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STOCHASTIC REALIZATION OF SPECTRAL DENSITY MATRICES
WHICH MAY POSSESS SYMMETRIES

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Abstract

A new procedure for stochastic realization of spectral density matrices as proposed in [6] is briefly reviewed. Afterwards we show how this approach may be used such that external symmetries are reflected in the stochastic realization.

1. A new procedure for stochastic realization and spectral factorization of spectral density matrices

In this section we will briefly review the procedure for stochastic realization as proposed in our paper [6]. We study the (weak) *stochastic realization* problem [3]:

Given a stationary zero mean gaussian p -dimensional vector process $y = \{y_t, t \in \mathbb{R}\}$ with a $p \times p$ rational spectral density matrix $\phi(s)$, construct a stationary zero mean Gauss-Markov n -dimensional vector process $x = \{x_t, t \in \mathbb{R}\}$ and a $p \times n$ matrix C such that the spectral density of Cx equals $\phi(s)$. We call (x, C) a *stochastic realization* of $\phi(s)$.

The above problem of constructing x and C is equivalent to the problem of generating $\phi(s)$ by passing white noise through a linear system (the shaping filter). Then we have to construct an $n \times n$ matrix A , an $n \times n$ matrix $Q = Q^T > 0$, and an $p \times n$ matrix C such that the stochastic differential equation

$$dx = Ax dt + B dw, \quad x(0) = x_0 \quad (1)$$

with A asymptotically stable, x_0 zero mean gaussian, $E\{x_0 x_0^T\} = Q$, $AQ + QA^T = -BB^T$ and $w = \{w_t, t \in [0, \infty)\}$ a normalized Wiener process independent of x_0 , yields a vector process Cx with spectral density equal to $\phi(s)$. We also call (A, Q, C) a stochastic realization of $\phi(s)$. Furthermore we require n , the dimension of the state space of a stochastic realization, to be as small as possible. If n is such we call (x, C) , or (A, Q, C) , a *minimal stochastic realization* of $\phi(s)$.

It is easy to see the spectral density of Cx with x generated by (1) is equal to $C(Is - A)^{-1}BB^T(-Is - A^T)^{-1}C^T$. Hence the stochastic realization problem for $\phi(s)$ is also equivalent to finding a transfer matrix $W(s)$, analytic in $\text{Re } s > 0$, such that $\phi(s) = W(s)W^T(-s)$. This is the *spectral factorization* problem.

The usual procedure for stochastic realization starts with the *partial fraction expansion* $\phi(s) = Z(s) + Z^T(-s)$, with $Z(s)$ analytic in $\text{Re } s > 0$. Then one continues by applying the Kalman-Yakubovich-Popov lemma to a minimal realization of the (positive real) transfer matrix $Z(s)$. In a certain sense this approach is natural since $Z(s)$ equals the (two-sided) Laplace transform of the autocorrelation matrix of a process y with spectral density $\phi(s)$. (Hence if the process is not given by $\phi(s)$ but by its autocorrelation matrix this is certainly the easiest way). On the other hand the partial fraction expansion requires the factorization of (high order) polynomials, which is a nonlinear problem of the same

level of difficulty as solving an algebraic Riccati equation. Motivated by this we have proposed in [6] another approach to stochastic realization of $\phi(s)$, which avoids the partial fraction expansion, and seems also theoretically of interest. However, which of the two approaches is algorithmically most advantageous remains a matter of study.

It is well-known that a rational spectral density matrix $\phi(s)$ of a process with integrable autocorrelation function is characterized by the following properties

- (i) $\phi(s) = \phi^T(-s)$ (this is usually called *para-hermitian*)
- (ii) $\phi(s)$ has no poles on the imaginary axis
- (iii) $\phi(i\omega) \geq 0, \forall \omega \in \mathbb{R}$
- (iv) $\lim_{s \rightarrow \infty} \phi(s) = 0$

Our approach is based on viewing $\phi(s)$ as a *transfer matrix* and the following key observations.

1. By i) $\phi(s)$ is a *Hamiltonian transfer matrix* [2,4]. It follows from the state space isomorphism theorem that a minimal realization (A, B, C) of $\phi(s)$ (i.e. $\phi(s) = C(Is - A)^{-1}B$) will satisfy.

$$\bar{A}^T J + J \bar{A} = 0 \quad (2)$$

$$\bar{B}^T J = \bar{C} \quad (3)$$

for some unique nonsingular matrix J satisfying $J = -J^T$. This already implies that the dimension of the state space of such a minimal realization is even, say $2n$.

2. By iii) the transfer matrix $\phi(s)$ is *passive* (for any input-output pair which corresponds to a closed path in state space, the L_2 inner product of input and output is nonnegative [9]). In fact iii) implies the existence of a $\Sigma = \Sigma^T, \det \Sigma \neq 0$, such that

$$\bar{A}^T \Sigma + \Sigma \bar{A} \leq 0 \quad (4)$$

$$\bar{B}^T \Sigma = \bar{C} \quad (5)$$

3. From (2) and (3) it follows that if Σ satisfies (4) and (5), then so does $J\Sigma^{-1}J$. Since the set of Σ 's satisfying (4) and (5) is convex and compact, and the map $\Sigma \rightarrow J\Sigma^{-1}J$ on the space of nonsingular symmetric matrices is continuous, we obtain by an application of Brouwer's fixed point theorem that there also exist Σ satisfying (4) and (5) and

$$\Sigma = J\Sigma^{-1}J \quad (6)$$

4. Now consider a solution Σ of (4), (5), (6) and the unique solution J of (2), (3). It can be proved that there exist bases of $\mathbb{R}^{2n}$ such that

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \quad (7)$$

In such a basis it follows from (2), (3), (4), (5) that $\bar{A}$, $\bar{B}$ and $\bar{C}$ have the form

$$\bar{A} = \begin{bmatrix} F & -P \\ -R & -F^T \end{bmatrix}; \quad \bar{B} = \begin{bmatrix} 0 \\ H^T \end{bmatrix}, \quad \bar{C} = (H \ 0) \quad (8)$$

with $P = P^T \geq 0$, $R = R^T \geq 0$. Moreover it follows from controllability of $(\bar{A}, \bar{B})$ that (F, P) is controllable.

We now have the basic ingredients for a stochastic realization of $\phi(s)$. Consider the Riccati equation associated to A in (8):

$$F^T K + KF - K^T P K + R = 0 \quad (9)$$

It follows from observation 4, and property ii) of $\phi(s)$ that (9) has a maximal symmetric solution K^+ and a minimal symmetric solution K^- , such that $K^+ - K^-$ is positive definite. By application of the symplectic transformation $K = \begin{pmatrix} -K^+ & 0 \\ 0 & I \end{pmatrix}$ to $\bar{A}, \bar{B}, \bar{C}$ we obtain

$$\bar{K} \bar{A} \bar{K}^{-1} = \begin{bmatrix} F - PK^+ & -P \\ 0 & -(F - PK^+)^T \end{bmatrix}, \quad \bar{K} \bar{B} = \begin{bmatrix} 0 \\ H^T \end{bmatrix}; \quad \bar{C} \bar{K}^{-1} = (H \ 0) \quad (10)$$

Now define $A := F - PK^+$, $C := H$, $Q = (K^+ - K^-)^{-1}$ and write $P = BB^T$. Then it is easy to see that

$$\bar{C}(I_s - \bar{A})^{-1} \bar{B} = C(I_s - A)^{-1} B B^T (-I_s - A^T)^{-1} C^T \quad (11)$$

and that the stochastic system

$$dx = Ax dt + B dw, \quad x(0) = x_0 \quad (12)$$

$$\tilde{y} = Cx$$

with $E\{x_0 x_0^T\} = Q$, generates a process $\tilde{y}$ with spectral density $\phi(s)$.

With respect to the computational side of the above procedure we make the following remarks (see [6] for details). Having fixed a Σ satisfying (4), (5), there exists an algorithm to compute a Σ satisfying (4), (5) and (6). The basis referred to in observation 3 can be directly computed from J and Σ . In fact $\text{Ker}(I - J^{-1}\Sigma)$ is the upper half of the state space, while $\text{Ker}(I + J^{-1}\Sigma)$ is the lower half. What remains to be done is the solution of the Riccati equation (9), especially K^+ . We notice that after having fixed a Σ satisfying (4), (5), (6) the stochastic realization (12) is uniquely determined up to a state space transformation. The possible freedom in choosing a Σ satisfying (4), (5), (6) corresponds to the partitioning of the transmission zeros of $\phi(s)$ into the zeros of $W(s)$ and $W^T(-s)$ or, equivalently, to the freedom in the choice of the covariance Q of the shaping filter.

2. Symmetries

Let y be again a zero mean gaussian vector process with spectral density $\phi(s)$. In this section we will show how we can adapt the algorithm for stochastic realization of $\phi(s)$ given in the previous section in such a way that symmetry properties of the stochastic process y are reflected in its state space realization (x, C) . We call two p -dimensional vector processes y and y' (stochastically) equivalent, denoted by $y \sim y'$, if the probability distributions of the vectors $(y_{t_1}, y_{t_2}, \dots, y_{t_k})$ and $(y'_{t_1}, y'_{t_2}, y'_{t_k})$ are equal for all choices of $t_1, t_2, \dots, t_k$. Equivalently, $y \sim y'$ if y and y' have the same spectral densities or autocorrelation functions.

Definition 1

Let $y := \{y_t, t \in \mathbb{R}\}$ be a p -dimensional vector process. An (external) symmetry for y is a nonsingular linear map $U: \mathbb{R}^p \rightarrow \mathbb{R}^p$ such that $Uy \sim y$. Let (x, C) be an n -dimensional realization of y . A symmetry for the com-

bined process (x, y) is a pair (S, U) , with $U: \mathbb{R}^p \rightarrow \mathbb{R}^p$ and $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ nonsingular linear maps, such that $(Sx, Uy) \sim (x, y)$. We will call such an $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ (hence $Sx \sim x$) an (internal) symmetry.

First we will treat the case that y is given by an autocorrelation function $R(t) = E\{y(t)y^T(0)\}$.

Proposition 2

Let y be a p -dimensional vector process with autocorrelation function R . Let U be an (external) symmetry for y . Then there exists a minimal realization (x, C) of R and, given this realization, a uniquely determined S such that (S, U) is a symmetry for (x, y) . This implies that we can choose the covariance Q of a minimal realization (A, Q, C) such that

$$SAS^{-1} = A, \quad SQS^T = Q, \quad CS^{-1} = UC. \quad (13)$$

Proof: From $y \sim Uy$ it follows that

$$R(t) = E\{y(t)y^T(0)\} = E\{Uy(t)y^T(0)U^T\} = UR(t)U^T.$$

Let (F, G, H) be a minimal n -dimensional realization of R , i.e. $R(t) = He^{Ft}G$. Then by the state space isomorphism theorem, $R(t) = UR(t)U^T$ implies the existence of a nonsingular $S: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$SFS^{-1} = F, \quad SG = GU^T, \quad HS^{-1} = UH \quad (14)$$

The set of solution $Q > 0$ of

$$FQ + QF^T \leq 0, \quad QH^T = G \quad (15)$$

is convex and compact. Moreover it follows from (14) that if Q is a solution of (15), so is SQS^T . Therefore since the map $Q \rightarrow SQS^T$ (viewed as a map on the symmetric matrices) is continuous, there exists a $Q > 0$ satisfying (15) such that $Q = SQS^T$. Take $A = F$, $C = H$, and we have obtained a minimal realization (A, Q, C) satisfying (13). $\square$

Therefore we see that the general statement that an external symmetry for y implies the existence of an internal symmetry for a certain class of minimal realizations of y , is easily proved if we use the autocorrelation function of y . We will now translate the property of symmetry to spectral density matrices and show how a minimal stochastic realization possessing a symmetry can be obtained using the algorithm proposed in Section 1.

Theorem 3

Let y be a p -dimensional vector process given by a spectral density matrix $\phi(s)$. U is a symmetry for y iff $U\phi(s)U^T = \phi(s)$. Then there exists a minimal stochastic realization (x, C) of $\phi(s)$ and a (unique) S such that (S, U) is a symmetry for (x, y) . Equivalently, if this realization is given by (A, Q, C) , equations (13) hold.

Proof. It can be easily seen that

$$\{y \sim Uy\} \Leftrightarrow \{R(t) = UR(t)U^T\} \Leftrightarrow \{\phi(s) = U\phi(s)U^T\}.$$

Let now $(\bar{A}, \bar{B}, \bar{C})$ be a minimal $2n$ -dimensional realization of $\phi(s)$, satisfying (see (2), (3))

$$\bar{A}^T J + J \bar{A} = 0, \quad \bar{B}^T J = \bar{C}, \quad \text{with } J = -J^T \quad (16)$$

Since $\phi(s) = U\phi(s)U^T$, the state space isomorphism theorem implies the existence of a unique nonsingular $S: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that

$$\bar{S} \bar{A} \bar{S}^{-1} = \bar{A}, \quad \bar{S} \bar{B} = \bar{B} U^T, \quad \bar{C} \bar{S}^{-1} = U \bar{C} \quad (17)$$

Moreover, $\bar{S}$ is necessarily symplectic, i.e. $\bar{S}^T J \bar{S} = J$. Indeed, it follows from (16) and (17) that for all $r \geq 0$ $J \bar{S} A^r \bar{B} = J \bar{A}^r \bar{B} = (-1)^r (\bar{A}^T)^r J \bar{B} U^T = -(-1)^r (\bar{A}^T)^r \bar{C} U^T = -(-1)^r (\bar{A}^T)^r (\bar{S}^T)^{-1} \bar{C}^T = -(-1)^r (\bar{S}^T)^{-1} (\bar{A}^T)^r J J^{-1} \bar{C}^T = (\bar{S}^T)^{-1} J \bar{A}^r \bar{B}$

and therefore, by controllability of $(\bar{A}, \bar{B})$, we see that $\bar{J}\bar{S} = (\bar{S}^T)^{-1}\bar{J}$, or $\bar{S}^T\bar{J}\bar{S} = \bar{J}$. It can now be checked (using (17)) that if $\Sigma = \Sigma^T$ is a solution of

$$\bar{A}^T\Sigma + \Sigma\bar{A} < 0, \quad \bar{B}^T\Sigma = \bar{C} \quad (18)$$

then so is $\bar{S}^T\Sigma\bar{S}$. From the passivity of $\phi(s)$ it follows that the set of Σ 's, satisfying $\Sigma = \Sigma^T$, $\det \Sigma \neq 0$ and (18), is nonempty. Moreover this set is convex and compact. Therefore, since the map $\Sigma \rightarrow \bar{S}^T\Sigma\bar{S}$ (with $\Sigma = \Sigma^T$, $\det \Sigma \neq 0$) is continuous there exists such a Σ satisfying (18) and $\Sigma = \bar{S}^T\Sigma\bar{S}$. However the set of Σ 's satisfying (18) and $\Sigma = \bar{S}^T\Sigma\bar{S}$ is again convex and compact. Furthermore if Σ is a solution of (18) and $\Sigma = \bar{S}^T\Sigma\bar{S}$, so is $\bar{J}\Sigma^{-1}\bar{J}$ (this follows from (16) and $\bar{S}^T\bar{J}\bar{S} = \bar{J}$). Hence, since the map $\Sigma \rightarrow \bar{J}\Sigma^{-1}\bar{J}$, with $\Sigma = \Sigma^T$ and $\det \Sigma \neq 0$, is continuous, there exists a $\Sigma = \Sigma^T$, $\det \Sigma \neq 0$, satisfying (18) as well as $\Sigma = \bar{S}^T\Sigma\bar{S}$ and $\Sigma = \bar{J}\Sigma^{-1}\bar{J}$.

Given such a Σ , there exists a basis (see equation (7)) of $\mathbb{R}^{2n}$ such that

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}.$$

From $\bar{S}^T\Sigma\bar{S} = \Sigma$ and $\bar{S}^T\bar{J}\bar{S} = J$ it follows that $\bar{S}J^{-1}\Sigma = J^{-1}\Sigma\bar{S}$. Therefore $\bar{S}$ commutes with $J^{-1}\Sigma$ and hence leaves the positive and negative eigenspaces of $J^{-1}\Sigma$ (corresponding respectively to the n eigenvalues of $J^{-1}\Sigma$ which are $+1$ and the n eigenvalues which are -1) invariant. Since $\bar{S}$ is symplectic, we obtain that in this basis $\bar{S}$ has the form

$$\bar{S} = \begin{bmatrix} S & 0 \\ 0 & (S^T)^{-1} \end{bmatrix}, \quad \text{with } S \text{ nonsingular.} \quad (19)$$

Now $\bar{A}, \bar{B}$ and $\bar{C}$ can be written in this basis as in equation (8). Then from (17) and (19) it follows that

$$SFS^{-1} = F, \quad SPS^T = P, \quad S^T RS = R, \quad HS^{-1} = UH \quad (20)$$

We now look at the Riccati equation associated to $\bar{A}$

$$F^T K + KF - K^T PK + R = 0 \quad (21)$$

We state separately

Lemma 4

Let K^+ and K^- be the maximal and minimal solution of (21), and let S satisfy (20). Then $S^T K^+ S = K^+$ and $S^T K^- S = K^-$.

Proof

Consider the usual partial ordering on symmetric matrices: $K_1 > K_2 \iff K_1 - K_2$ is nonnegative definite. Under this ordering the set K of symmetric solutions of (18) is partially ordered with unique maximum K^+ and unique minimum K^- ([8]). From (20) it follows that if $K \in K$ then also $S^T K S \in K$. Furthermore the map $K \rightarrow S^T K S$ preserves the ordering of K . Hence the maximum K^+ and the minimum K^- are invariant under this map. $\square$

Proof of Theorem 3 continued:

Let us again denote $A := F - PK^+$, $C := H$. Then it follows from Lemma 3 and (20) that $SAS^{-1} = A$ and $CS^{-1} = UC$. Moreover it follows from Lemma 3 that $S^T(K^+ - K^-)S = K^+ - K^-$, and hence, since $Q = (K^+ - K^-)^{-1}$, that $SQS^T = Q$. $\square$

Remark

An external symmetry U for a spectral density $\phi(s)$ can be also interpreted as a *Hamiltonian symmetry* (as defined in [5]) for the Hamiltonian transfer matrix $\phi(s)$. Notice that $\bar{S}$ is an internal Hamiltonian symmetry for $(\bar{A}, \bar{B}, \bar{C})$, and that S is a symmetry for the Riccati equation (21) (see also [7]).

It immediately follows from the proof of Theorem 3 how we can adapt the algorithm of Section 1 to obtain a minimal stochastic realization (A, Q, C) exhibiting an internal symmetry S if $\phi(s)$ has an external symmetry U . In fact there exists a unique $\bar{S}$ satisfying (17). Then we have to find a Σ satisfying (4), (5), (6) and $\Sigma = \bar{S}^T\Sigma\bar{S}$. Taking a basis as in (7) we obtain

$\bar{S} = \begin{pmatrix} S & 0 \\ 0 & (S^T)^{-1} \end{pmatrix}$, with S the resulting internal symmetry. From a computational point of view the easiest way is first to compute a solution Σ of (4), (5) and $\Sigma = \bar{S}^T\Sigma\bar{S}$, and then to take this solution as starting point for the algorithm defined in [6, equation (16)] to obtain a Σ also satisfying (6).

Instead of dealing with a *single* external symmetry it is of interest to consider a *group* of external symmetries U . It then follows from the state space isomorphism theorem that there exists an isomorphic group of symmetries $\bar{S}$ for $(\bar{A}, \bar{B}, \bar{C})$. In order to obtain a group of internal symmetries S for a stochastic realization (x, C) of $\phi(s)$ it is necessary and sufficient that there exists a Σ satisfying (4), (5) and $\bar{S}^T\Sigma\bar{S} = \Sigma$ for all symmetries $\bar{S}$. This is a very interesting problem. If the group G of $\bar{S}$ is abelian, it is clear that such a Σ exists. For take an arbitrary $\bar{S} \in G$. Then the set $\Sigma_{\bar{S}}$ of nonsingular Σ satisfying (4), (5) and $\bar{S}^T\Sigma\bar{S} = \Sigma$ is nonempty, and convex and compact. Take another symmetry $S_1 \in G$, and consider the map $\Sigma \rightarrow S_1^T\Sigma S_1$ for $\Sigma \in \Sigma_{\bar{S}}$. Since G is abelian $\bar{S}^T(S_1^T\Sigma S_1)\bar{S} = \bar{S}^T(\bar{S}^T\Sigma\bar{S})S_1 = \bar{S}_1^T\Sigma\bar{S}_1$ and it easily

follows that $\bar{S}_1^T\Sigma\bar{S}_1$ is again in $\Sigma_{\bar{S}}$. Hence by application of Brouwer's fixed point theorem to $\Sigma_{\bar{S}}$ and this map, we conclude that the set of Σ satisfying (4), (5) and $\bar{S}^T\Sigma\bar{S} = \Sigma = S_1^T\Sigma S_1$ is nonempty.

By continuing this process for a set of generators of G we finally obtain a Σ satisfying (4), (5) and $\bar{S}^T\Sigma\bar{S} = \Sigma$ for all $\bar{S} \in G$. However this proof does not work for non-abelian groups, and therefore it remains an open problem if for a non-abelian group of symmetries for $\phi(s)$ there exists a minimal stochastic realization of $\phi(s)$ displaying an isomorphic group of internal symmetries.

Now we turn attention to a more special kind of symmetry, namely *time-reversibility*.

Definition 5

Let $y := \{y_t, t \in \mathbb{R}\}$ be a p -dimensional vector process. We call y *time-reversible* if $y \sim Ry$ (with R defined by $(Ry)(t) := y(-t)$).

If the stochastic process y is given by its autocorrelation function R , it can be easily seen that time-reversibility is equivalent to $R = R^T$. The following result is then obtained in [10] (see also [1]).

Theorem 6

Let R be an autocorrelation function satisfying $R = R^T$. Then R has a minimal stochastic realization (A, Q, C) satisfying

$$A = MA^T, \quad CM = C, \quad Q = I \quad (22)$$

where M denotes a signature matrix (i.e. a diagonal matrix with elements $+1$ or -1 on the diagonal).

The above theorem hence effectively says that there exists a realization (x, C) of y such that $(x, y) \sim R(Mx, y)$.

If y is given by its spectral density $\phi(s)$, then time-reversibility corresponds to $\phi(s) = \phi(-s)$ (or, since $\phi(s) = \phi^*(-s)$, to $\phi(s) = \phi^T(s)$). In the following theorem we give a constructive proof how by using the approach sketched in Section 1 we can also obtain a stochastic realization satisfying (22).

Theorem 7

Let $\phi(s)$ be a spectral density satisfying $\phi(s) = \phi(-s)$. Then there exists a minimal stochastic realization

(A,Q,C) of $\phi(s)$ satisfying $A = MA^T M$, $CM = C$, $Q = I$, with M a signature matrix.

Proof

Let $(\bar{A}, \bar{B}, \bar{C})$ be a minimal $2n$ -dimensional realization of $\phi(s)$, satisfying $\bar{A}^T J + J \bar{A} = 0$, $\bar{B}^T J = \bar{C}$ (see (2.3)). Since $\phi(s) = \phi(-s)$, also $(-\bar{A}, -\bar{B}, \bar{C})$ is a minimal realization. Hence by the state space isomorphism theorem, there exists a unique $\bar{S} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ such that

$$\bar{S} \bar{A} \bar{S}^{-1} = -\bar{A}, \bar{S} \bar{B} = -\bar{B}, \bar{C} \bar{S}^{-1} = \bar{C}. \quad (23)$$

Furthermore, in [4] it is proved this $\bar{S}$ also satisfies

$$\bar{S}^T J \bar{S} = -J, \bar{S}^2 = I. \quad (24)$$

Now look at the solution $\Sigma = \Sigma^T$ of

$$\bar{A}^T \Sigma + \Sigma \bar{A} < 0, \quad \bar{B}^T \Sigma = \bar{C} \quad (25)$$

If Σ is a solution of (25), so is $-\bar{S}^T \Sigma \bar{S}$ by (23). Then since the map $\Sigma \rightarrow -\bar{S}^T \Sigma \bar{S}$ is continuous, and the solution set of (25) is convex and compact there exists a $\Sigma = \Sigma^T$ satisfying (25), such that $\Sigma = -\bar{S}^T \Sigma \bar{S}$. The set of Σ 's satisfying (25) and $\Sigma = -\bar{S}^T \Sigma \bar{S}$ is again convex and compact. Moreover if Σ satisfies (25) and $\Sigma = -\bar{S}^T \Sigma \bar{S}$, so does $J \Sigma^{-1} J$, by $\bar{S}^T J \bar{S} = -J$ and (2), (3). Since the map $\Sigma \rightarrow J \Sigma^{-1} J$ (viewed as a map on the space of symmetric nonsingular matrices), is continuous, this implies the existence of a $\Sigma = \Sigma^T$, $\det \Sigma \neq 0$, satisfying (25) and

$$\Sigma = -S^T \Sigma S, \quad \Sigma = J \Sigma^{-1} J \quad (26)$$

From $\Sigma = -S^T \Sigma S$ and $J = -\bar{S}^T J \bar{S}$ it follows that $\bar{S} J^{-1} \Sigma = J^{-1} \Sigma \bar{S}$. Therefore $\bar{S}$ leaves the positive and negative eigenspaces of $J^{-1} \Sigma$ invariant. Hence if we take a basis as in (7), $\bar{S} = \begin{pmatrix} S & 0 \\ 0 & -S^T \end{pmatrix}$, with $S^2 = I$. In this basis $\bar{A}$, $\bar{B}$ and $\bar{C}$ are as in (8). Then (23) implies

$$SFS = -F, SPS^T = P, S^T RS = R, HS = H \quad (27)$$

For the Riccati equation (9) associated to $\bar{A}$ we obtain

Lemma 7

Let K^+ and K^- be the maximal and minimal solution of (9), and let S satisfy (27). Then $S^T K^+ S = -K^-$ and $S^T K^- S = -K^+$.

Proof

If K satisfies (9), then so does $-S^T K S$, by (27). Furthermore the map $K \rightarrow -S^T K S$ reverses the partial ordering on the solution set of (9). Therefore the maximum K^+ is mapped onto the minimum and vice versa.

□

Proof of Theorem 6 continued

Define $F^+ = F - PK^+$, $F^- = F - PK^-$. It follows from Lemma 7 and (27) that

$$SF^+ = S(F - PK^+) = -FS + PK^- = -F^-S \quad (28)$$

and, since K^+ and K^- are solutions of (9), that:

$$(F^+)^T (K^+ - K^-) + (K^+ - K^-) F^- = 0 \quad (29)$$

(see [8]). Since $K^+ - K^- > 0$ we can find a nonsingular N such that $N^T (K^+ - K^-) N = I$. Then apply the basis-transformation $\bar{N} = \begin{pmatrix} N & 0 \\ 0 & (N^T)^{-1} \end{pmatrix}$. The Riccati equation associated to $\bar{N} \bar{A} \bar{N}^{-1}$ has maximal solution $N^T K^+ N$ and minimal solution $N^T K^- N$. Therefore in this new basis $Q^{-1} = N^T (K^+ - K^-) N = I$. From Lemma 7 it follows that $S^T (K^+ - K^-) S = K^+ - K^-$, and hence in this basis $S^T S = I$, or, since $S^2 = I$, $S = S^T$. Therefore there exists an orthogonal O such that $OSO^T = M$, with M a signature matrix. Apply the basis transformation $\bar{O} = \begin{pmatrix} O & 0 \\ 0 & (O^T)^{-1} \end{pmatrix}$,

This transforms $\bar{S} = \begin{pmatrix} S & 0 \\ 0 & -S^T \end{pmatrix}$ into $\begin{pmatrix} M & 0 \\ 0 & -M^T \end{pmatrix}$. Since O is orthogonal it leaves $K^+ - K^-$ invariant and therefore Q remains the identity. Finally write in this new basis $\bar{A}$, $\bar{B}$, $\bar{C}$ as in (8).

Then define $A = F - PK^+$, $C = H$, $Q = I$. Since $K^+ - K^- = I$ (29) yields $(F^+)^T + F^- = 0$, and therefore (28) is equivalent to (with $S = M$), $MF^+ = (F^+)^T M$, or (with $F^+ = A$), $A = MA^T M$. From (27) it follows (with $S = M$, $H = C$) that $CM = C$, and therefore the minimal realization (A, Q, C) satisfies (22).

□

Again we can easily adapt the algorithm of Section 1 if $\phi(s) = \phi(-s)$. Indeed there exists a unique $\bar{S}$ satisfying (23). Then we have to find a Σ satisfying (4), (5), (6) and $\Sigma = \bar{S}^T \Sigma \bar{S}$. In the basis corresponding to Σ as in (7) we have $\bar{S} = \begin{pmatrix} S & 0 \\ 0 & -S^T \end{pmatrix}$. Finally we have to apply a transformation which changes $K^+ - K^-$ (maximal and minimal solutions of (9)) into I , and S into a signature matrix M .

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