Discontinuous subgroups of $\text{PGL}_2(K)$

Marius van der Put* and Harm H. Voskuil

Received 22 November 2001
Communicated by Leonard Lipshitz

1. Introduction

Let $K$ be an algebraically closed field, which is complete with respect to a non-Archimedean valuation. A finitely generated discontinuous subgroup $\Gamma \subset \text{PGL}_2(K)$ has, acting on $\mathbb{P}^1(K)$, a compact set of limit points $\mathcal{L}$. Its set of ordinary points $\Omega = \mathbb{P}^1(K) \setminus \mathcal{L}$ is a rigid analytic subspace of $\mathbb{P}^1(K)$ on which $\Gamma$ acts discontinuously. The quotient $\Omega/\Gamma$ is known to be a smooth, complete, irreducible algebraic curve over $K$. In this paper we investigate the groups $\Gamma$ such that $\Omega/\Gamma \cong \mathbb{P}^1_K$. Let $\text{pr} : \Omega \to \mathbb{P}^1(K)$ denote the induced morphism of rigid spaces. A point $y \in \Omega$ will be called ramified if its stabilizer $\{ \gamma \in \Gamma \mid \gamma(y) = y \}$ is non-trivial. A branch point for $\Gamma$ is a point $x \in \mathbb{P}^1(K)$ such that $x = \text{pr}(y)$ for some ramified point $y$. The group $\Gamma$ has many normal subgroups $\Delta$ of finite index, such that $\Delta$ is a free group. The pair $(\Delta, \Gamma)$ induces a (ramified) Galois covering $X := \Omega/\Delta \to \Omega/\Gamma = \mathbb{P}^1_K$ of the projective line with $X$ a Mumford curve. Such a covering is called a Mumford covering (see also [7]) and every Mumford covering of the projective line is obtained by a pair $(\Delta, \Gamma)$. The branch points of the covering $X \to \mathbb{P}^1_K$ induced by $(\Delta, \Gamma)$ are the branch points of $\Gamma$.

The aim of this paper is to classify all possible groups $\Gamma$ as amalgams of certain finite trees of groups (see 3.12, 3.14, 4.10, 4.11) and to give a formula (see 5.3) for the number of branch points $\text{br}(\Gamma)$ of $\Gamma$. The results depend heavily on the characteristic $p_K$ of $K$ and the characteristic $p_k$ of the residue field $k$ of $K$. This classification turns out to be extraordinary rich and complicated. There are exceptional groups $\Gamma$ in case $p_K = 0$ and $p_k = 2, 3$ or 5, for which no reasonable classification seems to exist. We exclude these groups and restrict ourselves to ordinary groups (see 4.1 and 4.2).

A central role is played by subtrees of the (generalized) Bruhat–Tits tree of $\text{PGL}_2(K)$, which consists of the lattice classes in $K^2$. The finite subgroups $G$ of $\text{PGL}_2(K)$ are investigated and one associates to $G$ a finite tree $\text{Tree}(G)$ which captures the configuration of the ramification points of $\mathbb{P}^1_K \to \mathbb{P}^1_K/G$ (see Sections 2.4 and 2.5). Using these trees, all

* Corresponding author.
E-mail address: mvdput@math.rug (M. van der Put).

0021-8693/ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/S0021-8693(03)00398-3
\( \Gamma \) of the form \( G_1 \ast_{G_2} G_2 \) are classified (see 3.5 and 3.9). This part of the paper extends and completes earlier work of F. Herrlich [5].

The main purpose of Section 4 is to construct the tree \( T^c \) associated to an indecomposable, ordinary group \( \Gamma \) and to show that the finite tree of groups \( T^c / \Gamma \) has the correct properties. In Section 5, this finite tree of groups is used for the formula for \( \text{br}(\Gamma) \).

Examples and lists of Mumford coverings of \( \mathbf{P}_K^1 \), unramified outside \([0, 1, \infty]\) and for a field \( K \) of characteristic 0, have been given by Y. André [1] and F. Kato [6]. For these examples the corresponding groups \( \Gamma \) satisfy \( \text{br}(\Gamma) = 3 \). These groups are exceptional (see 5.5) and there is no overlap between [1,6] and the present paper. In [2] a list of Mumford coverings of \( \mathbf{P}_K^1 \) with two branch points is given for fields \( K \) of positive characteristic. In principle this list can be deduced from our classification of the trees \( T^c / \Gamma \) having \( \text{br}(\Gamma) = 2 \) (see 5.8). The methods of [2] however, are quite different from the ones developed here.

2. Trees and finite subgroups of \( \text{PGL}_2(K) \)

In this section the material on subtrees of the (generalized) Bruhat–Tits tree for the group \( \text{PGL}_2(K) \) is presented. More information and more detailed proofs can be found in [4] and [3]. The information on the action of finite subgroups on \( \mathbf{P}_K^1 \), needed in the sequel, is also provided in this section. We note that some of this material is already present in [2,5,6].

2.1. Lattices and trees

The valuation ring of \( K \) is denoted by \( K^0 \) and its maximal ideal by \( K^{00} \). The characteristic of \( K \) is denoted by \( p_K \) and that of its residue field \( k = K^0 / K^{00} \) by \( p_k \).

The field \( K \) is supposed to be algebraically complete and we suppose that \( p_k > 0 \) (\( p_k = 0 \) seems uninteresting). Any reduced algebraic variety over \( K \) (or over \( k \)) will be identified with its set of \( K \)-valued (or \( k \)-valued) points. A lattice \( M \subset K^2 \) is a free submodule over \( K^0 \) of rank two. Two lattices \( M_1, M_2 \) are called equivalent if there exists a \( \lambda \in K^* \) with \( M_1 = \lambda M_2 \). The equivalence class of the lattice \( M \) will be denoted by \([M]\). As usual \( \mathbf{P}_K^1(K) \) is identified with \( \mathbf{P}(K^2) \). For a given lattice \( M \) one has a reduction map:

\[
\text{red}_{[M]}: \mathbf{P}_K^1(K) = \mathbf{P}(K^2) \rightarrow \mathbf{P}(M \otimes k) = \mathbf{P}_k^1(k).
\]

This reduction map depends only on the class of \( M \). For three distinct points \( K v_1, K v_2, K v_3 \in \mathbf{P}_K^1(K) \) there is a unique lattice class \([M]\) such that the three points are mapped to three distinct points of \( \mathbf{P}(M \otimes k) = \mathbf{P}_k^1(k) \). One can describe \([M]\) explicitly by the following. Let \( \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 \) be the unique (up to a multiple) non-trivial linear relation between \( v_1, v_2, v_3 \). Then \( M \) is the \( K^0 \)-module generated by \( \lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3 \).

For a lattice class \([M_1] \neq [M]\) one defines \( \text{red}_{[M]}([M_1]) \in \mathbf{P}(M \otimes k) \) as follows. We may suppose that \( M = K^0 e_1 + K^0 e_2 \) and \( M_1 = K^0 e_1 + K^0 \pi e_2 \) for suitable \( e_1, e_2 \) and \( 0 < |\pi| < 1 \). Then \( \text{red}_{[M]}([M_1]) := \text{red}_{[M]}(Ke_1) \).
For a moment we replace the field \( K \) be a local field \( L \) with residue field \( \ell \) and generator \( \pi_L \) of the maximal ideal of \( L^0 \). The collection \( \mathcal{BT}(L) \) of all lattice classes in \( L^2 \) is a locally finite tree, called the Bruhat–Tits building of \( \mathbf{P}^1(L) \) or of \( \mathbf{PGL}_2(L) \), defined by

(a) The vertices of the tree are the lattice classes.

(b) \( [[M_1],[M_2]] \) is an edge if one can choose the lattices \( M_1, M_2 \) such that \( M_1 \supset M_2 \supset \pi_L \mathbf{M}_1 \) and \( M_2 \neq M_1, \pi_L \mathbf{M}_1 \).

Every vertex is contained in precisely \( 1 + \# \ell \) edges. The ends of \( \mathcal{BT}(L) \) are in a 1–1 correspondence with the points of \( \mathbf{P}^1(L) \).

We return now to the algebraically closed and complete valued field \( K \) and write \( \mathcal{BT} \), the “generalized Bruhat–Tits tree,” for the collection of all classes of lattices. This object is too large to be a (locally finite) tree but still has a tree-like structure in the following sense. Let \( [M_1] \neq [M_2] \) denote two classes of lattices. One can represent them by lattices \( M_1 \supset M_2 \) such that \( M_1/M_2 \) is isomorphic to \( K^0/\pi K \) for some \( \pi \) with \( 0 < |\pi| < 1 \). This representation is unique up to multiplication by some \( \lambda \in K^* \). The segment \( [[M_1],[M_2]] \), also denoted by \( \text{conv}([M_1],[M_2]) \), joining \([M_1],[M_2]\) consists of all \([M]\) such that

\[
M_1 \supset M \supset M_2.
\]

For any three distinct classes of lattices \( v_1, v_2, v_3 \) there exists a unique lattice class \( v_4 \) such that precisely one of the following statements holds:

(a) \( v_4 = v_i \) for some \( i \in \{1, 2, 3\} \) and \( v_i \in [v_j, v_k] \) where \( \{i,j,k\} = \{1, 2, 3\} \).

(b) \( v_4 \neq v_1, v_2, v_3 \) and for all \( i \neq j \in \{1, 2, 3\} \) one has \( [v_i, v_4] \cap [v_j, v_4] = [v_4] \).

This last property implies that the collection of all lattice classes does not contain a cycle. We will consider certain subsets of \( \mathcal{BT} \) which are actually locally finite trees.

1. **The tree \( T_F \) of a finite subset \( F \) of \( \mathbf{P}^1(K) \).**

For any finite set \( F \subset \mathbf{P}^1(K) \) of cardinality \( \geq 3 \). The vertices of \( T_F \) are the lattice classes \([M]\) such that the image \( \text{red}_{[M]}(F) \) consists of at least three points. For two vertices \([M_1] \neq [M_2]\) one considers the map \( \text{red}_{[M_1],[M_2]} := \text{red}_{[M_1]} \times \text{red}_{[M_2]} : \mathbf{P}^1(K) \to \mathbf{P}(M_1 \otimes k) \times \mathbf{P}(M_2 \otimes k) \). The image can be seen to be the union of the two lines \( \mathbf{P}^1(k) \times \{a_2\} \) and \( \{a_1\} \times \mathbf{P}^1(k) \) intersecting at \((a_1, a_2)\). The pair \([M_1],[M_2]\) is called an edge if there is no point \( f \in F \) with \( \text{red}_{[M_1],[M_2]} f = (a_1, a_2) \). This defines a graph which is actually a finite tree. One combines all reductions maps for the vertices \([M]\) of \( T_F \) to a map \( \text{red}_F = \prod [M] \text{red}_{[M]} : \mathbf{P}^1(K) \to \prod [M] \mathbf{P}(M \otimes k) \). The image of \( \text{red}_F \) is called \((\mathbf{P}^1, F)\) and has the following properties:

(a) \((\mathbf{P}^1, F)\) is a reduced variety over \( k \). Each irreducible component is a \( \mathbf{P}^1(k) \). Each singular point is a normal intersection of two irreducible components.

(b) The dual graph of \((\mathbf{P}^1, F)\) is defined as follows. Its set of vertices are the irreducible components and its set of edges are the singular points. The dual graph is the tree \( T_F \).

(c) The map \( \text{red}_F : F \to \text{red}_F(F) \) is bijective and \( R_F(F) \) consists of non singular points.
(d) Every irreducible component has at least three special points, i.e., intersections with other components or elements in red$_F (F)$. 

(2) The tree $T_F$ of a infinite subset $F$ of $\mathbf{P}^1 (K)$, having compact closure $\overline{F}$.

The vertices and the edges of tree $T_F$ are defined as in (1) above. Let $F^*$ denote the (compact) subset of the non-isolated points of $\overline{F}$. Put $\Omega = \mathbf{P}^1 (K) \setminus F^*$. One can combine the reductions with respect to all vertices $[M]$ to a reduction map red$_F : \Omega \to (\overline{\Omega}, \overline{F})$, where the latter is the direct limit of the $(\mathbf{P}^1, G)$ taken over all finite subsets $G$ of $F$. One can prove that $(\overline{\Omega}, \overline{F})$ is a locally finite reduced variety over $k$ satisfying the above properties (a)–(d), where one has to replace in (c) and (d) the set $F$ by $F \cap \Omega$. The dual graph of $(\overline{\Omega}, \overline{F})$ is the locally finite tree $T_F$. The points of $\overline{F}$ outside $\Omega$ are not mapped by red$_F$ to points in $(\overline{\Omega}, \overline{F})$. These points correspond with the ends of $T_F$. For more information and proofs, see [4].

(3) Discrete subsets $\mathcal{M}$ of $BT$ and the tree $T_{\mathcal{M}}$.

A set $\mathcal{M}$ of lattice classes will be called discrete if there exists a compact $F \subset \mathbf{P}^1 (K)$ such that for all $[M] \in \mathcal{M}$ the set red$_{[M]} (F)$ contains at least three elements. In other words $\mathcal{M}$ is a subset of the vertices of $T_F$ for a suitable compact $F$. A (locally finite) tree $T$ in $BT$ is defined by

(a) A finite or discrete subset $V$ of $BT$.
(b) An edge of $T$ is a pair $v_1, v_2 \in V, v_1 \neq v_2$ such that $[v_1, v_2] \cap V = \{v_1, v_2\}$.
(c) There are no cycles in the graph $T$.

From the discreteness of $V$ one concludes that every $v \in V$ is contained in only finitely many edges. Moreover property (c) can be seen to be equivalent to

(c') For any three distinct elements $v_1, v_2, v_3 \in V$, such that $v_i \notin \{v_j, v_k\}$ if $\{i, j, k\} = \{1, 2, 3\}$, the unique lattice class $[M]$ such that the red$_{[M]} (v_i)$, $i = 1, 2, 3$, are distinct, belongs to $V$.

It is not difficult to show that for any tree $T$ in $BT$ there exists a compact set $F \subset \mathbf{P}^1 (K)$ such that $T = T_F$. We note that this set $F$ is far from unique. Its set of non-isolated points $F^*$ is unique and will be called the set of limit points of $T$. In particular one can associate to $T$ a reduction red$_T : \Omega \to (\overline{\Omega}, \overline{T})$ with $\Omega : = \mathbf{P}^1 (K) \setminus F^*$ and $(\overline{\Omega}, \overline{T}) : = (\overline{\Omega}, \overline{F})$ having the properties of (2) above.

In general a discrete set of lattice classes $\mathcal{M}$ is not the set of vertices of a tree. One defines the tree $T_{\mathcal{M}}$ to be the smallest tree (contained in $BT$) such that $\mathcal{M}$ is contained in the set of its vertices. One constructs $T_{\mathcal{M}}$ as follows.

The convex hull conv$(\mathcal{M})$ is defined as the union $\bigcup \text{conv}([M_1], [M_2])$ taken over all $[M_1, [M_2]] \in \mathcal{M}$. The interior of conv$(\mathcal{M})$ consists of the $[M] \in \text{conv}(\mathcal{M})$ for which there exists a pair of equivalence classes $[M_1], [M_2] \in \text{conv}(\mathcal{M}), [M_1], [M_2] \neq [M]$, such that $[M] \in \text{conv}([M_1], [M_2])$. The set of lattice classes $V(\text{conv}(\mathcal{M})) \subset BT$ consist of:
(i) The $[M] \in \text{conv}(\mathcal{M})$ that are not contained in the interior of $\text{conv}(\mathcal{M})$.

(ii) The $[M] \in \text{conv}(\mathcal{M})$ for which there exist $[M_1], [M_2], [M_3] \in \text{conv}(\mathcal{M})$ that have the following two properties:

(a) $[M]$ is contained in the interior of both $\text{conv}([M_1], [M_2])$ and $\text{conv}([M_1], [M_3])$.

(b) $\text{conv}([M_1], [M_2]) \cap \text{conv}([M_1], [M_3]) = \text{conv}([M_1], [M])$.

The discrete set $V(\text{conv}(\mathcal{M}))$ already defines a locally finite tree in $BT$. The discrete set $\mathcal{M} \cup V(\text{conv}(\mathcal{M}))$ defines a locally finite tree in $BT$ which is easily seen to be $T_\mathcal{M}$.

Consider any $[M] \in BT$. Then we define a map $\text{red}_{[M]}$ from the edges $e \in T_{\mathcal{M} \cup [M]}$ that contain $[M]$ to $\mathcal{P}(\mathcal{M} \otimes k)$ by $\text{red}_{[M]}(e) = \text{red}_{[M]}([M'])$ if $e$ is the edge with vertices $[M]$, $[M']$. If $[M]$ happens to be a vertex $v$ of $T_\mathcal{M}$ then we will often write $\text{red}_v$ for the map $\text{red}_{[M]}$.

(4) The affinoid covering of $\Omega$ corresponding to a tree $T$.

As in (3) above we consider an infinite tree $T$ in $BT$ with set of limit points $\mathcal{L}$ and put $\Omega = \mathcal{P}^1(K) \setminus \mathcal{L}$. Let $\text{red}_T : \Omega \to (\Omega, T)$ denote the reduction. A vertex $v$ of $T$ corresponds to an irreducible component, say $L_v$, of $(\Omega, T)$. Let $L_v^*$ be obtained from $L_v$ by omitting the double points and define $X_v := \text{red}_T^{-1}(L_v^*)$. Then $X_v$ is an affinoid subset of $\mathcal{P}^1(K)$ with canonical reduction $L_v^*$. For an edge $e$ of $T$ there is a corresponding double point $d \in (\Omega, T)$ which is the intersection of two irreducible components, say $L_{v_1}$, $L_{v_2}$, where $v_1$, $v_2$ are the vertices of the edge $e$. Let $(L_{v_1} \cup L_{v_2})^*$ denote the union of $L_{v_1}$ and $L_{v_2}$ where one has omitted all double points different from $d$. One defines $X_e := \text{red}_T^{-1}(L_{v_1} \cup L_{v_2})^*$. This is an affinoid subset of $\Omega$. If the two vertices $v_1$, $v_2$ are not end vertices, then $(L_{v_1} \cup L_{v_2})^*$ is the canonical reduction of the affinoid $X_e$. The affinoid covering of $\Omega$ associated to $T$ is the admissible affinoid covering $\{X_v, X_e \mid \text{all } v, e\}$. In the case that $T$ has no extremal vertices, $\{X_v \mid \text{all } e\}$ is an admissible pure (or formal) affinoid covering of $\Omega$ and this covering defines again the reduction $(\Omega, T)$ of $\Omega$.

2.2. The tree of a finite subgroup $G$ of $\text{PGL}_2(K)$ and separating lattices

Suppose that the set of ramification points $\mathcal{F}$ for the action of $G$ on $\mathcal{P}^1(K)$ has cardinality at least 3. Then the tree $\text{Tree}(G)$ of $G$ is defined to be the tree of $\mathcal{F}$. The group $G$ acts on $(\mathcal{P}^1(K), \mathcal{F})$ and on $\text{Tree}(G)$. The quotient graph $\text{Tree}(G)/G$ is again a tree and there is a subtree $\text{Tree}(G)^*$ of $\mathcal{T}_G$ which is mapped bijectively to the quotient tree. We make $\text{Tree}(G)^*$ into a tree of groups by attaching to each vertex its stabilizer in $G$. The stabilizer of any edge is clearly the intersection of the stabilizers of its two vertices. The tree $\text{Tree}(G)$ is completely described by the tree of groups $\text{Tree}(G)^*$. In the sequel we will give a list of the finite subgroups $G$ and their associated trees of groups. The set of ramification points $\mathcal{F}$ is the union of at most three $G$-orbits (as we will see). In the pictures for the tree of groups we will indicate the position of the images of these orbits on $(\mathcal{P}^1(K), \mathcal{F})$.

Let $[M]$ be a $G$-invariant lattice. The action of $G$ on $[M]$ induces an action of $G$ on $\mathcal{P}(\mathcal{M} \otimes k) = \mathcal{P}^1(k)$. A $G$-invariant lattice $M$ is called separating if any two points of $\mathcal{F}$
belonging to distinct $G$-orbits are mapped to distinct points of $\mathbb{P}^1(k)$. This property has the following translation. Let $\mathbb{P}^1(k) \to \mathbb{P}^1(k)/G$ denote the quotient map. The images of the $G$-orbits of $\mathcal{F}$ are the branch points. There is also a quotient map $\mathbb{P}^1(k) \to \mathbb{P}^1(k)/G$ and an induced map $\phi: \mathbb{P}^1(k)/G \to \mathbb{P}^1(k)/G$. Then $[M]$ separates if $\phi$ is injective on the set of branch points. The questions that we want to answer are

(i) Is there only one class of invariant lattices?
(ii) Is there a separating lattice? If so, is its class unique?

The answers depend heavily on the group $G$ and the characteristics $p_K, p_k$. In the sequel we have to treat each case separately. In the calculations we will, without explicitly stating this, replace $G \subset \text{PGL}(2, K) = \text{PSL}(2, K)$ by its preimage $\tilde{G} \subset \text{SL}(2, K)$ and identify $G$ with $\tilde{G}/\{\pm 1\}$. The group $\tilde{G}$ actually acts on $K^2$ and can be represented by some matrices.

An element of $G$ is thus represented by a matrix modulo $\pm 1$.

2.3. Trees for $\Gamma \subset \text{PGL}_2(K)$ and indecomposable groups

As before, $\Gamma \subset \text{PGL}_2(K)$ denotes an infinite discontinuous group, which is finitely generated and satisfies $\Omega/\Gamma \cong \mathbb{P}^1(K)$. Let $L \subset \mathbb{P}^1(K)$ denote the set of the limits points of $\Gamma$. By definition $\Omega = \mathbb{P}^1(K) \setminus L$. There are three possibilities:

(i) $L$ consists of one point.

We may suppose that $L = \{\infty\}$. It can be seen that in this case $p = p_K > 0$ and that the group has the form $\{z \mapsto \zeta_m^i z + a \mid i = 0, \ldots, m - 1; a \in A\}$ where $p$ does not divide $m$ and $\zeta_m$ is a primitive $m$th-root of unity. Let $F_q$ denote the smallest finite field containing $\zeta_m$. Then $A$ is a $F_q$-linear subspace of $K$. Moreover, for every positive real number $R$ the set $\{a \in A \mid |a| \leq R\}$ is a finite dimensional $F_q$-linear space. We note that $\Gamma$ is not finitely generated and this example will play a minor role in the sequel.

(ii) $L$ consists of two points.

We may suppose that $L = [0, \infty]$. It can be seen that $\Gamma$ is conjugated to a group consists of the transformations $\{z \mapsto \zeta_m^i \pi^n z^{\pm 1} \mid i = 0, \ldots, m - 1; n \in \mathbb{Z}\}$, where $\zeta_m$ is a primitive $m$th-root of unity and $\pi \in K$ satisfies $0 < |\pi| < 1$. For $m > 1$, the group $\Gamma$ is the amalgam $D_m \ast_{C_1} D_m$ and for $m = 1$ one has $\Gamma = C_2 \ast C_2$. Let $\mathcal{F}$ denote the set of the fixed points of $\Gamma$ and let $T$ denote the corresponding tree of the reduction $(K^*, \mathcal{F})$. Then $T/\Gamma$ has two vertices and one edge. Let $T/\Gamma^*$ be a subtree of $T$ which maps bijectively to $T/\Gamma$. One makes $T/\Gamma^*$ into a tree of groups by assigning to each vertex and edge its stabilizer in $\Gamma$. By [8] the amalgam of this tree of groups is $\Gamma$. This is consistent with the above description of $\Gamma^*$ as an amalgam. Finally, we note that the number of branch points of the map $\Omega = K^* \to \Omega/\Gamma \cong \mathbb{P}^1(K)$ is equal to 4 if $p_K \neq 2$ and equal to 2 if $p_K = 2$. 
(iii) $\mathcal{L}$ is infinite, compact and has no isolated points.

We are mainly interested in this case. The infinite set $\mathcal{L}$ determines a reduction of $\Omega$ and a tree $T_\mathcal{L}$ on which $\Gamma$ acts faithfully. An inversion is defined as an edge $e$ with vertices $v_1$, $v_2$ such that some element in $\Gamma$ permutes $v_1$, $v_2$. If $\Gamma$ acts without inversion on $T_\mathcal{L}$ then one defines $T := T_\mathcal{L}$. If $\Gamma$ acts with inversions then $T$ is defined as the tree obtained from $T_\mathcal{L}$ by subdividing each edge where an inversion occurs. In general one can also enlarge $\mathcal{L}$ to $\mathcal{F}$ by adding to $\mathcal{L}$ a finite set of $\Gamma$-orbits of points of $\Omega$. The resulting tree $T_\mathcal{F}$ can be obtained from $T_\mathcal{L}$ by adding finitely many $\Gamma$-orbits of vertices and edges. Again $\Gamma$ acts on this new tree. The tree $T$ can be obtained as a $T_F$ for a suitable $F$.

The quotient $T/\Gamma$ is a finite graph and in fact it is a tree since $\Omega/\Gamma \cong P^1(K)$. There exists a finite subtree $T/\Gamma^*$ of $T$ which maps bijectively to $T/\Gamma$. We make $T/\Gamma^*$ into a tree of groups by assigning to each vertex and edge its stabilizer subgroup in $\Gamma$. According to [8], the group $\Gamma$ is equal to the amalgam of $T/\Gamma^*$. Moreover, the tree of groups $T/\Gamma^*$ does not depend on the choice of $T$ in an essential way (indeed, one tree is obtained from the other by a sequence of subdivisions and contractions).

The group $\Gamma$ and the tree of groups are called indecomposable if every vertex and every edge of $T/\Gamma^*$ has a non-trivial stabilizer.

In the general situation, one removes from $T/\Gamma^*$ the vertices and the edges with trivial stabilizers. There results a number of indecomposable trees of groups $T/\Gamma_i^*$, $i = 1, \ldots, s$. Each piece has an amalgam $\Gamma_i$ and $\Gamma$ is the free product of the indecomposable subgroups $\Gamma_i$. The group $\Gamma_i$ is again a discontinuous, finitely generated subgroup of $\PGL_2(K)$, has a set of limits points $\mathcal{L}_i \subset \mathcal{L}$. We note that $\Gamma_i$ can be a finite group. Put $\Omega_i = P^1_K \setminus \mathcal{L}_i$. Then $\Omega_i/\Gamma_i$ is isomorphic to $P^1(K)$ since the graph $T/\Gamma_i^*$ is a tree. This decomposition of $\Gamma$ as a free product of indecomposable groups is helpful for the study of the maximal finite subgroups of $\Gamma$ and their intersections. Indeed, according to [8], each finite subgroup of $\Gamma_i \cdots \Gamma_1$ is conjugated to a subgroup of $\Gamma_i$ for a unique $i$. Similarly, let $G_1, G_2 \subset \Gamma$ be two finite groups with $G_1 \cap G_2 \neq 1$. Then there is a unique $i$ and a $\gamma \in \Gamma$ such that $\gamma G_j \gamma^{-1} \subset \Gamma_i$ for $j = 1, 2$.

2.4. Finite subgroups of $\PGL_2(K)$ for $p_K = p > 0$

In the sequel $G \neq 1$ is a finite subgroup of $\PGL_2(K)$. We investigate various properties of $G$ and the morphism $\pi : P^1(K) \to P^1(K)/G \cong P^1(K)$.

Suppose that $G$ is a $p$-group. Let $h \in G$ be an element of order $p$ in the center of $G$. Then $h$ has a single fixed point which we may suppose to be $\infty$. For any $g \in G$ one has $hg(\infty) = gh(\infty) = g(\infty)$ and thus $g(\infty) = \infty$. Thus $G$ is contained in the group $\mathcal{V} = \mathcal{V}(K)$ and can be identified with a finite subgroup of $K$. The map $\pi$ has only one ramification point and only one branch point. The contribution of the ramification point for the Riemann–Hurwitz formula is $2(-1 + \#G)$.

Since there is only one ramification point one cannot make the reduction of $P^1_K$ w.r.t. this set. The group itself is a finite subgroup $A$ of $K$ and consists of the maps $z \mapsto z + a$ ($a \in A$). One considers a set $S \subset P^1_K$ consisting of one orbit and the point $\infty$. For this set one can make the reduction of $P^1_K$, its dual graph. The latter is divided by the
action of $A$ and yields a the following graph of groups, which is similar to the one of the
next case

```
[Diagram image]
```

Suppose that $G$ lies in a Borel subgroup, in which case we will call $G$ a group of Borel
type. We exclude the two cases: $G$ is a $p$-group and $G$ is a cyclic group. Let $A \subseteq G$ be
the unique $p$-Sylow subgroup of $G$ and let $m$ be the order of the cyclic group $G/A$. Then $G$
is the semi-direct product of $C_m$ and $A$. Then $A \subseteq G$ is a finite-dimensional vector space over
$F_{p'} = F_p(\xi)$ where $\xi$ is a primitive $m$th-root of unity. The group $G$ can be represented by
the set of matrices $\{(\xi^c, 1) \mid c \in (\xi) \subseteq F_{p'}, \ a \in A\}$. The set of ramification points is $A \cup \{\infty\}$. The stabilizer of each $a \in A$ is a cyclic group of order $m$. The stabilizer of $\infty$ is $G$. The
map $\pi$ has two branched points, namely $\pi(\infty)$ and $\pi(0) = \pi(A)$. The contribution of the
point $\infty$ for the Riemann–Hurwitz–Zeuthen formula is $(m + 1)p^n - 2$ where $p^n = #A$.
The group described above will be called type $B(n,m)$ where $n$ is the dimension of the
$F_p$-vector space $A$. Clearly $m$ divides $p^n - 1$ and $p^n - 1$. Moreover all groups of type
$B(n,m)$ are isomorphic (in general not by conjugation in $PGL_2(K)$).

One can form the reduction of $P^1$ with respect to the set of ramification points $A \cup \{\infty\}$
and its dual graph. The latter is divided out by the action of $G$. The result is a graph
of groups. The groups attached to the vertices and the edges are the stabilizer subgroups
of these objects. The set of absolute values $\{|a| \mid a \in A, a \neq 0\}$ is written as $v_1 < v_2 < \cdots < v_s$. Let $A_i = \{a \in A \mid |a| \leq v_i\}$. Then $A_i$ is a $F_q$-subspace of $A$. The picture is the following:

```
[Diagram image]
```

We remark that a similar analysis and picture can be made for an infinite discontinuous
subgroup $G$ of a Borel group (see 2.3 case (i)). This will only be used for discontinuous
groups which are not finitely generated (see 5.9). The group $G$ can be represented by the
matrices $\{(\xi^c, 1) \mid c \in (\xi) \subseteq F_{p'}, \ a \in A\}$, where $\xi$ is a primitive $m$th-root of unity and $F_{p'} = F_p(\xi)$. Further $A$ is an infinite discrete $F_{p'}$-subspace of $K$. Let $v_1 < v_2 < v_3 < \cdots$ denote the absolute values of the non zero elements of $A$ and let $A_i := \{a \in A_i \mid |a| \leq v_i\}$. Then each $A_i$ is a finite dimensional $F_{p'}$-vector space. The action of $G$ on $A^1$ is discontinuous; the set of the ramification points is $A$. The graph of groups obtained, similarly as above, has the following picture

```
[Diagram image]
```

Suppose that $G$ does not lie in a Borel subgroup, in other words $G$ is not of Borel type.

Proposition 2.1. Suppose that $G$ is not of Borel type. Then

1. $\pi$ has at most three branch points.
(2) \( \pi \) has two branch points if \( p \) divides \( \#G \). (Moreover, one branch point is wildly ramified and the other is tamely ramified.)

(3) \( \pi \) has three branch points if \( p \) does not divide \( \#G \).

(4) Suppose that \( \pi \) has three branch points. Then \( G \) is one of the groups \( D_n, A_4, S_4, A_5 \), provided that \( p \) does not divide the order of this group.

Proof. Let \( a_1, \ldots, a_s \) denote the branch points of \( \pi \). The ramification index of \( a_i \) is written as \( e_i p^{d_i} \) with \( e_i \) is not divisible by \( p \). The contribution of the points of \( P^1 \) above \( a_i \) to the Riemann–Hurwitz–Zeuthen formula is

\[
\frac{\#G}{e_i p^{d_i}} ((e_i + 1) p^{d_i} - 2).
\]

This formula yields

\[
2 - \frac{2}{\#G} = \sum_{i=1}^{s} \frac{(e_i + 1) p^{d_i} - 2}{e_i p^{d_i}}.
\]

The term \( \frac{(e_i+1)p^{d_i} - 2}{e_i p^{d_i}} \) is \( \geq 1 \) if \( d_i \neq 0 \) and is \( \geq 1/2 \) if \( d_i = 0 \). Moreover, if \( p \) divides the order of \( G \), then the fixed point of an element of order \( p \) in \( P^1 \) is mapped to some \( a_i \) and thus \( d_i \geq 1 \). This proves (1)–(3).

From the formula

\[
2 - \frac{2}{\#G} = \sum_{i=1}^{3} \frac{e_i - 1}{e_i}
\]

one derives, as in the characteristic zero situation, the possibilities for \( e_1, e_2, e_3 \) and the structure of the group \( G \). \( \square \)

Remarks. In [9] there is a list of the groups described in part (2) of Proposition 5.4, namely,

(a) \( p = 2 \), group \( D_n, (n, 2) = 1 \). The group can be given by \( z \mapsto \zeta_n^a z \), where \( \zeta_n \) is a primitive \( n \)th-root of unity and \( a = 0, \ldots, n - 1 \). The ramification points are: 0, \( \infty \) for the elements \( z \mapsto \zeta_n^a z \) and \( \{ \zeta_n^a | a = 0, \ldots, n - 1 \} \) for the elements of order two in \( D_n \).

The group \( D_n \) can also be realized as the subgroup of \( PSL_2(F_2^s) \), generated by the matrices \( \begin{pmatrix} a & 1 \\ 1 & a \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Here \( s \) is minimal such that \( \zeta_n \in F_{2^{s+1}} \) and \( a = \zeta_n + \zeta_n^{-1} \).

(b) \( p = 3 \) and group \( A_5 \). This group can be realized as the subgroup of \( PSL_2(F_3) \) which contains \( PSL_2(F_3) \).

(c) The group \( PGL(2, F_q) \). The ramification points are \( P(F_q) \) and \( P(F_q^2) \setminus P^1(F_q) \). The stabilizer of a point in the first orbit is a Borel subgroup. The stabilizer of a point in the second orbit is a cyclic subgroup of order \( q + 1 \).

(d) The group \( PSL(2, F_q) \). The situation is similar. The ramification points are \( P(F_q^2) \).

There are two orbits, namely, \( P(F_q) \) and \( P(F_q^2) \setminus P^1(F_q) \). The stabilizer of a point
in the first orbit is a Borel subgroup. The stabilizer of a point in the second orbit is a cyclic subgroup of order \((q + 1)/2\).

**Proposition 2.2.** Suppose that \(G\) is not of Borel type. The \(G\)-invariant lattices \(M\) are all equivalent. The reduction map

\[
\text{red}_{[M]}: \mathbb{P}^1(K) = \mathbb{P}(M) \to \mathbb{P}(M \otimes k) = \mathbb{P}^1(k)
\]

with respect to \(M\), induces an injective homomorphism \(G \to \text{PGL}_2(k)\). Moreover, \(\text{red}_{[M]}\) induces a bijective map from the ramification points of \(G\) on \(\mathbb{P}^1(K)\) to the ramification points of the action of \(G\) on \(\mathbb{P}^1(K)\). For any ramification point \(x \in \mathbb{P}^1(K)\), the stabilizer in \(G\) of \(x\) coincides with the stabilizer in \(G\) of \(\text{red}_{[M]}(x)\). In particular, the lattice \(M\) separates the branch points.

**Proof.** Suppose that \(G\) has two non-equivalent lattices \(M_1, M_2\). Then one may suppose that \(M_1 = K^0e_1 + K^0e_2 \supset M_2 = K^0e_1 + K^0\rho e_2\) for some element \(\rho\) with \(0 < |\rho| < 1\). For \(g \in G\) one considers the reduction modulo the maximal ideal \(K^{00}\) of \(K^0\) of the matrix of \(g\) w.r.t. the basis \(e_1, e_2\). This matrix can be represented by \(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\) with \(a \in k^\times\) and \(b \in k\). The homomorphism \(G \to k^\times/\{\pm 1\}\) has as image an \(m\)-cyclic group with \(p \nmid m\) and its kernel \(N\) is a \(p\)-group. Then \(G\) is a semi-direct product of a cyclic group \(C_m\) of order \(m\) and \(N\). The group \(N\) has a single ramification point. This is also the fixed point of the elements in \(C_m\). Thus \(G\) is of Borel type. This proves the first statement.

The kernel \(N\) of \(G\) consists of the \(g \in G\) such that \(g\) acts as the identity on \(M \otimes k\). Thus \(g = 1 + a\) where the linear map \(a\) maps \(M\) into \(\rho M\) for some \(\rho\) with \(0 < |\rho| < 1\). Then, since the group \(G\) is finite, for a suitable \(q\), a power of \(\rho\), one has \(g^q = 1\). Thus \(N\) is a normal \(p\)-group. If \(N \neq 1\), then all elements of \(N\) have a single fix point, say \(\infty\). Since \(N\) is normal, one has \(g(\infty) = \infty\) for all \(g \in G\). This contradicts the assumption that \(G\) is not of Borel type. The last part of the proposition follows from Proposition 2.1. \(\square\)

2.5. Finite subgroups of \(\text{PGL}_2(K)\) with \(pk = 0\)

The groups \(G\) are \(D_n, A_4, S_4, A_5\). The set of ramification points \(\mathcal{F}\) has three orbits \(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_\infty\) and correspond with the fibres above \(0, 1, \infty \in \mathbb{P}^1 / G = \mathbb{P}^1\). The ramification indices of \(0, 1, \infty\) are denoted by \(e_0, e_1, e_\infty\). The above groups correspond to the following triples \((e_0, e_1, e_\infty)\):

\[(2, 2, n); \quad (2, 3, 3); \quad (2, 3, 4); \quad (2, 3, 5).\]

For each case there is a reduction of \(\mathbb{P}^1_k\) with respect to the set of ramification points \(\mathcal{F}\). Its dual graph is divided by the action of the group and this produces a graph (tree) of groups. If \(p_k\) does not divide the order of the group then this graph is just one point with stabilizer the group. The pictures for the interesting cases, where \(p_k\) divides the order of the group are the following:
$A_5$ and $p_k = 2$. Unique invariant lattice, not separating.

$A_5$ and $p_k = 3$. Unique invariant lattice, not separating.

$A_5$ and $p_k = 5$. Unique invariant lattice, not separating.

$S_4$ and $p_k = 2$. Unique invariant lattice, not separating.

$S_4$ and $p_k = 3$. Unique invariant lattice, not separating.

$A_4$ and $p_k = 2$. Unique separating lattice. More invariant lattices.

$A_4$ and $p_k = 3$. Unique invariant lattice, not separating.

$D_n$, $n$ odd, $p_k = 2$. Unique invariant lattice, not separating.
\[ \mathcal{F}_0 \quad D_p \quad D_{p^2} \quad \mathcal{F}_{\infty} \quad D_n \quad D_{p^2} \quad \mathcal{F}_1 \quad D_n; \quad p_k > 2; \quad n = p_k^i m; \quad (m, p_k) = 1. \]

Unique separating lattice.

For \( m = 1 \) more invariant lattices.

\[ \mathcal{F}_0 \quad D_2 \quad D_4 \quad \mathcal{F}_{\infty} \quad D_{2^2} \quad D_{2^2} \quad D_{2^2} \quad \mathcal{F}_1 \quad D_{2^2} \quad \mathcal{F}_1 \quad D_2; \quad p_k = 2. \]

Unique separating lattice.

More invariant lattices.

\[ \mathcal{F}_0 \quad D_2 \quad D_4 \quad \mathcal{F}_{\infty} \quad D_{2^2} \quad D_{2^2} \quad D_{2^2} \quad \mathcal{F}_1 \quad D_2; \quad p_k = 2; \quad n = 2^i m; \quad (m, 2) = 1; \quad m > 1. \]

Unique invariant lattice, not separating.

**Comments on the calculations**

1. Suppose that there are at least two classes of invariant lattices for the finite (non-cyclic) group \( G \subset \text{PGL}_2(K) \). Then there are invariant lattices

\[ M_1 = K^0 e_1 + K^0 e_2 \supset M_2 = K^0 e_1 + K^0 \pi e_2 \]

for some \( \pi \in K \) with \( 0 < |\pi| < 1 \). This induces a homomorphism \( \psi : G \mapsto k^*/\{\pm 1\} \) given by \( \psi(g) \) satisfies \( g(e_1) \equiv \psi(g)e_1 \mod K_{00}M_1 \). For any element \( g \) in the kernel of \( \psi \) one has that \( g^{p_k} \) acts trivially on \( M_1 \otimes k \). This implies that the order of \( g \) is some power of \( p_k \) and that the kernel of \( \psi \) is a \( p_k \)-group and contains the commutator subgroup \( [G, G] \). The image \( \psi(G) \) is a cyclic group of order prime to \( p_k \). From this one easily derives the only possibilities for the pairs \( (G, p_k) \), namely \( (D_{p_k}, p_k) \) for \( p_k \geq 2 \) and \( (A_4, p_k = 2) \).

2. For each of the above groups \( G \) one needs an explicit representation of \( G \) acting on \( \mathbf{P}^1(K) \). Using this, one calculates \( \mathcal{F} \) and the reduction \( (\mathbf{P}^1(K), \mathcal{F}) \) and the tree of groups is derived from the latter.

3. **Realizable amalgams**

Let \( \Gamma \subset \text{PGL}_2(K) \) be a finitely generated, infinite, discontinuous group such that \( \Omega / \Gamma \cong \mathbf{P}^1(K) \). We are investigating the structure of \( \Gamma \) and the number of branch points \( \text{br}(\Gamma) \) of \( \Omega \mapsto \Omega / \Gamma \cong \mathbf{P}^1(K) \). According to 2.3, \( \Gamma \) is a free product of indecomposable groups \( \Gamma_i, i = 1, \ldots, s \). We start by proving that \( \text{br}(\Gamma) = \sum \text{br}(\Gamma_i) \). If \( \mathcal{L} \) consists of two points then \( \text{br}(\Gamma) = 4 \) if \( p_k \neq 2 \) and is equal to \( 2 \) if \( p_k = 2 \) (see 2.3). In the sequel we will suppose that \( \mathcal{L} \) has more than two points. As in 2.3, one considers the subdivision \( \mathcal{T} \) of the tree \( \mathcal{T}_\mathcal{L} \) on which the group \( \Gamma \) acts without inversions. Let \( \mathcal{T}_\mathcal{G} \) denote a chosen embedding of \( \mathcal{T} / \Gamma \) in \( \mathcal{T} \). This makes of \( \mathcal{T}_\mathcal{G} \) a tree of groups. Let \( \mathcal{T}_\mathcal{G}^* \) denote the subset of \( \mathcal{T}_\mathcal{G} \) consisting of the vertices and edges which have a non-trivial stabilizer. Then \( \mathcal{T}_\mathcal{G}^* \) is the disjoint union of finite trees \( \mathcal{T}_1, \ldots, \mathcal{T}_s \). Let \( \mathcal{L}_i \) denote the subgroup of \( \Gamma \) generated by the stabilizers of the edges and vertices of \( \mathcal{T}_i \). Then \( \mathcal{L}_i \) is an indecomposable group and \( \Gamma = \Gamma_1 \star \cdots \star \Gamma_s \). Let \( \mathcal{L}_i \) denote the set of the limit points of \( \Gamma_i \). Then \( \mathcal{L}_i := \mathbf{P}^1(K) \setminus \mathcal{L}_i \) is the set of ordinary points of \( \Gamma_i \) and \( \mathcal{L}_i / \Gamma_i \cong \mathbf{P}^1_{\mathcal{L}_i} \). The group \( \Gamma_i \) can be finite, in which case \( \mathcal{L}_i = \mathbf{P}^1(K) \). Since \( \mathcal{L}_i \subset \Gamma \) one has \( \mathcal{L}_i \subset \mathcal{L} \) and therefore \( \mathcal{L} \subset \mathcal{L}_i \).
Lemma 3.1. Any point $x \in \Omega_i$, which is fixed by a non-trivial finite subgroup of $\Gamma_i$ belongs to $\Omega$.

Proof. Let $G_x \subseteq \Gamma_i$ be the stabilizer of the point $x \in \Omega_i$. Since $G_x$ is finite, the group $G_x$ stabilizes a vertex $v \in T$. After replacing $x$ by $\gamma(x)$ for a suitable element $\gamma \in \Gamma_i$, we may assume that $v \in T_i \subseteq T \subseteq T$.

If $x$ is a limit point of $\Gamma$, then $x$ determines a unique end of $T$. Let $L \subseteq T$ be the halfline that starts in the vertex $v$ and that corresponds to the end $x$ of $T$. Clearly, $G_x$ stabilizes the halfline $L$.

Let $T_i \subseteq T$ be the subtree, whose edges and vertices are stabilized by non-trivial finite subgroups of $\Gamma_i$. Then $T_i/\Gamma_i = T_i$. The ends of $T_i$ correspond to the limit points of $\Gamma_i$. In particular, $L \subseteq T_i$ and $x$ is a limit point of $\Gamma_i$. This contradicts $x \in \Omega_i$. $\Box$

Proposition 3.2. With the above notations one has $\text{br}(\Gamma) = \sum_{i=1}^s \text{br}(\Gamma_i)$.

Proof. Let $x \in \Omega$ represent a branch point for $\Gamma$. The stabilizer $G_x$ in $\Gamma$ is then finite and non-trivial. There is a unique $i \in \{1, \ldots, s\}$ such that $\gamma G_x \gamma^{-1}$ lies in $\Gamma_i$ for a suitable $\gamma \in \Gamma$. Then $\gamma(x)$ represents the same branch point of $\Gamma$ and represents moreover a branch point for $\Gamma_i$ since $\Omega \subseteq \Omega_i$. On the other hand a branch point for $\Gamma_i$ is represented by a point $x \in \Omega_i$ and is also a branch point for $\Gamma$ according to Lemma 3.1. $\Box$

According to 3.2, the questions of this paper are reduced to indecomposable groups. We will make these problems more precise. A finite indecomposable tree of groups $(T, G)$ is a finite tree $T$ with the additional structure:

(a) for the vertices $v$ and edges $e$ there are associated finite, non-trivial, finite groups $G_v, G_e$ which can be realized as subgroups of $\text{PGL}_2(K)$,
(b) for every edge $e$ with vertices $v_1, v_2$ there are given injective homomorphisms $G_e \to G_{v_1}$ and $G_e \to G_{v_2}$. In the sequel we will just identify $G_e$ with a subgroup of $G_{v_1}$ and $G_{v_2}$.

The group of $(T, G)$, i.e., the amalgam of the tree of groups $(T, G)$, will be denoted by $\Gamma$. We recall from [8], that there is an abstract tree $\text{Tree}(T, G)$ on which $\Gamma$ acts. Its defining properties are:

- $T$ is a subtree of $\text{Tree}(T, G)$, and for every vertex $v$ or edge $e$ of $T$, the groups $G_v, G_e$ are the stabilizers in $\Gamma$ of $v$ and $e$. Moreover, the map $T \to \text{Tree}(T, G)/\Gamma$ is an isomorphism of trees.
- An embedding $\tau$ of $(T, G)$ in $\text{BT}$ is a map $\tau$ from the vertices of $T$ to lattice classes, and for every vertex $v$ an injective homomorphism $\tau_v : G_v \to \text{PGL}_2(K)$ such that:

  (a) The subtree of $\text{BT}$ generated by all $\tau(v)$ is isomorphic to $T$.
  (b) $\tau_v(G_v)$ stabilizes the lattice class $\tau(v)$.
  (c) For any edge $e = \{v_1, v_2\}$ the restrictions of $\tau_{v_1}$ and $\tau_{v_2}$ to $G_v$ coincide.
The embedding $\tau$ is called a realization if the induced homomorphism from the amalgam $\Gamma$ of $(T, G)$ to $\text{PGL}_2(K)$ is injective and moreover its image is a discontinuous subgroup of $\text{PGL}_2(K)$. For convenience we will identify $\Gamma$ with its image in $\text{PGL}_2(K)$.

Let a realization $\tau$ of $(T, G)$ be given. For a vertex $v$ of $T$, the stabilizer $F$ in $\tau(v)$ of $\tau(v)$ is equal to $\tau_v(G_v)$. Indeed, $F \supset \tau_v(G_v)$ and $\tau_v(G_v)$ is a maximal finite subgroup of $\Gamma$. Then a map $\tau: T \to \mathcal{B}T$ extends uniquely to a $\Gamma$-equivariant map $\tau^*$ from the vertices of Tree$(T, G)$ to $\mathcal{B}T$. This map is injective since any maximal finite subgroup of $\Gamma$ is conjugated to a group $G_v$ for a unique vertex $v$ of $T$. We note that the subtree $T$ of $\mathcal{B}T$, generated by the image of $\tau^*$, has in general more vertices. Moreover, for an edge $e = \{v_1, v_2\}$ of Tree$(T, G)$, the pair $\{\tau^*v_1, \tau^*v_2\}$ need not be an edge of $T$. Indeed, according to part (3) of Section 2.1, new vertices and edges will occur if the image of $\tau^*$ contains three lattice classes $[M_i]$, $i = 1, 2, 3$, which do not lie on a segment of $\mathcal{B}T$ and such that the unique lattice class $[M]$ determined by $\{[M_i]\}_{i=1,2,3}$ does not lie in the image of $\tau^*$. Nevertheless, $\Gamma$ acts on $T$ and this action has no inversions since $\Gamma$ has no inversions on Tree$(T, G)$. The quotient graph $T/\Gamma$ can be seen to be a finite tree. An embedding of this quotient graph in $T$ makes it into a finite tree of groups. The latter is essentially some subdivision of $(T, G)$.

**Problem 1.** Classify the realizable finite indecomposable trees of groups.

Since a given tree of groups can be changed by subdivision or contraction, this question can only be handled if we introduce the notion of contracted tree of groups. This notion, which will be defined in 3.16, seems at first sight to give a restriction on the subgroups of $\text{PGL}_2(K)$ under consideration.

**Problem 2.** What is $\text{br}(\Gamma)$ for a realizable contracted $(T, G)$?

**Problem 3.** Is every finitely generated, discontinuous, indecomposable $\Gamma \subset \text{PGL}_2(K)$ with $\Omega/\Gamma \cong \mathbb{P}^1_k$, isomorphic to the amalgam of a contracted, finite, indecomposable tree of groups $(T, G)$?

For the first question, F. Herrlich [5] has given several criteria. The condition (b) of Satz 1 of [5] can be formulated as follows: Let $\tau$ be an embedding of $(T, G)$ in $\mathcal{B}T$. Then $\tau$ is a realization if for every edge $e = \{v_1, v_2\}$ and every $[M] \in [\tau(v_1), \tau(v_2)]$, $[M] \neq [\tau(v_1), \tau(v_2)]$ and every

$$g \in \bigcup_v \tau_v(G_v)$$

such that $g \not\in \tau_{v_1}(G_{v_2}) = \tau_{v_2}(G_{v_2})$ one has that $g([M])$ does not lie in the convex hull in $\mathcal{B}T$ spanned by all $\tau(v)$. We will reformulate this criterion and give an independent proof for the case of edges. The latter will lead to a classification of all groups $\Gamma := G_1 *_{G_1} G_2$, with $1 \neq G_3 \neq G_1, G_2$, which are realizable as a discontinuous subgroup of $\text{PGL}_2(K)$. For a more general situation a criterion is formulated and proved which makes it possible to realize the contracted trees of groups by induction.
Theorem 3.3 (Herrlich’s criterion).

(1) (The criterion for edges.) Let $G_1, G_2$ be finite subgroups of $\text{PGL}_2(K)$ and let $G_3 \neq 1$ be a proper subgroup of $G_1 \ast G_2$. The natural homomorphism $\Gamma := G_1 \ast G_2 \to \text{PGL}_2(K)$ is a realization of $G_1 \ast G_2$ as a discontinuous subgroup if and only if:

1a) There are lattice classes $[M_1] \neq [M_2]$ such that $[M_i]$ is $G_i$-invariant for $i = 1, 2$.

1b) There is a lattice class $T \in [[M_1], [M_2]]$ having the property: if $\alpha \in G_1 \ast G_2$ satisfies $\alpha T = T$, then $\alpha \in G_1$.

(2) (A more general case.) Let a finite tree of groups $(T, G)$ be given and let $e = \{v_1, v_2\}$ be an edge. Let $(T^1, G^1), (T^2, G^2)$ denote the trees of groups obtained by deleting the edge $e$ and suppose $v_1 \in T^1, v_2 \in T^2$. An embedding $\tau$ of $(T, G)$ is a realization if:

2a) The restriction of the embedding $\tau$ to each $(T^1, G^1), (T^2, G^2)$ is a realization.

Let $\Gamma^1, \Gamma^2$ denote the resulting discontinuous subgroups of $\text{PGL}_2(K)$.

2b) There is a lattice class $V \in [\tau(v_1), \tau(v_2)]$ with $V \neq \tau(v_1), \tau(v_2)$, such that for $g_i \in \Gamma^i \setminus \tau(G_i), i = 1, 2$, one has that $V \neq g_1 V, g_2 V$ and $V \in [g_1 V, g_2 V]$.

Observation 3.4 (Invariant lattice classes). Suppose first that $p_K = 0$. Let $A \in \text{PGL}_2(K)$ be an element of order $m > 1$ with eigenvectors $e_1$ and $e_2$ in $K^2$. Then the lattice classes $[K^0e_1 + K^0\lambda e_2]$ with $\lambda \in K^*$ are invariant under $A$. We will call this infinite line the axis of $A$ (or of the group generated by $A$). If $m$ is not divisible by $p_K$, then there are no other invariant lattice classes. The same holds if $m$ is divisible by $p_K$, but not equal to some power of $p_K$. For $m = p_K^a$, a lattice class $[M]$ is invariant if and only if there is a lattice class $[M_1]$ of the form $M_1 = K^0e_1 + K^0\lambda e_2$ such that for a suitable choice of $M_2$, representing $[M]$, one has $M_1/M_2 = K^0/\pi K^0$ with $|\pi| \geq |\zeta_{p_K} - 1|$. Here $\zeta_{p_K}$ denotes a primitive $p_K$th root of unity. In geometric terms, $[M]$ is invariant under $A$ if and only if its “distance” to the axis of $A$ is less than or equal to $-\log |\zeta_{p_K} - 1|$.

Suppose now that $p_K = p_K = p > 0$. Let $A \in \text{PGL}_2(K)$ have finite order $m$, then either $m$ is not divisible by $p$ or $m = p$. In the first case $A$ has two independent eigenvectors $e_1, e_2$ and the set of invariant lattice classes is again the axis of $A$. In the second case, there is a basis $e_1, e_2$ for which $A$ has the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The set of the invariant lattice classes is $[[K^0e_1 + K^0(\alpha e_1 + b e_2)] | a, b \in K; \ 0 < |b| \leq 1]$. This set is a subtree of the tree of all lattice classes.

Proof. (1) Suppose that the amalgam is realizable. Then we consider invariant lattices $[M_1], [M_2]$ for the groups $G_1, G_2$. Since $\Gamma = G_1 \ast G_2$ contains hyperbolic elements, one has that $[M_1] \neq [M_2]$. Take an element $\alpha \in G_1 \setminus G_3$, then the set of elements in $[[M_1], [M_2]]$ which are invariant under $\alpha$ has the form $[[M_1], [M_1(\alpha)]]$ for some $[M_1(\alpha)] \neq [M_2]$. Indeed, the group generated by $\alpha$ and $G_2$ contains a hyperbolic element and thus $[M_2]$ is not stable under $\alpha$. Observation 3.4 proves the existence of $[M_1(\alpha)]$. Let $\alpha_1 \in G_1 \setminus G_3$ be such that the length of $[[M_1], [M_1(\alpha_1)]]$ is maximal. Similarly for $\beta \in G_2 \setminus G_3$ the set of $\beta$-invariant lattice classes in $[[M_1], [M_2]]$ is $[[M_2(\beta)], [M_2]]$ with $[[M_2(\beta)]] \neq [M_1]$. Let $\beta_1 \in G_2 \setminus G_3$ be such that the length of $[[M_2(\beta_1)], [M_2]]$ is maximal. If a lattice class $S$ lies in the intersection of $[[M_1], [M_1(\alpha_1)]]$ and $[[M_2(\beta_1)], [M_2]]$ then $S$ is invariant under $\alpha_1 \beta_1 \in \Gamma$. This element has infinite order, is therefore hyperbolic and has no invariant lattice class. So we conclude that the above intersection is empty and we
take for $T$ any element in $[[M_1(\alpha_1)], [M_2(\beta_1)]]$, different from the two endpoints of this segment. Clearly $T$ has the required property.

Suppose that conditions (1a) and (1b) are satisfied. We make the following observation:

If $g_i \in G_i \setminus G_3$, $i = 1, 2$, then $g_1 T \neq g_2 T$ and $T \in [g_1 T, g_2 T]$ (and, of course, $T \notin g_1 T, g_2 T$).

Indeed, the intersection $[[M_1], T] \cap [g_1[M_1], g_1 T]$ is equal to $[[M_1], S_1]$ with $S_1 \in [[M_1], T]$ and $S_1 \neq T$. Similarly, $[T, [M_2]] \cap [g_2 T, g_2 [M_2]] = [S_2, [M_2]]$ with $S_2 \in [T, [M_2]]$ and $S_2 \neq T$. This proves the observation.

Consider a word $w_1 \cdots w_1$ in $G_1 \ast G_2$, where $w_i \in G_1 \setminus G_3$ if $i$ is odd and $w_i \in G_2 \setminus G_3$ if $i$ is even. By induction on $s$, we will show that the sequence of lattice classes $T, w_1 T, w_2 w_1 T, \ldots, w_s w_{s-1} \cdots w_1 T$ are distinct consecutive points on the segment $[T, w_s \cdots w_1 T]$.

The statement is obvious for $s = 1$. The induction hypothesis says that $T, w_{s-1} T, \ldots, w_{1} T$ are consecutive points on the segment $[T, w_{s-1} \cdots w_1 T]$. By the observation, $w_{s}^{-1} T, T, w_{s-1} T$ are consecutive points on the segment $[w_{s}^{-1} T, w_{s-1} T]$. Since the collection of all lattice classes is a tree-like object, one finds that $w_{s}^{-1} T, T, w_{s-1} T, \ldots, w_1 T$ are consecutive points on the segment $[w_{s}^{-1} T, w_{s-1} \cdots w_1 T]$. Applying $w_1$ to the latter one obtains the statement for $s$. With the above notations one has that $T$ lies in the segment $[[M_1(\alpha_1)], [M_2(\beta_1)]]$ and is not an end point. Therefore there is a $c > 0$, such that each segment $[T, w_1 T], [T, w_{s-1} T], \ldots, [T, w_1 T]$ has length $\geq c$. This implies that the distance between $w_s \cdots w_1 T$ and $T$ is $\geq sc$.

We note that a similar statement holds for words $w_s \cdots w_1 \in G_1 \ast G_2$ with $w_i \in G_1 \setminus G_3$ for even $i$ and $w_i \in G_2 \setminus G_3$ for odd $i$.

The elements of $G_1 \ast G_2$ can be written in the form $w_s \cdots w_1$ with $w_1 \in G_1$, further $w_i \in G_1 \setminus G_3$ for odd $i > 1$ and $w_i \in G_2 \setminus G_3$ for even $i$. Suppose that a word $w_s \cdots w_1$ (as above) maps to $1 \in \text{PGL}_2(K)$. Then $w_s \cdots w_1 T = T$. What we have shown above implies that $w_1 \in G_3$. From $w_s \cdots w_2 T = T$ one concludes that $s = 1$ and $w_1 = 1$. Thus the natural homomorphism $G_1 \ast G_2 \to \text{PGL}_2(K)$ is injective.

The group $G_1 \ast G_2$ has a normal subgroup of finite index $N$ which is a finitely generated free group. The group $N$ is a Schottky group if every $\gamma \in N$, with $\gamma \neq 1$ is hyperbolic. In that case $G_1 \ast G_2$ is a discontinuous group. Take an element $\gamma \in N$, $\gamma \neq 1$. One can represent $\gamma$ by some word $w_s \cdots w_1$ as above. After replacing $\gamma$ by a conjugate we may suppose that this word is cyclically reduced which means that each $w_i$ lies in $G_1 \cup G_2$ and not in $G_3$. Moreover consecutive $w_i$’s are not in the same $G_j$ and $w_i, w_1$ are not in the same $G_j$. For every $n \geq 1$ the element $\gamma^n$ has length $ns$ and the distance of $T$ to $\gamma^n T$ is $\geq ns$. Suppose that $\gamma$ is not hyperbolic, then there is a fixed lattice class $S$ for $\gamma$. Let $d$ be the distance of $S$ to $T$. Then the distance of $\gamma^n T$ to $S$ is also $d$ and therefore the distance between $T$ and $\gamma^n T$ is bounded by $2d$. This contradiction shows that $\gamma$ is hyperbolic.

(2) One has to show that the canonical homomorphism $\Gamma^1 \ast_{\Gamma_1} \Gamma^2 \to \text{PGL}_2(K)$ is injective and that its image is a discontinuous group. The proof of (1) above remains valid if one replaces $G_1, G_2, G_3, T$ by $\Gamma^1, \Gamma^2, \tau(G_\epsilon), V$. 

In [5], a list is given of the amalgams $\Gamma = G_1 \ast G_3 G_2$ which can be realized as discontinuous subgroup of $\text{PGL}_2(K)$ with $p_K = 0$. Our list 3.5 is longer, probably because there
seems to be the additional condition in [5], that the quotient tree $T/\Gamma$ (see 2.3) has two vertices and one edge.

**Theorem 3.5.** All realizable amalgams $G_1 *_{G_3} G_2$ for $p_K = 0$ and $p_K > 0$. We will assume that $1 \neq G_3 \neq G_1, G_2$. Further $G_1 *_{G_3} G_2$ and $G_2 *_{G_3} G_1$ are considered as the same amalgam.

1. The only cases with cyclic $G_3$ are

$$C_{3m} *_{C_3} A_4, \ p_k = 3 \quad \text{and} \quad C_{2m} *_{C_2} D_n, \ p_k = 2, \ n \ \text{odd}.$$  

In the sequel we suppose that $G_1, G_2 \in \{A_5, S_4, A_4, D_n\}$.

2. $G_3 = C_m$ and $C_m$ is a maximal cyclic subgroup of both $G_1$ and $G_2$.

3. $D_{3m} *_{C_2} D_n, \ p_k = 2, \ n \ \text{odd}$ and $D_{3m} *_{C_3} A_4, \ p_k = 3$ for $m > 1$. In both cases $G_3$ is not maximal cyclic in $G_1$.

4. $p_k = 5$ and $A_5 *_{D_5} \{A_5, D_{3m}\}$ with $m > 1$.

5. $p_k = 3$ and $\{A_5, S_4\} *_{D_3} \{A_5, S_4, D_{3m}\}$ with $m > 1$.

6. $p_k = 2$ and $A_5 *_{A_4} \{A_5, S_4\}, \ S_4 *_{D_4} \{S_4, D_{4m}\}, \ D_{2n} *_{D_2} \{A_5, A_4, D_{2n}\}$ with $m, n > 1$ and odd $n$. Finally, $D_{2n} *_{D_2} S_4$ with odd $n$ and such that the image of $D_2$ in $S_4$ is not contained in $A_4$.

For a finite subgroup $G \subset \text{PGL}_2(K)$ we write $\text{br}(G)$ for the number of branch points of $P^1_K \rightarrow P^1_K/G$. For any realizable $\Gamma = G_1 *_{G_3} G_2$ is realizable the formula $\text{br}(\Gamma) = \text{br}(G_1) + \text{br}(G_2) - \text{br}(G_3)$ holds.

**Proof.** Suppose $p_K = 0$ and let $G$ be a finite, non-cyclic subgroup of $\text{PGL}_2(K)$. The reduction of $P^1(K)$ with respect to the set $\mathcal{F}$ of its ramification points defines a subtree $\text{Tree}(G)$ of the tree of $P^1(K)$, i.e., the tree of all lattice classes in $K^2$. One can reconstruct most of $\text{Tree}(G)$ from its quotient $\text{Tree}(G)/G$ and the additional data of stabilizers and images of the points of $\mathcal{F}$. For a cyclic subgroup $H$ of $G$ one can determine where the axis of $H$ lies with respect to $\text{Tree}(G)$. For a non-cyclic subgroup $H$ of $G$ one can also determine the position of $\text{Tree}(H)$ with respect to $\text{Tree}(G)$.

The general method for obtaining a realizable $G_1 *_{G_3} G_2$ is to embed the two trees $\text{Tree}(G_1), i = 1, 2,$ into the tree of $P^1(K)$ in a way compatible with the common subgroup $G_3$ and such that one can apply Herrlich’s criterion. In case $G_3$ is cyclic, this amounts to determining the $G_3$-axis for both $\text{Tree}(G_i), i = 1, 2,$ and placing the invariant lattices classes $[M_i], i = 1, 2,$ for $G_i$, $i = 1, 2,$ with respect to this $G_3$-axis such that Herrlich’s criterion can be applied. For non-cyclic $G_3$ the situation is similar, but more complicated.

1. Let $G_1 = C_{3\ell}$, $G_3 = C_\ell$ with $m, \ell > 1$. According to Theorem 3.3, $G_1$ must have less invariant lattice classes than $G_3$. Observation 3.4 yields that $\ell = p^3_k$. The $G_3$-axis is the $G_1$-axis and therefore this axis can, in $\text{Tree}(G_2)$, only intersect vertices with stabilizer $G_3$. This prevents $G_2$ from being cyclic. From the pictures of Section 2.3 one concludes that only $C_{3\ell} *_{C_3} A_4$ with $p_k = 3$ and $C_{2\ell} *_{C_3} A_4$ with $p_k = 2$ are candidates.

Let us consider the first case in more detail. The tree $\text{Tree}(A_4)$ has a central point $[M_2]$ with stabilizer $A_4$. Connected to this there are 4 vertices with stabilizers the 4 subgroups of order 3. One fixes one of those vertices, say $P$, and calls its stabilizer $C_3$. From the position of the images of $\mathcal{F}$ one can see that the $C_3$-axis in $\text{Tree}(A_4)$ only intersects in the vertex $P$. 
The lattice class $[M_1]$ is put anywhere on the $C_3$-axis. We have now to find the lattice class $T$ of Theorem 3.3 on the segment $[P, [M_2]]$. A small calculation shows that the distance between $P$ and $[M_2] = -\log |C_3 - 1|$. In particular, the vertex $[M_2]$ is not stable under any $\alpha \in G_1 \setminus G_3$. Hence $T$ exists and $C_{3k} \ast C_3 A_4$, $p_k = 3$ is realizable. The same proof works for the other candidate.

(2) Let $C_m$ be a maximal cyclic subgroup of $G_1$. Then the points on the $C_m$-axis in $\text{Tree}(G_1)$ which have a large enough distance to $[M_1]$, a chosen $G_1$-invariant lattice class, are only stabilized by $C_m$. The same holds if $C_m$ is a maximal cyclic subgroup of $G_2$. If one places $[M_1]$ and the $G_2$-invariant lattice $[M_2]$ at great enough distance then there is a $T$ on the $C_m$-axis, between $[M_1]$ and $[M_2]$, which is stabilized only by $C_m$.

(3) We suppose that $G_3 = C_6$ is cyclic but not maximally in $G_1$. Let $C_{m3}$ be the maximal cyclic subgroup of $G_1$ containing $G_3$. If $G_1 \ast G_1$, $G_2$ is realizable then also $C_{m3} \ast C_6$, $G_2$. From (1) one concludes that the only candidates are $D_{2m} \ast C_2 D_n$, $p_k = 2$, $n$ odd, and $D_{3m} \ast C_3 A_4$. The proof that these two groups are realizable is similar to the proof in (1).

(4) Now we consider non-cyclic $G_3$’s. Since $G_3$ stabilizes at least two lattice classes one concludes that $G_3 = D_{p_k}$ or $A_4$ with $p_k = 2$. In the latter case the only candidates are $G_1 \ast A_2$, $G_2$ with $G_1, G_2 \in \{A_5, S_4\}$. Using the pictures of Section 2.3 one concludes that the tree $\text{Tree}(A_4)$ fits in two ways into $\text{Tree}(S_4)$. The same holds for the embedding of $\text{Tree}(A_4)$ and $\text{Tree}(S_4)$ if one fixes $A_4$ as subgroup of $A_5$. Using this and Theorem 3.3, one obtains that $A_5 \ast A_4 \{A_5, S_4\}$, $p_k = 2$ are the only realizable amalgames with $G_3 = A_4$.

Next we suppose that $G_3 = D_{p_k}$ with $p_k > 2$. If both $G_1$ and $G_2$ are dihedral groups, say with fixed lattice classes $[M_1]$, $[M_2]$. The axis for an element $C$ of order 2 in $G_3$ is the horizontal line of the corresponding picture of Section 2.3. The same holds for $C$ considered as element of $G_1$ and $G_2$. In gluing $\text{Tree}(G_1)$ over $\text{Tree}(G_2)$, the invariant lattice classes for $G_1$ and $G_2$ are on the same position. This contradiction yields that we may suppose that $G_3$ is not a dihedral group and thus $G_3 = D_{p_k}$. For $p_k = 3, 5$ it can seen that all candidates are realizable.

In the last part of the proof $p_k = 2$. Consider first $G_3 = D_2$.

The essential part $E$ of the tree $\text{Tree}(D_2)$ is the set of vertices and edges which are invariant under $D_2$. The vertices of $E$ are denoted by $v_0, v_1, v_2, v_3$ and the edges of $E$ are $[v_0, v_i], i = 1, 2, 3$. The automorphism group $S_3$ of $D_2$ acts faithfully on $v_1$, $v_2$, $v_3$. For $G \in \{A_5, S_4, A_4, D_{2m}\}$ and a given embedding $D_2 \subset G$ there is an embedding of $\text{Tree}(D_2)$ in $\text{Tree}(G)$. For each of the $v_i$ we write $G(v_i)$ for the stabilizer of $v_i$ in the group $G$. We note that there is still the freedom of permuting $v_1$, $v_2$, $v_3$. Consider two embeddings $D_2 \subset G_1, G_2$. This induces groups $G_1(v_i)$ and $G_2(v_i)$ for each $i$. If $G_1 \ast G_2$, $G_2$ is realizable then necessarily for each $i$ at most one of the groups $G_j(v_i)$ is different from $D_2$. To show that this condition is also sufficient one considers an edge, say $[v_0, v_1]$, with $G_2(v_0) = D_2$ and $G_1(v_1) = D_2$. There is a point $T_1 \in [v_0, v_1]$ such that $\alpha \in G_1 \setminus D_2$ does not stabilize any point $P \neq T_1$ with $P \in [T_1, v_1]$. Similarly, there is a point $T_2$ such that $\beta \in G_2 \setminus D_2$ does not stabilize any point $P \neq T_2$ with $P \in [v_0, T_2]$. Now one has to verify that $[v_0, T_2] \cap [T_1, v_1]$ is a non-trivial segment. It happens that every case where the necessary condition is satisfied, this intersection is non-trivial. The details can easily be deduced from the information on the groups $G$. From the following table one can read off all realizable $G_1 \ast D_2 G_2$. A $\ast$ indicates that the corresponding group $G(v_i)$ is not equal
The first $S_4$ means that $D_2 \subset S_4$ is not contained in $A_4$ and the second $S_4$ indicates the opposite situation.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$D_{2n}$, odd $n$</th>
<th>$D_{2n}$, even $m$</th>
<th>$S_4$</th>
<th>$S_4$</th>
<th>$A_5$</th>
<th>$A_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_0$</td>
<td>$*$</td>
<td>$*$</td>
<td>$*$</td>
<td>$*$</td>
<td>$*$</td>
<td>$*$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$*$</td>
<td>$*$</td>
<td>$*$</td>
<td>$*$</td>
<td>$*$</td>
<td>$*$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$*$</td>
<td>$*$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Next we consider $G_3 = D_4$. The essential part $E$ of Tree$(D_4)$ will be the part stabilized by $D_4$. This $E$ has the same description as before. However one of the end vertices, say $v_1$, is “marked” by the position of the ramification points of order 4. An automorphism of $D_4$ permutes the other points $v_2, v_3$. For an embedding $D_4 \subset G$ one writes, as above, $G(v_1)$ for the stabilizer in $G$ of the vertex $v_1$. For $G = D_{4m}$ one finds, due to this marking, that $D_{4m}(v_1) = D_{4m}$. Moreover $D_{4m}(v_1) \neq D_4$ if and only if $m$ is even. Further $D_{4m}(v_2) = D_{4m}(v_3) = D_4$. In particular, for a realizable $G_1 * G_2$, one of the $G_i$ is not a dihedral group. With the previous notation one has $S_4(v_1) \neq D_4$ only if $i = 2$. This shows that (omitting a small verification concerning the length of certain segments) the candidates $S_4 * D_4 \{S_4, D_{4m}\}$ are indeed realizable.

Finally, $G_3 = D_2^s$ with $s \geq 3$ is seen to be impossible by methods similar to the case of $D_4$.

The formula $\text{br}(\Gamma) = \text{br}(G_1) + \text{br}(G_2) - \text{br}(G_3)$ in situation (2) of 3.5, is a special case of the main result of Section 5. We sketch a method which proves the formula in all cases of 3.5. A realization embeds $\Gamma$ as discontinuous subgroup of $\text{PGL}_2(K)$. Let $\mathcal{F}_i$, $i = 1, 2$, denote the ramification points of $G_i$. Let $\mathcal{F}$ be the union of all $\Gamma$-orbits of $\mathcal{F}_1 \cup \mathcal{F}_2$. One considers the tree $T_{\mathcal{F}}$ and the corresponding admissible affinoid covering $\{X_v, X_e \mid v \in \mathcal{V}, e \in \mathcal{E}\}$. A careful case by case, analysis is needed to locate the affinoids of this covering which contain the points $\mathcal{F} \cap \Omega$. As an example we consider $p_k = 2$ and $G_1 * G_2$, with $G_1, G_2 \cong A_5$ and $G_3 \cong A_4$. From the given construction of the realization one can read off the following data. The tree $T_{\mathcal{F}}$ can be seen to have four kinds of vertices $[M]$, namely with stabilizers $\Gamma_{[M]}$ conjugated to: (i) $G_1$, (ii) $G_2$, (iii) $G_3$, (iv) the subgroup of $G_3$, isomorphic to $D_2$.

$X_{[M]} \cap \mathcal{F}$ consists of two ramification points of order 5 in the cases (i) and (ii); is empty for case (iii) and consists of two ramification points of order 2 for case (iv). For an edge $e$, the set $X_e \cap \mathcal{F}$ is empty. We conclude that $\text{br}(\Gamma) = 3$. \( \square \)

**Corollary 3.6.** Suppose that $p_K = 0$. Let $\Gamma \subset \text{PGL}_2(K)$ be a finitely generated, discontinuous, infinite, indecomposable group such that $\Omega / \Gamma \cong \mathbb{P}^1(K)$. We exclude the following cases for the group $\Gamma$:

(a) $p_k = 2$ and $\Gamma$ contains $S_4$ or $A_5$ or $D_4$ with $n \neq 2^e$.
(b) $p_k = 3$ and $\Gamma$ contains $A_4$.
(c) $p_k = 5$ and $\Gamma$ contains $A_5$.

Then every maximal finite subgroup of $\Gamma$ is non-cyclic and has a unique separating lattice class. Every non-trivial intersection $H$ of distinct maximal finite subgroups $G_1, G_2$ is a
maximal cyclic subgroup of \( \Gamma \) and moreover the canonical homomorphism \( G_1 *_H G_2 \to \Gamma \) is injective.

**Proof.** Let \( G \) be a maximal finite subgroup and let \([M] \in \mathcal{T}\) be stabilized by \( G \). There is a finite subtree \( \mathcal{T} \) of \( \mathcal{T} \), containing \([M]\), which is mapped bijectively to \( \mathcal{T} / \Gamma \). This implies that \( \Gamma \) contains a non-trivial amalgam \( G_1 *_G G_2 \) with \( G = G_1 \). By 3.5, \( G \) is not cyclic and moreover \( G \) has a unique separating lattice class.

Consider two maximal finite subgroups \( G_1 \neq G_2 \) with \( H = G_1 \cap G_2 \neq 1 \). Let \([M_1]\), \([M_2]\) denote the separating lattice classes for \( G_1 \) and \( G_2 \). First suppose that \( H \) contains an element \( h \) of order not divisible by \( p_k \). Let \( h_0, h_1 \) denote the two fixed points of \( h \). They determine the axis \( L \) of \( h \) in \( \mathcal{T} / \mathcal{H} \). The group \( \Gamma_L \) consisting of the elements \( y \in \Gamma \) which have each point of \( L \) as fixed point is equal to \( \{ y \in \Gamma | y h_0 = h_0, y h_1 = h_1 \} \), is a discontinuous group, and has a maximal finite subgroup \( F \). This group is a maximal cyclic subgroup of \( \Gamma \). Since \([M_1]\) and \([M_2]\) are on \( L \) one has \( H \supset F \). In case \( H = F \) we are done. If \( H \neq F \), then \( H \cong D_{a^2} \) for some \( a \geq 1 \). Moreover \( G_1 \cong D_n \) and \( G_2 \cong D_m \) with \( p_k \mid n \) and \( p_k \mid m \). Further \([M_1]\) and \([M_2]\) are also separating lattices for the subgroup \( H \) of \( G_1 \) and of \( G_2 \). This contradicts the uniqueness of the separating lattice for \( H \).

Suppose now that every element of \( H \) has order a power of \( p_k \). Then \( H \) is cyclic. Further \( G_1 \) and \( G_2 \) are dihedral groups. From the pictures in Section 2.5 it follows that the \( H \)-axis \( L \) passes through the separating lattices \([M_1]\) and \([M_2]\). Further \( L \) is also pointwise invariant under the normal subgroups of index 2 (note that \( p_k \neq 2 \)) of \( G_1 \) and \( G_2 \). We conclude that \( H \) is maximal cyclic in both \( G_1 \) and \( G_2 \).

Finally, we want to show that the canonical homomorphism \( G_1 *_H G_2 \to \operatorname{PGL}_2(K) \) is injective. Let again \([M_1]\), \([M_2]\) denote the separating lattices for \( G_1 \) and \( G_2 \). Let \([M_3] \neq [M_1]\) be the vertex of \( \mathcal{T} \) on the segment \([M_1],[M_2]\) in \( \mathcal{T} / \mathcal{H} \), closest to \([M_1]\), such that the stabilizer \( \Gamma_{[M_1]} \) of \([M_3]\) is a maximal finite subgroup of \( \Gamma \) and \([M_2]\) is its separating lattice. For any \([M] \neq [M_1],[M_2]\) on the segment \([M_1],[M_2]\) the stabilizer \( \Gamma_{[M]} \) is not a maximal finite subgroup of \( \Gamma \) or it is a maximal finite subgroup but \([M]\) is not separating for \( \Gamma_{[M]} \). Clearly \( \Gamma_{[M]} \supset H \). Suppose that \([M] \neq [M_1],[M_3],[M] \in ([M_1],[M_3]) \) has the property that \( \Gamma_{[M]} \) is not contained in either \( G_1 \) or \( G_3 \). The group \( \Gamma_{[M]} \) is contained in \( (\text{or equal to}) \) a maximal finite subgroup \( G_4 \) with separating lattice \([M_4]\) \( \neq [M] \). The intersection \( G_1 \cap G_4 \) is maximal cyclic, contains \( H \) and is therefore equal to \( H \). As seen above, \([M_1]\) and \([M_4]\) lie on the axis of the cyclic group \( H \). Similarly \([M_3]\) and \([M_4]\) lie on the axis of \( H \). However the vertices \([M_1]\), \([M_3]\), \([M_4]\) do not lie on a line of \( \mathcal{BT} \). We conclude that \( \Gamma_{[M]} \) lies in either \( G_1 \) or \( G_3 \) for every \([M] \in ([M_1],[M_3]) \).

For every \( g \in G_1 \setminus H \) there is a lattice \([M(g)] \in ([M_1],[M_3]) \) such that \( g \) stabilizes all \([M] \in ([M_1],[M_3)]) \) and does not stabilize any other \([M] \) on \([M_1],[M_3]) \). Let \([M_1],[M_3]) \) denote the union of the \([M_1],[M_3]) \) for all \( g \in G_1 \setminus H \). Then \( \Gamma_{[M]} \subset G_1 \) if and only if \([M] \in ([M_1],[M_3]) \) or \( \Gamma_{[M]} = H \). Similarly, there is an \([M_4]) \in ([M_1],[M_3]) \) such that \( \Gamma_{[M]} \subset G_3 \) if and only if \([M] \in ([M_4]) \) or \( \Gamma_{[M]} = H \). Suppose that there is no \([M] \) with \( \Gamma_{[M]} = H \), then \([M_4],[M_3]) \) is the union of the segments \([M_1],[M_4]), ([M_4],[M_3]), ([M_3],[M_4]) \) and thus the intersection \([M_4],[M_4]) \cap ([M_4],[M_4]) \) is not empty. A lattice class \([M] \) in this intersection satisfies \( \Gamma_{[M]} \subset G_1 \cap G_4 = H \). This contradiction shows that \( \Gamma_{[M]} = H \) holds for some \([M] \in ([M_1],[M_3]) \). According to the first part of Theorem 3.3 the homomorphism \( G_1 *_H G_2 \to \operatorname{PGL}_2(K) \) is injective. \( \square \)
In the sequel of this section we suppose that $p_k = p_k = p > 0$ and our aim is to compute the list of all $G_1 \ast_{G_1} G_2$ which can be realized as discontinuous subgroup of $\text{PGL}_2(K)$.

**Proposition 3.7.** Suppose that $p_K = p > 0$. Let $\Gamma \subset \text{PGL}_2(K)$ be a finitely generated, discontinuous, infinite indecomposable group such that $\Omega/\Gamma \cong P^1(K)$. Then every maximal finite subgroup of $\Gamma$ is non-cyclic.

**Proof.** We may restrict our attention to a realizable $\Gamma = G_1 \ast_{G_1} G_2$ and prove that $G_1$ cannot be cyclic. We identify $G_i$ for $i = 1, 2, 3$ with its image $H_i \subset \text{PGL}_2(K)$. Suppose that $G_1$ is cyclic. Let $[M_1] \neq [M_2]$ denote $G_1$ and $G_2$ stable classes of lattices. We make the following observations:

1. Let $g \in \text{PGL}_2(K)$ have order $l$, $1 < l < \infty$, then $l = p$ or $p \mid l$.
2. Let $g \in \text{PGL}_2(K)$ have order $l \geq 2$, not divisible by $p$. Let $e_1$, $e_2$ be a basis of $K^2$ consisting of eigenvectors. The collection of all $g$-stable lattice classes is $[[K^0 e_1 + K^0 e_2] | \lambda \in K^*]$. This is an “infinite line” in the tree of all classes of lattices.

The first observation prevents $G_3$ of having order $p$. The second observation implies that $G_1$ and $G_2$ have the same infinite line of stable classes of lattices. This line passes through $[M_1]$ and $[M_2]$, which contradicts Herrlich’s criterion. □

**Observation 3.8** ($\rho$-Sylow subgroups). Let $G$ be a finite group which has an embedding in $\text{PGL}_2(K)$ and let $C_m \subset G$ be a maximal cyclic subgroup with $m > 1$ and $m$ not divisible by $p$. We fix an embedding $\phi : C_m \rightarrow \text{PGL}_2(K)$, say with image $\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} | a^m = 1 \}$. The group $\phi(C_m)$ normalizes two Borel groups $B_+ = \{ (t)_* | t \}_{\text{sym}}$, $B_- = \{ (t)_* | t \}_{\text{anti}}$ and their unipotent subgroups $U_+ = \{ (1)^*_+ | t \}_{\text{sym}}$, $U_- = \{ (1)^*_+ | t \}_{\text{anti}}$. Let $\psi : G \rightarrow \text{PGL}_2(K)$ be an embedding that extends $\phi$. Define $U_{\pm}(G) = \{ g \in G | \psi(g) \in U_{\pm} \}$. A group $U_{\pm}(G)$ is either trivial (i.e., $= \{ 1 \}$) or a $p$-Sylow subgroup of $G$ normalized by $C_m$. For $m > 2$, the groups $U_{\pm}(G)$ do not depend on $\psi$. For $m = 2$ one can change $\psi$ into $\tilde{\psi}$ given by $\tilde{\psi}(g) = \psi(g^{-1})^\ast$, where $\ast$ means the transposed w.r.t. a basis of eigenvectors for $\phi(C_m)$. Thus for $m = 2$ one cannot distinguish between the two groups $U_{\pm}(G)$.

We give now a list of all possible pairs $C_m \subset G$ and $U_{\pm}(G)$:

1. $B(n,m)$ with $m > 1$. Precisely one non-trivial $U_{\pm}(G)$.
2. $\text{PGL}_2(F_q^d)$ with $q > 2$, $m = q - 1$. Both $U_{\pm}(G)$ are non-trivial.
3. $\text{PGL}_2(F_q^d)$ and $m = q + 1$. Both $U_{\pm}(G)$ are trivial.
4. $\text{PSL}_2(F_q^d)$ with $p \neq 2$ and $m = (q - 1)/2 > 1$. Both $U_{\pm}(G)$ are non-trivial.
5. $\text{PSL}_2(F_q^d)$, $p \neq 2$ and $m = (q + 1)/2$. Both $U_{\pm}(G)$ are trivial.
6. $A_5$, $p = 3$ and $m = 5$. Both $U_{\pm}(G)$ are trivial.
7. $A_5$, $p = 3$ and $m = 2$. Both $U_{\pm}(G)$ are non-trivial.
8. $D_{11}$, $p = 2$, $m = \ell$ odd. Both $U_{\pm}(G)$ are trivial.
9. $G \in \{ D_n, A_4, S_4, A_5 \}$, $p \mid |G|, m$ maximal cyclic subgroup. Both $U_{\pm}(G)$ are trivial.

This observation and the list will be used in the formulation of the next propositions.

**Proposition 3.9.** Suppose $p_K = p > 0$. Consider an amalgam $\Gamma = G_1 \ast_{G_1} G_2$ with $G_1, G_2$ isomorphic to subgroups of $\text{PGL}_2(K)$, which are not of Borel type. The only cases where $\Gamma$ can be realized are
(1) $G_3$ is a maximal cyclic subgroup of both $G_1$ and $G_2$ of order $m > 1$ prime to $p$. Moreover all the groups $U_{\pm}(G_1)$, $U_{\pm}(G_2)$ are trivial.

(2) In addition for $p = 3$, the group $\text{PSL}_2(\mathbb{F}_1) \ast_{G_1} \text{PSL}_2(\mathbb{F}_1)$.

(3) In addition for $p = 2$, the groups $D_\ell \ast_{C_2} D_m$ with odd $\ell$, $m$.

**Proof.** Suppose that $G = G_1 \ast_{G_1} G_2$ is realizable as a discontinuous group. For convenience we identify $G_i$, $i = 1, 2, 3$, with their images in $\text{PGL}_2(K)$. Let $[M_i]$, $i = 1, 2$, be the lattice classes invariant under $G_i$. Then $[M_1] \neq [M_2]$ are both stable under $G_3$ and thus $G_3$ lies in a Borel group $B \subset \text{PGL}_2(K)$.

(1) Suppose that $p$ does not divide the order of $G_3$. Then $G_3$ is cyclic of order $m > 1$ with $p \nmid m$. By Theorem 3.3, $G_3$ is a maximal cyclic subgroup of both $G_1$ and $G_2$ (and thus of $\Gamma$). Suppose that, say, both $U_+(G_1)$, $U_+(G_2)$ are non-trivial. These groups lie in a common Borel subgroup of $\text{PGL}_2(K)$ and generate a finite subgroup of $\Gamma$. This subgroup must be conjugated to a subgroup of either $G_1$ or $G_2$. Since this is clearly not the case, this possibility is excluded.

Suppose that both $U_{\pm}(G_1)$ are non-trivial. The two invariant lattices have the form $M_1 = K^0 e_1 + K^0 e_2$ and $M_2 = K^0 e_1 + K^0 \pi e_2$, where $e_1, e_2$ is a basis of $K^2$ consisting of eigenvectors for the group $C_m$. After possibly interchanging $e_1$ and $e_2$ one finds that $U_+(G_1)$ stabilizes $M_2$. Therefore the group $G \subset \Gamma$ generated by $U_+(G_1)$ and $G_2$ stabilizes $M_2$. By assumption $\Gamma$ is discontinuous and so $G$ is a finite group. However no conjugate of $G$ is contained in either $G_1$ or $G_2$. This contradicts that $\Gamma = G_1 \ast_{G_1} G_2$.

In view of the list in Observation 3.8 we conclude that the groups $U_{\pm}(G_1)$, $U_{\pm}(G_2)$ are trivial.

In order to show that the conditions are sufficient, we apply Herrlich's criterion. One fixes an embedding of $G_3 = C_m$ in $\text{PGL}_2(K)$. Let $e_1$, $e_2$ denote two independent eigenvectors of $C_m$. One chooses the lattices $M_1 = K^0 e_1 + K^0 e_2$ and $M_2 = K^0 \pi e_1 + K^0 e_2$ and embeddings of $G_1, G_2$ such that for $i = 1, 2$ the lattice $M_i$ is invariant under $G_i$ and with $0 < |\pi| < 1$. The conditions of Theorem 3.3 are satisfied as is easily seen from Proposition 2.1 and direct computation.

(2) Suppose that $p$ divides the order of $G_3$. Suppose that the groups $G_i$, $i = 1, 2$, are isomorphic to $\text{PGL}_2(\mathbb{F}_{q_1})$ where $H$ denotes $G$ or $S$ and $q_1, q_2$ are powers of $p$. Choose a basis $e_1, e_2$ of $K^2$ such that $G_1$ is the subgroup $\text{PGL}(\mathbb{F}_{q_1} e_1 + \mathbb{F}_{q_1} e_2)$ of $\text{PGL}_2(K)$ and such that $G_3$ contains the matrix $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ w.r.t. this basis. The group $G_2$ can then be identified with $\text{PGL}(\mathbb{F}_{q_2} e_1 + \mathbb{F}_{q_2} e_2)$ for some $e_3 = a e_1 + b e_2$. Using that $G_3 \subset G_2$ one finds that $b \in \mathbb{F}_{q_2}$.

We may then suppose that $b = 1$. Moreover $a \neq 0$ and $|a| > 1$. Indeed, otherwise the lattice $K^0 e_1 + K^0 e_2$ is stabilized by both $G_1$ and $G_2$ and thus the group generated by $G_1$ and $G_2$ cannot be an infinite discontinuous group. The intersection of $G_1$ and $G_2$ is easily seen to be $\{(1,0) \mid y \in \mathbb{F}_{q_1} \cap \mathbb{F}_{q_2}\}$. Let $B$ be the Borel subgroup of $\text{PGL}_2(K)$ which contains the element $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \in G_3$. Then $B \cap G_1$ and $B \cap G_2$ lie in $B$ and generate a finite subgroup $T$ of $\Gamma$. A conjugate of $T$ should lie in either $G_1$ or $G_2$. Suppose that a conjugate of $T$ lies in $G_2$. Then $T \subset G_2$. Thus $B \cap G_1 \subset B \cap G_2 = T$ and $G_3 \supset B \cap G_1$. Since the elements of $G_3$ have order 1 or $p$, the same must hold for $B \cap G_1$. This is only possible for $p = 2$ and $G_1 = \text{PSL}_2(\mathbb{F}_2) \cong D_3$ or $p = 3$ and $G_1 = \text{PSL}_2(\mathbb{F}_3) \cong A_4$. 


The element \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) stabilizes both lattices \( M_1 = K^0e_1 + K^0e_2 \) and \( M_2 = K^0e_1 + K^0(ae_1 + e_2) \). Then also \( B \cap G_2 \) stabilizes \( M_1 \). The subgroup \( G \) of \( \Gamma \) generated by \( G_1 \) and \( B \cap G_2 \) is finite since \( \Gamma \) is discontinuous and \( G \) stabilizes \( M_1 \). Therefore \( G \) is finite and a conjugate of \( G \) must lie in either \( G_1 \) or \( G_2 \). This can only be the case when \( G = G_1 \) and we conclude that there are only two possibilities left, namely \( \text{PSL}_2(\mathbb{F}_2) \) and a conjugate of \( D_\ell \) using Borel subgroups of \( \text{PGL}_2 \) group \( \text{PSL}_2 \) \( \text{PSL}_2(\mathbb{F}_2) \).

Take an element \( \pi \in K \) with \( 0 < |\pi| < 1 \) and consider the groups

\[
G_1 = \text{PSL}(\mathbb{F}_2e_1 + \mathbb{F}_2e_2), \quad G_2 = \text{PSL}(\mathbb{F}_2\pi e_1 + \mathbb{F}_2(e_1 + \pi e_2))
\]

and the lattices

\[
M_1 = K^0e_1 + K^0e_2 = K^0(e_1 + \pi e_2) \quad \text{and} \quad M_2 = K^0\pi e_1 + K^0(e_1 + \pi e_2) = K^0(e_1 + \pi e_2) + K^0\pi^2 e_2.
\]

For \( i = 1, 2 \) the lattice \( M_i \) is stable under \( G_i \). Any lattice class \([M_3] \neq [M_1], [M_2]\) lying in the segment \([M_1], [M_2]\) has the form \( M_3 = K^0(e_1 + \pi e_2) + K^0\lambda e_2 \) where \( |\pi^2| < |\lambda| < 1 \). One easily verifies that the only non-trivial element in \( G_1 \cup G_2 \) which stabilizes \([M_1]\) is represented by the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) with respect to the basis \( e_1, e_2 \). Therefore the conditions of Theorem 3.3 are satisfied and we conclude that \( \text{PSL}_2(\mathbb{F}_2) \ast_{C_2} \text{PSL}_2(\mathbb{F}_2) \) can be realized as a discontinuous subgroup of \( \text{PGL}_2(K) \). The same method works for the group \( \text{PSL}_2(\mathbb{F}_3) \ast_{C_3} \text{PSL}(\mathbb{F}_3) \).

(3) For \( p = 3 \) a new possibility for \( G_1 \) or \( G_2 \) occurs, namely the group \( A_5 \). Arguments using Borel subgroups of \( \text{PGL}_2(K) \), as in (2) above, exclude this possibility.

(4) For \( p = 2 \) the new possibility for \( G_1 \) or \( G_2 \) is \( D_\ell \) with odd \( \ell \). Arguments using Borel subgroups, as in (2), exclude all possibilities, except for \( D_\ell \ast_{C_2} D_m \) with odd \( m \). For these groups one can verify the conditions of Theorem 3.3. This exhaust all combinations. \( \square \)

**Proposition 3.10.** Suppose that \( p_K = p > 0 \), that \( G_1 \) has type \( B(n, m) \) with \( n > 0 \) and \( m \geq 1 \). The list of the amalgams \( \Gamma = G_1 \ast_{G_3} G_2 \) which can be realized as discontinuous subgroups of \( \text{PGL}_2(K) \) is the following:

1. \( m > 1 \), \( G_3 = C_m \). Fix an embedding of \( C_m \) in \( \text{PGL}_2(K) \) and let \( U_+(G_1) \) be the normal \( p \)-Sylow group of \( G_1 \). Then \( G_3 \) is a maximal cyclic subgroup of \( G_2 \) and \( U_+(G_2) \) is trivial. For \( m = 2 \), this condition can also be formulated as: one of the groups \( U_\pm(G_2) \) is trivial.

2. \( m \geq 1 \) and \( G_2 = \text{PHL}_2(\mathbb{F}_q) \) with \( H \in \{ G, S \} \). Let \( B \) denote the Borel subgroup of \( \text{PHL}_2(K) \). Then \( G_3 = B(\mathbb{F}_q) \) and \( m = q - 1 \) if \( H = G \) and \( m = (q - 1)/2 \) if \( H = S \) and \( p \neq 2 \).

3. In addition for \( p = 3 \), the groups \( B(n, 2) \ast_{D_1} A_5 \).

4. In addition for \( p = 2 \), the groups \( B(n, 1) \ast_{C_2} D_\ell \) with odd \( \ell \).

**Proof.** (i) The proof of part (1) is similar to the one of part (2) of Theorem 3.5. Now we have to consider the case that \( p \) divides the order of \( G_3 \).
(ii) Suppose first that $G_2$ is also of type $B(\tilde{n}, \tilde{m})$. Then $G_1$ and $G_2$ lie in the same Borel subgroup $B$ of $\text{PGL}_2(K)$ and generate therefore a finite group, which is not isomorphic to $G_1 \ast G_3 G_2$.

We conclude that $G_2$ is not of type $B(\tilde{n}, \tilde{m})$. Since $p$ divides the order of $G_2$, the remark following Proposition 2.1 gives the possibilities for $G_2$.

(iii) Suppose that $G_2$ has the form $\text{PGL}(F_q e_1 + F_q e_2)$, where $e_1$, $e_2$ is a basis of $K^2$ over $K$. This basis is chosen such that some element of $G_3$ has the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The unique Borel subgroup $B \subset \text{PGL}_2(K)$ which contains $G_3$ consists of all matrices with $e_1$ as eigenvector. Also $G_1 \subset B$ and the subgroup generated by $G_1$ and $B(F_q)$ is finite. Since $G_1$ is a maximal finite subgroup of $G_1 \ast G_3 G_2$ one has that $G_1 \supset B(F_q)$ and therefore $G_3 = B(F_q)$. In particular, $m$ is a multiple of $q - 1$.

Write $m = d(q - 1)$ with $d \geq 1$. Let $[M_1] \neq [M_2]$ denote the lattice classes stabilized by $G_1$ and $G_2$. Take an element $a \in G_1$ with order $d(q - 1)$. Then $a^d$ stabilizes both $[M_1]$ and $[M_2]$. Since the order of $a$ is not divisible by $p$, also $a$ stabilizes $[M_2]$ and the group generated by $a$ and $G_2$ stabilizes $[M_2]$. Since we have supposed that $G_1 \ast G_3 G_2$ is realizable, this implies that $a \in G_2$. We conclude that $m = q - 1$. The same reasoning holds for $G_2 = \text{PSL}_2(F_q)$. We conclude that the amalgams in part (2) of the present proposition are the only candidates.

In order to show that the candidates pass the test of Theorem 3.3, we consider an example. The general case can be treated in the same way. For this example we take $G_2 = \text{PGL}(F_q e_1 + F_q e_2)$ and we take for $G_1$ the group, given by matrices \((\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a \in F_q^*, b \in F_q + \pi^{-1}F_q)\) with respect to the basis $e_1$, $e_2$. Here $0 < |\pi| < 1$. Consider the two lattices $M_1 = K^0 e_1 + K^0 \pi e_2$ and $M_2 = K^0 e_1 + K^0 e_2$, invariant under respectively $G_1$ and $G_2$. The verification of the conditions of Theorem 3.3 is immediate.

(iv) For $p = 3$, the new possibility is $B(n, m) \ast C_3 A_5$, where $3$ divides the order of $G_3$. Suppose that this group is realizable. As in part (iii) of the proof one finds that $G_3 = D_3$ (i.e., the intersection of a Borel group with $A_5$) and $m = 2$. The verification that $B(n, 2) \ast C_3 A_5$ satisfies the conditions of Theorem 3.3 is similar to the verification in (iii).

(v) For $p = 2$ we have to consider the possibility $B(n, m) \ast C_3 D_\ell$ with odd $\ell$. Since $G_3 \subset D_\ell$, one must have $G_3 = C_2$. Let $[M_1] \neq [M_2]$ denote the lattice classes, stabilized by respectively $G_1$ and $G_2$. Let $A = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ belong to $G_3 = C_2$ and let $\zeta_n$ denote a primitive $m$th-root of unity. Then some $B = (\begin{pmatrix} \zeta_n & 1 \\ 0 & 1 \end{pmatrix})$ belongs to $G_1$ and $C = BAB^{-1} = (\begin{pmatrix} \zeta_n & 1 \\ 0 & 1 \end{pmatrix}) \in G_1$. Clearly $C$ also stabilizes $[M_2]$ and the group generated by $C$ and $G_2$ is finite since we have supposed that $G_1 \ast G_3 G_2$ is realizable. Since $G_2$ is a maximal finite subgroup of $G_1 \ast G_3 G_2$ one has $m = 1$. We conclude that the only candidate is $B(n, 1) \ast C_2 D_1$. As in (iii), one verifies the conditions of 3.3. □

Corollary 3.11. The list of discontinuous groups $G_1 \ast G_3 G_2$ for $p_\mathbb{K} = p > 0$.

1. $G_3 = C_m$, $m > 1$, $p \nmid m$, $C_m$ maximal cyclic subgroup of both $G_1$, $G_2$ and satisfying the condition on the groups $U_\pm(G_1)$, $U_\pm(G_2)$ of 3.9 and 3.10. Observation 3.8 provides all possibilities.

2. $B(n, q - 1) \ast B(F_q) \text{PGL}_2(F_q)$.

3. $B(n, (q - 1)/2) \ast B(F_q) \text{PSL}_2(F_q)$ for $p \neq 2$. 


For $p = 3$ additionally: $\text{PSL}_2(\mathbb{F}_3) \ast \text{PSL}_2(\mathbb{F}_3)$ and $B(n, 2) \ast D_3, A_5$.

(5) For $p = 2$ additionally: $D_4 \ast C_2, D_m, D_4 \ast C_2, B(n, 1)$ with odd $\ell, m$. (For $q = 2$, $B(n, 1) \ast C_2, D_3$ coincides with $B(n, q - 1) \ast B(\mathbb{F}_q)$.)

The next two theorems extend the above results to more complicated finite indecomposable trees of groups.

**Theorem 3.12.** Contracted finite, indecomposable trees for $p_K = 0$. Suppose $p_K = 0$. Let the finite tree of groups $(T, G)$ satisfy the conditions (a)–(f) below. Then $(T, G)$ can be realized in $BT$.

(a) $G_v \in \{D_8, A_4, S_4, A_5\}$ for every vertex $v$ of $T$.
(b) If $p_k \neq 2$ and $p_k \mid |G_v|$, then $G_v$ is a dihedral group.
(c) If $p_k = 2$, then $G_v \in \{A_4, D_2^s\}$ with $s \geq 1$.
(d) $G_v$ is a maximal cyclic subgroup of $G_v$, if $e$ is an edge of $v$.
(e) If a maximal cyclic subgroup $H$ of $G_v$ is equal to some $G_e$, then there is at most one edge $e' \neq e$ with $G_{e'} = H$. Moreover, no conjugate $H' \neq H$ of $H$ is equal to $G_{e''}$ for some edge $e''$ of the vertex $v$.
(f) If there are two edges $e \neq e'$ with vertex $v$ and $G_e = G_{e'} = H$, then the normalizer of $H$ in $G_v$ is $H$ itself.

**Proof.** We will prove by induction on the number of edges of $(T, G)$ that a realization $\tau$ exists. By Theorem 3.5 we may suppose that there are at least two edges. Let $e = \{v_1, v_2\}$ be an edge. Define the trees of groups $(T^i, G^i)$, $i = 1, 2$, obtained by deleting $e$ from $(T, G)$ and such that $v_i$ is a vertex of $T^i$. First we embed $H := G_v$ in $\text{PGL}_2(K)$. Then one chooses two lattice classes $[M_1], [M_2]$ on the axis $L \subset BT$ of $G_v$ having a large enough distance. By induction the $(T^i, G^i)$, $i = 1, 2$, are realized in $BT$ such that $v_1, v_2$ are mapped to $[M_1], [M_2]$. Now we verify condition (2b) of 3.3.

Let $I^i \subset \text{PGL}_2(K)$ be the realization of the amalgam of $(T^i, G^i)$. The group $H^i := \{g \in I^i \mid gL = L\}$ has a subgroup of index 1 or 2 consisting of the elements of $I^i$ commuting with $H$. By assumption $I^i$ is the amalgam of $(T^i, G^i)$. The elements of the amalgam of $(T^i, G^i)$ can uniquely be represented by (suitably chosen) reduced words, as in the case of an amalgam of the form $G_1 \ast G_3, G_2$. Using this and the properties (d), (e) and (f), one shows that $\{\gamma \in I^i \mid \gamma \text{ commutes with } H\}$ is a finite group. Hence $H^i$ is finite, too.

One considers the subtree $T^i$ of $BT$ generated by the $I^i$-orbits of all the embedded vertices of $(T^i, G^i)$). We claim that $S' := T^i \cap L$ is a finite set. Suppose that $S'$ is infinite. The tree $T^i$ is generated by the set of vertices $\{\gamma v \mid \gamma \in I^i, v \text{ a vertex of } T^i\}$. It follows that there are also infinitely many elements of this set lying on $L$. Hence there exists a vertex $v$ of $T^i$ and there are infinitely many $\gamma \in I^i$ such that $H \subset G_{\gamma v}$. Thus there are infinitely many $\gamma \in I^i$ with $\gamma H \gamma^{-1} \subset G_v$. The latter yields the contradiction that $\{\gamma \in I^i \mid \gamma \text{ commutes with } H\}$ is infinite. (In particular, the two fixed points of $H = G_v$ are not limit points for $I^i$.)

The distance between $[M_1], [M_2]$ on $L$ is taken large enough and thus there exists a lattice class $V$ on $[[M_1], [M_2]]$, such that $S^1 \subset ([M_1], V), S^2 \subset (V, [M_2])$ and $gIV = V$.
with \(g_i \in \Gamma'\) implies \(g_i \in H\). For any subtree \(S\) of \(BT\) there is an obvious projection \(\text{pr}_S : BT \to S\). We write \(\text{pr}_1\) for the projection on the subtree \(T'\). Clearly, \(\text{pr}_1(V)\) lies in \(S' \cap \{\mathcal{M}_1, V\}\).

For \(g_1 \in \Gamma' \setminus H\) one has \(\text{pr}_1(g_1 V) = g_1 \text{pr}_1(V)\) and the distance of \(V\) to \(T'\) is the same as the distance of \(g_1 V\) to \(T'\). If \(V\) lies on \([g_1 V, g_1 \text{pr}_1(V)]\), then \(V = g_1 V\), which is excluded by the choice of \(g_i\). The path \([g_1 V, g_1 \text{pr}_1(V)]\) followed by the path in \(T'\) from \(g_1 \text{pr}_1(V)\) to \(S'\) does not contain \(V\). Thus \(\text{pr}_2(g_1 V)\) lies on the left hand side of \(V\). Similarly, for any \(g_2 \in \Gamma' \setminus H\), the \(\text{pr}_2(g_2 V)\) lies in on the right hand side of \(V\). We conclude that \(g_1 V \neq V \neq g_2 V\) and that \(V \in [g_1 V, g_2 V]\). This is condition (2b) of Theorem 3.3. □

Remarks 3.13.

1. It is an exercise to show that condition (f) is equivalent to: Let \(x_1, x_2 \in \mathbb{P}^1(K)\) denote the two fixed points of the maximal cyclic subgroup \(H\) of \(G_v\). Then the images of \(x_1, x_2\) under the canonical map \(\mathbb{P}^1(K) \to \mathbb{P}^1(K) / G_v \cong \mathbb{P}^1(K)\) are distinct.

2. Consider for \(n > 2\), the amalgam \(D_n \ast_{\mathbb{C}} D_n \ast_{\mathbb{C}} D_n\). An embedding of this group in \(\text{PGL}_2(K)\) is easily seen to be conjugated to a group with generators \(\tau, \sigma_i, i = 1, 2, 3\), where \(\tau z = \zeta z\) and \(\zeta\) is a primitive \(n\)th root of unity, \(\sigma_i z = a_i z^{-1}\) with "independent" \(a_1, a_2, a_3 \in K^*\). This group is not discontinuous because it contains the elements \(z \mapsto \frac{a_1}{a_2} z\) and \(z \mapsto \frac{a_2}{a_3} z\). This example explains condition (f) of the theorem. A similar example shows that condition (e) is needed in the theorem.

3. Suppose \(p_k > 3\). Let \((T, G)\) denote the tree of groups with vertices \(v_1, v_2, v_3\), edges \(e_1 = [v_1, v_2], e_2 = [v_2, v_3]\) and groups \(G_{v_1} = G_{v_2} = D_3, G_{v_3} = A_4\) and \(G_{e_1} = G_{e_2} = C_3\) with the obvious embeddings in \(G_{v_i}\) and \(G_{e_i}\) and any embedding in \(G_v\). Then \((T, G)\) is not realizable. Let \((T, G)'\) denote the same tree of groups but with \(G_{v_2}, G_{v_3}\) interchanged. Then \((T, G)'\) is realizable. We note that the two trees of groups have the same amalgam!

Theorem 3.14. Contracted finite, indecomposable trees for \(p_k = p > 0\). Let \(p_k = p > 0\). Suppose that the finite tree of groups \((T, G)\) satisfies the conditions below, then \((T, G)\) is realizable in \(BT\).

If \(p \geq 5\), then

(i) Any vertex group \(G_v\) is isomorphic to a finite non-cyclic subgroup of \(\text{PGL}_2(K)\). In the sequel we will view the \(G_v\)'s and \(G_e\)'s as subgroups of \(\text{PGL}_2(K)\) and write \(\phi_v : \mathbb{P}^1_K \to \mathbb{P}^1_K / G_v\) for the canonical morphism.

(ii) For any edge \(e = [v_1, v_2]\) one has \(1 \neq G_e \neq G_{v_1}, G_{v_2}\) and \(G_e\) is of Borel type. If \(p \mid \#G_e\), then the group \(G_{v_i}\) is of Borel type for precisely one \(i \in \{1, 2\}\).

(iii) Suppose that the vertex group \(G_v\) is not of Borel type, then

(a) For any edge \(e\) of \(v\) the group \(G_e\) is a ramification group of \(\phi_v\).

(b) Suppose that a ramification group \(H \subset G_v\) of \(\phi_v\) is equal to \(G_e\). Then there is at most one edge \(e' \neq e\) with \(G_{e'} = H\). Moreover, no conjugate \(H' \neq H\) of \(H\) is equal to some \(G_{e'}\).
(c) If the edges \( e \neq e' \), with vertex \( v \), satisfy \( H := G_v = G_{v'} \), then \( H \) has two fixed points \( x_1 \neq x_2 \in P^1(K) \) and \( \phi_v(x_1) \neq \phi_v(x_2) \).

(iv) If the vertex group \( G_v \) is of Borel type and is not a \( p \)-group, then \( v \) is an extremal vertex.

(v) If the vertex group \( G_v \) is a \( p \)-group, then \( G_v \cong C_p \) for every edge \( e \) containing \( v \).

If \( p = 3 \), then we admit also vertices \( v \) with \( G_v = C_3 \). Let \( \{v, v_i\}_{i=1, \ldots, m} \) denote the edges of \( v \). We require that \( m \geq 2 \), that \( G_{v_i} = \text{PSL}_2(\mathbb{F}_3) \) and \( G_{\{v, v_i\}} = C_3 \) for all \( i \). Moreover we exclude edges \( e = \{v_1, v_2\} \) with groups \( G_{v_1}, G_{v_2} \cong \text{PSL}_2(\mathbb{F}_3) \) and \( G_e \cong C_3 \).

If \( p = 2 \), then we admit also vertices \( v \) with \( G_v = C_2 \). Let \( \{v, v_i\}_{i=1, \ldots, m} \) denote the edges of \( v \). We require that \( m \geq 2 \), that \( G_{v_i} = D_\ell_i \) with odd \( \ell_i \) and that \( G_{\{v, v_i\}} = C_2 \) for all \( i \). Moreover we exclude edges \( e = \{v_1, v_2\} \) with groups \( G_{v_1} \cong D_\ell_i, G_{v_2} \cong D_\ell_i, G_e \cong C_2 \) and odd \( \ell, \ell' \).

Remarks 3.15 (The special features for \( p_K = 2, 3 \)).

(1) For \( p_K = 2 \) and every \( m \geq 2 \) the tree of groups \((T, G)\) with vertices \( v, v_1, \ldots, v_m \), edges \( e_i = \{v_i, v\}, i = 1, \ldots, m \), and groups \( G_v = G_{v_i} = C_2, i = 1, \ldots, m \), and \( G_{v_i} = D_\ell_i, i = 1, \ldots, m \), with odd \( \ell_i \)'s can be realized in \( BT \). One can prove this as follows: Let \( \ell \) be the l.c.m. of \( \ell_1, \ldots, \ell_m \). Take \( n > 1 \) sufficiently large. The tree of groups with vertices \( v_1, v_2 \), edge \( e = \{v_1, v_2\} \) and groups \( G_{v_1} = D_\ell, G_e = C_2, G_{v_2} = C_n^2 \) has a realization \( \tau \) according to 3.10 part (4). In \( BT \) one considers the locally finite subtree \( T \) generated by the images of \( \tau(v_1), \tau(v_2) \) under the action of \( \Gamma = D_\ell \rtimes C_2 \). The vertex \( \tau(v_2) \) has \( \#(C_n^2/C_2) \) edges. The stabilizer of each edge is the same group \( C_2 \). The stabilizer of each end point \( \neq \tau(v_2) \) of an edge, is isomorphic to \( D_\ell \). We select now \( m \) of those edges \( e_1, \ldots, e_m \) and consider for each \( i \) a subgroup \( G_i = D_\ell_i \) of \( D_\ell \) which contains the fixed subgroup \( C_2 \) of \( C_n^2 \). This is an embedding of \((T, G)\) in \( BT \). It is a realization, since the homomorphism of the amalgam of \((T, G)\) to \( D_\ell \rtimes C_2 \) is injective and the latter group is discontinuous.

The same method can be used to realize the tree of groups \((T, G)\) with vertices \( v, v_1, \ldots, v_m \), edges \( e_i = \{v_i, v\}, i = 1, \ldots, m \), and groups \( G_v = C_2 \) (any \( n \geq 1 \)), \( G_{v_i} = C_2, i = 1, \ldots, m \), and \( G_{v_i} = \text{PSL}_2(\mathbb{F}_3), i = 1, \ldots, m \), can be realized in \( BT \).

Similar to (1) above, one proves this by means of the realizable amalgam \( \text{PSL}_2(\mathbb{F}_3) \rtimes C_3 \) \( C_3^k \) with say \( k > 1 \) sufficiently large.

(2) For \( p_K = 3 \) and every \( m \geq 2 \) the tree of groups with vertices \( v, v_1, \ldots, v_m \), edges \( e_i = \{v_i, v\}, i = 1, \ldots, m \), and groups \( G_v = C_3^n \) (any \( n \geq 1 \)), \( G_{v_i} = C_3, i = 1, \ldots, m \), and \( G_{v_i} = \text{PSL}_2(\mathbb{F}_3), i = 1, \ldots, m \), can be realized in \( BT \).

The general idea for the formulation of Theorem 3.14 is that the tree of groups \((T, G)\) can be realized and that no contraction of an edge in \((T, G)\) is possible. We make a small exception for these rule, namely for technical reasons we allow that \( m = 2 \) in part (v) of Theorem 3.14. For the special cases \( p_K = p = 2, 3 \) one would like to contract the vertex with group \( C_p \) and all its edges. However, for \( m > 2 \) this introduces cycles and the new object is no longer a tree of groups.

According to 3.9 and 3.10, only for \( p_K = 2, 3 \) this special feature can occur.
Proof. The existence of a realization of \((T, G)\) is proved by induction on the number of edges of \(T\). By 3.11, we may suppose that \(T\) has at least two edges. For an edge \(e\) with \(p \nmid \#G_e\), one can apply the method of the proof of 3.12.

Suppose that \(p\) divides \(\#G_e\) for every edge \(e\), that \(G_e \neq G_v\) if \(e\) is an edge of \(v\) and that for no vertex \(v\) the group \(G_v\) is a \(p\)-group. Consider an edge \(e = \{v_1, v_2\}\). By (ii), \(G_{v_1} \cong B(n, m)\) with \(m > 1\) and \(G_{v_2}\) is not of Borel type. By (iv), \(v_1\) is an extremal edge.

Let \((T^1, G^1)\) be the tree of groups obtained by deleting \(v_1\) and \(e\). This tree is given a realization. It is not difficult to see that \(G_{v_1}\) can be embedded in \(\text{PGL}_2(K)\), such that condition (2b) of Theorem 3.3 is satisfied. In particular, for \(p_K \geq 5\) the theorem is proved.

For \(p_K = 2\) or 3, one first makes realizations of the subtrees of \((T, G)\) which have the form described in Remarks 3.15, part (1) or (2). By induction and with 3.3, one can complete this to a realization of all of \((T, G)\).

Definition 3.16. A contracted finite, indecomposable tree of groups is a tree of groups satisfying the conditions of Theorems 3.12 or 3.14 (depending on \(p_K\) and \(p_k\)).

4. The trees \(T, T^c, T^\dagger\) associated to \(\Omega\) and \(\Gamma\)

As before, we consider an infinite, finitely generated discontinuous group \(\Gamma \subset \text{PGL}_2(K)\) such that \(\Omega/\Gamma\) is isomorphic to \(\mathbb{P}^1_K\). The aim of this section to find the structure of \(\Gamma\). As we have seen, it suffices to consider an indecomposable group \(\Gamma\) such that its set of limit points has more than two elements. Let \(T\) denote the tree associated to \(\Gamma\), defined in 2.3. The quotient \(T/\Gamma\) is a finite tree and we fix an embedding of \(T/\Gamma\) into \(T\). This makes \(T/\Gamma\) into a tree of groups. Since \(\Gamma\) is indecomposable every vertex and edge of \(T/\Gamma\) has a non-trivial stabilizer. In general, this tree of groups does not satisfy the properties stated in Theorems 3.12 or 3.14. Another tree \(T^c\), on which \(\Gamma\) acts, is constructed directly from the group \(\Gamma\). Eventually, it will be shown that the tree of groups \(T^c/\Gamma\) has the properties of Theorems 3.12 or 3.14. In other words the structure of the above groups \(\Gamma\) has been established. In order to link \(T^c\) with a pure affinoid covering of \(\Omega\), we will have to consider still another tree \(T^\dagger\). There are exceptional groups \(\Gamma\) for which this construction does not work. The exceptional groups occur for \(p_K = 0\) and \(p_k = 2, 3, 5\). For these groups it seems rather difficult to find a structure theorem and a general formula for the number of branch points \(\text{br}(\Gamma)\) of \(\Gamma\). In the paper [6] the exceptional groups with \(\text{br}(\Gamma) = 3\) are studied. We will introduce the notion of ordinary group \(\Gamma\) (which excludes the exceptional groups), carry out the constructions of \(T^c\) and \(T^\dagger\) and prove that the tree of groups \(T^c/\Gamma\) satisfies the properties of Theorems 3.12 or 3.14. In the sequel of this section, \(\Gamma\) will denote (unless otherwise stated) a finitely generated, infinite, indecomposable, discontinuous subgroup of \(\text{PGL}_2(K)\). Moreover we will assume that the set of its limit points \(\mathcal{L}\) is infinite. For the omitted case, where \(\Gamma\) has two limit points, \(\text{br}(\Gamma)\) is known, see 2.3.

Definition 4.1. The group \(\Gamma\) will be called ordinary if every maximal finite subgroup, which is not of Borel type, has a separating lattice class. A few consequences of the property “ordinary” are
(i) No maximal finite subgroup of $\Gamma$ is cyclic.

For $p_K = 0$ this follows from Theorem 3.5, since for $p_k = 3$ the group $A_4$ is excluded and for $p_k = 2$ the group $D_n$ with odd $n$ is excluded. For $p_K > 0$, this is proven in Proposition 3.7.

(ii) Let $H$ be a maximal finite subgroup, which is not of Borel type, and $H'$ is any maximal finite subgroup, not conjugated to $H$ in $\Gamma$, with $H \cap H' \neq 1$. Then there exists a Borel group $B \subset \text{PGL}_2(K)$ such that $H \cap H' = H \cap B$.

For $p_K = 0$ this follows from Theorem 3.5 since for $p_k = 2, 3$ or 5, the maximal finite subgroups for which (ii) does not hold are excluded. For $p_K > 0$ and $H'$ not of Borel type, (ii) follows from 3.9. For $p_K > 0$ and $H'$ of Borel type, statement (ii) follows from 3.10.

(iii) Let $H$ be a maximal finite subgroup of $\Gamma$, which is not of Borel type. Then the ramification groups of the map $\varphi_H : \mathbb{P}_K^1 \to \mathbb{P}_K^1/H$ are the subgroups $B \cap H \neq 1$ with $B \subset \text{PGL}_2(K)$ a Borel group.

For $p_K = 0$ this follows from (i) above. For $p_K > 0$, this follows from (ii) above.

The proposition below shows that ordinary groups are very common indeed.

**Proposition 4.2.** The group $\Gamma$ is ordinary if and only if one of the following statements holds:

(i) $p_K = p > 0$ or $p_K = 0$ and $p_k > 5$.

(ii) $p_K = 0$, $p_k = 3, 5$ and every finite non-cyclic subgroup $H \subset \Gamma$ with $p_k \mid \#H$ is a dihedral group.

(iii) $p_K = 0$, $p_k = 2$ and every maximal finite, non-cyclic subgroup of $\Gamma$ is isomorphic to either $A_4$ or $D_{2s}$ for some $s \geq 1$.

We note that this assumption on $\Gamma$ implies that any two maximal finite, non-cyclic, non-conjugated subgroups $H, H'$ of $\Gamma$ have either intersection $\{1\}$ or their intersection is a maximal cyclic subgroup of both $H$ and $H'$.

**Proof.** As in the proof of 3.6 one shows that it suffices to consider $\Gamma$’s of the form $G_1 \ast_{G_1} G_2$. For these groups the statements follow from the properties of the finite subgroups considered in Section 2 and the classification of the discontinuous groups of the form $G_1 \ast_{G_1} G_2$, given in Section 3. $\square$

**Definition 4.3** (The graph $T^c$). The group $\Gamma$ is supposed to be ordinary. The collection of the maximal finite subgroups of $\Gamma$ is denoted by $\text{max}_1 \Gamma$. We associate to $\Gamma$ a graph $T^c$ on which the group $\Gamma$ acts.

The vertices of the graph $T^c$ are the following finite subgroups $H \subset \Gamma$:
(v1) $H \in \max \Gamma$.
(v2) If $p_K = p = 2$ or $3$, then a $p$-cyclic subgroup $H$ of $\Gamma$ is a vertex if:
(a) If $H \subset H' \in \max \Gamma$, then $H'$ is not of Borel type.
(b) The group $H$ is contained in at least two elements of $\max \Gamma$.

The edges $\{v_1, v_2\}$ of $T^c$ are defined by

(e1) There exist lattice classes $[M_1], [M_2]$ with stabilizers $v_1$ and $v_2$, such that for every $[M] \in [[M_1], [M_2]]$ the stabilizer $\Gamma_{[M]}$ is contained in $v_1$ or $v_2$.
(e2) In case $p_K = p = 2$ we exclude that the triple $(v_1, v_1 \cap v_2, v_2)$ is isomorphic to $(D_\ell, C_2, D_\ell)$.
(e3) In case $p_K = p = 3$ we exclude that the triple $(v_1, v_1 \cap v_2, v_2)$ is isomorphic to $(\text{PSL}_2(F_3), C_3, \text{PSL}_2(F_3))$.

4.3.1. Comments on (e1)

We note that there is a rather subtle point in the formulation of (e1). One would like to state that $[M_1], [M_2]$ are vertices of $T$. This is true except for the cases $p_K = p = 2$ or $3$, $H \cong C_p$ satisfying (v2) and such that $H$ is contained in precisely two maximal finite subgroups $H_1, H_2$ of $\Gamma$. In this situation (e1) prescribes that $[H_1, H]$ and $[H, H_2]$ are edges of $T^c$. Indeed, let $[N_i]$ denote the unique $H_i$-invariant lattice class for $i = 1, 2$. Then $[N_i], i = 1, 2$, are vertices of $T$. It is not difficult to show that there exists a lattice class $[M]$ in $[[N_1], [N_2]]$ with stabilizer $H$. Thus $[H_1, H]$ and $[H, H_2]$ are edges of $T^c$. However $[M]$ need not be a vertex of $T$. In view of this one may replace (e1) by the condition that one of the two lattice classes $[M_1], [M_2]$ belongs to $T$.

In order to see that property (e1) of an edge $e = \{v_1, v_2\}$ is natural, we will show that $H := v_1 \cap v_2$ is non-trivial and that the homomorphism $v_1 *_{H} v_2 : \Gamma \rightarrow \Gamma$ is injective. We may of course suppose that $v_1 \nsubseteq v_2$ and $v_2 \nsubseteq v_1$. Every lattice class $[M] \in [[M_1], [M_2]]$ has a non-trivial stabilizer $\Gamma_{[M]}$, since $\Gamma$ is indecomposable. Moreover this group is contained in either $v_1$ or $v_2$. It suffices to produce $[M] \in [[M_1], [M_2]]$ with $\Gamma_{[M]}$ contained in both $v_i$. Indeed, then $1 \neq \Gamma_{[M]} = v_1 \cap v_2$, and we can apply part (1) of Theorem 3.3.

Suppose that for no $[M] \in [[M_1], [M_2]]$ the group $\Gamma_{[M]}$ is contained in both $v_i$. For any $g \in v_1$, $g \notin v_2$ one defines $[M_1(g)] \in [[M_1], [M_2]]$ by: for $[M] \in [[M_1], [M_2]]$ one has $g \in \Gamma_{[M]}$ if and only if $[M] \in [[M_1], [M_1(g)]]$. Let $[[M_1], [M_1^*]]$ denote the union of all such $[[M_1], [M_1(g)]]$. Then $[M] \in [[M_1], [M_2]]$ has the property $\Gamma_{[M]} \subset v_1$ if and only $[M] \in [[M_1], [M_1^*]]$. There is a $[M_2^*] \in [[M_1], [M_2]]$ with the similar property w.r.t. $v_2$. These two segments cover $[[M_1], [M_2]]$ and have empty intersection. This is a contradiction.

4.3.2. Further comments on the definition

Part (v1) of the definition is natural, too. The additions (v2), (e2) and (e3) have their origin in the special features for $p_K = p = 2, 3$ (see 3.15). In particular, if one omits the extra vertices of (v2), then $T^c$ will in general have cycles and will not be a tree.

We remark moreover that no two maximal finite subgroups $H_1 \neq H_2$ in part (b) of (v2) are conjugated in $\Gamma$. This can be seen as follows: Let $H_1 \neq H_2$ be two maximal finite subgroups of $\Gamma$ containing $H \cong C_p$. Write $[M_i], i = 1, 2$, for the unique invariant
lattice class of the group $H_i$. In general, property (e1) does not hold for the segment $[[M_1], [M_2]]$. However this segment has a subdivision in segments $[[N_i], [N_{i+1}]]$ such that the stabilizer $T_i$ of each $[N_i]$ in $\Gamma$ is a maximal finite subgroup and each $[[N_i], [N_{i+1}]]$ satisfies condition (e1). According to 4.3.1, $T_i \ast T_i \cap T_{i+1} \ast T_{i+1}$ is a realizable amalgam. Each group $T_i$ is a maximal finite subgroup of $\Gamma$ containing $H \cong C_p$. Condition (v2), combined with Proposition 3.9 yields that $T_i \ast T_i \cap T_{i+1} \ast T_{i+1}$ is $D_\ell \ast D_\ell$ if $p = 2$ and is equal to $\text{PSL}_2(F_3) \ast C_p \text{PSL}_2(F_3)$ for $p = 3$. Hence each maximal finite subgroup of $\Gamma$ containing $H$ is a $D_\ell$ for $p = 2$ or a $\text{PSL}_2(F_3)$ for $p = 3$. Let $B \subset \text{PGL}_2(K)$ be the unique Borel group containing $H$. Then there exists a maximal finite subgroup, say $H_3$, of $\Gamma$ containing $\Gamma \cap B$. Since $H_3$ is either a $D_\ell$ or a $\text{PSL}_2(F_3)$, one concludes that $\Gamma \cap B = H$.

Now suppose that there exists $\gamma \in \Gamma$ with $\gamma H_1 \gamma^{-1} = H_2$. According to Proposition 2.1, the morphism $P^1 \to P^1 / H_1$ has two branch points and precisely one of them is wild. Thus $H_1$ contains a wild ramification group $R_i \supset H$ and every wild ramification group in $H_1$ is $H_i$-conjugated with $R_i$. Then $R_i = H_i \cap B$. Now $\gamma_1 \gamma^{-1}$ is a wild ramification group of $H_2$ and is therefore equal to $\delta R_2 \delta^{-1}$ for some $\delta \in H_2$. Then $\tilde{\gamma} := \delta^{-1} \gamma \in \Gamma$ satisfies $\tilde{\gamma} H_1 \tilde{\gamma}^{-1} = H_2$ and $\tilde{\gamma} R_1 \tilde{\gamma}^{-1} = R_2$. Hence $\tilde{\gamma} \in B \cap \Gamma = H$. This yields the contradiction $H_1 = H_2$.

4.3.3. The action of $\Gamma$ on the vertices of $T^c$

This action is defined by $\gamma(v) = \gamma H \gamma^{-1}$ for $v \in H$ a vertex and $\gamma \in \Gamma$. We write $\Gamma_v$ for $\{ \gamma \in \Gamma | \gamma H \gamma^{-1} = H \}$. Suppose that $v = H$ is a maximal finite subgroup of $\Gamma$. If $H$ is not of Borel type then it has a unique separating lattice class $[M]$. The group $\Gamma_v$ stabilizes $[M]$ and is therefore finite and hence $\Gamma_v = H$. If $H$ is of Borel type then it is contained in a unique Borel group $B$ and $\Gamma_v$ is easily seen to be $\Gamma \cap B$. The latter group is an increasing union of its finite subgroups. Since $\Gamma$ is finitely generated we conclude that $\Gamma \cap B$ is in fact a finite group. By the maximality of $H$ we have again that $\Gamma_v = H$.

Suppose now that $v = H$ is not maximal. Then $p_K = p = 2, 3$ and $H \cong C_p$ satisfies (a) and (b) of (v2). As before, $\Gamma_v$ is equal to the fixed group $\Gamma \cap B$ where $B$ is the unique Borel subgroup which contains $H$. Choose two maximal finite subgroups $H_1 \neq H_2$ containing $H$. We may suppose that $H_1 \supset \Gamma_v$. Let $[M_1], [M_2]$ denote the unique invariant lattice classes for $H_1, H_2$. Suppose that $H_1, H_2$ satisfy property (e1), then $H_1 \ast H_2 \cap H \subset \Gamma$. Since $H \subset H_1 \cap H_2$, we conclude by Proposition 3.9 that $H_1 = D_\ell$ for $p = 2$ and $H_1 = \text{PSL}_2(F_3)$ for $p = 3$, because $H \subset H_1 \cap H_2$. By construction $\Gamma_v \subset H_1 \cap B$. The group $H_1 \cap B$ is easily seen to be $C_p$. Thus $\Gamma_v = H$.

In the opposite situation, one considers $[M] \in [[M_1], [M_2]]$, closest to $[M_1]$ such that $\Gamma|_{[M]}$ is not contained in any of the $H_i$. Let $H_3$ be a maximal finite subgroup of $\Gamma$ containing $\Gamma|_{[M]}$ and let $[M_3]$ denote its unique invariant lattice class. Then $H_1, H_3$ satisfy property (e1). As before this implies that $\Gamma_v = H$.

The aim of this section is to show that the graph $T^c$ is actually a locally finite tree and that $T^c / \Gamma$ is a contracted, indecomposable finite tree of groups.

Lemma 4.4. The notions and notations are those of Section 2.1. Let the set of lattice classes $M$ consists of finitely many $\Gamma$-orbits. Then $M$ is discrete and $\Gamma$ acts on the tree $T_M$. Moreover,
(i) The ends of the tree $T_M$ are in bijective correspondence with the points of the limit set $\mathcal{L}$ for the group $\Gamma$.

(ii) For $x \in \Omega$ there exists a unique equivalence class $[M] \in \text{conv}(\mathcal{M})$ such that $\text{red}_M(x) \neq \text{red}_M(e)$ for all edges $e \in T_M \cup [M]$ which have $[M]$ as vertex.

**Proof.** The discreteness of $\mathcal{M}$ and (i) follow easily from Section 2.1. Write $T$ for $T_M$. The reduction map $\text{red}_T : \Omega \to (\Omega, T)$ maps $x$ to some point $\text{red}_T(x)$. If $\text{red}_T(x)$ lies on a single irreducible component of $(\Omega, T)$, then the corresponding lattice class $[M]$, which is a vertex of $T_M$, has the required property. Suppose that $\text{red}_T(x)$ is a double point, lying on the intersection of two irreducible components $L_{\nu_1}, L_{\nu_2}$ corresponding to vertices $v_1, v_2$ of $T_M$. Then one chooses points $y_1, y_2 \in \Omega$ with images on the non singular points of $L_{\nu_1}$ and $L_{\nu_2}$. Let $[M]$ denote the lattice class given by the triple $y_1, x, y_2$. One easily sees that $[M] \in \text{conv}(\mathcal{M}(T))$ is the equivalence class that is closest to $B$ if $[M]$ has property (ii) of Lemma 4.4 with respect to $x = x_B$.

**Definition 4.5.** Let $B$ be a Borel subgroup of $\text{PGL}_2(K)$ and let $x_B \in \mathbb{P}_K^1$ be the unique point that is fixed by $B$. We suppose that $x_B$ is not a limit point for $\Gamma$, i.e., $x_B \notin \mathcal{L}$. We recall that $T$ is a subdivision of the tree $T_{\mathcal{L}}$ (possibly needed in order to let $\Gamma$ act without inversions).

Let $\mathcal{M}(T)$ be the set of equivalence classes corresponding to the vertices of $T$. We say that an equivalence class $[M] \in \text{conv}(\mathcal{M}(T))$ is the equivalence class that is closest to $B$ if $[M]$ has property (ii) of Lemma 4.4 with respect to $x = x_B$.

**Lemma 4.6.** Let $x \in \mathbb{P}_K^1$ be a fixed point of some element in $\Gamma$ of finite order. Let $G_x \subset \Gamma$ denote the stabilizer of $x$ and $H_x \subset G_x$ the maximal finite subgroup of $G_x$. Then

1. $x \in \Omega$ if and only if $H_x = G_x$.
2. If $p_K = 0$, then $H_x$ is a maximal finite cyclic subgroup of $\Gamma$.
3. If $p_K = p > 0$ and $p$ divides $\# H_x$, then $x \in \Omega$ and $H_x = G_x = \Gamma \cap B$, where $B$ is the Borel group associated to $x$.

**Proof.** The group $G_x$ is a discontinuous subgroup of a Borel group. If $p_K = 0$, then $G_x$ does not contain unipotent elements $\neq 1$ and this implies that $G_x$ has the form $\{ (a, 0) \mid a \in A \}$ and $A \subset K^*$ a certain discontinuous subgroup. The group $H_x$ is then equal to $\{ (a, 0) \mid a \in B \}$, where $B$ is a finite subgroup of $K^*$. Thus $H_x$ is cyclic. If $p_K = 0 > 0$ and $G_x$ does not contain unipotent elements $\neq 1$, then again $H_x$ is a cyclic group of order not divisible by $p$. If $p_K = p > 0$ and $G_x$ does contain a unipotent element $\neq 1$ (or equivalently $G_x$ contains an element of order $p$), then one easily sees that $G_x$ is the filtered union of finite groups. Since $\Gamma$ is finitely generated, this implies that $G_x$ itself is finite.

If $x \in \Omega$, then clearly $G_x$ is finite. Suppose that $x \notin \Omega$. We fix an embedding of $T_{\Gamma} := T / \Gamma$ in $T$. After replacing $x$ by a $\Gamma$-conjugate we may suppose that $H_x$ stabilizes a vertex $v$ of $T_{\Gamma}$. Then $H_x$ also stabilizes the half line $L$ starting with $v$ in the “direction” $x$. Infinitely many $\Gamma$-conjugates of some vertex $w$ of $T_{\Gamma}$ lie on the half line $L$. Let $G$ denote the stabilizer of $w$. Then for infinitely many $\gamma \in \Gamma$ one has $\gamma H_x \gamma^{-1} \subset G$. This has as consequence that the group $\{ \gamma \in \Gamma \mid \gamma h \gamma^{-1} = h \text{ for all } h \in H_x \}$ is infinite. Then also $G_x$ is infinite. The reasoning above also implies statements (2) and (3). \qed
Definition 4.7 (The tree $T^\dagger$). The graph $T^\perp$ is an abstract graph on which the group $\Gamma$ acts. However, one associates to each vertex $v \in T^\perp$ a lattice class $[M_v]$ as follows:

(i) If $v$ is not of Borel type, then $[M_v]$ is the unique separating lattice class for the group $v$.
(ii) If $v$ is a subgroup of a Borel subgroup $B \subset PGL_2(K)$, then according to 4.6 the fixed point $x$ of $B$ lies in $\Omega$ and one defines $[M_v]$ to be the unique lattice class in $\text{conv}(\mathcal{M}(T))$ that is closest to $B$.

Put $\mathcal{M}(T^\perp) = \{ [M_v] \mid v \text{ vertex of } T^\perp \}$. We define $T^\dagger := T_{\mathcal{M}(T^\perp)}$. By $\mathcal{M}(T^\dagger)$ we denote the set of lattice classes corresponding to the vertices of the tree $T^\dagger$. We recall that $\mathcal{M}(T^\dagger) = \mathcal{M}(T^\perp) \cup \text{V(\text{conv}(\mathcal{M}(T^\perp)))}$.

Lemma 4.8. Let $v = H$ be a vertex of $T^\perp$ and $[M_v]$ its associated lattice class. Then $v = \Gamma_v$ coincides with the stabilizer of the lattice class $[M_v]$.

Proof. For any $\gamma \in \Gamma$ one has $\gamma v := \gamma H \gamma^{-1}$ and $[M_{\gamma v}] = [\gamma M_v]$. Therefore $\Gamma_v$ is a subgroup of the stabilizer $\widetilde{\Gamma}_v$ of $[M_v]$. If $H$ is a maximal finite subgroup then clearly $\widetilde{\Gamma}_v = \Gamma_v = H$.

Suppose that $pK = p = 2, 3$ and $H \cong C_p$. Let $B$ denote the unique Borel group containing $H$ and $x_B \in \Omega$ the fixed point of this Borel group. Let $H_1$, $H_2$ denote two, non-conjugated, maximal finite subgroups of $\Gamma$ containing $H$ and let $[M_1]$, $[M_2]$ denote their separating lattices. If $[M_1] = [M_2]$ then the images of $x_B$ and $[M_2]$ in $P(M_v \otimes_{K^0} k)$ are distinct. If $[M_2] = [M_v]$ then the images of $x_B$ and $[M_1]$ in $P(M_v \otimes_{K^0} k)$ are distinct. If $[M_1] \neq [M_2]$, then the images of $x_B$, $[M_1]$, $[M_2]$ in $P(M_v \otimes_{K^0} k)$ are distinct. Thus in all cases $H$ has at least two fixed points in $P(M_v \otimes_{K^0} k)$. Thus $H$ acts trivially on $P(M_v \otimes_{K^0} k)$. By Proposition 2.2, the group $\widetilde{\Gamma}_v$ is of Borel type.

Thus $\widetilde{\Gamma}_v \neq H = \Gamma_v$. Let $H_3 \supset \widetilde{\Gamma}_v$ be a maximal finite subgroup of $\Gamma$. As in 4.3, one concludes for that $H_3$ is $D_\ell$ (with odd $\ell$) for $p = 2$ and $H_3 = PSL_2(F_3)$ for $p = 3$. In the second case $\widetilde{\Gamma}_v = PSL_2(F_3)$ and is a maximal finite subgroup of $\Gamma$. In the first case one finds that $\widetilde{\Gamma}_v = D_m$ for some divisor $m$ of $\ell$. In both cases $\widetilde{\Gamma}_v$ is not of Borel type, which yields a contradiction. Therefore $\widetilde{\Gamma}_v = \Gamma_v = H$. □

Theorem 4.9. Let $\Gamma$ be ordinary. Then the graph $T^\perp$ is a tree.

Proof. Suppose that $T^\perp$ contains three vertices $\{v_1, v_2, v_3\}$ such that $\{v_i, v_j\}$ is an edge for all $i < j$. For convenience we write also $v_i$ for the lattice class $[M_{v_i}]$ in $T^\perp$ and $\Gamma_i$ for the stabilizer of $v_i$. The minimal subtree $T$ of $BT$ containing $\{v_1, v_2, v_3\}$ has vertices $\{v, v_1, v_2, v_3\}$ and edges $\{v, v_i\}$ for $i = 1, 2, 3$. The vertex $v$ is defined as $[v_1, v_2] \cap [v_1, v_3]$. By construction $T \subset T^\perp$. Let $S$ be the stabilizer of $v$ in $\Gamma$. For $i < j$ one has $S \subset \Gamma_i$ or $S \subset \Gamma_j$, and $S \supset \Gamma_i \cap \Gamma_j \neq 1$, since $\{v_i, v_j\}$ is an edge. Suppose that $S$ is contained in every $\Gamma_i$ then $S = \Gamma_1 \cap \Gamma_i$ for all $i < j$ and $S = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. Suppose that $S$ is not contained in, say, $\Gamma_1$, then $S \subset \Gamma_2 \cap \Gamma_3$ and consequently $S = \Gamma_2 \cap \Gamma_3$. From $S \subset \Gamma_1 \cap \Gamma_2$ we conclude that $H := \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \neq 1$. If $H$ contains an element of order not divisible by $p_k$, then the vertices $v_1, v_2, v_3$ lie on a line in $BT$, namely the axis of that element. We conclude that the order of $H$ is a power of $p_k$. 
Consider the case \( p_K = 0 \), \( p_k > 2 \). Since \( \Gamma^* \) is ordinary, every \( \Gamma_i \) is a dihedral group. \( H \) is a cyclic group and \( H \) is normal in every \( \Gamma_i \). From the description in Section 2 of the tree of \( D_L \) where \( p_k \not\mid \ell \), one concludes that the unique separating lattice class for \( D_L \) lies on the axis of any non-trivial subgroup of \( C_L \subset D_L \). One finds the contradiction that \( v_{1}, v_{2}, v_{3} \) lie on a segment in \( BT \).

Consider the case \( p_K = 0 \), \( p_k = 2 \). If every \( \Gamma_i \cong A_4 \), then \( H = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \) is isomorphic to \( C_2 \). Let \( L \subset BT \) denote the \( H \)-axis. From the description of the tree of \( A_4 \) in Section 2.5, one concludes that the lattice classes \( v_1, v_2, v_3 \) are not on \( L \). Their projections \( pr_{L}(v_1), pr_{L}(v_2), \) \( pr_{L}(v_3) \) are distinct, since for \( i < j \) the subgroup generated by \( \Gamma_i \) and \( \Gamma_j \) is equal to \( \Gamma_i \ast H \Gamma_j \). Moreover, the stabilizer of \( pr_{L}(v_i) \) contains the unique subgroup of \( v_i \), which is isomorphic to \( D_2 \). Suppose that \( pr_{L}(v_2) \in [pr_{L}(v_1), pr_{L}(v_3)] \). Then \( v_1, v_3 \) contains \( pr_{L}(v_2) \) and the stabilizer of this lattice class is not contained in \( v_1 \) or \( v_3 \). This contradicts the assumption that \( v_{1}, v_{3} \) is an edge of \( T^* \). Similar arguments rule out all the other possible situations for \( p_K = 0, p_k = 2 \).

Consider the case \( p_K = p > 0 \). For \( i < j \) the group \( \Gamma_i \ast_{\Gamma_0 \cap \Gamma_j} \Gamma_j \) is a subgroup of \( \Gamma \) and thus a realizable amalgam. Moreover \( H \) is a \( p \)-group. According to Corollary 3.11 this leaves only the possibilities \( p = 3 \), \( H = C_3 \) and each \( \Gamma_i \cong PSL_2(\mathbb{F}_3) \) or \( p = 2 \), \( H = C_2 \) and each \( \Gamma_i \) is a dihedral group \( D_{\ell_i} \) with odd \( \ell_i \). This is excluded by (e2) and (e3) of Definition 4.3.

Now we consider the case where \( T^c \) contains a “circle” with consecutive vertices \( v_1, v_2, ..., v_{s}, v_1 \) with \( s > 3 \) and \( s \) minimal. We will use a “cyclic” notation for the vertices, i.e., \( v_{i+1} = v_i \) for all \( i \in \mathbb{Z} \). Let \( T \subset BT \) denote the smallest tree containing \( v_1, ..., v_{s} \). By construction \( T \subset T^\dagger \). For convenience we will write \( \Gamma_i := \Gamma_{v_i} \). First we will show that the extremal vertices of \( T \) are precisely \( \{v_1, ..., v_{s}\} \).

A vertex \( v \not\in \{v_1, ..., v_{s}\} \) of \( T \) has at least three edges. Hence every extremal vertex of \( T \) is some \( v_{a} \). Suppose that some \( v_b \) is not an extremal edge of \( T \). Then there are extremal edges \( v_{a}, v_{b}, v_{c} \) of \( T \) such that \( v_b \) lies in the segment \( \{v_{a}, v_{c}\} \). We may suppose that \( 1 \leq a < b < c \leq s \). Let \( pr : T \to [v_{a}, v_{c}] \) denote the projection. This means that \( pr(v_{d}) \) is the point \( [v_{a} \cup v_{c}] \cap [v_{d}, v_{c}] \). For neighbours \( v_{d}, v_{d+1} \) with \( \ell_2 \neq d, d+1 \) the point \( v_b \) is not contained in \( [pr(v_{d}), pr(v_{d+1})] \). Suppose the opposite, then \( v_b \) lies on \( [v_{d}, v_{d+1}] \) and \( \Gamma_b \) is contained in either \( \Gamma_{d} \) or \( \Gamma_{d+1} \). Then \( \Gamma_b \) is not a maximal finite subgroup and we are in the situation \( \Gamma_b = C_{p} \) with \( p_K = p = 2 \) or 3. Then \( \Gamma_{d} \cap \Gamma_{d+1} = C_{p} \) and by 3.11, this only holds if \( p = 2 \) and one of the groups \( \Gamma_{d}, \Gamma_{d+1} \) is equal to \( B(n, 1) \) with \( n > 1 \). The latter is excluded by part (2) of Definition 4.3.

One concludes that \( pr(v_{d}) \not\in [v_{a}, v_{b}] \) for \( d \) with \( a \leq d \leq b \) or \( c \leq d \leq a + s \). The same reasoning yields that \( pr(v_{d}) \not\in [v_{b}, v_{c}] \) if \( b \leq d < a + s \). Therefore \( pr(v_{d}) = v_b \) for \( d \) with \( c < d \leq a + s \). If \( c < a + s - 1 \), then \( v_b \) lies on \( [v_{a+s-1}, v_{a+s}] \). If \( c = a + s - 1 \), then \( v_b \) lies on \( [v_c, v_b] \). In both cases one finds a contradiction as above. We conclude that the extremal edges of \( T \) are \( \{v_1, ..., v_{s}\} \).

In the following we view the tree \( T \) as a topological space by identifying each edge with a copy of \([0, 1] \subset \mathbb{R} \). It is convenient to assume that \( |K^*| = \mathbb{R}_{>0} \) (this can be achieved by replacing \( K \) be a larger complete, algebraically closed extension). Then all the points on the topological tree \( T \) correspond with lattice classes. For \( i \in \{1, ..., s\} \) one considers the subset \( T_i^* \) of \( T \) corresponding to lattice classes \([M]\) such that its stabilizer \( \Gamma_i[M] \) is contained in \( \Gamma_i \). The complement of \( T_i^* \) is the union over the elements \( g \in \bigcup_{j=1}^{s} \Gamma_j, g \not\in \Gamma_i \) of the set.
Theorem 4.10. Suppose that $V(g) = \{ [M] \in T \mid g \in \Gamma_{[M]} \}$. Since each $V(g)$ is closed, one has that $T^*_v$ is open. Let $T_i$ be the connected component of $T^*_v$ containing $v_i$. The sets $T_i$ have the following properties:

1. $T_i$ is convex in the sense that with $a, b \in T_i$ also $[a, b] \subset T_i$.
2. $[v_i, v_{i+1}] \subset T_i \cup T_{i+1}$ and $[v_i, v_{i+1}] \cap T_i \cap T_{i+1}$ is not empty.
3. $\bigcup_i T_i = T$ since every point of $T$ lies on $[v_i, v_{i+1}]$ for some $i$.

If the intersection $T_{i_0,i_1,...,i_l}$ of the $T_{i_0}, \ldots, T_{i_l}$ is non-empty then it is convex. Suppose that each $T_i$ meets only $T_{i-1}$ and $T_{i+1}$. Then the union of the segments $[v_i, v_{i+1}]$ produces a circle in $T$, which is impossible. Hence there are $i < j$ with $T_i \cap T_j \neq \emptyset$ and such that $[v_i, v_j]$ is not an edge. It follows that $[v_i, v_j] \subset T_i \cup T_j$ and that $T_i \ast T_j \ast T_j' \ast T_j''$ is either $D_\ell \ast C_{2\ell}$ or $D_\ell \ast C_{2\ell}$ with odd $\ell$, $\ell' \neq pK = p \neq 2$ or $PSL_2(F_3) \ast C_3 \ast PSL_2(F_3)$ and $pK = p = 3$.

We consider first the case $pK = p = 2$. Let $pr: T \rightarrow [v_i, v_j]$ denote the projection of the tree $T$ on the segment $[v_i, v_j] \subset T$. It follows from the structure of the subgroups of dihedral group that for any $[M] \in [v_i, v_j]$ with $[M] \neq v_i, v_j$ that $\Gamma_{[M]} = C_2$. Now $pr(v_{i+1}) \neq v_i, v_j$ and one concludes that $T_i \cap T_j \neq \emptyset$. Similarly $T_i \cap T_j \neq \emptyset$ and $n' > 1$ in conflict with (v1) and (v2). The proof for the case $pK = p = 3$ is completely similar. □

Theorem 4.10. Suppose that $pK = 0$. The tree $T^c$ has properties (i)–(iii) below. As a consequence the tree of groups $T^c := T^c / \Gamma$, obtained by embedding $T^c$ in $\Gamma$, has the properties of Theorem 3.12.

(i) The map $v \mapsto \Gamma_v$, from the set of vertices of $T^c$ to the set of maximal finite subgroups of $\Gamma$, is a bijection.
(ii) The stabilizer of an edge $e \in T^c$ is a maximal finite cyclic subgroup of $\Gamma$.
(iii) Let $v \in T^c$ be a vertex. Then the following two statements hold:

(a) For any maximal cyclic subgroup $H \subset \Gamma_v$, there are at most two edges $e$ with vertex $v$, such that $\Gamma_e = H$.
(b) Suppose that $pK > 2$. If two distinct edges $e' = [v, v']$, $e'' = [v, v'']$ have the same stabilizer $H \subset \Gamma_v$, then red$_v(e') \neq$ red$_v(e'')$.

Proof. (i) follows from the definition of $T^c$ and (ii) is proved in Corollary 3.6.

(iii) Let $H$ be a maximal cyclic subgroup of $\Gamma_v$. Suppose that $pK \nmid \#H$. The group $H$ acts faithfully on $P(M_v \otimes k)$ and has there two fixed points which are the images $\bar{a}, \bar{b}$ of the two fixed points $a, b$ of $H$ on $P^1(k)$.

An edge $e' = [v, v']$ with $\Gamma_{e'} \subset H$ have the property that $\psi_v(e') = \psi_v([M_{e'}])$ is invariant under $H$ and thus $\psi_v([M_{e'}]) = \bar{a}$ or $\bar{b}$. Suppose that two edges $e' = [v, v'], e'' = [v, v'']$ have $\Gamma_{e'} = \Gamma_{e''} = H$ and $\psi_v([M_{e'}]) = \psi_v([M_{e''}]) = \bar{a}$. Then the three separating lattices $[M_{e'}], [M_{e''}], [M_v]$ lie on the axis of $H$ in $BT$ and $[M_v]$ does not lie in between $[M_{e'}]$ and $[M_{e''}]$. Thus, say, $[M_{e'}]$ lies in between $[M_v]$ and $[M_{e''}]$. This contradicts the assumption that $[v, v']$ is an edge.

Suppose that the order of $H$ is divisible by $pK > 2$. Then $\Gamma_v \cong D_\ell$ with $pK \mid \ell$ and $H \cong C_\ell$. Let again $e' = [v, v']$ be an edge. Then $\Gamma_{e'}$ is also isomorphic to $D_\ell$. Moreover
the separating lattices \([M_\ell], [M_{e'}]\) lie an the axis of \(H\). The latter follows from the picture in Section 2, concerning the groups \(D_\ell\) with \(p_K|\ell\). For a second edge \(e'' = \{v, v''\}\) the same holds. Moreover, as above, \([M_\ell]\) must lie in between \([M_{e'}]\) and \([M_{e''}]\) on the axis of \(H\).

Suppose that \(p_K = 2\) and \(#H = 2^s\) with \(s > 1\). Every maximal finite subgroup \(w\) of \(\Gamma\), containing \(H\), is isomorphic to \(D_2^s\). Let \(L \subset B^T\) denote the axis of \(H\). For maximal finite subgroups \(w_1 \neq w_2\) of \(\Gamma\), containing \(H\), one has that \(w_1 * H w_2 \to \Gamma\) is an injection. This implies that the projections \(pr_L([M_{w_1}]), pr_L([M_{w_2}])\) are distinct. As a consequence there are at most two edges \(e, e'\) of \(w\) with \(H \equiv G_e \equiv G_{e'}\). In this case however, \(\text{red}_e(e) = \text{red}_e(e')\).

Suppose \(p_K = 2\) and \(H \cong C_2\). Let \(L \subset B^T\) denote the axis of \(H\). For any maximal finite subgroup \(w\) of \(\Gamma\), which has \(H\) as maximal cyclic subgroup, one has that \([M_{w}]\) does not lie on \(L\). Moreover, for two such groups \(w_1 \neq w_2\), the argument above shows that the projections \(pr_L([M_{w_1}]), pr_L([M_{w_2}])\) are distinct. As a consequence there are at most two edges \(e, e'\) of \(v\) with \(H \equiv G_e \equiv G_{e'}\). Again in this situation \(\text{red}_e(e) = \text{red}_e(e')\).

Clearly, \(T^c\) has the properties (a)–(e) of Theorem 3.12. Suppose that \(e_i = \{v_1, v\}, i = 1, 2\), are two edges in \(T^c\) with \(\Gamma_{v_1} = H \subset \Gamma_v\). If \(g \in \Gamma_v\) satisfies \(gHg^{-1} = H\) and \(g \notin H\), then \(g(e_1) = e_2\). This contradicts the definition of \(T^c\) and we conclude that (f) of Theorem 3.12 is also valid. We note that property (iii)(b) has not been used here. □

**Theorem 4.11.** Suppose that \(p_K = p > 0\). The tree \(T^c\) satisfies the properties (1)–(5) below. As a consequence, the tree of groups \(T^c := T^c/\Gamma\) satisfies the properties of Theorem 3.14.

1a) The map \(v \mapsto \Gamma_v\) is a bijection between the vertices of \(T^c\), with \(\Gamma_v \not\cong C_p\), and the maximal finite subgroups of \(\Gamma\).

1b) Only for \(p = 2, 3\), the group \(\Gamma\) may contain a maximal finite subgroup \(G\) which is a \(p\)-group. In that case \(G \cong B(n, 1)\) with \(n > 1\).

1c) Only for \(p = 2, 3\), the tree \(T^c\) may contain vertices \(v\) with \(\Gamma_v \cong C_p\). In which case, the map \(v \mapsto \Gamma_v\) yields a bijection between the vertices \(v\) with \(\Gamma_v\) not a finite maximal subgroup of \(\Gamma\), and the subgroups \(H \subset \Gamma\) satisfying (\(\alpha2\)) of Definition 4.3.

2a) The stabilizer \(\Gamma_e\) of an edge \(e = \{v_1, v_2\}\) of \(T^c\) is a non-trivial group of Borel type.

2b) If \(\Gamma_v\) is a \(p\)-group then \(\Gamma_v \cong C_p\) and \(p = 2, 3\). Furthermore, after interchanging \(v_1, v_2\) if necessary, one has that \(\Gamma_{v_1}\) is a \(p\)-group. If \(p = 2\), then \(\Gamma_{v_2} \equiv D_\ell\) with odd \(\ell\). If \(p = 3\), then \(\Gamma_{v_2} \equiv \text{PSL}_2(\mathbb{F}_3)\).

2c) If \(\Gamma_v\) contains a \(p\)-group, then \(\Gamma_v\) is of Borel type for precisely one \(i\).

3) Let \(v \in T^c\) be a vertex such that the stabilizer \(\Gamma_v\) of \(v\) is not of Borel type and let \(e\) be an edge of \(v\). Then the following holds:

3a) \(\Gamma_e \subset \Gamma_v\) is a ramification group of the map \(\varphi_{\Gamma_v} : \mathbb{P}^1_K \to \mathbb{P}^1_K/\Gamma_v\).

3b) If \(e' \neq e\) is an edge of \(v\) and \(\Gamma_e \equiv \Gamma_{e'}\), then \(\text{red}_e(e) \neq \text{red}_{e'}(e')\).

4) If the stabilizer \(\Gamma_v\) of a vertex \(v \in T^c\) is of Borel type and \(\Gamma_v\) is not a \(p\)-group, then \(\Gamma_v\) acts transitively on the edges \(e\) that contain \(v\).

5) If \(\Gamma_v\) is a \(p\)-group, then for all edges \(e, e' \ni v\) one has \(\Gamma_{e} = \Gamma_{e'} \equiv C_p\). Furthermore, if \(\Gamma_v \equiv C_p\), then the vertex \(v\) is contained in at least two edges of \(T^c\).

**Proof.** (1) follows from the construction of \(T^c\) and 3.11.
(2a) The group $\Gamma_e$ is non-trivial because $\Gamma$ is indecomposable and it is of Borel type since it stabilizes at least two lattice classes.

(2b), (2c) and (3a) follow from 3.11 and conditions (e2), (e3) of 4.3.

(3b) Let $e = \{v, v'\}$ be an edge such that $p \mid \#\Gamma_e$ and let $B$ be the unique Borel group containing $\Gamma_e$. By (2c), $\Gamma_{v'}$ is of Borel type and actually equal to $\Gamma \cap B$. In particular, $e$ is determined by $\Gamma_{v'}$.

If $p \nmid \#\Gamma_e$, then $\Gamma_e$ is a maximal cyclic group of order prime to $p$. The group $\Gamma_e$ has to fixed points $a, b \in \mathbb{P}^1(K)$ and $e$ is determined by $\Gamma_{v'}$ and a choice of one of the fixed points. One has to prove that the images of $a, b$ in $\mathbb{P}(M_e \otimes k)$ are distinct. This follows from the classification in Proposition 2.1.

(4) Assume that $\Gamma_{v}$ is of Borel type $B(n, m)$ with $m > 1$ and let $e = \{v, v'\}$ be an edge of $T^c$. From Corollary 3.11 and (e2), (e3) of Definition 4.3, one can read off the possibilities for $\Gamma_{v} \ast_{\Gamma_e} \Gamma_{v'}$. Statement (iv) translates into: “the possible subgroups $\Gamma_e$ are conjugated in $B(n, m)$.”

The case $\Gamma_e \cong C_m$ is valid since all cyclic subgroups of order $m$ in $B(n, m)$ are conjugated. The remaining possibilities for $\Gamma_{v} \ast_{\Gamma_e} \Gamma_{v'}$ are

(a) $B(n, q - 1) \ast_{B(F_q)} \text{PGL}_2(F_q)$ with $q \neq 2$.
(b) $B(n, \frac{q - 1}{2}) \ast_{B(F_q)} \text{PSL}_2(F_q)$ and $p \neq 2$.
(c) $B(n, 2) \ast_{D_3} A_5$ and $p = 3$.

Consider case (a) and let $e_1 = \{v, v_1\}$ be an edge. Let $f_1, f_2$ be a basis of $K^2$ over $K$, such that $\Gamma_{v_1} = \text{PGL}(F_q, f_1 + F_q, f_2)$ and $\Gamma_{v_1}$ is the Borel subgroup consisting of the elements which leave the line $F_q.f_1$ invariant. On this basis, the group $B(n, q - 1)$ consists of the matrices $\{(\begin{smallmatrix}a & b \\ 0 & 1 \end{smallmatrix}) \mid a \in F_q, b \in V\}$ where $V \subset K$ is a finite-dimensional vector space over $F_q$. One has that $F_q \subset V$ and $F_q \neq V$. As in the proof of Proposition 3.10 one verifies that the condition that $\Gamma_{v} \ast_{\Gamma_e} \Gamma_{v_1}$ is a realizable amalgam is equivalent to $V = F_q \oplus W$, where the $F_q$-vector space $W$ has the property $|w| > 1$ for every $w \in W$, $w \neq 0$. The lattice class $[M_e]$ in $T^c$ associated to $\Gamma_v$ is given by $M_e = K^0f_1 + K^0f_2$, where $\lambda \in K$ is chosen such that $|\lambda| = \max_{v \in V} |v|$. The lattice class $[M_{v_1}]$ associated to $\Gamma_{v_1}$ is given by $M_{v_1} = K^0f_1 + K^0f_2$. One observes that the distance between the two lattice classes depends only on the group $\Gamma_v$. Let $e_2 = \{v, v_2\}$ be another edge of $v$. After conjugation with an element $g \in \Gamma_v = B(n, q - 1)$ we may suppose that the subgroup $C_{q-1} = \{(\begin{smallmatrix}a & b \\ 0 & 1 \end{smallmatrix}) \mid a \in F_q^*\}$ of $B(n, q - 1)$ belongs to $\Gamma_{v_2}$. The intersection $\Gamma_v \cap \Gamma_{v_2}$ has the form $\{(\begin{smallmatrix}a & b \\ 0 & 1 \end{smallmatrix}) \mid a \in F_q^*, b \in Z\}$, where $Z$ is a 1-dimensional vector space over $F_q$. Then $[M_{v_1}], [M_{v_2}]$ lie on the axis of the group $C_{q-1}$, on the same side of $[M_e]$ and with the same distance to $[M_e]$. We conclude that $[M_{v_1}] = [M_{v_2}]$ and $v_1 = v_2$. This proves that $\Gamma_v$ acts transitively on the edges of $v$ in $T^c$. The cases (b) and (c) can be handled in the same way.

(5) $\Gamma_{v}$ is a $B(n, 1)$. For $n > 1$, the only possibilities for an edge $e = \{v, v'\}$ produce the amalgams $B(n, 1) \ast_{\Gamma_e} D_\ell$ with odd $\ell$ and $p = 2$ (see 3.11). By Definition 4.3, an edge $e = \{v, v'\}$ for the case $n = 1$ is only possible with $\Gamma_{v'} \cong D_\ell$ with odd $\ell$ and $p = 2$ or $\Gamma_{v'} \cong \text{PSL}_2(F_3)$ and $p = 3$. 

---

270

Finally, the verification that $T^c$ satisfies the properties of 3.14, is straightforward. We note that some of the stated properties of $T^c$ are superfluous for this verification. □

**Observations 4.12** (Relations between the trees $T$, $T^c$ and $T^\dagger$). As before, we suppose that $\Gamma$ is finitely generated, discontinuous, indecomposable, ordinary and its set of limit points $L$ has more than two elements.

1. $\Gamma$ acts on the tree $T_L$ without inversion and thus $T = T_L$.

   $L$ is also the set of equivalence classes of the ends of the tree $T^c$. From 4.10 and 4.11 it follows that no inversion is possible.

2. The tree $T^c$ has no extremal vertices.

   This follows from the descriptions 3.12 and 3.14 of $T^c := T^c/\Gamma$, proved in 4.10 and 4.11.

3. $T$ and $T^\dagger$ have the “same” classes of infinite ends. Moreover every vertex of $T$ is also a vertex of $T^\dagger$.

   The infinite ends are, for both trees, in bijection with the limit points $L$. A vertex $[M]$ of $T$ is determined by three points of $L$. The corresponding three ends of $T^\dagger$ determine a vertex of $T^\dagger$, which coincides with $[M]$.

4. $T = T^\dagger$ if and only if $[M_v]$ belongs to $T$ for every vertex $v$ of $T^c$.

5. If $p_K = p \geq 5$, then $T = T^\dagger$.

One has to verify that $[M_v]$, attached to a maximal finite subgroup $v$ of $\Gamma$, belongs to $T$. If $v$ is not of Borel type, then $v$ has a unique invariant lattice in $BT$. This lattice is equal to $[M_v]$ and is also equal to the fixed point of $v$ on the tree $T$. Let $v$ be a Borel type $B(n, m)$, then $m > 1$. For a suitable basis $e_1, e_2$ of $K^2$, one can represent $v$ by the collection of matrices $\{(\zeta^a) | \zeta \in F_q^*, a \in A\}$, where $A \subset K$ is a finite-dimensional vector space over $F_q$ and moreover $\max_{a \in A} |a| = 1$. For this representation, the fixed point of $v$ is $\infty$. The invariant lattice classes can be represented by $M = K^0e_1 + K^0\lambda e_2$ with $\lambda \in K^*$ and $|\lambda| \leq 1$. For the lattice $M_1 := K^0e_1 + K^0\lambda e_2$ the image under red$_{[M_1]}$ of the set $A \cup \{\infty\}$ of ramification points of $B(n, m)$ has at least three points. For the other lattices classes $[M]$, this image consists of the two points $\text{red}_{[M]}0, \text{red}_{[M]}\infty$. The vertex $[M_v]$ has at least two edges $e_1, e_2$ in $T^\dagger$. The three points $\text{red}_{[M_v]}e_1, \text{red}_{[M_v]}e_2, \text{red}_{[M_v]}\infty$ are distinct and lie in the image of the ramification points of $v$. This proves that $[M_v] = [M_1]$. The vertex $v$ has at least three edges in $T^c$. This implies that red$_{[M_1]}L$ consists of at least three points and $[M_1] \in T$.

6. For $p_K = 2, 3$ one considers the set $F \subset \Omega$ consisting of the points $x$ for which there exists a maximal finite subgroup $H \subset \Gamma$, which is a $p$-group, or a group $H \cong C_p$. 
having property (v2) of Definition 4.3, such that \( x \) is the fixed point of \( H \). Then \( T^\dagger \)
coincides with the tree \( T_{F(x)\mathcal{L}} \) defined in Section 2.1.

This follows readily from the definition of \([M_v]\) for \( v = H \) and \( H \) as above.

(7) Let \( p_k = 0, p_k > 2 \) and let \( v \) be a maximal finite subgroup of \( \Gamma \). If \( v \not\cong D_\ell \) with \( \ell \) a power of \( p_k \), then \([M_v]\) belongs to \( T \).

Suppose that \( v \cong D_\ell \) and \( \ell \) is a power of \( p_k \). We choose a coordinate \( z \) for \( P^1(K) \) such that \( v \) consists of the transformations

\[
\{z \mapsto \zeta^a z^b \mid 0 \leq a < \ell, \ b = \pm 1\}
\]

where \( \zeta \) is a primitive \( \ell \)-th root of unity. The three lattice classes in \( BT \), invariant under \( v \) are \([M_1], [M_2], [M_{-1}]\), given by the ramification points \( \{1, \zeta, \zeta^{-1}\}, \{1, -1, \infty\} \) and \( \{-1, -\zeta, -\zeta^{-1}\} \). Let \( \text{red}_v \) denote the reduction \( P^1(K) \rightarrow P(M_\ell \otimes k) \).

Then \( \text{red}_v([\text{edges of } v]) \) is a subset of \( \{1, -1, 0, \infty\} \), the images under \( \text{red}_v \) of the ramification points of \( v \). Further, \( \text{red}_v(L) \subset \text{red}_v([\text{edges of } v]) \) consists of \( \Gamma_v \)-orbits.

There are the following possibilities:

(a) \( \text{red}_v(L) = \{1, -1, 0, \infty\} \). Then \([M_1], [M_2], [M_{-1}]\) \( \in T \).

(b) \( \text{red}_v(L) = \{1, -1\} \). Then \([M_1], [M_{-1}] \in T, [M_v] \notin T \) and \([M_v] \in T^\dagger \) is not an extremal vertex of \( T^\dagger \).

(c) \( \text{red}_v(L) = \{0, \infty\} \) with \( \delta = \pm 1 \). Then \([M_0], [M_\delta] \in T, [M_\delta] \notin T^\dagger \).

(d) \( \text{red}_v(L) = \{0, \infty\} \). Then \([M_1], [M_{-1}] \notin T^\dagger, [M_v] \notin T \) and \([M_v] \) is an extremal edge of \( T^\dagger \).

(e) \( \text{red}_v(L) = \{0\} \) with \( \delta = \pm 1 \). Then \([M_0], [M_v] \notin T, [M_{\delta}] \notin T^\dagger \) and \([M_v] \) is an extremal vertex of \( T^\dagger \).

The proof is a straightforward computation. One concludes that \( T^\dagger \) is obtained from \( T \) by possibly a subdivision of edges (occurs only in case (b)) and by possibly attaching extremal vertices (occurs only in cases (d) and (e)). The corresponding situation for \( p_k = 2 \) is somewhat different.

(8) \( T^c \) is a contraction of \( T^\dagger \) in the following sense:

(a) \( v \mapsto [M_v] \) is an injective map from the vertices of \( T^c \) to those of \( T^\dagger \).

(b) \( \{v_1, v_2\} \) is an edge of \( T^c \) if and only if for every \([M] \in [M_{v_1}, M_{v_2}]\) the group \( \Gamma_{[M]} \) is contained in \( v_1 \) or \( v_2 \) and no other \([M_v]\) lies in \([M_{v_1}], [M_{v_2}]\).

(c) \( T^c \) and \( T^\dagger \) have the “same” classes of infinite ends.

This follows from the definition of \( T^\dagger \) and the fact that \( T^c \) is a tree.

(9) In general, \( T^c \neq T^\dagger \).

This is illustrated by the example \( \Gamma := \text{PGL}_2(F_q) *_{B(F_q)} B(n, q - 1) \) for \( p \geq 5 \).
5. Counting the number of branch points

In the sequel we will, unless otherwise stated, assume that $\Gamma$ is a finitely generated, discontinuous, indecomposable, ordinary subgroup of $\text{PGL}_2(K)$ such that $\Omega/\Gamma \cong \mathbb{P}^1(K)$. Moreover we suppose that its set of limit points contains more than two points.

For the counting of the number of branch points $\text{br}(\Gamma)$ we will need to know the location of the ramification points in $\Omega$, i.e., the points in $\Omega$ having a non-trivial stabilizer in $\Gamma$. In the previous section, a detailed description of the trees $T$, $T^c$ and $T^\dagger$ was obtained for $\Gamma$. According to Section 2.1 part (4), the tree $T^\dagger$ yields an admissible affinoid covering $\{X_v, X_e, v, e \in \Omega\}$. The result which makes counting possible is

**Theorem 5.1.** The ramification points of the map $\Omega \to \Omega/\Gamma$ are contained in the union of the affinoids $X_v, v \in \Omega$ corresponding to the vertices $[M_v]$ of $T^\dagger$ with $v$ a vertex of $T^c$.

Let $x = x_1 \in \Omega$ be a ramification point such that the order of the group $H := \{\gamma \in \Gamma : \gamma(x) = x\}$ is not divisible by $p_K$. Then $H$ is cyclic and the other fixed point $x_2$ in $\mathbb{P}^1(K)$ of $H$ also belongs to $\Omega$.

**Proof.** Let $T_H \subset T^\dagger$ be the subtree consisting of the vertices and edges which are invariant under $H := \{\gamma \in \Gamma : \gamma(x) = x\}$. We claim that $T_H$ is a finite tree. Indeed, $T^\dagger/\Gamma$ is a finite tree. Let $G$ denote the stabilizer of $[M]$ in $\Gamma$. Then there are infinitely many $\gamma \in \Gamma$ such that $H \subset \gamma G \gamma^{-1}$. It follows that there are infinitely many $\gamma \in \Gamma$ with $\gamma H \gamma^{-1} \subset G$ and also infinitely many $\gamma \in \Gamma$ which commute with $H$. Now Lemma 4.6 yields the contradiction that $x$ is a limit point.

Suppose that the order of $H$ is not divisible by $p_K$. Then clearly, $H$ is cyclic. If the second fixed point $x_2$ of $H$ is a limit point, then this point determines a halfline in $T^\dagger$ which is invariant under $H$. Since $T_H$ is finite, this is not possible and $x_2 \in \Omega$.

(1) The case $p_K = 0$ and $p_k \neq 2$.

Let $x \in \Omega$ be a ramification point for $\Gamma$. Its stabilizer $H$ is a maximal finite cyclic group subgroup of $\Gamma$ with fixed points $x = x_1, x_2 \in \Omega$. Put $m = \#H$.

(1a) Suppose that $m$ is not a power of $p_k$ and $m \neq 2$. $T_H$ is equal to the intersection of the axis of $H$ in $\mathcal{B}T$ with $T^\dagger$ and has the form $\{[M_1], \ldots, [M_s]\}$ with $s \geq 1$. If $[M] \in T_H$ is equal to a $[M_v]$ with $v$ a vertex of $T^c$, then $v$ is not isomorphic to $D_t$ with $t$ a power of $p_k$. It follows that the reduction map $\text{red}_{[M]} : \mathbb{P}^1(K) \to \mathbb{P}(M \otimes k)$ is injective on the set of the ramified points of $\mathbb{P}^1(K)$ for the group $v$. In particular, $\text{red}_{[M]}(x_1) \neq \text{red}_{[M]}(x_2)$. If $[M] \in T_H$ does not have the above form, then $[M]$ has at least three edges in the direction of vertices of the form $[M_v]$. It follows that $[M]$ lies in a segment $\{[M_v] \in T_H \mid v, v' \text{ vertices of } T^c\}$. Again $\text{red}_{[M]}(x_1) \neq \text{red}_{[M]}(x_2)$. For an edge $e = [M_j], [M_{j+1}]$ there is a $j \in [1, 2]$ with $\text{red}_{[M_j]}(e) = \text{red}_{[M_j]}(x_j)$ and $\text{red}_{[M_{j+1}]}(e) = \text{red}_{[M_{j+1}]}(x_j)$. For an extremal vertex $[M]$ of $T_H$ one has $\text{red}_{[M]}(e) \neq \text{red}_{[M]}(x_1), \text{red}_{[M]}(x_2)$ for every edge which does not belong to $T_H$. Now we conclude:
For \( s = 1 \) one has \([M_1] = [M_v]\) for some vertex \( v \) of \( T^c \) and \( x_1, x_2 \in X_v\). For \( s > 1 \), one writes \([M_1] = [M_{v_1}]\) and \([M_2] = [M_{v_2}]\) with \( v_1, v_2 \) vertices of \( T^c \) and (say) \( x_1 \in X_{v_1}, x_2 \in X_{v_2}\).

(1b) Suppose that \( m = 2 \). As in case (1a), \( T_H \) lies on the axis of \( H \) and has the form \([[N_1], \ldots, [N_s]]\) with \( s \geq 1 \). An extremal vertex of \( T_H \) is again a \([M_v]\) for some \( v \in T^c\).

A new possibility would be that \([N_1] = [M_v]\) for \( v \equiv D_\ell \) with \( \ell \) a power of \( p_k \). We compare this with 4.12 part (7). There are two other lattice classes invariant under \( v \) and hence under \( H \), namely \([M_{-1}]\) and \([M_1]\). At least one of them does not belong to \( T^c \). If neither belongs to \( T^c \) (this is case (d)) then \( x_1, x_2 \in X_v\). If one of the \([[M_\pm 1]]\) belongs to \( T^c \) (this is case (e)), then \( s > 1 \) and one of the points \( x_1, x_2 \) belongs to \( X_v\). The other fixed point of \( H \) lies in \( X_v\), where \( v' \) is the vertex of \( T^c \) satisfying \([M_{v'}] = [N_x]\).

(1c) Suppose that \( m \) is a power of \( p_k \). Suppose that \( H \) is contained in at least two maximal finite subgroups \( v_1, v_2 \). Both groups are isomorphic to \( D_m \) and have separating invariant lattices \([M_1], [M_2] \in T_H\). After changing \([M_2]\) if necessary, one may suppose that the stabilizer of every \([M] \in [[M_1], [M_2]]\) is contained in \( v_1 \) or \( v_2 \). By Theorem 3.3, \( v_1 \ast H v_2 \) is a subgroup of \( H \). The points \( x_1, x_2 \) are limit points for this subgroup and we find a contradiction. Thus \( H \) is contained in a single maximal finite subgroup \( v \) and the points \( x_1, x_2 \) are lying in \( X_v\) (according to the tree of \( D_{p_k}^s \) in Section 2).

(2) The case \( p_K = 0, p_k = 2 \).

Let \( x \in \Omega \) be a ramification point for \( \Gamma \). Its stabilizer \( H \) is a cyclic group of order \( m = 2, 3 \) or \( 2^s \) with \( s > 1 \). For \( m = 3 \), every maximal finite subgroup \( v \) of \( \Gamma \), containing \( H \) is isomorphic to \( A_4 \). The proof of (1a) can be copied in this situation.

For \( m = 2^s \) with \( s > 1 \), every maximal finite subgroup \( v \supset H \) of \( \Gamma \) is isomorphic to \( D_{2^s} \). As in (1c) one shows that \( H \) is contained in only one maximal finite subgroup \( v \) and that \( x_1, x_2 \in X_v\).

For \( m = 2 \), we consider the axis \( L \subset B\Gamma \) of \( H \) and the finite collection \( V \) of all maximal finite subgroup \( v \) of \( \Gamma \) such that \( H \) is maximal cyclic in \( v \). We may suppose that \( V \) consists of more than one element. For any two groups \( v, v' \in V \) one has that \( v \ast H v' \) is a realizable amalgam. The proof of Theorem 3.5 shows that the lattice class \([M_v]\) does not lie on \( L \). Moreover, the projections \( \{ \text{pr}_L([M_v]) \mid v \in V \} \) have to be distinct. There are \( v_1, v_2 \in V \) such that \( \text{pr}_L(v_1), \text{pr}_L(v_2) \) are extremal vertices of the set \( \{ \text{pr}_L([M_v]) \mid v \in V \} \). Then (say) \( x_1 \in X_{v_1} \) and \( x_2 \in X_{v_2} \).

(3) The case \( p_K = p \geq 5 \).

If the stabilizer \( H \) of \( x \in \Omega \) has an order \( n \) not divisible by \( p \), then the method of (1a) can be applied to prove that the two fixed points lie in affinoids \( X_v \) with \( v \in T^c \).

Suppose that the stabilizer \( H \) of \( x \in \Omega \) contains an element of order \( p \). If \( H \) is a maximal finite subgroup of \( \Gamma \), then \( H \cong B(n, m) \) with \( m > 1 \) since we have excluded \( p = 2, 3 \). Then \( v = H \) is a vertex of \( T^c \). By the definition of \([M_v]\) one has that \( \text{red}_{[M_v]}(x) \) is different from \( \text{red}_{[M_v]}(e) \) for every edge of \([M_v]\) in \( T^c \). It follows that \( x \in X_v\).

If \( H \) is not a maximal finite subgroup, then a maximal finite subgroup \( v \supset H \) of \( \Gamma \) must be isomorphic to \( \text{PGL}_2(F_q) \) or \( \text{PSL}_2(F_q) \). Again, \( H \cong B(n, m) \) with \( m > 1 \) and
Moreover \( H \) is a maximal proper subgroup of \( v \). The lattice classes of \( BT \), stabilized by \( H \), form a half line (see 4.12 part (5)). Suppose that \( H \) is contained in another maximal finite subgroup \( v' \), then \([M_\alpha], [M_{v'}]\) lies on this half line and we may suppose that for any \([M] \in [[M_\alpha], [M_{v'}]]\), \([M] \neq [M_\alpha], [M_{v'}]\) its stabilizer \( \Gamma[M] > H \) is not a maximal finite subgroup. If \( \Gamma[M] \neq H \), then \( \Gamma[M] \) lies in a third maximal finite subgroup \( v'' \). Also \( v'' \) lies on this halfline and \( H \) is a maximal proper subgroup of \( v, v', v'' \). This yields a contradiction and we conclude that \( \Gamma[M] \) must be \( H \). Theorem 3.1 implies that \( v \ast_H v' \) is a realizable amalgam. By 3.11 and \( p \neq 2, 3 \) this is not possible. Thus \( v \) is the only maximal finite subgroup containing \( H \). Then for every edge \( e \) of \( v \), the group \( \Gamma_e \) is distinct from \( H \). Thus \( x \in X_v \).

(4) The case \( p_K = p = 2, 3 \).

Let \( x \in \Omega \) be a ramification point for \( \Omega \) with stabilizer \( H \) in \( \Gamma \). If \( H \) is not a \( p \)-group, then, as in (3) above, one can show that \( x \in X_v \) for some vertex \( v \) of \( T^c \). If \( H \cong C_p \) and \( H \) is contained in only one maximal finite subgroup \( v \) of \( \Gamma \), then \( v \) is a vertex of \( T^c \) and \( x \in X_v \). Any other \( p \)-group \( H \) is a maximal \( p \)-group and \( H \) is a vertex of \( T^c \). Let \([M_H]\) be the associated lattice class. By definition, the reduction map \( \text{red} : \Omega \to (\Omega, T^c) \) has the property that \( \text{red}(x) \) lies on only one irreducible component, namely the one corresponding with \([M_H]\). Thus \( x \in X_H \). □

**Definition 5.2.** Now we define the objects and numbers which will appear in the formulas for the number of branch points \( \text{br}(\Gamma) \) of \( \Gamma \). Let \( \text{Max}(i) \), \( i = 2, 3 \), be the set of conjugacy classes of maximal finite subgroups \( H \subset \Gamma \) such that \( \text{br}(H) = i \). Put \( \text{max}(i) = \# \text{Max}(i) \).

For \( p_K = p = 2, 3 \) we consider \( \text{Maxp} := \) the set of conjugacy classes of the subgroups \( H \subset \Gamma \) such that \( H \) is a maximal \( p \)-group and, moreover, \( H \) intersects at least two maximal finite, non-conjugated, subgroups \( H_1, H_2 \subset \Gamma \) such that \( H \cap H_1 = H \cap H_2 \cong C_p \). For \( \alpha \in \text{Maxp} \), represented by the group \( H \subset \Gamma \), one puts \( d_\alpha := \# \{ \beta \in \text{Max}(2) \mid \exists(H_1 \in \beta) \cap H_1 \cong C_p \} \). We note that \( d_\alpha \) is equal to the number of edges in \( T^c := T^c/\Gamma \) of the vertex \( v \in T^c \) such that \( \Gamma_v \) belongs to \( \alpha \). Further we define \( \text{maxp} := \sum_{\alpha \in \text{Maxp}} (d_\alpha - 1) \).

For \( p_K = 0 \), \( \text{Max}^c \) denotes the set of conjugacy classes of maximal finite cyclic subgroups of \( \Gamma \). For \( p_K = p > 0 \), \( \text{Max}^c \) is the set of conjugacy classes of maximal finite cyclic subgroups \( H \subset \Gamma \) such that \( p \nmid \# H \).

For a class \( \alpha \in \text{Max}^c \), represented by \( H \), we define the integer \( m_\alpha := \# \{ \beta \in \text{Max}(2) \cup \text{Max}(3) \mid \exists(H_1 \in \beta) \cap H_1 \subset \Gamma_H \} \). Put \( \text{maxc} := \sum_{\alpha \in \text{Max}^c, m_\alpha \neq 0} (m_\alpha - 1) \).

We note that \( m_\alpha - 1 \) gives the number of edges \( e \) in \( T^c \) such that the stabilizer \( \Gamma_e \) contains a maximal cyclic subgroup contained in the conjugacy class \( \alpha \). Further, \( \text{maxc} \) is equal the number of edges in \( T^c \) such that \( \Gamma_e \) is not a \( p \)-group.

For a finite group \( G \) acting on some space \( A \), we will write \( \text{br}(G, A) \) for the number of branch points of the map \( A \to A/G \). Moreover, we will write \( \text{br}(G) \) for \( \text{br}(G, P^1(K)) \).

**Theorem 5.3.** Let \( \Gamma \subset \text{PGL}_2(K) \) be a finitely generated infinite discontinuous, indecomposable and ordinary group with \( \Omega/\Gamma \cong P^1_K \). We fix an embedding of \( T^c := T^c/\Gamma \) into \( T^c \). Then the number of branch points of \( \Gamma \) satisfies
(1) $\text{br}(\Gamma) = \sum_v \text{vertex of } T^c \text{ } \text{br}(\Gamma_v) = \sum_e \text{edge of } T^c \text{ } \text{br}(\Gamma_e)$.

(2) $\text{br}(\Gamma) = 3 \cdot \text{max}(3) + 2 \cdot \text{max}(2) - \text{maxp} - 2 \cdot \text{maxc}$.

(3) $\text{br}(\Gamma) = \text{max}(3) + \text{maxp} + 2$.

In particular, $\text{maxp} = 0$ if $p_K \neq 2, 3$ and $\text{br}(\Gamma) = 2 + \#\{v \text{ vertex of } T^c\}$ if $p_K = 0$ or $p_K \geq 5$.

**Proof.** We note that formulas (2) and (3) use only the structure of $\Gamma$. We will first show that formula (1) implies the other two formulas.

For $p_K = 0$ one has $\text{br}(\Gamma_v) = 3$. For $p_K = p > 0$ one has:

$$\text{br}(\Gamma_v) = 3 \quad \text{if and only if} \quad p \mid \#\Gamma_v,$$

$$\text{br}(\Gamma_v) = 2 \quad \text{if and only if} \quad p \mid \#\Gamma_v \text{ and } \Gamma_v \text{ is not a } p\text{-group},$$

$$\text{br}(\Gamma_v) = 1 \quad \text{if and only if} \quad \Gamma_v \text{ is a } p\text{-group (occurs only for } p = 2, 3).$$

Let $N_1$ (resp. $N_{e,1}$) be the number of vertices $v$ (resp. edges $e$) in $T^c$ such that $\Gamma_v$ (resp. $\Gamma_e$) is a $p\text{-group}$. Then $\text{br}(\Gamma) = 3 \cdot \text{max}(3) + 2 \cdot \text{max}(2) + N_1 - 2 \cdot \text{maxc} - N_{e,1}$. The vertices $v \in T^c$ with $\Gamma_v$ a $p\text{-group}$, not involved in $\mathcal{M}_{\text{maxp}}$, are extremal vertices. Hence $N_1 + \text{maxp} = N_{e,1}$. This implies (2). Formula (3) is obtained by noting that $\text{max}(3) + \text{max}(2) + N_1 - \text{maxc} - N_{e,1} = 1$, since $T^c$ is a tree.

Now we prove formula (1). By 5.1, the set of the ramification points of $\Omega \to \Omega/\Gamma$ is the disjoint union of the sets $\text{Ram}(\Gamma_v, X_v)$ with $v$ a vertex of $T^c$, consisting of the ramification points for the groups $\Gamma_v$ acting upon the affinoid set $X_v$ attached to the vertex $v$. Then $\text{br}(\Gamma) = \sum_v \text{br}(\Gamma_v, X_v)$, where the sum is taken over the vertices of $T^c$. We recall that $\psi_v$ is the reduction map $\mathcal{P}^1(K) \to \mathcal{P}(M_v \otimes k)$. The set $\text{Ram}(\Gamma_v, \mathcal{P}^1(K))$ of the ramification points of $\Gamma_v$ acting upon $\mathcal{P}^1(K)$ is the disjoint union of $\text{Ram}(\Gamma_v, X_v)$ and the sets $\text{Ram}(\Gamma_e, \text{red}_v^{-1} \text{red}_v(e))$, taking over the edges $e$ of $v$ in $T^c$, consisting of the ramification points of $\Gamma_e$ on the set $\text{red}_v^{-1} \text{red}_v(e)$. One obtains the formula

$$\text{br}(\Gamma_v, X_v) = \text{br}(\Gamma_v) = \sum_e \text{br}(\Gamma_e, \text{red}_v^{-1} \text{red}_v(e)),$$

where the sum is taken over all edges $e$ of $v$ in $T^c$. Let $H$ be a finite subgroup of $\Gamma$. Then formula (1) follows from

$$\sum_e \text{br}(\Gamma_e) = \sum_{e, v} \text{br}(\Gamma_e, \text{red}_v^{-1} \text{red}_v(e)),$$

where the first sum is taken over the edges $e$ of $T^c$ with $\Gamma_e = H$ and the second sum is taken over the same edges and the vertices $v_i$ in $T^c$ of those edges.

The verification of ($\ast$) for $p_K = 0$, follows easily from the assumption $\Gamma$ ordinary and $[M_1]$ is the separating lattice class if $v \cong D_4$ with $\ell$ a power of $p_k$.

For $p_K = p > 0$, all cases which do not involve a vertex $v$ with $\Gamma_v$ a $p\text{-group}$, follow by straightforward computation from the possibilities given by Corollary 3.11 and (e2), (e3).
of Definition 4.3. The remaining cases are $p = 2, 3$, $H \cong C_p$ and there exists a vertex $v$ of $T^c$ with $H \subset v$ and $v$ is a $p$-group. Let $e_i = \{v, v_i\}$, $i = 1, \ldots, s$, denote the edges of $v$ in $T^c$. Every $\Gamma_{v_i}$ is a dihedral group if $p = 2$ and is $\cong \text{PSL}_2(F_3)$ if $p = 3$. Now $\text{br}(\Gamma_{v_i}) = 1$, $\text{br}(\Gamma_{e_i}, \text{red}^{-1}_{v_i} \text{red}_{v_i}(e_i)) = 1$, $\text{br}(\Gamma_{e_i}, \text{red}^{-1}_{v_i} \text{red}_{v_i}(e_i)) = 0$ for all $i$. This proves (•). □

Remark 5.4. In [6, Theorem 1], Kato has given a list of finitely generated, infinite discontinuous subgroups $\Gamma \subset \text{PGL}_2(K)$ with $\Omega/\Gamma \cong \mathbb{P}^1_K$ with $\text{br}(\Gamma) = 3$ for the case $p_K = 0$. In particular, he shows that such groups only exist if $p_K \leq 5$. In the corollary below, we recover this part of his result.

Corollary 5.5. Suppose $p_K = 0$. Let $\Gamma \subset \text{PGL}_2(K)$ be a finitely generated, infinite, discontinuous group. Let $v$ denote any maximal finite subgroup of $\Gamma$. Assume that every indecomposable component of $\Gamma$ is ordinary. According to 4.2 this is equivalent to:

1. If $p_K > 2$ and $p_K \mid \#v$, then $v$ is a dihedral group, and
2. Suppose that $v$ is not an extremal vertex and $p \mid \#v$. Let $e_i = \{v_i, v\}$, $i = 1, \ldots, s$, denote the edges of $v$. The only possibilities are
   a. $\Gamma_v$ is not of Borel type, $s = 2$ and (after renumbering) $p \mid \#e_1$, $p \nmid \#e_2$, and $\Gamma_{v_1}$ is of Borel type.
   b. $p = 2, 3$, $\Gamma_v$ is a $p$-group and all $\Gamma_{e_i} \cong C_p$.
3. If $p$ divides the order of every vertex group of $T^c$ and (2b) holds for no vertex of $T^c$ which has at least two edges, then $T^c$ has two extremal vertices and at most four vertices.
4. Suppose that $\Gamma_v$ is of Borel type for every vertex $v$ of $T^c$. Then $\Gamma$ is isomorphic to $B(n_1, m) *_{C_m} B(n_2, m)$ with $m > 1$ and $n_1, n_2 > 1$.

Proof. (1) follows at once from part (4) of Theorem 4.11.
(2) Suppose that $Γ_v$ is not of Borel type. Since $[M_t]$ is separating and $br(Γ_v) = 2$, one has $s = 2$. Moreover, one of the branch points is wildly ramified and the other is tamely ramified, see Proposition 2.1. Hence $p | #Γ_{v_1}$ and $p | #Γ_{v_2}$. By 3.11 and (e2), (e3) of Definition 4.3, $Γ_{v_1}$ is of Borel type (and can even be a $p$-group).

Suppose that $Γ_v$ is of Borel type. Then $Γ_v = B(n, 1)$ and $p = 2, 3$ and all $Γ_v \cong C_p$, by 3.11.

(3) By assumption, every vertex $v$ of $T^c$, which is not extremal, satisfies $p | #Γ_v$, $Γ_v$ is a $p$-group and not of Borel type by (1). Hence $v$ has two edges and $T^c$ has two extremal vertices. Let $v_1, \ldots, v_s$ denote the vertices of $T^c$ and let $\{v_i, v_{i+1}\}, i = 1, \ldots, s - 1$, be the edges. Then $s \leq 4$ since $p | #Γ_{v_i}$ and $Γ_{v_i}$ is not a $p$-group and even not of Borel type for $i = 2, \ldots, s - 1$.

(4) Follows at once from 3.11 and (3) above. ∎

**Remark 5.7.** In [2, Proposition 4.6], Cornelissen et al. have determined all finitely generated discontinuous subgroups $Γ \subset \text{PGL}_2(K)$ with $br(Γ) = 2$ for the case where $p_K = p > 0$. The proposition below recovers part of their result.

**Proposition 5.8.** Let $p_K = p > 0$ and let $Γ$ be infinite. We assume that $br(Γ) = 2$. If $Γ$ is indecomposable, then we fix an embedding of $T^c := T^c/Γ$ into $T^c$. Then the following holds:

(i) If $Γ$ is indecomposable, then $p|#Γ_v$ for all vertices $v$ of $T^c$ and one of the following statements holds:
   (a) $Γ \cong B(n_1, m) * C_m B(n_2, m)$ with $m, n_1, n_2 > 1$.
   (b) There is precisely one vertex $v$ of $T^c$ such that $Γ_v$ is not of Borel type.
   (c) Precisely two vertices $v_1, v_2$ of $T^c$ have groups $Γ_v$ which are not of Borel type.

(ii) If $Γ$ is decomposable, then $Γ$ is a free amalgam of two $p$-groups.

**Proof.** (i) Suppose that $Γ$ is indecomposable, then $br(Γ) = \text{max}(3) + \text{max}p + 2$. Since $br(Γ) = 2$, one must have that $\text{max}(3) = \text{max}p = 0$. In particular, for every vertex $v$ of $T^c$, the order of $Γ_v$ is divisible by $p$ and if $v$ is not an extremal vertex then $Γ_v$ is not a $p$-group. One applies now Corollary 5.6.

(ii) $Γ$ is the free amalgam of two discontinuous groups $Γ_1, Γ_2 \subset Γ$ with $br(Γ_1) = br(Γ_2) = 1$. Both $Γ_1$ and $Γ_2$ are clearly finite $p$-groups. ∎

**Remark 5.9.** (Discontinuous groups which are not finitely generated). Let $p_K = p > 0$. There are many natural examples of discontinuous groups $Γ \subset \text{PGL}_2(K)$ such that $\Omega/Γ$ is isomorphic to $\text{PGL}_2|S$, where $S$ is a finite, non-empty set. E.g., let $A = \mathbb{F}_p[t] \subset K$ with $|t| > 1$. Then $Γ = \text{PGL}_2(A)$ is a discontinuous group such that $Ω/Γ \cong A^1_K$ and in particular $Γ$ is not finitely generated.

We will indicate how one can extend the results of Sections 4 and 5 to discontinuous groups $Γ$ as above. As before, one associates a tree $T$ to $Ω$. In particular, the definition of indecomposable group $Γ$ still makes sense and one can decompose $Γ$ as a free product $Γ_1 * \cdots * Γ_s$ of indecomposable groups of the same type. For an indecomposable $Γ$ as
above, one defines the tree $T^c$ as follows. The vertices $v$ of $T^c$ are the following subgroups of $\Gamma$:

(i) Maximal finite subgroups of $\Gamma$.
(ii) For $p = 2, 3$, subgroups $H \subset \Gamma$, $H \cong C_p$ satisfying (v2) of Definition 4.3.
(iii) Infinite stabilizers $\Gamma_y$ of points $y \in \mathbb{P}^1(K)$.

We note that an infinite stabilizer $\Gamma_y$ is isomorphic to a semi-direct product $N \rtimes A$, where $N \subset K$ is an infinite discontinuous group and $A \subset K^*$ is a finite group. An edge of the tree $T^c$ is a pair of vertices $v_1, v_2 \in T^c$ for which there exist vertices $v_1, v_2 \in T$ such that one of the following holds:

(i) The stabilizer $\Gamma_v$ is a non-cyclic subgroup of $\Gamma_v$ for $i = 1, 2$. Furthermore, the stabilizer of any vertex between $v_1$ and $v_2$ is contained in $\Gamma_{v_1}$ or $\Gamma_{v_2}$ and, moreover, $\Gamma_{v_1} \cap \Gamma_{v_2} \neq C_p$.
(ii) All elements of either $\Gamma_{v_1}$ or $\Gamma_{v_2}$ have order $p$ and $\Gamma_{v_1} \cap \Gamma_{v_2} \cong C_p$.

To the vertices of the tree $T^c$ one cannot always associate lattice classes. If the stabiliser $\Gamma_v$ of a vertex $v \in T^c$ is infinite, then one associates to $v$ the line $L_v$ in $V$ that corresponds to the unique point $y \in \mathbb{P}_K$ that is stabilized by the group $\Gamma_v$. If the group $\Gamma_v$ is finite, then one associates to $v$ a lattice class $[M_v]$ as before. The lattice classes contained in the convex hull of all lattices classes $[M_v]$ and all lines $L_v$ with $v \in T^c$ a vertex, define again a locally finite tree $T^c$.

We note that the tree $T^c$ is not locally finite. Indeed, if the stabilizer $\Gamma_v$ of a vertex $v \in T^c$ is infinite, then the vertex $v$ is contained in infinitely many edges. The tree $T^c / \Gamma$, however, is still finite. One can verify that the proofs of 4.9, 4.10, et cetera, remain valid, mutatis mutandis.

For the number of branch points $br(\Gamma)$ of $\Omega \to \Omega / \Gamma$ one finds again $br(\Gamma) = \sum_i br(\Gamma_i)$ holds. Let $E(\Gamma)$ denote the number of ends of the tree $T / \Gamma$. Then $E(\Gamma)$ is the cardinality of $S$. One has $E(\Gamma) = \sum_i E(\Gamma_i)$.

For an indecomposable $\Gamma$ we will give the formula for $br(\Gamma)$. We fix an embedding of $T^c / \Gamma$ into $T^c$ and let $T^{nB} \subset T^c / \Gamma$ consist of the vertices $v \in T^c / \Gamma$ such that $\Gamma_v$ is not contained in a Borel subgroup of $\text{PGL}_2(K)$. Then $T^{nB}$ is either empty or a disjoint union of subtrees of $T^c / \Gamma$.

Suppose that $T^{nB} = \emptyset$. Then $T^c / \Gamma$ consists of at most two vertices. If $T^c / \Gamma$ consists of a single vertex, then we do not define a group $\Gamma^{nB}$. In that case $\Gamma$ is an infinite discontinuous subgroup of a Borel subgroup of $\text{PGL}_2(K)$ and $E(\Gamma) = 1$. Moreover, $br(\Gamma) = 0$ if all elements of $\Gamma$ have order equal to $p$. Otherwise, $br(\Gamma) = 1$. If $T^c / \Gamma$ consist of a single edge $e$, then we put $\Gamma^{nB} := \Gamma_e$.

Suppose that $T^{nB} \neq \emptyset$. Then $\Gamma^{nB}$ denotes the group generated by the groups $\Gamma_v$ with $v \in T^{nB}$. For the $\Gamma_v$ for which $\Gamma_v \subset \Gamma$ is well-defined, one has $br(\Gamma) = br(\Gamma^{nB}) = E(\Gamma)$. We remark that $\Gamma^{nB}$ is finitely generated and indecomposable and thus $br(\Gamma^{nB})$ is given by Theorem 5.3.
References