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The Painlevé VI tau-function of Kerr-AdS5

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The Painlevé VI τ -function of Kerr-AdS₅

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Chapter 1

Introduction

This thesis is a collection and adaptation of the original work portrayed in [27–29]. To get an appreciation for the reasons behind this dissertation, we will first illustrate the relevant role of the quasi-normal modes in black holes physics, and then present the isomonodromy method to treat linear perturbations of matter fields propagating in a five dimensional Kerr-AdS black hole. Yet, the method applies to different space-times, as well as other physical systems.

Quasi-normal modes in General Relativity

The last decade has produced stunning results for General Relativity (GR). After the LIGO-Virgo collaboration reported the first direct detection of gravitational waves and the first direct observation of a binary black hole merger GW150914 [1, 2], we have seen for the first¹ time ever the Event Horizon Telescope (EHT) image of the supermassive black hole at the center of Messier 87 (M87) galaxy [7–12].

The gravitational-wave events detected by LIGO are characterized by three phases: (1) inspiral, (2) merger, and (3) ringdown. During most of the inspiral, the distance between the binary components is large so that they can be treated in the post-Newtonian approximation, whereas numerical relativity simulations must be performed to generate the waveform expected through the coalescence,

¹This is one consequence of breakthrough discoveries, so many “first” in the first paragraph.

i.e., in the merger phase. See Fig. 1.1 for a reconstruction of the waveform of the astrophysical signal.

Figure 1.1: The Coherent WaveBurst (cWB) algorithm searches for gravitational wave transients and provides a first estimation of the event parameters and sky location. The pipeline identifies coincident events in data from the two LIGO detectors (L1 and H1) and reconstructs the gravitational wave signal (red) associated with these events. In this search, BayesWave (BW) pipeline distinguishes GW signals from glitches in the detectors and is run as a follow-up analysis for candidate events first identified by cWB. On the y-axis, Sigma is a measure of the amplitude in terms of the number of noise standard deviations. Adapted from [3].

In the ringdown (post-merger) phase, the remnant black hole (BH) resulting from the merger of the binary system is initially highly perturbed, and with the emission of gravitational waves it relaxes to a final stationary Kerr black hole configuration. The dominant part of the gravitational waves emitted as the black hole settles down can be described as a sum over a countably infinite set of damped sinusoids, each characterized by an amplitude, phase, frequency and damping time.

Gravitational waves detectors respond to a linear combination of the radiation in the two polarization modes of the incident gravitational waves. In terms of the transverse traceless gauge metric perturbation h_{ij} the observable

$h(t)$ may be written in the form

$$h(t) \simeq \text{Re} \left[\sum_{n,\ell,m} A_{n\ell m} e^{-i(\omega_{n\ell m} t + \phi_{n\ell m})} \right] \quad (1.1)$$

where the summation indices characterize the particular mode. For Kerr the symmetry is axisymmetric and the appropriate decomposition of the perturbation is given by the spin-2 spheroidal harmonics [157]. The amplitudes $A_{n\ell m}$ and phases $\phi_{n\ell m}$ depend on the initial conditions and the relative orientation of the detector and the source [23, 33, 63]; however, the complex frequency $\omega_{n\ell m}$ depends only on the intrinsic parameters characterizing the black hole: i.e., its mass M and angular momentum aM^2 .

Each mode has a complex frequency $\omega_{n\ell m} = 2\pi f_{n\ell m} + i(1/\tau_{n\ell m})$, whose real part is related to the oscillation frequency and the imaginary part gives the inverse of the damping time, that is uniquely determined by the mass and angular momentum of the black hole. These complex frequencies form the so-called quasi-normal modes (QNMs).

QNMs are therefore labeled by a set of discrete numbers related with the isometries of the space-time: the spin-weighted spheroidal indices (ℓ, m) and an overtone index n , which sorts the modes by their decay time. For a more detailed discussion on quasi-normal modes in gravitational physics, the reader is referred to [32, 107, 110, 135].

For instance, by measuring the natural frequencies of a Schwarzschild BH one can infer its mass, since this solution is fully described by one parameter. The first gravitational quasi-normal mode frequency that corresponds to the fundamental ($n = 0$) quadrupole ($\ell = 2$) mode is $M\omega = 0.37367 - 0.08896i$, measured in units of the black hole mass M . Thus, a one solar mass black hole has a ringing frequency $f = 12 \text{ kHz}^2$, and a damping timescale, due to gravitational wave emission, of $\tau = 3.74 \times 10^{-4} \text{ s}$.

Four-dimensional Kerr black holes depend only on two parameters, i.e., their mass and their spin angular momentum, and we can estimate both the mass and spin of the final state by computing the leading (least-damped) gravitational mode with frequency f_{022} and damping time τ_{022} , whereas further subleading modes provide multiple independent consistency checks of the Kerr metric,

²The frequency is given in geometrical units ($c = G = 1$) and the conversion factor to Hertz is $(c^3/(GM_\odot)) \times (M_\odot/M)$.

since the QNMs are generically different in extensions of GR. In addition, the detection of the first overtone ($n = 1$) in the ringdown phase can shed light on black hole alternatives for very compact objects and a way to distinguish them from a Kerr black hole [109]. We refer to [25] for a roadmap over the challenges of gravitational waves detection and General Relativity.

In fact, behind this gravitational context we can find an intertwining between powerful numerical procedures that yield an accurately numerics and analytical solutions based on the presence of symmetries in the physical system under investigation.

Black Holes in Higher Dimensions

While the relevance of the recent events mentioned above is undeniable, it is also true that classical General Relativity in more than four dimensions has been established as an interesting laboratory to study extensions and underlying mathematical structure of Einstein's theory and new black-holes solutions. Black holes in higher dimensions can shed light on which properties such as uniqueness, spherical topology, dynamical stability, and the laws of black hole mechanics, are peculiar to four-dimensions and which of them are universal properties of the theory (the dimension independent ones). See [62, 93] and references therein.

One of the most attractive motivations for studying higher-dimensional black holes comes from string theory, which inevitably requires more than four-dimensions. Successful examples include: i) a microscopic description of the black hole entropy [152], ii) the AdS/CFT correspondence [6]. Here, 'AdS' stands for Anti-de Sitter space and 'CFT' for conformal field theory. Within this framework black hole solutions in an asymptotically AdS describe thermal states of the corresponding CFT at the boundary, with the temperature given by the Hawking temperature of the black hole [160, 161]. In addition, a linear perturbation will induce a small deviation from the equilibrium, and the decay of the perturbation corresponds to the return to thermal equilibrium. Thus one can compute the relaxation times in the strongly coupled CFT by equating them to the imaginary part of the eigenfrequencies [34, 94, 138]. Namely, the quasi-normal modes of the fluctuation are related to the poles of the retarded Green's function on the conformal side, providing insights on the transport coefficients and on the quasi-particle spectrum. There have been many studies

of QNMs for various types of perturbations on several background solutions in asymptotically AdS, and we refer to [32] for further discussions.

The gauge/gravity duality has flourished as a new framework to construct phenomenological gravity duals in extra-dimensions aimed to predict thermodynamic and transport properties of strongly coupled gauge theories in the large N limit [144]. For instance, an interesting realization of AdS/CFT suggests that a Reissner-Nordström-AdS black hole can be used as the gravitational dual of the transition from normal state to superconducting state in the boundary field theory [80, 83, 84]. Further examples in the context of condensed matter applications include the study of non-Fermi liquids, strange metal phase of the cuprate superconductors and the quantum Hall effect [85, 118].

Moreover, rotating black holes in AdS have been discussed in holographic models for rotating quark-gluon plasmas [125, 126, 132], as well as rotating superconductors [150].

In this thesis we turn our attention to a specific background, the five-dimensional Kerr-AdS black hole [86]. This solution describes a rotating black hole with two independent angular momenta embedded in a five dimensional anti-de Sitter space-time.

By the AdS/CFT duality, perturbations on the Kerr-AdS₅ black hole serve as a tool to study the associated CFT thermal state [87, 114] with a sufficiently general set of Lorentz charges (mass and angular momenta). Furthermore, linear perturbations are relevant for our understanding of many physical processes in the vicinity of a stationary black hole, such as propagation, scattering and stability.

Separability and Painlevé transcendents

Full separation of variables in the Schwarzschild geometry follows from the isometries generated by Killing vector fields. As a result, due to the spherical symmetry the decomposition into spherical harmonics is suitable to be used as a basis for the angular dependence for perturbations with generic spin. However, this is not the case for the four-dimensional Kerr metric since the total angular momentum is no longer conserved. Surprisingly, in this case there exists another integral of motion, nowadays known as the Carter's constant, associated with an irreducible rank-two Killing tensor. It was demonstrated that this tensor

ensures full separation of variables not only for the Hamilton-Jacobi equation but also for the Klein-Gordon equation in the Kerr space-time [44, 45]. Electromagnetic and gravitational perturbations were decoupled using the Newman-Penrose formalism [156, 157], and it was later shown that there exists a new fundamental object that encodes the hidden symmetries of the Kerr geometry, the so called Killing-Yano tensor.

In the five-dimensional Kerr-(A)dS space-time, the separability of the scalar wave equation is guaranteed at the expense of a second rank Killing tensor $K_{\mu\nu}$ [113], while for spinors, such separation of the Dirac equation follows from the existence of an anti-symmetric Killing-Yano tensor [162, 164]. Nevertheless, the appropriate separation scheme for vector and tensor perturbations in dimensions $D > 4$ remained elusive.

A remarkable progress on the separability of Maxwell equations in rotating black hole space-times has been recently achieved by Lunin. The separability relies on the existence of a Killing-Yano tensor and the introduction of an arbitrary parameter μ , along with the separation constant, in a new ansatz proposed in [120] for the vector potential of the electromagnetic field. This method works for Myers-Perry black holes, as well as Kerr-(A)dS in arbitrary dimensions and has filled a long-standing gap in the literature. Afterwards, Frolov, Krtouš and Kubizňák showed that Lunin’s ansatz can be written in terms of the principal tensor (a non-degenerate closed conformal Killing-Yano 2-form) and generalized to massive vector perturbations in Kerr-NUT-(A)dS black hole space-times in any number of dimensions [66, 112]. We refer to [68] for the reader that might be interested in hidden symmetries of rotating black holes in higher dimensions.

The question about separability plays an important role in black hole perturbation theory. It reduces partial differential equations to a set of ordinary differential equations (ODEs), which can be solved either analytically or by numerical methods. In this context, the method of isomonodromic deformations was developed from early extensions of the WKB method using monodromy techniques [129, 133] to compute analytically highly damped BH QNMs, and explored also in [47, 48] to obtain the scattering matrix of a scalar field in the Kerr black hole. In [136] – see also [43, 137] – the isomonodromy method was introduced as an approach to study linear perturbations on rotating black holes in four dimensions with a cosmological constant. The method has deep ties to integrable systems and the Riemann-Hilbert problem in complex analysis, re-

lating scattering coefficients to monodromies of a flat holomorphic connection of a certain matricial differential system associated to the Painlevé VI (PVI) equation. For the Heun equation related to the Kerr-de Sitter and Kerr-anti-de Sitter black holes, the solution for the scattering problem has been given in terms of transcendental equations involving the isomonodromic τ -function of the Painlevé VI transcendent.

In addition, the PVI τ -function can be thought of as a correlation function between primary fields of a two-dimensional conformal field theory with central charge $c = 1$, through the Alday–Gaiotto–Tachikawa (AGT) conjecture [13, 14, 71]. In the latter work, the authors have provided series expansion for the aforementioned function in terms of the $c = 1$ conformal blocks, expanding the early work by Jimbo [102]. More recently, the authors of [38, 75] have reformulated this isomonodromic function in terms of the determinant of a certain class of Fredholm operators³.

The isomonodromic τ -functions of the Painlevé transcendents have proven successful to describe diverse physical systems depending on the expansion about different critical points and the character of the singularities.

One can find the conformal mapping accessory parameter for simply connected domains, as well as for unbounded domains by performing the asymptotic expansion of the PVI τ -function around a convenient critical point, i.e., at $t = \{0, 1, \infty\}$ [17]. In this context, the extremal limit of the Kerr-de Sitter black hole in [137] is then described by an asymptotic expansion around $t = 0$.

On the other hand, the character of the singularities in a ODE has been treated with different Fuchsian systems and has led to explore all the other Painlevé equations, see Appendix B and the references therein. Only recently it has been realized that the emptiness formation probability in the XY spin chain can be given in terms of the Painlevé V (PV) equation [18]. Analogously, the Rabi model in the Bargmann representation is described by a confluent Heun equation and can be analyzed via isomonodromic deformations of the associated Fuchsian system [42]. These two physical systems possess the same number and type of singularities: two regular singular points and one irregular singular point of Poincaré rank 1 and, furthermore, can be exactly solved in terms of the PV τ -function.

³We will see that this formulation has computational advantages over the Conformal Blocks expansion and will allow us to numerically solve the transcendental equations posed by the quasi-normal modes with high accuracy.

As reflected in the title of this thesis, we will be mainly interested in the Painlevé VI τ -function as the solution of the eigenvalue problem of the radial and angular equations derived from the dynamics of scalar and vector field perturbations in Kerr-AdS₅.

Outline of the thesis

We shall begin by explaining the metric of the Kerr-AdS₅ black hole solution in Chapter 2. Its thermodynamic properties, conserved quantities and asymptotic geometries relevant in Einstein gravity are also briefly discussed in Section 2.1 and Section 2.2, respectively. Since we are interested in scalar and vector perturbations (described in Chapter 5) on this background, we proceed to write the Klein-Gordon equation in this background and derive the decoupled system of radial and angular differential equations in Section 2.3. These equations can be reduced to the canonical form of the Heun Equation, a second order differential equation with four regular singular points.

In Chapter 3, we explore the method of isomonodromic deformations. We analyse the map between a Fuchsian system with regular singular points and its monodromy representation in Section 3.2. This correspondence is not bijective for $n \geq 3$, and admits a description in terms of a family of equations satisfying a zero curvature condition with a given monodromy data, the Schlesinger equations. In Section 3.3, we examine the isomonodromic deformations of a Fuchsian system with four regular singular points, that leads to the Painlevé VI equation.

In Section 3.4, we present the isomonodromic τ -function of the Painlevé VI transcendent, as well as its asymptotic expansion. The relation between the accessory parameter and the Painlevé VI τ -function is discussed in subsection 3.4.3.

In Chapter 4 we start to present the results of the thesis. We show the explicit calculation of the separation constant as the result of evaluating the logarithmic derivative of the angular PVI τ -function for slow rotation or near equally rotating black holes in Section 4.2; then, we carry out a numerical analysis on the radial PVI τ -function to compute fundamental quasi-normal modes for Schwarzschild-AdS₅, while varying the size of the event horizon, and compare with the Frobenius method and Quadratic Eigenvalue Problem (QEP) in Section 4.3.

In Section 4.4, we examine numerically the quasi-normal modes for Kerr-AdS₅ as a function of the size of the outer horizon, and give an asymptotic formula for the quasi-normal modes in the subcase where the field does not carry any azimuthal angular momenta $m_1 = m_2 = 0$ (and therefore the orbital angular momentum quantum number ℓ is even) in the small BH limit. The appearance of superradiant modes for ℓ odd is discussed through some numerical evidence and the asymptotic expansion of the τ -function.

Following the ansatz proposed in [120] for the separability of the Maxwell equations in Kerr-AdS₅, the role of the introduction of an arbitrary μ parameter is studied in terms of the isomonodromic deformations in Chapter 5.

In Section 5.2, we introduce the elements to decouple the Maxwell equations in terms of a scalar function and bring the radial and angular ODEs into the Heun form. One can see that μ is related by a Möbius transformation with an apparent singularity in the deformed Heun equation. Subsequently, the initial conditions on the isomonodromic τ -function of the Painlevé VI are written in Section 5.4. A numerical analysis is also presented in subsection 5.4.2 for ultraspinning black holes. This regime is described by an expansion of the angular PVI τ -function around $t = 1$, and allows to solve the complex system of transcendental equations.

Finally, we conclude in Chapter 6 and present the future perspectives of this work. In Appendix A we describe the Fredholm determinant formulation of the PVI τ function, reviewing work done in [75] and Appendix B is devoted to Painlevé equations.

Chapter 2

Kerr-AdS₅ Black Hole

The Kerr-AdS₅ space-time is a solution to the Einstein's equations with negative cosmological constant, which describes a rotating black hole with two independent angular momenta within a five dimensional anti-de Sitter background.

The metric was obtained by Hawking, Hunter and Taylor-Robinson in [86], and subsequently generalized to arbitrary dimensions, with multiple rotation parameters by Gibbons, Lü, Pope and Page [76, 77], which provided a formal proof of the solution in [86]. It is given by

$$\begin{aligned} ds^2 = & -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a_1 \sin^2 \theta}{\Xi_1} d\phi - \frac{a_2 \cos^2 \theta}{\Xi_2} d\psi \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a_1 dt - \frac{(r^2 + a_1^2)}{\Xi_1} d\phi \right)^2 \\ & + \frac{1 + r^2 \ell^{-2}}{r^2 \rho^2} \left(a_1 a_2 dt - \frac{a_2 (r^2 + a_1^2) \sin^2 \theta}{\Xi_1} d\phi - \frac{a_1 (r^2 + a_2^2) \cos^2 \theta}{\Xi_2} d\psi \right)^2 \\ & + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left(a_2 dt - \frac{(r^2 + a_2^2)}{\Xi_2} d\psi \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2, \end{aligned} \quad (2.1)$$

where

$$\begin{aligned}
\Delta_r &= \frac{1}{r^2}(r^2 + a_1^2)(r^2 + a_2^2)(1 + r^2\ell^{-2}) - 2M \\
&= \frac{1}{r^2}(r^2 - r_0^2)(r^2 - r_-^2)(r^2 - r_+^2), \\
\Delta_\theta &= 1 - a_1^2\ell^{-2}\cos^2\theta - a_2^2\ell^{-2}\sin^2\theta, \\
\rho^2 &= r^2 + a_1^2\cos^2\theta + a_2^2\sin^2\theta, \\
\Xi_1 &= 1 - \frac{a_1^2}{\ell^2}, \quad \Xi_2 = 1 - \frac{a_2^2}{\ell^2},
\end{aligned} \tag{2.2}$$

a_1 and a_2 are two independent rotation parameters related with the angular momenta, as well as M is associated to the BH mass. The metric satisfies $R_{\mu\nu} = -4\ell^{-2}g_{\mu\nu}$ and from now on, we assume that the AdS radius $\ell = 1$. The determinant of the metric is

$$\sqrt{-g} = \frac{r\rho^2 \sin\theta \cos\theta}{\Xi_1\Xi_2}. \tag{2.3}$$

The horizons of the black hole are obtained from the equation $\Delta_r = 0$, which for $M > 0$, $a_1^2, a_2^2 < 1$ guarantees two real roots r_-, r_+ , the inner and the outer horizon of the black hole respectively, whereas r_0 is purely imaginary:

$$r_0^2 = -(1 + a_1^2 + a_2^2 + r_-^2 + r_+^2). \tag{2.4}$$

We point out that the time translational and rotational (bi-azimuthal) isometries of the space-time (2.1) are defined by the Killing vector fields^{1,2}

$$k = \partial_t, \quad m = \partial_\phi, \quad n = \partial_\psi, \tag{2.5}$$

which can be used to construct a co-rotating Killing field

$$\chi = \partial_t + \Omega_{a_1}(r_+)\partial_\phi + \Omega_{a_2}(r_+)\partial_\psi, \tag{2.6}$$

that becomes null on the outer Killing horizon at $r = r_+$. Notice that in the limit $a_i \rightarrow 0$, one recovers the Schwarzschild-AdS₅ metric, and adding $M \rightarrow 0$, the space corresponds to empty AdS₅.

¹The component notation of the time-like Killing vector is $k^\mu = \delta_t^\mu$, for instance.

²We recall that any vector ξ_μ that satisfies $\nabla_{(\mu}\xi_{\nu)} = 0$ is known as a Killing vector field.

2.1 Black Hole Thermodynamics

The outer horizon, defined as the largest root r_+ of $\Delta_r = 0$, corresponds to an event horizon. The area of the Kerr-AdS black hole is the surface at $r = r_+$ given by

$$A = \int d^3x \sqrt{g_{3d}}|_{r=r_+} \quad (2.7)$$

$$= \frac{(r_+^2 + a_1^2)(r_+^2 + a_2^2)}{r_+ \Xi_1 \Xi_2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \int_0^{\pi/2} d\theta \sin \theta \cos \theta, \quad (2.8)$$

therefore, we obtain

$$A = 2\pi^2 \frac{(r_+^2 + a_1^2)(r_+^2 + a_2^2)}{r_+ \Xi_1 \Xi_2}. \quad (2.9)$$

Here the integration is computed over the volume of the unit 3-sphere³ and g_{3d} is the induced metric at the horizon (at fixed time and $r = r_+$).

Similarly to the zero angular momentum observer in four dimensions, we can define a five-velocity unit vector, u^μ , for a locally non-rotating observer that is orthogonal to the azimuthal Killing vectors in (2.5). Then we require that

$$\begin{aligned} m \cdot u = 0 & \Rightarrow u^t g_{t\phi} + u^\phi g_{\phi\phi} + u^\psi g_{\phi\psi} = 0, \\ n \cdot u = 0 & \Rightarrow u^t g_{t\psi} + u^\phi g_{\phi\psi} + u^\psi g_{\psi\psi} = 0, \end{aligned} \quad (2.10)$$

and defining the angular velocities as

$$\Omega_{a_1} = \frac{u^\phi}{u^t}, \quad \Omega_{a_2} = \frac{u^\psi}{u^t}, \quad (2.11)$$

we can solve the system (2.10) for $\Omega_{a_1}, \Omega_{a_2}$ in terms of the metric components,

$$\Omega_{a_1} = \frac{a_1 \Xi_1 [M (r^2 + a_2^2) \Delta_\theta - \Delta_r \rho^2 \Xi_2]}{M (r^2 + a_1^2) (r^2 + a_2^2) \Delta_\theta + \Delta_r \rho^2 \Xi_1 \Xi_2}, \quad (2.12)$$

$$\Omega_{a_2} = \frac{a_2 \Xi_2 [M (r^2 + a_1^2) \Delta_\theta - \Delta_r \rho^2 \Xi_1]}{M (r^2 + a_1^2) (r^2 + a_2^2) \Delta_\theta + \Delta_r \rho^2 \Xi_1 \Xi_2}. \quad (2.13)$$

³Note that in five dimensions $0 \leq \theta \leq \pi/2$, due to the cosine direction parametrization of the 3-sphere.

Note that the angular velocities (2.12) and (2.13) at the outer event horizon reduce to

$$\Omega_{a_1,+} = \Omega_{a_1}(r_+) = \frac{a_1 \Xi_1}{r_+^2 + a_1^2}, \quad \Omega_{a_2,+} = \Omega_{a_2}(r_+) = \frac{a_2 \Xi_2}{r_+^2 + a_2^2}, \quad (2.14)$$

where we have used the fact that $\Delta_r(r_+) = 0$. Nevertheless, the angular velocities do not vanish at the asymptotic boundary, a remarkable feature of rotating black holes in AdS, different from the asymptotically flat case, where $\Omega_\infty = 0$. Instead, $\Omega_{a_i,\infty} = -a_i$, $i = 1, 2$, which imply that the observer is rotating with respect to the boundary. This issue can be solved by a coordinate transformation

$$\bar{t} = t, \quad \bar{\phi} = \phi - \Omega_{a_1,\infty} t, \quad \bar{\psi} = \psi - \Omega_{a_2,\infty} t. \quad (2.15)$$

Then, the angular velocities entering the thermodynamic relations measured by a static observer are

$$\Omega_{a_i} = \Omega_{a_i,+} - \Omega_{a_i,\infty} = \frac{a_i(1 + r_+^2)}{r_+^2 + a_i^2}, \quad i = 1, 2. \quad (2.16)$$

Analytic continuation of the Lorentzian metric by $t \rightarrow -i\tau$, $a_1 \rightarrow ia_1$ and $a_2 \rightarrow ia_2$ yields the Euclidean section, whose regularity at $r = r_+$ imposes certain periodicity in the Euclidean variables, $\tau \sim \tau + \beta$, $\phi \sim \phi + i\beta\Omega_{+,a_1}$ and $\psi \sim \psi + i\beta\Omega_{+,a_2}$, where the inverse of Hawking temperature β is given by

$$\beta = \frac{1}{T} = \frac{4\pi(r_+^2 + a_1^2)(r_+^2 + a_2^2)}{r_+^2 \Delta'_r(r_+)}, \quad (2.17)$$

where $\Delta'_r(r_+)$ means the derivative of Δ_r in (2.2) with respect to r evaluated at the outer event horizon. In Figure 2.1 we show the behaviour of the Hawking temperature, for different non-vanishing rotation parameters, as a function of the size of the black hole. We see that the temperature approaches to zero as the radius goes to zero.

Despite the fact of the unanimity about the definition of area, Hawking temperature and angular velocities, there has been some debate on the calculation of the total mass and the angular momenta in asymptotically AdS space-times⁴, due to the appearance of divergent terms, and because of the

⁴These quantities are defined unambiguously using the Komar approach in asymptotically flat space-times [108].

Figure 2.1: Temperature as a function of the outer horizon, with $a_1 = a_2 = a$. From bottom to top: $a = 1/3, 0.25, 1/8, 0.05, 0.02$. Adapted from [46]

rotation. Nevertheless, several approaches for conserved charges have been proposed: the construction of Ashtekar, Magnon and Das (ADM) based on the electric part of the Weyl tensor [19, 20], the Komar integrals in asymptotically AdS [106, 122], the Hamiltonian charge by Henneaux and Teitelboim [88], the “pseudotensor” approach of Abbott and Deser [4], the covariant phase space formalism used by Hollands et al. in [92], and the counterterm subtraction method [24, 55, 89, 140, 141]. For a detailed comparison between these different definitions, we recommend [51, 92].

In particular, one needs to proceed with considerable care since in asymptotically AdS space-times there is an additional subtlety regarding the appropriate definition of a timelike Killing vector. In such a rotating frame, an arbitrary linear combination of ∂_t , ∂_ϕ , and ∂_ψ will result in a conserved charge that is a linear combination of the total mass and the angular momenta. Notice that the discrepancies in [22, 86] with respect to [78, 141], arise precisely because they calculated the energy using a Killing vector field ∂_t , which is rotating at infinity. In contrast, the suitable non-rotating time-like Killing field is

$$\partial_{\bar{t}} = \partial_t - a_1 \partial_\phi - a_2 \partial_\psi. \quad (2.18)$$

We follow the AMD proposal for the computation of the total mass and the angular momenta. The definition of a conserved quantity $Q[\xi]$, associated to any asymptotic Killing field ξ in an asymptotically-AdS spacetime, involves an integral of certain components of the Weyl tensor over a codimension-2 sphere lying on the conformal boundary. If $\bar{C}^\mu_{\nu\rho\sigma}$ is the Weyl tensor of the conformally rescaled metric $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, and $\bar{n}_\mu = \partial_\nu \Omega$, then in d dimensions one defines

$$\bar{\mathcal{E}}^\mu_\nu = \Omega^{3-d} \bar{n}^\rho \bar{n}^\sigma \bar{C}^\mu_{\nu\rho\sigma} \quad (2.19)$$

as the electric part of the Weyl tensor on the conformal boundary, and $Q[\xi]$ is then given by

$$Q[\xi] = \frac{1}{8\pi(d-3)} \oint_\Sigma \bar{\mathcal{E}}^\mu_\nu \xi^\nu d\bar{\Sigma}_\mu, \quad (2.20)$$

where $d\bar{\Sigma}_\mu$ is the area element of the $(d-2)$ -sphere section of the conformal boundary. In five dimensions, the expression (2.19) reduces to

$$\bar{\mathcal{E}}^\mu_\nu = \frac{1}{\Omega^2} \bar{g}^{\rho\alpha} \bar{g}^{\sigma\beta} \bar{n}_\alpha \bar{n}_\beta \bar{C}^\mu_{\rho\nu\sigma} = \frac{1}{\Omega^6} g^{\rho r} g^{\sigma r} \bar{n}_r \bar{n}_r C^\mu_{\rho\nu\sigma}, \quad (2.21)$$

where the indices on $\bar{n}_\mu$ are raised and lowered with respect to the rescaled metric $\bar{g}^{\mu\nu}$, and $\bar{C}^\mu_{\rho\nu\sigma} = C^\mu_{\rho\nu\sigma}$. After some careful manipulations, one finds that the metric on the boundary has the form

$$ds^2 = r^2 \left[-\frac{\Delta_\theta}{\Xi_1 \Xi_2} dt^2 + \frac{1}{\Delta_\theta} d\theta^2 + \frac{\sin^2 \theta}{\Xi_1} d\phi^2 + \frac{\cos^2 \theta}{\Xi_2} d\psi^2 + \mathcal{O}\left(\frac{1}{r^4}\right) \right], \quad (2.22)$$

with conformal factor defined as $\Omega = \frac{1}{r}$. Then the rescaled metric $\bar{g}_{\mu\nu}$ can be read off from (2.22) as follows

$$d\bar{s}^2 = -\frac{\Delta_\theta}{\Xi_1 \Xi_2} dt^2 + \frac{1}{\Delta_\theta} d\theta^2 + \frac{\sin^2 \theta}{\Xi_1} d\phi^2 + \frac{\cos^2 \theta}{\Xi_2} d\psi^2. \quad (2.23)$$

In particular, the conserved charges associated to the Killing vectors (2.5) of the metric (2.1) can be obtained from

$$\begin{aligned} Q[\partial_\phi] &= \frac{1}{16\pi} \oint_\Sigma \bar{\mathcal{E}}^t_\phi d\bar{\Sigma}_t, & Q[\partial_\psi] &= \frac{1}{16\pi} \oint_\Sigma \bar{\mathcal{E}}^t_\psi d\bar{\Sigma}_t, \\ Q[\partial_t] &= \frac{1}{16\pi} \oint_\Sigma \bar{\mathcal{E}}^t_t d\bar{\Sigma}_t. \end{aligned} \quad (2.24)$$

By a straightforward calculation, it turns out that the leading order term, as $r \rightarrow \infty$, for the relevant components of the Weyl tensor of the physical metric (2.1) are

$$\begin{aligned} C^t_{rtr} &= \frac{6M}{r^6} + \mathcal{O}\left(\frac{1}{r^8}\right), \\ C^t_{r\phi r} &= -\frac{8Ma_1 \sin^2 \theta}{\Xi_1 r^6} + \mathcal{O}\left(\frac{1}{r^8}\right), \\ C^t_{r\psi r} &= -\frac{8Ma_2 \cos^2 \theta}{\Xi_2 r^6} + \mathcal{O}\left(\frac{1}{r^8}\right). \end{aligned} \quad (2.25)$$

The electric components of the Weyl tensor, defined on the conformal boundary, are therefore given by

$$\begin{aligned} \bar{\mathcal{E}}^t_t &= 6M, \\ \bar{\mathcal{E}}^t_\phi &= -\frac{8Ma_1 \sin^2 \theta}{\Xi_1}, \quad \bar{\mathcal{E}}^t_\psi = -\frac{8Ma_2 \cos^2 \theta}{\Xi_2}. \end{aligned} \quad (2.26)$$

The area element $d\bar{\Sigma}_t = \bar{u}_t d\Sigma$ is the spacelike hypersurface defined on the conformal boundary (2.23) and $\bar{u}_t$ a unit normal timelike vector. Thus, we shall have

$$d\bar{\Sigma}_t = \frac{\sin \theta \cos \theta}{\Xi_1 \Xi_2} d\theta d\phi d\psi. \quad (2.27)$$

Performing the integration, we find the conserved charges associated to ∂_ϕ and ∂_ψ

$$\begin{aligned} Q[\partial_\phi] &= -\frac{1}{16\pi} \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \frac{8Ma_1}{\Xi_1^2 \Xi_2} \sin^3 \theta \cos \theta \\ &= -\frac{\pi Ma_1}{2\Xi_1^2 \Xi_2}, \end{aligned} \quad (2.28a)$$

$$\begin{aligned} Q[\partial_\psi] &= -\frac{1}{16\pi} \int_0^{2\pi} d\psi \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \frac{8Ma_2}{\Xi_1 \Xi_2^2} \cos^3 \theta \sin \theta \\ &= -\frac{\pi Ma_2}{2\Xi_1 \Xi_2^2}, \end{aligned} \quad (2.28b)$$

and the corresponding angular momenta of the black hole can be taken from the following relations

$$Q[\partial_\phi] = -\mathcal{J}_\phi, \quad Q[\partial_\psi] = -\mathcal{J}_\psi. \quad (2.29)$$

Thus the conformal mass, calculated with respect to (2.18) a non-rotating time-like Killing vector, is given by

$$\begin{aligned} Q[\partial_t - a_1\partial_\phi - a_2\partial_\psi] &= Q[\partial_t] - a_1Q[\partial_\phi] - a_2Q[\partial_\psi], \\ \mathcal{M} &= Q[\partial_t] + a_1\mathcal{J}_\phi + a_2\mathcal{J}_\psi, \\ \mathcal{M} &= \frac{\pi M(2\Xi_1 + 2\Xi_2 - \Xi_1\Xi_2)}{4\Xi_1^2\Xi_2^2}, \end{aligned} \tag{2.30}$$

where $Q[\partial_t] = \frac{3\pi M}{4\Xi_1\Xi_2}$. This result (2.30) agrees precisely with the mass obtained in [78] and satisfies the first law of thermodynamics

$$d\mathcal{M} = T dS + \Omega_{a_1} d\mathcal{J}_\phi + \Omega_{a_2} d\mathcal{J}_\psi. \tag{2.31}$$

2.2 Asymptotic Geometries

Let us review two important emergent geometries from (2.1):

The asymptotically global AdS_5

The asymptotic structure of the metric (2.1) is involved. One of the reasons relies on the non-vanishing value of the angular velocities at spatial infinity. While the second reason can be inferred by the different metrics that can be realized on the conformal boundary depending on the choice of the conformal factor. See, for instance (2.22).

Nevertheless, by introducing the change of coordinates

$$\begin{aligned} \Xi_1 y^2 \sin^2 \hat{\theta} &= (r^2 + a_1^2) \sin^2 \theta, \\ \Xi_2 y^2 \cos^2 \hat{\theta} &= (r^2 + a_2^2) \cos^2 \theta, \\ \hat{\phi} = \phi + a_1 t, \quad \hat{\psi} = \psi + a_2 t, \quad \hat{t} &= t, \end{aligned} \tag{2.32}$$

and some useful relations

$$\begin{aligned} 1 + y^2 &= \frac{1 - a_1^2 \cos^2 \theta - a_2^2 \sin^2 \theta}{\Xi_1 \Xi_2} (1 + r^2), \\ y^2 (1 - a_1^2 \sin^2 \hat{\theta} - a_2^2 \cos^2 \hat{\theta}) &= r^2 + a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta \end{aligned} \tag{2.33}$$

that can be inverted to yield the asymptotic relations:

$$\begin{aligned}
1 + r^2 &= (1 - a_1^2 \sin^2 \hat{\theta} - a_2^2 \cos^2 \hat{\theta})(1 + y^2) + \frac{(a_1^2 - a_2^2)^2 \sin^2 \hat{\theta} \cos^2 \hat{\theta}}{1 - a_1^2 \sin^2 \hat{\theta} - a_2^2 \cos^2 \hat{\theta}} + \mathcal{O}\left(\frac{1}{r^2}\right), \\
1 - a_1^2 \cos^2 \theta - a_2^2 \sin^2 \theta &= \frac{\Xi_1 \Xi_2}{(1 - a_1^2 \sin^2 \hat{\theta} - a_2^2 \cos^2 \hat{\theta})} + \mathcal{O}\left(\frac{1}{r^2}\right),
\end{aligned}
\tag{2.34}$$

we arrive to

$$\begin{aligned}
d\hat{s}^2 &= -\left(1 + y^2\right) dt^2 + \frac{dy^2}{1 + y^2} + y^2 \left(d\hat{\theta}^2 + \sin^2 \hat{\theta} d\hat{\phi}^2 + \cos^2 \hat{\theta} d\hat{\psi}^2\right) \\
&\quad + \frac{2M}{y^2 \Delta_{\hat{\theta}}^3} \left(dt - a_1 \sin^2 \hat{\theta} d\hat{\phi} - a_2 \cos^2 \hat{\theta} d\hat{\psi}\right)^2 + \dots,
\end{aligned}
\tag{2.35}$$

where

$$\Delta_{\hat{\theta}} = 1 - a_1^2 \sin^2 \hat{\theta} - a_2^2 \cos^2 \hat{\theta}.
\tag{2.36}$$

The asymptotic metric reduces to the line element of global AdS_5 plus a correction term given by the mass and rotation parameters of the black hole. In our case, we may take

$$\Omega = \frac{1}{y},
\tag{2.37}$$

so that the boundary is given by $y = \infty$ and the metric on the conformal boundary for this choice of conformal factor is different from (2.23), and given by

$$d\bar{s}^2 = -dt^2 + d\hat{\theta}^2 + \sin^2 \hat{\theta} d\hat{\phi}^2 + \cos^2 \hat{\theta} d\hat{\psi}^2.
\tag{2.38}$$

In other words, the conformal boundary of the bulk space-time is the static Einstein universe $\mathbb{R} \times S^3$ [78].

The near-horizon limit of the extremal Kerr- AdS_5

An interesting property of a rotating black hole is that it has an extreme configuration where the temperature vanishes but the entropy remains finite. Bardeen and Horowitz [26] showed that the near-horizon limit of the extremal Kerr black hole is a space-time similar to $\text{AdS}_2 \times S^2$, and is called the near-horizon extreme Kerr geometry (NHEK). For asymptotically AdS rotating black holes, their near-horizon extremal geometries – the NHEK-AdS – have been explicitly derived by Lü, Mei and Pope [119].

The extremal limit occurs when the function Δ_r in (2.2) has a double zero at the outer horizon, which we denote by $r = r_*$, i.e. when

$$\Delta_r(r_*) = 0, \quad \Delta'_r(r_*) = 0, \quad (2.39)$$

implying that the Hawking temperature (2.17) vanishes, or equivalently, the coalescence between the inner and outer horizons to a single horizon at $r = r_*$. If one expands Δ_r up to quadratic order around r_* , one finds

$$\begin{aligned} \Delta_r &= V(r - r_*)^2 + \mathcal{O}\left((r - r_*)^3\right), \\ V &= \frac{1}{2}\Delta''_r(r_*) = \frac{4(3a_1^2a_2^2r_*^2 - r_*^6 + a_1^4a_2^2 + a_1^2a_2^4)}{r_*^4(2r_*^2 + a_1^2 + a_2^2)}. \end{aligned} \quad (2.40)$$

To describe the near-horizon geometry of the extremal Kerr-AdS metric we make the following coordinate transformations

$$\begin{aligned} r &= r_*(1 + \lambda y), & t &= \frac{\tau}{2\pi r_* T'_H \lambda}, \\ \phi &= \phi_1 + \Omega_{1,*}t, & \psi &= \phi_2 + \Omega_{2,*}t, \\ \Omega_{1,*} &= \frac{a_1 \Xi_1}{r_*^2 + a_1^2}, & \Omega_{2,*} &= \frac{a_2 \Xi_2}{r_*^2 + a_2^2}, \end{aligned} \quad (2.41)$$

where λ is a scaling parameter, and $\Omega_{i,*}$ are the angular velocities defined at the horizon. The quantity T'_H is the derivative of Hawking temperature with respect to the outer horizon evaluated at $r_+ = r_*$,

$$T'_H = \left. \frac{\partial T_H}{\partial r_+} \right|_{r_+=r_*} = \frac{r_*^2 V}{2\pi(r_*^2 + a_1^2)(r_*^2 + a_2^2)}. \quad (2.42)$$

Taking the limit $\lambda \rightarrow 0$, we obtain the near-horizon geometry of the form

$$\begin{aligned} ds^2 &= A(\theta) \left(-y^2 d\tau^2 + \frac{dy^2}{y^2} \right) + F(\theta) d\theta^2 + B_1(\theta) e_1^2 + B_2(\theta) (e_2 + C(\theta) e_1)^2, \\ e_1 &= d\phi_1 + k_1 y d\tau, & e_2 &= d\phi_2 + k_2 y d\tau, \end{aligned} \quad (2.43)$$

with Δ_θ defined in (2.2) and

$$\begin{aligned}
k_1 &= \frac{2a_1\Xi_1(r_\star^2 + a_2^2)}{Vr_\star(r_\star^2 + a_1^2)}, & k_2 &= \frac{2a_2\Xi_2(r_\star^2 + a_1^2)}{Vr_\star(r_\star^2 + a_2^2)}, \\
\rho_\star^2 &= r_\star^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta, \\
A(\theta) &= \frac{\rho_\star^2}{V}, & F(\theta) &= \frac{\rho_\star^2}{\Delta_\theta}, \\
B_1(\theta) &= \frac{\Delta_\theta \sin^2 \theta (r_\star^2 + a_1^2)^2}{\Xi_1^2 \rho_\star^2} \left[1 + \frac{a_2^2 (1 + r_\star^2) \sin^2 \theta}{\Delta_\theta r_\star^2 + a_1^2 (1 + r_\star^2) \cos^2 \theta} \right], \\
B_2(\theta) &= \frac{\Delta_\theta \cos^2 \theta (r_\star^2 + a_2^2)^2}{\Xi_2^2 \rho_\star^2} \left[1 + \frac{a_1^2 (1 + r_\star^2) \cos^2 \theta}{\Delta_\theta r_\star^2} \right], \\
C(\theta) &= \frac{a_1 a_2 \Xi_2 (r_\star^2 + a_1^2) (1 + r_\star^2) \sin^2 \theta}{\Xi_1 (r_\star^2 + a_2^2) (\Delta_\theta r_\star^2 + a_1^2 (1 + r_\star^2) \cos^2 \theta)}. \tag{2.44}
\end{aligned}$$

One recognizes that the term inside the parenthesis in the metric (2.43) is analogous to AdS_2 in the Poincaré patch with a horizon at $y = 0$. Then the geometry related to (2.43) is a warped and twisted product of $\text{AdS}_2 \times \text{S}^3$, with isometry $SL(2, \mathbb{R}) \times U(1) \times U(1)$.

A further coordinate transformation leads to the NHEK-AdS metric in global coordinates

$$\begin{aligned}
ds^2 &= A(\theta) \left(-(1 + r^2) dt^2 + \frac{dr^2}{1 + r^2} \right) + F(\theta) d\theta^2 + B_1(\theta) e_1^2 + B_2(\theta) (e_2 + C(\theta) e_1)^2, \\
e_1 &= d\phi_1 + k_1 r dt, & e_2 &= d\phi_2 + k_2 r dt. \tag{2.45}
\end{aligned}$$

Note that A, F, B_1, B_2 and C are only functions of θ , while k_1 and k_2 are related with the inverse of Frolov-Thorne temperatures associated with the CFTs for each azimuthal angle. For a more detailed discussion on the Kerr/CFT correspondence we recommend [54] and references therein.

In [119] it was shown the agreement between the microscopic entropy resulting from each chiral two-dimensional CFT associated to each rotation plane and the Bekenstein-Hawking entropy of the extremal rotating black hole, following the so-called Kerr/CFT correspondence [81].

2.3 Scalar Perturbations

We are going to consider the dynamics of GR in space-times with cosmological constant Λ , described by the Einstein-Hilbert action [32],

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} (g^{\mu\nu} R_{\mu\nu} - 2\Lambda) + S_m, \quad (2.46)$$

where G_5 is the gravitational constant, and S_m represents the action of the matter fields $\{\Psi_i\}$ coupled to gravity. The equations of motion for the fields $g_{\mu\nu}$ and Ψ_i are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (2.47a)$$

$$\frac{\delta S_m}{\delta \Psi_i} = 0, \quad (2.47b)$$

where $T_{\mu\nu}$ is the stress-energy tensor associated to the matter fields.

Now, consider perturbation of the fields of the form

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}, \quad \Psi_i = \tilde{\Psi}_i + \Phi_i, \quad (2.48)$$

where we assume that $h_{\mu\nu}$ and Φ_i are small perturbations. Thus substituting the ansatz (2.48) into (2.47a) and (2.47b) and neglecting quadratic and higher order powers of the perturbation fields, we are left with a set of linear equations for $h_{\mu\nu}$ and Φ_i , which are coupled. However, if we set $\tilde{\Psi}_i = 0$, we observe that the linearized equations of motion for $h_{\mu\nu}$ and Φ_i decouple, and thus fluctuations $h_{\mu\nu}$ can be consistently set to zero. In such a case, the dynamics of the generic small perturbations of the matter fields is equivalent to studying the test fields Φ_i in the background metric $\tilde{g}_{\mu\nu}$.

For a real scalar field we have

$$S_m = -\frac{1}{2} \int d^5x \sqrt{-\tilde{g}} \left(\tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \mu^2 \Phi^2 \right), \quad (2.49)$$

which leads to the Klein-Gordon (KG) equation in the given background metric (2.1)

$$\frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left(\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \partial_\nu \Phi \right) - \mu^2 \Phi = 0, \quad (2.50)$$

where $\sqrt{-\tilde{g}}$ is the determinant of the metric (2.70) and μ parametrizes the mass of the field.

The separation of variables is achieved due to the presence of hidden symmetries in the form of Killing tensors, as it was shown in [67]. These then guarantee the separability of the geodesic equation, the Klein-Gordon equation and also the Dirac equation [162, 164]. In Chapter 5 we address the separation of the Maxwell's equations.

Then the Klein-Gordon equation (2.50) in the background (2.1) is separable by the factorization

$$\Phi = R(r)S(\theta)e^{-i\omega t + im_1\phi + im_2\psi}, \quad (2.51)$$

where $\omega \in \mathbb{C}$ is the frequency of the mode, and $m_1, m_2 \in \mathbb{Z}$ are the azimuthal components of the mode's angular momentum.

2.3.1 Angular and Radial Heun equation

By means of (2.51) we are left with two decoupled ordinary differential equations for the angular and radial functions. The angular equation is given by

$$\begin{aligned} & \frac{1}{\sin \theta \cos \theta} \frac{d}{d\theta} \left(\sin \theta \cos \theta \Delta_\theta \frac{dS(\theta)}{d\theta} \right) - \left[\omega^2 + \frac{(1 - a_1^2)m_1^2}{\sin^2 \theta} + \frac{(1 - a_2^2)m_2^2}{\cos^2 \theta} \right. \\ & \quad \left. - \frac{(1 - a_1^2)(1 - a_2^2)}{\Delta_\theta} (\omega + m_1 a_1 + m_2 a_2)^2 + \mu^2 (a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta) \right] S(\theta) \\ & = -\lambda S(\theta), \end{aligned} \quad (2.52)$$

where λ is the separation constant. By two consecutive coordinate transformations $\chi = \sin^2 \theta$, and $u = \chi/(\chi - \chi_0)$, one can take the four singularities of (2.52) to be located at

$$u = 0, \quad u = 1, \quad u = u_0 = \frac{a_2^2 - a_1^2}{a_2^2 - 1}, \quad u = \infty, \quad (2.53)$$

and the indicial exponents⁵ are

$$\alpha_0 = \pm \frac{m_1}{2}, \quad \alpha_1 = \frac{1}{2} \left(2 \pm \sqrt{4 + \mu^2} \right), \quad \alpha_{u_0} = \pm \frac{m_2}{2}, \quad (2.54)$$

$$\alpha_\infty = \pm \frac{1}{2} (\omega + a_1 m_1 + a_2 m_2), \quad (2.55)$$

where the exponents $\alpha_1^\pm$ of the ODE at $u = 1$, which correspond to $\Delta/2$, $(4 - \Delta)/2$ respectively, are related to the dimension Δ^6 of a CFT primary field $\mathcal{O}$ on the boundary [142, 160].

By applying the following transformation

$$S(u) = u^{m_1/2} (u - u_0)^{m_2/2} (u - 1)^{\Delta/2} Y(u) \quad (2.56)$$

we bring (2.52) to the canonical Heun equation form

$$\frac{d^2 Y}{du^2} + \left(\frac{1 + m_1}{u} + \frac{\Delta - 1}{u - 1} + \frac{1 + m_2}{u - u_0} \right) \frac{dY}{du} + \left(\frac{q_- q_+}{u(u - 1)} - \frac{u_0(u_0 - 1)Q_0}{u(u - 1)(u - u_0)} \right) Y = 0 \quad (2.57)$$

with the q_- , q_+ and the accessory parameter Q_0 given by

$$\begin{aligned} q_- q_+ &= \frac{1}{4} \left((m_1 + m_2 + \Delta)^2 - \beta^2 \right), \quad \beta = \omega + a_1 m_1 + a_2 m_2, \\ 4u_0(u_0 - 1)Q_0 &= - \frac{\omega^2 + a_1^2 \Delta(\Delta - 4) - \lambda}{a_2^2 - 1} - u_0 \left[(m_2 + \Delta - 1)^2 - m_2^2 - 1 \right] \\ &\quad - (u_0 - 1) \left[(m_1 + m_2 + 1)^2 - \beta^2 - 1 \right]. \end{aligned} \quad (2.58)$$

One notes that (2.57) has the same AdS spheroidal harmonics form as the problem in four dimensions [31, 52]. Also, we have that u_0 in (2.53) is close to zero for $a_2 \simeq a_1$, the equal rotation limit.

The radial equation reads as follows,

$$\begin{aligned} \frac{1}{rR(r)} \frac{d}{dr} \left(r \Delta_r \frac{dR(r)}{dr} \right) &- \left[\lambda + \mu^2 r^2 + \frac{1}{r^2} (a_1 a_2 \omega - a_2 (1 - a_1^2) m_1 - a_1 (1 - a_2^2) m_2)^2 \right] + \\ &+ \frac{(r^2 + a_1^2)^2 (r^2 + a_2^2)^2}{r^4 \Delta_r} \left(\omega - \frac{m_1 a_1 (1 - a_1^2)}{r^2 + a_1^2} - \frac{m_2 a_2 (1 - a_2^2)}{r^2 + a_2^2} \right)^2 = 0, \end{aligned} \quad (2.59)$$

⁵Defined as the asymptotic behavior of the function near the singular points $S(u) \simeq (u - u_i)^{\alpha_i}$ or $S(u) \simeq u^{-\alpha_\infty}$ for the point at infinity.

⁶ $\Delta = 2 + \sqrt{4 + \mu^2}$, in five dimensions.

which again has four regular singular points, located at the roots of $r^2\Delta_r(r^2)$ and infinity. The indicial exponents $\beta_i^\pm$ are defined analogously to the angular case. Schematically, they are given by

$$\beta_k = \pm \frac{1}{2}\theta_k, \quad k = +, -, 0, \quad \text{and} \quad \beta_\infty = \frac{1}{2}(2 \pm \theta_\infty), \quad (2.60)$$

which in terms of the temperatures and angular velocities

$$\theta_\nu = \frac{i}{2\pi} \left(\frac{\omega - m_1\Omega_{k,1} - m_2\Omega_{k,2}}{T_k} \right), \quad \theta_\infty = 2 - \Delta, \quad (2.61)$$

where θ_ν , $\nu = 0, -, +, \infty$ are the single monodromy parameters. The choice of root is tied to the boundary conditions satisfied by standing waves, as we will see in Sec.2.3.2. To bring this equation to the canonical Heun form, we perform the change of variables⁷,

$$z = \frac{r^2 - r_-^2}{r^2 - r_0^2}, \quad R(z) = z^{-\theta_-/2}(z - z_0)^{-\theta_+/2}(z - 1)^{\Delta/2}F(z), \quad (2.62)$$

where

$$z_0 = \frac{r_+^2 - r_-^2}{r_+^2 - r_0^2}. \quad (2.63)$$

Then, after some algebra, one can check that the function $F(z)$ obeys the equation

$$\frac{d^2F}{dz^2} + \left[\frac{1 - \theta_-}{z} + \frac{\Delta - 1}{z - 1} + \frac{1 - \theta_+}{z - z_0} \right] \frac{dF}{dz} + \left(\frac{\kappa_- \kappa_+}{z(z - 1)} - \frac{z_0(z_0 - 1)K_0}{z(z - 1)(z - z_0)} \right) F(z) = 0, \quad (2.64)$$

where

$$\begin{aligned} \kappa_- \kappa_+ &= \frac{1}{4} \left[(\theta_- + \theta_+ - \Delta)^2 - \theta_0^2 \right] \\ 4z_0(z_0 - 1)K_0 &= -\frac{\lambda + \mu^2 r_-^2 - \omega^2}{r_+^2 - r_0^2} - (z_0 - 1)[(\theta_- + \theta_+ - 1)^2 - \theta_0^2 - 1] \\ &\quad - z_0 [2(\theta_+ - 1)(1 - \Delta) + \Delta(\Delta - 4) + 2]. \end{aligned} \quad (2.65)$$

⁷Note that, with this choice of variables, we have that at infinity, the radial solution will behave as $R(z) \sim z^{-\theta_0/2}$

2.3.2 Solution of the radial and angular equations

Equations (2.57) and (2.64) are written in the canonical form of the Heun equation:

$$y''(z) + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_{t_0}}{z-t_0} + \frac{1-\theta_1}{z-1} \right) y'(z) + \left[\frac{\kappa_-\kappa_+}{z(z-1)} - \frac{t_0(t_0-1)K_0}{z(z-1)(z-t_0)} \right] y(z) = 0. \quad (2.66)$$

Both angular and radial equations can be solved as a series expansion of the hypergeometric functions whose coefficients satisfy the three term recurrence relations, similar in spirit to the Leaver's continued fraction method [123, 153, 154].

In Chapter 4, we compare two well-established numerical methods in GR: the matching method and the Quadratic Eigenvalue Problem. The first method relies on the matching of two Frobenius solutions constructed at the horizon and the boundary, while the second method discretizes the differential equations using a pseudo-spectral grid [57].

In order to solve the associated boundary value problem, we need to define the boundary conditions that are physically relevant. For instance, in asymptotically AdS space-times, a generic perturbation can reach the spatial infinity, in finite time, and come back to interact with the black hole. Such interaction can trigger (superradiant) instabilities at the linear level [36]. Then we typically want to choose boundary conditions that preserve the asymptotic boundary metric.

We are interested in solutions for (2.57) which satisfy

$$Y(u) = \begin{cases} 1 + \mathcal{O}(u), & u \rightarrow 0, \\ 1 + \mathcal{O}(u - u_0), & u \rightarrow u_0, \end{cases} \quad (2.67)$$

which will set a quantization condition for the separation constant λ . For the radial equation with $\mu^2 > 0$, the conditions that $R(z)$ corresponds to a purely ingoing wave at the outer horizon $z = z_0$ and normalizable at the boundary $z = 1$ are translated in terms of $F(z)$ as follows⁸

$$F(z) = \begin{cases} 1 + \mathcal{O}(z - z_0), & z \rightarrow z_0, \\ 1 + \mathcal{O}(z - 1), & z \rightarrow 1, \end{cases} \quad (2.68)$$

⁸The computation of the accessory parameters and the boundary conditions of the radial equation are slightly different with respect to those shown in [28]. We have chosen a more suitable Möbius transformation for the asymptotic expansion of the Painlevé VI τ -function

where $F(z)$ is a regular function at the boundaries. This condition will enforce the quantization of the (not necessarily real) frequencies ω , which will correspond to the quasi-normal modes.

Before discussing the isomonodromic deformation theory in the next Chapter, we proceed to introduce a toy model computation of the eigenmodes in AdS_5 . One might think that a scalar field propagating in pure AdS, in five dimensions, describes the simplest exercise where the Klein-Gordon equation can be separated and solved in terms of special functions. However, the asymptotic analysis of the solutions, as well as the choice of the boundary conditions remain conceptually the same in more complicated backgrounds.

Embedding a black hole solution will increase the number of singularities due to the appearance of event horizons. Then the study of the singularities, where are located and their character, becomes the essence of the analysis of the resulting differential equations [148]. In particular, we are interested in ODEs that possess four regular singular points.

2.3.3 Waves in AdS_5

We consider the Klein-Gordon equation for a massive scalar field in a pure AdS_5 background. The space-time metric in global coordinates is given by

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 \left[d\theta^2 + \sin^2 \theta \left(d\phi^2 + \sin^2 \phi d\psi^2 \right) \right] \quad (2.69)$$

where the determinant of the metric is given by

$$\sqrt{-g} = r^3 \sin^2 \theta \sin \phi. \quad (2.70)$$

Then we can write down (2.50) in the form

$$\begin{aligned} \frac{1}{r^3} \frac{\partial}{\partial r} \left(r^3 (1+r^2) \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \left[\frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin^2 \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left(\frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial \Phi}{\partial \phi} \right) \right. \right. \\ \left. \left. + \frac{1}{\sin^2 \phi} \frac{\partial^2 \Phi}{\partial \psi^2} \right) \right] - \frac{1}{1+r^2} \frac{\partial^2 \Phi}{\partial t^2} - \mu^2 \Phi = 0. \end{aligned} \quad (2.71)$$

We can simplify this equation by defining

$$\Phi(t, r, \theta, \phi, \psi) = e^{-i\omega t} R(r) \mathcal{Y}_{\ell m p}(\theta, \phi, \psi), \quad (2.72)$$

where we assume that $\omega > 0$, as matter of simplification. One recognizes that the angular dependence in (2.71) is related with the Laplace-Beltrami operator of the unit 3-sphere. The eigenfunctions are the hyperspherical harmonics $\mathcal{Y}_{\ell m \rho}(\theta, \phi, \psi)$ ⁹, with eigenvalues determined by the equation

$$\Delta \mathcal{Y}_{\ell m \rho}(\theta, \phi, \psi) = -\ell(\ell + 2) \mathcal{Y}_{\ell m \rho}(\theta, \phi, \psi), \quad (2.74)$$

with $\ell \geq 0$, $0 \leq m \leq \ell$ and $-m \leq \rho \leq m$. Then, inserting the angular eigenvalues into the field equation lead us to the following ODE for the radial function $R(r)$:

$$\frac{1}{r} \frac{d}{dr} \left(r^3 (1 + r^2) \frac{dR}{dr} \right) + \left(\frac{\omega^2 r^2}{1 + r^2} - \ell(\ell + 2) - \mu^2 r^2 \right) R(r) = 0. \quad (2.75)$$

To find the analytical solution of this equation, one first introduces a new radial coordinate,

$$x = 1 + r^2, \quad 1 \leq x \leq \infty, \quad (2.76)$$

where the boundaries of the AdS space are located at $x = 1$ and $x = \infty$. Then, we have

$$x(1 - x) \frac{d^2 R}{dx^2} + (1 - 3x) \frac{dR}{dx} - \left[\frac{\omega^2}{4x} + \frac{\ell(\ell + 2)}{4(1 - x)} - \frac{\mu^2}{4} \right] R = 0. \quad (2.77)$$

Through the definition

$$R(x) = x^{\omega/2} (1 - x)^{\ell/2} F(x), \quad (2.78)$$

one can verify that the function $F(x)$ satisfies the equation

$$x(1 - x) \frac{d^2 F}{dx^2} + [\gamma - (\alpha + \beta + 1)x] \frac{dF}{dx} - \alpha\beta F(x) = 0, \quad (2.79)$$

⁹These harmonics can be written as

$$\mathcal{Y}_{\ell m \rho}(\theta, \phi, \psi) = \sqrt{\frac{2^{2m+1}(\ell - m)!(1 + \ell)}{\pi(1 + m + \ell)!}} m! \sin^m \theta C_{\ell - m}^{m+1}(\cos \theta) Y_m^\rho(\phi, \psi), \quad (2.73)$$

where $Y_m^\rho(\phi, \psi)$ are the spherical harmonics of the 2-sphere, and $C_{\ell - m}^{m+1}(\cos \theta)$ are the Gegenbauer polynomials [21].

with the identifications

$$\begin{aligned}\alpha &= \frac{1}{2} \left(2 + \ell + \omega - \sqrt{4 + \mu^2} \right), \\ \beta &= \frac{1}{2} \left(2 + \ell + \omega + \sqrt{4 + \mu^2} \right), \\ \gamma &= 1 + \omega.\end{aligned}\tag{2.80}$$

This is the hypergeometric differential equation – the Fuchsian equation¹⁰ with three regular singular points –, whose general solution in the neighborhood of $x = \infty$ is given by

$$F(x) = C x^{-\alpha} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, 1/x) + D x^{-\beta} F(\beta, \beta - \gamma + 1, \beta - \alpha + 1, 1/x).\tag{2.83}$$

So, one finds that the most general solution for $R(x)$ is

$$\begin{aligned}R(x) &= C x^{\omega/2 - \alpha} (1 - x)^{\ell/2} F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, 1/x) + \\ &D x^{\omega/2 - \beta} (1 - x)^{\ell/2} F(\beta, \beta - \gamma + 1, \beta - \alpha + 1, 1/x).\end{aligned}\tag{2.84}$$

The boundary condition that one must impose is the following: we require that the scalar field vanishes at $x \rightarrow \infty$, because the AdS space behaves effectively as a reflecting box. Using the property $F(a, b, c, 0) = 1$, the asymptotic solution has the form

$$R(x) \sim C(-1)^{\ell/2} x^{(\Delta-4)/2} + D(-1)^{\ell/2} x^{-\Delta/2}\tag{2.85}$$

and thus we must set $C = 0$. This choice selects the normalizable modes for $\Delta \geq 4$. Since we are interested in the small r limit, i.e. $x \rightarrow 1$, we can express the resulting solution in (2.84) at $x = \infty$ as a linear combination of the hypergeometric functions around $x = 1$ given by

¹⁰Consider the second order linear differential equation

$$\frac{d^2 u(z)}{dz^2} + p(z) \frac{du(z)}{dz} + q(z) u(z) = 0.\tag{2.81}$$

If all its singular points are regular singular points, the equation is of *Fuchsian type*. An equation of Fuchsian type therefore only has regular singular points in the complex plane (including the point at infinity). This implies that the functions $p(z)$ and $q(z)$ are rational functions as

$$p(z) = \sum_{r=1}^n \frac{c_r}{z - a_r}, \quad q(z) = \sum_{r=1}^n \frac{d_r}{(z - a_r)^2} + \sum_{r=1}^n \frac{f_r}{z - a_r}.\tag{2.82}$$

$$\begin{aligned}
F(\beta, \beta - \gamma + 1, \beta - \alpha + 1, 1/x) &= x^{\beta - \gamma + 1} (x - 1)^{\gamma - \beta - \alpha} \\
&\times \frac{\Gamma(\beta - \alpha + 1) \Gamma(\beta + \alpha - \gamma)}{\Gamma(\beta) \Gamma(\beta - \gamma + 1)} F(1 - \alpha, 1 - \beta, \gamma - \beta - \alpha + 1, 1 - x) \\
&+ x^\beta \frac{\Gamma(\beta - \alpha + 1) \Gamma(\gamma - \beta - \alpha)}{\Gamma(1 - \alpha) \Gamma(\gamma - \alpha)} F(\beta, \alpha, \beta + \alpha - \gamma + 1, 1 - x),
\end{aligned} \tag{2.86}$$

we find that in the limit $x \rightarrow 1$, or equivalently $r^2 \rightarrow 0$, the asymptotic behavior has the form

$$\begin{aligned}
R \sim D \Gamma(\Delta - 1) &\left[\frac{(-1)^{\ell - \omega/2} \Gamma(\ell + 1)}{\Gamma(\frac{1}{2}(\omega + \ell + \Delta)) \Gamma(\frac{1}{2}(\Delta + \ell - \omega))} r^{-\ell - 2} \right. \\
&\left. + \frac{(-1)^{\ell + \omega/2} \Gamma(-\ell - 1)}{\Gamma(\frac{1}{2}(\Delta - \ell - \omega - 2)) \Gamma(\frac{1}{2}(\Delta + \omega - \ell - 2))} r^\ell \right].
\end{aligned} \tag{2.87}$$

The first term of the solution (2.87) diverges, since $r^{-\ell - 2} \rightarrow \infty$ for small values of r . Then, in order to have a regular solution at the origin of the AdS space ($r = 0$), we must demand that $\Gamma(\frac{1}{2}(\Delta + \ell - \omega)) \rightarrow \infty$ as well. This occurs when the argument of the gamma function is a non-positive integer, $\Gamma(-n) = \Gamma(\frac{1}{2}(\Delta + \ell - \omega))$ with $n = 0, 1, 2, \dots$. Therefore, the requirement of regularity allows us to select the frequencies that might propagate in the AdS background. These are given by the discrete spectrum

$$\omega_{n,\ell} = 2n + \ell + \Delta = 2n + \ell + \sqrt{4 + \mu^2} + 2, \tag{2.88}$$

which agrees with known results [37, 142] and reduces in the massless case to $\omega_{n,\ell} = 2n + \ell + 4$. As we can see the spectrum of eigenfrequencies is real, which will correspond to normal modes. Turning to the existence of a black hole in this space-time changes the boundary at the origin of AdS. Instead, the proper boundary is defined at the outer event horizon.

In this scenario, the natural eigenfrequencies are complex numbers whose real parts give the oscillation frequencies, while the imaginary parts describe the damping of the modes. These complex frequencies form the so-called quasi-normal spectrum of the system.

In Section 3.2, we present the relation between the Hypergeometric equation (2.79) and the 2×2 Fuchsian system with three regular singular points and their monodromy data.

Chapter 3

Isomonodromic τ -function

In this Chapter we discuss the idea behind isomonodromic deformations and the Painlevé VI equation. We start with an overview of the monodromy group, i.e., the group of linear transformations over canonical paths encircling the singular points of a complex function and its connection with a linear system of ODEs with rational coefficients in the complex plane.

The fascinating problem of reconstructing an ODE from its monodromy group naturally leads to the Riemann-Hilbert (RH) problem, which is equivalent to the inverse monodromy problem. Section 3.2 presents the map between a Fuchsian system with $n + 1$ regular singular points and its monodromy representation. We will see that this correspondence is no longer one-to-one for $n \geq 3$.

Then, in Section 3.3, we consider the isomonodromic deformation of a Fuchsian system with four regular singular points, that leads to the Painlevé VI equation [70]. This can be thought of as the introduction of an apparent singularity in the associated second order differential equation, that makes manifest the Hamiltonian structure connected with the isomonodromic deformation equations [147] and allows a consistent definition of the τ -function in the Jimbo, Miwa and Ueno sense [103].

In Section 3.4, we introduce the definition of the Painlevé VI τ -function in terms of $c = 1$ conformal blocks series expansion and discuss its asymptotic behavior near to a critical point. By means of this transcendental function the accessory parameter, related to the Heun equation, can be computed pertur-

batively. We close with some comments about the structure of the conformal blocks expansion in subsection 3.4.3.

Finally, Section 3.5 summarizes key concepts presented throughout the Chapter.

3.1 The monodromy data

In simple terms, the monodromy of a (generally multivalued) holomorphic function describes how the function changes if we continue it analytically around a loop γ encircling some singular point.

For example, if we take the differential equation

$$z \frac{df}{dz} = \alpha f, \quad \alpha \in \mathbb{C}, \quad (3.1)$$

on the punctured complex plane $\mathbb{C} - \{0\}$, its solution is the function $f(z) = az^\alpha$, $a \in \mathbb{C}$, which under analytic continuation along a path γ which loops once counter-clockwise around the origin (e.g. $z \rightarrow e^{2\pi i}z$) is transformed into $e^{2\pi i\alpha}az^\alpha$.

Let us consider the case $\alpha = 1/2$, and $a = 1$, then $f(z) = \sqrt{z}$, $z \neq 0$. By letting $z = re^{i\vartheta}$, and $\vartheta = \theta + 2\pi n$, we have

$$f(z) = r^{1/2}e^{i(\theta+2\pi n)/2} \quad (3.2)$$

where $0 \leq \theta \leq 2\pi$ and n is an integer. For a given value z , the function $f(z)$ takes two possible values depending on n even and n odd. Namely, for $n = 1$, $f(z)$ does not return to its original value, instead one has $f(e^{2\pi i}z) = -\sqrt{z}$. But after two turns $n = 2$, we can recover the same function. The point $z = 0$ is called a branch point¹. A point is a branch point if the multivalued function $f(z)$ is discontinuous upon traversing a small circuit around this point.

In order to get the Riemann surface of $\sqrt{z}$, we take two copies of the complex plane and cut them along the closed positive axis $z \geq 0$. We put these sheets one above another, turn the upper one along the real axis and glue the boundaries, see Figure 3.1.

¹It should be noted that the point $z = \infty$ is also a branch point

Figure 3.1: Riemann surface for the function $f(z) = \sqrt{z}$. The colors are assigned according to the argument values on each branch.

Thus, the pole at $z = 0$ in the differential equation (3.1) corresponds to a branch point in the solution.

Along the same lines, the general solution of a linear ordinary differential equation with rational coefficients is generally multivalued [148]. The starting point is a homogeneous linear ODE, say of order N , that is most conveniently written as a set of N coupled linear ODEs of first order:

$$\frac{d}{dz}y(z) = A(z)y(z). \quad (3.3)$$

Here, $y(z)$ is an N -component vector, and $A(z)$ is an $N \times N$ matrix, rational in z . It is convenient to deal with *fundamental matrix solutions* $\Phi(z)$, which are

$N \times N$ matrices built of N linearly independent solutions, which constitute the columns of $\Phi(z)$. Then, $\Phi(z)$ satisfies the same equation

$$\frac{d}{dz}\Phi(z) = A(z)\Phi(z). \quad (3.4)$$

The N vector solutions are linearly independent if their *Wronskian*

$$W(\Phi, z) = \det \Phi(z) \quad (3.5)$$

does not vanish identically, and we have $A(z) = \left[\frac{d\Phi}{dz} \right] \Phi(z)^{-1}$ from (3.4).

If $A(z)$ is analytic at $z = z_0$, so is $\Phi(z)$. The matrix $A(z)$ has singular points located at a_ν ($\nu = 1, \dots, n$) and $a_\infty = \infty$, where $\Phi(z)$ is generally multivalued. Then one can introduce the monodromy of $\Phi(z)$ and see its behaviour under analytic continuation around its singular points.

The *monodromy group* of the linear system (3.4) is defined as a representation of the fundamental group $\pi_1(\mathbb{CP}^1 - \{a_1, \dots, a_\infty\})$, which can be obtained via solutions of the system as follows:

Consider a base point $a_0 \in \mathbb{CP}^1 - \{a_1, \dots, a_\infty\}$ and a matrix $\Phi_0 \in GL(N, \mathbb{C})$ such that $\Phi(a_0) = \Phi_0$. Under an analytic continuation along a loop $\gamma \in \pi_1$,

$$\Phi(z) \mapsto \Phi_\gamma(z). \quad (3.6)$$

Both the matrix functions $\Phi(z)$ and $\Phi_\gamma(z)$ are fundamental solutions of the same linear ODE (3.4). Therefore, exist an invertible constant matrix M such that

$$\Phi_\gamma(z) = \Phi(z)M. \quad (3.7)$$

We note that by fixing a_0 and Φ_0 , the matrix M can depend on the loop γ ,

$$M \equiv M(\gamma), \quad (3.8)$$

then, the correspondence

$$\rho : \gamma \mapsto M(\gamma) \quad (3.9)$$

is a linear representation of the fundamental group of the punctured Riemann sphere $\pi_1(\mathbb{CP}^1 - \{a_1, \dots, a_\infty\})$.

A representation (3.9) is called a *monodromy representation* of the system (3.4). Given $A(z)$, the subgroup

$$\mathfrak{M} \equiv \{M(\gamma), \gamma \in \pi_1\} \subset GL(N, \mathbb{C}) \quad (3.10)$$

is called the *monodromy group* of the linear system (3.4).

Let $\gamma_1, \gamma_2, \dots, \gamma_\infty$ denote the usual set of generators of the fundamental group π_1 , then the matrices

$$M_\nu \equiv M(\gamma_\nu), \quad \nu = 1, 2, \dots, \infty, \quad (3.11)$$

form a set of generators for the monodromy group $\mathfrak{M}$. They are usually called *monodromy matrices*.

It is always possible to choose the γ_ν 's in such a way that the product $\gamma_1 \cdots \gamma_n \gamma_\infty$ is homotopic to a point. Then the following monodromy constraint holds:

$$M_\infty \cdots M_2 M_1 = \mathbb{1}. \quad (3.12)$$

Suppose that the rational matrix function $A(z)$ has only simple poles. Then all singular points of (3.4) are Fuchsian², and the equation itself is called *Fuchsian system*.

We can write (3.4) with $n + 1$ (regular) singular points as follows

$$\frac{d\Phi}{dz} = \sum_{\nu=1}^n \frac{A_\nu}{z - a_\nu} \Phi(z), \quad (3.13)$$

where the A_ν 's are given $N \times N$ matrices. If the matrix

$$A_\infty = - \sum_{\nu=1}^n A_\nu \quad (3.14)$$

does not vanish, the point $a_\infty = \infty$ is a Fuchsian singularity. We assume that all eigenvalues, which satisfy a non-resonancy condition, are distinct modulo non-zero integers³. This in turn implies that all matrices A_ν are diagonalizable.

We introduce the diagonalizations of A_ν by means of the matrices G_ν , such that

$$A_\nu = G_\nu \Theta_\nu G_\nu^{-1}, \quad \det G_\nu \neq 0, \quad (3.15)$$

and, we assume that A_∞ is diagonal, so that $G_\infty = \mathbb{1}$.

²The point z_0 is called a Fuchsian singular point if the coefficient matrix $A(z)$ has a simple pole at the point z_0 , if $z_0 \neq \infty$, or, if $z_0 = \infty$, it has a simple zero at z_0 .

³For a discussion about the resonant case [64].

The local behaviour of the fundamental matrix solution near the regular singular points can be written as

$$\begin{aligned}\Phi(z \rightarrow a_\nu) &= G_\nu \left(\sum_{m=0}^{\infty} \Phi_{\nu,m} (z - a_\nu)^m \right) (z - a_\nu)^{\Theta_\nu} C_\nu, \quad \text{with} \quad \Phi_{\nu,0} = \mathbb{1}, \\ \Phi(z \rightarrow \infty) &= G_\infty \left(\sum_{m=0}^{\infty} \Phi_{\infty,m} z^{-m} \right) z^{-\Theta_\infty} C_\infty.\end{aligned}\tag{3.16}$$

The connection matrices C_ν are determined by the Fuchsian system (3.13), the initial conditions and the choice of diagonalizations in (3.15). By analytic continuation we have

$$\begin{aligned}\Phi\left((z - a_\nu)e^{i2\pi} + a_\nu\right) &= G_\nu \left(\sum_{m=0}^{\infty} \Phi_{\nu,m} (z - a_\nu)^m e^{i2\pi m} \right) (z - a_\nu)^{\Theta_\nu} e^{i2\pi\Theta_\nu} C_\nu, \\ &= \Phi(z) C_\nu^{-1} e^{i2\pi\Theta_\nu} C_\nu.\end{aligned}\tag{3.17}$$

Then, (3.7) and (3.17) give us the monodromy matrices

$$M_\nu = C_\nu^{-1} e^{i2\pi\Theta_\nu} C_\nu, \quad \nu = 1, \dots, \infty,\tag{3.18}$$

Following [105] some definitions are in order:

- The monodromy data is given by

$$\mathbf{M} = \{(M_1, \dots, M_n, M_\infty) \mid M_\infty M_n \cdots M_1 = \mathbb{1}\}.\tag{3.19}$$

- The singular data of the Fuchsian system (3.13) reads

$$\mathbf{A} = \left\{ (A_1, \dots, A_n, A_\infty) \mid A_\infty = - \sum_{\nu=1}^n A_\nu \right\}.\tag{3.20}$$

- The extended monodromy data

$$\mathbb{M} = \{a_1, \dots, a_n, a_\infty; \mathbf{M}\}.\tag{3.21}$$

Because the monodromy matrices in (3.18) are defined up to conjugation, it is appropriate to work with their traces, which are invariant quantities with

respect to the overall diagonal conjugation, written in terms of the characteristic exponents $\alpha_\nu^\pm$. Namely, one has that

$$\mathrm{Tr} M_\nu = 2e^{i\pi(\alpha_\nu^+ + \alpha_\nu^-)} \cos \pi (\alpha_\nu^+ - \alpha_\nu^-). \quad (3.22)$$

The sum of the roots of the characteristic equation $\alpha_\nu^\pm$ can be thought of as an Abelian “charge”: its value does not matter for the determination of the entries of the monodromy matrix M_ν . From the ODE perspective, the value of the characteristic exponents can be modified with a s-homotopic transformation⁴, like (2.56) and (2.62). We will consider $\alpha_\nu^+ + \alpha_\nu^- = 0$ from now on, and define $\sigma_\nu = \alpha_\nu^+ - \alpha_\nu^- = \theta_\nu$, then

$$p_\nu = 2 \cos \pi \theta_\nu. \quad (3.24)$$

Another set of conjugation-invariant quantities are the composite monodromies

$$p_{\mu\nu} = \mathrm{Tr} M_\mu M_\nu = 2 \cos \pi \sigma_{\mu\nu}, \quad \sigma_{\mu\nu} = \{\sigma_{0t}, \sigma_{1t}, \sigma_{01}\} \quad (3.25)$$

where $p_{\mu\nu} = p_{\nu\mu}$ and $\mu \neq \nu$. From (3.18) we obtain

$$M_\mu M_\nu = C_\mu^{-1} e^{i2\pi\Theta_\mu} C_\mu C_\nu^{-1} e^{i2\pi\Theta_\nu} C_\nu. \quad (3.26)$$

Tracing (3.26) and defining $E_{\mu\nu} = C_\mu C_\nu^{-1}$, we note that

$$\mathrm{Tr} M_\mu M_\nu = \mathrm{Tr} \left(E_{\mu\nu}^{-1} e^{i2\pi\Theta_\mu} E_{\mu\nu} e^{i2\pi\Theta_\nu} \right), \quad (3.27)$$

by the cyclic property of the trace. Since we can choose a basis where the composite monodromy is diagonalizable, $E_{\mu\nu}$ can be given in the $GL(2, \mathbb{C})$ form

$$E_{\mu\nu} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det E = ad - cb \neq 0. \quad (3.28)$$

⁴Let $u(z)$ be a solution to (2.81) with n regular singular points in the finite complex plane, located at $z = a_r, r = 1, 2, \dots, n$ and one regular singular point at $z = \infty$, and introduce a new function $\nu(z)$ by multiplication of powers of the following form:

$$u(z) = \nu(z) \prod_{r=1}^n (z - a_r)^{\rho_r} \quad (3.23)$$

where $\rho_r, r = 1, 2, \dots, n$ are arbitrary complex numbers. This multiplication leads to a change of the differential equation, but the location of the singular points and their singular classification remain. The roots of the characteristic equation, however, shift or are displaced.

Unfortunately, the matrix $E_{\mu\nu}$ is in general a very complicated function of the parameters of the differential equation we started with. On the other hand, the boundary conditions demand a specific behaviour at the critical points, such that one or more connection coefficients – entries of $E_{\mu\nu}$ – vanish. For $E_{\mu\nu}$ lower triangular or upper triangular, the expression (3.27) reduces to

$$\text{Tr} M_\mu M_\nu = 2 \cos \pi(\sigma_\mu + \sigma_\nu). \quad (3.29)$$

Then it is a straightforward calculation to verify that the parameters of the composite monodromy between a_ν and a_μ introduced above will satisfy the quantization condition

$$\sigma_{\mu\nu} = \sigma_\mu + \sigma_\nu + 2n, \quad n \in \mathbb{Z}. \quad (3.30)$$

If the composite monodromy parameter $\sigma_{\mu\nu}$ satisfies the equation (3.30), then the monodromy matrices M_ν and M_μ commute, and hence can be put simultaneously into the lower triangular form. This in turn implies that there exist solutions with the desired behaviour at both points a_ν and a_μ .

Translating this to the boundary conditions for $S(u)$ and $R(z)$ in (2.67) and (2.68), respectively we find that, if one can compute the composite monodromy parameter $\sigma_{\mu\nu}$ in terms of the quantities in each ODE (2.57) and (2.64), we have

$$\sigma_{0u_0}(m1, m2, \beta, \Delta, u_0, \lambda_\ell) = m_1 + m_2 + 2j, \quad j \in \mathbb{Z}, \quad (3.31)$$

$$\sigma_{1z_0}(\theta_k, \Delta, z_0, \omega_n, \lambda_\ell) = \theta_+ + \Delta - 2 + 2n, \quad n \in \mathbb{Z}. \quad (3.32)$$

The first quantization condition defines regularity of the solutions between the regular singular points $u = 0$ and $u = u_0$ in the angular equation, while the second one describes an incoming wave at the horizon $z = z_0$ and regularity at the boundary $z = 1$ of the radial equation.

In the particular case of the Fuchsian system with four regular singular points, we will see that its monodromy group is defined as a representation of the fundamental group of the four-punctured Riemann sphere.

Moreover, we can construct an irreducible representation for the monodromy matrices – up to conjugation – as follows, see [100] for more details on this choice. When $\sigma = \sigma_{0t} \neq 0$, we have

$$\sigma \notin \mathbb{Z}, \quad \sigma \pm \theta_0 \pm \theta_t \notin \mathbb{Z}, \quad \sigma \pm \theta_1 \pm \theta_\infty \notin \mathbb{Z}, \quad (3.33)$$

although we will have a few words about the last condition below.
The monodromy matrices are

$$M_0 = \frac{1}{i \sin \pi \sigma} \begin{pmatrix} \cos \pi \theta_t - e^{i\pi\sigma} \cos \pi \theta_0 & s_i [\cos \pi \theta_0 - \cos \pi(\theta_t - \sigma)] \\ s_i^{-1} [\cos \pi(\theta_t + \sigma) - \cos \pi \theta_0] & e^{-i\pi\sigma} \cos \pi \theta_0 - \cos \pi \theta_t \end{pmatrix} \quad (3.34a)$$

$$M_t = \frac{i}{\sin \pi \sigma} \begin{pmatrix} \cos \pi \theta_0 - e^{i\pi\sigma} \cos \pi \theta_t & s_i e^{i\pi\sigma} [\cos \pi(\theta_t - \sigma) - \cos \pi \theta_0] \\ s_i^{-1} e^{-i\pi\sigma} [\cos \pi \theta_0 - \cos \pi(\theta_t + \sigma)] & e^{-i\pi\sigma} \cos \pi \theta_t - \cos \pi \theta_0 \end{pmatrix} \quad (3.34b)$$

$$M_1 = \frac{i}{\sin \pi \sigma} \begin{pmatrix} e^{-i\pi\sigma} \cos \pi \theta_1 - \cos \pi \theta_\infty & s_e e^{i\pi\sigma} [\cos \pi \theta_\infty - \cos \pi(\theta_1 + \sigma)] \\ s_e^{-1} e^{-i\pi\sigma} [\cos \pi(\theta_1 - \sigma) - \cos \pi \theta_\infty] & \cos \pi \theta_\infty - e^{i\pi\sigma} \cos \pi \theta_1 \end{pmatrix} \quad (3.34c)$$

$$M_\infty = \frac{i}{\sin \pi \sigma} \begin{pmatrix} e^{-i\pi\sigma} \cos \pi \theta_\infty - \cos \pi \theta_1 & s_e [\cos \pi(\theta_1 + \sigma) - \cos \pi \theta_\infty] \\ s_e^{-1} [\cos \pi \theta_\infty - \cos \pi(\theta_1 - \sigma)] & \cos \pi \theta_1 - e^{i\pi\sigma} \cos \pi \theta_\infty \end{pmatrix} \quad (3.34d)$$

and verify

$$M_t M_0 = \begin{pmatrix} e^{i\pi\sigma} & 0 \\ 0 & e^{-i\pi\sigma} \end{pmatrix}, \quad M_\infty M_1 = \begin{pmatrix} e^{-i\pi\sigma} & 0 \\ 0 & e^{i\pi\sigma} \end{pmatrix}, \quad (3.35)$$

$$M_\infty M_1 M_t M_0 = \mathbb{1},$$

in a basis which diagonalizes $M_0 M_t$. The parameters s_i, s_e in (3.34) are then related as follows

$$s \equiv \frac{s_i}{s_e}. \quad (3.36)$$

Notice that the generators of the monodromy group M_ν depend not only on the local monodromies θ_ν , but also on the composite monodromy σ and s . The trace functions (3.24) and (3.30) satisfy the quartic equation,

$$p_0 p_t p_1 p_\infty + p_{0t} p_{1t} p_{01} - (p_0 p_t + p_1 p_\infty) p_{0t} - (p_t p_1 + p_0 p_\infty) p_{1t} - (p_0 p_1 + p_t p_\infty) p_{01} \\ + p_{0t}^2 + p_{1t}^2 + p_{01}^2 + p_0^2 + p_t^2 + p_1^2 + p_\infty^2 = 4, \quad (3.37)$$

the so-called *Fricke-Jimbo relation*. For fixed choices of $\theta_0, \dots, \theta_\infty$ in (3.24) equation (3.37) describes the character variety as a cubic surface in $\mathbb{C}^3$ [155].

This surface admits a parametrization in terms of coordinates (σ, s) of the

form

$$\begin{aligned}
\sin^2 \pi \sigma \cos \pi \sigma_{1t} &= \cos \pi \theta_0 \cos \pi \theta_\infty + \cos \pi \theta_t \cos \pi \theta_1 \\
&\quad - \cos \pi \sigma (\cos \pi \theta_0 \cos \pi \theta_1 + \cos \pi \theta_t \cos \pi \theta_\infty) \\
&\quad - \frac{1}{2} (\cos \pi \theta_\infty - \cos \pi (\theta_1 - \sigma)) (\cos \pi \theta_0 - \cos \pi (\theta_t - \sigma)) s \\
&\quad - \frac{1}{2} (\cos \pi \theta_\infty - \cos \pi (\theta_1 + \sigma)) (\cos \pi \theta_0 - \cos \pi (\theta_t + \sigma)) s^{-1}.
\end{aligned} \tag{3.38}$$

We close by noting that for the special case of interest where $\sigma_{1t} = \theta_1 + \theta_t + 2n$, $n \in \mathbb{Z}$, the expressions above are still valid.

3.2 Riemann-Hilbert Problem

The Riemann-Hilbert (RH) problem, also known as Hilbert's twenty-first problem, posed the question whether it is possible to construct a Fuchsian system (3.13) with given singular points and monodromy.

Namely, one has to analyse if the map between the following spaces is bijective:

$$\left\{ (A_1, \dots, A_\infty) \mid A_\infty = - \sum_{\nu=1}^n A_\nu \right\} \xrightarrow{\text{RH}_a} \left\{ (M_1, \dots, M_\infty) \mid M_\infty \cdots M_1 = \mathbb{1} \right\} \tag{3.39}$$

where $M_\nu = \rho(\gamma_\nu) \in GL(2, \mathbb{C})$. Then, the Riemann-Hilbert problem becomes: given a point $\mathbf{M} = (M_1, \dots, M_\infty)$ on the RHS of (3.39), are there matrices $\mathbf{A} = (A_1, \dots, A_\infty)$ with (3.14) on the LHS such that $\text{RH}_a(\mathbf{A}) = \mathbf{M}$?

Instead of explicitly solving the Riemann-Hilbert problem directly, we are interested in the the first non-trivial case and how one soon becomes embroiled in isomonodromic deformation equations 3.3. Then we shall now consider the 2×2 Fuchsian system containing regular singular points only, and increase successively the number of regular singular points until the map is no more a one-to-one correspondence.

Two regular singular points

Consider the system

$$\frac{d\Phi}{dz} = \frac{A_0}{z}\Phi \quad (3.40)$$

where $z = 0$ is a regular singular point. Notice that the transformation $u = 1/z$ leads to

$$\frac{d\Phi}{du} = \frac{A_\infty}{u}\Phi, \quad A_\infty = -A_0, \quad (3.41)$$

with $z = \infty$ being our second regular singular point.

A fundamental solution $\Phi(z)$ is

$$\Phi(z) = z^{A_0}. \quad (3.42)$$

Under analytic continuation around the singular point, $\Phi(z)$ transforms like

$$\Phi(ze^{i2\pi}) = z^{A_0}e^{i2\pi A_0} = \Phi(z)M_0. \quad (3.43)$$

The corresponding monodromy group is generated by only one monodromy matrix,

$$M_\infty M_0 = \mathbb{1} \quad \therefore \quad M_0 = M_\infty^{-1} = e^{i2\pi A_0}. \quad (3.44)$$

The inverse monodromy problem is uniquely solvable by $A_0 = \frac{1}{i2\pi} \ln M_0$.

Three regular singular points

Using a Möbius transformation⁵ we can fix the singular points at $a_\nu = \{0, 1, \infty\}$, thus the Fuchsian system reads

$$\frac{d\Phi}{dz} = \left[\frac{A_0}{z} + \frac{A_1}{z-1} \right] \Phi, \quad (3.46)$$

and $A_\infty = -A_0 - A_1$. We can assume that the system is traceless, and, for simplicity, the matrices A_0, A_1 and A_∞ have non-zero eigenvalues, $\pm\theta_0, \pm\theta_1$ and

⁵The general form of a Möbius transformation is given by

$$w(z) = \frac{az+b}{cz+d}, \quad (3.45)$$

where a, b, c, d are any complex numbers satisfying $ad - bc \neq 0$.

$\pm\theta_\infty$, respectively. The singularity at $z = \infty$ plays a role of the normalization point of the fundamental matrix solution $\Phi(z)$, thus (3.15) implies

$$A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}. \quad (3.47)$$

Consider the most general form of A_0 and A_1

$$A_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A_1 = \begin{pmatrix} e & f \\ g & h \end{pmatrix}. \quad (3.48)$$

The traceless condition, $\text{Tr} A_\nu = 0$, gives

$$\begin{aligned} a + d &= 0, \\ e + h &= 0, \end{aligned} \quad (3.49)$$

then the matrices A_0 and A_1 have the form

$$A_0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad A_1 = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}, \quad (3.50)$$

while substituting (3.50) in $A_0 + A_1 + A_\infty = 0$ gives

$$\begin{aligned} a + e + \theta_\infty &= 0, \\ b + f &= 0, \\ c + g &= 0, \end{aligned} \quad (3.51)$$

where by notation $a = w$, and clearly

$$A_0 = \begin{pmatrix} w & b \\ c & -w \end{pmatrix}, \quad A_1 = \begin{pmatrix} -\theta_\infty - w & -b \\ -c & \theta_\infty + w \end{pmatrix}. \quad (3.52)$$

Finally, we diagonalize A_0 and A_1 to obtain

$$\begin{aligned} b c &= -(w - \theta_0)(w + \theta_0), \\ b &= -u(w - \theta_0), \quad c = \frac{1}{u}(w + \theta_0), \\ A_0 &= \begin{pmatrix} w & -u(w - \theta_0) \\ \frac{1}{u}(w + \theta_0) & -w \end{pmatrix}, \quad A_1 = \begin{pmatrix} -\theta_\infty - w & u(w - \theta_0) \\ -\frac{1}{u}(w + \theta_0) & \theta_\infty + w \end{pmatrix}, \end{aligned} \quad (3.53)$$

where $w = \frac{\theta_1^2 - \theta_0^2 - \theta_\infty^2}{2\theta_\infty}$. Thus, the matrices A_i are parametrized by $\theta_0, \theta_1, \theta_\infty$ and u , with the dimension of the space given by the free parameters,

$$\dim \mathbf{A} = 4. \quad (3.54)$$

At the same time, the full space of the monodromy data (3.19) can be represented by the monodromy matrices M_0, M_1 and M_∞ of the system (3.46), which satisfy the equations

$$\begin{aligned} \det M_\nu &= \mathbb{1}, & \nu &= 0, 1, \infty, \\ M_\infty M_1 M_0 &= \mathbb{1}, \\ M_\infty &= \begin{pmatrix} e^{i2\pi\theta_\infty} & 0 \\ 0 & e^{-i2\pi\theta_\infty} \end{pmatrix}. \end{aligned}$$

According to (3.55), the number of independent parameters of the monodromy group $\mathbf{M}$ is equal to 4, i.e., the dimension of the space is

$$\dim \mathbf{M} = 4. \quad (3.55)$$

Therefore, $\dim \mathbf{M} = \dim \mathbf{A}$, as one expects a one-to-one correspondence between the 2×2 Fuchsian system with three regular singular points and its monodromy data. For instance, the monodromy data concerning to equation (2.79) are given by

$$\theta_0 = \Delta - 2, \quad \theta_1 = 1 + \ell, \quad \theta_\infty = 1 + \omega. \quad (3.56)$$

Four regular singular points

The Fuchsian system is now

$$\frac{d\Phi}{dz} = \left[\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1} \right] \Phi \quad (3.57)$$

where the singularities are located at $a_\nu = \{0, t, 1, \infty\}$, and the change of variable $u = 1/z$ gives

$$A_\infty = -(A_0 + A_t + A_1). \quad (3.58)$$

The A_ν matrices are

$$A_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A_t = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, \quad A_1 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad A_\infty = \begin{pmatrix} u & v \\ w & z \end{pmatrix}. \quad (3.59)$$

We assume that the matrices A_ν are traceless and their eigenvalues are non-zero, such that

$$A_\nu = G_\nu \Theta_\nu G_\nu^{-1}, \quad \det G_\nu \neq 0, \quad \Theta_\nu = \begin{pmatrix} \theta_\nu & 0 \\ 0 & -\theta_\nu \end{pmatrix}, \quad \nu = 0, t, 1, \infty, \quad (3.60)$$

and, by fixing $G_\infty = \mathbb{1}$, one has A_∞ of the form

$$A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}. \quad (3.61)$$

This parametrization will be different from the one used in [103] by a gauge transformation of the A_ν 's matrices, but this will be developed further in subsection 3.3.2.

The condition, $\text{Tr } A_\nu = 0$, implies:

$$a + d = 0, \quad e + h = 0, \quad p + s = 0, \\ A_0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad A_t = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}, \quad A_1 = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \quad (3.62)$$

while (3.58) introduces the following constraints

$$a + e + p = -\theta_\infty, \\ b + f + q = 0, \\ c + g + r = 0. \quad (3.63)$$

Following (3.53) for three regular singular points case, we can obtain the A_ν matrices in the form

$$A_\nu = \begin{pmatrix} w_\nu & -u_\nu(w_\nu - \theta_\nu) \\ \frac{1}{u_\nu}(w_\nu + \theta_\nu) & -w_\nu \end{pmatrix}, \quad \nu = 0, t, 1. \quad (3.64)$$

Therefore, the set of A_ν matrices can be parametrized by $\theta_\infty, w_\nu, u_\nu, \theta_\nu$ and two constraint equations

$$\begin{aligned}\sum_\nu u_\nu (w_\nu - \theta_\nu) &= 0, \\ \sum_\nu \frac{1}{u_\nu} (w_\nu + \theta_\nu) &= 0,\end{aligned}\tag{3.65}$$

which imply that the complex dimension of the manifold $\mathbf{A}$ is

$$\dim \mathbf{A} = 8.\tag{3.66}$$

Now, we consider the space of monodromy matrices M_ν , $\nu = 0, t, 1, \infty$, associated to (3.57). The relevant monodromy data, which give the $SL(2, \mathbb{C})$ representation of the respective loops γ_ν , satisfy the equations

$$\begin{aligned}\det M_\nu &= \mathbb{1}, \quad \nu = 0, t, 1, \infty, \\ M_\infty M_1 M_t M_0 &= \mathbb{1}, \\ M_\infty &= \begin{pmatrix} e^{i2\pi\theta_\infty} & 0 \\ 0 & e^{-i2\pi\theta_\infty} \end{pmatrix}.\end{aligned}\tag{3.67}$$

Here, we can reduce the number of elements in M_ν by using (3.67), in such a way that the complex dimension of the manifold of the monodromy data of the system (3.57) equals 7, then

$$\dim \mathbf{M} = \dim \mathbf{A} - 1.\tag{3.68}$$

In other words, for $n = 3$ (4 regular singularities) one finds the simplest non-trivial case for the Riemann-Hilbert map. Given a monodromy data, one expects a one-parameter family of equations (3.57) in the space $\mathbf{A}$. Nevertheless, one may introduce the idea of deforming the Fuchsian system while preserving the monodromies, such that the matrices A_ν satisfy non-linear differential equations, now known as the Schlesinger equations and described in 3.3.1.

3.3 Isomonodromic deformation for Painlevé VI

The modern theory of the isomonodromic deformations was developed in the pioneering work of Jimbo, Miwa and Ueno [103, 105], although its origin goes back to the classical paper of Fuchs [70], Garnier [73, 74] and Schlesinger [147].

The isomonodromic deformations associated to (3.13) were first considered in the works of Schlesinger [147]. The idea is to vary the position of the singularities a_ν and matrices A_ν but preserve the monodromies M_ν invariant. In other words, we wish to keep invariant the monodromy data as we deform equation (3.13) through some deformation parameters.

We recall that the 2×2 Fuchsian system with four regular singular points located at $z = \{0, t, 1, \infty\}$ can be written as

$$\frac{d\Phi}{dz} = A(z)\Phi(z) = \left[\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1} \right] \Phi(z), \quad (3.69)$$

$$A_\infty = -(A_0 + A_t + A_1) = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}.$$

Specifically, in (3.69) the unique deformation parameter is given by t .

3.3.1 The Schlesinger system

Now, since we are interested in properties of the solutions of (3.69), which depend solely on the monodromy data we are free to change the parameters of the equations as long as they do not change the monodromy data. The isomonodromic deformations, parametrized by a change of t , view $A(z)$ in (3.69) as the “ z -component” of a flat holomorphic connection $\mathcal{A}$. The “ t -component” can be guessed immediately:

$$\mathcal{A}_z = A(z), \quad \mathcal{A}_t = -\frac{A_t}{z-t}, \quad (3.70)$$

and the flatness condition⁶

$$\partial_t \mathcal{A}_z - \partial_z \mathcal{A}_t + [\mathcal{A}_z, \mathcal{A}_t] = 0 \quad (3.71)$$

gives us the Schlesinger equations

$$\begin{aligned} \frac{\partial A_0}{\partial t} &= -\frac{1}{t}[A_0, A_t], & \frac{\partial A_1}{\partial t} &= -\frac{1}{t-1}[A_1, A_t], \\ \frac{\partial A_t}{\partial t} &= \frac{1}{t}[A_0, A_t] + \frac{1}{t-1}[A_1, A_t]. \end{aligned} \quad (3.72)$$

⁶A Pfaffian system is completely integrable if and only if it satisfies a compatibility condition.

The system (3.72) represents a set of differential relations for the entries of the matrices A_ν . When integrated, these equations will define a family of flat holomorphic connections $\mathcal{A}(z, t)$ with the same monodromy data, parametrized by a possibly complex parameter t . The set of corresponding $A(z, t)$ will be called the isomonodromic family.

Jimbo, Miwa and Ueno [103, 105] showed that this Schlesinger system is equivalent to a second order non-linear differential equation, the Painlevé VI equation, assuming the normalization of A_∞ in (3.69). We can show that the off-diagonal element of $A(z)$ in (3.69) can be written in the form

$$A_{12}(z) = \frac{k(z - \lambda)}{z(z - 1)(z - t)}, \quad (3.73)$$

where

$$\begin{aligned} k &= tu_0(w_0 - \theta_0) - (1 - t)u_1(w_1 - \theta_1), \\ k\lambda &= tu_0(w_0 - \theta_0). \end{aligned} \quad (3.74)$$

Then, $\lambda = \lambda(t)$ is a single zero of $A_{12}(z)$ and satisfies the Painlevé VI equation. See Appendix B, for a brief exposition on Painlevé equations.

On the other hand, the Painlevé VI equation can be studied with a different approach if we notice that the Schlesinger equations also present a Hamiltonian structure which is connected to the Garnier system $\mathcal{G}_1$ [91, 124].

3.3.2 From the Garnier system to the Heun Equation

The fundamental matrix solution of (3.4) is composed by vectors of linearly independent solutions

$$\Phi(z) = \begin{pmatrix} y_1(z) & w_1(z) \\ y_2(z) & w_2(z) \end{pmatrix}, \quad (3.75)$$

where each column of the matrix function $\Phi(z)$ satisfies the system of linear differential equations,

$$\frac{dy_j}{dz} = \sum_{k=1}^2 A_{jk}(z) y_k, \quad j = 1, 2 \quad (3.76)$$

and it is straightforward to check that $y_j(z)$, $j = 1, 2$ satisfies the following second order differential equation

$$y'' - (\text{Tr} A + \partial_z \log A_{12}) y' + (\det A - \partial_z A_{11} + A_{11} \partial_z \log A_{12}) y = 0, \quad (3.77)$$

that depends on the entries of $A(z)$. In 3.3 we have defined the $A_{12}(z)$ element, therefore by replacing it into (3.77), this equation becomes

$$y'' + p(z) y' + q(z) y = 0 \quad (3.78a)$$

$$p(z) = \frac{1 - \text{Tr} A_0}{z} + \frac{1 - \text{Tr} A_1}{z - 1} + \frac{1 - \text{Tr} A_t}{z - t} - \frac{1}{z - \lambda}, \quad (3.78b)$$

$$\begin{aligned} q(z) = & \frac{\det A_0}{z^2} + \frac{\det A_1}{(z - 1)^2} + \frac{\det A_t}{(z - t)^2} \\ & + \frac{\kappa}{z(z - 1)} - \frac{t(t - 1)H}{z(z - 1)(z - t)} + \frac{[A_\infty]_{11}}{z(z - 1)} + \frac{\lambda(\lambda - 1)\mu}{z(z - 1)(z - \lambda)} \\ & - \frac{t(t - 1)}{z(z - 1)(z - t)} \left[\frac{[A_0]_{11} + [A_t]_{11}}{t} + \frac{[A_1]_{11} + [A_t]_{11}}{t - 1} + \frac{[A_t]_{11}}{\lambda - t} \right], \end{aligned} \quad (3.78c)$$

where

$$\begin{aligned} \kappa &= \det A_\infty - \det A_0 - \det A_1 - \det A_t, \\ \mu &= \frac{[A_0]_{11}}{\lambda} + \frac{[A_t]_{11}}{\lambda - t} + \frac{[A_1]_{11}}{\lambda - 1}, \\ H &= -\frac{1}{t} \text{Tr} A_0 \text{Tr} A_t - \frac{1}{t - 1} \text{Tr} A_1 \text{Tr} A_t + \frac{1}{t} \text{Tr}(A_0 A_t) + \frac{1}{t - 1} \text{Tr}(A_1 A_t). \end{aligned} \quad (3.79)$$

The expression inside the brackets in (3.78c) can be written in a more convenient form

$$\frac{[A_0]_{11} + [A_t]_{11}}{t} + \frac{[A_1]_{11} + [A_t]_{11}}{t - 1} + \frac{[A_t]_{11}}{\lambda - t} = \frac{\lambda(\lambda - 1)\mu}{t(t - 1)} + \frac{(\lambda - t)[A_\infty]_{11}}{t(t - 1)}, \quad (3.80)$$

and, then we define

$$K = H + \frac{\lambda(\lambda - 1)\mu}{t(t - 1)} + \frac{(\lambda - t)[A_\infty]_{11}}{t(t - 1)}. \quad (3.81)$$

Assuming $\det A_i = 0$ and $\text{Tr} A_i = \theta_i$ for $i = 0, t, 1$, and a different parametrization for the diagonal matrix $A_\infty = \text{diag}(\kappa_1, \kappa_2)$, the resulting equation will be

slightly different from the Heun equation, but has five singularities

$$y'' + \left(\frac{1-\theta_0}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_t}{z-t} - \frac{1}{z-\lambda} \right) y' + \left[\frac{\kappa_1(1+\kappa_2)}{z(z-1)} - \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda-1)\mu}{z(z-1)(z-\lambda)} \right] y = 0, \quad (3.82)$$

and we will refer to this equation as the *deformed Heun equation*.

Hence, λ will appear as an extra singular point of (3.82), which does not correspond to the poles of $A(z)$. A direct calculation shows that this singular point has indicial exponents $(0, 2)$, with no logarithmic tails, and hence corresponds to an *apparent singularity*. The monodromy matrix is therefore trivial, $M_\lambda = \mathbb{1}$.

The absence of logarithmic behavior at $z = \lambda$ results in the following algebraic relation between K, μ, λ , and t :

$$K(\mu, \lambda, t) = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \left[\mu^2 - \left(\frac{\theta_0}{\lambda} + \frac{\theta_1}{\lambda-1} + \frac{\theta_t-1}{\lambda-t} \right) \mu + \frac{\kappa_1(1+\kappa_2)}{\lambda(\lambda-1)} \right]. \quad (3.83)$$

Notice that $K(\mu, \lambda, t)$ is a rational function and corresponds exactly to the Painlevé VI Hamiltonian [139]. It can be thought of as a one-dimensional classical dynamic system, where μ , λ , and t denote the particle momentum, coordinate, and time⁷, respectively. It can be shown [101, 139] that the classical motion of the particle is determined by the Hamilton equations

$$\frac{d\lambda}{dt} = \frac{\partial K}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial K}{\partial \lambda}, \quad (3.84)$$

ensuring that monodromy data of the differential equation (3.82) stay constant under variations of the parameters (λ, μ) satisfying (3.84). The coordinates (λ, μ) are Darboux coordinates for the natural symplectic structure of the Garnier system $\mathcal{G}_1$ [155].

It can be verified that the expressions above are equivalent to a non-linear second order differential equation for $\lambda(t)$, the so-called *Painlevé VI equation*

⁷The physical interpretation of t depends on the application. For instance, in the 2D Ising model [61], $t \equiv \sinh^4(2J\beta)$ is the temperature.

[70]:

$$\begin{aligned} \lambda'' = & \frac{1}{2} \left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t} \right) (\lambda')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t} \right) \lambda' + \\ & + \frac{\lambda(\lambda-1)(\lambda-t)}{2t^2(t-1)^2} \left[(\theta_\infty - 1)^2 - \frac{\theta_0^2 t}{\lambda^2} + \frac{\theta_1^2(t-1)}{(\lambda-1)^2} - \frac{(\theta_t^2 - 1)t(t-1)}{(\lambda-t)^2} \right], \end{aligned} \quad (3.85)$$

where we set $\theta_\infty = \kappa_1 - \kappa_2$. Then, the Painlevé VI equation is equivalent to the Hamiltonian system, called the Painlevé VI system. This statement extends to the other Painlevé equations [101].

It has been known since the pioneering work of the Kyoto school in the 1980's – see [101] for the general review and [43] for the specific case we consider here – that the isomonodromic flow defined in terms of (3.84) is Hamiltonian, and can be conveniently defined by the isomonodromic τ -function

$$\frac{d}{dt} \log \tau(t) = \frac{1}{t} \text{Tr}(\hat{A}_0 \hat{A}_t) + \frac{1}{t-1} \text{Tr}(\hat{A}_1 \hat{A}_t), \quad (3.86)$$

where the $\hat{A}_i$ matrices are traceless. For convenience, we have changed to a gauge where $\text{Tr} A_i = \theta_i$. Thus, the τ -function can be related to the parameters of (3.82) by doing $\hat{A}_i \rightarrow A_i - \frac{1}{2} \text{Tr} A_i \mathbb{1}$,

$$\frac{d}{dt} \log \tau(t) = \frac{1}{t} \text{Tr}(A_0 A_t) + \frac{1}{t-1} \text{Tr}(A_1 A_t) - \frac{1}{2t} \theta_0 \theta_t - \frac{1}{2(t-1)} \theta_1 \theta_t. \quad (3.87)$$

Replacing (3.87) and (3.79) in (3.81) gives us the relation between the PVI τ -function and the accessory parameter of (3.82) in the following way

$$\frac{d}{dt} \log \tau(t) = K + \frac{1}{2t} \theta_0 \theta_t + \frac{1}{2(t-1)} \theta_1 \theta_t - \frac{\lambda(\lambda-1)\mu}{t(t-1)} - \frac{(\lambda-t)\kappa_1}{t(t-1)}, \quad (3.88)$$

or, more precisely

$$t(t-1) \frac{d}{dt} \log \tau(t) = t(t-1) K + \frac{(t-1)}{2} \theta_0 \theta_t + \frac{t}{2} \theta_1 \theta_t - \lambda(\lambda-1)\mu - (\lambda-t)\kappa_1. \quad (3.89)$$

A second condition for the τ -function can be written by taking the derivative of (3.87) and making use of the Schlesinger equations (3.72). It reads

$$\frac{d}{dt} \left[t(t-1) \frac{d}{dt} \log \tau(t) \right] = \text{Tr}(A_0 A_t) + \text{Tr}(A_1 A_t) - \frac{1}{2} \theta_0 \theta_t - \frac{1}{2} \theta_1 \theta_t. \quad (3.90)$$

By setting the initial conditions of the canonical variables $(\lambda(t), \mu(t))$ at $t = t_0$, we find that the dynamical system flows, while the monodromy data is preserved, and reduces (3.82) to the Heun equation (2.66). One might think of these conditions as the initial conditions for the Schlesinger equations [136]. Then, we have

$$t = t_0, \quad \lambda(t_0) = t_0, \quad \theta_t = \theta_{t_0} - 1, \quad (3.91)$$

where the apparent singularity coalesces into the singular point t_0 . We notice that (3.82) and (3.83) enforce the following relation among $\mu(t)$, the Hamiltonian and the accessory parameter K_0

$$t = t_0, \quad K(t_0, \lambda, \mu) - \mu(t_0) = K_0 \quad \Rightarrow \quad \mu(t_0) = -\frac{K_0}{\theta_t}, \quad (3.92)$$

$$\mu(t_0) = \frac{K_0}{1 - \theta_{t_0}}$$

and finally, $\kappa_1(1 + \kappa_2) = \kappa_- \kappa_+$, which implies $\theta_\infty = \theta_{\infty_0} + 1$. Then we recognize the Heun equation (2.66) as an element of a family of an isomonodromically deformed system.

The initial conditions for $(\lambda(t_0), \mu(t_0))$ can be translated in terms of the isomonodromic τ -function in the following form:

$$\left. \frac{d}{dt} \log \tau(t; \{\vec{\theta}, \vec{\sigma}\}^-) \right|_{t=t_0} = \frac{(\theta_{t_0} - 1)\theta_1}{2(\theta_{t_0} - 1)} + \frac{(\theta_{t_0} - 1)\theta_0}{2\theta_{t_0}} + K_0, \quad (3.93)$$

$$\left. \frac{d}{dt} \left[t(t-1) \frac{d}{dt} \log \tau(t; \{\vec{\theta}, \vec{\sigma}\}^-) \right] \right|_{t=t_0} = \frac{1}{2} (\theta_{t_0} - 1) (\theta_{\infty_0} - \theta_{t_0} + 2),$$

where the parameters – i.e. the monodromy data – of the τ -function are given by

$$\{\vec{\theta}, \vec{\sigma}\}^- = \{\theta_0, \theta_t = \theta_{t_0} - 1, \theta_1, \theta_\infty = \theta_{\infty_0} + 1; \sigma_{0t} = \sigma_{0t_0} - 1, \sigma_{1t} = \sigma_{1t_0} - 1\}. \quad (3.94)$$

The conditions in (3.93) can be understood as determining the initial value problem for the dynamical system defined by (3.84).

We plan to apply (3.93) for the problem of scalar perturbations, namely to the radial equation (2.64) and the angular equation (2.57) (the initial conditions obtained for the problem of vector perturbations will be described by an extra term due to the presence of an apparent singularity, and will be discussed

in detail in Chapter 5). Then, two systems of transcendental equations are written in terms of the black hole parameters and can be solved for the angular eigenvalue problem and the quasi-normal modes frequencies.

The theory of isomonodromic deformations gives rise to non-linear special functions such as the Painlevé transcendents, which appear in numerous problems of mathematical physics. Moreover, it has recently been shown [17] that casting the accessory parameter problem in terms of the monodromy data gives a higher numerical precision when compared with shooting methods.

3.4 Conformal Blocks expansion

This Section is devoted to present the isomonodromic τ -function of the Painlevé VI equation and its asymptotic expansion. A first attempt can be found in the seminal work by Jimbo in 1982, where the first three terms of the asymptotic expansion were obtained in terms of the monodromy [102].

In 2012, Gamayun, Iorgov and Lisovyy discovered that the τ -function of the Painlevé VI equation can be expressed as a sum of $c = 1$ conformal blocks, multiplied by some structure constants written in terms of Barnes functions and s being an instanton counting parameter [71, 72]. Their formula gives the general solution of the Painlevé VI equation.

This relation has been proved in two independent ways: one approach in [96] considers the operator-valued monodromy of conformal blocks with additional level 2 degenerate insertions. At $c = 1$, the Fourier transform of such conformal blocks reduces their “quantum” monodromy to $SL(2, \mathbb{C})$ matrices that solve the associated Fuchsian system. The second approach [30] uses embedding of two copies of the Virasoro algebra into the super-Virasoro algebra extended by Majorana fermions to prove certain bilinear relations for 4-point conformal blocks, equivalent to Painlevé VI equation.

More recently, Gavrylenko and Lisovyy derived the Fredholm determinant representation of the isomonodromic τ -function, in the case of the Fuchsian system with n regular singular points on the Riemann sphere and generic monodromy in $GL(N, \mathbb{C})$, via a decomposition of the n -punctured sphere into pairs of pants. We strongly recommend to go through [75] for a thorough discussion of the pants decomposition of the Riemann-Hilbert problem.

Consider now the complete Conformal Blocks expansion of the Painlevé VI τ -function, given in [71, 72], around the critical point $t = 0$:

$$\tau(t) = \sum_{n \in \mathbb{Z}} C(\vec{\theta}, \sigma + 2n) s^n t^{\frac{1}{4}((\sigma+2n)^2 - \theta_0^2 - \theta_t^2)} \mathcal{B}(\vec{\theta}, \sigma + 2n; t), \quad (3.95)$$

where the parameters in this function are related to the monodromy data $\{\vec{\theta}, \vec{\sigma}\} = \{\theta_0, \theta_t, \theta_1, \theta_\infty; \sigma_0, \sigma_t\}$ and we assume that the real part of σ obeys

$$0 \leq \operatorname{Re} \sigma < 1. \quad (3.96)$$

The parameters σ and s play the role of two integration constants. The structure constants are given in terms of Barnes functions

$$C(\vec{\theta}, \sigma) = \prod_{\alpha, \beta = \pm} \frac{G(1 + \frac{1}{2}(\theta_1 + \alpha\theta_\infty + \beta\sigma)) G(1 + \frac{1}{2}(\theta_t + \alpha\theta_0 + \beta\sigma))}{G(1 + \sigma) G(1 - \sigma)} \quad (3.97)$$

where $G(z)$ is defined by the solution of the functional equation $G(z+1) = \Gamma(z)G(z)$, with $G(1) = 1$ and $\Gamma(z)$ is the Euler gamma function. The last term in (3.95), $\mathcal{B}(\vec{\theta}, \sigma; t)$ coincides with the $c = 1$ Virasoro conformal block function, where the simple monodromies represent the conformal dimensions of the fields in a four-point correlation function, and the composite monodromy can be interpreted as the conformal dimension of the intermediate channel. Via the AGT conjecture [14], the conformal blocks are explicitly given by combinatorial series in terms of Nekrasov functions, which implies

$$\mathcal{B}(\vec{\theta}, \sigma; t) = (1-t)^{\frac{1}{2}\theta_1\theta_t} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{B}_{\lambda, \mu}(\vec{\theta}, \sigma) t^{|\lambda|+|\mu|} \quad (3.98a)$$

$$\begin{aligned} \mathcal{B}_{\lambda, \mu}(\vec{\theta}, \sigma) &= \prod_{(i, j) \in \lambda} \frac{((\theta_t + \sigma + 2(i-j))^2 - \theta_0^2)((\theta_1 + \sigma + 2(i-j))^2 - \theta_\infty^2)}{16h_\lambda^2(i, j)(\lambda'_j - i + \mu_i - j + 1 + \sigma)^2} \\ &\times \prod_{(i, j) \in \mu} \frac{((\theta_t - \sigma + 2(i-j))^2 - \theta_0^2)((\theta_1 - \sigma + 2(i-j))^2 - \theta_\infty^2)}{16h_\lambda^2(i, j)(\mu'_j - i + \lambda_i - j + 1 - \sigma)^2}, \end{aligned} \quad (3.98b)$$

where the sum is over Young tableaux λ and μ contained in $\mathbb{Y}$, the set of all such diagrams which represent ordered partitions of integers, and denoted by $\mathcal{B}_{\lambda, \mu}$. $|\lambda|$ and $|\mu|$ are number of indistinguishable boxes, or the size of the diagram. For

each box situated at (i, j) in λ , λ_i are the number of boxes at row i of λ , and λ'_j corresponds to the number of boxes at column j of λ ; $h(i, j) = \lambda_i + \lambda'_j - i - j + 1$ is the hook length of the box at (i, j) . For the sake of clarity, let us show some conformal blocks contributions,

$$\begin{aligned} \mathcal{B}_{\emptyset, \emptyset} &= 1, \\ \mathcal{B}_{\square, \emptyset} &= \frac{((\theta_t + \sigma)^2 - \theta_0^2)((\theta_1 + \sigma)^2 - \theta_\infty^2)}{16\sigma^2}, \\ \mathcal{B}_{\square, \square} &= \frac{((\theta_t + \sigma)^2 - \theta_0^2)((\theta_1 + \sigma)^2 - \theta_\infty^2)((\theta_t - \sigma)^2 - \theta_0^2)((\theta_1 - \sigma)^2 - \theta_\infty^2)}{256(1 + \sigma)^2(1 - \sigma)^2}, \end{aligned} \quad (3.99)$$

$$\begin{aligned} \mathcal{B}_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}, \emptyset} &= \frac{1}{36864(\sigma + 1)^2(\sigma - 1)^2\sigma^2}((\theta_t + \sigma)^2 - \theta_0^2)((\theta_1 + \sigma)^2 - \theta_\infty^2) \\ &((\theta_t + \sigma - 2)^2 - \theta_0^2)((\theta_1 + \sigma - 2)^2 - \theta_\infty^2)((\theta_t + \sigma + 2)^2 - \theta_0^2)((\theta_1 + \sigma + 2)^2 - \theta_\infty^2). \end{aligned}$$

Finally, the parameter s is given in terms of the monodromy data by:

$$s = \frac{(w_{1t} - 2p_{1t} - p_{0t}p_{01}) - (w_{01} - 2p_{01} - p_{0t}p_{1t})\exp(\pi i\sigma_{0t})}{(2\cos\pi(\theta_t - \sigma_{0t}) - p_0)(2\cos\pi(\theta_1 - \sigma_{0t}) - p_\infty)} \quad (3.100)$$

where

$$w_{0t} = p_0p_t + p_1p_\infty, \quad w_{1t} = p_1p_t + p_0p_\infty, \quad w_{01} = p_0p_1 + p_tp_\infty. \quad (3.101)$$

Combinatorial expansions of the type of (3.95) can also be found around the two remaining critical points $t = 1, \infty$, as their role is analogous to that of $t = 0$. For instance. the series expansion around $t = 1$ is obtained when one makes the following exchanges

$$t \longleftrightarrow 1 - t, \quad \theta_0 \longleftrightarrow \theta_1, \quad \sigma_{0t} \longleftrightarrow \sigma_{1t} \quad (3.102)$$

and, in the definition of s , one must replace $e^{i\pi\sigma_{0t}} \rightarrow e^{-i\pi\sigma_{1t}}$.

3.4.1 Asymptotic expansion of PVI τ -function

Following the approach in [115] – inspired by [117] – we are able to expand the isomonodromic τ -function for t sufficiently close to zero. Let us define the τ -function as follows,

$$\tau(\{\vec{\theta}, \vec{\sigma}\}; t) = t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_t^2)} (1 - t)^{\frac{1}{2}\theta_1\theta_t} \hat{\tau}(t), \quad (3.103)$$

where $\hat{\tau}(t)$ will be a series in t that depends on the structure constant and the conformal blocks,

$$\hat{\tau}(t) = \sum_{n \in \mathbb{Z}} C(\vec{\theta}, \sigma + 2n) t^{n^2 + n\sigma} s^n \mathcal{B}(\vec{\theta}, \sigma + 2n; t). \quad (3.104)$$

We can rewrite the $c = 1$ blocks (3.98a) in terms of the levels L

$$(1 - t)^{-\frac{1}{2}\theta_1\theta_1} \mathcal{B}(\vec{\theta}, \sigma + 2n; t) = \sum_{L=0}^{\infty} \mathcal{B}_L^{(n)}(\vec{\theta}, \sigma) t^L, \quad (3.105)$$

where

$$\mathcal{B}_L^{(n)}(\vec{\theta}, \sigma) = \sum_{|\lambda| + |\mu| = L} \mathcal{B}_{\lambda, \mu}^{(n)}(\vec{\theta}, \sigma + 2n) \quad (3.106)$$

is a restricted sum at level L of the coefficients (3.98b). For instance, if $n = 0$ the expansion in powers of L is given by

$$\begin{aligned} (1 - t)^{-\frac{1}{2}\theta_1\theta_1} \mathcal{B} &= \mathcal{B}_{\emptyset, \emptyset} + (\mathcal{B}_{\square, \emptyset} + \mathcal{B}_{\emptyset, \square}) t \\ &+ \left(\mathcal{B}_{\square\square, \emptyset} + \mathcal{B}_{\square, \square} + \mathcal{B}_{\square, \emptyset} + \mathcal{B}_{\emptyset, \square} + \mathcal{B}_{\emptyset, \square\square} \right) t^2 \\ &+ \left(\mathcal{B}_{\square\square\square, \emptyset} + \mathcal{B}_{\square\square, \emptyset} + \mathcal{B}_{\square, \square} + \mathcal{B}_{\square, \emptyset} + \mathcal{B}_{\emptyset, \square} + \mathcal{B}_{\emptyset, \square\square} \right. \\ &\quad \left. + \mathcal{B}_{\square, \square\square} + \mathcal{B}_{\square, \square\square} + \mathcal{B}_{\emptyset, \square\square} + \mathcal{B}_{\emptyset, \square\square} + \mathcal{B}_{\emptyset, \square\square\square} \right) t^3 + \dots \end{aligned} \quad (3.107)$$

Normalizing $\hat{\tau}(t)$ ⁸⁹ in (3.103), the Painlevé VI τ -function reads

$$\begin{aligned} \tau(t) &= C_0 t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_1^2)} (1 - t)^{\frac{1}{2}\theta_1\theta_1} \left\{ 1 + \left[\mathcal{B}_{\square, \emptyset}^{(0)} + \mathcal{B}_{\emptyset, \square}^{(0)} + \frac{C_1}{C_0} s t^\sigma + \frac{C_{-1}}{C_0} \frac{1}{s t^\sigma} \right] t \right. \\ &+ \left[\frac{C_{-1}}{C_0} \frac{1}{s t^\sigma} \left(\mathcal{B}_{\square, \emptyset}^{(-1)} + \mathcal{B}_{\emptyset, \square}^{(-1)} \right) + \left(\mathcal{B}_{\square\square, \emptyset}^{(0)} + \mathcal{B}_{\square, \square}^{(0)} + \mathcal{B}_{\emptyset, \square}^{(0)} + \mathcal{B}_{\emptyset, \square\square}^{(0)} \right) \right. \\ &\quad \left. \left. + \frac{C_1}{C_0} s t^\sigma \left(\mathcal{B}_{\square, \emptyset}^{(1)} + \mathcal{B}_{\emptyset, \square}^{(1)} \right) \right] t^2 + \mathcal{O}(t^3) \right\} \end{aligned} \quad (3.108)$$

⁸Instead of $C(\vec{\theta}, \sigma + 2n)$, we denote the structure constant as C_n

⁹We divide by C_0 , therefore the zeroth order term of the $\hat{\tau}$ expansion depends only on the conformal blocks $\mathcal{B}_{\emptyset, \emptyset}$ given in (3.99)

It can be shown that the following relation between the structure constant holds [102, 115]:

$$\frac{C_1}{C_0} = -\frac{(\theta_0^2 - (\theta_t - \sigma)^2)(\theta_\infty^2 - (\theta_1 - \sigma)^2)}{16\sigma^2(1 + \sigma)^2} \frac{\Gamma^2(1 - \sigma)\Gamma(1 + \frac{1}{2}(\theta_1 - \theta_\infty + \sigma))}{\Gamma^2(1 + \sigma)\Gamma(1 + \frac{1}{2}(\theta_1 - \theta_\infty - \sigma))} \frac{\Gamma(1 + \frac{1}{2}(\theta_1 + \theta_\infty + \sigma))\Gamma(1 + \frac{1}{2}(\theta_t - \theta_0 + \sigma))\Gamma(1 + \frac{1}{2}(\theta_t + \theta_0 + \sigma))}{\Gamma(1 + \frac{1}{2}(\theta_1 + \theta_\infty - \sigma))\Gamma(1 + \frac{1}{2}(\theta_t - \theta_0 - \sigma))\Gamma(1 + \frac{1}{2}(\theta_t + \theta_0 - \sigma))}. \quad (3.109)$$

For $n < 0$, we change $\sigma \rightarrow -\sigma$ in the formula above. Up to a common multiplier, C_n 's essentially coincide with the chiral parts of the corresponding structure constants in time-like Liouville theory [82, 146, 167].

Notice that the Γ factors and s in (3.108) define $\hat{s}$ in [102], here called κ . An expression for the ratio of Painlevé VI structure constants in terms of Pochhammer symbols can be found in appendix B of [115]. Plugging the above relation into (3.108), we get

$$\begin{aligned} \tau(t) = C_0 t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_t^2)} (1 - t)^{\frac{1}{2}\theta_1\theta_t} \\ \times \left\{ 1 + \left[\frac{\theta_1\theta_t}{2} + \frac{(\theta_0^2 - \theta_t^2 - \sigma^2)(\theta_\infty^2 - \theta_1^2 - \sigma^2)}{8\sigma^2} \right. \right. \\ \left. - \frac{(\theta_0^2 - (\theta_t - \sigma)^2)(\theta_\infty^2 - (\theta_1 - \sigma)^2)}{16\sigma^2(1 + \sigma)^2} \kappa t^\sigma \right. \\ \left. - \frac{(\theta_0^2 - (\theta_t + \sigma)^2)(\theta_\infty^2 - (\theta_1 + \sigma)^2)}{16\sigma^2(1 - \sigma)^2} \frac{1}{\kappa t^\sigma} \right] t + \dots \Big\}. \end{aligned} \quad (3.110)$$

The expression (3.110) can be inserted in (3.93) to solve the associated eigenvalue problem. However, the complex structure of the higher order terms in the Conformal Blocks expansions (3.107) seems to be unavoidable in the computation of the eigenfrequencies.

On the other hand, an analytic expansion for κ can be constructed if one recognizes the second derivative condition of (3.93) in terms of certain τ -function with the shifted monodromies, and we refer to the incoming Section for a more detailed discussion.

3.4.2 The tale of the composite monodromy

As we will see, the second derivative condition in (3.93) can be used to find an analytic expansion for κ , which is related to $s(\sigma, t_0)$, the parameter s in the Nekrasov expansion (3.95). Then, by substituting the asymptotic expansion into the first derivative condition of the τ -function, one can obtain the composite monodromy as a function of t_0 . In this case, the accessory parameter K_0 is given as a perturbative expansion for small t_0 , as was done originally by [117] and more recently using the isomonodromic τ -function formalism in [115].

The second condition in (3.93) is related to an associated τ -function, with shifted monodromy arguments via the so-called “Toda equation” – see proposition 4.2 in [139], or [17] for a sketch of the proof. It reads

$$\frac{d}{dt} \left[t(t-1) \frac{d}{dt} \log \tau(t) \right] - \frac{\theta_t (\theta_\infty - \theta_t)}{2} = c \frac{\tau^+(t) \tau^-(t)}{\tau^2(t)} \quad (3.111)$$

where $c \in \mathbb{C}$ is a non-zero constant. We know that the left hand side is equal to zero at $t = t_0$, and this implies that either t_0 is a zero of $\tau^+(t)$ or it is a root of the $\tau^-(t)$ function. The divergence at t_0 in the denominator, $\tau(t_0) \rightarrow \infty$ can be ruled out, since the τ -function is holomorphic except at its singular points $t = \{0, 1, \infty\}$ [127]. $\tau^\pm(t)$ are defined with shifted monodromy arguments

$$\theta_i^\pm = \{\theta_0, \theta_t \pm 1, \theta_1, \theta_\infty \mp 1\}, \quad \sigma_{ij}^\pm = \{\sigma_{0t} \pm 1, \sigma_{1t} \pm 1, \sigma_{01}\}. \quad (3.112)$$

Therefore, with help from the Toda equation, the second condition in (3.93) can be more succinctly phrased as

$$\tau^+(\theta_0, \theta_{t_0}, \theta_1, \theta_{\infty_0}, \sigma_{0t_0}, \sigma_{1t_0}; t_0) = 0. \quad (3.113)$$

Notice that the τ function is quasi-periodic with respect to shifts of σ_{0t} by even integers $\sigma_{0t} \rightarrow \sigma_{0t} + 2p$:

$$\tau(t; \{\vec{\theta}\}, \{\sigma_{0t} + 2p, \sigma_{1t}\}) = s^{-p} \tau(t; \{\vec{\theta}\}, \{\sigma_{0t}, \sigma_{1t}\}), \quad p \in \mathbb{Z}. \quad (3.114)$$

This means that, upon inverting the equations (3.31) and (3.32), we will obtain, rather than the σ_{0t} associated to the system, a parameter, which we will call σ , related to σ_{0t} by the shift $\sigma_{0t} = \sigma + 2p$. Let us digress over the consequences of this periodicity by analyzing the structure of the expansion (3.95). Schematically,

$$\tau(t_0) = t_0^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_t^2)} \sum_{m \in \mathbb{Z}} P(\sigma + 2m; t_0) s^m t_0^{m^2 + m\sigma}, \quad (3.115)$$

where $P(\sigma + 2m; t_0)$ is analytic in t_0 , and to find the zero of (3.113) is useful to define $X = st_0^\sigma$, making the expansion analytic in t_0 and meromorphic in X . We can now solve (3.113) and thus define $X(\sigma, t_0)$ in terms of σ as a series in t_0 . Let us classify these solutions by their leading term:

$$X_p(\sigma; t_0) \equiv s_p t_0^\sigma = t_0^{2p+1} (x_0 + x_1 t_0 + x_2 t_0^2 + \dots) \quad (3.116)$$

Depending on the sign of $\text{Re } \sigma$, the leading term will depend on t_0 or t_0^{-1} . We will suppose $\text{Re } \sigma > 0$ for the discussion. The “fundamental” solution X_0 is written as (see (A.6))

$$\begin{aligned} X_0(\sigma; t_0) = & \frac{\Gamma^2(1 + \sigma)}{\Gamma^2(1 - \sigma)} \frac{\Gamma(1 + \frac{1}{2}(\theta_{t_0} + \theta_0 - \sigma))\Gamma(1 + \frac{1}{2}(\theta_{t_0} - \theta_0 - \sigma))}{\Gamma(1 + \frac{1}{2}(\theta_{t_0} + \theta_0 + \sigma))\Gamma(1 + \frac{1}{2}(\theta_{t_0} - \theta_0 + \sigma))} \\ & \times \frac{\Gamma(1 + \frac{1}{2}(\theta_1 + \theta_{\infty_0} - \sigma))\Gamma(1 + \frac{1}{2}(\theta_1 - \theta_{\infty_0} - \sigma))}{\Gamma(1 + \frac{1}{2}(\theta_1 + \theta_{\infty_0} + \sigma))\Gamma(1 + \frac{1}{2}(\theta_1 - \theta_{\infty_0} + \sigma))} Y(\sigma; t_0) \end{aligned} \quad (3.117)$$

with

$$\begin{aligned} Y(\sigma; t_0) = & \left[\frac{((\theta_{t_0} + \sigma)^2 - \theta_0^2)((\theta_1 + \sigma)^2 - \theta_{\infty_0}^2)}{16\sigma^2(\sigma - 1)^2} t_0 \right] \\ & \times \left(1 - (\sigma - 1) \frac{(\theta_0^2 - \theta_{t_0}^2)(\theta_1^2 - \theta_{\infty_0}^2) + \sigma^2(\sigma - 2)^2}{2\sigma^2(\sigma - 2)^2} t_0 + \mathcal{O}(t_0^2) \right) \end{aligned} \quad (3.118)$$

Solutions with $\text{Re } \sigma < 0$ can be obtained sending σ to $-\sigma$ and inverting the term in square brackets in the expression for Y . Solutions with higher value for p will also be of interest. These will have the leading term of order t_0^{2p+1} and can be obtained from the quasi-periodicity property (3.114), which translates to a shifting property for X_p . From the generic structure (3.115) above we have

$$\sum_{m \in \mathbb{Z}} P(\sigma + 2m; t_0) X^m t_0^{m^2} = \tilde{X}^{-p} t_0^{-p^2} \sum_{m \in \mathbb{Z}} P(\tilde{\sigma} + 2m; t_0) \tilde{X}^m t_0^{m^2} \quad (3.119)$$

where

$$\tilde{\sigma} = \sigma - 2p, \quad X = \tilde{X} t_0^{2p}. \quad (3.120)$$

By this property, assuming $\text{Re } \sigma > 0$, we have that a solution $X_p(\sigma; t_0)$ for (3.113) with leading term of higher order in t_0 can be obtained from a fundamental solution of leading order t_0 with shifted σ :

$$X_p(\sigma; t_0) = t_0^{2p} X_0(\sigma - 2p; t_0) \quad (3.121)$$

This allows us to construct a class of solutions for the conditions (3.93) which are generic enough for our purposes. From $X_p(\sigma; t_0)$ or $Y(\sigma; t_0)$ we can define the parameter κ entering the expansion (3.110):

$$\kappa(t_0; \{\vec{\theta}, \vec{\sigma}\}) = Y(\sigma; t_0) t_0^{-\sigma}. \quad (3.122)$$

More explicitly, up to second order in t_0 the expression is

$$\begin{aligned} \kappa = t_0^{-\sigma} & \left[\frac{((\theta_1 + \sigma)^2 - \theta_{\infty_0}^2)((\theta_{t_0} + \sigma)^2 - \theta_0^2)}{16\sigma^2(1 - \sigma)^2} t_0 \right] \\ & \times \left(1 + (1 - \sigma) \frac{(\theta_0^2 - \theta_{t_0}^2)(\theta_1^2 - \theta_{\infty_0}^2) + \sigma^2(\sigma - 2)^2}{2\sigma^2(\sigma - 2)^2} t_0 + \mathcal{O}(t_0^2) \right) \end{aligned} \quad (3.123)$$

valid when $\text{Re } \sigma > 0$ and $\sigma \rightarrow -\sigma$ for $\text{Re } \sigma < 0$, and the family of parameters s_p :

$$s_p = X_p(\sigma; t_0) t_0^{-\sigma} = X_0(\sigma - 2p; t_0) t_0^{-\sigma + 2p}, \quad (3.124)$$

with X_0 given in terms of Y as above. The knowledge of both parameters $s_p(\sigma, t_0)$ and $\sigma(t_0)$ is sufficient to determine the monodromy data by (3.100).

3.4.3 The accessory parameter K_0 and Catalan numbers

We can now proceed to compute the accessory parameter K_0 by substituting κ back in the first equation in (3.93), noting that (3.122) is an unshifted expansion, because it has been defined from (3.113), which cancels out the shifts in the monodromy data. Using the fact that $s(\{\vec{\theta}, \vec{\sigma}\}^-) = s(\{\vec{\theta}, \vec{\sigma}\})$, we can derive the right $\kappa(\{\vec{\theta}, \vec{\sigma}\}^-)$ for the first initial condition,

$$\kappa(\{\vec{\theta}, \vec{\sigma}\}^-) = - \frac{16\sigma^2(\sigma - 1)^2}{((\theta_{t_0} + \sigma)^2 - \theta_0^2)((\theta_{\infty_0} - \sigma + 1)^2 - (\theta_1 + 1)^2)} \kappa(\{\vec{\theta}, \vec{\sigma}\}). \quad (3.125)$$

Then, the expansion for the accessory parameter K_0 is given by

$$\begin{aligned}
4t_0 K_0 = & (\sigma - 1)^2 - (\theta_0 + \theta_{t_0} - 1)^2 + \left[2(\theta_1 - 1)(\theta_{t_0} - 1) \right. \\
& + \frac{((\sigma - 1)^2 + \theta_{t_0}^2 - \theta_0^2 - 1)((\sigma - 1)^2 + \theta_1^2 - \theta_{\infty_0}^2 - 1)}{2\sigma(\sigma - 2)} \Big] t_0 \\
& + \left[\frac{13}{32}\sigma(\sigma - 2) + 2(\theta_1 - 1)(\theta_{t_0} - 1) - \frac{1}{32}(5 + 14(\theta_0^2 + \theta_{\infty_0}^2) - 18(\theta_{t_0}^2 + \theta_1^2)) \right. \\
& + \frac{((\theta_0 - 1)^2 - \theta_{t_0}^2)((\theta_0 + 1)^2 - \theta_{t_0}^2)((\theta_1 - 1)^2 - \theta_{\infty_0}^2)((\theta_1 + 1)^2 - \theta_{\infty_0}^2)}{32(\sigma + 1)(\sigma - 3)} \\
& - \frac{((\theta_0^2 - \theta_{t_0}^2)(\theta_1^2 - \theta_{\infty_0}^2) + 8)^2 - 2(\theta_0^2 + \theta_{t_0}^2)(\theta_1^2 - \theta_{\infty_0}^2)^2 - 2(\theta_0^2 - \theta_{t_0}^2)^2(\theta_1^2 + \theta_{\infty_0}^2)}{32\sigma(\sigma - 2)} \\
& \left. + \frac{2}{\sigma(\sigma - 2)} + \frac{(\theta_0^2 - \theta_{t_0}^2)^2(\theta_1^2 - \theta_{\infty_0}^2)^2}{64} \left(\frac{1}{\sigma^3} - \frac{1}{(\sigma - 2)^3} \right) \right] t_0^2 + \mathcal{O}(t_0^3).
\end{aligned} \tag{3.126}$$

for $\text{Re } \sigma > 0$. The corresponding expression for $\text{Re } \sigma < 0$ can be obtained by sending $\sigma \rightarrow -\sigma$. The higher order corrections can be consistently computed from the series derived in [115]. Note that, since any solution for X in the series (3.121) will yield the same value for s in (3.95), and hence the same value for K_0 , the difference between σ and σ_{0t} is tied to which terms of the expansion are dominant, and depends on the particular value for s and t_0 . The generic structure of the conformal block expansion, of which K_0 is the semi-classical limit, was discussed at some length in the classical CFT literature [166, 168]. The relevant facts for our following discussion, given the generic expansion:

$$4t_0 K_0 = k_0 + k_1 t_0 + k_2 t_0^2 + \dots + k_n t_0^n + \dots \tag{3.127}$$

are as follows: k_n is a rational function of the monodromy parameters, the numerator is a polynomial in the “external” parameters θ_i and σ , and the denominator is a polynomial of σ alone; Secondly, k_n is invariant under the reflection $\sigma \leftrightarrow 2 - \sigma$. Thirdly, k_n has simple poles at $\sigma = 3, 4, \dots, n + 1$ and $\sigma = -1, -2, \dots, -n + 1$, poles of order $2n - 1$ at $\sigma = 0, 2$ and is analytic at $\sigma = 1$. Fourthly, the leading order term of k_n near $\sigma \simeq 2$ is (for $n \geq 1$)

$$k_n = -4\mathbf{C}_{n-1} \frac{(\theta_0^2 - \theta_{t_0}^2)^n (\theta_1^2 - \theta_{\infty_0}^2)^n}{16^n (\sigma - 2)^{2n-1}} + \dots, \quad \mathbf{C}_n = \frac{1}{n+1} \binom{2n}{n}, \tag{3.128}$$

where $\mathbf{C}_n$ is the n -th Catalan number. A similar structure exists for the fundamental solution $X_0(\sigma; t_0)$, or, rather, $Y(\sigma; t_0)$:

$$Y(\sigma; t_0) = \chi_1 t_0 + \chi_2 t_0^2 + \dots \quad (3.129)$$

with leading order for each χ_n given by (for $n \geq 3$)

$$\chi_n = -\mathbf{C}_{n-2} \frac{((\theta_{t_0} + \sigma)^2 - \theta_0^2)((\theta_1 + \sigma)^2 - \theta_{\infty_0}^2)}{16\sigma^2(1 - \sigma)^2} \frac{(\theta_1^2 - \theta_{\infty_0}^2)^{n-1}(\theta_0^2 - \theta_{t_0}^2)^{n-1}}{4^{n-1}\sigma^{2(n-1)}(\sigma - 2)^{2(n-1)}} + \dots, \quad (3.130)$$

where the implicit terms are of order $\mathcal{O}((\sigma - 2)^{-2n+3})$ or higher.

Recently, in [41] the same pole structure has been found for the accessory parameter of the confluent Heun equation derived from the Teukolsky master equation for the scattering problem by a four dimensional Kerr black hole. In this case, the analytical solution is given by the Painlevé V τ -function, which can be computed in terms of irregular conformal blocks due to the presence of an irregular singular point of Poincaré rank 1 at $z = \infty$ [72, 98, 116, 131].

3.5 Discussion

We have discussed that the monodromy group of a Fuchsian system with four regular singular points (3.57) is defined as a representation of the fundamental group of the four-punctured Riemann sphere, and labeled by seven invariants in trace coordinates. The Fricke-Jimbo relation (3.37) reduces them to six parameters: four monodromies at the singularities and two composite monodromies $\{\vec{\theta}, \vec{\sigma}\} = \{\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma_{0t}, \sigma_{1t}\}$. Instead, following [102, 115], if we fix the simple monodromies, the monodromy group can be parametrized by (σ_{0t}, s) .

The deformation of a linear ordinary differential equation while preserving its associated monodromy representation leads to a matrix $A(z)$ that is a rational function of the deformation parameter and satisfies a set of completely integrable non-linear partial differential equations (3.72), also known as the isomonodromic deformation equations or the Schlesinger equations.

For any solution of the Schlesinger equations, the 1-form

$$\omega = \sum_{\mu < \nu} \text{Tr} A_\mu A_\nu d \log(a_\mu - a_\nu) \quad (3.131)$$

is closed [103,105]. This allows for the definition of a τ -function as $\omega = d \log \tau$.

In addition, we have shown in Section 3.3, that assuming the residue matrix A_∞ to be a diagonal matrix, leads to a single zero of the off-diagonal element of $A(z)$ as:

$$A_{12}(z) = \frac{k(z - \lambda)}{z(z - 1)(z - t)}$$

then, $\lambda \equiv \lambda(t)$ satisfies the Painlevé VI equation.

On the other hand, one might think of as the point $z = \lambda$ corresponds to the introduction of an apparent singularity in the associated second order differential equation of (3.69), that makes manifest the Hamiltonian structure of the deformed Heun equation (3.82) and is connected with the isomonodromic deformation equations (3.72).

The canonical coordinates of the Garnier system $(\lambda(t), \mu(t))$ at $t = t_0$, determine the initial conditions for the isomonodromic flow, given by (3.83) and (3.84). Namely, one can arrive to the Heun equation by the coalescence of the apparent singularity with one of the singular points, while the monodromy data is preserved. When written in terms of the associated Painlevé VI τ -function, these conditions define a well-posed initial value problem (3.93).

Finally, we have derived the expressions (3.123) and (3.126), which are one of the main results of this Chapter. The separation constant λ_ℓ can be computed directly from (3.126) and (2.58), for instance. The radial case is more complicated due to the complex structure of the monodromies (2.61), therefore we need to construct an analytic expansion for $s(\sigma, t_0)$ and $\sigma(t_0)$ at small t_0 to solve the QNMs frequencies.

Chapter 4

Quasi-normal modes of the five dimensional Kerr-AdS black hole

In this chapter¹, we find the quasi-normal modes of the Kerr-AdS₅ black hole. Instead of solving the boundary value problem for the radial Heun equation (2.64), we use the conditions derived in (3.93) for the initial value problem of the PVI τ -function.

First we summarize the monodromy data for the radial and angular τ -functions. Then, the separation constant is found directly from the logarithmic derivative of the angular τ -function at $t_0 = u_0$ and some comments are presented afterwards. Once we have an expression for the separation constant we can address the computation of the quasi-normal modes using the two initial conditions for the radial Painlevé VI τ -function at $t_0 = z_0$.

In Section 4.3 we review the quasi-normal modes for Schwarzschild-AdS₅. This space-time appears as a special limit of the Kerr-AdS₅ geometry when the rotation parameters go to zero. While the angular differential equation reduces to the Laplace-Beltrami operator of the 3-sphere, the radial equation remains of the Heun type and can be studied with the τ -function formalism. Although these results are well known for almost two decades [94], we can consider them as a test for the new method.

¹This chapter is mostly an adaption of [27]

We also study the double rotating black hole for varying size of the outer horizon r_+ , the mass of the scalar field and the rotation parameters. Numerical evidence suggests that this approach achieves a higher accuracy than other numerical methods such as the matching method or the pseudo-spectral collocation method, for small and intermediate black holes with different rotation parameters.

Finally, following the expressions (3.123) and (3.126) in the small black hole limit, we present a perturbative expression for the fundamental QNM frequency. As the horizon radius decreases, the fundamental mode approaches to its pure AdS value (2.88) [142], with a first correction of order $\mathcal{O}(r_+^2)$. We analyse the modes of even and odd orbital quantum number ℓ , and find that for even ℓ (and therefore $m_1 = m_2 = 0$) the quasi-normal modes are stable for $r_+ \sim 10^{-3}$, without developing an instability (in the sense that the imaginary part of its complex frequency becomes positive). Nevertheless, the modes with odd ℓ do exhibit a regime of superradiance, in agreement with [39, 40, 56, 159].

4.1 Radial and angular τ -functions

The monodromy data for the radial and angular Painlevé VI τ -function are given explicitly by

	t_0	θ_0	θ_{t_0}	θ_1	θ_{∞_0}
$\tau_{Rad}(t)$	z_0	θ_-	θ_+	$2 - \Delta$	θ_0
$\tau_{Ang}(t)$	u_0	$-m_1$	$-m_2$	$2 - \Delta$	β

For u_0 or z_0 sufficiently close to a critical value of the Painlevé VI τ -function, both the Nekrasov expansion and the Fredholm determinant will converge fast. It makes sense then to explore solutions with this property. For $u_0 \rightarrow 0$, we have either $a_1 \rightarrow a_2$, an equally rotating or $a_1, a_2 \rightarrow 0$ a slowly rotating black hole. Furthermore, for $z_0 \ll 1$, we consider either the near-extremal limit $r_+ \rightarrow r_-$ or a small black hole limit $r_+, r_- \rightarrow 0$. This can be seen easily from the exact expressions

$$z_0 = \frac{r_+^2 - r_-^2}{r_+^2 - r_0^2}, \quad u_0 = \frac{a_2^2 - a_1^2}{a_2^2 - 1}. \quad (4.1)$$

We focus on small $r_+ \ll 1$ and intermediate black holes $r_+ \sim 1$. Big black holes in principle can be described if we can integrate efficiently the τ -function when $z_0 \rightarrow 1/2^2$.

The physical boundary conditions (3.31) and (3.32) play a preponderant role in the asymptotic expansion of the τ -function. For instance, the angular τ -function is directly related with σ_{0u_0} , thus one finds an expression for the separation constant from the derivative of the accessory parameter. On the other hand, σ_{1z_0} is related to the asymptotic expansion around $z_0 = 1$, which is a non-physical limit from the black hole side, $r_+^2 \rightarrow r_0^2$. Then we need to find σ_{0z_0} in terms of the monodromy data (4.16).

4.2 The separation constant

The separation constant can be calculated from the tau function expansion by imposing the quantization condition (3.31). For equal rotation parameters, $a_1 = a_2$, the Heun equation reduces to a hypergeometric, and an analytic expression in terms of finite combinations of elementary functions can be obtained [15]. This result can be recovered with the Painlevé VI τ -function by taking the asymptotic expansion near to 0. The leading term of (3.126) gives

$$\lambda = \left(1 - a_1^2\right) \left[(m_1 + m_2 + 2j)(m_1 + m_2 + 2j - 2) - 2\omega(a_1 m_1 + a_1 m_2) - (a_1 m_1 + a_1 m_2)^2 \right] + a_1^2 \omega^2 + a_1^2 \Delta(\Delta - 4) \quad (4.2)$$

which agrees the literature if we set the integer labeling the angular mode as

$$\ell = -(m_1 + m_2 + 2j). \quad (4.3)$$

For generic angular parameters, the monodromy data of the angular equation (2.57) are composed of the indicial exponents (2.55), with the single monodromy parameters, $\vec{\alpha} = \{\alpha_0, \alpha_{u_0}, \alpha_1, \alpha_\infty\}$ and the composite monodromy parameters $\vec{\sigma} = \{\sigma_{0u_0}, \sigma_{1u_0}\}$. Using naively the accessory parameter expansion (3.126), the separation constant (4.2) can be written in the form up to second order in u_0

²By the time of writing this thesis a representation in continued fractions for the τ -function has been formulated and surprisingly this description possesses a high accuracy with $r_+ > 1$

(remember that $\beta = \omega + a_1 m_1 + a_2 m_2$):

$$\begin{aligned}
\lambda_\ell \simeq & \omega^2 + \ell(\ell+2) - \beta^2 - \frac{a_1^2 + a_2^2}{2} \left(\ell(\ell+2) - \beta^2 - \Delta(\Delta-4) \right) \\
& - \frac{(a_1^2 - a_2^2)(m_1^2 - m_2^2)}{2\ell(\ell+2)} \left(\ell(\ell+2) - \beta^2 + \Delta(\Delta-4) + 4 \right) \\
& - \frac{(a_1^2 - a_2^2)^2}{1 - a_2^2} \left[\frac{(\ell(\ell+2) + m_2^2 - m_1^2)(\ell(\ell+2) + \Delta(\Delta-4) + 4 - \beta^2)}{2\ell(\ell+2)} \right. \\
& - \frac{13}{32}\ell(\ell+2) + \frac{1}{32} \left(5 + 14(m_1^2 + \beta^2) - 18(m_2^2 + (2-\Delta)^2) \right) \\
& - \frac{((m_1+1)^2 - m_2^2)((1-m_1)^2 - m_2^2)((1-\Delta)^2 - \beta^2)((3-\Delta)^2 - \beta^2)}{32(\ell-1)(\ell+3)} \\
& + \frac{((m_1^2 - m_2^2)((2-\Delta)^2 - \beta^2) + 8)^2 - 64 - 2(m_1^2 + m_2^2)((2-\Delta)^2 - \beta^2)^2}{32\ell(\ell+2)} \\
& - \frac{2(m_1^2 - m_2^2)^2((2-\Delta)^2 + \beta^2)}{32\ell(\ell+2)} - \frac{(m_1^2 - m_2^2)^2((2-\Delta)^2 - \beta^2)^2}{64} \\
& \left. \times \left(\frac{1}{(\ell+2)^3} - \frac{1}{\ell^3} \right) \right] + \mathcal{O}((a_1^2 - a_2^2)^3) \tag{4.4}
\end{aligned}$$

This expression reduces to the ones found in [15] when $a_1 \simeq a_2$. It also agrees with the expression in [52], up to higher order terms. The computation of (4.4) to higher order, however, is not so straightforward due to the existence of the pole structure of the accessory parameter expansion mentioned above. Given the quantization condition on the composite monodromy σ_{0u_0} by (3.31), one cannot make sense of the terms in (3.126) directly, and a careful limit of the tau function expansion (3.95) has to be taken. The particular case $\sigma = 0$ was considered by [102], and logarithm terms $\log t$ appear in the expansion. We expect this to induce logarithm terms in the accessory parameter as well, but, given that the expression above will suffice to our purposes, we will leave a detailed discussion of λ_ℓ to future work.

With an expression for the separation constant we can address the computation of the quasi-normal modes using the two initial conditions for the radial Painlevé VI τ -function at $t_0 = z_0$. We will explore this in the next Section and compare with numerical results obtained from well-established methods in numerical relativity.

4.3 Quasi-normal modes for Schwarzschild-AdS₅

In the limit $a_i \rightarrow 0$, one recovers the Schwarzschild-AdS₅ metric, and one expects that the resulting radial differential equation coming from the Klein-Gordon equation for massive scalar fields (2.59) can be reduced to the standard form of the Heun equation as well. The exponents θ_k are given by

$$\theta_+ = \frac{i}{2\pi} \frac{\omega}{T}, \quad \theta_- = 0, \quad \theta_0 = \frac{1}{2\pi} \frac{\omega}{T} \frac{\sqrt{1+r_+^2}}{r_+}, \quad \theta_\infty = 2 - \Delta, \quad (4.5)$$

where $2\pi T = 2\pi T_+ = (1 + 2r_+^2)/r_+$ is the temperature of the black hole, given by (2.17) by setting $a_1 = a_2 = r_- = 0$. The mass of the black hole is given by $\mathcal{M} = \frac{1}{2}r_+^2(1+r_+^2)$. We note that the system of coordinates is different from [151], and the singular point at $r = r_+$ is mapped by (2.62) to $z_0 = r_+^2/(1 + 2r_+^2)$.

Likewise, the angular equation (2.52) reduces to a standard hypergeometric form. The angular eigenvalues can be seen to be the usual SO(4) Casimir: $\lambda_\ell = \ell(\ell + 2)$. In terms of ω, Δ, r_+ , the accessory parameter K_0 in (2.65) is

$$K_0 = -\frac{\omega^2}{4(1+r_+^2)} + \frac{1+2r_+^2}{1+r_+^2} \left[\frac{\ell(\ell+2)}{4r_+^2} + \frac{\Delta(\Delta-2)}{4} \right] + \frac{i\omega}{2r_+} \frac{1+r_+^2(2-\Delta)}{1+r_+^2}. \quad (4.6)$$

This, along with the quantization condition for the radial monodromies (3.32), provide through (3.93) an implicit solution for the quasi-normal modes ω_n along with the composite monodromy σ_{0t} .

In order to test the method, we present the numerical solution for the fundamental quasi-normal mode $n = 0, \ell = 0$ (s-wave) case and compare with known methods, the pseudo-spectral method with a Chebyshev-Gauss-Lobatto grid to solve the associated Quadratic Eigenvalue Problem (QEP), and the usual numerical matching method based on the Frobenius expansion of the solution near the horizon and spatial infinity³.

The Frobenius method implements the smoothness on the first derivative at the matching point of the two series solutions constructed with 15 terms, at the horizon and the boundary [128]. On the other hand, the pseudo-spectral method relies on a grid with 120 points between 0 and 1. For a more comprehensive reading, we recommend [16, 57]. The results are shown in Tables 4.1 and 4.2.

³It should be noted that the Frobenius method is in spirit similar to the old combinatorical approach for the Painlevé VI τ -function given by Jimbo [102].

r_+	z_0	ω_0
0.005	2.49988×10^{-5}	$3.9998498731325748 - 1.5044808171834238 \times 10^{-6}i$
0.01	9.99800×10^{-5}	$3.9993983005189682 - 1.2123793015872442 \times 10^{-5}i$
0.05	2.48756×10^{-3}	$3.9844293869590734 - 1.7525974895168137 \times 10^{-3}i$
0.1	9.80392×10^{-3}	$3.9355764849860639 - 1.7970664179740506 \times 10^{-2}i$
0.2	3.70370×10^{-2}	$3.7906778316981978 - 0.1667439940917780i$
0.4	0.121212	$3.7173879743704008 - 0.7462495474087164i$
0.6	0.209302	$3.8914015767067012 - 1.3656095289384492i$

Table 4.1: The massless scalar field s-wave $\ell = 0$ and fundamental $n = 0$ quasi-normal modes ω_0 in Schwarzschild-AdS₅ for some values of r_+ . The results were obtained using the Fredholm determinant expansion for the τ -function with $N = 16$.

r_+	Frobenius	QEP
0.005	$3.9998498731325743 - 1.5044808171845522 \times 10^{-6}i$	$3.9998483860043481 - 2.8895543908757586 \times 10^{-5}i$
0.01	$3.9993983005189876 - 1.2123793015712405 \times 10^{-5}i$	$3.9993981402971502 - 2.3439366987252536 \times 10^{-5}i$
0.05	$3.9844293869590911 - 1.7525974895155961 \times 10^{-3}i$	$3.9844293921364538 - 1.7526437924554161 \times 10^{-3}i$
0.1	$3.9355764849860673 - 1.7970664179739766 \times 10^{-2}i$	$3.9355763694852816 - 1.7970671629389028 \times 10^{-2}i$
0.2	$3.7906778316982394 - 0.1667439940917505i$	$3.7906771832980760 - 0.1667441392742093i$
0.4	$3.7173879743704317 - 0.7462495474087220i$	$3.7173988607936563 - 0.7462476412816416i$
0.6	$3.8914015767126869 - 1.3656095289361863i$	$3.8913340701538795 - 1.3656086881322822i$

Table 4.2: The same quasi-normal modes frequencies ω_0 computed using numerical matching from Frobenius solutions (with 15 terms) and Quadratic Eigenvalue Problem (with 120 point-lattice).

The Schwarzschild-AdS case has been considered before [94, 138, 151, 158, 159], and should be thought of as a test of the new method. Even without optimized code⁴, the Fredholm determinant evaluation of the Painlevé VI τ -function provides a faster way of computing the frequencies than either the numerical matching and the QEP method. Convergence is particularly fast for small $z_0 \sim 10^{-5}$, which can provide at least 14 significant digits for the fundamental frequencies.

In Figure 4.1, we show the dependence of the fundamental quasi-normal mode frequency on r_+ , as well as the composite monodromy σ_{0t} . The fact that the latter has a zero limit for small black holes will be relevant in the next Section.

Figure 4.1: In the first row, the dependence of the real and imaginary part of the fundamental quasi-normal mode frequency ω_0 at $\ell = 0$ with respect to the radius r_+ for Schwarzschild-AdS₅ BHs ($a_{1,2} = 1 \times 10^{-9}$, $\mu = 0$). In the second, the dependence of the composite parameter σ_{0t} with r_+ .

⁴Implementations in Python for both the Nekrasov expansion and the Fredholm determinant can be provided upon request.

4.4 Small black holes via isomonodromy

The fast convergence and high accuracy of the τ function calculation is suitable for the study of small black holes. Turning our attention to Kerr-AdS₅, we consider spinning black holes of different angular momenta and radii. In view of holographic applications, we make use of an extra parameter given by the mass of the scalar field scattered by the black hole. Numerical results are presented in Table 4.3⁵.

One can use the initial condition for the first derivative in (3.93) and (3.113) to determine an asymptotic formulas for the composite monodromy parameters σ and s as functions of the frequency. In the spirit of establishing the occurrence of instabilities, it is worth looking at the small black hole limit. To better parametrize this limit, let us define

$$a_1^2 = \epsilon_1 r_+^2, \quad a_2^2 = \epsilon_2 r_+^2, \quad (4.7)$$

with the understanding that r_+^2 is a small number. The three parameters r_+^2, ϵ_1 and ϵ_2 are sufficient to express the other roots of Δ_r as follows:

$$r_-^2 = \frac{1 + (1 + \epsilon_1 + \epsilon_2)r_+^2}{2} \left(\sqrt{1 + \frac{4\epsilon_1\epsilon_2 r_+^2}{(1 + (1 + \epsilon_1 + \epsilon_2)r_+^2)^2}} - 1 \right), \quad (4.8)$$

$$-r_0^2 = \frac{1 + (1 + \epsilon_1 + \epsilon_2)r_+^2}{2} \left(\sqrt{1 + \frac{4\epsilon_1\epsilon_2 r_+^2}{(1 + (1 + \epsilon_1 + \epsilon_2)r_+^2)^2}} + 1 \right). \quad (4.9)$$

Since we want $r_-^2 \leq r_+^2$, the ϵ_i will satisfy

$$\epsilon_1 \epsilon_2 \leq 1 + (2 + \epsilon_1 + \epsilon_2)r_+^2 \simeq 1, \quad (4.10)$$

and we remind the reader that $\epsilon_{1,2}$ are also constrained by the extremality condition $a_i < 1$. The space of allowed $\epsilon_{1,2}$ is illustrated in Figure 4.2.

⁵In the table values, we have neglected some precision in the results for the sake of clarity, but we can provide more accurate values upon request.

r_+	z_0	τ function	Frobenius
0.00200	4.0×10^{-8}	$3.9999043938966996 - 3.9179009496192059 \times 10^{-7}i$	$3.9999043938967028 - 3.9179009496196828 \times 10^{-7}i$
0.02185	0.000476717	$3.9970574292783057 - 1.3247529381539807 \times 10^{-4}i$	$3.9970574292783089 - 1.3247529381539848 \times 10^{-4}i$
0.04169	0.001732030	$3.9892155345505937 - 9.8235136622117991 \times 10^{-4}i$	$3.9892155345505973 - 9.8235136622118292 \times 10^{-4}i$
0.06154	0.003758101	$3.9760894388470440 - 3.4698629309308322 \times 10^{-3}i$	$3.9760894388470473 - 3.4698629309308430 \times 10^{-3}i$
0.08138	0.006536145	$3.9575448170177743 - 8.8770413086742446 \times 10^{-3}i$	$3.9575448170177776 - 8.8770413086742730 \times 10^{-3}i$
0.10123	0.010040760	$3.9339314599984108 - 1.8761575868127569 \times 10^{-2}i$	$3.9339314599984140 - 1.8761575868127629 \times 10^{-2}i$
0.12107	0.014240509	$3.9062924890993003 - 3.4636778810775676 \times 10^{-2}i$	$3.9062924890993034 - 3.4636778810775789 \times 10^{-2}i$
0.14092	0.019098605	$3.8762906043241960 - 5.7537333581688194 \times 10^{-2}i$	$3.8762906043241993 - 5.7537333581688376 \times 10^{-2}i$
0.18061	0.030620669	$3.8166724096683002 - 1.2480348073108545 \times 10^{-1}i$	$3.8166724096683035 - 1.2480348073108582 \times 10^{-1}i$
0.22030	0.044236431	$3.7668353453284391 - 2.1574723769724682 \times 10^{-1}i$	$3.7668353453284420 - 2.1574723769724741 \times 10^{-1}i$
0.29968	0.076131349	$3.7116288122171590 - 4.3786490332401062 \times 10^{-1}i$	$3.7116288122171622 - 4.3786490332401161 \times 10^{-1}i$
0.37906	0.111610120	$3.7104224042819611 - 6.8107859662243775 \times 10^{-1}i$	$3.7104224042819692 - 6.8107859662244147 \times 10^{-1}i$
0.49813	0.165833126	$3.7816024214536172 - 1.0519267755676109i$	$3.7816024214748239 - 1.0519267755684242i$
0.61720	0.216209245	$3.9134030353146323 - 1.4181181443831172i$	$3.9134030400737264 - 1.4181181441373386i$
0.73627	0.260096962	$4.0879586460765776 - 1.7776344225896197i$	$4.0879588168442726 - 1.7776344550831753i$

Table 4.3: Fundamental modes for Kerr-AdS₅, at $\ell = m_1 = m_2 = 0$, assuming $a_1 = 0.002$, $a_2 = 0.00199$ and the mass of the scalar field is $\mu = 7.96 \times 10^{-8}$. See Figure 4.3.

Figure 4.2: The space of parameters defined by $\epsilon_1 \epsilon_2 < 1$ corresponds to the grey area. The dashed lines represent the extremal black holes where $r_+ = r_-$, for $r_+ = 0.002, 0.2, 0.25, 0.333, 0.5, 1$ with increasing dash density. The curve $r_+ = 1$, the one closest to the right upper corner, is drawn for comparison.

We will focus on the case $m_1 = m_2 = 0$ (and therefore ℓ even) in order to keep the expressions reasonably short. It will be convenient to leave z_0 implicit at times,

$$z_0 = \frac{r_+^2 - r_-^2}{r_+^2 - r_0^2} = \frac{1 + (3 + \epsilon_1 + \epsilon_2)r_+^2 - \sqrt{(1 + (1 + \epsilon_1 + \epsilon_2)r_+^2)^2 + 4\epsilon_1\epsilon_2r_+^2}}{1 + (3 + \epsilon_1 + \epsilon_2)r_+^2 + \sqrt{(1 + (1 + \epsilon_1 + \epsilon_2)r_+^2)^2 + 4\epsilon_1\epsilon_2r_+^2}}, \quad (4.11)$$

which asymptotes as $z_0 = (1 - \epsilon_1\epsilon_2)r_+^2 + \mathcal{O}(r_+^4)$. The expansions of the single monodromy parameters are, up to terms of order $\mathcal{O}(r_+^3)$:

$$\theta_0 = \omega \left(1 - \frac{3}{2}(1 + \epsilon_1)(1 + \epsilon_2)r_+^2 + \dots \right) \quad (4.12)$$

$$\theta_+ = i\omega \frac{(1 + \epsilon_1)(1 + \epsilon_2)}{1 - \epsilon_1\epsilon_2} r_+ + \dots \quad (4.13)$$

$$\theta_- = -i\omega \frac{(1 + \epsilon_1)(1 + \epsilon_2)}{1 - \epsilon_1\epsilon_2} \sqrt{\epsilon_1\epsilon_2} r_+ + \dots \quad (4.14)$$

The single monodromy parameters can be seen to have the structure:

$$\theta_- = -i\phi_- r_+, \quad \theta_+ = i\phi_+ r_+, \quad (4.15)$$

where $\phi_{\pm}$ are real and positive for real and positive ω . We also observe that θ_0 is parametrically close to the frequency ω , and the correction is negative for positive r_+ .

We now proceed to solve for the composite monodromy parameter $\sigma_{\ell} \equiv \sigma_{0z_0}(\ell)$ using the series expansion (3.126). For even $\ell \geq 2$, the first correction is

$$\begin{aligned} \sigma_{\ell} &\equiv \ell + 2 - \nu_{\ell} r_+^2 \\ &= \ell + 2 - \frac{(1 + \epsilon_1)(1 + \epsilon_2)}{4(\ell + 1)} (3\omega^2 + 3\ell(\ell + 2) - \Delta(\Delta - 4)) r_+^2 + \mathcal{O}(r_+^4), \quad \ell \geq 2, \end{aligned} \quad (4.16)$$

and, due to the pole structure of (3.127), naive series inversion will yield the expansion for σ up to order $r_+^{2\ell}$. The case $\ell = 0$ is then special, and will be dealt with shortly. One can see from (3.124) that, for $p = 0$, the monodromy parameter s will behave asymptotically as $z_0^{-\sigma}$, diverging for small z_0 . Changing the value of p will change this behaviour. Changing the value of p means shifting the argument σ that enters the definition of $X_0(\sigma, t_0)$ in (3.121) and therefore of $Y(\sigma, t_0)$ in (3.117). Let us call $Y_{\ell, 2p}$ the expression in (3.118) for generic p and $\sigma \simeq 2 + \ell$. The expression for $p = 0$ is given by

$$\begin{aligned} Y_{\ell, 0} \equiv Y(\sigma_{\ell}; z_0) &= -(1 - \epsilon_1 \epsilon_2) \frac{\omega^2 - (\Delta - \ell - 4)^2}{16(\ell + 1)^2} \\ &\quad \times \left(1 + \frac{2i}{\ell + 2} \phi_+ r_+ \right) r_+^2 + \dots, \quad \ell \geq 2. \end{aligned} \quad (4.17)$$

We point out that this value is actually independent of p , except when $2p = \ell$, as we will see below. We anticipate, from (3.122), that $Y_{\ell, p}$ for $2p < \ell$ will yield a larger value for s_{ℓ} for smaller r_+ . We also remark that s_{ℓ} will have a non-analytic expansion in r_+ , due to the term $z_0^{-\sigma_{\ell}}$. Finally, from the expansion we conclude that $Y_{\ell, p}$ has an imaginary part of subleading order.

4.4.1 $\ell = 0$

The “s-wave” case is singular since the leading behavior of $\sigma - 2$ is of order r_+^2 . The expansion (3.127) does not converge in general due to the denomina-

tor structure of the coefficients κ_n . For the small r_+ black hole application, however, we are really dealing with a scaling limit where

$$\theta_- = \varphi_- \sqrt{z_0}, \quad \theta_+ = \varphi_+ \sqrt{z_0}, \quad \text{and} \quad \sigma = 2 - \nu z_0 \quad (4.18)$$

have finite limits for $\varphi_{\pm}$ and ν as $z_0 \rightarrow 0$. Because of the poles of increasing order in σ in (3.127), in the $\ell = 0$ case one has to resum the whole series in order to compute ν .

Thankfully the task is amenable due to the fact that, in the scaling limit, the term of order z_0 in each of the factors $k_n t_0^n$ in the expansion (3.127) comes from the leading order pole (3.128)

$$k_n z_0^n = -4\mathbf{C}_{n-1} \frac{(\varphi_-^2 - \varphi_+^2)^n (\theta_1^2 - \theta_{\infty}^2)^n}{16^n \nu^{2n-1}} z_0 + \mathcal{O}(z_0^2). \quad (4.19)$$

The series can be resummed using the generating function for the Catalan numbers

$$1 + x + 2x^2 + 5x^3 + \dots = \sum_{n=0}^{\infty} \mathbf{C}_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x} \quad (4.20)$$

and the result for ν readily written

$$\begin{aligned} 4z_0 K_0(\ell = 0) + (\theta_+ + \theta_- - 1)^2 + 2(\theta_1 - 1)(\theta_+ - 1) \frac{z_0}{z_0 - 1} \\ = 1 + \frac{1}{2}(\theta_1^2 - \theta_{\infty}^2)z_0 - 2\nu z_0 \sqrt{1 + \frac{(\varphi_+^2 - \varphi_-^2)(\theta_1^2 - \theta_{\infty}^2)}{4\nu^2}} + \mathcal{O}(z_0^2). \end{aligned} \quad (4.21)$$

A similar procedure allows us to compute the parameter $Y(\nu) \equiv Y(2 - \nu z_0; z_0)$ up to order $z_0^{3/2}$:

$$Y(\nu) = -z_0(1 + \varphi_+ \sqrt{z_0}) \frac{\theta_{\infty}^2 - (\theta_1 + 2)^2}{64} \left(1 + \sqrt{1 + \frac{(\varphi_+^2 - \varphi_-^2)(\theta_1^2 - \theta_{\infty}^2)}{4\nu^2}} \right)^2 + \dots \quad (4.22)$$

For the application to the $\ell = 0$ case of the scalar field, we will use the notation (4.15), and again use $\sigma_0 = 2 - \nu_0 r_+^2$. In terms of the black hole parameters, ν_0 has a surprisingly simple form:

$$\nu_0 = \frac{1}{4}(1 + \epsilon_1)(1 + \epsilon_2) \sqrt{(3\omega^2 - \Delta(\Delta - 4))^2 - 4\omega^2(\omega^2 - (\Delta - 2)^2)} + \mathcal{O}(r_+^2), \quad (4.23)$$

and

$$Y_{0,0} \equiv Y(\sigma_0; z_0) = -(1 - \epsilon_1 \epsilon_2) r_+^2 (1 + i\phi_+ r_+) \frac{\omega^2 - (\Delta - 4)^2}{64} \\ \times \left(1 + \frac{3\omega^2 - \Delta(\Delta - 4)}{\sqrt{(3\omega^2 - \Delta(\Delta - 4))^2 - 4\omega^2(\omega^2 - (\Delta - 2)^2)}} \right)^2 + \dots \quad (4.24)$$

Finally, let us define the shifted $Y_{\ell,\ell}$ for $2p = \ell$. Since the shifted argument $\sigma - 2p$ is close to 2, we need the same scaling limit as above in (4.22). The result is

$$Y_{\ell,\ell} \equiv Y(\sigma_\ell - \ell; z_0) = -(1 - \epsilon_1 \epsilon_2) r_+^2 (1 + i\phi_+ r_+) \frac{\omega^2 - (\Delta - 4)^2}{64} \\ \times \left(1 + \sqrt{1 + \frac{4(\ell + 1)^2 \omega^2 (\omega^2 - (\Delta - 2)^2)}{(3\omega^2 + 3\ell(\ell + 2) - \Delta(\Delta - 4))^2}} \right)^2 + \dots, \quad (4.25)$$

where ν_ℓ is taken from (4.16).

To sum up, we exhibit the overall structure for small r_+ :

$$\sigma_\ell = \ell + 2 - \nu_\ell r_+^2 + \dots, \quad (4.26)$$

$$Y_{\ell,\ell} = -(1 - \epsilon_1 \epsilon_2) \vartheta_\ell (1 + i\phi_+ r_+) r_+^2 + \dots, \quad (4.27)$$

where ν_ℓ and ϑ_ℓ have non-zero limits as $r_+ \rightarrow 0$, corrections of order r_+^2 , and, most importantly, are positive for ω real and greater than $\Delta - 4$.

4.4.2 The quasi-normal modes

Implementation of the quantization condition (3.32) can be done with the formula (3.38). This yields a transcendental equation for ω whose solutions will give all complex frequencies for the radial quantization condition. These include negative-real part frequencies, as well as non-normalizable modes. Since we are interested in positive real-part frequencies, we will consider a small correction to the vacuum AdS_5 result [6, 142]

$$\omega_{n,\ell} = \Delta + 2n + \ell + \eta_{n,\ell} r_+^2, \quad (4.28)$$

under the hypothesis that $\eta_{n,\ell}$ has a finite limit as $r_+ \rightarrow 0$. One notes by (4.12) that θ_0 and ω are perturbatively close, so $\eta_{n,\ell}$ can be calculated perturbatively from the expansion of θ_0 . We will assume that Δ is *not* an integer.

The parametrization (4.28) allows us to expand (3.124) as a function of r_+ . The procedure is straightforward: We use $Y_{\ell,\ell}$ from (4.27), as it gives the right asymptotic behavior, to compute X_0 using (3.117) and then the s parameter (3.124). To second order in r_+ we have

$$s_{n,\ell} = -\frac{16\Gamma(n+\ell/2+1)\Gamma(\Delta-2+n+\ell/2)}{\Gamma(n+\ell/2+3)\Gamma(\Delta+n+\ell/2)} \times \frac{\nu_\ell^2}{(1-\epsilon_1\epsilon_2)^2(\phi_+^2-\phi_-^2)} \left(1-i\phi_+r_+ + 2i\frac{\phi_+\nu_\ell r_+}{\phi_+^2-\phi_-^2}\right) Y_{\ell,\ell} r_+^{-2+2\nu_\ell r_+^2}, \quad (4.29)$$

and the leading behavior for the parameter $s_{n,\ell}$ given $Y_{\ell,\ell}$ in (4.27) is

$$s_{n,\ell} = \Sigma_{n,\ell} \left(1 + \frac{2i\nu_{n,\ell}r_+}{(1+\epsilon_1)(1+\epsilon_2)(\Delta+2n+\ell)}\right) r_+^{2\nu_{n,\ell}r_+^2} + \dots, \quad (4.30)$$

where we defined $\nu_{n,\ell}$ as the correction for σ as in (4.27) calculated at the vacuum frequency $\nu_\ell(\omega = \Delta + 2n + \ell)$. Finally,

$$\Sigma_{n,\ell} = \frac{16\Gamma(n+\ell/2+1)\Gamma(\Delta-2+n+\ell/2)}{\Gamma(n+\ell/2+3)\Gamma(\Delta+n+\ell/2)} \frac{\nu_{n,\ell}^2 \vartheta_{n,\ell}}{(1+\epsilon_1)^2(1+\epsilon_2)^2(\Delta+2n+\ell)^2}, \quad (4.31)$$

again, with $\vartheta_{n,\ell} = \vartheta(\omega = \Delta + 2n + \ell)$. We also note that $\Sigma_{n,\ell}$ is real and positive under the same conditions as (4.27). Moreover, the choice of p , implicit in $Y_{\ell,\ell}$ guarantees that $s_{n,\ell}$ has a finite limit as $r_+ \rightarrow 0$, although its dependence on r_+ is non-analytic.

Equation (3.38) can now be used, setting $\cos \pi \sigma_{1t} = \cos \pi(\theta_1 + \theta_t)$ for the radial parameters, to find a perturbative equation for $\eta_{n,\ell}$. We expand each of the terms in (3.38) using (4.15) as well as

$$\theta_0 = \omega_0 - \beta r_+^2, \quad \omega_0 = \Delta + 2n + \ell, \quad \text{and} \quad \sigma = 2 + \ell - \nu_\ell r_+^2. \quad (4.32)$$

Now, the following two relations hold

$$\begin{aligned} & \sin^2 \pi \sigma \cos \pi(\theta_1 + \theta_t) - \cos \pi \theta_0 \cos \pi \theta_\infty \\ & - \cos \pi \theta_t \cos \pi \theta_1 + \cos \pi \sigma (\cos \pi \theta_0 \cos \pi \theta_1 + \cos \pi \theta_t \cos \pi \theta_\infty) = \\ & \frac{\pi^3}{2} \sin(\pi \Delta) (\phi_+^2 - \phi_-^2) \left(\beta + \frac{2i\nu_\ell^2 r_+}{(1+\epsilon_1)(1+\epsilon_2)\omega_0} \right) r_+^4 + \dots \end{aligned} \quad (4.33)$$

Figure 4.3: In the first row, the real and imaginary part of the fundamental mode ω_0 at $\ell = m_1 = m_2 = 0$, given as functions of r_+ for small Kerr-AdS₅ BHs. In the second, the dependence of the composite parameter $2 - \sigma$ with r_+ .

$$\begin{aligned}
 & -\frac{1}{2}(\cos \pi \theta_\infty - \cos \pi(\theta_1 \pm \sigma))(\cos \pi \theta_0 - \cos \pi(\theta_t \pm \sigma)) = \\
 & \frac{\pi^3}{2} \sin(\pi \Delta)(\phi_+^2 - \phi_-^2) \left(\frac{\beta \pm \nu}{2} \right) \left(1 \pm \frac{2i\nu_\ell r_+}{(1 + \epsilon_1)(1 + \epsilon_2)\omega_0} \right) r_+^4 + \dots \quad (4.34)
 \end{aligned}$$

We can now proceed to calculate the first correction to the eigenfrequencies (4.28). By using the approximations (4.33) and (4.34) above we find the correction to θ_0 for each of the modes n, ℓ

$$\beta_{n,\ell} = \nu_{n,\ell} \frac{\Sigma_{n,\ell} + 1}{\Sigma_{n,\ell} - 1} + 4i \frac{\nu_{n,\ell}^2}{(1 + \epsilon_1)(1 + \epsilon_2)(\Delta + 2n + \ell)} \frac{\Sigma_{n,\ell}}{(\Sigma_{n,\ell} - 1)^2} r_+ + \mathcal{O}(r_+^2 \log r_+). \quad (4.35)$$

Finally, after some laborious calculations, we find

$$\eta_{n,\ell} = -\frac{(1+\epsilon_1)(1+\epsilon_2)}{2} \left[\frac{Z_{n,\ell}}{2(\ell+1)} - 3(\Delta+2n+\ell) \right] - \frac{i}{4}(2n+\ell+2)(1+\epsilon_1)(1+\epsilon_2)(\Delta+2n+\ell)(2\Delta+2n+\ell-2)r_+ + \mathcal{O}(r_+^2 \log r_+), \quad \ell \geq 2, \quad (4.36)$$

with

$$Z_{n,\ell}^2 = (3(\Delta+2n+\ell)^2 + 3\ell(\ell+2) - \Delta(\Delta-4))^2 + 4(\ell+1)^2(\Delta+2n+\ell)^2(2n+\ell+1)(2\Delta+2n+\ell-2). \quad (4.37)$$

For $\ell = 0$, the form of the correction is slightly different. Repeating the calculation, we see that $\eta_{n,\ell=0}$ has the simpler form

$$\eta_{n,0} = -\frac{(1+\epsilon_1)(1+\epsilon_2)}{4} (3(\Delta+2n-1)^2 - (\Delta-2)^2 + 1) - i(n+1)(1+\epsilon_1)(1+\epsilon_2)(\Delta+2n)(\Delta+n-1)r_+ + \mathcal{O}(r_+^2 \log r_+). \quad (4.38)$$

We note that both the real and imaginary parts of the corrections $\eta_{n,\ell}$ are negative, the real part of order r_+^2 as expected, and the imaginary part of order r_+^3 . We stress that we are taking $m_1 = m_2 = 0$.

In the midst of the calculation, we see that the imaginary part of $\eta_{n,0}$ has the same sign as the imaginary part of θ_+ , which in turn is essentially the entropy intake of the black hole as it absorbs a quantum of frequency ω and angular momenta m_1 and m_2 :

$$\theta_+ = \frac{i}{2\pi} \delta S = \frac{i}{2\pi} \frac{\omega - m_1 \Omega_{+,1} - m_2 \Omega_{+,2}}{T_+} \quad (4.39)$$

giving the same sort of window for unstable modes parameters m_1, m_2 as in superradiance, so a closer look at higher values for $m_{1,2}$ is perhaps in order for future work. A full consideration of linear perturbations of the five-dimensional Kerr-AdS black hole, involving higher spin [120, 163], can be done within the same theoretical framework presented here, and will be left for the future. We close by observing that the expressions (4.36) and (4.38) above seem to represent a distinct limit than the results in [15] – which are, however, restricted to $\Delta = 4$ – and therefore not allowing for a direct comparison.

4.4.3 Some words about the ℓ odd case

Let us illustrate the parameters for the subcase $m_1 = \ell, m_2 = 0$. The single monodromy parameters admit the expansion

$$\theta_0 = \omega + \sqrt{\epsilon_1} \ell r_+ - \frac{3}{2}(1 + \epsilon_1)(1 + \epsilon_2)\omega r_+^2 + \dots \quad (4.40)$$

$$\theta_+ = -i\ell \frac{\sqrt{\epsilon_1}(1 + \epsilon_2)}{1 - \epsilon_1\epsilon_2} + i\omega \frac{(1 + \epsilon_1)(1 + \epsilon_2)}{1 - \epsilon_1\epsilon_2} r_+ + \dots \quad (4.41)$$

$$\theta_- = i\ell \frac{\sqrt{\epsilon_2}(1 + \epsilon_1)}{1 - \epsilon_1\epsilon_2} - i\omega \frac{(1 + \epsilon_1)(1 + \epsilon_2)}{1 - \epsilon_1\epsilon_2} \sqrt{\epsilon_1\epsilon_2} r_+ + \dots, \quad (4.42)$$

with all of them finite and non-zero as $r_+ \rightarrow 0$. As usual, $\theta_{\pm}$ are purely imaginary whereas θ_0 is real for real ω . These properties hold for any value of m_1 and m_2 .

For $\ell \geq 1$ and odd, the composite monodromy parameters are found much in the same way as the case $\ell \geq 2$ considered above, by inverting (3.126). In the following we set $\omega_0 = \Delta + 2n + \ell$ as the limit of the frequency as $r_+ \rightarrow 0$. We have for the composite monodromy parameter

$$\sigma_{\ell} = 2 + \ell - \nu_{\ell} r_+^2 + \mathcal{O}(r_+^4), \quad (4.43)$$

with ν_{ℓ} , defined as in (4.30), now for $\ell > 1$

$$\nu_{\ell} = (1 + \epsilon_1)(1 + \epsilon_2) \frac{3\omega_0^2 + 3\ell(\ell + 2) - \Delta(\Delta - 4)}{4(\ell + 1)}, \quad \ell \geq 3. \quad (4.44)$$

For $\ell = 1$, finding ν_1 from condition (3.93) requires going to higher order in z_0 , due to the pole at $\sigma = 3$ in the expansion (3.126),

$$\begin{aligned} \nu_1 &= \frac{(1 + \epsilon_1)(1 + \epsilon_2)}{32} (3\omega_0^2 + 9 - \Delta(\Delta - 4)) \\ &\times \left(2 + \frac{1}{3} \sqrt{34 - 8 \frac{2\Delta^4 - 16\Delta^3 + (50 - 3\omega_0^2)\Delta^2 + 12(\omega_0^2 - 6)\Delta - 36\omega_0^2}{(3\omega_0^2 + 9 - \Delta(\Delta - 4))^2}} \right). \end{aligned} \quad (4.45)$$

For the following discussion, we take from this calculation that the ν_{ℓ} 's are real and greater than 1 for $\Delta > 1$, which we will assume to hold for any m_1 and m_2 . Apart from these properties, the particular form for ν_{ℓ} will be left implicit.

Given ν_ℓ , we can use the same procedure as in the even ℓ case to compute the s parameter. Again, in order to have a finite $r_+ \rightarrow 0$ limit, we take $p = (\ell + 1)/2$. After some calculations, we have

$$s_{n,\ell} = 1 + 2\nu_\ell r_+^2 \log r_+ + \Xi_{n,\ell} \nu_\ell r_+^2 + \mathcal{O}(r_+^4 (\log r_+)^2) \quad (4.46)$$

with

$$\begin{aligned} \Xi_{n,\ell} = 2\gamma + \log(1 - \epsilon_1 \epsilon_2) + \Psi\left(\frac{1 + \theta_+ + \theta_-}{2}\right) + \Psi\left(\frac{1 + \theta_+ - \theta_-}{2}\right) \\ + \Psi\left(\frac{3 + 2n + \ell}{2}\right) + \Psi\left(\frac{3 - 2\Delta - 2n - \ell}{2}\right), \end{aligned} \quad (4.47)$$

where $\Psi(z)$ is the digamma function, and $\gamma = -\Psi(1)$ the Euler-Mascheroni constant. In the definition above we have already set $\theta_0 = \Delta + 2n + \ell - \beta_{n,\ell} r_+^2$, but as we can see from (4.46), now we need $s_{n,\ell}$ to second order in the expansion parameter r_+ . We again assume that Δ is in general *not* an integer, since this is irrelevant for the determination of the imaginary part of the frequency. However, having Δ integer will change the behavior of the real part of the correction to the eigenfrequency with respect to r_+ .

We note that $s_{n,\ell}$ is non-analytic, and therefore the expansion for $\beta_{n,\ell}$ will include terms like $\log r_+$. We expand (3.38), with $\sigma_{1t} = \theta_+ + \Delta - 2$ (up to an even integer) to fourth order and find as first approximation to the correction to the frequency

$$\eta_{n,\ell} = \dots + \nu_{n,\ell} \left(\frac{\nu_{n,\ell} + 1}{\nu_{n,\ell} - 1} + \Xi_{n,\ell} \right) \frac{r_+^2}{\log r_+} + \dots, \quad (4.48)$$

where the terms left out are real, stemming from the relation between θ_0 and ω .

From (4.48), any possible imaginary part for the eigenfrequency will then come from the imaginary part of $\Xi_{n,\ell}$. The latter can be calculated by using the reflexion property of the digamma function

$$\Im \Xi_{n,\ell} = -\frac{i\pi}{2} \left(\tan \frac{\pi}{2} (\theta_+ + \theta_-) + \tan \frac{\pi}{2} (\theta_+ - \theta_-) \right), \quad (4.49)$$

or, in terms of m_1 and m_2 ,

$$\Im \Xi_{n,\ell} = \frac{\pi}{2} \tanh \left(\frac{\pi}{2} \frac{\sqrt{\epsilon_1} - \sqrt{\epsilon_2}}{1 + \sqrt{\epsilon_1 \epsilon_2}} (m_1 - m_2) \right) + \frac{\pi}{2} \tanh \left(\frac{\pi}{2} \frac{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}}{1 - \sqrt{\epsilon_1 \epsilon_2}} (m_1 + m_2) \right). \quad (4.50)$$

We then see that the imaginary part of $\Xi_{n,\ell}$ can have any sign, a strong indication that the ℓ odd modes are unstable. Numerical support for this is included in Fig. 4.4, in which we use an arbitrary-precision Python code (capped at 50 decimal places) to show a slightly positive imaginary part for the resonant frequency at $r \lesssim 0.02$. We point out that, indeed, instabilities in asymptotically anti-de Sitter spaces are expected from general grounds [79], and odd ℓ instabilities for the massless case ($\Delta = 4$) were found in [15].

Figure 4.4: Real (left) and imaginary parts (right) of the fundamental $n = 0$ quasi-normal modes ω_0 at $\ell = 1$ and $m_1 = \pm 1$, given as functions of r_+ for small Kerr-AdS₅ black holes ($a_1 = a_2 = 0.002$ and $\Delta = 4$). The dashed line corresponds to $\text{Im}(\omega_0) = 0$, to exhibit more clearly the appearance of superradiant instabilities (seen in the insets).

4.5 Discussion

In this chapter we have applied the isomonodromy method to derive asymptotic expressions for the separation constant for the angular equation (angular eigenvalue) in (4.4) as well as the frequencies for the scalar quasi-normal modes in a five-dimensional Kerr-AdS background in the limit of small black holes, see in particular (4.36) and (4.38). The numerical analysis carried out for the Schwarzschild-AdS and Kerr-AdS cases showed that the τ function approach has advantages when compared to standard methods, in terms of faster process-

ing times. For ℓ even, the correction to the vacuum AdS frequencies is negative with negative imaginary part for $\Delta > 1$, the scalar unitarity bound, showing no instability in the range studied. For ℓ odd, there are strong indications for instability due to the general structure of the corrections in (4.50). In particular, for $\ell = 1$, the numerical results shown in Fig. 4.4 exhibit an unstable mode for $r_+ \leq 0.02$ and nearly equal rotational parameters. We plan to address the phase space of instabilities and holographic consequences in future work.

The method relies on the construction of the τ -function of the PVI transcendent proposed in the literature following the AGT conjecture. The conditions in (3.93) translate the accessory parameters in the ODEs governing the propagation of the field – themselves depending on the physical parameters – into monodromy parameters, and the quantization condition (3.31) allows us to derive the angular separation constant (4.4). In turn, the quantization condition for the radial equation (3.32), through series solutions for the composite monodromy parameters s and σ , allows us to solve for the eigenfrequencies $\omega_{n,\ell}$, even in the generic complex case.

The interpretation of the ODEs involved as the level 2 null vector condition of semi-classical Liouville field theory allows us to conclude that all descendants are relevant for the calculation of the monodromy parameters, even though for angular momentum parameter $\ell \geq 2$, one can consider just the conformal primary (first channel) for the parameter σ_{0t} .

The scaling limit resulting from this analysis gives the monodromy parameter σ in (4.16). For the parameter σ_{1t} , the requisite of a smooth $r_+ \rightarrow 0$ limit forces us to consider the asymptotics of the whole series (3.126), thus involving all descendants. This means that naive matching of the solution obtained from the near horizon approximation to the asymptotic solution near infinity is not a suitable tool for dealing with small black holes. For the composite monodromy parameter σ_{1t} , more suitably parametrized by s in (3.38), the requirement of a finite $r_+ \rightarrow 0$ limit allows us to select the solutions (4.30) for ℓ even and (4.46) for ℓ odd. Although finite in the small black hole limit, the s parameter has a non-analytic expansion in terms of $r_+^m (\log r_+)^n$.

For the s-wave ($\ell = 0$) calculations, we had to consider a scaling limit in (3.126) where the Liouville momenta associated to θ_+ and θ_- go to zero as r_+ , at the same time as z_0 and $\sigma - 2$ scales as r_+^2 . The formulas (4.21) and (4.22) are reminiscent of the light-light-heavy-heavy limit of Witten diagrams for conformal blocks [90]. It would be interesting to understand the CFT meaning of

this limit.

The Toda equation, which allows us to interpret the second condition (3.93) on the Painlevé τ function, also merits further study. As for the first condition, we note that it provides the accessory parameters for both the angular and radial equations – Q_0 in (2.57) and K_0 in (2.64), respectively – as the derivative of the logarithm of the τ function for each system. On the other hand, these accessory parameters are both related to the separation constant of the Klein-Gordon equation, as can be verified through (2.58) and (2.65). Including these terms in the definition of a τ function for the angular and radial systems, we can represent the fact that the separation constant is the same for (2.57) and (2.64) as the condition

$$\frac{d}{du_0} \log \tau_{\text{angular}} = \frac{d}{dz_0} \log \tau_{\text{radial}}, \quad (4.51)$$

which in turn can be interpreted as a thermodynamical equilibrium condition. Given the usual interpretation of the τ function as the generating functional of a quantum theory, the elucidation of this structure can shed light on the space-time approach to conformal blocks.

Chapter 5

Vector Perturbations of Kerr-AdS₅

5.1 Introduction

Black holes in higher dimensions [62, 93] are important to understand aspects of the gauge/gravity correspondence, with the ultimate goal a better understanding of theories with non-trivial infrared fixed points. On the other hand, a better understanding of general relativity is an interesting goal *per se*, with a clear view on the generic properties and the features which are special to four dimensions well worth pursuing.

Black hole solutions are particularly distinguished by their integrable structure. The first example is the four-dimensional vacuum solution given by the Kerr geometry, which can be found explicitly even though its isometries – time translation and axial rotation – do not warrant integrability of the equations in the Liouville sense. The solution was generalized to non-zero cosmological constant by Carter [45], and to higher even dimensions $D = 2n$ by Myers and Perry [130], characterized by n conserved charges. In odd dimensions, they were constructed in [86] for the particular case $D = 5$ and then, generically, in [76, 77]. The family of solutions present an integrable set of null congruences, and the integrability of the solutions themselves can be ascribed to the existence of higher-rank tensors, satisfying an analogue of the Killing equation for isometries, the so-called Killing tensors.

This integrable structure, called *hidden symmetries*, allows not only for the construction of the solutions, but also for separability of the scalar and spinor wave equations [68]. For spin 1 fields, however, the situation is murkier. The separation of Maxwell’s equations in four dimensions, obtained first by Teukolsky [156], is a result of the existence of a Killing-Yano conserved tensor. In higher dimensions, the separation was achieved by Lunin [120] at the expense of the introduction of an arbitrary parameter μ . This new technique was dubbed “ μ -separability” in [69]. The new parameter is related to the existence of different polarizations of the electromagnetic and Proca fields [59, 112], as well as the higher p -form generalization considered in [121]. Because of this, the treatment of tensor fields in these black hole backgrounds is quite different from the scalar case. The introduction of this extra separability parameter brings in further questions, related to which physical requirements should fix its value, as in the determination of scattering coefficients, angular eigenvalues and the frequency quasi-normal modes.

Coming from a different perspective, the separability of the scalar wave equation in the subcase of a five dimensional black hole with a negative cosmological constant – Kerr-AdS₅ – was tied to the construction of two flat holomorphic connections in a previous article by the authors [28], related to the solutions of the angular and radial differential equations. There, the purpose was solely “dynamical”: flat holomorphic connections have a residual gauge-symmetry which allows for solving the connection problem of the differential equations [27].

The residual gauge symmetry, known as “isomonodromy transformations” in the theory of ordinary differential equations [101], is realized in the angular and radial equations by the presence of an extra singular point in the Fuchsian equations, whose monodromy properties are trivial. This extra apparent singularity can be moved around the complex plane, and the isomonodromy transformation forces a functional dependence between the position of the apparent singularity and the positions of the other singularities, which was found to be the celebrated Painlevé transcendent of the sixth type.

In the scalar case, these extra singularities play an auxiliary role in the actual solution of the problem: quantities such as scattering amplitudes and the quasi-normal modes depend solely on the monodromy data. One can then compute them at any point of the isomonodromic flow, with the coincident point where the apparent singularity merges with one of the remaining singularities

being particularly convenient.

The present chapter is mostly an adaptation of [29]. We study the μ -separability in the particular case of the spin 1 field in a generic Kerr-AdS₅ black hole in order to further elucidate the role of the μ parameter. As we will see in Section 5.2 this is directly related by a Möbius transformation to the Painlevé transcendent, and parametrizes the position of the apparent singularity of both the radial and angular equations. This leads us to the conclusion that the role of μ in higher dimensions is different to that in four dimensions, where it can be eliminated by a change of parametrization in the corresponding equations.

In Section 5.3, we review the trick of “deforming” the Heun equation by adding an extra, apparent singularity. However, in the case considered here, the presence of the apparent singularity is mandatory. In fact, the boundary conditions for angular eigenvalues and quasi-normal modes can be written in terms of the monodromy data, and hence can be thought of as isomonodromy invariants. In Section 5.4, assuming this invariance, we are able to fix the parameter μ through a consistency condition of the isomonodromic flow in the radial and angular equations. We then proceed to a short numerical analysis of the solution proposed and close with a short discussion and prospects in Section 5.5.

5.2 Maxwell perturbations on Kerr-AdS₅

This particular form of the metric (2.1) allows to define a orthonormal 1-form basis e^{A1}

$$e^0 = \sqrt{\frac{\Delta}{r^2 + x^2}} \left(dt - \frac{a_1 \sin^2 \theta}{1 - a_1^2} d\phi - \frac{a_2 \cos^2 \theta}{1 - a_2^2} d\psi \right), \quad (5.1a)$$

$$e^1 = \sqrt{\frac{r^2 + x^2}{\Delta}} dr, \quad (5.1b)$$

$$e^2 = \sqrt{\frac{r^2 + x^2}{1 - x^2}} d\theta, \quad (5.1c)$$

¹ $e^A = e_\mu^A dx^\mu$, the Lorentz indices run as follows $A = \{0, 1, 2, 3, 4\}$.

$$e^3 = \sqrt{\frac{1-x^2}{r^2+x^2}} \frac{\sin \theta \cos \theta}{x} \left((a_2^2 - a_1^2) dt + \frac{a_1(r^2 + a_1^2)}{1 - a_1^2} d\phi - \frac{a_2(r^2 + a_2^2)}{1 - a_2^2} d\psi \right), \quad (5.1d)$$

$$e^4 = \frac{a_1 a_2}{rx} \left(dt - \frac{(r^2 + a_1^2) \sin^2 \theta}{a_1(1 - a_1^2)} d\phi - \frac{(r^2 + a_2^2) \cos^2 \theta}{a_2(1 - a_2^2)} d\psi \right), \quad (5.1e)$$

which, then, allows us to write

$$ds^2 = -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2. \quad (5.2)$$

The inverse metric has a similar factorization

$$g^{\mu\nu} \partial_\mu \partial_\nu = -(e_0)^2 + (e_1)^2 + (e_2)^2 + (e_3)^2 + (e_4)^2, \quad (5.3)$$

where

$$e_0 = \sqrt{\frac{1}{\Delta(r^2 + x^2)}} \frac{(r^2 + a_1^2)(r^2 + a_2^2)}{r^2} \left(\partial_t + \frac{a_1(1 - a_1^2)}{r^2 + a_1^2} \partial_\phi + \frac{a_2(1 - a_2^2)}{r^2 + a_2^2} \partial_\psi \right), \quad (5.4a)$$

$$e_1 = \sqrt{\frac{\Delta}{r^2 + x^2}} \partial_r, \quad (5.4b)$$

$$e_2 = \sqrt{\frac{1 - x^2}{r^2 + x^2}} \partial_\theta, \quad (5.4c)$$

$$e_3 = \frac{1}{\sqrt{(1 - x^2)(r^2 + x^2)}} \frac{\sin \theta \cos \theta}{x} \left((a_1^2 - a_2^2) \partial_t + \frac{a_1(1 - a_1^2)}{\sin^2 \theta} \partial_\phi - \frac{a_2(1 - a_2^2)}{\cos^2 \theta} \partial_\psi \right), \quad (5.4d)$$

$$e_4 = -\frac{a_1 a_2}{rx} \left(\partial_t + \frac{(1 - a_1^2)}{a_1} \partial_\phi + \frac{(1 - a_2^2)}{a_2} \partial_\psi \right). \quad (5.4e)$$

Following [120], to separate the radial and angular equation for the gauge field, we need to construct a special frame with a pair of real null vectors, a pair of complex null vectors and a space-like unit vector orthogonal to all others. They are given as $k, \ell, m, \bar{m}, k$ as follows

$$\begin{aligned} \ell &= \sqrt{\frac{r^2 + x^2}{\Delta}} (e_0 + e_1) \\ &= \frac{(r^2 + a_1^2)(r^2 + a_2^2)}{r^2 \Delta} \left(\partial_t + \frac{a_1(1 - a_1^2)}{r^2 + a_1^2} \partial_\phi + \frac{a_2(1 - a_2^2)}{r^2 + a_2^2} \partial_\psi \right) + \partial_r, \end{aligned} \quad (5.5a)$$

$$\begin{aligned}
n &= \frac{1}{2} \sqrt{\frac{\Delta}{r^2 + x^2}} (e_0 - e_1) \\
&= \frac{(r^2 + a_1^2)(r^2 + a_2^2)}{2r^2(r^2 + x^2)} \left(\partial_t + \frac{a_1(1 - a_1^2)}{r^2 + a_1^2} \partial_\phi + \frac{a_2(1 - a_2^2)}{r^2 + a_2^2} \partial_\psi \right) - \frac{\Delta}{2(r^2 + x^2)} \partial_r,
\end{aligned} \tag{5.5b}$$

$$\begin{aligned}
m &= \frac{1}{\sqrt{2}} \frac{r - ix}{\sqrt{r^2 + x^2}} (e_2 + ie_3) \\
&= \frac{\sqrt{1 - x^2}}{\sqrt{2}(r + ix)} \left[\partial_\theta + i \frac{\sin \theta \cos \theta}{x(1 - x^2)} \left((a_1^2 - a_2^2) \partial_t + \frac{a_1(1 - a_1^2)}{\sin^2 \theta} \partial_\phi - \frac{a_2(1 - a_2^2)}{\cos^2 \theta} \partial_\psi \right) \right],
\end{aligned} \tag{5.5c}$$

$$\begin{aligned}
\bar{m} &= (m)^* = \frac{1}{\sqrt{2}} \frac{r + ix}{\sqrt{r^2 + x^2}} (e_2 - ie_3) \\
&= \frac{\sqrt{1 - x^2}}{\sqrt{2}(r - ix)} \left[\partial_\theta - i \frac{\sin \theta \cos \theta}{x(1 - x^2)} \left((a_1^2 - a_2^2) \partial_t + \frac{a_1(1 - a_1^2)}{\sin^2 \theta} \partial_\phi - \frac{a_2(1 - a_2^2)}{\cos^2 \theta} \partial_\psi \right) \right],
\end{aligned} \tag{5.5d}$$

$$k = -\frac{a_1 a_2}{rx} \left(\partial_t + \frac{(1 - a_1^2)}{a_1} \partial_\phi + \frac{(1 - a_2^2)}{a_2} \partial_\psi \right). \tag{5.5e}$$

The first four elements of the list $\ell, n, m, \bar{m}$ are null vectors – a null tetrad – whereas k is orthogonal and space-like unit vector. Now we define the null transformed “light-cone” basis

$$\begin{aligned}
\ell_+ &= \ell, & \ell_- &= -\frac{2(r^2 + x^2)}{\Delta} n, \\
m_+ &= \sqrt{2}(r + ix)m, & m_- &= \sqrt{2}(r - ix)\bar{m} = (m_+)^*,
\end{aligned} \tag{5.6}$$

leaving k unchanged. Now (ℓ_+, ℓ_-) do not depend on the polar angle θ , and (m_+, m_-) do not depend on the radial coordinate r .

5.2.1 Separation of variables for Maxwell equations

In a particular coordinate basis $\{x^\mu\}$, the source-free Maxwell equations for a massless vector field can be written as

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0, \quad \text{with} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{5.7}$$

which unfortunately are *not* separable in the background (2.1). Again following [120], we can achieve separability by introducing a parameter μ , and define two classes of solutions, corresponding to two different polarizations, called electric and magnetic modes:

$$\ell_{\pm}^a A_a^{(el)} = \pm \frac{\mu r}{\mu \mp ir} \ell_{\pm}^a \nabla_a \Psi, \quad m_{\pm}^a A_a^{(el)} = \pm \frac{i\mu x}{\mu \pm x} m_{\pm}^a \nabla_a \Psi, \quad k^a A_a^{(el)} = 0; \quad (5.8a)$$

$$\ell_{\pm}^a A_a^{(mgn)} = \pm \frac{1}{r \pm i\mu} \ell_{\pm}^a \nabla_a \Psi, \quad m_{\pm}^a A_a^{(mgn)} = \mp \frac{i}{x \pm \mu} m_{\pm}^a \nabla_a \Psi, \quad (5.8b)$$

$$k^a A_a^{(mgn)} = \lambda \Psi,$$

where Ψ is a scalar function which, as we will see below, satisfies a separable equation. We note that the covariant derivative is always applied to scalars, so they are independent of the Christoffel connection. Writing

$$\Psi = e^{-i\omega t + i\tilde{m}_1 \phi + i\tilde{m}_2 \psi} \Phi(r) S(x), \quad (5.9)$$

we can express the components of the potential A_a explicitly in the “light-cone” basis (5.6) and (5.8a). For instance, for the electric solution $A_a^{(el)}$ ($A_{\mu}^{(el)} = A_a^{(el)} (\partial_{\mu})^a$):

$$A_t^{(el)} = \frac{\Psi}{(r^2 + \mu^2)} \left\{ \frac{\mu^2 r \Delta}{(r^2 + x^2)} \frac{\Phi'(r)}{\Phi(r)} + \frac{S'(x)}{S(x)} \frac{\mu^2 (r^2 + \mu^2) (1 - x^2) \sqrt{(a_1^2 - x^2)(x^2 - a_2^2)}}{(r^2 + x^2)(x^2 - \mu^2)} + \frac{\mu}{(x^2 - \mu^2)} \left[\omega (r^2 + \mu^2)(x^2 - a_2^2) + \omega (r^2 + a_1^2)(a_2^2 - \mu^2) - a_1 \tilde{m}_1 (a_2^2 - \mu^2) - a_2 \tilde{m}_2 (a_1^2 - \mu^2) \right] \right\}, \quad (5.10a)$$

$$A_r^{(el)} = \frac{i\Psi}{(r^2 + \mu^2)} \left[r^2 \mu \frac{\Phi'(r)}{\Phi(r)} - \frac{\mu^2 (r^2 + a_1^2)(r^2 + a_2^2)}{r \Delta} \left(\omega - \frac{a_1 \tilde{m}_1}{r^2 + a_1^2} - \frac{a_2 \tilde{m}_2}{r^2 + a_2^2} \right) \right], \quad (5.10b)$$

$$A_{\theta}^{(el)} = \frac{i\Psi}{(x^2 - \mu^2)} \left[\mu x^2 \frac{S'(x)}{S(x)} - \frac{\mu^2 \sqrt{(a_1^2 - x^2)(x^2 - a_2^2)}}{(1 - x^2)} \left(\omega - \frac{a_1 \tilde{m}_1}{(a_1^2 - x^2)} - \frac{a_2 \tilde{m}_2}{(a_2^2 - x^2)} \right) \right], \quad (5.10c)$$

$$A_{\phi}^{(el)} = -\frac{a_1 \Psi}{(1 - a_1^2)(r^2 + \mu^2)} \left[\frac{\Delta \mu^2 r (x^2 - a_1^2)}{(r^2 + x^2)(a_2^2 - a_1^2)} \frac{\Phi'(r)}{\Phi(r)} - \frac{\mu^2 (r^2 + a_1^2)(r^2 + \mu^2)}{(r^2 + x^2)} \right. \\ \times \frac{(1 - x^2) \sqrt{(a_1^2 - x^2)(x^2 - a_2^2)}}{(a_2^2 - a_1^2)(x^2 - \mu^2)} \frac{S'(x)}{S(x)} + \frac{\mu}{(x^2 - \mu^2)} \left(a_1 \tilde{m}_1 (\mu^2 - a_2^2) \right. \\ \left. + a_1 \tilde{m}_1 (r^2 + a_2^2) \frac{(a_2^2 - x^2)}{(a_2^2 - a_1^2)} - a_2 \tilde{m}_2 (r^2 + a_1^2) \frac{(x^2 - a_1^2)}{(a_2^2 - a_1^2)} \right. \\ \left. + \omega (a_2^2 - \mu^2)(r^2 + a_1^2) \frac{(x^2 - a_1^2)}{(a_2^2 - a_1^2)} \right) \left. \right], \quad (5.10d)$$

$$A_{\psi}^{(el)} = -\frac{a_2 \Psi}{(1 - a_2^2)(r^2 + \mu^2)} \left[\frac{\Delta \mu^2 r (a_2^2 - x^2)}{(r^2 + x^2)(a_2^2 - a_1^2)} \frac{\Phi'(r)}{\Phi(r)} - \frac{\mu^2 (r^2 + a_2^2)(r^2 + \mu^2)}{(r^2 + x^2)} \right. \\ \times \frac{(1 - x^2) \sqrt{(a_1^2 - x^2)(x^2 - a_2^2)}}{(x^2 - \mu^2)(a_1^2 - a_2^2)} \frac{S'(x)}{S(x)} + \frac{\mu}{(x^2 - \mu^2)} \left(a_2 \tilde{m}_2 (\mu^2 - a_1^2) \right. \\ \left. + a_2 \tilde{m}_2 (r^2 + a_1^2) \frac{(x^2 - a_1^2)}{(a_2^2 - a_1^2)} - a_1 \tilde{m}_1 (r^2 + a_2^2) \frac{(a_2^2 - x^2)}{(a_2^2 - a_1^2)} \right. \\ \left. - \omega (r^2 + a_2^2)(\mu^2 - a_1^2) \frac{(a_2^2 - x^2)}{(a_2^2 - a_1^2)} \right) \left. \right]. \quad (5.10e)$$

Now, the equations for $\Phi(r)$ and $S(x)$ can be written as two separate equations, coupled by a separation constant C_m and the parameter μ

$$\frac{D_r}{r} \frac{d}{dr} \left[\frac{r Q_r^2 (\Delta - R)}{D_r} \frac{d\Phi}{dr} \right] + \left\{ \frac{2\tilde{\Lambda}}{D_r} + \frac{R^2 \tilde{W}_r^2}{r^4 Q_r^2 (\Delta - R)} - \frac{a_1^2 a_2^2 D_r}{r^2} \tilde{\Omega}^2 + \mu^2 C_m D_r \right\} \Phi(r) = 0, \quad (5.11a)$$

$$\frac{D}{x} \frac{d}{dx} \left[\frac{Q^2 H}{x D} \frac{dS}{dx} \right] + \left\{ \frac{2\tilde{\Lambda}}{D} - \frac{H \tilde{W}^2}{Q^2 x^2} + \frac{a_1^2 a_2^2 D}{x^2} \tilde{\Omega}^2 + \mu^2 C_m D \right\} S(x) = 0, \quad (5.11b)$$

and the functions and constants given by

$$R = (r^2 + a_1^2)(r^2 + a_2^2), \quad Q_r^2 = \frac{R(1 + r^2) - 2Mr^2}{r^2(\Delta - R)}, \quad (5.12)$$

$$D_r = 1 + \frac{r^2}{\mu^2}, \quad \tilde{W}_r = \omega - \frac{\tilde{m}_1 a_1}{r^2 + a_1^2} - \frac{\tilde{m}_2 a_2}{r^2 + a_2^2},$$

$$H = (a_1^2 - x^2)(a_2^2 - x^2), \quad Q^2 = 1 - x^2, \quad (5.13)$$

$$D = 1 - \frac{x^2}{\mu^2}, \quad \tilde{W} = \omega - \frac{\tilde{m}_1 a_1}{a_1^2 - x^2} - \frac{\tilde{m}_2 a_2}{a_2^2 - x^2},$$

$$\tilde{\Lambda} = \frac{(a_1^2 - \mu^2)(a_2^2 - \mu^2)}{\mu^3} \left(\omega - \frac{\tilde{m}_1 a_1}{a_1^2 - \mu^2} - \frac{\tilde{m}_2 a_2}{a_2^2 - \mu^2} \right), \quad \tilde{\Omega} = \omega - \frac{\tilde{m}_1}{a_1} - \frac{\tilde{m}_2}{a_2}. \quad (5.14)$$

The subscript m in the separation constant C_m is an integer index and will be discussed in detail in Section 5.4.

The equations above determine the electric polarization, in the sense described in (5.8a), for the potential. The corresponding equations for the magnetic polarizations were also worked out in [120], and it is eventually found that the function Ψ defined through (5.8b) also satisfies (5.11a) and (5.11b), provided the separation parameter μ is substituted by $1/\mu$. Given μ , the value for λ in (5.8b) is fixed to $\tilde{\Omega}/\mu$. The details can be checked in [120] – although we note the slight change of notation $\omega(\text{there}) = -\omega(\text{here})$, $a_{1,2}(\text{there}) = -a_{1,2}(\text{here})$, $M(\text{there}) = 2M(\text{here})$ and $P_0(\text{there}) = \mu^2 C_m(\text{here})$. We also note that, because of the periodicity of the coordinates ϕ and ψ , we have $\tilde{m}_i = (1 - a_i^2)m_i$, with m_i integers.

Explicitly, the radial and the angular equations are

$$\begin{aligned} & \frac{r^2 + \mu^2}{r} \frac{d}{dr} \left[\frac{r \Delta}{(r^2 + \mu^2)} \frac{d\Phi}{dr} \right] + \left\{ -\frac{(r^2 + \mu^2)}{\mu^2 r^2} (a_1 a_2 \omega - (1 - a_1^2) m_1 a_2 \right. \\ & - (1 - a_2^2) m_2 a_1)^2 + \frac{(r^2 + a_1^2)^2 (r^2 + a_2^2)^2}{r^4 \Delta} \left(\omega - \frac{a_1 (1 - a_1^2) m_1}{r^2 + a_1^2} - \frac{a_2 (1 - a_2^2) m_2}{r^2 + a_2^2} \right)^2 \\ & - \frac{2(a_1^2 - \mu^2)(a_2^2 - \mu^2)}{\mu(r^2 + \mu^2)} \left(\omega - \frac{a_1 (1 - a_1^2) m_1}{a_1^2 - \mu^2} - \frac{a_2 (1 - a_2^2) m_2}{a_2^2 - \mu^2} \right) \\ & \left. + C_m (r^2 + \mu^2) \right\} \Phi(r) = 0, \end{aligned} \quad (5.15)$$

and

$$\begin{aligned}
& \frac{(\mu^2 - x^2)}{x} \frac{d}{dx} \left[\frac{(1 - x^2)(a_1^2 - x^2)(a_2^2 - x^2)}{x(\mu^2 - x^2)} \frac{dS}{dx} \right] + \left\{ C_m(\mu^2 - x^2) \right. \\
& - \frac{(a_1^2 - x^2)(a_2^2 - x^2)}{x^2(1 - x^2)} \left(\omega - \frac{a_1(1 - a_1^2)m_1}{a_1^2 - x^2} - \frac{a_2(1 - a_2^2)m_2}{a_2^2 - x^2} \right)^2 \\
& - \frac{2(a_1^2 - \mu^2)(a_2^2 - \mu^2)}{\mu(\mu^2 - x^2)} \left(\omega - \frac{a_1(1 - a_1^2)m_1}{a_1^2 - \mu^2} - \frac{a_2(1 - a_2^2)m_2}{a_2^2 - \mu^2} \right) \\
& \left. + \frac{(\mu^2 - x^2)}{\mu^2 x^2} (a_1 a_2 \omega - a_2(1 - a_1^2)m_1 - a_1(1 - a_2^2)m_2)^2 \right\} S(x) = 0
\end{aligned} \tag{5.16}$$

where now the values r_+^2, r_-^2 and r_0^2 are defined, following [27], as the roots of Δ ,

$$\Delta = \frac{(1 - r^2)(r^2 + a_1^2)(r^2 + a_2^2)}{r^2} - 2M = \frac{(r^2 - r_0^2)(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}. \tag{5.17}$$

5.2.2 The radial and angular systems

The radial equation (5.15) can be brought to a standard form by making a Möbius transformation

$$z = \frac{r^2 - r_-^2}{r^2 - r_0^2}, \quad \text{with} \quad z_0 = \frac{r_+^2 - r_-^2}{r_+^2 - r_0^2}, \tag{5.18}$$

followed by introducing a new radial function regular at the horizon and the boundary,

$$\Phi(z) = z^{-\alpha_-} (z - 1)^{\alpha_\infty} (z - z_0)^{-\alpha_+} R(z). \tag{5.19}$$

The exponents α_k are related to the monodromy parameters θ_k as

$$\alpha_k = \pm \frac{1}{2} \theta_k, \quad k = +, -, 0 \quad \text{and} \quad \alpha_\infty = \frac{1}{2} \left(1 \pm \sqrt{1 - C_m} \right), \tag{5.20}$$

which in turn are given in terms of the physical parameters by

$$\theta_k = \frac{i}{2\pi} \left(\frac{\omega - m_1 \Omega_{k,1} - m_2 \Omega_{k,2}}{T_k} \right), \quad \theta_1 = -\sqrt{1 - C_m}. \tag{5.21}$$

We note that, just like the scalar case [28], and in the four-dimensional Teukolsky master equation [41], the monodromy parameters θ_+ and θ_- , respectively

associated to the outer and inner horizons at $r = r_+$ and $r = r_-$ are proportional to the variation of the black hole entropy as a quantum of energy ω and angular momenta m_1 and m_2 passes through the horizon.

With these definitions, the radial equation becomes

$$\begin{aligned} \frac{d^2 R}{dz^2} + \left[\frac{1 - \theta_-}{z} + \frac{1 - \theta_1}{z - 1} + \frac{1 - \theta_+}{z - z_0} - \frac{1}{z - z_\star} \right] \frac{dR}{dz} \\ + \left(\frac{\kappa_+ \kappa_-}{z(z - 1)} + \frac{z_0(z_0 - 1)K_0}{z(z - 1)(z - z_0)} + \frac{z_\star(z_\star - 1)K_\star}{z(z - 1)(z - z_\star)} \right) R(z) = 0, \end{aligned} \quad (5.22)$$

with the parameters as

$$z_\star = \frac{r_-^2 + \mu^2}{r_0^2 + \mu^2}, \quad (5.23a)$$

$$\kappa_+ \kappa_- = \frac{1}{4}((\theta_- + \theta_+ + \theta_1 - 1)^2 - \theta_0^2), \quad (5.23b)$$

$$\begin{aligned} 4z_0 K_0 = & (\theta_- + \theta_+ + \theta_1 - 1)^2 - \theta_0^2 - 2\theta_- \theta_1 + 2\theta_1 - 2 - \frac{2(1 - \theta_1)\theta_+}{(z_0 - 1)} + \frac{\omega^2}{(r_-^2 - r_0^2)} \\ & - \frac{a_1^2 a_2^2 \tilde{\Omega}^2}{\mu^2(r_-^2 - r_0^2)} + \frac{2z_\star \theta_+}{(z_0 - z_\star)} + \frac{2(z_\star - 1)\mu^3 \omega}{(r_+^2 - r_0^2)(r_-^2 - r_0^2)(z_0 - z_\star)} + \frac{2z_0(1 - \theta_1)}{(z_0 - 1)} \\ & + \frac{2(z_\star - 1)a_1^2 a_2^2 \tilde{\Omega}}{\mu(r_+^2 - r_0^2)(r_-^2 - r_0^2)(z_0 - z_\star)} + C_m + \frac{(z_0 - z_\star)}{(z_0 - 1)(z_\star - 1)} C_m \\ & - \frac{2(z_\star - 1)\mu((a_1^2 + a_2^2)\omega - a_1(1 - a_1^2)m_1 - a_2(1 - a_2^2)m_2)}{(r_+^2 - r_0^2)(r_-^2 - r_0^2)(z_0 - z_\star)}, \end{aligned} \quad (5.23c)$$

$$\begin{aligned} 4z_\star K_\star = & -\frac{2(z_\star - 1)a_1^2 a_2^2 \tilde{\Omega}}{\mu(r_+^2 - r_0^2)(r_-^2 - r_0^2)(z_0 - z_\star)} - \frac{2(z_\star - 1)\mu^3 \omega}{(r_+^2 - r_0^2)(r_-^2 - r_0^2)(z_0 - z_\star)} \\ & + \frac{2(z_\star - 1)\mu((a_1^2 + a_2^2)\omega - a_1(1 - a_1^2)m_1 - a_2(1 - a_2^2)m_2)}{(r_+^2 - r_0^2)(r_-^2 - r_0^2)(z_0 - z_\star)} \\ & + 2\theta_- - \frac{2z_\star \theta_+}{(z_0 - z_\star)} - \frac{2z_\star(1 - \theta_1)}{(z_\star - 1)}. \end{aligned} \quad (5.23d)$$

The differential equation (5.22) is Fuchsian, with 5 regular singular points at $z = \{0, z_0, z_\star, 1, \infty\}$. It is sometimes called the deformed Heun equation, because, as we will see below, the singular point at $z_\star$ is *apparent*: the indicial

coefficients are $\{0, 2\}$ and, due to an algebraic relation between the parameters, there are no logarithmic tails². Then the monodromy property of the solution around this point is trivial. The position of this apparent singularity is related to the parameter μ by a Möbius transformation as it can be seen in (5.23a). Finally, we note that the deformed Heun equation (5.22) depends on μ only through $z_\star, K_0, K_\star$.

The angular equation (5.16) can be brought to the same form (5.22) by the Möbius transformation

$$u = \frac{x^2 - a_1^2}{x^2 - 1}, \quad \text{with} \quad u_0 = \frac{a_2^2 - a_1^2}{a_2^2 - 1}. \quad (5.24)$$

The resulting equation is again Fuchsian with 5 regular singular points, located at $u = \{0, u_0, u_\star, 1, \infty\}$, and the characteristic exponents are

$$\beta_0^\pm = \pm \frac{m_1}{2}, \quad \beta_1^\pm = \frac{1}{2} \left(1 \pm \sqrt{1 - C_m} \right), \quad \beta_{u_0}^\pm = \pm \frac{m_2}{2}, \quad (5.25)$$

$$\beta_\star = \{0, 2\}, \quad \beta_\infty^\pm = \pm \frac{1}{2} \varsigma = \pm \frac{1}{2} (\omega + a_1 m_1 + a_2 m_2). \quad (5.26)$$

We can check that, once more, the point at $u = u_\star$ is an apparent singularity due to an algebraic relation between the parameters.

Finally, the angular equation (5.16) can be brought to a canonical form by the transformation

$$S(u) = u^{m_1/2} (u - 1)^{(1-\theta_1)/2} (u - u_0)^{m_2/2} Y(u), \quad (5.27)$$

which leads to the deformed Heun form (5.32),

$$\begin{aligned} \frac{d^2 Y}{du^2} + \left[\frac{1 + m_1}{u} + \frac{1 - \theta_1}{u - 1} + \frac{1 + m_2}{u - u_0} - \frac{1}{u - u_\star} \right] \frac{dY}{du} \\ + \left(\frac{q_+ q_-}{u(u - 1)} + \frac{u_0(u_0 - 1)Q_0}{u(u - 1)(u - u_0)} + \frac{u_\star(u_\star - 1)Q_\star}{u(u - 1)(u - u_\star)} \right) Y(u) = 0, \end{aligned} \quad (5.28)$$

²The name of K. Heun is usually connected to the Fuchsian equation with 4 regular singular points. The generic differential equation with 5 regular singular points has no widespread name, although it was associated to F. Klein and M. Bôcher in the classic treatise of E. L. Ince [95].

with the accessory parameters given by

$$u_\star = \frac{\mu^2 - a_1^2}{\mu^2 - 1}, \quad (5.29a)$$

$$q_+ q_- = \frac{1}{4}((m_1 + m_2 + 1 - \theta_1)^2 - (\omega + a_1 m_1 + a_2 m_2)^2), \quad (5.29b)$$

$$\begin{aligned} 4u_0 Q_0 = & (m_1 + m_2 + 1 - \theta_1)^2 - (\omega + a_1 m_1 + a_2 m_2)^2 + 2m_1 \theta_1 - 2(1 - \theta_1) \\ & + \frac{2u_0(1 - \theta_1)}{(u_0 - 1)} - \frac{a_1^2 a_2^2 \tilde{\Omega}^2}{\mu^2(1 - a_1^2)} + \frac{\omega^2}{1 - a_1^2} - \frac{2u_\star m_2}{(u_0 - u_\star)} + \frac{2(u_\star - 1)\mu^3 \omega}{(1 - a_1^2)(1 - a_2^2)(u_0 - u_\star)} \\ & + \frac{2m_2(1 - \theta_1)}{(u_0 - 1)} + \frac{2(u_\star - 1)a_1^2 a_2^2 \tilde{\Omega}}{\mu(1 - a_1^2)(1 - a_2^2)(u_0 - u_\star)} + C_m + \frac{(u_0 - u_\star)}{(u_0 - 1)(u_\star - 1)} C_m \\ & - \frac{2(u_\star - 1)\mu((a_1^2 + a_2^2)\omega - a_1(1 - a_1^2)m_1 - a_2(1 - a_2^2)m_2)}{(1 - a_1^2)(1 - a_2^2)(u_0 - u_\star)}, \end{aligned} \quad (5.29c)$$

$$\begin{aligned} 4u_\star Q_\star = & -\frac{2u_\star(1 - \theta_1)}{(u_\star - 1)} - 2m_1 + \frac{2u_\star m_2}{(u_0 - u_\star)} \\ & - \frac{2(u_\star - 1)\mu^3 \omega}{(1 - a_1^2)(1 - a_2^2)(u_0 - u_\star)} - \frac{2(u_\star - 1)a_1^2 a_2^2 \tilde{\Omega}}{\mu(1 - a_1^2)(1 - a_2^2)(u_0 - u_\star)} \\ & + \frac{2(u_\star - 1)\mu((a_1^2 + a_2^2)\omega - a_1(1 - a_1^2)m_1 - a_2(1 - a_2^2)m_2)}{(1 - a_1^2)(1 - a_2^2)(u_0 - u_\star)}. \end{aligned} \quad (5.29d)$$

We see again that the position of the apparent singularity at $u = u_\star$ is related to the parameter μ through a Möbius transformation (5.29a) and that the single monodromy parameters m_1, m_2, θ_1 and $\varsigma = \omega + a_1 m_1 + a_2 m_2$ do not depend on μ . The initial proposal of μ -separability in [120] generated some discussion about the interpretation of the parameter [66, 69]. In order to add to that, we need to take a detour and write about isomonodromy.

5.3 Conditions on the Painlevé VI system

The most natural setting to describe isomonodromy is the theory of flat holomorphic connections. The exposition here follows the monograph by Iwasaki *et al.*, [101], with some additions suited to our purposes. Consider the matricial

system of differential equations on a single complex variable

$$\frac{d\Phi(z)}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) \Phi(z), \quad A_\infty = -A_0 - A_1 - A_t = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}. \quad (5.30)$$

Choosing A_∞ diagonal comes at the expense of fixing a basis for the fundamental solution $\Phi(z)$. Let us parametrize it as

$$\Phi(z) = \begin{pmatrix} y_1(z) & y_2(w) \\ w_1(z) & w_2(w) \end{pmatrix}. \quad (5.31)$$

It is a straightforward exercise to see that the differential equation satisfied by the first row of $\Phi(z)$ is

$$y_i''(z) - \left(\text{Tr} A(z) + \frac{A'_{12}(z)}{A_{12}(z)} \right) y_i'(z) + \left(\det A(z) - A'_{11}(z) + A_{11}(z) \frac{A'_{12}(z)}{A_{12}(z)} \right) y_i(z) = 0. \quad (5.32)$$

Furthermore, with A_∞ diagonal, we have

$$A_{12}(z) = \frac{(A_0)_{12}}{z} + \frac{(A_1)_{12}}{z-1} + \frac{(A_t)_{12}}{z-t} = \frac{k(z-\lambda)}{z(z-1)(z-t)}, \quad (5.33)$$

so, for the matricial system (5.30), the associated scalar equation (5.32) will be Fuchsian, with 5 singular points at $z = \{0, 1, t, \lambda, \infty\}$, exactly the type encountered in the radial and angular systems, given by equations (5.22) and (5.28), respectively.

From this formulation it seems clear that, from the matricial system (5.30) perspective, the singularity at $z = \lambda$ is a consequence of our choice of gauge $A_\infty = \text{diag}(\kappa_1, \kappa_2)$. As a matter of fact, we can see that there is a residual gauge symmetry that moves λ , as discovered by Jimbo, Miwa and Ueno [103]. Let us introduce the parametrization for the coefficient matrices A_i

$$A_i = \begin{pmatrix} p_i + \hat{\theta}_i & -q_i p_i \\ \frac{p_i + \hat{\theta}_i}{q_i} & -p_i \end{pmatrix}, \quad (5.34)$$

which is the most general for the gauge choice where $\text{Tr} A_i = \hat{\theta}_i$ and $\det A_i = 0$, $i = 0, 1, t$ are fixed. The parameters p_i, q_i are subject to extra constraints. The diagonal terms of $A_\infty = -(A_0 + A_1 + A_t)$ are,

$$\kappa_1 = \frac{\hat{\theta}_\infty - \hat{\theta}_0 - \hat{\theta}_1 - \hat{\theta}_t}{2}, \quad \kappa_2 = \frac{-\hat{\theta}_\infty - \hat{\theta}_0 - \hat{\theta}_1 - \hat{\theta}_t}{2}. \quad (5.35)$$

Let us now define, along with λ ,

$$\eta = A_{11}(z = \lambda) = \frac{p_0 + \hat{\theta}_0}{\lambda} + \frac{p_1 + \hat{\theta}_1}{\lambda - 1} + \frac{p_t + \hat{\theta}_t}{\lambda - t}. \quad (5.36)$$

We will now solve for p_i and q_i in terms of λ and η^3 . The solution will also depend on an extra parameter k , which can be made equal to one by conjugation of all the A_i by a diagonal matrix, the particular value of k will not enter into (5.32). The explicit solutions for p_i and q_i are given as [103]

$$q_0 = \frac{k\lambda}{tp_0}, \quad q_1 = -\frac{k(\lambda - 1)}{(t - 1)p_1}, \quad q_t = \frac{k(\lambda - t)}{t(t - 1)p_t}, \quad (5.37)$$

with k undefined and

$$p_0 = \frac{\lambda}{t\hat{\theta}_\infty} \left(\lambda(\lambda - 1)(\lambda - t)\tilde{\eta}^2 + (\hat{\theta}_1(\lambda - t) + t\hat{\theta}_t(\lambda - 1) - 2\kappa_2(\lambda - 1)(\lambda - t))\tilde{\eta} \right. \\ \left. + \kappa_2^2(\lambda - t - 1) - \kappa_2(\hat{\theta}_1 + t\hat{\theta}_t) \right), \quad (5.38a)$$

$$p_1 = -\frac{\lambda - 1}{(t - 1)\hat{\theta}_\infty} \left(\lambda(\lambda - 1)(\lambda - t)\tilde{\eta}^2 + ((\hat{\theta}_1 + \hat{\theta}_\infty)(\lambda - t) + t\hat{\theta}_t(\lambda - 1) \right. \\ \left. - 2\kappa_2(\lambda - 1)(\lambda - t))\tilde{\eta} + \kappa_2^2(\lambda - t) - \kappa_2(\hat{\theta}_1 + t\hat{\theta}_t) - \kappa_1\kappa_2 \right), \quad (5.38b)$$

$$p_t = \frac{\lambda - t}{t(t - 1)\hat{\theta}_\infty} \left(\lambda(\lambda - 1)(\lambda - t)\tilde{\eta}^2 + (\hat{\theta}_1(\lambda - t) + t(\hat{\theta}_t + \hat{\theta}_\infty)(\lambda - 1) \right. \\ \left. - 2\kappa_2(\lambda - 1)(\lambda - t))\tilde{\eta} + \kappa_2^2(\lambda - 1) - \kappa_2(\hat{\theta}_1 + t\hat{\theta}_t) - t\kappa_1\kappa_2 \right), \quad (5.38c)$$

$$\tilde{\eta} = \eta - \frac{\hat{\theta}_0}{\lambda} - \frac{\hat{\theta}_1}{\lambda - 1} - \frac{\hat{\theta}_t}{\lambda - t}. \quad (5.38d)$$

In terms of λ and η , the equation (5.32) is written as

$$y_i''(z) + p(z)y_i'(z) + q(z)y_i(z) = 0, \\ p(z) = \frac{1 - \hat{\theta}_0}{z} + \frac{1 - \hat{\theta}_1}{z - 1} + \frac{1 - \hat{\theta}_t}{z - t} - \frac{1}{z - \lambda}, \\ q(z) = \frac{\kappa_1(\kappa_2 + 1)}{z(z - 1)} + \frac{t(t - 1)K}{z(z - 1)(z - t)} + \frac{\lambda(\lambda - 1)\eta}{z(z - 1)(z - \lambda)}, \quad (5.39)$$

³We have changed the notation to avoid any confusion with the separation parameter μ . However, in the previous Chapters we refer to η as μ in the isomonodromic deformations theory

with λ and η as above and

$$K = -H - \frac{\lambda(\lambda-1)}{t(t-1)}\eta - \frac{\lambda-t}{t(t-1)}\kappa_1 + \frac{\hat{\theta}_0\hat{\theta}_t}{2t} + \frac{\hat{\theta}_1\hat{\theta}_t}{2(t-1)}, \quad (5.40)$$

where H will be relevant to us in the following

$$H = \frac{1}{t}\text{Tr}(A_0A_t) + \frac{1}{t-1}\text{Tr}(A_1A_t) - \frac{\hat{\theta}_0\hat{\theta}_t}{2t} - \frac{\hat{\theta}_1\hat{\theta}_t}{2(t-1)}. \quad (5.41)$$

We now note that the singularity at $z = \lambda$ in (5.32) is apparent, and then K , λ and η satisfy an algebraic constraint. Translating this constraint to H ,

$$H = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \left(\eta^2 - \left(\frac{\hat{\theta}_0}{\lambda} + \frac{\hat{\theta}_1}{\lambda-1} + \frac{\hat{\theta}_t}{\lambda-t} \right) \eta + \frac{\kappa_1\kappa_2}{\lambda(\lambda-1)} \right) + \frac{\hat{\theta}_0\hat{\theta}_t}{2t} + \frac{\hat{\theta}_1\hat{\theta}_t}{2(t-1)}. \quad (5.42)$$

From the gauge field perspective, we can think of $A(z) = [\partial_z \Phi(z)]\Phi(z)^{-1}$ as a flat connection, whose observables are traces of non-contractible Wilson loops. In the language of complex analysis, holonomy is represented by monodromy matrices, M_0 , M_t , M_1 and M_∞ associated to loops around each singular point of the system (5.30). These matrices are defined up to conjugation and constrained by the fact that the composition of the monodromies over all singular points is a contractible curve:

$$M_\infty M_1 M_t M_0 = \mathbb{1}. \quad (5.43)$$

The gauge-invariant observables are the traces of the matrices $\text{Tr } M_i = 2 \cos \pi \hat{\theta}_i$ ⁴, and the traces of two of the composite monodromies:

$$\text{Tr } M_0 M_t = 2 \cos \pi \sigma_{0t}, \quad \text{Tr } M_t M_1 = 2 \cos \pi \sigma_{1t}. \quad (5.44)$$

The third combination $\text{Tr } M_0 M_1$ is related to these two by a polynomial identity (Fricke-Jimbo relation), involving the $\hat{\theta}_i$, as can be seen in [102].

⁴Here, we have performed a normalization of the solution $\Phi(z)$ so that its determinant is constant equal to one. One can check that this just subtracts from each coefficient matrix A_i its trace.

The gauge-invariant quantities $\hat{\theta}_0, \hat{\theta}_t, \hat{\theta}_1, \hat{\theta}_\infty, \sigma_{0t}, \sigma_{1t}$ are called *monodromy data*, associated to the system (5.30) – or, alternatively, to the equation (5.32). Of those, the single monodromy parameters $\hat{\theta}_i$ can be read directly from the differential equation, whereas σ_{0t} and σ_{1t} are not readily available. On the other hand, they comprise the information needed from the equation to solve the scattering problem [136], or to find quasi-normal modes [27]. Therefore, finding them is a problem of interest.

Solving for σ_{0t}, σ_{1t} makes use of a residual gauge symmetry of (5.30), which changes the position of the singularity at $z = t$. The zero curvature condition $\partial_z \partial_t \Phi(z, t) = \partial_t \partial_z \Phi(z, t)$ forces the coefficient matrices A_i to satisfy the *Schlesinger equations*

$$\begin{aligned} \frac{\partial A_0}{\partial t} &= -\frac{1}{t}[A_0, A_t], & \frac{\partial A_1}{\partial t} &= -\frac{1}{t-1}[A_1, A_t], \\ \frac{\partial A_t}{\partial t} &= \frac{1}{t}[A_0, A_t] + \frac{1}{t-1}[A_1, A_t], \end{aligned} \quad (5.45)$$

which, when written in terms of λ and η , result in the Painlevé VI transcendent.

In a seminal paper [102], Jimbo derived asymptotic expansions for the Painlevé VI transcendent in terms of monodromy data, written in a slightly different guise:

$$\frac{\partial}{\partial t} \log \tau(\hat{\theta}, \vec{\sigma}; t) = H - \frac{1}{2t} \hat{\theta}_0 \hat{\theta}_t - \frac{1}{2(t-1)} \hat{\theta}_1 \hat{\theta}_t, \quad (5.46)$$

where τ is a function of the monodromy data and of t , called the isomonodromic time. In another big development, [71] gave the full expansion for τ , given generic monodromy arguments, in terms of Nekrasov functions, with the structure

$$\tau(\hat{\theta}, \vec{\sigma}; t) = \sum_{n \in \mathbb{Z}} C(\hat{\theta}, \sigma_{0t} + 2n) [s(\hat{\theta}, \vec{\sigma})]^{n t^{\frac{1}{4}} \sigma_{0t}^2 + n(\sigma_{0t} + n)} \mathcal{B}(\hat{\theta}, \sigma_{0t} + 2n; t), \quad (5.47)$$

where the Nekrasov functions $\mathcal{B}$ are analytic in t , see 3.4. We refer to [28] for details. These functions were introduced as the instanton partition function of four-dimensional $\mathcal{N} = 2$ SU(N) Yang-Mills [134] coupled to matter multiplets, and were related to two dimensional conformal blocks by the Alday-Gaiotto-Tachikawa conjecture [14], later proved by Alba, Fateev, Litvinov and Tarnopolsky [13]. The relation then comes full circle to help solve classical field propagation in five dimensional space-times.

With the full expansion given in [71, 72], connection formulas for the expansions at different singular points were given [97], and a Fredholm determinant formulation was constructed [75], which is well-suited to numerical calculations [17].

The monodromy problem, which is the original formulation of the classical Riemann-Hilbert problem, consists in finding the full set of monodromy data from the parameters in the equation (5.39). For our purposes, the latter will consist of the single monodromy data $\{\hat{\theta}_i\}$ and the parameters λ, η – remember that K is related to them by (5.40) and (5.42). These conditions are best written in terms of the ζ function defined in [139] (called $\sigma(t)$ there),

$$\zeta(t) = t(t-1) \frac{\partial}{\partial t} \log \tau(\hat{\theta}, \vec{\sigma}; t) = (t-1) \text{Tr} A_0 A_t + t \text{Tr} A_1 A_t - \frac{t-1}{2} \hat{\theta}_0 \hat{\theta}_t - \frac{t}{2} \hat{\theta}_1 \hat{\theta}_t. \quad (5.48)$$

In terms of $\zeta(t)$, the Schlesinger equations read

$$\frac{d\zeta}{dt}(t) = -\text{Tr}(A_t(A_t + A_\infty)) - \frac{1}{2}(\hat{\theta}_0 + \hat{\theta}_1)\hat{\theta}_t, \quad \frac{d^2\zeta}{dt^2}(t) = -\frac{\text{Tr}(A_\infty[A_0, A_t])}{t(t-1)}. \quad (5.49)$$

The strategy to recover the monodromy data from (5.39) is now clear: given the differential equation, which is parametrized by particular values for λ_0, η_0 , and the monodromy time t_0 , one can find the coefficient matrices A_i – up to overall conjugation – by solving for p_i 's and q_i 's using the formula above. Given (5.40) and (5.42), as well as the formulas for the entries of A_i above, we can then compute the derivatives of ζ at $t = t_0$

$$\begin{aligned} \zeta(t_0) = \lambda_0(\lambda_0 - 1)(\lambda_0 - t_0) & \left[\eta_0^2 - \left(\frac{\hat{\theta}_0}{\lambda_0} + \frac{\hat{\theta}_1}{\lambda_0 - 1} + \frac{\hat{\theta}_t}{\lambda_0 - t_0} \right) \eta_0 + \frac{\kappa_1 \kappa_2}{\lambda_0(\lambda_0 - 1)} \right] \\ & + \frac{t-1}{2} \hat{\theta}_0 \hat{\theta}_t + \frac{t}{2} \hat{\theta}_1 \hat{\theta}_t, \quad (5.50a) \end{aligned}$$

$$\begin{aligned} \frac{d\zeta}{dt}(t_0) = -\frac{\lambda_0(\lambda_0 - 1)(\lambda_0 - t_0)^2}{t_0(t_0 - 1)} & \left[\eta_0^2 - \left(\frac{\hat{\theta}_0}{\lambda_0} + \frac{\hat{\theta}_1}{\lambda_0 - 1} + \frac{\hat{\theta}_t - \hat{\theta}_\infty}{\lambda_0 - t_0} \right) \eta_0 \right. \\ & \left. + \frac{\kappa_1^2}{(\lambda_0 - t_0)^2} \right] - \frac{\lambda_0 - 1}{t_0 - 1} \kappa_1 \hat{\theta}_0 - \frac{\lambda_0}{t_0} \kappa_1 \hat{\theta}_1 - \kappa_1 \kappa_2 + \frac{1}{2}(\hat{\theta}_0 + \hat{\theta}_1)\hat{\theta}_t, \quad (5.50b) \end{aligned}$$

and the second derivative can be written in terms of $\zeta(t)$ and $\zeta'(t)$, resulting in the “ σ -form” of the Painlevé VI equation

$$\begin{aligned} & \zeta'(t(t-1)\zeta'')^2 + [2\zeta'(t\zeta' - \zeta) - (\zeta')^2 - \frac{1}{16}(\hat{\theta}_t^2 - \hat{\theta}_\infty^2)(\hat{\theta}_0^2 - \hat{\theta}_1^2)]^2 \\ &= (\zeta' + \frac{1}{4}(\hat{\theta}_t + \hat{\theta}_\infty)^2)(\zeta' + \frac{1}{4}(\hat{\theta}_t - \hat{\theta}_\infty)^2)(\zeta' + \frac{1}{4}(\hat{\theta}_0 + \hat{\theta}_1)^2)(\zeta' + \frac{1}{4}(\hat{\theta}_0 - \hat{\theta}_1)^2). \end{aligned} \quad (5.51)$$

In principle, the initial conditions for $\zeta(t)$ at $t = t_0$ above determine $\zeta(t)$ uniquely through the differential equation. The function can then be inverted to recover σ_{0t} and σ_{1t} .

We note that the change of the parameters λ and η with respect to t , along the isomonodromy solution is a gauge transformation in the sense that the monodromy data is kept invariant, therefore

$$\delta\lambda = \frac{\partial K}{\partial \eta} \delta t, \quad \delta\eta = -\frac{\partial K}{\partial \lambda} \delta t, \quad (5.52)$$

is a residual gauge transformation.

5.4 Formal solution to the radial and angular systems

5.4.1 Writing the boundary conditions in terms of monodromy data

In terms of the Painlevé VI τ -function, or rather the ζ function defined in (5.48), the parameters of ODE comprise an initial value problem for the ζ function. The case of angular and radial equations above is a little more involved since the parameters are coupled. Let us start with the following identification

	t_0	$\hat{\theta}_0$	$\hat{\theta}_t$	$\hat{\theta}_1$	$\hat{\theta}_\infty$	K	λ_0	η_0
$\zeta_{\text{Rad}}(t)$	z_0	θ_-	θ_+	θ_1	θ_0	$K_0(\mu, C_m)$	$z_\star(\mu)$	$K_\star(\mu)$
$\zeta_{\text{Ang}}(t)$	u_0	$-m_1$	$-m_2$	θ_1	ς	$Q_0(\mu, C_m)$	$u_\star(\mu)$	$Q_\star(\mu)$

where we highlighted the dependence of the parameters of the radial and angular systems on μ and the separation constant C_m . It is a straightforward

exercise to show that the corresponding quantities K , λ_0 , η_0 and t_0 for the radial and angular systems are not independent, satisfying (5.40) and (5.42). This fact shows that the singularity at λ_0 for both radial and angular systems is apparent, as anticipated in Sec. 5.2.2.

For the angular system, we want to solve the eigenvalue problem. So, in principle, (5.50) gives a condition on the generic solution of the Painlevé equation (5.51), given by (5.47), in which we read the two monodromy parameters

$$\sigma_{0,u_0;\text{Ang}}(\omega, \mu, C_m), \quad \text{and} \quad \sigma_{u_0,1;\text{Ang}}(\omega, \mu, C_m), \quad (5.53)$$

where we omitted the dependence on m_1, m_2, a_1, a_2 .

As discussed in [27, 28], the condition that the solutions of the angular differential equation (5.16) are well-behaved both at the North and South poles of the sphere x, ϕ, ψ can be written in terms of the monodromy parameters. Let us now review this construction.

Let $y_{1,2;i}(z)$ be (normalized) solutions of the deformed Heun equation (5.32) associated to a fundamental matrix $\Phi_i(z)$ whose monodromy matrix is diagonal at a chosen regular singular point $z = z_i$:

$$\Phi_i((z - z_i)e^{2\pi i} + z_i) = \Phi_i(z)e^{\pi i \hat{\theta}_i \sigma_3}. \quad (5.54)$$

Note that this implies that the solutions $y_{1,2;i}(z)$ have different behavior asymptotizing z_i , with $y_{1,i}(z)/y_{2,i}(z) \propto (z - z_i)^{\theta_i}$ as $z \rightarrow z_i$. The analogous solution at a different singular point $z = z_j$, $\Phi_j(z)$, is connected to $\Phi_i(z)$ by a constant matrix E_{ij}

$$\Phi_j(z) = \Phi_i(z)E_{ij}, \quad (5.55)$$

called the *connection matrix* between z_i and z_j . It is straightforward to see that, if a given solution $y(z)$ of (5.32) has a definite behavior, in the sense that it asymptotes one of the solutions at $z = z_i$, say $y_{1,i}(z)$, and one of the solutions at $z = z_j$, say $y_{1,j}(z)$, then the connection matrix E_{ij} must be triangular. This in turn implies that the monodromy matrix of $\Phi_i(z)$ around $z = z_j$, generically of the form $M_j = E_{ij}e^{\pi i \hat{\theta}_j \sigma_3}E_{ij}^{-1}$, is also triangular, and then the composite monodromy parameter σ_{ij} will satisfy

$$2 \cos \pi \sigma_{ij} = \text{Tr} M_i M_j = 2 \cos \pi (\hat{\theta}_i + \hat{\theta}_j) \longrightarrow \sigma_{ij} = \hat{\theta}_i + \hat{\theta}_j + 2m, \quad m \in \mathbb{Z}. \quad (5.56)$$

It is also straightforward to show that the converse is also true: if the composite monodromy σ_{ij} satisfies (5.56), then the connection matrix is triangular.

Coming back to the angular system, the condition that the solutions are well-behaved at the North pole ($x = a_2^2$, or $u = 0$) and at the South pole ($x = a_1^2$, or $u = u_0$) means

$$\sigma_{0,u_0;\text{Ang}}(\omega, \mu, C_m) = m_1 + m_2 + 2m, \quad m \in \mathbb{Z} \quad (5.57)$$

which defines the separation constant as an integer family of functions of μ and ω : $C_m(\mu, \omega)$. We will overlook issues of existence and uniqueness for the purposes of this exposition.

Now, for the radial system, again the solution of the isomonodromic equation (5.51) with the initial conditions (5.50) will define the corresponding two composite monodromy parameters, associated to paths encircling the singularities at $z = 0, z_0$ and $z = z_0, 1$, respectively

$$\sigma_{0,z_0;\text{Rad}}(\omega, \mu, C_m), \quad \sigma_{z_0,1;\text{Rad}}(\omega, \mu, C_m). \quad (5.58)$$

where we omit the dependence on the other physical parameters M, a_1, a_2, m_1, m_2 . It is customary to substitute in this condition the expression for the separation constant C_m obtained from the angular equation. We will postpone this step for now.

As an aside, note that the interest in the radial and angular systems, apart from finding the actual form of the radial wavefunctions – whose local expansions can be obtained from Frobenius method – usually consists of the scattering problem and the quasinormal modes problem. They can both be cast in terms of the monodromy parameters problem, with now the relevant composite monodromy parameter involving the outer horizon $r = r_+$, or $z = z_0$ and the conformal boundary at $r = \infty$, or $z = 1$. The transmission coefficient, for instance, is [43]

$$|\mathcal{T}|^2 = \left| \frac{\sin \pi \theta_+ \sin \pi \theta_0}{\sin \frac{\pi}{2}(\sigma_{z_0,1;\text{Rad}} - \theta_+ + \theta_1) \sin \frac{\pi}{2}(\sigma_{z_0,1;\text{Rad}} + \theta_+ - \theta_1)} \right|, \quad (5.59)$$

which poses an interpretation problem for μ : since in principle the electric and magnetic modes (5.8a) exhaust the 3 polarizations of the photons in five dimensions – with 1 of them electric and 2 magnetic as argued in [120] – the fact that the scattering coefficient depends on the extra parameter μ seems spurious.

This redundancy also arises in the calculation of quasi-normal modes from the radial system, whose method of solution mirrors that of the angular eigenvalue. The requirement that the radial wavefunction is “purely outgoing” at $r = r_+$ and “purely ingoing” at $r = \infty$ can be cast, by the same arguments put forward above, in terms of the quantization condition for the composite monodromy parameter

$$\sigma_{z_0,1;\text{Rad}}(\omega, \mu) = \theta_+(\omega) + \sqrt{1 - C_m(\omega, \mu)} + 2n, \quad n \in \mathbb{Z}. \quad (5.60)$$

This condition defines implicitly the modes $\omega_{n,m}(\mu)$ as a function of the radial and azimuthal quantum numbers n and m . Again, it seems rather unphysical that there will be a 1-parameter family of vector quasi-normal modes in the space-time.

The redundancy is solved by the intertwining of both radial and angular systems. Note that, in both angular and radial equations, (5.28) and (5.22), the function that parametrizes the position of the apparent singularity, represented by λ in the generic deformed Heun equation (5.39) is essentially, up to a global conformal transformation, the parameter μ :

$$\lambda_{\text{Ang}} = u_\star = \frac{\mu^2 - a_1^2}{\mu^2 - 1}, \quad \lambda_{\text{Rad}} = z_\star = \frac{r_-^2 + \mu^2}{r_0^2 + \mu^2}. \quad (5.61)$$

Using the properties of the Painlevé VI equation, the function λ can be computed using the ζ function defined in (5.48) by resorting to the demonstration in [139],

$$\begin{aligned} \frac{1}{\lambda - t} = & -\frac{1}{2} \left(\frac{1}{t} + \frac{1}{t-1} \right) \\ & + \frac{\hat{\theta}_\infty t(t-1)\zeta'' + (\zeta' + \frac{1}{4}(\hat{\theta}_t^2 - \hat{\theta}_\infty^2))((2t-1)\zeta' - 2\zeta + \frac{1}{4}(\hat{\theta}_0^2 - \hat{\theta}_1^2)) + \frac{1}{4}\hat{\theta}_\infty^2(\hat{\theta}_0^2 - \hat{\theta}_1^2)}{2t(t-1)(\zeta' + \frac{1}{4}(\hat{\theta}_t - \hat{\theta}_\infty)^2)(\zeta' + \frac{1}{4}(\hat{\theta}_t + \hat{\theta}_\infty)^2)}, \end{aligned} \quad (5.62)$$

using the Hamiltonian properties of the isomonodromic system. Equation (5.62) as well as analogues for the other Painlevé transcendents, can be found in [71].

With the help of (5.62), we can lift the ambiguity, using the procedure that we can now describe. The angular and radial systems define four monodromy parameters as stated above. Of these, the quantization condition for the angular solutions and the quasi-normal modes will set two,

$$\sigma_{0,u_0;\text{Ang}}(\omega, \mu, C_m), \quad \sigma_{z_0,1;\text{Rad}}(\omega, \mu, C_m) \quad (5.63)$$

which can be used to implicitly define $\omega(\mu)$ and $C_m(\mu)$ as functions of the redundancy parameter μ . Now, with this substitution, one can write the two remaining monodromy parameters as functions of μ

$$\sigma_{u_0,1;\text{Ang}}(\omega(\mu), \mu, C_m(\mu)), \quad \sigma_{0,z_0;\text{Rad}}(\omega(\mu), \mu, C_m(\mu)). \quad (5.64)$$

These four one-parameter families of monodromy parameters can now be fed into (5.47) to define two one-parameter families of ζ functions, one for the angular system and one for the radial system. Calling $\lambda_{\text{Ang}}(\mu)$ and $\lambda_{\text{Rad}}(\mu)$ the respective functions defined by the right hand side of (5.62), we find an extra condition by requiring that the parameter μ defined by both systems is equal:

$$\mu^2 = \frac{\lambda_{\text{Ang}}(\mu) - a_1^2}{\lambda_{\text{Ang}}(\mu) - 1} = \frac{r_-^2 - \lambda_{\text{Rad}}(\mu)r_0^2}{\lambda_{\text{Rad}}(\mu) - 1}, \quad (5.65)$$

which can be seen as a consistency condition for both isomonodromic systems defined by the angular and radial equations. One can rephrase this property in different ways, and we will find below that using the expression for the second derivative of ζ given by (5.49) as a fifth condition, along with the values of the monodromy parameters above is a more computationally efficient approach. This condition, along with the corresponding quantization conditions for the angular (5.57) and radial system (5.60) are sufficient to determine all the separation constants, as well as the frequencies for the quasi-normal modes. This procedure is in stark contrast to the role of μ in the four-dimensional case studied previously [59], where it can be eliminated by a simple change of variables.

We should point out, however, that the implicit definitions for the quantities ω, C_m, μ presented above may not be single valued, which will then allow for orbits of μ with disconnected components. In particular, one can indeed have more than one solution to (5.65). In the next Section, we will take the solution closest to u_0 , due to the nature of the Nekrasov expansion. It is an open question whether a different choice will lead to different physics. We will leave the study of these subtleties for future work.

5.4.2 Quasinormal modes from the radial system

The conditions put forth in the last Section provide an exact, procedural solution to the quasinormal modes for the vector perturbations. However, the

analytical treatment of these conditions is of very limited scope at this moment, given the five transcendental equations one has to deal with. One may, however, resort to numerical implementations of the τ function.

In this Section we are going to describe the numerical treatment of eigenfrequencies for ultraspinning $a_1 \rightarrow 1$ black holes. This amounts to solving numerically in this limit the equations listed previously for $\{\omega, \sigma_{0,z_0;Rad}, \sigma_{z_0,1;Rad}, \mu, C_m, \sigma_{u_0,1;Ang}, \sigma_{0,u_0;Ang}\}$. For $a_1 \rightarrow 1$, the angular system is better served by the expansion of the τ function around $u_0 = 1$, which was given in [71]. Recently, in [49] a similar analysis was performed for four dimensional Kerr-Newman and Kerr-Sen black holes. Nevertheless, our problem presents an extra difficulty related to the presence of μ in the radial and angular equations, which is not the case in four dimensions where the separation parameter μ can be absorbed by a redefinition of the separation constant C_m , as can be seen in [59, 66]. See also [50] for a similar discussion.

In Figure 5.1, we display the results, with the rotation parameter $a_2 = 0.001$ and the size of the outer horizon $r_+ = 0.05$. The ultraspinning black hole regime considered is $0.99 \leq a_1 \leq 0.99999$, and the values of the quantum numbers are set at $\ell = 2, m_1 = m_2 = 0$. The frequencies found are stable and increase monotonically with a_1 . The value of μ has a more complicated behavior with a_1 , but it should be kept in mind that the change comes in the tenth decimal place and may be affected by numerical errors. Even with this caveat, the approximation of the angular τ function improves as $a_1 \rightarrow 1$ and the expansion of the radial τ function should be valid as long as $|\theta_+ z_0| \ll 1$, with typical values in the range $|\theta_+ z_0| \sim 10^{-2}$. In our analysis, we have used the Nekrasov expansion (5.47) truncating the number of channels $n \leq N_c = 7$ and the number of levels in the conformal block $\mathcal{B}$ to $m \leq N_b = 7$. For $z_0 \approx 10^{-2}$, we estimate at least 10 digits accuracy.

Note that the quasi-normal frequency ω_0 increases until $a_1 = 0.999$, where it shows an asymptote, and possibly a non polynomial behavior in $1 - a_1$. A better understanding of the physics behind this analysis is indeed well-deserved, particularly the prospect of superradiant modes [27]. We will, however, leave these aspects to future work.

As a preliminary test of the results above, we have constructed the angular eigenfunctions $Y(u)$ using the values for $\{\omega, C_m, \mu\}$ obtained using isomonodromy for a few values of a_1 and plotted in Figure 5.2. The construction is the standard Frobenius method, where expansions for $Y(u)$ at both points

Figure 5.1: Quasinormal frequencies ω_0 , separation constant C_m and the parameter μ , as given functions of a_1 . Note that the change of the separation parameter μ in the range considered is very small, even when the parameter a_1 comes very close to the extremal value 1.

$u = 0$ and $u = u_0 \approx 1$ and matching at middle point are performed for the value of the function as well as 15 derivatives. The asymptotic behaviour in Figure 5.2 is as expected, and we could verify the values of the parameters obtained to at least 10 digits. Unfortunately, the construction for the radial eigenfunctions is much more computationally demanding and a detailed analysis is also postponed to future work.

Figure 5.2: Numerical eigenfunctions of the angular equation (5.28) for different values of a_1 . These were obtained by matching the Frobenius expansion at two of the singular points, $u = 0$ and $u = u_0$, with 16 terms. The values of μ were chosen by requiring consistency with the radial system.

5.5 Discussion

In this Chapter we studied the role of the separation parameter μ , introduced in [120] to allow for the separation of Maxwell equations in a five-dimensional Kerr-(anti) de Sitter background. We saw that μ is related to an apparent singularity of the resulting angular and radial differential equations. Specifically, the position of the apparent singularity is related to μ by a simple Möbius transformation. After translating the boundary conditions for the radial equation (5.60) and angular equation (5.57) in terms of the monodromy data, we could outline a method to find the separation constant and the quasi-normal modes frequencies. The separation parameter μ is fixed by a consistency condition between the τ functions for the radial and angular systems (5.65). We have then checked the procedure numerically by considering small $r_+ \ll 1$ and ultraspinning black holes $a_2 = 0.001$, $a_1 \lesssim 1$.

We should point out now that, unlike the scalar case studied in [27], the use of the τ function and the isomonodromic deformations method for the vector case is not just a numerically more efficient way for computing the quasi-normal modes. The monodromy language allows us to define the quantities involved in a way independent of μ and hence to decouple the conditions necessary to solve the problem. It is interesting to notice that the introduction of the apparent singularity mirrors the remark by Poincaré that apparent singularities are necessary in ordinary differential equations if we want to solve the problem of finding the parameters of the ordinary differential equation whose solutions are associated to generic monodromy parameters [101]. We hope that the results presented here can help to elucidate the geometrical structure behind the introduction of μ . We also expect that the method presented here will help with the solution for the p -form fields in the same background, which were shown to lead to separable equations in [121].

One can deduce from the analysis that the separation parameter μ in higher dimensions plays a more prominent role than in the four dimensional case, where it can be eliminated by a suitable change of variables. The case studied here, that of “electric” polarizations, as defined in [120] is related to the “magnetic” polarizations by an inversion $\mu \rightarrow 1/\mu$. Although this inversion could be interpreted as a gauge transformation of the electric mode, we note that it nevertheless modifies the asymptotic behaviour of the field, so a more careful analysis is in order. At any rate, the results above found that μ should assume a discrete set of values, at least to allow solutions for the quasi-normal modes. In turn, the latter fact opens up the possibility of studying the different polarizations by exploring the symmetries of the Painlevé system. Another outstanding problem is the relation among the different definitions of polarizations in the literature [112]. We leave the exploration of these issues for future work.

The fact that the separation parameter μ parametrizes the position of the extra apparent singularity for both the angular and the radial equations seems an indication that perhaps the matrix system (5.30) plays a more prominent role than previously thought. That the particular choice of parameters, including μ , is able to decouple the equations mirrors the treatment of conformal blocks of conformal field theories in higher ($D > 2$) dimensions [143] where there exists a particular set of coordinates which factorizes the partial wave expansion as a product of two hypergeometric functions (at least in $D = 4$, see [58]).

Given the holographic aspect of perturbations of AdS_5 space, perhaps the

last point is more than an analogy. One may hope that the results here can be of use for the study of conformal bootstrap and conformal perturbation theory in four dimensions. On a more immediate direction, a systematic study of the quasinormal modes for generic black hole parameters as well as the near extremal case, with the prospect of instabilities, is also necessary.

Chapter 6

Conclusions and outlook

We conclude with a summary of the main results and an outlook to possible future research directions.

This thesis is devoted to the study of scalar and vector perturbations on Kerr-AdS₅ black hole by means of the isomonodromic deformations method. The isomonodromic deformation of the Fuchsian system with four regular singular points leads to the introduction of an extra singularity in the associated second order differential equation. By demanding that this singularity is an apparent one, we find the Painlevé VI equation. The Heun equation is then derived as a particular solution of the confluence between the apparent singularity and one regular singular point. Namely, we have shown that the boundary value problem related to the radial and angular ODEs can be converted into an initial value problem in terms of the Painlevé VI τ -function for each system (3.93).

The logarithmic derivative in the angular case is related with the calculation of the separation constant for slowly rotating black holes. The radial case is more intricate since the composite monodromy given by the quantization condition (3.32) is related to the asymptotic expansion around $z_0 = 1$ ¹, while for small black holes the asymptotic expansion is given at $z_0 = 0$. In fact, this issue can be circumvented by recasting the second derivative of the τ -function (3.93) in terms of an isomonodromic τ -function with shifted monodromies via the so-called “Toda equation” (3.111). This in turn provides a

¹Notice that this is not the case in [137], where the frequencies can be extracted directly from the first condition for the τ -function.

way to compute an analytic expansion for κ in (3.110) that one can replace into the accessory parameter expansion (3.126) to obtain a perturbative expression for the composite monodromy $\sigma_{0,z_0}(r_+)$.

Finally, an asymptotic formula for the quasi-normal frequencies (4.28) has been found in subsection 4.4.2. We note that for the s-wave mode ($\ell = 0$), its imaginary part is negative and describes a stable mode. For even ℓ , the first correction to the vacuum AdS_5 frequencies is of order $\mathcal{O}(r_+^2)$ with negative imaginary part for $\Delta > 1$, i.e., these modes are stable as one decreases the size of the outer horizon. On the other hand, numerical evidence shown in Figure 6.1 and the structure of corrections in (4.50) suggest that for odd ℓ , QNMs frequencies develop an instability for small black holes.

Figure 6.1: The imaginary part of the first quasi-normal mode frequency ω_0 at $\ell = 1, m_1 = 1$ and $m_2 = 0$ for small Kerr- AdS_5 black holes with equal rotation parameters and massless scalar field, $\Delta = 4$.

Our study suggests that the PVI τ function seems to be a reliable tool – both numerically and analytically – to study connection problems for Fuchsian equations, in particular scattering and resonance problems for a wide class of black holes. The implementation of more realistic elements such as spinorial fields and flavor should be still mathematically tractable by the tools devel-

oped here. Mathematically, the Dirac equation on Kerr-AdS₅ has an entirely analogous treatment in terms of Heun's equations as described in [162, 164].

We should point out now that, unlike the scalar case studied in [27], the use of the τ -function and the isomonodromic deformations method for the vector case is not just a numerically more efficient way for computing the quasi-normal modes. The monodromy language allows us to define the quantities involved in a way independent of μ and hence to decouple the conditions necessary to solve the problem. We hope that the results in this paper can help to elucidate the geometrical structure behind the introduction of μ .

The analytical tools developed here, especially the inspection of (3.93), may help understand the constraints posed by holography onto the Renormalization Group flow in a generic setting. As important properties of non-Abelian gauge theories, such as confinement and the conformal window, depend crucially on the specific way the theory flows from the UV to the IR, it is important to understand in detail which flows can be holographically described.

One can study the extremal limit in the Kerr-AdS₅ background. The coalescence between the inner and the outer horizon produces an irregular singular point, which can be thought of as the reduction from the Painlevé VI to the Painlevé V τ -function². This in turn implies that the Heun equation reduces to an ODE with two regular singular points and one irregular singular point, the so-called confluent Heun equation [148]. It is worth mentioning that this is completely different from the near-horizon extremal limit proposed by Bardeen and Horowitz in [26], where the resulting singular point of the collision among the inner and outer horizon, remains a regular one. Therefore, a proper description of the extremal limit in terms of the Painlevé transcendents is well-deserved.

An interesting realization occurs if instead of considering a Fuchsian system with four regular singular points, we replace one regular singular point by one irregular singular point at infinity, for instance. The resulting equation is known as the generalized Heun Equation and describes both radial and angular equations for a massive fermion in the Kerr-Newman black hole background [111]. The general form of the generalised Heun Equation in the mathematical liter-

²In [116], a series representation of the Painlevé V τ -function in terms of irregular conformal blocks has been derived via the confluence of two singularities.

ature has the form [145]:

$$y''(z) + \left(\frac{1-\mu_0}{z} + \frac{1-\mu_1}{z-1} + \frac{1-\mu_2}{z-a} + \alpha \right) y'(z) + \frac{\beta_0 + \beta_1 z + \beta_2 z^2}{z(z-1)(z-a)} y(z) = 0 \quad (6.1)$$

where $a \in \mathbb{C}$ and $\mu_0, \mu_1, \mu_2, \alpha, \beta_0, \beta_1, \beta_2$ are arbitrary complex numbers. The regular singular points are located at $z = \{0, 1, a\}$, with exponents $\{0, \mu_0\}, \{0, \mu_1\}$ and $\{0, \mu_2\}$ respectively, while at infinity we have an irregular singularity of rank 1. Therefore, we expect that an isomonodromic deformation approach can shed some light on the existence of a τ -function that in principle could be described by irregular conformal blocks inside the Fredholm determinant [27, 75].

Summary

The gravitational-wave events detected by LIGO are stunning results for General Relativity. The merger of two rotating black holes is one of the most violent events in the universe. Nevertheless, after relaxation and the emission of gravitational waves, perturbation theory works fine again and we can describe some aspects of the nature of the resulting single black hole and its progenitors, in terms of a set of discrete complex frequencies – the quasi-normal modes (QNMs).

Starting in the 1970s theoretical work led to different approaches to compute the black hole QNMs depending on the type of perturbation, the separability of the equations of motion and the specific background under consideration. Some methods that have been historically used include: i) the Wentzel-Kramers-Brillouin (WKB) approximation, ii) the continued-fraction method, iii) the Ferrari-Mashhoon method, iv) the Frobenius series and v) the quadratic eigenvalue problem (QEP).

In this respect, analytical solutions that exploit the presence of space-time symmetries with the aid of numerical relativity have produced not only accurate predictions of the gravitational waveforms, but also provided a wealth of information concerning the black hole’s surroundings, such as matter distribution and its radiative properties. By the same token, the study of space-times in dimensions greater than four has given insights into the underlying mathematical structure of Einstein’s theory and its extensions, providing as well a consistent test for the stability of new black-hole solutions.

This thesis focuses on the study of scalar and vector fields propagating in a Kerr-AdS₅ space-time. The latter is a vacuum solution to the Einstein’s equations with negative cosmological constant, which describes a rotating black hole with two independent angular momenta within a five dimensional anti-de

Sitter background. On the other hand, the formalism developed here can be used to treat linear perturbations in different space-times and other physical systems as well. Here, we summarize our results.

For the Kerr-AdS₅ black hole, the separation of variables leads to radial and angular second order ordinary differential equations (ODEs) with four regular singular points. In order to study this type of differential equations, we use the method of isomonodromic deformations. This method relies on the existence of families of Fuchsian systems with the same monodromy data that can be isomonodromically deformed and lead to the Painlevé equations, a set of nonlinear second order ODEs in the complex plane with the Painlevé property. Their solutions are called the Painlevé transcendents.

At the same time, the Painlevé transcendents solve the connection problem for the Heun type of equations, and provide new tools to explore several problems in Mathematics and Physics, from the Rabi model in quantum optics to the general mathematical problem of conformal mappings of simply connected domains. Indeed, one might consider to apply the derivations of this dissertation to solve other eigenvalue problems.

Namely, we will see that a Fuchsian system with four regular singular points can be deformed while preserving its monodromy data, where the isomonodromic equations are related to the Painlevé VI (PVI) equation. Moreover, the isomonodromic deformation can be thought of as the introduction of an apparent singularity in the deformed second order ODE, that makes manifest the Hamiltonian structure of the system and allows for a consistent definition of the PVI τ -function in the Jimbo, Miwa and Ueno sense.

Recently, it has been shown that the PVI τ -function can be written in terms of a certain correlation function between primary fields of a two-dimensional conformal field theory with central charge $c = 1$, and a series expansion in terms of conformal blocks has been provided by Gamayun, Iorgov and Lisovyy. By means of the τ -function we can reformulate the eigenvalue problem of the radial (angular) Heun equation into an initial value problem of the corresponding τ -function. For example, the first initial condition for the angular system solves the separation constant problem of the angular Heun equation. Once we have an asymptotic expression for the separation constant, we can address the computation of the quasi-normal modes using the two initial conditions for the radial Painlevé VI τ -function, performing the series expansion around the convenient singular point, i.e. at $t = \{0, 1, \infty\}$. In our application, we will see that

near-equally rotating, as well as the small black holes limit can be described by expansions of the associated PVI τ -functions around $t = 0$.

We begin by analysing the quasi-normal modes for Schwarzschild-AdS₅ as a test for the proposed method. Subsequently, we study scalar perturbations in the Kerr-AdS₅ black hole varying the radius of the outer horizon r_+ , the mass of the scalar field and the rotation parameters. Numerical evidence suggests that this approach achieves a higher accuracy than other numerical methods such as the matching method or the pseudo-spectral collocation method, for small and intermediate black holes ($r_+ < 1$) with near-equal rotation parameters.

Afterwards, we compute a perturbative expression for the QNMs frequencies of the Kerr-AdS₅ black hole. As the radius of the outer horizon decreases, the frequency approaches its pure AdS value with a first correction of order $\mathcal{O}(r_+^2)$. We analyse the scalar modes for the s-wave case ($\ell = 0$) and even orbital quantum number ($\ell > 0$ with $m_1 = m_2 = 0$). They turn out to be stable for small r_+ black holes, i.e., $r_+ \sim 10^{-3}$. Instead, modes with odd ℓ do exhibit a regime of superradiance in the small r_+ limit, in the sense that the imaginary part of their complex frequencies becomes positive, such that the black hole is unstable.

Finally, we address vector perturbations in our favorite background, following a recent remarkable progress on the separability of Maxwell equations in rotating black hole space-times in arbitrary dimensions. The separability relies on the existence of a Killing-Yano tensor and the introduction of an arbitrary parameter μ , along with the separation constant, in a new Ansatz for the vector field of the electromagnetic tensor. Namely, after separating the Maxwell equations in terms of a scalar function and bringing the radial and angular ODEs into the Heun form, one can see that μ is related by a Möbius transformation with an apparent singularity in the deformed Heun equation. With the help of the isomonodromy method, we can derive the initial conditions for the isomonodromic PVI τ -function and perform a numerical analysis for ultra-spinning black holes. The latter regime is described by an expansion of the angular PVI τ -function around $t = 1$, and allows to solve the complex system of transcendental equations.

We conclude that in higher dimensions μ behaves differently than in four dimensions, where it can be eliminated by a change of parametrization in the corresponding equations. We also note that in the scalar case, the apparent singularity plays an auxiliary role in the actual solution of the problem: quantities

such as scattering amplitudes and the quasi-normal modes depend only on the monodromy data. One can then compute them at any point of the isomonodromic flow, with the coincident point where the apparent singularity merges with one of the remaining singularities being particularly convenient. Yet, in the Maxwell case, the presence of the apparent singularity is mandatory.

As reflected in this thesis, we have shown that the PVI τ -function can be regarded as a reliable and powerful tool to solve the eigenvalue problem for the radial and angular Heun equations, derived from the dynamics of scalar and vector fields in Kerr-AdS₅.

Samenvatting

De door LIGO gedetecteerde zwaartekrachtgolven zijn verbluffende bevestigingen van de Algemene Relativiteitstheorie. De fusie van twee roterende zwarte gaten is een van de meest energierijke gebeurtenissen in het universum. Niettemin, na relaxatie en het uitzenden van zwaartekrachtgolven, werkt perturbatietheorie weer prima en kunnen we enkele aspecten van de aard van het resulterende enkele zwarte gat en zijn voorlopers, in termen van een reeks discrete complexe frequenties - de quasi-normale modi (QNM's) bestuderen.

Beginnend in de jaren zeventig leidde het theoretische werk tot verschillende benaderingen om eigenschappen van het zwarte gat te berekenen afhankelijk van het type verstoring, de scheidbaarheid van de bewegingsvergelijkingen en specifieke achtergrond die wordt overwogen. Sommige methoden die vroeger gebruikt zijn bevatten onder meer: i) de Wentzel-Kramers-Brillouin (WKB) benadering, ii) de kettingbreuk methode, iii) de Ferrari-Mashhoon-methode, iv) de Frobenius-reeks en v) het kwadratisch eigenwaardeprobleem (QEP).

In dit opzicht geven analytische oplossingen die gebruik maken van de aanwezigheid van ruimte-tijd symmetrie met behulp van numerieke relativiteitstheorie niet alleen nauwkeurige voorspellingen van de gravitatiegolf vormen, maar leveren ze ook een schat aan informatie over de omgeving van het zwarte gat, zoals de verdeling van materie en zijn stralingseigenschappen. Door dezelfde token, de studie van ruimtetijden in dimensies groter dan vier heeft inzicht gegeven in de onderliggende wiskundige structuur van Einsteins theorie en haar uitbreidingen, evenals een test voor de stabiliteit van nieuwe zwartegatoplossingen.

Dit proefschrift richt zich op de studie van scalaire en vectorvelden die propageren in een Kerr-AdS₅ ruimte tijd. Dit laatste is een vacuümplossing voor de Einsteins vergelijkingen met een negatieve kosmologische constante, die

een roterend zwart gat beschrijft met twee onafhankelijke grote momenta binnen een vijfdimensionale anti-de Sitter-achtergrond. Aan de andere kant kan het formalisme dat hier is ontwikkeld, gebruikt worden om lineaire verstoringen op andere ruimte tijden te behandelen en ook ander fysiek systeem. Hier vatten we onze resultaten samen.

Voor het Kerr-AdS₅-zwarte gat leidt de scheiding van variabelen tot radiale en hoekige gewone differentiaalvergelijkingen (ODE's) van de tweede orde met vier reguliere singuliere punten. Om dit soort differentiaalvergelijkingen te bestuderen, gebruiken we de methode van isomonodroom vervormingen. Deze methode is gebaseerd op het bestaan van families van Fuchsiaanse systemen met dezelfde monodromie gegevens die isomonodromisch kunnen worden vervormd en kunnen leiden tot de Painlevé-vergelijkingen, een reeks niet-lineaire tweede orde ODE's in het complexe vlak met de Painlevé-eigenschap. Deze oplossingen worden de transcendenten van Painlevé genoemd.

Tegelijkertijd lossen de Painlevé-transcendenten het verbindingsprobleem op voor de Heun-type vergelijkingen, en biedt het nieuwe tools om verschillende problemen in de wiskunde en de natuurkunde te verkennen, van het Rabi-model in de kwantumoptica tot het algemene wiskundige probleem van hoekgetrouwe toewijzingen van eenvoudig verbonden domeinen. Inderdaad, men zou de afleidingen in dit proefschrift toe kunnen passen om andere eigenwaarde problemen op te lossen.

We zullen namelijk zien dat een Fuchsiaans systeem met vier regelmatige singuliere punten kan zijn vervormd met behoud van de monodromie gegevens, waar de isomonodromische vergelijkingen zijn gerelateerd aan de vergelijking van Painlevé VI (PVI). Bovendien kan de isomonodrome vervorming worden gezien als de introductie van een schijnbare singulariteit in de vervormde tweede orde ODE, die de Hamiltoniaanse structuur van het systeem manifesteert en die het toelaat om een consistente definitie van de PVI τ -functie in de zin van Jimbo, Miwa en Ueno.

Onlangs is aangetoond dat de PVI τ -functie kan worden geschreven in termen van een zekere correlatiefunctie tussen primaire velden van een tweedimensionaal hoekgetrouwe veldentheorie met centrale lading $c = 1$, en een reeksuitbreiding in termen van conforme blokken is gegeven door Gamayun, Iorgov en Lisovyy. Door middel van de τ -functie kunnen we het eigenwaardeprobleem van de radiale (hoekige) Heun-vergelijking herformuleren in een initiale waardeprobleem van de corresponderende τ -functie. Bijvoorbeeld de eerste

beginvoorwaarde voor het hoeksysteem lost het probleem van de scheiding constante van de hoek vergelijking van Heun op. Zodra we een asymptotische uitdrukking hebben voor de scheiding constante, kunnen we de berekening van de quasi-normale modi met behulp van de twee beginvoorwaarden voor de radiale Painlevé VI τ -functie, waarbij de reeksuitbreiding wordt uitgevoerd rond het handige enkelvoud punt, d.w.z. op $t = \{0, 1, \infty\}$. In onze toepassing zullen we zien dat bijna gelijke rotatie, evenals kleine zwarte gaten kunnen worden beschreven door uitbreidingen van de bijbehorende PVI τ -functies rond $t = 0$.

We beginnen met het analyseren van de quasi-normale modi voor Schwarzschild-AdS₅ als test voor de voorgestelde methode. Vervolgens bestuderen we scalaire verstoringen in de Kerr-AdS₅ zwart gat variërend van de straal van de buitenste horizon r_+ , de massa van het scalaire veld en de rotatie parameters. Numeriek bewijs suggereert dat deze benadering een hogere nauwkeurigheid heeft dan andere numerieke methoden, zoals de koppeling methode of de pseudo-spectrale collocatie methode, voor kleine en middelgrote zwarte gaten ($r_+ < 1$) met vrijwel gelijke rotatieparameters.

Daarna berekenen we een storende uitdrukking voor de QNM-frequenties van de Kerr-AdS₅ zwart gat. Als de straal van de buitenste horizon afneemt, benadert de frequentie dat van pure AdS met een eerste correctie van orde $\mathcal{O}(r_+^2)$. We analyseren de scalaire modi voor het s-golf-geval ($\ell = 0$) en zelfs met orbitaal kwantumgetal ($\ell > 0$ met $m_1 = m_2 = 0$). Ze blijken stabiel te zijn voor kleine r_+ zwarte gaten, d.w.z. $r_+ \sim 10^{-3}$. In plaats daarvan vertonen modi met oneven ℓ een regime van super radiantie in de kleine r_+ limiet, in de zin dat het denkbeeldige deel van hun complexe frequenties positief wordt, zodat het zwarte gat onstabiel is.

Ten slotte pakken we vector verstoringen in onze favoriete achtergrond aan, na een recent opmerkelijke vooruitgang bij de scheidbaarheid van de Maxwell-vergelijkingen in roterend zwart gat ruimtetijden van willekeurige dimensies. De scheidbaarheid berust op het bestaan van een Killing-Yano-tensor en de introductie van een willekeurige parameter μ , samen met de scheidingsconstante, in een nieuwe Ansatz voor het vectorveld van de elektromagnetische veldtensor. Namelijk, na het scheiden van de Maxwell-vergelijkingen in termen van een scalaire functie en het omschrijven van de radiale and hoekige ODE's naar de Heun-vorm, kan men zien dat μ gerelateerd is door een Möbius transformatie met een schijnbare singulariteit in de vervormde Heun-vergelijking. Met behulp van de isomonodromie methode kunnen we de beginvoorwaarden voor

de isomonodrome PVI τ -functie en kunnen we een numerieke analyse uitvoeren voor ultra spinnende zwarte gaten. Dit laatste regime wordt beschreven door een uitbreiding van de hoekige PVI τ -functie rond $t = 1$, wat het mogelijk maakt om het complexe systeem van transcendentale vergelijkingen op te lossen.

We concluderen dat parameter μ zich in hogere dimensies anders gedraagt dan in vier dimensies, waar het kan worden geëlimineerd door een wijziging van parametrisatie in de overeenkomstige vergelijkingen. We merken ook op dat in het scalaire geval de schijnbare singulariteit een ondersteunende rol speelt in de daadwerkelijke oplossing van het probleem: grootheden zoals verstrooiing amplitudes en de quasi-normale modi zijn alleen afhankelijk van de monodromie gegevens. Men kan ze dan berekenen op elk punt van de isomonodrome stroom, met het samenvallende punt waar de schijnbare singulariteit versmelt met een van de resterende singulariteiten, wat bijzonder goed uitkomt. Echter is in de Maxwell zaak de aanwezigheid van de schijnbare singulariteit verplicht.

Zoals dit proefschrift laat zien, hebben we aangetoond dat de PVI τ -functie kan worden beschouwd als een betrouwbaar en krachtig hulpmiddel om het eigenwaardeprobleem voor de radiale en hoekige Heun-vergelijkingen op te lossen, afgeleid van de dynamiek van scalaire en vectorvelden in Kerr-AdS₅.

Resumo

As detecções de ondas gravitacionais pelo LIGO são resultados deslumbrantes para a teoria da Relatividade Geral de Einstein. A fusão de dois buracos negros rotacionando é um dos eventos mais violentos no universo. Por outro lado, depois do relaxamento e da emissão de ondas gravitacionais, a teoria de perturbações pode descrever alguns aspectos da natureza do buraco negro resultante e dos seus progenitores, caracterizados em termos de um conjunto discreto de frequências complexas – os modos quase-normais (QNMs).

A partir de 1970, distintas abordagens teóricas levaram a diferentes métodos para calcular os QNMs produzidos por diferentes campos de matéria, dependendo da separabilidade das equações de movimento e o espaço-tempo especificamente considerado. Alguns métodos historicamente usados incluem: i) a aproximação Wentzel-Kramers-Brillouin (WKB), ii) o método da fração continuada, iii) o método de Ferrari-Mashoonn, iv) as séries de Frobenius e v) o problema quadrático de autovalores (QEP).

Portanto, as soluções analíticas que exploram a presença de simetrias do espaço-tempo com ajuda da relatividade numérica tem produzido predições precisas sobre o espectro de ondas gravitacionais, a informação concernente as vizinhanças de um buraco negro, a distribuição de matéria e suas propriedades radiativas. Da mesma maneira, o estudo das equações de Einstein em dimensões maiores que quatro proporciona novos pontos de vista sobre a estrutura matemática da teoria da Relatividade Geral e suas extensões, assim como testes para comprovar a estabilidade de novas soluções encontradas.

Nesta tese, nós focamos no estudo de campos escalares e vetoriais que se propagam no espaço-tempo de Kerr-AdS₅. Esta solução descreve um buraco negro em rotação, com dois momentos angulares independentes e mergulhado num espaço-tempo anti-de Sitter em 5 dimensões. Por outro lado, o formalismo

desenvolvido neste trabalho pode ser usado no tratamento de perturbações lineares em diferentes espaços-tempo, assim como em outros sistemas físicos. Na sequência resumimos nossos principais resultados.

No caso do buraco negro de Kerr-AdS₅, a separação de variáveis nos leva a duas equações diferenciais ordinárias (EDOs) de segunda ordem para a parte radial e angular, com quatro pontos singulares regulares. Para estudar este tipo de equação diferencial, utilizamos o método das deformações isomonodrômicas. A metodologia aplicada fundamenta-se na existência de famílias de sistemas Fuchsianos com os mesmos dados de monodromia e que podem ser deformados isomonodromicamente, originando as equações de Painlevé – um conjunto de equações diferenciais não lineares de segunda ordem no plano complexo e com a propriedade de Painlevé. As soluções destas equações são chamadas transcendentais de Painlevé.

Ao mesmo tempo, os transcendentais de Painlevé resolvem o problema da conexão das equações tipo Heun e fornecem novas ferramentas para explorar vários problemas na Matemática e na Física, desde o modelo de Rabi em óptica quântica, até o problema matemático dos mapeamentos conformes de domínios simplesmente conectados. Além disso, é possível considerar a aplicação das derivações desta tese na resolução de outros problemas de autovalores.

Particularmente, verificamos que um sistema Fuchsiano com quatro pontos singulares regulares pode ser deformado preservando-se seus dados de monodromia, de modo que as equações isomonodrômicas estejam relacionadas à equação de Painlevé VI (PVI). Adicionalmente, a deformação isomonodrômica pode ser pensada como a introdução de uma singularidade aparente na EDO de segunda ordem deformada, evidenciando a estrutura hamiltoniana do sistema e permitindo a definição consistente da função τ de PVI à la Jimbo, Miwa e Ueno.

Recentemente foi demonstrado que a função τ de PVI pode ser escrita como a função de correlação entre campos primários de uma teoria de campos conformes em duas dimensões, com carga central $c = 1$, cuja expansão em série está descrita em termos dos blocos conformes, segundo Gamayun, Iorgov e Lisovsky. Por meio da função τ , podemos reformular o problema de autovalores da equação de Heun radial (angular) em um problema de valor inicial para a função τ correspondente. Por exemplo, a primeira condição inicial para o sistema angular resolve o problema da constante de separação da equação de Heun angular. Uma vez que temos uma expressão assintótica para a constante de se-

paração, podemos abordar o cálculo dos modos quase-normais usando as duas condições iniciais para a função τ radial de Painlevé VI, realizando a expansão da série ao redor do ponto singular conveniente, ou seja, em $t = \{0, 1, \infty\}$. No caso estudado, os buracos negros rotacionando com velocidades quase iguais e os buracos negros pequenos podem ser descritos por expansões das funções τ de PVI associadas próximas de $t = 0$.

Os modos quase-normais para Schwarzschild-AdS₅ foram analisados como um teste para o método proposto. Seguidamente, estudamos perturbações escalares no buraco negro de Kerr-AdS₅, variando o raio do horizonte externo r_+ , a massa do campo escalar e os parâmetros de rotação. A evidência numérica encontrada sugere que esta abordagem atinge uma precisão maior que outros métodos numéricos, como o método de Frobenius ou o método de colocação pseudo-espectral, considerando buracos negros pequenos e intermediários ($r_+ < 1$), cujos parâmetros de rotação são quase iguais.

Posteriormente, calculamos uma expressão perturbativa para as frequências dos QNMs do buraco negro Kerr-AdS₅. Conforme o raio do horizonte externo diminui, a frequência aproxima-se do valor de AdS puro, com uma primeira correção de ordem $\mathcal{O}(r_+^2)$. Nós analisamos os modos escalares para o caso da onda s ($\ell = 0$) e modos com número quântico orbital par ($\ell > 0$ e $m_1 = m_2 = 0$). Estes modos são estáveis para os buracos negros com r_+ pequeno, ou seja, $r_+ \sim 10^{-3}$. Entretanto, os modos com ℓ ímpar exibem um regime de superradiância no limite de r_+ pequeno, apresentando a parte imaginária da frequência positiva e tornando o buraco negro instável.

Finalmente, abordamos as perturbações vetoriais em nosso espaço-tempo favorito, seguindo um recente e notável progresso na separabilidade das equações de Maxwell, considerando buracos negros rotacionando em dimensões arbitrárias. Esta separabilidade depende da existência do tensor de Killing-Yano e da introdução de um parâmetro arbitrário μ , junto com a constante de separação, gerando um novo Ansatz para o campo vetorial do tensor eletromagnético. Depois de separar as equações de Maxwell em termos de uma função escalar e escrever as EDOs radiais e angulares na forma de Heun, foi possível observar que μ está relacionado com uma singularidade aparente na equação de Heun deformada por meio de uma transformação de Möbius. Usando o método da isomonodromia, podemos derivar as condições iniciais para a função τ de PVI e realizar uma análise numérica para buracos negros rotacionando rapidamente. Este último regime pode ser descrito por uma expansão da função τ

angular de PVI em torno de $t = 1$, permitindo resolver o sistema de equações transcendentais.

Concluimos que em dimensões mais altas, o parâmetro μ se comporta de maneira diferente que em quatro dimensões, onde pode ser eliminado por uma mudança de variáveis nas equações correspondentes. Também é possível notar que no caso escalar, a singularidade aparente desempenha um papel auxiliar na solução do problema: quantidades como amplitudes de espalhamento e os modos quase-normais dependem apenas dos dados de monodromia. Pode-se computá-los em qualquer ponto do fluxo isomonodrômico, sendo o ponto coincidente onde a singularidade aparente se funde com uma das singularidades restantes de escolha conveniente. Contudo, no caso de Maxwell, a presença da singularidade aparente se torna obrigatória.

Desta forma, mostra-se que a função τ de PVI pode ser considerada uma ferramenta robusta e confiável na resolução do problema de autovalores das equações de Heun radial e angular, derivadas da dinâmica de campos escalares e vetoriais em Kerr-AdS₅.

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Appendices

Appendix A

Fredholm determinant

The Fredholm determinant representation for the PVI τ -function uses the usual Riemann-Hilbert problem formulation in terms of Plemelj (projection) operators and jump matrices. The idea is to introduce projection operators which act on the space of (pair of) functions on the complex plane to give analytic functions with prescribed monodromy (Cauchy-Riemann operators). Details can be found in [75]. One should point out that the two expansions agree as functions of t up to a multiplicative constant.

$$\tau(t) = \text{const.} \cdot t^{\frac{1}{4}(\sigma^2 - \theta_0^2 - \theta_i^2)} (1-t)^{-\frac{1}{2}\theta_i\theta_1} \det(\mathbb{1} - AD), \quad (\text{A.1})$$

where the Plemelj operators A, D act on the space of pairs of square-integrable functions defined on $\mathcal{C}$, a circle on the complex plane with radius $R < 1$:

$$\begin{aligned} (Ag)(z) &= \oint_{\mathcal{C}} \frac{dz'}{2\pi i} A(z, z') g(z'), & (Dg)(z) &= \oint_{\mathcal{C}} \frac{dz'}{2\pi i} D(z, z') g(z'), \\ g(z') &= \begin{pmatrix} f_+(z') \\ f_-(z') \end{pmatrix} \end{aligned} \quad (\text{A.2})$$

with kernels given, for $|t| < R$, explicitly by

$$\begin{aligned} A(z, z') &= \frac{\Psi(\theta_1, \theta_\infty, \sigma; z) \Psi^{-1}(\theta_1, \theta_\infty, \sigma; z') - \mathbb{1}}{z - z'}, \\ D(z, z') &= \Phi(t) \frac{\mathbb{1} - \Psi(\theta_t, \theta_0, -\sigma; t/z) \Psi^{-1}(\theta_t, \theta_0, -\sigma; t/z')}{z - z'} \Phi^{-1}(t). \end{aligned} \quad (\text{A.3})$$

The parametrix $\Psi(z)$ and the “gluing” matrix $\Phi(t)$ are

$$\Psi(\alpha_1, \alpha_2, \alpha_3; z) = \begin{pmatrix} \phi(\alpha_1, \alpha_2, \alpha_3; z) & \chi(\alpha_1, \alpha_2, \alpha_3; z) \\ \chi(\alpha_1, \alpha_2, -\alpha_3; z) & \phi(\alpha_1, \alpha_2, -\alpha_3; z) \end{pmatrix}, \quad \Phi(t) = \begin{pmatrix} t^{-\sigma/2} \kappa^{-1/2} & 0 \\ 0 & t^{\sigma/2} \kappa^{1/2} \end{pmatrix}, \quad (\text{A.4})$$

with ϕ and χ given in terms of Gauss’ hypergeometric function:

$$\begin{aligned} \phi(\alpha_1, \alpha_2, \alpha_3; z) &= {}_2F_1\left(\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3), \frac{1}{2}(\alpha_1 - \alpha_2 + \alpha_3); \alpha_3; z\right) \\ \chi(\alpha_1, \alpha_2, \alpha_3; z) &= \frac{\alpha_2^2 - (\alpha_1 + \alpha_3)^2}{4\alpha_3(1 + \alpha_3)} z {}_2F_1\left(1 + \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3), 1 + \frac{1}{2}(\alpha_1 - \alpha_2 + \alpha_3); 2 + \alpha_3; z\right). \end{aligned} \quad (\text{A.5})$$

Finally, κ is a known function of the monodromy parameters:

$$\begin{aligned} \kappa = s \frac{\Gamma^2(1 - \sigma)}{\Gamma^2(1 + \sigma)} \frac{\Gamma(1 + \frac{1}{2}(\theta_t + \theta_0 + \sigma))\Gamma(1 + \frac{1}{2}(\theta_t - \theta_0 + \sigma))}{\Gamma(1 + \frac{1}{2}(\theta_t + \theta_0 - \sigma))\Gamma(1 + \frac{1}{2}(\theta_t - \theta_0 - \sigma))} \times \\ \frac{\Gamma(1 + \frac{1}{2}(\theta_1 + \theta_\infty + \sigma))\Gamma(1 + \frac{1}{2}(\theta_1 - \theta_\infty + \sigma))}{\Gamma(1 + \frac{1}{2}(\theta_1 + \theta_\infty - \sigma))\Gamma(1 + \frac{1}{2}(\theta_1 - \theta_\infty - \sigma))}. \end{aligned} \quad (\text{A.6})$$

One drawback in these expansions (3.95) and (A.1) is that they assume that σ_{1t} is in the fundamental domain of the cosine. The consequence of this fact is that, with the range of z_0 necessary for the study of the radial equation, from equation (3.32) and the asymptotic expansion of the tau function, we are only able to determine the fundamental mode $n = 0$. For the angular equation, there is no such drawback, but the parameter s is singular. Meaningful limits can be obtained by cancelling the factors in the denominator of s with poles of the Barnes function from the structure constants $C(\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma + 2n)$.

For the numerical implementation, we write the matrix elements of A and D in the Fourier basis z^n , truncated up to order N . Again, the structure of the matrix elements A_{mn} and D_{mn} can be found in [75]. This truncation gives τ up to terms $\mathcal{O}(t^N)$, and, unlike the Nekrasov expansion, can be computed in polynomial time. The formulation does in principle allow for calculation for arbitrary values of t , by evaluating the integrals in (A.2) as Riemann sums using quadratures [35], so there are good perspectives for using the method outlined here for more generic configurations.

Appendix B

Painlevé equations

The Painlevé equations are now regarded as “non-linear special functions”, being non-linear analogs of the classical special functions. Indeed Iwasaki et al. [101] characterize the Painlevé equations as “the most important non-linear differential equations”. There are important applications of Painlevé equations in different fields of Mathematics and Physics, and we refer to [53, 64] for a review of the modern theory of Painlevé functions and their applications. It is worth mentioning the black hole scattering [136], conformal mapping [17], Hele-Shaw process [65], Ising model [165], impenetrable Bose gas [104], Rabi model [42], spin chains [18], topological quantum field theory [60].

The Painlevé equations were discovered about a hundred years ago by Painlevé and his colleagues whilst studying a problem posed by Picard. Picard asked which second order differential equations of the form

$$\frac{d^2u}{dx^2} = F\left(u, \frac{du}{dx}, x\right), \quad (\text{B.1})$$

where F is rational in u and du/dx and locally analytic in x , have the property that the solutions have no movable branch points, i.e., the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation; this is now known as the *Painlevé property*.

Painlevé et al. showed that there were 50 canonical equations of form (B.1). However, only six of them can not be reduced to linear ODEs or solved in terms

of elliptic functions. They were called later Painlevé equations. In standard form they are written as

- Painlevé I (PI):

$$\frac{d^2u}{dx^2} = 6u^2 + x, \quad (\text{B.2})$$

- Painlevé II (PII):

$$\frac{d^2u}{dx^2} = 2u^3 + xu + \alpha, \quad (\text{B.3})$$

- Painlevé III (PIII):

$$\frac{d^2u}{dx^2} = \frac{1}{u} \left(\frac{du}{dx} \right)^2 - \frac{1}{x} \left(\frac{du}{dx} \right) + \frac{\alpha u^2}{x} + \frac{\beta}{x} + \gamma u^3 + \frac{\delta}{u}, \quad (\text{B.4})$$

- Painlevé IV (PIV):

$$\frac{d^2u}{dx^2} = \frac{1}{2u} \left(\frac{du}{dx} \right)^2 + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha)u + \frac{\beta}{u}, \quad (\text{B.5})$$

- Painlevé V (PV):

$$\begin{aligned} \frac{d^2u}{dx^2} = & \left(\frac{1}{2u} + \frac{1}{u-1} \right) \left(\frac{du}{dx} \right)^2 - \frac{1}{x} \left(\frac{du}{dx} \right) + \frac{(u-1)^2}{x^2} \left(\alpha u + \frac{\beta}{u} \right) \\ & + \frac{\gamma u}{x} + \frac{\delta u(u+1)}{u-1}, \end{aligned} \quad (\text{B.6})$$

- Painlevé VI (PVI):

$$\begin{aligned} \frac{d^2u}{dx^2} = & \frac{1}{2} \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x} \right) \left(\frac{du}{dx} \right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x} \right) \frac{du}{dx} \\ & + \frac{u(u-1)(u-x)}{x^2(x-1)^2} \left(\alpha + \frac{\beta x}{u^2} + \frac{\gamma(x-1)}{(u-1)^2} + \delta \frac{x(x-1)}{(u-x)^2} \right), \end{aligned} \quad (\text{B.7})$$

where $\alpha, \beta, \gamma, \delta$ are complex numbers. For example, the PVI parameters can be defined in terms of (3.85) as

$$(\alpha, \beta, \gamma, \delta)_{\text{VI}} = \left(\frac{1}{2}(\theta_\infty - 1)^2, -\frac{1}{2}\theta_0^2, \frac{1}{2}\theta_1^2, \frac{1}{2}(1 - \theta_t^2) \right). \quad (\text{B.8})$$

It turns out that the PVI equation contains the first five by a limiting procedure, carried out by first transforming u and x appropriately in terms of a suitable (small) parameter, and then taking limits of the parameter to zero [72, 101]. In PIII case, one also distinguishes three sub-cases

$$\text{PIII(D6): } \gamma\delta \neq 0,$$

$$\text{PIII(D7): } \gamma = 0, \quad \alpha\delta \neq 0 \quad \text{or} \quad \delta = 0, \quad \beta\gamma \neq 0$$

$$\text{PIII(D8): } \gamma = \delta = 0$$

The solutions of PI–PVI are called the *Painlevé transcendents*. Painlevé equations appear as ODE reductions of various non-linear integrable partial differential equations (PDEs) admitting soliton solutions and therefore they describe non-linear wave phenomena [5]

Korteweg-de Vries equation $\Rightarrow$ PI,

modified Korteweg-de Vries equation $\Rightarrow$ PII,

Sine-Gordon equation $\Rightarrow$ PIII,

non-linear Schrödinger equation $\Rightarrow$ PIV,

Ernst equation $\Rightarrow$ PV,

three wave resonant interaction equation $\Rightarrow$ PVI.

One of the main reasons for studying Painlevé functions is that they describe the isomonodromic deformations of a linear system of ODEs with rational coefficients and hence admit the so-called Riemann-Hilbert representation [72, 97, 99, 100].

Each of the Painlevé equations can be written as a Hamiltonian system

$$\frac{dq}{dx} = \frac{\partial \mathcal{H}_J}{\partial p}, \quad \frac{dp}{dx} = -\frac{\partial \mathcal{H}_J}{\partial q}, \quad J = \text{I}, \dots, \text{VI}, \quad (\text{B.9})$$

for a suitable Hamiltonian function $\mathcal{H}_J(q, p, x)$ [103, 149].

Curriculum vitae

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List of Publications

1. J. Barragán, D. Mateo, M. Pi, F. Salvat, M. Barranco, and H. J. Maris. Electron photo-ejection from bubble states in liquid ^4He . *Journal of Low Temperature Physics*, 171(3):171–177, 2013.
2. J. Barragán Amado, B. Carneiro da Cunha, and E. Pallante. On the Kerr-AdS/CFT correspondence. *JHEP*, 08:094, 2017.
3. J. Barragán Amado, B. Carneiro da Cunha, and E. Pallante. Scalar quasinormal modes of Kerr-AdS₅. *Phys. Rev. D*, 99:105006, May 2019.
4. J. Barragán Amado, B. Carneiro da Cunha, and E. Pallante. Vector perturbations of Kerr-AdS₅ and the Painlevé VI transcendent. *JHEP*, 04:155, 2020.

5. J. Barragán Amado, B. Carneiro da Cunha, and E. Pallante. Remarks on holographic models of the Kerr-AdS₅ geometry. *To appear soon*.

Conferences and Workshops

1. *On the asymptotic behaviour of the two-point correlation functions*, ‘Dutch Research School of Theoretical Physics’, São Paulo, Brazil (Jan 2015).
2. *The Heun Equation*, ‘Dutch Research School of Theoretical Physics’, Dalfsen, Netherlands (Jan 2016).
3. *On the Kerr-AdS/CFT correspondence*, ‘School on AdS/CMT correspondence’, São Paulo, Brazil (April 2017).
4. *Quasinormal modes of Kerr-AdS₅ via Isomonodromic Deformations*, ‘X Black Holes’ workshop, Aveiro, Portugal (Dec 2017).

Technical Strengths

Computer Languages
Languages

Python, Mathematica
Spanish, English, Portuguese, Russian

Bibliography

- [1] B. P. Abbott et al. GW150914: First results from the search for binary black hole coalescence with Advanced LIGO. *Phys. Rev.*, D93(12):122003, 2016.
- [2] B. P. Abbott et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.*, 116(6):061102, 2016.
- [3] B. P. Abbott et al. Observing gravitational-wave transient GW150914 with minimal assumptions. *Phys. Rev.*, D93(12):122004, 2016. [Addendum: *Phys. Rev.* D94,no.6,069903(2016)].
- [4] L. Abbott and S. Deser. Stability of gravity with a cosmological constant. *Nuclear Physics B*, 195(1):76 – 96, 1982.
- [5] M. Ablowitz and P. Clarkson. *Solitons, nonlinear evolution equations and inverse scattering*, volume 149. 1991.
- [6] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz. Large N field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000.
- [7] K. Akiyama et al. First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole. *Astrophys. J.*, 875(1):L1, 2019.
- [8] K. Akiyama et al. First M87 Event Horizon Telescope Results. II. Array and Instrumentation. *Astrophys. J.*, 875(1):L2, 2019.
- [9] K. Akiyama et al. First M87 Event Horizon Telescope Results. III. Data Processing and Calibration. *Astrophys. J.*, 875(1):L3, 2019.

- [10] K. Akiyama et al. First M87 Event Horizon Telescope Results. IV. Imaging the Central Supermassive Black Hole. *Astrophys. J.*, 875(1):L4, 2019.
- [11] K. Akiyama et al. First M87 Event Horizon Telescope Results. V. Physical Origin of the Asymmetric Ring. *Astrophys. J.*, 875(1):L5, 2019.
- [12] K. Akiyama et al. First M87 Event Horizon Telescope Results. VI. The Shadow and Mass of the Central Black Hole. *Astrophys. J.*, 875(1):L6, 2019.
- [13] V. A. Alba, V. A. Fateev, A. V. Litvinov, and G. M. Tarnopolskiy. On combinatorial expansion of the conformal blocks arising from AGT conjecture. *Lett.Math.Phys.*, 98:33–64, 2011.
- [14] L. F. Alday, D. Gaiotto, and Y. Tachikawa. Liouville Correlation Functions from Four-dimensional Gauge Theories. *Lett. Math. Phys.*, 91:167–197, 2010.
- [15] A. N. Aliev and O. Delice. Superradiant Instability of Five-Dimensional Rotating Charged AdS Black Holes. *Phys. Rev.*, D79:024013, 2009.
- [16] T. Andrade. Holographic Lattices and Numerical Techniques. 2017.
- [17] T. Anselmo, R. Nelson, B. Carneiro da Cunha, and D. G. Crowdy. Accessory parameters in conformal mapping: exploiting the isomonodromic tau function for painlevé vi. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 474(2216), 2018.
- [18] F. Ares and J. Viti. Emptiness formation probability and Painlevé V equation in the XY spin chain. 2019.
- [19] A. Ashtekar and S. Das. Asymptotically Anti-de Sitter space-times: Conserved quantities. *Classical and Quantum Gravity*, 17(2):L17–L30, Jan. 2000.
- [20] A. Ashtekar and A. Magnon. Asymptotically anti-de Sitter space-times. *Classical and Quantum Gravity*, 1:L39–L44, July 1984.
- [21] J. Avery. *Hyperspherical harmonics: applications in quantum theory*. Reidel Texts in the Mathematical Sciences A Graduate-Level Book Series. Springer, Dordrecht, 1989.

- [22] A. M. Awad and C. V. Johnson. Higher dimensional Kerr - AdS black holes and the AdS / CFT correspondence. *Phys. Rev.*, D63:124023, 2001.
- [23] V. Baibhav, E. Berti, and V. Cardoso. LISA parameter estimation and source localization with higher harmonics of the ringdown. *Phys. Rev. D*, 101(8):084053, 2020.
- [24] V. Balasubramanian and P. Kraus. A Stress tensor for Anti-de Sitter gravity. *Commun. Math. Phys.*, 208:413–428, 1999.
- [25] L. Barack et al. Black holes, gravitational waves and fundamental physics: a roadmap. *Class. Quant. Grav.*, 36(14):143001, 2019.
- [26] J. M. Bardeen and G. T. Horowitz. The Extreme Kerr throat geometry: A Vacuum analog of $AdS(2) \times S^{*2}$. *Phys. Rev.*, D60:104030, 1999.
- [27] J. Barragán Amado, B. Carneiro da Cunha, and E. Pallante. Scalar quasinormal modes of Kerr- AdS_5 . *Phys. Rev. D*, 99:105006, May 2019.
- [28] J. Barragán Amado, B. Carneiro da Cunha, and E. Pallante. On the Kerr-AdS/CFT correspondence. *JHEP*, 08:094, 2017.
- [29] J. Barragán Amado, B. Carneiro da Cunha, and E. Pallante. Vector perturbations of Kerr- AdS_5 and the Painlevé VI transcendent. *JHEP*, 04:155, 2020.
- [30] M. A. Bershtein and A. I. Shchekkin. Bilinear equations on Painlevé τ functions from CFT. *Commun. Math. Phys.*, 339(3):1021–1061, 2015.
- [31] E. Berti, V. Cardoso, and M. Casals. Eigenvalues and eigenfunctions of spin-weighted spheroidal harmonics in four and higher dimensions. *Phys. Rev. D*, 73:024013, Jan 2006.
- [32] E. Berti, V. Cardoso, and A. O. Starinets. Quasinormal modes of black holes and black branes. *Class. Quant. Grav.*, 26:163001, 2009.
- [33] E. Berti, V. Cardoso, and C. M. Will. Gravitational-wave spectroscopy of massive black holes with the space interferometer lisa. *Phys. Rev. D*, 73:064030, Mar 2006.
- [34] D. Birmingham, I. Sachs, and S. N. Solodukhin. Conformal field theory interpretation of black hole quasinormal modes. *Phys. Rev. Lett.*, 88:151301, 2002.

- [35] F. Bornemann. On the numerical evaluation of Fredholm determinants. *Mathematics of Computation*, 79(270):871–915, 2010.
- [36] R. Brito, V. Cardoso, and P. Pani. *Superradiance: Energy Extraction, Black-Hole Bombs and Implications for Astrophysics and Particle Physics*, volume 906. Springer, 2015.
- [37] C. P. Burgess and C. A. Lütken. Propagators and effective potentials in anti-de Sitter space. *Physics Letters B*, 153(3):137–141, Mar. 1985.
- [38] M. Cafasso, P. Gavrylenko, and O. Lisovyy. Tau functions as Widom constants. 2017.
- [39] V. Cardoso and O. J. C. Dias. Small Kerr-anti-de Sitter black holes are unstable. *Phys. Rev.*, D70:084011, 2004.
- [40] V. Cardoso, Ó. J. C. Dias, G. S. Hartnett, L. Lehner, and J. E. Santos. Holographic thermalization, quasinormal modes and superradiance in Kerr-AdS. *JHEP*, 04:183, 2014.
- [41] B. Carneiro da Cunha and J. P. Cavalcante. Confluent conformal blocks and the Teukolsky master equation. 2019.
- [42] B. Carneiro da Cunha, M. C. de Almeida, and A. R. de Queiroz. On the Existence of Monodromies for the Rabi model. 2015.
- [43] B. Carneiro da Cunha and F. Novaes. Kerr-de sitter greybody factors via isomonodromy. *Phys. Rev. D*, 93:024045, Jan 2016.
- [44] B. Carter. Global structure of the Kerr family of gravitational fields. *Phys. Rev.*, 174:1559–1571, 1968.
- [45] B. Carter. Hamilton-Jacobi and Schrodinger separable solutions of Einstein’s equations. *Commun. Math. Phys.*, 10(4):280–310, 1968.
- [46] B. M. N. Carter and I. P. Neupane. Thermodynamics and stability of higher dimensional rotating (Kerr) AdS black holes. *Phys. Rev.*, D72:043534, 2005.
- [47] A. Castro, J. M. Lapan, A. Maloney, and M. J. Rodriguez. Black Hole Monodromy and Conformal Field Theory. *Phys. Rev.*, D88:044003, 2013.

- [48] A. Castro, J. M. Lapan, A. Maloney, and M. J. Rodriguez. Black Hole Scattering from Monodromy. *Class. Quant. Grav.*, 30:165005, 2013.
- [49] R. Cayuso, O. J. C. Dias, F. Gray, D. Kubizňák, A. Margalit, J. E. Santos, R. Gomes Souza, and L. Thiele. Massive vector fields in Kerr-Newman and Kerr-Sen black hole spacetimes. 2019.
- [50] R. Cayuso, F. Gray, D. Kubizňák, A. Margalit, R. Gomes Souza, and L. Thiele. Principal Tensor Strikes Again: Separability of Vector Equations with Torsion. *Phys. Lett.*, B795:650–656, 2019.
- [51] W. Chen, H. Lü, and C. N. Pope. Mass of rotating black holes in gauged supergravities. *Phys. Rev. D*, 73:104036, May 2006.
- [52] H. T. Cho, A. S. Cornell, J. Doukas, and W. Naylor. Scalar spheroidal harmonics in five dimensional Kerr-(A)dS. *Prog. Theor. Phys.*, 128:227–241, 2012.
- [53] P. A. Clarkson. Painlevé equations—nonlinear special functions. *Journal of Computational and Applied Mathematics*, 153(1):127 – 140, 2003. Proceedings of the 6th International Symposium on Orthogonal Polynomials, Special Functions and their Applications, Rome, Italy, 18-22 June 2001.
- [54] G. Compère. The Kerr/CFT correspondence and its extensions. *Living Rev. Rel.*, 15:11, 2012. [Living Rev. Rel.20,no.1,1(2017)].
- [55] S. de Haro, S. N. Solodukhin, and K. Skenderis. Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence. *Commun. Math. Phys.*, 217:595–622, 2001.
- [56] O. J. C. Dias and J. E. Santos. Boundary Conditions for Kerr-AdS Perturbations. *JHEP*, 10:156, 2013.
- [57] O. J. C. Dias, J. E. Santos, and B. Way. Numerical Methods for Finding Stationary Gravitational Solutions. *Class. Quant. Grav.*, 33(13):133001, 2016.
- [58] F. A. Dolan and H. Osborn. Conformal four point functions and the operator product expansion. *Nucl. Phys.*, B599:459–496, 2001.

- [59] S. R. Dolan. Instability of the Proca field on Kerr spacetime. *Phys. Rev.*, D98(10):104006, 2018.
- [60] B. Dubrovin. Geometry of 2-D topological field theories. *Lect. Notes Math.*, 1620:120–348, 1996.
- [61] G. V. Dunne. Resurgence, Painleve Equations and Conformal Blocks. 2019.
- [62] R. Emparan and H. S. Reall. Black Holes in Higher Dimensions. *Living Rev. Rel.*, 11:6, 2008.
- [63] E. E. Flanagan and S. A. Hughes. Measuring gravitational waves from binary black hole coalescences: 1. Signal-to-noise for inspiral, merger, and ringdown. *Phys. Rev. D*, 57:4535–4565, 1998.
- [64] A. Fokas, A. Its, A. Novokshenov, A. Kapaev, and V. Novokshenov. *Painleve Transcendents: The Riemann-Hilbert Approach*. Mathematical surveys and monographs. American Mathematical Society, 2006.
- [65] A. S. FOKAS and S. TANVEER. A hele-shaw problem and the second painlevé transcendent. *Mathematical Proceedings of the Cambridge Philosophical Society*, 124(1):169–191, 1998.
- [66] V. P. Frolov, P. Krtouš, D. Kubizňák, and J. E. Santos. Massive Vector Fields in Rotating Black-Hole Spacetimes: Separability and Quasinormal Modes. *Phys. Rev. Lett.*, 120:231103, 2018.
- [67] V. P. Frolov, P. Krtouš, and D. Kubizňák. Separability of Hamilton-Jacobi and Klein-Gordon Equations in General Kerr-NUT-AdS Spacetimes. *JHEP*, 02:005, 2007.
- [68] Frolov, Valeri and Krtouš, Pavel and Kubizňák, David. Black holes, hidden symmetries, and complete integrability. *Living Reviews in Relativity*, 20(1):6, 2017.
- [69] Frolov, Valeri P. and Krtouš, Pavel. Duality and μ separability of Maxwell equations in Kerr-NUT-(A)dS spacetimes. *Phys. Rev. D*, 99(4):044044, 2019.

- [70] R. Fuchs. Über lineare homogene differentialgleichungen zweiter ordnung mit drei im endlichen gelegenen wesentlich singulären stellen. *Mathematische Annalen*, 70(4):525–549, Dec 1911.
- [71] O. Gamayun, N. Iorgov, and O. Lisovyy. Conformal field theory of Painlevé VI. *JHEP*, 10:038, 2012. [Erratum: JHEP10,183(2012)].
- [72] O. Gamayun, N. Iorgov, and O. Lisovyy. How instanton combinatorics solves Painlevé VI, V and IIIs. *J. Phys.*, A46:335203, 2013.
- [73] R. Garnier. Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes. *Annales scientifiques de l'École Normale Supérieure*, 3e série, 29:1–126, 1912.
- [74] R. Garnier. étude de l'intégrale générale de l'équation vi de m. painlevé dans le voisinage de ses singularités transcendentes. *Annales scientifiques de l'École Normale Supérieure*, 3e série, 34:239–353, 1917.
- [75] P. Gavrylenko and O. Lisovyy. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions. 2016.
- [76] G. W. Gibbons, H. Lu, D. N. Page, and C. N. Pope. Rotating black holes in higher dimensions with a cosmological constant. *Phys. Rev. Lett.*, 93:171102, 2004.
- [77] G. W. Gibbons, H. Lu, D. N. Page, and C. N. Pope. The General Kerr-de Sitter metrics in all dimensions. *J. Geom. Phys.*, 53:49–73, 2005.
- [78] G. W. Gibbons, M. J. Perry, and C. N. Pope. The First law of thermodynamics for Kerr-anti-de Sitter black holes. *Class. Quant. Grav.*, 22:1503–1526, 2005.
- [79] S. R. Green, S. Hollands, A. Ishibashi, and R. M. Wald. Superradiant instabilities of asymptotically anti-de Sitter black holes. *Class. Quant. Grav.*, 33(12):125022, 2016.
- [80] S. S. Gubser. Breaking an Abelian gauge symmetry near a black hole horizon. *Phys. Rev.*, D78:065034, 2008.

- [81] M. Guica, T. Hartman, W. Song, and A. Strominger. The Kerr/CFT Correspondence. *Phys. Rev.*, D80:124008, 2009.
- [82] D. Harlow, J. Maltz, and E. Witten. Analytic Continuation of Liouville Theory. *JHEP*, 12:071, 2011.
- [83] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz. Building a Holographic Superconductor. *Phys. Rev. Lett.*, 101:031601, 2008.
- [84] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz. Holographic Superconductors. *JHEP*, 12:015, 2008.
- [85] S. A. Hartnoll, J. Polchinski, E. Silverstein, and D. Tong. Towards strange metallic holography. *JHEP*, 04:120, 2010.
- [86] S. W. Hawking, C. J. Hunter, and M. Taylor. Rotation and the AdS / CFT correspondence. *Phys. Rev.*, D59:064005, 1999.
- [87] S. W. Hawking and H. S. Reall. Charged and rotating AdS black holes and their CFT duals. *Phys. Rev.*, D61:024014, 2000.
- [88] M. Henneaux and C. Teitelboim. Asymptotically anti-de sitter spaces. *Communications in Mathematical Physics*, 98(3):391–424, 1985.
- [89] M. Henningson and K. Skenderis. The Holographic Weyl anomaly. *JHEP*, 07:023, 1998.
- [90] E. Hijano, P. Kraus, E. Perlmutter, and R. Snively. Witten Diagrams Revisited: The AdS Geometry of Conformal Blocks. *JHEP*, 01:146, 2016.
- [91] N. Hitchin. Geometrical aspects of schlesinger’s equation. *Journal of Geometry and Physics*, 23(3):287 – 300, 1997.
- [92] S. Hollands, A. Ishibashi, and D. Marolf. Comparison between various notions of conserved charges in asymptotically AdS-spacetimes. *Class. Quant. Grav.*, 22:2881–2920, 2005.
- [93] G. T. Horowitz, editor. *Black holes in higher dimensions*. Cambridge Univ. Pr., Cambridge, UK, 2012.
- [94] G. T. Horowitz and V. E. Hubeny. Quasinormal modes of AdS black holes and the approach to thermal equilibrium. *Phys. Rev.*, D62:024027, 2000.

- [95] E. L. Ince. *Ordinary Differential Equations*. Dover, 1956.
- [96] N. Iorgov, O. Lisovyy, and J. Teschner. Isomonodromic tau-functions from Liouville conformal blocks. *Commun. Math. Phys.*, 336(2):671–694, 2015.
- [97] N. Iorgov, O. Lisovyy, and Y. Tykhyy. Painlevé VI connection problem and monodromy of $c = 1$ conformal blocks. *JHEP*, 1312:029, Aug. 2013.
- [98] A. Its, O. Lisovyy, and Y. Tykhyy. Connection problem for the sine-Gordon/Painlevé III tau function and irregular conformal blocks. 2014.
- [99] A. Its and A. Prokhorov. Connection problem for the tau-function of the sine-gordon reduction of painlevé-iii equation via the riemann-hilbert approach. *International Mathematics Research Notices*, page rnv375, Jan 2016.
- [100] A. R. Its, O. Lisovyy, and A. Prokhorov. Monodromy dependence and connection formulae for isomonodromic tau functions. *Duke Math. J.*, 167(7):1347–1432, 05 2018.
- [101] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida. *From Gauss to Painlevé: A Modern Theory of Special Functions*, volume 16 of *Aspects of Mathematics E*. Braunschweig, 1991.
- [102] M. Jimbo. Monodromy Problem and the boundary condition for some Painlevé equations. *Publ. Res. Inst. Math. Sci.*, 18:1137–1161, 1982.
- [103] M. Jimbo and T. Miwa. Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. ii. *Physica D: Nonlinear Phenomena*, 2(3):407 – 448, 1981.
- [104] M. Jimbo, T. Miwa, M. Sato, and Y. Mori. Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent. *Physica D*, 1:80–158, 1980.
- [105] M. Jimbo, T. Miwa, and K. Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. i. general theory and τ -function. *Physica D: Nonlinear Phenomena*, 2(2):306–352, 1981.

- [106] D. Kastor. Komar Integrals in Higher (and Lower) Derivative Gravity. *Class. Quant. Grav.*, 25:175007, 2008.
- [107] K. D. Kokkotas and B. G. Schmidt. Quasinormal modes of stars and black holes. *Living Rev. Rel.*, 2:2, 1999.
- [108] A. Komar. Covariant conservation laws in general relativity. *Phys. Rev.*, 113:934–936, Feb 1959.
- [109] R. Konoplya and A. Zhidenko. Detection of gravitational waves from black holes: Is there a window for alternative theories? *Phys. Lett.*, B756:350–353, 2016.
- [110] R. A. Konoplya and A. Zhidenko. Quasinormal modes of black holes: From astrophysics to string theory. *Rev. Mod. Phys.*, 83:793–836, 2011.
- [111] G. V. Kraniotis. The massive Dirac equation in the Kerr-Newman-de Sitter and Kerr-Newman black hole spacetimes. *J. Phys. Comm.*, 3:035026, 2019.
- [112] P. Krtouš, V. P. Frolov, and D. Kubizňák. Separation of Maxwell equations in Kerr–NUT–(A)dS spacetimes. *Nucl. Phys.*, B934:7–38, 2018.
- [113] H. K. Kunduri and J. Lucietti. Integrability and the Kerr-(A)dS black hole in five dimensions. *Phys. Rev.*, D71:104021, 2005.
- [114] K. Landsteiner and E. Lopez. The Thermodynamic potentials of Kerr - AdS black holes and their CFT duals. *JHEP*, 12:020, 1999.
- [115] M. Lencsés and F. Novaes. Classical Conformal Blocks and Accessory Parameters from Isomonodromic Deformations. *JHEP*, 04:096, 2018.
- [116] O. Lisovsky, H. Nagoya, and J. Roussillon. Irregular conformal blocks and connection formulae for painlevé v functions. *Journal of Mathematical Physics*, 59(9):091409, Sep 2018.
- [117] A. Litvinov, S. Lukyanov, N. Nekrasov, and A. Zamolodchikov. Classical Conformal Blocks and Painleve VI. *JHEP*, 07:144, 2014.
- [118] H. Liu, J. McGreevy, and D. Vegh. Non-Fermi liquids from holography. *Phys. Rev.*, D83:065029, 2011.

- [119] H. Lu, J. Mei, and C. N. Pope. Kerr/CFT Correspondence in Diverse Dimensions. *JHEP*, 04:054, 2009.
- [120] O. Lunin. Maxwell’s equations in the Myers-Perry geometry. *JHEP*, 12:138, 2017.
- [121] O. Lunin. Excitations of the Myers-Perry Black Holes. *JHEP*, 10:030, 2019.
- [122] A. Magnon. On Komar integrals in asymptotically anti-de Sitter space-times. *J. Math. Phys.*, 26:3112–3117, 1985.
- [123] S. Mano, H. Suzuki, and E. Takasugi. Analytic solutions of the Teukol-sky equation and their low frequency expansions. *Prog. Theor. Phys.*, 95:1079–1096, 1996.
- [124] M. Mazzocco. The geometry of the classical solutions of the Garnier systems. *International Mathematics Research Notices*, 2002(12):613–646, 01 2002.
- [125] B. McInnes. Angular Momentum in QGP Holography. *Nucl. Phys.*, B887:246–264, 2014.
- [126] B. Mcinnes. Applied holography of the AdS_5 –Kerr space–time. *Int. J. Mod. Phys.*, A34(24):1950138, 2019.
- [127] T. Miwa. Painleve property of monodromy preserving deformation equations and the analyticity of τ functions. *Publications of the Research Institute for Mathematical Sciences*, 17:703–721, sep 1981.
- [128] C. Molina, P. Pani, V. Cardoso, and L. Gualtieri. Gravitational signature of Schwarzschild black holes in dynamical Chern-Simons gravity. *Phys. Rev.*, D81:124021, 2010.
- [129] L. Motl and A. Neitzke. Asymptotic black hole quasinormal frequencies. *Adv. Theor. Math. Phys.*, 7(2):307–330, 2003.
- [130] R. C. Myers and M. J. Perry. Black Holes in Higher Dimensional Space-Times. *Annals Phys.*, 172:304, 1986.
- [131] H. Nagoya. Irregular conformal blocks, with an application to the fifth and fourth Painlevé equations. *J. Math. Phys.*, 56(12):123505, 2015.

- [132] A. Nata Atmaja and K. Schalm. Anisotropic Drag Force from 4D Kerr-AdS Black Holes. *JHEP*, 04:070, 2011.
- [133] A. Neitzke. Greybody factors at large imaginary frequencies. 2003.
- [134] N. A. Nekrasov. Seiberg-Witten prepotential from instanton counting. *Adv. Theor. Math. Phys.*, 7(5):831–864, 2003.
- [135] H.-P. Nollert. TOPICAL REVIEW: Quasinormal modes: the characteristic ‘sound’ of black holes and neutron stars. *Classical and Quantum Gravity*, 16(12):R159–R216, Dec. 1999.
- [136] F. Novaes and B. Carneiro da Cunha. Isomonodromy, Painlevé transcendents and scattering off of black holes. *JHEP*, 07:132, 2014.
- [137] F. Novaes, C. Marinho, M. Lencsés, and M. Casals. Kerr-de Sitter Quasinormal Modes via Accessory Parameter Expansion. *JHEP*, 05:033, 2019.
- [138] A. Nunez and A. O. Starinets. AdS / CFT correspondence, quasinormal modes, and thermal correlators in N=4 SYM. *Phys. Rev.*, D67:124013, 2003.
- [139] K. Okamoto. Studies on Painlevé Equations. *Ann. Mat. Pura Appl.*, 146(1):337–381, 1986.
- [140] R. Olea. Regularization of odd-dimensional AdS gravity: Kounterterms. *JHEP*, 04:073, 2007.
- [141] I. Papadimitriou and K. Skenderis. Thermodynamics of asymptotically locally ads spacetimes. *Journal of High Energy Physics*, 2005(08):004, 2005.
- [142] J. Penedones. TASI lectures on AdS/CFT. In *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015*, pages 75–136, 2017.
- [143] D. Poland, S. Rychkov, and A. Vichi. The Conformal Bootstrap: Theory, Numerical Techniques, and Applications. *Rev. Mod. Phys.*, 91:015002, 2019.

- [144] G. Policastro, D. T. Son, and A. O. Starinets. The Shear viscosity of strongly coupled $N=4$ supersymmetric Yang-Mills plasma. *Phys. Rev. Lett.*, 87:081601, 2001.
- [145] R. Schäfke and D. Schmidt. The connection problem for general linear ordinary differential equations at two regular singular points with applications in the theory of special functions. *SIAM Journal on Mathematical Analysis*, 11(5):848–15, 09 1980.
- [146] R. Schiappa and N. Wyllard. An $A(r)$ threesome: Matrix models, 2d CFTs and 4d $N=2$ gauge theories. *J. Math. Phys.*, 51:082304, 2010.
- [147] L. Schlesinger. Über eine klasse von differentialsystemen beliebiger ordnung mit festen kritischen punkten. *Journal für die reine und angewandte Mathematik*, 141:96–145, 1912.
- [148] S. Slavyanov and W. Lay. *Special Functions: A Unified Theory Based on Singularities*.
- [149] S. Y. Slavyanov. Painlevé equations as classical analogues of Heun equations. *Journal of Physics A Mathematical General*, 29:7329–7335, Nov. 1996.
- [150] J. Sonner. A Rotating Holographic Superconductor. *Phys. Rev.*, D80:084031, 2009.
- [151] A. O. Starinets. Quasinormal modes of near extremal black branes. *Phys. Rev.*, D66:124013, 2002.
- [152] A. Strominger and C. Vafa. Microscopic origin of the Bekenstein-Hawking entropy. *Phys. Lett.*, B379:99–104, 1996.
- [153] H. Suzuki, E. Takasugi, and H. Umetsu. Perturbations of Kerr-de Sitter black hole and Heun’s equations. *Prog. Theor. Phys.*, 100:491–505, 1998.
- [154] H. Suzuki, E. Takasugi, and H. Umetsu. Analytic solutions of Teukolsky equation in Kerr-de Sitter and Kerr-Newman-de Sitter geometries. *Prog. Theor. Phys.*, 102:253–272, 1999.
- [155] J. Teschner. Classical conformal blocks and isomonodromic deformations. 2017.

- [156] S. A. Teukolsky. Rotating black holes: Separable wave equations for gravitational and electromagnetic perturbations. *Phys. Rev. Lett.*, 29(16):1114, 1972.
- [157] S. A. Teukolsky. Perturbations of a rotating black hole. 1. Fundamental equations for gravitational electromagnetic and neutrino field perturbations. *Astrophys. J.*, 185:635–647, 1973.
- [158] N. Uchikata and S. Yoshida. Quasinormal modes of a massless charged scalar field on a small Reissner-Nordstrom-anti-de Sitter black hole. *Phys. Rev.*, D83:064020, 2011.
- [159] N. Uchikata, S. Yoshida, and T. Futamase. Scalar perturbations of Kerr-AdS black holes. *Phys. Rev.*, D80:084020, 2009.
- [160] E. Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.
- [161] E. Witten. Anti-de Sitter space, thermal phase transition, and confinement in gauge theories. *Adv. Theor. Math. Phys.*, 2:505–532, 1998.
- [162] S.-Q. Wu. Separability of the massive Dirac’s equation in 5-dimensional Myers-Perry black hole geometry and its relation to a rank-three Killing-Yano tensor. *Phys. Rev.*, D78:064052, 2008.
- [163] S.-Q. Wu. Separability of massive field equations for spin-0 and spin-1/2 charged particles in the general non-extremal rotating charged black holes in minimal five-dimensional gauged supergravity. *Phys. Rev.*, D80:084009, 2009.
- [164] S.-Q. Wu. Symmetry operators and separability of the massive Dirac’s equation in the general 5-dimensional Kerr-(anti-)de Sitter black hole background. *Class. Quant. Grav.*, 26:055001, 2009. [Erratum: *Class. Quant. Grav.* 26,189801(2009)].
- [165] T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch. Spin spin correlation functions for the two-dimensional Ising model: Exact theory in the scaling region. *Phys. Rev. B*, 13:316–374, 1976.
- [166] A. B. Zamolodchikov. Conformal Symmetry In Two-Dimensions: An Explicit Recurrence Formula For The Conformal Partial Wave Amplitude. *Commun. Math. Phys.*, 96:419–422, 1984.

- [167] A. B. Zamolodchikov. Three-point function in the minimal Liouville gravity. 2005. [Theor. Math. Phys.142,183(2005)].
- [168] A. B. Zamolodchikov and A. B. Zamolodchikov. Conformal field theory and 2-D critical phenomena. 3. Conformal bootstrap and degenerate representations of conformal algebra. 1990.