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Schaft, A.J. van der; Maschke, B.M.

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MATHEMATICAL MODELING OF CONSTRAINED HAMILTONIAN SYSTEMS

A.J. VAN DER SCHAFT* and B.M. MASCHKE**

*Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands, e-mail: a.j.vanderschaft@math.utwente.nl

**Lab. d'Automatisme Industriel, Conservatoire National des Arts et Métiers, 21 rue Pinel, F-75013 Paris, France

Abstract. Network modelling of unconstrained energy conserving physical systems leads to an intrinsic generalized Hamiltonian formulation of the dynamics. Constrained energy conserving physical systems are directly modelled as implicit Hamiltonian systems with regard to a generalized Dirac structure on the space of energy variables, which can be rewritten (under a nondegeneracy condition on the internal energy) as an explicit generalized Hamiltonian system on the constrained state space. This is specialized to mechanical systems with kinematic constraints.

Key Words. mechanical systems, constraints, Hamiltonian dynamics

In [1] it has been shown how by using a generalized bond graph formalism the dynamics of non-resistive physical systems (belonging to different domains, i.e., electrical, mechanical, hydraulical etc.) can be given an *intrinsic* Hamiltonian formulation. Subsequently in [6] the interaction of non-resistive physical systems with their environment has been formalized by the inclusion of external ports, naturally yielding two conjugated sets of external variables: the inputs u represented as generalized flow sources, and the outputs y which are the corresponding conjugated efforts. The Hamiltonian dynamics of the network model is described as follows. Let $x = (x_1, \dots, x_n)$ denote the vector of energy variables (which are assumed to be independent) associated with all the energy storing elements, $H(x)$ be the total stored energy, and $J(x)$ the modulated gyrator (in bond graph terminology) which is associated with the network topology (interconnection structure of the network). Since the interconnections are all power conserving $J(x)$ satisfies the important property

$$J(x) = -J^T(x), \text{ for all } x \quad (1)$$

Finally, let the column vectors $g_j(x)$ denote the (state modulated) transformers describing the influence of the external flow sources $u_j, j = 1, \dots, m$. The conjugated efforts, denoted by y_j , are given as

$$y_j = g_j^T(x) \frac{\partial H}{\partial x}(x), \quad j = 1, \dots, m \quad (2)$$

with

$$\frac{\partial H}{\partial x}(x) := \left[\frac{\partial H}{\partial x_1}(x), \dots, \frac{\partial H}{\partial x_n}(x) \right]^T$$

the column vector of partial derivatives of H (i.e., the efforts conjugated to the flows $\dot{x}_i = \frac{dx_i}{dt}, i = 1, \dots, n$). The Hamiltonian dynamics of the network model is then given as

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + \sum_{j=1}^m g_j(x) u_j \quad (3)$$

$$y_j = g_j^T(x) \frac{\partial H}{\partial x}(x), \quad j = 1, \dots, m$$

or more compactly, with $g(x)$ the $n \times m$ matrix with columns $g_j(x), j = 1, \dots, m$, and $u := (u_1, \dots, u_m)^T, y := (y_1, \dots, y_m)^T$

$$\begin{aligned} \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x) u \\ \Sigma : \quad y &= g^T(x) \frac{\partial H}{\partial x}(x) \end{aligned} \quad (4)$$

Σ is called a *port-controlled generalized Hamiltonian system* with Hamiltonian H , generalized Poisson structure matrix $J(x)$, and input matrix $g(x)$. It immediately follows from (1) and (4) that along the trajectories of Σ

$$\frac{d}{dt} H = u^T y \quad (5)$$

which expresses *energy conservation*. (Note that $u^T y$ is the external power applied to the system.) Mathematically this is further formalized as follows. In general, the vector $x = (x_1, \dots, x_n)$ is a

local coordinate vector for the state space manifold $\mathcal{X}$ (in the simplest case equal to $\mathbb{R}^n$). We assume that the entries of $J(x)$ and $g(x)$ are smooth functions of x . The matrix $J(x)$ defines a *generalized Poisson bracket* $\{, \}$ on $\mathcal{X}$ by letting in local coordinates x the Poisson bracket of two arbitrary smooth functions F and G on $\mathcal{X}$ be defined as

$$\{F, G\}(x) := \frac{\partial F^T}{\partial x}(x) J(x) \frac{\partial G}{\partial x}(x), \quad (6)$$

$$F, G : \mathcal{X} \rightarrow \mathbb{R}$$

Clearly, this bracket satisfies the properties

$$\{F, G\} = -\{G, F\}, \quad \forall F, G \quad (7)$$

$$\{F, G \cdot H\} = \{F, G\} \cdot H + G \cdot \{F, H\} \quad (8)$$

$$\forall F, G, H$$

Conversely, a bracket $\{, \}$ satisfying (7) and (8) defines a structure matrix $J(x)$ satisfying (1) by defining in local coordinates $(x_1, \dots, x_n)$, $J_{ij}(x) := \{x_i, x_j\}$. However, $\{, \}$ defined by $J(x)$ satisfying (1) is called a *generalized Poisson bracket* on $\mathcal{X}$ since it need not necessarily satisfy the third defining property of a true Poisson bracket;

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0, \quad \forall F, G, H \quad (9)$$

In fact, (9) is equivalent to the following "integrability" condition on $J(x)$:

$$\sum_{l=1}^n \left[J_{lj}(x) \frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x) \frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x) \frac{\partial J_{ji}}{\partial x_l}(x) \right] = 0, \quad i, j, k = 1, \dots, n \quad (10)$$

as follows by substituting the coordinate functions x_i, x_j, x_k in (9). The property (9) or equivalently (10) is called the *Jacobi identity*.

We have seen that the Jacobi-identity is not important for the energy conservation property (5); however it is important at least in the following sense.

Theorem 1 (see e.g. [3]) Let $\{, \}$ be a generalized Poisson bracket on $\mathcal{X}$ with structure matrix $J(x)$ satisfying (1). Suppose $J(x)$ has constant rank $2k$ in a neighborhood of a point $x_0 \in \mathcal{X}$. Then there exist local coordinates $\tilde{x} = (q_1, \dots, q_k, p_1, \dots, p_k, r_1, \dots, r_\ell)$, $2k + \ell = n$, about x_0 such that in these new coordinates

$$J(\tilde{x}) = \begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (11)$$

if and only if $\{, \}$ satisfies the Jacobi-identity (9).

The coordinates $(q_1, \dots, q_k, p_1, \dots, p_k, r_1, \dots, r_\ell)$ in the above theorem are called *canonical*; note that in these coordinates the Hamiltonian dynamics (4) for $u = 0$ amounts to

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q, p, r) & i = 1, \dots, k \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q, p, r) & i = 1, \dots, k \\ \dot{r}_j &= 0 & j = 1, \dots, \ell \end{aligned} \quad (12)$$

which are almost the *standard Hamiltonian equations* of motion (except for the conserved quantities $r_1, \dots, r_\ell$).

In previous papers we have treated three classes of port-controlled generalized Hamiltonian systems, namely *LC-circuits* [5], mechanical systems with kinematic constraints [9,7], and multi-body systems [8]. In the present paper we wish to focus on the modeling and analysis of *constrained* generalized Hamiltonian systems, and we will show how the above mentioned work on mechanical systems with constraints fits within this general treatment. The constraints under consideration are *effort-constraints* modeled by setting the outputs (efforts) of some external ports equal to zero, i.e.

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + \sum_{j=1}^m g_j(x) u_j \quad (13)$$

$$0 = y_j = g_j^T(x) \frac{\partial H}{\partial x}(x), \quad j = 1, \dots, m$$

Here the variables $u_1, \dots, u_m$ are the corresponding external flows which take values such that the constraints $y_j = 0$ are satisfied for all time. For simplicity of notation we will not consider extra controlled flow sources with extra conjugated efforts (which do not have to be zero). Thus the efforts of all the external ports are set equal to zero and represent the (power-continuous) constraints imposed on the system.

Furthermore, we will throughout assume that for every initial condition x_0 with $g^T(x_0) \frac{\partial H}{\partial x}(x_0) = 0$ there exists a unique $u(t)$, $t \geq 0$, such that the solution $x(t)$ of (13) will satisfy $g^T(x(t)) \frac{\partial H}{\partial x}(x(t)) = 0$, for all $t \geq 0$. This can be ensured as follows. First we note that the output $y_j = g_j^T(x) \frac{\partial H}{\partial x}(x)$ can be more concisely rewritten as $y_j = L_{g_j} H(x)$, with L_X denoting the Lie-(directional) derivative along X . For any function $F : \mathcal{X} \rightarrow \mathbb{R}$ the vector field determined by F and $J(x)$ as

$$\dot{x} = J(x) \frac{\partial F}{\partial x}(x) \quad (14)$$

will be denoted as $\dot{x} = X_F(x)$, and is called the *Hamiltonian vector field* with respect to J and

Hamiltonian F . We note the useful identity

$$L_{X_F} G = \frac{\partial^T G}{\partial x}(x) J(x) \frac{\partial F}{\partial x}(x) = \{G, F\}(x) \quad (15)$$

$$F, G : \mathcal{X} \rightarrow \mathbb{R}$$

with $\{, \}$ the bracket determined by J .

Using the above, the time-derivative of the constraint function $y_j = L_{g_j} H$ along (13) is given as

$$\dot{y}_j = \{L_{g_j} H, H\}(x) + \sum_{i=1}^m L_{g_j} L_{g_i} H(x) u_i \quad (16)$$

Assumption 2 The $m \times m$ matrix

$[L_{g_j} L_{g_i} H(x)]_{i,j=1,\dots,m}$ is invertible for all $x \in \mathcal{X}$ such that $L_{g_j} H(x) = 0$, $j = 1, \dots, m$.

It immediately follows from (16) and Assumption 2 that for every x_0 satisfying the constraints $L_{g_j} H(x_0) = 0$, $j = 1, \dots, m$, there exists a unique $u(t)$, $t \geq 0$, such that $L_{g_j} H(x(t)) = 0$, $j = 1, \dots, m$, i.e., the constraints are satisfied for all $t \geq 0$. Thus we may define the *constrained state space*

$$\mathcal{X}_c = \{x \in \mathcal{X} \mid L_{g_j} H(x) = 0, j = 1, \dots, m\} \quad (17)$$

and for every $x \in \mathcal{X}_c$ there exists a unique $u(\cdot)$ such that the motion $x(\cdot)$ remains in $\mathcal{X}_c$. We now wish to describe this resulting dynamics on $\mathcal{X}_c$. In order to do so we employ the notion of a Dirac structure [1,2], already used in [10] for describing implicit generalized Hamiltonian systems.

Let us first define *constant* Dirac structures on *vector spaces*. Since we will only deal with finite-dimensional systems we denote the vector spaces by $\mathcal{V}^n$, with n its dimension.

Definition 3 ([1], [2]) A (constant) *Dirac structure* on $\mathcal{V}^n$ is an n -dimensional subspace $L \subset \mathcal{V}^n \times (\mathcal{V}^n)^*$ with the property that

$$\langle x|y' \rangle + \langle x'|y \rangle = 0, \quad (18)$$

$$\forall (x, y), (x', y') \in L,$$

with $\langle | \rangle$ denoting the natural pairing between $\mathcal{V}^n$ and $(\mathcal{V}^n)^*$.

For our purposes the following equivalent formulation is more suited [10].

Proposition 4 A (constant) Dirac structure on $\mathcal{V}^n$ is an n -dimensional subspace $L \subset \mathcal{V}^n \times (\mathcal{V}^n)^*$ with the property that

$$\langle x|y \rangle = 0, \text{ for all } (x, y) \in L \quad (19)$$

We have the following useful representation of Dirac structures. For linear maps $F : \mathcal{V}^n \rightarrow$

$\mathcal{W}^n, E : (\mathcal{V}^n)^* \rightarrow \mathcal{W}^n$ let us define $F + E : \mathcal{V}^n \times (\mathcal{V}^n)^* \rightarrow \mathcal{W}^n$ as

$$(x, y) \in \mathcal{V}^n \times (\mathcal{V}^n)^* \xrightarrow{F+E} F(x) + E(y) \in \mathcal{W}^n \quad (20)$$

Proposition 5 [10]

- (i) Every Dirac structure $L \subset \mathcal{V} \times (\mathcal{V}^n)^*$ can be written as $L = \ker(F + E)$ for certain linear maps $F : \mathcal{V}^n \rightarrow \mathcal{W}^n, E : (\mathcal{V}^n)^* \rightarrow \mathcal{W}^n$. Furthermore, any such E and F satisfy

$$EF^* + FE^* = 0 \quad (21)$$

- (ii) Every n -dimensional subspace $L = \ker(F + E)$ defined by linear maps $F : \mathcal{V}^n \rightarrow \mathcal{W}^n, E : (\mathcal{V}^n)^* \rightarrow \mathcal{W}^n$ satisfying (21) defines a Dirac structure.

Remark 6 If F is invertible, then L is simply given by the *graph* of the linear map $-F^{-1}E$, which is *skew-symmetric* by (21).

Dirac structures on manifolds are now defined as follows. Let $\mathcal{X}$ be a manifold with tangent bundle $T\mathcal{X}$ and cotangent bundle $T^*\mathcal{X}$. We define $T\mathcal{X} \oplus T^*\mathcal{X}$ as the (smooth) vector bundle over $\mathcal{X}$ with fiber at each $x \in \mathcal{X}$ given by $T_x\mathcal{X} \times T_x^*\mathcal{X}$.

Definition 7 ([1], [2]) A *generalized Dirac structure* on a manifold $\mathcal{X}$ is given by a smooth vector subbundle $L \subset T\mathcal{X} \oplus T^*\mathcal{X}$ such that the linear space $L(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$ is a Dirac structure (in the sense of Definition 4.1) on $T_x\mathcal{X}$, for every $x \in \mathcal{X}$.

Using Proposition 5 we immediately obtain the following representation of generalized Dirac structures on manifolds.

Proposition 8 Let L be a generalized Dirac structure on an n -dimensional manifold $\mathcal{X}$. Let $x = (x_1, x_2, \dots, x_n)$ be local coordinates for $\mathcal{X}$. Then locally there exist $n \times n$ matrices $E(x)$ and $F(x)$ depending smoothly on x such that

$$L(x) = \{(f, e) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid \quad (22)$$

$$F(x)f + E(x)e = 0\}$$

with

$$E(x)F^T(x) + F(x)E^T(x) = 0 \quad (23)$$

Conversely, any pair $E(x), F(x)$ satisfying (23) and such that $[F(x), E(x)]$ has rank n , locally defines a generalized Dirac structure on $\mathcal{X}$.

Remark 9 If $F(x)$ is invertible for all $x \in \mathcal{X}$, then $J(x) := -F^{-1}(x)E(x)$ is the structure matrix of

a generalized Poisson bracket.

The Dirac structures of Definition 7 are called "*generalized*", because they do not necessarily satisfy the following integrability condition replacing the Jacobi-identity for Poisson structures.

Definition 10 [2] A generalized Dirac structure L on $\mathcal{X}$ is *integrable* if

$$\begin{aligned} < L_{f_1} e_2 | f_3 > + < L_{f_2} e_3 | f_1 > + \\ & + < L_{f_3} e_1 | f_2 > = 0 \end{aligned} \quad (24)$$

for all vectorfields f_1, f_2, f_3 and all one-forms e_1, e_2, e_3 such that $(f_i(x), e_i(x)) \in L(x)$ for every $x \in \mathcal{X}, i = 1, 2, 3$.

Implicit generalized Hamiltonian systems *without* external ports are now defined [10] by a generalized Dirac structure L on the space of energy variables $\mathcal{X}$, together with a Hamiltonian $H : \mathcal{X} \rightarrow \mathbb{R}$, by requiring that the implicit dynamics is given as

$$(f(x) = \dot{x}, e(x) = dH(x)) \in L(x), \quad \forall x \in \mathcal{X} \quad (25)$$

with $\dot{x} = \frac{dx}{dt} \in T_x \mathcal{X}$ the velocity (flow) at state x . Note that (25) expresses conservation of energy, since by Proposition 4 $\frac{dH}{dt} = < dH(x) | \dot{x} > = 0$ for $(\dot{x}, dH(x)) \in L(x)$.

Analogously to Theorem 1 it can be shown [1], [2] that if the generalized Dirac structure is *integrable* then there exist locally coordinates $(q_1, \dots, q_k, p_1, \dots, p_k, r_1, \dots, r_\ell, s_1, \dots, s_m)$ for $\mathcal{X}$ such the implicit Hamiltonian dynamics is given as

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q, p, r, s) & i &= 1, \dots, k \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q, p, r, s) & i &= 1, \dots, k \\ \dot{r}_j &= 0 & j &= 1, \dots, \ell \\ 0 &= \frac{\partial H}{\partial s_j}(q, p, r, s) & j &= 1, \dots, m \end{aligned}$$

Remark 11 In [10] it is shown how general *LC*-circuits (not assuming independence of the capacitors and inductors) lead to implicit generalized Hamiltonian systems.

Now let us come back to the effort-constrained Hamiltonian system (13). Define the distribution G on $\mathcal{X}$ as

$$G(x) = \text{span}\{g_1(x), \dots, g_m(x)\} \quad (26)$$

and its annihilating co-distribution $\text{ann}G$ as

$$\begin{aligned} \text{ann}G(x) &= \text{span}\{\alpha(x) \mid \alpha \text{ one-form on} \\ \mathcal{X} \text{ s.t. } \alpha(x)g_j(x) &= 0, \quad j = 1, \dots, m\} \end{aligned} \quad (27)$$

(For simplicity we assume *throughout* that $\dim G(x) = m$ for every x .)

Proposition 12 Consider the n -dimensional manifold $\mathcal{X}$ with structure matrix $J(x)$ and constant-dimensional distribution G on $\mathcal{X}$. Then

$$\begin{aligned} L(x) &= \{(f, e) \in T_x \mathcal{X} \times T_x^* \mathcal{X} \mid \\ e &\in \text{ann}G(x), \quad f = J(x)e \text{ modulo } G(x)\} \end{aligned} \quad (28)$$

defines a generalized Dirac structure on $\mathcal{X}$.

Proof It is easy to see that $< f | e > = 0$ for all $(f, e) \in L(x)$. Furthermore it is readily checked that $\dim L(x) = n$. $\square$

Remark 13 It can be actually shown (cf. [1], Proposition 1.1.5) that every (regular) generalized Dirac structure can be represented as in (28) for some structure matrix $J(x)$ and distribution G .

The crucial observation is that the effort-constrained dynamics (13) can be rewritten as the *implicit* generalized Hamiltonian system

$$\left(\dot{x}, \frac{\partial H}{\partial x}(x)\right) \in L(x) \quad (29)$$

with $L(x)$ given by (28). Indeed, $(\dot{x}, \frac{\partial H}{\partial x}(x)) \in L(x)$ is equivalent to

$$\begin{aligned} (i) \quad \frac{\partial H}{\partial x}(x) &\in \text{ann}G(x) \Leftrightarrow g^T(x) \frac{\partial H}{\partial x}(x) = 0 \\ (ii) \quad \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) \text{ modulo } G(x) \Leftrightarrow \\ \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)u, \quad \text{for some } u \end{aligned} \quad (30)$$

Thus to every effort-constrained generalized Hamiltonian system there corresponds a generalized Dirac structure and an implicit generalized Hamiltonian system on $\mathcal{X}$. (Conversely, by Remark 13, there corresponds to every Dirac structure and implicit generalized Hamiltonian system on $\mathcal{X}$ an effort-constrained generalized Hamiltonian system!).

Next question is how to model the resulting dynamics on the constrained state space $\mathcal{X}_c$ as a generalized Hamiltonian system. First, for every $x \in \mathcal{X}$ we define the canonical (linear) projection

$$P(x) : T_x \mathcal{X} \rightarrow T_x \mathcal{X} / G(x) \quad (31)$$

with its dual

$$P^*(x) : (T_x \mathcal{X} / G(x))^* = \text{ann}G(x) \rightarrow T_x^* \mathcal{X} \quad (32)$$

which is nothing else than the natural embedding of $\text{ann}G(x)$ as a subspace in $T_x^* \mathcal{X}$. Then, consider for $x_c \in \mathcal{X}_c$ the *restriction* of $P(x_c)$ to $T_{x_c} \mathcal{X}_c \subset T_{x_c} \mathcal{X}$, denoted as $P^r(x_c)$

$$P^r(x_c) : T_{x_c} \mathcal{X}_c \rightarrow T_{x_c} \mathcal{X} / G(x_c) \quad (33)$$

We claim that $P^r(x_c)$ is invertible. Clearly the dimensions of $T_{x_c}\mathcal{X}_c$ and $T_{x_c}\mathcal{X}/G(x_c)$ are equal, so we only have to prove that $P^r(x_c)$ is injective. Suppose there exists $0 \neq v \in T_{x_c}\mathcal{X}_c$ such that $P^r(x_c)v = 0$, or equivalently $v \in G(x_c)$. This would imply $L_v L_g H(x_c) = 0, j = 1, \dots, m$, whereas $v \in G(x_c)$, which contradicts Assumption 2. Thus we may define the invertible map

$$R(x_c) := [P^r(x_c)]^{-1} : T_{x_c}\mathcal{X}/G(x_c) \rightarrow T_{x_c}\mathcal{X}_c \quad (34)$$

Now consider the diagram

$$\begin{array}{ccccc} T_{x_c}^* \mathcal{X}_c & \xrightarrow{R^*(x_c)} & \text{ann} G(x_c) & \xrightarrow{P^*(x_c)} & T_{x_c}^* \mathcal{X} \\ \downarrow J(x_c) & & \downarrow P(x_c) & & \downarrow R(x_c) \\ T_{x_c} \mathcal{X} & \xrightarrow{P(x_c)} & T_{x_c} \mathcal{X}/G(x_c) & \xrightarrow{R(x_c)} & T_{x_c} \mathcal{X}_c \end{array} \quad (35)$$

and define the structure matrix $J_c(x_c)$ on $\mathcal{X}_c$ as the composition

$$J_c(x_c) := R(x_c)P(x_c)J(x_c)P^*(x_c)R^*(x_c) \quad (36)$$

It follows that the *implicit* generalized Hamiltonian system (29) on $\mathcal{X}$ may be equivalently described as the generalized Hamiltonian system on $\mathcal{X}_c$ given as

$$\dot{x}_c = J_c(x_c) \frac{\partial H_c}{\partial x_c}(x_c) \quad (37)$$

with $H_c : \mathcal{X}_c \rightarrow \mathbb{R}$ the restriction of H to $\mathcal{X}_c$. (Note that $R^*(x_c) \frac{\partial H_c}{\partial x_c}(x_c) = \frac{\partial H}{\partial x}(x_c)$ since $\frac{\partial H}{\partial x}(x_c) \in \text{ann} G(x_c)$!)

Let us apply the above theory to mechanical systems with kinematic constraints [9]. We have the Hamiltonian formulation

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} + A(q)\lambda \quad (38)$$

$$0 = A^T(q)\dot{q} = A^T(q) \frac{\partial H}{\partial p}(q, p)$$

with $A(q)\lambda$ the reaction forces corresponding to the constraints $A^T(q)\dot{q} = 0$ ($\text{rank } A(q) = k \leq n$). An explicit expression for P and R is easily obtained in this case. Indeed, following [9], we define new coordinates $(q, \tilde{p})$ by

$$\begin{aligned} \tilde{p}^1 &:= S^T(q)p \in \mathbb{R}^{n-k} \\ \tilde{p}^2 &:= A^T(q)p \in \mathbb{R}^k \end{aligned} \quad (39)$$

where $S(q)$ is an $n \times (n-k)$ matrix of rank $n-k$ such that

$$A^T(q)S(q) = 0 \quad (40)$$

Define the projection

$$(q, \tilde{p}^1, \tilde{p}^2) \mapsto (q, \tilde{p}^1), \quad (41)$$

then $P(x_c)$ is nothing else than the differential of this expression, i.e.

$$\begin{aligned} P(x_c) &= \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_{n-k} & 0 \end{bmatrix}, \\ R(x_c) &= \begin{bmatrix} I_n & 0 \\ 0 & I_{n-k} \end{bmatrix}, \end{aligned} \quad (42)$$

if we take $q, \tilde{p}^1$ as coordinates for $\mathcal{X}_c = \{(q, \tilde{p}) \mid A^T(q) \frac{\partial H}{\partial \tilde{p}}(q, \tilde{p}) = 0\}$. Thus we only have to compute the structure matrix $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ in the coordinates q, p in the new coordinates $q, \tilde{p}$. In [9] it is shown that this transformed structure matrix is of the form

$$\begin{bmatrix} O_n & S(q) & * \\ -S^T(q) & [-p^T[S_i, S_j](q)]_{i,j=1,\dots,n-k} & * \\ * & * & * \end{bmatrix} \quad (43)$$

with $[S_i, S_j]$ denoting the Lie bracket of the i -th and j -th column S_i and S_j of $S(q)$. In view of (36) and (42) it immediately follows that the structure matrix J_c on $\mathcal{X}_c$ is given as (43) where the blocks denoted by $*$ are deleted and where p satisfies $A^T(q) \frac{\partial H}{\partial p}(q, p) = 0$. As a result we obtain the generalized Hamiltonian system on $\mathcal{X}_c$

$$\dot{x}_c = J_c(x_c) \frac{\partial H_c}{\partial x_c}(x_c), \quad (44)$$

in accordance with [9].

Conclusions It has been shown that effort-constrained generalized Hamiltonian systems can be equivalently rewritten as an implicit generalized Hamiltonian system with respect to a generalized Dirac structure determined by the generalized Poisson structure of the unconstrained system as well as the transformer matrix corresponding to the external ports modeling the effort constraints. Furthermore, under a non-degeneracy condition on the internal energy (Assumption 2), the implicit generalized Hamiltonian system has been rewritten as an explicit generalized Hamiltonian system on the constrained state space. It has been demonstrated how the previously treated case of mechanical systems with kinematic constraints fits within this approach. Next step is to extend the approach to multi-body systems [8], and to use the Hamiltonian format for control purposes, in the spirit of [7].

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