SUPERMEMBRANES AND ELEVEN-DIMENSIONAL SUPERGRAVITY

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We construct an action for a supermembrane propagating in $d=11$ supergravity background. Using the constraints of $d=11$ curved superspace, we show that the action is invariant under Siegel-type transformations recently generalized by Hughes, Li and Polchinski. The transformation parameter is a world-volume scalar and $d=11$ spacetime spinor. We also discuss the general problem of the coupling of $n$-dimensional extended objects to $d$-dimensional supergravity.

1. Now that we have become accustomed to the notion that strings should replace particles, it is natural to investigate the properties of higher-dimensional extended objects, in particular of membranes since they are the simplest extended objects, and they might describe strings in an appropriate limit.

In 1962 Dirac [1] put forward a theory of an extended electron based on the idea of a relativistic membrane. In 1976, Collins and Tucker [2] studied the classical and quantum mechanics of free relativistic membranes. A year later a locally supersymmetric and reparametrization-invariant action for a spinning membrane was constructed by Howe and Tucker [3]. The action describes anti de Sitter supergravity coupled to a number of scalar multiplets in three dimensions. It is the membrane analog of the Neveu–Schwarz–Ramond formulation of the spinning string theory.


The generalization of the Hughes et al. model to $n$-extended objects propagating in flat $d$-dimensional superspace is evident. All that is required is the existence of a closed super $(n+2)$-form given by

$$ H = E^a E^a E^a \ldots E^a (\gamma_{a_1 \ldots a_n})_{a \phi} , $$

where $(E^a, E^\alpha)$ are the basis one-forms in superspace. This form is closed provided that the following $\Gamma$-matrix identity holds:

$$ (\gamma^a)_{\alpha \beta} (\gamma^a_{\alpha_1 \ldots \alpha_n})_{a \phi} = 0 . $$

The purpose of this note is to construct Hughes et al. -type actions describing the propagation of an $n$-extended object in $d$-dimensional curved superspace. We give a general formula for the action and the transformation rules, whose consistency requires, among other things (see below), the existence of a closed $(n+2)$-form in curved superspace. Thus we
expect that the \( n \)-extended objects under consideration can consistently propagate only in \( d \leq 11 \) supergravities whose superspace formulation involves a closed \((n+2)\)-form. We further expect that such forms exist in supergravity theories in which a closed bosonic \((n+2)\)-form occurs. As far as we know, the following possibilities exist (we include the Yang-Mills couplings whenever possible):

The dual formulation of \( d=10 \), \( N=1 \) supergravity involves a closed seven-form. Its dimensional reduction on a \((10-d)\)-dimensional torus leads to real closed \((d-3)\)-forms in \( d \)-dimensional supergravities. (These are \( N=1 \) supergravities in \( d=8, 9, 10; N=2 \) in \( d=7 \) and \( N=2 \) or 4 in \( d=6 ) [9]. Apart from these, there is: (i) A real closed four-form in \( d=11 \), \( N=1 \) supergravity, (ii) a real closed three-form in non-chiral \( d=10 \), \( N=2 \) supergravity, (iii) a complex closed three-form in chiral \( d=10 \), \( N=2 \) or 4 supergravity.

Excluding Yang-Mills couplings, as is well known, closed super three-forms exist in \( d=3, 4, 6, \) and 10.

Considering the case of the membranes, from the above list it follows that the candidate dimensions are 7 and 11. Since the superspace formulation of \( d=7 \), \( N=2 \) supergravity is not known at present, we are led to consider the supermembrane propagating in eleven-dimensional spacetime.

Our result is the construction of an action which describes a consistent coupling of \( d=11 \) supergravity to a supermembrane. In particular the Kalb-Ramond-like third rank antisymmetric tensor field of \( d=11 \) supergravity is needed for the superspace description of \( d=11 \) supergravity \([10]\].

In the following we focus our attention on the description of the supermembrane action in \( d=11 \). The extension to the case of \( n \)-extended objects is given in the appendix.

2. We propose the following action for a supermembrane coupled to \( d=11 \) supergravity:

\[
S = \int d^2 \xi \left( \frac{1}{2} \sqrt{-g} g^{ij} E_i^a E_j^b \eta_{ab} + \epsilon^{ijk} E_i^a E_j^b E_k^c B_{CBA} - \frac{1}{2} \sqrt{-g} \right). \tag{3}
\]

Here \( i=0, 1, 2 \) labels the coordinates \( \xi^i = (\tau, \sigma, \rho) \) of the world volume with metric \( g_{ij} \) and signature \((-,-,+)\). The super three-form \( B \) is needed for the superspace description of \( d=11 \) supergravity \([10]\). For the Levi-Civita symbol \( \epsilon^{ijk} \) we use the same conventions as in ref. \([11]\). In (3) we have used the notation

\[
E_i^a = (\partial_i Z^M) E_M^a, \tag{4}
\]

where \( Z^M(\xi) \) are the superspace coordinates, and \( E_M^a(Z) \) is the supervielbein.

Note that the action has a cosmological constant with a fixed magnitude. This is so that the field equation of the metric \( g_{ij} \) gives the embedding equation

\[
g_{ij} = E_i^a E_j^b \eta_{ab} \equiv T_{ij}. \tag{5}
\]

We require that the action \( S \) is invariant under a fermionic gauge transformation of the form \([5]\)

\[
\delta E^a = 0, \quad \delta E^a = (1 + \Gamma)^{\alpha \beta} \kappa^a, \tag{7}
\]

\[
\delta g_{ij} = 2 [X_{ij} - g_{ij} X^k X_k] (n-1) \tag{6}
\]

\((n=2 \text{ for membrane})\),

where the transformation parameter \( \kappa^a(\xi) \) is a 32 component Majorana spinor, and a world-volume scalar, and

\[
\delta E^a = \delta Z^M E_M^a. \tag{8}
\]

\[
\Gamma^{\alpha \beta} = (1/6 \sqrt{-g}) \epsilon^{ijk} E_i^a E_j^b E_k^c (\gamma_{abc})^{\alpha \beta} \tag{9}
\]

Here \( \gamma^a(a=0, 1, \ldots, 10) \) are the Dirac matrices in eleven dimensions. \( X_{ij} \) is a function of \( E_i^a \) which will be determined by the invariance of the action. The choice of \( \delta g_{ij} \) is due to the fact that, given a variation of the action of the form \( \delta S = T_{ij} X^u \), and writing this variation as

\[
T_{ij} X^u = g_{ij} X^u + (T_{ij} - g_{ij}) X^u, \tag{10}
\]

the second term on the right-hand side cancels \( (\delta S/\delta g_{ij}) \delta g_{ij} \). Thus we are left with the first term on the right-hand side, which equals the left-hand side upon the use of (5). Effectively, this means that whenever we encounter a variation of the form \( T_{ij} X^u \), we can use eq. (5), provided that we add \( X^u \) to \( \delta g_{ij} \) as in (6).

The matrix \( \Gamma^{\alpha \beta} \) occurring in (8) satisfies the property

\[
\Gamma^{\alpha \beta} \Gamma^{\beta \delta} = (T_i^i T_i^j T_j^k) \delta^{\alpha \delta} \equiv \gamma^{\alpha \delta} \tag{11}
\]

The normalization in (8) is chosen such that upon the use of the equation \( T_{ij} = g_{ij} \), the matrix \( \Gamma^{\alpha \beta} \) satisfies the relation \( \Gamma^{\alpha \beta} \Gamma^{\beta \gamma} = \delta^{\alpha \gamma} \).
Now using (6) the variation of the action (3) is (we consider a closed supermembrane and therefore discard the surface terms)

$$\delta S = \int \mathrm{d}^2 \xi \left[ \sqrt{-g} g^{ij} (\delta E^\beta E_i \gamma^T \gamma^a E_{ja} + \sqrt{-g} g^{ij} (\delta E^\beta E_i \gamma^T \gamma^a E_{ja} + 
$$

$$+ \epsilon_{ijkl} E_i \gamma^a E_k \gamma^b \{ \delta E^a \} H_{abcdef} - \frac{1}{2} \sqrt{-g} \delta g^{ij} (T^{ij} - \frac{1}{2} g^{ij} T - \frac{1}{2} g^{ij}) \right].$$

(11)

The torsion two-form $T^a$ and the four-form field strength $H$ are defined by (our superspace conventions are those of Howe [12])

$$T^a = dE^a + E^b \Omega_b^a = \frac{1}{2} E^b E^c T_{CB}^a,$$

$$H = dB = \epsilon^{ab} E^a E^b D_{E^c}. (12)$$

We now organize the terms in (11) according to the number of one-forms $E^a$ they contain. Those with three $E^a$ and two $E^a$ come only from the Wess-Zumino term. They must vanish separately, and this requires the constraints

$$H_{a[\gamma \delta]} = 0. (13)$$

The cancellation of the terms linear in $E^a$ lead to the constraints

$$T^a_{a \beta} = \gamma^{a \beta}, (14)$$

$$H_{a \beta ab} = - \frac{1}{2} (\gamma_{ab})_{a \beta}, (15)$$

while the cancellation of the terms not containing $E^a$ require the constraint

$$\gamma_{(a} T_{b) \alpha} = \eta_{ab} A_\alpha, (16)$$

$$H_{a b c \alpha} = - \frac{1}{2} A_\alpha (\gamma_{abc})^\beta \gamma^\beta, (17)$$

Here $A_\alpha$ is an arbitrary spinor superfield which is vanishing in $d=11$ [10].

It is important to realize that in obtaining (14)–(17) we have used the identity

$$\delta E^a = \Gamma^a_{\beta \gamma} E^\beta \delta E^\gamma + (1 - \Gamma^2) \kappa^a. (18)$$

Using this identity in the variation of the kinetic term, the terms arising from $\Gamma^a_{\beta \gamma}$ in (18) can be shown to cancel similar terms coming from the variation of the Wess-Zumino term, modulo terms which cancel by an appropriate variation of $g_{ij}$. [Using the argument below (8) once.] In the remaining terms coming from $(1 - \Gamma^2)$, we use the argument given below (8) repeatedly to compute further contributions to $\delta g_{ij}$. Thus we find the result

$$X_{ij} = - \frac{1}{4} \epsilon_{ijkl} E_k a E_l b (\gamma_{ab})_{a \beta} \delta E^\beta E_j c$$

$$+ \frac{1}{2} \delta_{\gamma \delta} (T^{ij} - \frac{1}{2} g^{ij} T - \frac{1}{2} g^{ij} )$$

$$+ i \leftrightarrow j. (19)$$

In summary, the action (3) is invariant under (6) provided that (13)–(17) hold, and $X^{ij}$ is given by (19). In addition, the following Bianchi identities must hold:

$$DT^a = - E^b \wedge R_{BA}, DH = 0. (20)$$

The generalization of the results of this section to the general case of $n$-extended objects in $d$-dimensional supergravity is straightforward. The result is given in the appendix.

3. We observe that the superspace constraints of $d=11$ supergravity given by Cremmer and Ferrara [10] and Brink and Howe [10] do provide a solution to (13)–(17) and the Bianchi identities (20), with $A_\alpha = 0$.

In conclusion, we have shown that there exists a consistent coupling of a closed supermembrane to eleven-dimensional supergravity. (Note that it is natural to consider a closed supermembrane in eleven dimensions, since there are no matter multiplets in this dimension).

4. There are several directions in which the present work can be extended. We shall name a few.

Firstly, it is of interest to study the quantization of the supermembrane model in eleven dimensions. In particular, the question of whether massless gauge fields can possibly arise is a challenging one. Although usually one encounters difficulties in finding massless excitations of a membrane [13], it is encouraging that, here, we have a spacetime supersymmetric membrane action.

Secondly, it is natural to consider the dimensional reduction of our model from eleven- to ten-dimensional spacetime, and at the same time from three-dimensional world volume to a two-dimensional...
world sheet. It would be interesting to see what kind of $d=10$ string theories could possibly emerge in an infinitely thin membrane limit.

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Appendix. In this appendix we construct the action for an $n$-extended object propagating in $d$-dimensional supergravity background. We also give the transformation rules, and the constraints on the background.

The action is

$$S = \int d^2 \xi \left[ \frac{i}{2} \sqrt{-g} g^{ij} E_i a E_j b \eta_{ab} \right.$$\hspace{1cm} (A1)

$$+ \epsilon^{i_1 \ldots i_n} E_{i_1} a_1 \ldots E_{i_{n+1}} a_{n+1} B_{a_{n+1} \ldots a_1}$$

$$- \frac{1}{4} (n-1) \sqrt{-g} \right] .$$

The transformation rules are those in (6), where the matrix $\Gamma^a_\beta$ is now given by

$$\Gamma^a_\beta = \left[ \eta/(n+1)! \right] \sqrt{-g}$$

$$\times \epsilon^{i_1 \ldots i_n} E_{i_1} a_1 \ldots E_{i_{n+1}} a_{n+1} \eta_{\alpha a} (\gamma_{a_1 \ldots a_{n+1}})^\alpha \beta ,$$

(A2)

where $\eta$ is given by

$$\eta = (-1)^{(n+1)(n-2)/4} .$$

(A3)

Invariance of the action (A1) is ensured by imposing the following set of constraints:

$$T^a_\alpha \beta = (\gamma^a)_\alpha \beta , \quad \eta_{\alpha (a} T^c_\beta b) = \eta_{ab} A_\alpha ,$$

(A4)

$$H_{a a_{n+1} \ldots a_1} = (\eta/(n+1)!)(\gamma_{a_1 \ldots a_{n+1}}).$$

(A5)

$$H_{a a b \ldots a_{n+1}} = [\eta(-1)^n/(n+1)!](\gamma_{a_1 \ldots a_{n+1}}) \eta_{ab} ,$$

and by taking $X_\mu$ occurring in (6) to be

$$X_\mu = \left(-\eta/2n!\right)$$

$$\times \epsilon_{i_1 k_1 \ldots k_n} E_i a_1 \ldots E_{i_n} a_n (\gamma_{a_1 \ldots a_n})^\alpha \beta E_j^a E_j^\alpha$$

$$+ \frac{1}{4} \kappa^\beta [E_\alpha^a (\gamma^a)_{\alpha \beta} E^a + (n+1) A_\beta]$$

$$\times g_{ij} (T^{k_1} k_1 \ldots T^{k_n} k_n) + \delta^{k_1} k_1 T^{k_2} k_2 \ldots T^{k_n} k_n$$

$$+ \delta^{k_1} k_1 \delta^{k_{n-1}} k_{n-1} (T^{k_n} k_n)$$

$$- \frac{1}{4} \delta E_\beta \Lambda_\beta g_{ij} + (i \leftrightarrow j) .$$

(A6)

References