairing choices with nonzero elements, and the number of descriptive vector nonzero elements is more than 1. Hence, based on [Remark 4](#), $\Lambda = I$ does not refer to an essentially triangular matrix in this case.

Further analysis can be made by looking at (13) and (14).

$$H(z) = z^m + \frac{1}{\Pi} (a_{m-2}z^{m-2} + \dots + a_1z + a_0). \quad (19)$$

Therefore, as the product of diagonal elements approaches zero, the spectral radius, and consequently the structured singular value of the REM become arbitrarily large. In fact, strong interaction exists when the product of the selected pairs is close to zero, while that of some other pairing choices is not.

Hence, in [Example 2](#), strong interaction is expected for small values of $\alpha\beta$. Assuming the desired open-loop transfer function of individual loops as $1/s$, the resulting diagonal controller fails to stabilize the plant for $\alpha\beta < 0.25$.

Fig. 1. The magnitude of $\mu(E)$: solid line, and RGA number: dash-dot line.

Example 3. Consider the following 3×3 plant,

$$G(s) = \begin{bmatrix} \frac{2}{1+s} & \frac{0.5}{1+s} & \frac{3}{1+s} \\ \frac{2}{1+10s} & \frac{1}{1+s} & \frac{0.5}{1+10s} \\ 0 & \frac{3}{1+s} & \frac{3}{1+s} \end{bmatrix}.$$

The static RGA for the above plant is:

$$\Lambda = \begin{bmatrix} 0.37 & -0.12 & 0.75 \\ 0.63 & 0.50 & -0.12 \\ 0 & 0.62 & 0.37 \end{bmatrix}.$$

It is seen that, at zero frequency, the RGA implies strong interactions among the loops. Nevertheless, at higher frequencies, the values of g_{21} and g_{23} tend to zero much faster than the other elements; hence, the condition of [Theorem 1](#) approximately holds at these frequencies, and the descriptive vector is close to the ideal form described in (16). Therefore, without any calculation, it is expected to have small $\mu(E)$, and RGA close to the Identity Matrix at relatively high frequencies. To verify this, $\mu(E)$ and the RGA number are sketched in a range of frequencies (see [Fig. 1](#)).

Note that the RGA number, for a diagonal pairing, is defined as ([Skogestad & Postlethwaite, 2005](#))

$$\text{RGA number} = \|\Lambda - I\|_{\text{sum}}, \quad (20)$$

where $\|\cdot\|_{\text{sum}}$ is the sum norm, computed by adding up the absolute magnitude of all matrix elements.

The figure shows that at zero frequency the RGA is far from the Identity matrix and $\mu(E(0)) > 1$, whereas at higher frequencies, the RGA number is relatively small, and $\mu(E) < 1$.

5. Interaction analysis using the Jury Algorithm

Computing $H(z)$, given by (13), at $s = 0$ gives

$$\bar{H}(z) = \frac{1}{\bar{g}_{11}\bar{g}_{22} \dots \bar{g}_{mm}} \begin{bmatrix} \bar{g}_{11}z & \bar{g}_{12} & \dots & \bar{g}_{1m} \\ \bar{g}_{21} & \bar{g}_{22}z & \dots & \bar{g}_{2m} \\ \dots & \dots & \dots & \dots \\ \bar{g}_{m1} & \bar{g}_{m2} & \dots & \bar{g}_{mm}z \end{bmatrix} \quad (21)$$

where $\bar{g}_{ij}$ denotes the steady-state value of $g_{ij}(s)$. A subtle point is that $\bar{H}(z)$ can be interpreted as the characteristic polynomial of a discrete time plant. Hence, the constraint $\rho(E(0)) < 1$ is equivalent to the stability of that plant. Consequently, the Jury Algorithm ([Ogata, 1995](#)) can be adopted to check the spectral radius condition. In what follows, computation of the Niederlinski index, relative gains, and REM are carried out at steady-state.

¹ The given matrix is a generalization of a counterexample given by [Johnson and Shapiro \(1986\)](#). Though the example is hypothetical, the above matrix with a minor perturbation can be matched to a physical blending process (http://www.nt.ntnu.no/users/skoge/book/typo/RGA_counterexample.txt).

Proposition 2. For an $m \times m$ plant, we have $\rho(E(0)) < 1$ only if $NI > 0$.

Proof. According to the Jury Algorithm, a necessary condition for stability of $\bar{H}(z)$ is $\bar{H}(1) > 0$. Besides, replacing $z = 1$ in (21) obtains the Niederlinski Index. So, positive NI is a necessary condition for $\rho(E(0)) < 1$.

Here, the interaction analysis of 2×2 and 3×3 plants is provided.

Two inputs–two outputs plant:

The polynomial $\bar{H}(z)$ for a 2×2 plant is

$$\bar{H}(z) = \frac{1}{\bar{g}_{11}\bar{g}_{22}}(\bar{g}_{11}\bar{g}_{22}z^2 - \bar{g}_{12}\bar{g}_{21}), \quad \text{and} \quad \rho(E(0)) < 1 \quad (22)$$

results in $|\frac{\bar{g}_{12}\bar{g}_{21}}{\bar{g}_{11}\bar{g}_{22}}| < 1$ which in terms of the RGA is stated as (Grosdidier & Morari, 1986):

$$\lambda_{11} > 0.5. \quad (23)$$

It is worth mentioning that the above result can be easily generalized in a range of frequencies. Therefore, it is desired to have $|\lambda_{11}(j\omega)| > 0.5$ in the frequency range of interest. $\square$

Three inputs–three outputs plant:

Theorem 2. For a reordered 3×3 plant, $\rho(E(0)) < 1$ if and only if, the following constraints hold simultaneously

$$\begin{aligned} \text{(i) } NI > 0 \quad \text{(ii) } \text{Trace}(\Lambda) &> \frac{2}{NI} + 0.5 \\ \text{(iii) } |e_1| < 1 \quad \text{(iv) } \left| \frac{e_2}{1 - e_1^2} \right| &< 1 \end{aligned}$$

where

$$e_1 = NI \times (\text{Trace}(\Lambda) - 1) - 2 \quad (24)$$

$$e_2 = NI \times \text{Trace}(\Lambda) - 3. \quad (25)$$

Proof. See the Appendix.

Based on above constraints, to have $\rho(E(0)) < 1$, the point $(NI, \text{Trace}(\Lambda))$ should be located inside the area shown in Fig. 2. In addition, the farther the point $(NI, \text{Trace}(\Lambda))$ is located from the depicted area, the more interaction is available in the plant. Thus, with the help of Fig. 2, we can gain an estimation of the interaction by an instant look to the values of the RGA and the NI. It is well known that large RGA elements are inappropriate due to robustness considerations. As it can be seen from Fig. 2, the desired area gets narrower as the value of $\text{Trace}(\Lambda)$ increases. Thus, a small perturbation in the plant would drive the point $(NI, \text{Trace}(\Lambda))$ out of the promising area. Finally, note that for triangular plants, the values of NI and $\text{Trace}(\Lambda)$ are equal to 1 and 3, respectively. Therefore, for triangular plants, the point $(NI, \text{Trace}(\Lambda))$ is perfectly inside the desired area.

Computing the structured singular value implies high computational load. The main incentive to compute $\mu(E(0))$, rather than $\rho(E(0))$, is that $\mu(E(0)) < 1$ guarantees the DIC. However, this advantage is waived for 3×3 plants since $\rho(E(0)) < 1$ is also sufficient to ensure DIC. This is stated in the following proposition. $\square$

Proposition 3. A three inputs–three outputs plant with positive diagonal RGA elements is DIC if

$$\rho(E(0)) < 1. \quad (26)$$

Proof. A 3×3 plant with positive NI and diagonal RGA elements is DIC if (Yu & Fan, 1990):

$$\sqrt{\lambda_{11}} + \sqrt{\lambda_{22}} + \sqrt{\lambda_{33}} > 1. \quad (27)$$

Fig. 2. Schematic view of the area implying $\rho(E(0)) < 1$.

Note that the above condition is also necessary to have DIC (Yu & Fan, 1990), providing that we ignore the cases where $\sqrt{\lambda_{11}} + \sqrt{\lambda_{22}} + \sqrt{\lambda_{33}}$ is identically equal to 1 (Campo & Morari, 1994).

According to the first and third constraints in Theorem 2, $\rho(E(0)) < 1$ implies $NI > 0$ and $\text{Trace}(\Lambda) > 1$. Now, if any of diagonal RGA elements is larger than one, the inequality (27) trivially holds. On the other hand, if all diagonal elements are equal or smaller than one, then:

$$\sqrt{\lambda_{11}} + \sqrt{\lambda_{22}} + \sqrt{\lambda_{33}} \geq \lambda_{11} + \lambda_{22} + \lambda_{33}. \quad (28)$$

Hence, $\text{Trace}(\Lambda) > 1$ implies $\sqrt{\lambda_{11}} + \sqrt{\lambda_{22}} + \sqrt{\lambda_{33}} > 1$, and the plant is DIC. $\square$

Example 4. Consider the following 3×3 plant (Skogestad & Postlethwaite, 2005):

$$G(0) = \begin{bmatrix} -5 & 1 & 2 \\ 4 & 2 & -1 \\ -3 & -2 & 6 \end{bmatrix}.$$

The NI is equal to 1.25 for the above plant. The RGA is

$$\Lambda = \begin{bmatrix} 0.67 & 0.28 & 0.05 \\ 0.53 & 0.64 & -0.17 \\ -0.2 & 0.08 & 1.12 \end{bmatrix},$$

and $\text{Trace}(\Lambda) = 2.43$. So, all the constraints of Theorem 2 hold as: $1.25 > 0$, $2.43 > 2.1$, $|-0.22| < 1$, $|0.04| < 1$.

Hence, $\rho(E(0)) < 1$. Consequently, a detuneable diagonal controller with integral action can be easily designed for the above plant, and diagonal pairing is promising in this case. Note that the above result can also be deduced by Fig. 2, since the point $(1.25, 2.43)$ is inside the promising area.

Remark 5. To check the proposed constraints for pairings other than diagonal, we need not permute the plant to compute the value of $\text{Trace}(\Lambda)$. That is because any permutation of rows and columns in plant, results in the same permutation in the RGA (Hovd & Skogestad, 1992). Consequently, the value of $\text{Trace}(\Lambda)$ is simply computed by the summation of corresponding RGA elements for any pairing choice. On the contrary, computation of NI generally requires that the plant should be permuted first so that the selected pairs are along the diagonal. Based on Definition 5, the number of necessary permutation to locate the chosen pairs along the diagonal is the same as the inversion number. Thus, to avoid the permutation, one can alternatively use the following formula to compute the Niederlinski Index for a pairing other than diagonal.

$$NI = (-1)^p \frac{\det(G)}{p} \quad (29)$$

Table 1

Comparing the two alternate pairings in Example 5.

	$ e_1 $	$ e_2 $
Pairing (a)	52	78
Pairing (b)	1.5	0.75

where γ denotes the number of inversions corresponding to the selected pairing, and P is the product of transfer function elements corresponding to that pairing choice.

Note that the above formula is not limited to 3×3 plants, and it is applicable for the general case of $m \times m$ multivariable plants.

Here, the discussed points and results are partially summarized in the following corollary.

Corollary 1. *If a three input–three output plant satisfies the constraints of Theorem 2, or alternatively the point $(NI, \text{Trace}(\Lambda))$ is located inside the area depicted in Fig. 2, then*

- (i) *The plant with selected control configuration is DIC.*
- (ii) *The interaction is small and satisfactory performance is anticipated with decentralized control.*

Note that, as it will be observed in Example 5, interaction in a plant enjoying the DIC may still be considerable and result in a poor closed-loop performance.

In diagonal and triangular plants, RGA is equal to the Identity Matrix, and the NI is equal to 1. Hence, according to (24) and (25), the values of e_1 , and e_2 are equal to zero, and the third and fourth constraints are satisfied in a best way (i.e. with maximum margin). This point intuitively provides the insight that the magnitude of e_1 , and e_2 can be interpreted as *error terms* from an ideal plant. It also implies that to have relatively small interaction, not only the corresponding RGA elements but also the NI should be close to 1.

Example 5. Consider a 3×3 plant with (Hovd & Skogestad, 1992):

$$G(s) = \frac{(1-s)}{(1+5s)^2} \begin{bmatrix} 1 & -4.19 & -25.96 \\ 6.19 & 1 & -25.96 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\Lambda(s) = \begin{bmatrix} 1 & 5 & -5 \\ -5 & 1 & 5 \\ 5 & -5 & 1 \end{bmatrix}.$$

There are two alternate pairings with positive RGA elements: (a) the diagonal pairing, and (b) $\{(u_2, y_1), (u_3, y_2), (u_1, y_3)\}$. Note that according to (27), the plant is DIC for both pairings. The diagonal pairing seems to be the best since it implies pairing on RGA elements equal to 1. However, the NI corresponding to diagonal pairing is equal to 27, which is far from unity. Hence, we should not expect little interaction by diagonal pairing. In addition, we have: $\text{Trace}(\Lambda^{(a)}) = 3$ and $\text{Trace}(\Lambda^{(b)}) = 15$ for the first and second pairing choices, respectively. Also, $NI^{(a)} = 27$ and $NI^{(b)} = 0.25$. Note that the value of $NI^{(b)}$ can be conveniently computed using (29) and $\gamma = 2$. It can be easily verified that both pairings satisfy the necessary conditions, i.e. the first two constraints in Theorem 2. In addition, it is easy to check that the fourth constraint is violated by pairing (a), but satisfied by pairing (b). However, neither of the pairings satisfies the third constraint, and we can never have $\rho(E(0)) < 1$. It means achievable closed-loop performance by a decentralized controller may be limited. The values of the error terms, given by (24) and (25), are summarized in Table 1.

Thus, far less interaction is expected by pairing (b). This conclusion is also verified by Hovd and Skogestad (1992). They found that pairing (a) leads to significantly worse closed-loop responses than pairing (b).

6. Conclusions

In this paper, the Niederlinski Index, the Relative Gain array, and the characteristic equation of the REM are described in terms of a vector called descriptive vector. The descriptive vector shed lights on the underlying relation between the conventional interaction analysis tools.

In addition, the characteristic polynomial of the REM is analyzed. It is discussed that interaction-free cases are identifiable based on the number of nonzero elements in the descriptive vector. The Jury Algorithm is adopted to provide some insight for interaction analysis. Specially, for 3×3 plants, it is shown that satisfactory performance is anticipated when both the corresponding RGA elements and the Niederlinski Index are close to 1. Based on the Jury Algorithm, simple conditions are derived for interaction analysis of 3×3 plants. Those conditions are also adopted to identify the best pairing. The simplicity and effectiveness of the proposed conditions are verified through explanatory examples.

Appendix

Proof of Theorem 2. The polynomial $\bar{H}(z)$, given by (21), for a 3×3 plant is represented as:

$$\bar{H}(z) = \frac{1}{\Pi} (\Pi z^3 + a_1 z + a_0). \quad (30)$$

According to the Jury Algorithm, all roots of $\bar{H}(z)$ are located inside the unit circle if and only if, the following constraints hold simultaneously:

$$\begin{aligned} & \text{(i) } \bar{H}(1) > 0 \quad \text{(ii) } \bar{H}(-1) < 0 \\ & \text{(iii) } \left| \frac{a_0}{\Pi} \right| < 1 \quad \text{(iv) } \left| \frac{a_1}{\Pi} \right| < \left| 1 - \frac{a_0^2}{\Pi^2} \right|. \end{aligned}$$

Consider the descriptive vector of a 3×3 plant as

$$v = [\bar{g}_{11}\bar{g}_{22}\bar{g}_{33}, \bar{g}_{11}\bar{g}_{23}\bar{g}_{32}, \bar{g}_{12}\bar{g}_{21}\bar{g}_{33}, \bar{g}_{13}\bar{g}_{22}\bar{g}_{31}, \bar{g}_{12}\bar{g}_{23}\bar{g}_{31}, \bar{g}_{13}\bar{g}_{21}\bar{g}_{32}]^T. \quad (31)$$

Based on Proposition 1, the coefficients a_1 and a_0 in terms of the descriptive vector elements are stated as

$$a_1 = -(v_2 + v_3 + v_4) \quad (32)$$

$$a_0 = v_5 + v_6. \quad (33)$$

The RGA elements can be described in terms of descriptive vector elements as

$$\begin{aligned} \lambda_{11} \det(G) &= v_1 - v_2, & \lambda_{22} \det(G) &= v_1 - v_4, \\ \lambda_{33} \det(G) &= v_1 - v_3, & \lambda_{12} \det(G) &= -v_3 + v_5, \\ \lambda_{13} \det(G) &= -v_4 + v_6. \end{aligned}$$

Hence,

$$a_1 = -(v_2 + v_3 + v_4) = (\lambda_{11} + \lambda_{22} + \lambda_{33}) \det(G) - 3v_1.$$

Considering $v_1 = \Pi$, we obtain

$$a_1 = \Gamma \times \det(G) - 3\Pi \quad (34)$$

where $\text{Trace}(\Lambda)$ is denoted by Γ .

In addition, $a_0 = v_5 + v_6 = (\lambda_{12} + \lambda_{13}) \det(G) + v_3 + v_4$

$$\Rightarrow a_0 = (\lambda_{12} + \lambda_{13}) \det(G) + 2v_1 - (\lambda_{22} + \lambda_{33}) \det(G)$$

$$\Rightarrow a_0 = (1 - \lambda_{11}) \det(G) + 2v_1 - (\Gamma - \lambda_{11}) \det(G)$$

which yields

$$a_0 = 2\Pi - (\Gamma - 1) \det(G). \quad (35)$$

Now, based on (34) and (35), we define e_1 and e_2 as

$$e_1 = NI \times (\Gamma - 1) - 2 = -\frac{a_0}{\Pi} \quad (36)$$

$$e_2 = NI \times \Gamma - 3 = \frac{a_1}{\Pi} \quad (37)$$

$$(i) \bar{H}(1) > 0.$$

Based on Proposition 2, the constraint is equivalent to:

$$NI > 0. \quad (38)$$

$$(ii) \bar{H}(-1) < 0.$$

Based on (30), $\frac{1}{\Pi}(-\Pi - a_1 + a_0) < 0$. Using (34) and (35), $\frac{1}{\Pi}(4\Pi - 2\Gamma \times \det(G) + \det(G)) < 0$ which yields

$$\Gamma > \frac{2}{NI} + 0.5. \quad (39)$$

$$(iii) \left| \frac{a_0}{\Pi} \right| < 1.$$

According to (36), this constraint is equivalent to:

$$|e_1| < 1 \quad (40)$$

$$(iv) \left| \frac{a_1}{\Pi} \right| < \left| 1 - \frac{a_0^2}{\Pi^2} \right|.$$

Using (36) and (37), the above constraint is simplified to:

$$\left| \frac{e_2}{1 - e_1^2} \right| < 1. \quad \square \quad (41)$$

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