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The Scalar Theory of Nonradiating Partially Coherent Sources.

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The existence of nonradiating sources is crucial for the uniqueness of related inverse source problems. Sources can be determined from the emitted radiation only up to their nonradiating part ⁽¹⁾. While sufficient and necessary conditions are known for nonradiating deterministic sources ⁽²⁾, the corresponding problem for stochastic sources and fields of arbitrary degree of coherence has not been studied hitherto. In this letter we present a simple Fourier transform criterion which allows us to check whether given source correlations are nonradiating. Moreover we devise a general procedure that allows the construction of such correlations. The known deterministic results are reproduced in the coherent limit (*i.e.* factorizing correlations).

We start with the time-independent inhomogeneous wave equation in three dimensions

$$(1) \quad (\nabla^2 + k^2) \psi(\mathbf{r}) = \varrho(\mathbf{r}), \quad \mathbf{r} \in \mathbf{R}^3,$$

with the stochastic scalar field variable

$$(2) \quad \psi(\mathbf{r}, t) = \psi(\mathbf{r}) \exp[-i\omega t]$$

and the stochastic source distribution

$$(3) \quad \varrho(\mathbf{r}, t) = \varrho(\mathbf{r}) \exp[-i\omega t].$$

We assume that $\varrho(\mathbf{r})$ is confined to a finite domain $D \subset \mathbf{R}^3$. The unique solution of eq. (1) satisfying Sommerfelds radiation condition at infinity is

$$(4) \quad \psi(\mathbf{r}; k) = \int_D G(\mathbf{r}, \mathbf{r}'; k) \varrho(\mathbf{r}') d\mathbf{r}'$$

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(1) B. J. HOENDERS: *Inverse Source Problems in Optics*, edited by H. P. BALTES (Heidelberg, 1978), p. 41.

(2) J. K. COHEN and N. BLEISTEIN: *Journ. Nucl. Phys.*, **13**, 194 (1977).

with

$$(5) \quad G(\mathbf{r}, \mathbf{r}'; k) = |\mathbf{r} - \mathbf{r}'|^{-1} \exp [ik|\mathbf{r} - \mathbf{r}'|] .$$

The first-order correlation of this field reads ^(3,4)

$$(6) \quad \langle \psi(\mathbf{r}_1; k) \psi^*(\mathbf{r}_2; k) \rangle = \iint_D d\mathbf{r}'_1 d\mathbf{r}'_2 G(\mathbf{r}_1, \mathbf{r}'_1; k) G^*(\mathbf{r}_2, \mathbf{r}'_2; k) \langle \varrho(\mathbf{r}_1) \varrho^*(\mathbf{r}_2) \rangle ,$$

where the brackets $\langle \dots \rangle$ denote an ensemble average. The source correlation $\langle \varrho(\mathbf{r}_1) \varrho^*(\mathbf{r}_2) \rangle$ is called *nonradiating* if the field correlation (6) is identically zero for all $\mathbf{r}_1$ or/and $\mathbf{r}_2$ outside D . As we show below, this definition is equivalent to the requirement that the field correlation (6) vanishes in the far zone.

From eq. (6) and the asymptotic expansion

$$(7) \quad G(\mathbf{r}, \mathbf{r}'; k) \simeq r^{-1} \exp [ikr] \exp [-iks\mathbf{r}']$$

with $\mathbf{s} = \mathbf{r}/r$, $r = |\mathbf{r}|$, we find that a *necessary condition for a nonradiating stochastic source* is the vanishing of the far zone correlations:

$$(8) \quad \langle \varrho(k\mathbf{s}_1) \varrho^*(k\mathbf{s}_2) \rangle = \iint_D d\mathbf{r}_1 d\mathbf{r}_2 \exp [-ik(\mathbf{s}_1\mathbf{r}_1 - \mathbf{s}_2\mathbf{r}_2)] \langle \varrho(\mathbf{r}_1) \varrho^*(\mathbf{r}_2) \rangle = 0$$

for all $\mathbf{s}_1 = \mathbf{r}_1/r_1$ and $\mathbf{s}_2 = \mathbf{r}_2/r_2$.

Now we show that the condition (8) is also *sufficient*. Using Bauer's expansion ⁽⁵⁾

$$(9) \quad \exp [-iks\mathbf{r}] = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l (-i)^l j_l(k|\mathbf{s}|\mathbf{r}) Y_l^m(\theta_s, \varphi_s) Y_l^{m*}(\theta_r, \varphi_r)$$

with the spherical Bessel functions j_l and with (θ_s, φ_s) and (θ_r, φ_r) denoting the polar angles of the vectors $\mathbf{s}$ and $\mathbf{r}$, respectively, and using the linear independence of the spherical harmonic Y_l^m , we rewrite condition (8) in terms of discrete variables, *viz.*

$$(10) \quad c_{l,l',m,m'} = \iint_D d\mathbf{r}'_1 d\mathbf{r}'_2 j_l(kr'_1) j_{l'}(kr'_2) Y_l^{m*}(\theta'_1, \varphi'_1) Y_{l'}^{m'}(\theta'_2, \varphi'_2) \langle \varrho(\mathbf{r}'_1) \varrho^*(\mathbf{r}'_2) \rangle = 0$$

for all l, l', m, m' . Multiplying eq. (10) by

$$(11) \quad Y_l^m(\theta_1, \varphi_1) Y_{l'}^{m'*}(\theta_2, \varphi_2) h_l^{(1)}(kr_1) h_{l'}^{(1)*}(kr_2) ,$$

summing over all l, l', m, m' , and using the expansion $(r' < r)$

$$(12) \quad |\mathbf{r} - \mathbf{r}'|^{-1} \exp [ik|\mathbf{r} - \mathbf{r}'|] = 2ik \sum_{l=0}^{\infty} \sum_{m=-l}^l j_l(kr') h_l^{(1)}(kr) Y_l^m(\theta, \varphi) Y_l^{m*}(\theta', \varphi') ,$$

⁽³⁾ J. K. KLAUDER and E. C. G. SUDARSHAN: *Foundamentals of Quantum Optics*, sect. I (New York, N. Y., 1968).

⁽⁴⁾ M. J. BERAN and G. B. PARRENT jr.: *Theory of Partial Coherence* (Englewood Cliffs, N. J., 1964).

⁽⁵⁾ G. N. WATSON: *A Treatise on the Theory of Bessel Functions*, 2nd ed., subsect. 4'82 (Cambridge, Mass., 1966).

with the spherical Hankel functions of the first kind $h_l^{(1)}$, we find that the field correlation (6) vanishes outside the source domain D . Thus we have shown that the necessary condition (8) is also sufficient.

Now we describe how a source correlation fulfilling condition (8) can be constructed. Consider a twice continuously differentiable function $h(\mathbf{r}_1, \mathbf{r}_2)$ which vanishes outside D and is nonzero inside D . Then the source correlation

$$(13) \quad \langle \varrho(\mathbf{r}_1) \varrho^*(\mathbf{r}_2) \rangle \equiv (\nabla_1^2 + k^2)(\nabla_2^2 + k^2)h(\mathbf{r}_1, \mathbf{r}_2)$$

is nonradiating, as can be seen by inserting (13) into the criterion (8). Alternatively, the correlation (13) can also be shown to be nonradiating by using Green's theorem and eq. (6). Moreover, every nonradiating source correlation necessarily is of the form (13); for example take

$$(14) \quad h(\mathbf{r}_1, \mathbf{r}_2) = \langle \psi(\mathbf{r}_1) \psi^*(\mathbf{r}_2) \rangle.$$

Condition (8) or the equivalent set of conditions (10) can be interpreted as follows: only those source correlations are nonradiating whose projections on the Hilbert space vectors

$$j_l(kr_1)j_{l'}(kr_2)Y_l^m(\theta_1, \varphi_1)Y_{l'}^{m'*}(\theta_2, \varphi_2)$$

vanish. Therefore any correlation $\langle \varrho(\mathbf{r}_1) \varrho^*(\mathbf{r}_2) \rangle$ on $D \otimes D$, from which the projections have been subtracted, are nonradiating. Such source correlations are responsible for the nonuniqueness in the related inverse problem.

The above theory can be extended to *higher order* coherence theory. The corresponding necessary and sufficient condition reads

$$(15) \quad \int \dots \int d\mathbf{r}_1 \dots d\mathbf{r}_{2n} \exp[ik(\mathbf{s}_1 \mathbf{r}_1 + \dots + \mathbf{s}_n \mathbf{r}_n - \mathbf{s}_{n+1} \mathbf{r}_{n+1} - \dots - \mathbf{s}_{2n} \mathbf{r}_{2n})] \cdot \\ \cdot \langle \varrho(\mathbf{r}_1) \dots \varrho(\mathbf{r}_n) \varrho^*(\mathbf{r}_{n+1}) \dots \varrho^*(\mathbf{r}_{2n}) \rangle = 0$$

for all directions $\mathbf{s}_1, \dots, \mathbf{s}_{2n}$.

The corresponding theory of nonradiating stochastic *first-order current correlations* can be developed along the same lines in the framework of the theory of partially coherent electromagnetic fields. The necessary and sufficient condition corresponding to condition (8) reads

$$(16) \quad \langle \hat{\mathbf{j}}^T(k\mathbf{s}_1) \hat{\mathbf{j}}^{T*}(k\mathbf{s}_2) \rangle = 0$$

for all directions $\mathbf{s}_1, \mathbf{s}_2$, where $\hat{\mathbf{j}}^T(\mathbf{s}) = \mathbf{s} \times \mathbf{s} \times \hat{\mathbf{j}}(\mathbf{s})$, with $\hat{\mathbf{j}}(\mathbf{s})$ denoting the Fourier transform of the stochastic current $\mathbf{j}(\mathbf{r})$.

We point out that the above results are valid only for three-dimensional source correlations. We observe that *two-dimensional field* correlations on a finite surface always radiate as can easily be deduced from a definition analogous to (6) by virtue of continuity arguments.