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Research

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One contribution to a special feature 'Quantum gravity, branes and M-theory' dedicated to Michael J. Duff on the occasion of his 70th birthday, organized by Leron Borsten, Alessio Marrani, Christopher Pope and Kellogg Stelle.

Non-relativistic limits and three-dimensional coadjoint Poincaré gravity

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We show that a recently proposed action for three-dimensional non-relativistic gravity can be obtained by taking the limit of a relativistic Lagrangian that involves the coadjoint Poincaré algebra. We point out the similarity of our construction with the way that three-dimensional Galilei gravity and extended Bargmann gravity can be obtained by taking the limit of a relativistic Lagrangian that involves the Poincaré algebra. We extend our results to the anti-de Sitter case and we will see that there is a chiral decomposition at both the relativistic and non-relativistic level. We comment on possible further generalizations.

1. Introduction

Lie algebra contractions are a useful technique to obtain non-relativistic (NR) symmetries from relativistic ones [1]. Within this context, the Galilei algebra can be understood as an Inönü–Wigner contraction of the Poincaré algebra when the speed of light goes to infinity. This procedure can be generalized to obtain the Bargmann algebra [2,3], which allows one to properly implement the NR limit of a relativistic particle at the level of the action.¹

¹Note that when considering a $(p + 1)$ -dimensional extended object there are $p + 1$ possible NR limits [4,5].

The Inönü–Wigner contraction can be generalized to increase the number of Lie algebra generators by considering Lie algebra expansions [6–8]. Indeed, NR expansions of the Poincaré algebra lead to an infinite family of extensions of the Galilei algebra [9], which have been shown to underlie the large c expansion of General Relativity [10,11]. This method has been extended to the case of strings [9,12] and the case of a non-vanishing cosmological constant [13]. Another way to obtain such a sequence of NR symmetries is by considering suitable quotients of a Galilean free Lie algebra construction [14].

Concerning applications of these algebras to physical systems, one should distinguish between (i) limits of sigma model actions for particles, strings or p-branes and (ii) limits of target space actions leading to NR gravity. In the first case, it is only known how to define NR limits for the Galilei and Bargmann algebra, but not for further extensions of these symmetries. As far as gravitational actions are concerned, it is possible to construct gravitational actions invariant under extensions of the Bargmann algebra using the Lie algebra expansion method. However, to obtain these actions as NR limits of relativistic action is more subtle owing to the possible appearance of infinities. For example, the formulation of an NR limit of the four-dimensional Einstein–Hilbert action leading to a finite action of Newton–Cartan gravity is a well-known open problem; see, for example, [15,16] and references therein.²

In the case of three dimensions, where gravity can be formulated as a Chern–Simons (CS) gauge theory [18,19], the limit can be studied in a more transparent way. In Bergshoeff & Rosseel [20], it was shown that a consistent NR limit of three-dimensional Einstein gravity involving an extra CS term with two $u(1)$ gauge fields is given by an NR CS action invariant under the extended Bargmann algebra [21]. The inclusion of a cosmological constant in this NR gravity theory has been studied in detail in [22,23].

It is natural to address the question of whether the NR symmetries and corresponding gravity actions of [9–11,24] can be obtained as the NR limit of an enlarged Poincaré symmetry algebra and a corresponding gravity theory, respectively. Concerning the algebra, a natural possible candidate is the so-called coadjoint Poincaré symmetry. This algebra and some of its contractions have been studied in [25] to obtain the p-brane Galilei algebra [5,14,26–28]. These contractions could be useful to obtain the NR string theories of [17,29–31] as the limit of a relativistic string theory with an enlarged relativistic space–time symmetry algebra. In this paper, we will show that the coadjoint Poincaré algebra indeed provides a relativistic counterpart of the NR algebra introduced in [10].

On the other hand, the algebra found in [24] can be obtained from the coadjoint Poincaré algebra plus two $u(1)$ generators, leading to two central extensions. We will refer to this algebra as the enhanced Bargmann algebra. Furthermore, in [24], it was shown that the corresponding gravity action can be obtained as a limit of a CS action based on the $\mathfrak{iso}(2,1) \oplus \mathbb{E}_3 \oplus u(1)^2$ algebra. In this paper, we show that the same action can be obtained from a fully relativistic CS theory invariant under the direct sum of the coadjoint Poincaré algebra and two $u(1)$ factors. Furthermore, it is possible to obtain extended Bargmann gravity and Galilei gravity as alternative NR limits of the same action without generating infinities.

We point out the similarity of our construction with the way in which (2+1)-dimensional Galilei gravity [16] and extended Bargmann gravity [20,21,23] can be obtained as a limit of a relativistic CS action invariant under the Poincaré algebra. The results here obtained can be generalized in several ways. First, we discuss the extension to the case of the coadjoint anti-de Sitter (AdS) algebra by including a cosmological constant and show that it is possible to define a chiral decomposition at both the relativistic and NR level. This decomposition is the analogue of the $sl(2\mathbb{R})$ formulation of AdS_3 CS gravity [18,19] or the chiral decomposition of AdS_3 invariant dynamical systems and their NR counterparts [32–34]. Second, in the outlook, we argue that the coadjoint Poincaré algebra can be defined as a particular relativistic expansion of the Poincaré symmetry. Based on this fact, we suggest a generalization of our construction to extensions of the

²For a recent proposal for an action for Newtonian gravity, see [10]. Note that the limit exists in the case of four-dimensional string Newton–Cartan gravity, but only after a term has been added to the Einstein–Hilbert action [17]. In §4, we will give a new example of a finite limit of the Einstein–Hilbert action without the need to add an extra term.

coadjoint Poincaré algebra that result from using bigger semigroups [8] in the expansion of the Poincaré algebra.

The organization of the paper is as follows: in §2, we consider contractions of the Poincaré algebra and the enlarged Poincaré $\oplus \mathfrak{u}(1)^2$ algebra. We consider the corresponding NR limit of the CS actions that are based upon these two algebras and show that they give rise to Galilei and extended Bargmann gravity. In §3, we repeat the same analysis but now for the coadjoint Poincaré and the enlarged coadjoint Poincaré $\oplus \mathfrak{u}(1)^2$ algebras. In particular, we show that this time the CS actions lead to the actions of not only Galilei and extended Bargmann gravity but also to the action of enhanced Bargmann gravity [24]. In §4, we discuss the coadjoint AdS algebra and generalize our results to include a cosmological constant. In the Conclusion, we speculate how the results obtained in this paper can be generalized to construct gravity actions based on further extensions of the enhanced Bargmann algebra.

2. NR limits and the Poincaré algebra

In this section, we consider how the action for three-dimensional Galilei gravity [16] and the action for extended Bargmann gravity [20,21,23] can be obtained by taking the limit of specific relativistic actions. This section serves as a warming-up exercise for the next section, where we will go beyond extended Bargmann gravity and reproduce the enhanced Bargmann gravity action of [24] as the limit of a relativistic CS action invariant under the coadjoint Poincaré algebra following the same construction that we perform in this section.

(a) The Poincaré algebra

Our starting point is the D -dimensional Poincaré algebra of space–time translations $\tilde{P}_A$ and Lorentz transformations $\tilde{J}_{AB}$ ($A = 0, 1, \dots, D-1$)

$$[\tilde{J}_{AB}, \tilde{P}_C] = 2\eta_{C[B}\tilde{P}_{A]} \quad \text{and} \quad [\tilde{J}_{AB}, \tilde{J}_{CD}] = 4\eta_{[A[C}\tilde{J}_{D]B}], \quad (2.1)$$

where η_{AB} is the (mostly plus) Minkowski metric. Splitting the relativistic indices in temporal and spatial values as $A = (0, a)$, where $a = 1, \dots, D-1$, we can write

$$\tilde{J}_{AB} = \{\tilde{J}_{0a} \equiv \tilde{G}_a, \tilde{J}_{ab}\} \quad \text{and} \quad \tilde{P}_A = \{\tilde{P}_0 \equiv \tilde{H}, \tilde{P}_a\}, \quad (2.2)$$

and the commutation relations (2.1) take the form

$$[\tilde{G}_a, \tilde{H}] = \tilde{P}_a, \quad (2.3a)$$

$$[\tilde{G}_a, \tilde{P}_b] = \delta_{ab}\tilde{H}, \quad (2.3b)$$

$$[\tilde{G}_a, \tilde{G}_b] = \tilde{J}_{ab}, \quad (2.3c)$$

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = 4\delta_{[a[c}\tilde{J}_{d]b]}, \quad (2.3d)$$

$$[\tilde{J}_{ab}, \tilde{G}_c] = 2\delta_{c[b}\tilde{G}_{a]} \quad (2.3e)$$

and

$$[\tilde{J}_{ab}, \tilde{P}_c] = 2\delta_{c[b}\tilde{P}_{a]}. \quad (2.3f)$$

From these relations, we see that the Galilei algebra can be obtained by means of the following rescaling of the Poincaré generators:³

$$\tilde{J}_{ab} = J_{ab}, \quad (2.4a)$$

$$\tilde{H} = H, \quad (2.4b)$$

³This scaling of the generators is different from the scaling used in [35] and resembles more closely those of [36]. Other scalings are also possible [20,21,23]. This is related to the fact that the commutation relations of the Poincaré algebra are invariant under the rescaling $\tilde{H} = \lambda H$ and $\tilde{P}_a = \lambda \tilde{P}_a$ (e.g. [36]). At the field theory level, this corresponds to the fact that two different scalings can differ by an overall scaling of the Lagrangian that can be absorbed by a scaling of Newton's constant [9].

$$\tilde{G}_a = \varepsilon G_a \quad (2.4c)$$

and

$$\tilde{P}_a = \varepsilon P_a, \quad (2.4d)$$

and taking the limit $\varepsilon \rightarrow \infty$, which leads to

$$[G_a, H] = P_a, \quad (2.5a)$$

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]}, \quad (2.5b)$$

$$[J_{ab}, G_c] = 2\delta_{c[b}G_{a]} \quad (2.5c)$$

and

$$[J_{ab}, P_c] = 2\delta_{c[b}P_{a]}. \quad (2.5d)$$

In $D = 2 + 1$ dimensions, we can rewrite the Poincaré commutation relations (2.1) by defining the dual generators

$$\tilde{J}_{AB} \equiv \epsilon_{AB}^C \tilde{J}_C \quad (2.6)$$

in the following way:

$$[\tilde{J}_A, \tilde{J}_B] = \epsilon_{AB}^C \tilde{J}_C \quad \text{and} \quad [\tilde{J}_A, \tilde{P}_B] = \epsilon_{AB}^C \tilde{P}_C, \quad (2.7)$$

where we have defined the epsilon-tensor such that $\epsilon_{012} = -1$. Using the dual generator (2.6) requires us to replace the relation for $\tilde{J}_{AB}$ in (2.2) by

$$\tilde{J}_A = \{\tilde{J}_a \equiv \tilde{G}_a, \tilde{J}_0 \equiv \tilde{J}\}, \quad (2.8)$$

and to replace

$$J_{ab} = -\epsilon_{ab} J \quad \text{and} \quad G_a \rightarrow -\epsilon_a^b G_b \quad (2.9)$$

in the contraction (2.4). This leads to the following form for the Galilei algebra in three space-time dimensions:

$$[J, G_a] = -\epsilon_a^b G_b, \quad (2.10a)$$

$$[J, P_a] = -\epsilon_a^b P_b \quad (2.10b)$$

and

$$[H, G_a] = -\epsilon_a^b P_b, \quad (2.10c)$$

where the two-dimensional epsilon-tensor is defined such that $\epsilon_{ab} \equiv -\epsilon_{0ab} \Rightarrow \epsilon_{12} = 1$.

In 2+1 dimensions, we can define a gravity theory invariant under the full Poincaré algebra by considering the CS action [18,19],

$$S = \int \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle, \quad (2.11)$$

where the gauge field A takes values in the Poincaré algebra,

$$A = \Omega^A \tilde{J}_A + E^A \tilde{P}_A. \quad (2.12)$$

We consider a non-degenerate invariant bilinear form of the Poincaré algebra,

$$\langle \tilde{J}_A \tilde{P}_B \rangle = \alpha_1 \eta_{AB}, \quad (2.13)$$

with arbitrary parameter $\alpha_1 \neq 0$.

The curvature two-form of the Poincaré algebra reads

$$R = dA + A^2 = R^A(\tilde{J})\tilde{J}_A + R^A(\tilde{P})\tilde{P}_A, \quad (2.14)$$

where

$$R^A(\tilde{J}) = d\Omega^A + \frac{1}{2}\epsilon_{BC}^A \Omega^B \Omega^C \quad \text{and} \quad R^A(\tilde{P}) = dE^A + \epsilon_{BC}^A \Omega^B E^C. \quad (2.15)$$

Explicitly, up to boundary terms, one finds the three-dimensional version of the Einstein–Hilbert term as the CS action for Poincaré gravity, i.e.

$$S_{\text{Poincaré}} = 2\alpha_1 \int E^A R^B(\tilde{J}) \eta_{AB}. \quad (2.16)$$

Using (2.4) and (2.9), the connection A can be expressed as

$$A = \omega J + \omega^a G_a + \tau H + e^a P_a, \quad (2.17)$$

where the NR gauge fields are related to the relativistic ones by

$$\Omega^0 = \omega, \quad (2.18a)$$

$$\Omega^a = \frac{1}{\varepsilon} \omega^a, \quad (2.18b)$$

$$E^0 = \tau \quad (2.18c)$$

$$\text{and} \quad E^a = \frac{1}{\varepsilon} e^a. \quad (2.18d)$$

Using these relations, the Poincaré gravity action (2.16) takes the form

$$S_{\text{Poincaré}} = 2\alpha_1 \int \left[-\tau R(J) + \frac{1}{\varepsilon^2} e^a R_a(G) \right], \quad (2.19)$$

where

$$R(J) = d\omega \quad \text{and} \quad R^a(G) = d\omega^a - \epsilon_b^a \omega \omega^b \quad (2.20)$$

are the components of the NR curvature two-form of the Galilei algebra corresponding to spatial rotations and Galilean boosts, respectively.

(i) Galilei gravity

In the limit that $\varepsilon \rightarrow \infty$, the action for Poincaré gravity reduces to the action for Galilei gravity [16],

$$S_{\text{Galilei}} = -2\kappa \int \tau R(J), \quad (2.21)$$

where we have set $\alpha_1 = \kappa$, which plays the role of the gravitational coupling constant. We note that in this limit the invariant tensor (2.13) gives the NR invariant bilinear form⁴

$$\langle JH \rangle = -\kappa, \quad (2.22)$$

which defines a degenerate invariant bilinear form for the Galilei algebra.

⁴We note that the most general invariant tensor for the Galilei algebra is degenerate and is given by

$$\langle J, J \rangle = -\beta_1, \quad \langle J, H \rangle = -\beta_2, \quad \langle H, H \rangle = -\beta_3,$$

with $\beta_1, \beta_2, \beta_3$ arbitrary parameters. The first term comes from the relativistic invariant form $\langle \tilde{J}_A \tilde{J}_B \rangle = \beta_1 \eta_{AB}$, which is a degenerate invariant form for the Poincaré algebra (see equation (4.18)). The second term is the invariant tensor given in (2.22), while the third term is purely NR and does not follow from a relativistic invariant tensor upon contraction.

(b) The Poincaré $\oplus \mathfrak{u}(1)^2$ algebra

In arbitrary dimensions, the contraction (2.4) can be generalized by considering the direct sum of the Poincaré algebra (2.1) and a $\mathfrak{u}(1)$ generator $\tilde{M}$ [37] (see also [38]). This can be done by replacing the relation (2.4b) by

$$\tilde{H} = \frac{1}{2}H + \varepsilon^2 M \quad \text{and} \quad \tilde{M} = \frac{1}{2}H - \varepsilon^2 M. \quad (2.23)$$

This contraction leads to the Bargmann algebra, which corresponds to the universal central extension of the Galilei algebra [39], and enlarges (2.5) by adding the commutation relation

$$[G_a, P_b] = \delta_{ab} M. \quad (2.24)$$

In three dimensions, one can go further and endow the Bargmann algebra with a second central extension by considering the direct sum of the Poincaré algebra (2.7) and two $\mathfrak{u}(1)$ generators $\tilde{M}$ and $\tilde{S}$ [20] and replacing the relation (2.4a) by

$$\tilde{J} = \frac{1}{2}J + \varepsilon^2 S \quad \text{and} \quad \tilde{S} = \frac{1}{2}J - \varepsilon^2 S. \quad (2.25)$$

Using (2.23) and (2.25) and keeping the rescaling of the Galilean boosts and spatial translations generators⁵ as in (2.4c,d), the limit $\varepsilon \rightarrow \infty$ gives the extended Bargmann algebra [2,3]. Indeed, the corresponding inverse relations are given by

$$J = \tilde{J} + \tilde{S}, \quad (2.26a)$$

$$S = \frac{1}{2\varepsilon^2}(\tilde{J} - \tilde{S}), \quad (2.26b)$$

$$G_a = \frac{1}{\varepsilon}\tilde{G}_a, \quad (2.26c)$$

$$H = \tilde{H} + \tilde{M}, \quad (2.26d)$$

$$M = \frac{1}{2\varepsilon^2}(\tilde{H} - \tilde{M}) \quad (2.26e)$$

$$\text{and} \quad P_a = \frac{1}{\varepsilon}\tilde{P}_a. \quad (2.26f)$$

Using these relations, we find that, in the limit that $\varepsilon \rightarrow \infty$, the Poincaré $\oplus \mathfrak{u}(1)^2$ algebra reduces to the commutation relations of (2.10) plus

$$[G_a, P_b] = \epsilon_{ab} M \quad \text{and} \quad [G_a, G_b] = \epsilon_{ab} S, \quad (2.27)$$

which corresponds to the double central extension of the Galilei algebra, where M and S are central charge generators.

The action for Poincaré $\oplus \mathfrak{u}(1)^2$ gravity is given by the same CS action (2.11) as before, but with a gauge connection A that now takes values in the Poincaré $\oplus \mathfrak{u}(1)^2$ algebra,

$$A = \Omega^A \tilde{J}_A + E^A \tilde{P}_A + a_1 \tilde{S} + a_2 \tilde{M}. \quad (2.28)$$

Accordingly, the invariant tensor (2.13) has to be supplemented with the following invariant form for the Abelian generators:

$$\langle \tilde{M} \tilde{S} \rangle = \alpha_2, \quad (2.29)$$

where α_2 is an arbitrary parameter. Using relations (2.26), the gauge connection A can be expressed as

$$A = \omega J + \omega^a G_a + \tau H + e^a P_a + m M + s S, \quad (2.30)$$

⁵ As in the Galilei case, the scalings do not coincide with those used in [20]. This fact is the result of an invariance under scaling of momenta of the Poincaré algebra; see footnote 3.

where the NR gauge fields are related to the relativistic ones as follows:

$$\Omega^0 = \omega + \frac{1}{2\varepsilon^2} s, \quad (2.31a)$$

$$\Omega^a = \frac{1}{\varepsilon} \omega^a, \quad (2.31b)$$

$$a_1 = \omega - \frac{1}{2\varepsilon^2} s, \quad (2.31c)$$

$$E^0 = \tau + \frac{1}{2\varepsilon^2} m, \quad (2.31d)$$

$$E^a = \frac{1}{\varepsilon} e^a \quad (2.31e)$$

and
$$a_2 = \tau - \frac{1}{2\varepsilon^2} m. \quad (2.31f)$$

Using the above relations, the Poincaré $\oplus \mathfrak{u}(1)^2$ gravity action (2.11) takes the form

$$S_{\text{Poincaré} \oplus \mathfrak{u}(1)^2} = 2 \int \left[(-\alpha_1 + \alpha_2) \tau R(J) + \frac{\alpha_1}{\varepsilon^2} e^a R_a(G) - \frac{(\alpha_1 + \alpha_2)}{2\varepsilon^2} (mR(J) + \tau R(S)) + \mathcal{O}(\varepsilon^{-4}) \right], \quad (2.32)$$

where, apart from (2.20), we have defined

$$R(S) = ds + \frac{1}{2} \epsilon_{ab} \omega^a \omega^b, \quad (2.33)$$

which is the curvature two-form associated with the central charge generator S of the extended Bargmann algebra. We now consider two different sets of values for the parameters α_1 and α_2 .

(i) Galilei gravity

For general $\alpha_1 \neq \alpha_2$, we re-obtain, after taking the limit $\varepsilon \rightarrow \infty$, the Galilei gravity action (2.21) constructed in the previous subsection.

(ii) Extended Bargmann gravity

In the case that we consider

$$\alpha_1 = \alpha_2 = \varepsilon^2 \kappa, \quad (2.34)$$

we obtain an enhancement of Galilei gravity. In fact, the first term in (2.32) vanishes and, taking the limit $\varepsilon \rightarrow \infty$, we obtain the action for extended Bargmann gravity [20,21,23]

$$S_{\text{EBG}} = 2\kappa \int [e^a R_a(G) - mR(J) - \tau R(S)]. \quad (2.35)$$

This result is consistent with the known non-degenerate invariant bilinear form for the extended Bargmann algebra,

$$\langle J, M \rangle = -\kappa, \quad (2.36a)$$

$$\langle S, H \rangle = -\kappa \quad (2.36b)$$

and
$$\langle G_a, P_b \rangle = \kappa \delta_{ab}, \quad (2.36c)$$

which follows from (2.26), (2.29) and (2.34) in the limit $\varepsilon \rightarrow \infty$. Note that the equations of motion of (2.35) lead to the vanishing of all curvatures.

3. NR limits and the coadjoint Poincaré algebra

It is natural to ask oneself if the NR actions of [9–11,24] can be obtained from a relativistic action with an enlarged Poincaré symmetry algebra. If that is the case, what is the symmetry algebra? In this section, we address the question in the (2+1)-dimensional case and show how the action

of [24] can be obtained as the NR limit of a relativistic CS action, in almost the same way that the action for extended Bargmann gravity is obtained as the limit of a relativistic CS action invariant under the Poincaré algebra. In order to do this, we extend the algebras used in the previous section as follows:

$$\text{Poincaré} \rightarrow \text{coadjoint Poincaré} \quad (3.1)$$

and

$$\text{Poincaré} \oplus \mathfrak{u}(1)^2 \rightarrow \text{coadjoint Poincaré} \oplus \mathfrak{u}(1)^2. \quad (3.2)$$

As we will see, the relativistic symmetry behind this construction is the coadjoint Poincaré algebra, and it is only in the second case (3.2) that we obtain the action of [24]. We will now discuss these two cases separately.

(a) The coadjoint Poincaré algebra

In the following, we will consider the following extension of the Poincaré algebra:

$$[\tilde{J}_{AB}, \tilde{P}_C] = 2\eta_{C[B}\tilde{P}_{A]}, \quad (3.3a)$$

$$[\tilde{J}_{AB}, \tilde{J}_{CD}] = 4\eta_{[A[C}\tilde{J}_{D]B]}, \quad (3.3b)$$

$$[\tilde{J}_{AB}, \tilde{T}_C] = 2\eta_{C[B}\tilde{T}_{A]}, \quad (3.3c)$$

$$[\tilde{J}_{AB}, \tilde{S}_{CD}] = 4\eta_{[A[C}\tilde{S}_{D]B]} \quad (3.3d)$$

and

$$[\tilde{S}_{AB}, \tilde{P}_C] = 2\eta_{C[B}\tilde{T}_{A]}. \quad (3.3e)$$

We will refer to this algebra as coadjoint Poincaré algebra since it defines a consistent action of the Poincaré algebra on its dual space (see appendix in [25]). However, as we will see in the following, the invariant tensor associated with this algebra is not the standard bilinear form that follows from the general double-extension construction [40–42].⁶ It is important to note that another definition of the coadjoint algebra can be given in terms of the infinitesimal coadjoint representation associated with a semi-direct product group [43–45], where the existence of a non-degenerate invariant metric is guaranteed and corresponds to a particular case of a double-extended algebra [40,41]. In the subsequent sections, we will restrict ourselves to the (2+1)-dimensional case, where it is straightforward to show that the two definitions mentioned above are isomorphic.

Now we divide the generators into space and time components by splitting the indices in the form $A = \{0, a\}$, which yields (2.2) together with

$$\tilde{S}_{AB} = \{\tilde{S}_{0a} \equiv \tilde{B}_a, \tilde{S}_{ab}\} \quad \text{and} \quad \tilde{T}_A = \{\tilde{T}_0 \equiv \tilde{M}, \tilde{T}_a\}, \quad (3.4)$$

and consider the following contraction of the coadjoint Poincaré algebra:

$$\tilde{J}_{ab} = J_{ab}, \quad (3.5a)$$

$$\tilde{H} = H, \quad (3.5b)$$

$$\tilde{G}_a = \frac{\varepsilon}{2}G_a - \frac{\varepsilon^3}{2}B_a, \quad (3.5c)$$

$$\tilde{P}_a = \frac{\varepsilon}{2}P_a - \frac{\varepsilon^3}{2}T_a, \quad (3.5d)$$

$$\tilde{S}_{ab} = -\varepsilon^2 S_{ab}, \quad (3.5e)$$

$$\tilde{M} = -\varepsilon^2 M, \quad (3.5f)$$

$$\tilde{B}_a = -\varepsilon G_a - \varepsilon^3 B_a \quad (3.5g)$$

and

$$\tilde{T}_a = -\varepsilon P_a - \varepsilon^3 T_a. \quad (3.5h)$$

⁶We acknowledge Roberto Casalbuoni and Axel Kleinschmidt for discussions about general coadjoint actions.

Note that this contraction is different from the $k = 1$ contraction of the coadjoint Poincaré algebra discussed in [25]. In the limit $\epsilon \rightarrow \infty$ this leads to the following commutation relations for the NR generators:

$$[G_a, H] = P_a, \quad (3.6a)$$

$$[G_a, M] = T_a, \quad (3.6b)$$

$$[B_a, H] = T_a, \quad (3.6c)$$

$$[G_a, P_b] = \delta_{ab} M, \quad (3.6d)$$

$$[G_a, G_b] = S_{ab}, \quad (3.6e)$$

$$[S_{ab}, G_c] = 2\delta_{c[b} B_{a]}, \quad (3.6f)$$

$$[S_{ab}, P_c] = 2\delta_{c[b} T_{a]}, \quad (3.6g)$$

$$[J_{ab}, S_{cd}] = 4\delta_{[a[c} S_{d]b]}, \quad (3.6h)$$

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c} J_{d]b]}, \quad (3.6i)$$

and

$$[J_{ab}, X_c] = 2\delta_{c[b} X_{a]}, \quad (3.6j)$$

where we have used the collective notation $X_a = \{G_a, P_a, B_a, T_a\}$. This corresponds to the algebra found by Hansen *et al.* [10], which can be obtained as a quotient of the infinite-dimensional NR expansion of the Poincaré algebra given in [11,14]. Dividing out this algebra by the generators $\{B_a, T_a\}$ leads to a $D > 3$ version of the extended Bargmann algebra (see appendix A).

As in the Poincaré case, in three space–time dimensions we can dualize the generator of rotations as in (2.6), together with a similar definition for $\tilde{S}_{AB}$,

$$\tilde{S}_{AB} = \epsilon_{AB}^C \tilde{S}_C. \quad (3.7)$$

This allows us to rewrite the coadjoint Poincaré algebra as (2.7) plus

$$[\tilde{J}_A, \tilde{S}_B] = \epsilon_{AB}^C \tilde{S}_C, \quad [\tilde{J}_A, \tilde{T}_B] = \epsilon_{AB}^C \tilde{T}_C, \quad [\tilde{S}_A, \tilde{P}_B] = \epsilon_{AB}^C \tilde{T}_C. \quad (3.8)$$

In order to evaluate the NR limit, the equation (2.8) has to be supplemented with

$$\tilde{S}_A = \left\{ \tilde{S}_0 \equiv \tilde{S}, \tilde{S}_a \equiv \tilde{B}_a \right\}, \quad (3.9)$$

while the definition (3.7) implies (2.9) and

$$S_{ab} = -\epsilon_{ab} S \quad \text{and} \quad B_a \rightarrow -\epsilon_a^b B_b. \quad (3.10)$$

Using these redefinitions, the (2+1)-dimensional version of the algebra (3.6) takes the form of (2.10) plus the following commutation relations:

$$[J, B_a] = -\epsilon_a^b B_b, \quad (3.11a)$$

$$[J, T_a] = -\epsilon_a^b T_b, \quad (3.11b)$$

$$[S, G_a] = -\epsilon_a^b B_b, \quad (3.11c)$$

$$[S, P_a] = -\epsilon_a^b T_b, \quad (3.11d)$$

$$[G_a, G_b] = \epsilon_{ab} S, \quad (3.11e)$$

$$[M, G_a] = -\epsilon_a^b T_b, \quad (3.11f)$$

$$[G_a, P_b] = \epsilon_{ab} M \quad (3.11g)$$

and

$$[H, B_a] = -\epsilon_a^b T_b. \quad (3.11h)$$

We can construct a three-dimensional NR gravity theory with this symmetry by starting with the CS action (2.11) invariant under the coadjoint Poincaré algebra, where now A is a connection

taking values on the coadjoint Poincaré algebra, i.e.

$$A = \Omega^A \tilde{J}_A + E^A \tilde{P}_A + \Sigma^A \tilde{S}_A + L^A \tilde{T}_A. \quad (3.12)$$

We will consider the following invariant tensor:

$$\langle \tilde{J}_A \tilde{T}_B \rangle = \gamma_1 \eta_{AB}, \quad \langle \tilde{S}_A \tilde{P}_B \rangle = \gamma_1 \eta_{AB}, \quad \langle \tilde{J}_A \tilde{P}_B \rangle = \gamma_2 \eta_{AB}, \quad (3.13)$$

where γ_1 and γ_2 are arbitrary parameters. This invariant tensor is non-degenerate for $\gamma_1 \neq 0$.

The corresponding curvature two-form reads

$$R = dA + A^2 = R^A(\tilde{J}) \tilde{J}_A + R^A(\tilde{P}) \tilde{P}_A + R^A(\tilde{S}) \tilde{S}_A + R^A(\tilde{T}) \tilde{T}_A, \quad (3.14)$$

where $R^A(\tilde{J})$ and $R^A(\tilde{P})$ are given in (2.15) and

$$R^A(\tilde{S}) = d\Sigma^A + \epsilon_{BC}^A \Omega^B \Sigma^C \quad \text{and} \quad R^A(\tilde{T}) = dL^A + \epsilon_{BC}^A \left(\Omega^B L^C + \Sigma^B E^C \right). \quad (3.15)$$

The CS action for coadjoint Poincaré gravity is given by

$$S_{\text{Coad-Poincaré}} = 2\gamma_1 \int \left[E^A R_A(\tilde{S}) + L^A R_A(\tilde{J}) \right] + 2\gamma_2 \int E^A R_A(\tilde{J}). \quad (3.16)$$

Note that the term with γ_2 gives the Einstein–Hilbert action in 2+1 dimensions.

We next study the contraction (3.5) at the level of the CS action. Using the redefinitions (3.5), the gauge connection (3.12) can be written as

$$A = \omega J + \omega^a G_a + \tau H + e^a P_a + sS + b^a B_a + mM + t^a T_a, \quad (3.17)$$

where the NR fields are related to the relativistic ones by

$$\Omega^0 = \omega, \quad (3.18a)$$

$$\Omega^a = \frac{1}{\varepsilon} \omega^a - \frac{1}{\varepsilon^3} b^a, \quad (3.18b)$$

$$\Sigma^0 = -\frac{1}{\varepsilon^2} s, \quad (3.18c)$$

$$\Sigma^a = -\frac{1}{2\varepsilon} \omega^a - \frac{1}{2\varepsilon^3} b^a, \quad (3.18d)$$

$$E^0 = \tau, \quad (3.18e)$$

$$E^a = \frac{1}{\varepsilon} e^a - \frac{1}{\varepsilon^3} t^a, \quad (3.18f)$$

$$L^0 = -\frac{1}{\varepsilon^2} m \quad (3.18g)$$

and

$$L^a = -\frac{1}{2\varepsilon} e^a - \frac{1}{2\varepsilon^3} t^a, \quad (3.18h)$$

and in terms of which the CS action takes the form

$$S_{\text{Coad-Poincaré}} = 2 \int \left[-\gamma_2 \tau R(J) + \frac{(-\gamma_1 + \gamma_2)}{\varepsilon^2} e^a R_a(G) + \frac{(\gamma_1 - \gamma_2)}{\varepsilon^2} \tau R(S) + \frac{\gamma_1}{\varepsilon^2} m R(J) + \frac{\gamma_2}{\varepsilon^2} \tau ds + \mathcal{O}(\varepsilon^{-4}) \right]. \quad (3.19)$$

In the following, we consider two different sets of values for the parameters γ_1 and γ_2 in this action.

(i) Galilei gravity

For general $\gamma_1 \neq \gamma_2$, we re-obtain, after taking the limit $\varepsilon \rightarrow \infty$, the Galilei gravity action (2.21) constructed in the previous subsection.

(ii) Extended Bargmann gravity

On the other hand, setting

$$\gamma_2 = 0, \quad \gamma_1 = -\kappa \varepsilon^2, \quad (3.20)$$

we obtain the extended Bargmann gravity action of [20,21,23],

$$S_{\text{Extended-Bargmann}} = 2\kappa \int [e^a R_a(G) - \tau R(S) - mR(J)]. \quad (3.21)$$

Substituting, for the choice of parameters (3.20), the expansion (3.5) into expression (3.13) for the invariant tensor leads the known invariant bilinear form of extended Bargmann gravity given in equation (2.36). Note that this invariant form is non-degenerate with respect to the extended Bargmann algebra but it is degenerate with respect to the bigger algebra (3.11).

We have not been able to define other sets of values of the parameters leading to a finite action. In particular, the three-dimensional counterpart of the NR gravity action proposed in [10] cannot be obtained as an NR limit of the CS action corresponding to the coadjoint Poincaré algebra.

(b) The coadjoint Poincaré $\oplus \mathfrak{u}(1)^2$ algebra

Similarly to what happens in the Galilean case, in arbitrary dimensions it is possible to generalize the contraction (3.5) by considering the direct sum of the coadjoint Poincaré algebra (3.3) and a $\mathfrak{u}(1)$ generator $\tilde{Y}$. This is implemented by replacing the relations (3.5b,f) by

$$\tilde{H} = \frac{1}{2}H - \varepsilon^4 Y, \quad \tilde{Y} = \frac{1}{2}H + \varepsilon^4 Y, \quad \tilde{M} = -\varepsilon^2 M - \varepsilon^4 Y. \quad (3.22)$$

In fact, this leads to an extension of the algebra (3.6) by the commutation relations

$$[G_a, T_b] = \delta_{ab} Y \quad \text{and} \quad [B_a, P_b] = \delta_{ab} Y. \quad (3.23)$$

As happens in the case of the Bargmann algebra, in three space–time dimensions one can generalize the previous result to include a second central extension by considering the direct sum of the coadjoint Poincaré algebra in 2+1 dimensions and two $\mathfrak{u}(1)$ generators $\tilde{Y}$ and $\tilde{Z}$. This is done by considering (3.22) and replacing relations (3.5a,e) by

$$\tilde{J} = \frac{1}{2}J - \varepsilon^4 Z, \quad \tilde{Z} = \frac{1}{2}J + \varepsilon^4 Z, \quad \tilde{S} = -\varepsilon^2 S - \varepsilon^4 Z. \quad (3.24)$$

Using (3.22), (3.24) and keeping the definitions (3.5c–h) for the generators $\tilde{G}_a, \tilde{P}_a, \tilde{B}_a$ and $\tilde{T}_a$, the limit $\varepsilon \rightarrow \infty$ yields the bosonic algebra presented in [24] in the context of a novel supersymmetric NR gravity theory. This can be seen by considering the inverse relations

$$J = \tilde{J} + \tilde{Z}, \quad (3.25a)$$

$$G_a = \frac{1}{\varepsilon} \left(\tilde{G}_a - \frac{1}{2} \tilde{B}_a \right), \quad (3.25b)$$

$$S = -\frac{1}{\varepsilon^2} \left(\tilde{S} - \frac{1}{2} \tilde{J} + \frac{1}{2} \tilde{Z} \right), \quad (3.25c)$$

$$B_a = -\frac{1}{\varepsilon^3} \left(\tilde{G}_a + \frac{1}{2} \tilde{B}_a \right), \quad (3.25d)$$

$$Z = -\frac{1}{2\varepsilon^4} (\tilde{J} - \tilde{Z}), \quad (3.25e)$$

$$H = \tilde{H} + \tilde{Y}, \quad (3.25f)$$

$$P_a = \frac{1}{\varepsilon} \left(\tilde{P}_a - \frac{1}{2} \tilde{T}_a \right), \quad (3.25g)$$

$$M = -\frac{1}{\varepsilon^2} \left(\tilde{M} - \frac{1}{2} \tilde{H} + \frac{1}{2} \tilde{Y} \right), \quad (3.25h)$$

$$T_a = -\frac{1}{\varepsilon^3} \left(\tilde{P}_a + \frac{1}{2} \tilde{T}_a \right) \quad (3.25i)$$

$$\text{and} \quad Y = -\frac{1}{2\varepsilon^4} (\tilde{H} - \tilde{Y}). \quad (3.25j)$$

One can see that, in the limit $\varepsilon \rightarrow \infty$, these generators satisfy the commutation relations (2.10) and (3.11) plus

$$[G_a, B_b] = \epsilon_{ab} Z, \quad [G_a, T_b] = \epsilon_{ab} Y, \quad [B_a, P_b] = \epsilon_{ab} Y, \quad (3.26)$$

which define the double central extension [40,41] of the three-dimensional version of the algebra (3.6). Moreover, in the free algebra construction [14], this algebra can be obtained as the three-dimensional version of a suitable quotient of an infinite-dimensional expansion of the Poincaré algebra (see appendix A).

The corresponding NR gravity theory can be obtained as a contraction of a CS action invariant under the coadjoint Poincaré $\oplus \mathfrak{u}(1)^2$ algebra. This theory can be defined by supplementing the connection (3.12) with two extra Abelian gauge fields, which we denote by a_1 and a_2 , i.e.

$$A = \Omega^A \tilde{J}_A + E^A \tilde{P}_A + \Sigma^A \tilde{S}_A + L^A \tilde{T}_A + a_1 \tilde{Z} + a_2 \tilde{Y}. \quad (3.27)$$

Using (3.25) the connection can be expressed as

$$A = \omega J + \omega^a G_a + \tau H + e^a P_a + s S + b^a B_a + m M + t^a T_a + y Y + z Z, \quad (3.28)$$

where the NR fields are related to the relativistic ones by

$$\Omega^0 = \omega + \frac{1}{2\varepsilon^2} s - \frac{1}{2\varepsilon^4} z, \quad (3.29a)$$

$$\Omega^a = \frac{1}{\varepsilon} \omega^a - \frac{1}{\varepsilon^3} b^a, \quad (3.29b)$$

$$\Sigma^0 = -\frac{1}{\varepsilon^2} s, \quad (3.29c)$$

$$\Sigma^a = -\frac{1}{2\varepsilon} \omega^a - \frac{1}{2\varepsilon^3} b^a, \quad (3.29d)$$

$$a_1 = \omega - \frac{1}{2\varepsilon^2} s + \frac{1}{2\varepsilon^4} z, \quad (3.29e)$$

$$E^0 = \tau + \frac{1}{2\varepsilon^2} m - \frac{1}{2\varepsilon^4} y, \quad (3.29f)$$

$$E^a = \frac{1}{\varepsilon} e^a - \frac{1}{\varepsilon^3} t^a, \quad (3.29g)$$

$$L^0 = -\frac{1}{\varepsilon^2} m, \quad (3.29h)$$

$$L^a = -\frac{1}{2\varepsilon} e^a - \frac{1}{2\varepsilon^3} t^a \quad (3.29i)$$

$$\text{and} \quad a_2 = \tau - \frac{1}{2\varepsilon^2} m + \frac{1}{2\varepsilon^4} y. \quad (3.29j)$$

In order to explicitly compute the CS gravity action, the invariant tensor (3.13) has to be supplemented with

$$\langle \tilde{Y} \tilde{Z} \rangle = \gamma_3, \quad (3.30)$$

where we have introduced an extra arbitrary parameter γ_3 . The corresponding CS action then takes the form

$$S_{\text{Coad-Poincaré} \oplus \mathfrak{u}(1)^2} = 2 \int \left[(-\gamma_2 + \gamma_3) \tau R(J) + \frac{(2\gamma_1 - \gamma_2 - \gamma_3)}{2\varepsilon^2} mR(J) + \frac{(\gamma_1 - \gamma_2)}{\varepsilon^2} \tau R(S) \right. \\ + \frac{(\gamma_2 - \gamma_3)}{2\varepsilon^2} \tau ds + \frac{(-\gamma_1 + \gamma_2)}{\varepsilon^2} e^a R_a(G) + \frac{(\gamma_2 + \gamma_3)}{2\varepsilon^4} yR(J) + \frac{\gamma_2}{\varepsilon^4} \tau R(Z) \\ + \frac{(3\gamma_1 - \gamma_2)}{2\varepsilon^4} mR(S) + \frac{3(\gamma_1 - \gamma_2)}{2\varepsilon^4} \epsilon_{ab} e^a \omega^b s + \frac{(-\gamma_2 + \gamma_3)}{2\varepsilon^4} \tau dz \\ \left. + \frac{(-2\gamma_1 + \gamma_2 + \gamma_3)}{4\varepsilon^4} m ds - \frac{\gamma_2}{\varepsilon^4} (e^a R_a(B) + t^a R_a(G)) + \mathcal{O}(\varepsilon^{-6}) \right], \quad (3.31)$$

where we have used (2.20) and (2.33) and defined

$$R(Z) = dz + \epsilon_{ab} \omega^a b^b \quad \text{and} \quad R^a(B) = db^a + \epsilon_b^a (s \omega^b + \omega b^b). \quad (3.32)$$

We now consider three different sets of values for the parameters.

(i) Galilei gravity

For general $\gamma_2 \neq \gamma_3$, we re-obtain, after taking the limit $\varepsilon \rightarrow \infty$, the Galilei gravity action (2.21).

(ii) Extended Bargmann gravity

If we set $\gamma_2 = \gamma_3 = 0$ and implement the rescaling $\gamma_1 = -\varepsilon^2 \kappa$, the action (3.31) leads to no divergent terms, allowing us to recover, in the limit that $\varepsilon \rightarrow \infty$, the extended Bargmann gravity action of [20,21,23] given in (2.35). The same result can be recovered by setting $\gamma_1 = 0$ and $\gamma_2 = \gamma_3 = \varepsilon^2 \kappa$. This second option was to be expected since the invariant tensor (3.13) matches that of the Poincaré $\oplus \mathfrak{u}(1)^2$ algebra (3.30) when $\gamma_1 = 0$. Similarly, the corresponding CS action (3.31) then reduces to the Einstein–Hilbert action in three dimensions plus two $\mathfrak{u}(1)$ fields. Note that this choice yields a degenerate invariant tensor for the coadjoint Poincaré algebra.

(iii) Enhanced Bargmann gravity

Setting

$$\gamma_1 = \gamma_2 = \gamma_3 = -\varepsilon^4 \kappa, \quad (3.33)$$

and taking the limit $\varepsilon \rightarrow \infty$, we obtain the following enhancement of extended Bargmann gravity

$$S_{\text{Enhanced-Bargmann}} = 2\kappa \int (e^a R_a(B) + t^a R_a(G) - \tau R(Z) - yR(J) - mR(S)), \quad (3.34)$$

which is precisely the action that has been studied in [24].

Substituting (3.25) and (3.33) into the invariant tensor (3.30), and taking the limit that $\varepsilon \rightarrow \infty$, reduces to the following NR invariant tensor:

$$\langle S, M \rangle = \kappa, \quad (3.35a)$$

$$\langle Z, H \rangle = \kappa, \quad (3.35b)$$

$$\langle J, Y \rangle = \kappa, \quad (3.35c)$$

$$\langle G_a, T_b \rangle = -\kappa \delta_{ab} \quad (3.35d)$$

and

$$\langle B_a, P_b \rangle = -\kappa \delta_{ab}, \quad (3.35e)$$

which is non-degenerate.

4. NR limits and the coadjoint AdS algebra

The coadjoint AdS algebra in D dimensions can be obtained by supplementing the coadjoint Poincaré commutation relations (3.3) with

$$[\tilde{P}_A, \tilde{P}_B] = \frac{1}{\ell^2} \tilde{J}_{AB} \quad \text{and} \quad [\tilde{P}_A, \tilde{T}_B] = \frac{1}{\ell^2} \tilde{S}_{AB}, \quad (4.1)$$

where ℓ is the AdS radius. Naturally, in 2+1 dimensions, this algebra can be written as commutation relations (2.7), (3.8) and (4.1) with $\tilde{J}_{AB} = \epsilon_{AB} \tilde{J}$, $\tilde{S}_{AB} = \epsilon_{AB} \tilde{S}$. By adding two $\mathfrak{u}(1)$ generators $\tilde{Y}$ and $\tilde{Z}$, one can consider the contraction defined by (3.25). This defines an NR limit of coadjoint $\text{AdS}_3 \oplus \mathfrak{u}(1)^2$ given by the commutation relations of (2.10) and (3.11) plus

$$[H, P_a] = -\frac{1}{\ell^2} \epsilon_a^b G_b, \quad (4.2a)$$

$$[M, P_a] = -\frac{1}{\ell^2} \epsilon_a^b B_b, \quad (4.2b)$$

$$[H, T_a] = -\frac{1}{\ell^2} \epsilon_a^b B_b, \quad (4.2c)$$

$$[P_a, P_b] = \frac{1}{\ell^2} \epsilon_{ab} S \quad (4.2d)$$

and

$$[P_a, T_b] = \frac{1}{\ell^2} \epsilon_{ab} Z, \quad (4.2e)$$

which agrees with the result of [13].

The corresponding CS action (2.11) follows from considering a connection A of the form (3.12) (now taking values in the coadjoint AdS_3 algebra) and the invariant tensor (3.13). In this case, the components $R^A[\tilde{J}]$ and $R^A[\tilde{S}]$ of the curvature two-form (3.14) have an extra term proportional to the cosmological constant,

$$R^A[\tilde{J}] = d\Omega^A + \frac{1}{2} \epsilon_{BC}^A \left(\Omega^B \Omega^C + \frac{1}{\ell^2} E^B E^C \right) \quad (4.3)$$

and

$$R^A[\tilde{S}] = d\Sigma^A + \epsilon_{BC}^A \left(\Omega^B \Sigma^C + \frac{1}{\ell^2} E^B L^C \right),$$

while $R^A[\tilde{P}]$ and $R^A[\tilde{T}]$ have the same form given in (2.15) and (3.15), respectively. Evaluating the CS action leads to

$$\begin{aligned} S_{CS} = & 2\gamma_2 \int \left[E^A R_A[\tilde{J}] - \frac{1}{3\ell^2} \epsilon_{ABC} E^A E^B E^C \right] \\ & + 2\gamma_1 \int \left[E^A R_A[\tilde{S}] + L^A R_A[\tilde{J}] - \frac{1}{\ell^2} \epsilon_{ABC} E^A E^B L^C \right]. \end{aligned} \quad (4.4)$$

Remarkably, the coadjoint AdS_3 algebra can be written as the direct sum of two $\mathfrak{iso}(2,1)$ algebras,

$$[\mathcal{J}_A^\pm, \mathcal{J}_A^\pm] = \epsilon_{AB}^C \mathcal{J}_C^\pm, \quad [\mathcal{J}_A^\pm, \mathcal{P}_B^\pm] = \epsilon_{AB}^C \mathcal{P}_C^\pm, \quad [\mathcal{P}_A^\pm, \mathcal{P}_B^\pm] = 0, \quad (4.5)$$

where these two sets of Poincaré generators are related to the original basis $\{\tilde{J}_A, \tilde{P}_A, \tilde{S}_A, \tilde{T}_A\}$ as

$$\mathcal{J}_A^\pm = \frac{1}{2} (\tilde{J}_A \pm \ell \tilde{P}_A) \quad \text{and} \quad \mathcal{P}_A^\pm = \frac{1}{2} \left(\frac{1}{\ell} \tilde{S}_A \pm \tilde{T}_A \right). \quad (4.6)$$

This result is a generalization of the well-known result that, for the pure AdS_3 case, $\mathfrak{so}(2,2) = \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$. This can also be understood from the fact that the Poincaré algebra in 2+1 dimensions $\mathfrak{iso}(2,1)$ is isomorphic to the coadjoint $\mathfrak{so}(2,1)$ algebra [46].

The action (4.4) can be alternatively expressed in terms of two independent sets of gauge fields that follow from the redefinition of the Lie algebra generators (4.6), i.e.

$$E_{\pm}^A = \ell \Sigma^A \pm L^A \quad \text{and} \quad \Omega_{\pm}^A = \Omega^A \pm \frac{1}{\ell} E^A. \quad (4.7)$$

In this case, the connection can be written in terms of the Poincaré generators (4.5) as

$$A = A^+ + A^- \quad \text{and} \quad A^{\pm} = \Omega_{\pm}^A \mathcal{J}_A^{\pm} + E_{\pm}^A \mathcal{P}_A^{\pm}. \quad (4.8)$$

Similarly, the curvature takes the simple form

$$R = R^+ + R^-, \quad R^{\pm} = dA^{\pm} + A^{\pm 2} = R_{\pm}^A [\mathcal{J}] \mathcal{J}_A^{\pm} + R_{\pm}^A [\mathcal{P}] \mathcal{P}_A^{\pm}, \quad (4.9)$$

where

$$R_{\pm}^A [\mathcal{J}^{\pm}] = d\Omega_{\pm}^A + \frac{1}{2} \epsilon_{BC}^A \Omega_{\pm}^B \Omega_{\pm}^C \quad \text{and} \quad R_{\pm}^A [\mathcal{P}^{\pm}] = dE_{\pm}^A + \epsilon_{BC}^A \Omega_{\pm}^B E_{\pm}^C. \quad (4.10)$$

In this chiral basis, the invariant tensor (3.13) reads

$$\langle \mathcal{J}_A^{\pm} \mathcal{P}_B^{\pm} \rangle = \pm \frac{\gamma_1}{2} \eta_{AB} \quad \text{and} \quad \langle \mathcal{J}_A^{\pm} \mathcal{J}_B^{\pm} \rangle = \pm \frac{\ell \gamma_2}{2} \eta_{AB}, \quad (4.11)$$

which means the CS action splits as

$$S_{\text{CS}}[A] = S[A^+] - S[A^-], \quad (4.12)$$

where

$$S[A^{\pm}] = \gamma_1 \int E_{\pm}^A R_A^{\pm} [\mathcal{J}^{\pm}] + \frac{\ell \gamma_2}{2} \int \left[\eta_{AB} \Omega_{\pm}^A d\Omega_{\pm}^B + \frac{1}{3} \epsilon_{ABC} \Omega_{\pm}^A \Omega_{\pm}^B \Omega_{\pm}^C \right]. \quad (4.13)$$

We can consider $\gamma_2 = 0$ and interpret the coadjoint AdS_3 gravity action as the sum of two Einstein–Hilbert terms without interactions. Making this choice it is clear that one could add one $\mathfrak{u}(1)$ generator to each copy of Einstein gravity and define the NR limit of coadjoint $\text{AdS}_3 \oplus \mathfrak{u}(1)^2$ gravity in three dimensions as two copies of the Bargmann algebra by following the approach of [20,21] on each independent chiral sector. Nevertheless, this would lead to a degenerate invariant form after performing the NR limit.⁷

However, as we will see in the following, the CS term for Ω^A in (4.13) will be important to define a different NR limit of coadjoint AdS_3 gravity that connects with the results previously shown in the coadjoint Poincaré case.

(a) NR limit of coadjoint $\text{AdS}_3 \oplus \mathfrak{u}(1)^2$ gravity

The NR limit of coadjoint AdS_3 gravity in three dimensions can be studied in the same way as we did previously in the case of a vanishing cosmological constant. Before starting the analysis, it is important to recall what happens in the case of AdS_3 invariant CS gravity, where the Einstein–Hilbert action (2.19) is modified in the form

$$S_{\text{AdS}} = 2\alpha_1 \int \left(E^A R^B(\tilde{J}) - \frac{1}{\ell^2} \epsilon_{ABC} E^A E^B E^C \right). \quad (4.14)$$

Now the curvature component $R^A(\tilde{J})$ is not given by (2.15) but by

$$R^A(\tilde{J}) = d\Omega^A + \frac{1}{2} \epsilon_{BC}^A \left(\Omega^B \Omega^C + \frac{1}{\ell^2} E^B E^C \right), \quad (4.15)$$

while the torsion two-form $R^A(\tilde{P})$ keeps the same form as in (2.15).

The contraction of the AdS algebra gives the Newton–Hooke algebra [47]. However, the first term of the expansion of CS gravity corresponds to Galilei gravity (2.21) exactly as in the Poincaré case. The reason for this is that, in 2+1 dimensions, $R(J) = d\omega$ also in the Newton–Hooke case. The

⁷Furthermore, it is interesting to note that one could use the action (4.13) with $\gamma_2 = 0$ and add two Abelian fields to each copy of the Einstein–Hilbert action, which would lead to two copies of extended Bargmann gravity as the NR limit of coadjoint $\text{AdS}_3 \oplus \mathfrak{u}(1)^4$ gravity and to a non-degenerate NR invariant tensor.

next step is to add two $u(1)$ generators, which requires us to use the connection (2.28) and the invariant tensor (2.13) together with (2.29). At the level of the action, this implies that one has to add a term of the form

$$\int (a_1 da_2 + a_2 da_1), \quad (4.16)$$

with global factor α_2 to the action (4.14). Then, for $\varepsilon \rightarrow \infty$, the contraction (2.31) leads to Galilean gravity when $\alpha_1 \neq \alpha_2$, and to extended Bargmann–Newton–Hooke gravity [22,23]

$$S_{\text{Extended-BNH}} = 2\kappa \int \left[e^a R_a(G) - \tau R(S) - mR(J) + \frac{1}{\ell^2} \epsilon_{ab} \tau e^a e^b \right] \quad (4.17)$$

for $\alpha_1 = \alpha_2 = \varepsilon^2 \kappa$. The corresponding invariant tensor in this case is non-degenerate and given by (2.36). A more general non-degenerate invariant tensor for the extended Bargmann–Newton–Hooke algebra, which becomes degenerate in the extended Bargmann limit, comes from considering the relativistic invariant tensor for AdS,

$$\langle \tilde{J}_A \tilde{J}_B \rangle = \beta_1 \eta_{AB} \quad \text{and} \quad \langle \tilde{P}_A \tilde{P}_B \rangle = \frac{\beta_1}{\ell^2} \eta_{AB}, \quad (4.18)$$

in the contraction (3.25). We have not considered this case here.

Now that we have reviewed the well-known AdS case, we turn our attention to the novel case of the coadjoint AdS_3 algebra. As happens in the coadjoint Poincaré case, expressing the action (4.4) in terms of the NR fields (3.18) does not lead to new NR actions. Indeed, Galilean gravity (2.21) is recovered for $\gamma_2 \neq \gamma_3$ and extended Bargmann–Newton–Hooke gravity (4.17) follows from choosing $\gamma_1 = -\varepsilon^2 \kappa$, $\gamma_2 = 0$. Also, these CS theories are associated with degenerate invariant tensors for the coadjoint AdS_3 algebra.

As done in §3b, in order to obtain a new NR gravity action from the coadjoint AdS_3 algebra, we incorporate two Abelian fields into the theory by considering a connection of the form (3.27) and the invariant tensor formed by (3.13) and (3.30). This leads to the action (4.4) supplemented with the term (4.16) with global factor γ_3 . Expressing the action in terms of NR gauge fields using (3.29), one can consider the same three different sets of values for the parameters γ_1 , γ_2 and γ_3 considered in §3b.

(i) Galilei gravity

In the case $\gamma_2 \neq \gamma_3$, the limit $\varepsilon \rightarrow \infty$ leads to the action for Galilei gravity (2.21).

(ii) Extended Bargmann–Newton–Hooke gravity

The choice $\gamma_2 = \gamma_3 = 0$ together with the rescaling $\gamma_1 = -\varepsilon^2 \kappa$ leads to the action for extended Bargmann–Newton–Hooke gravity (4.17). This result can also be obtained by setting $\gamma_1 = 0$ and $\gamma_2 = \gamma_3 = \varepsilon^2 \kappa$.

(iii) Enhanced Bargmann–Newton–Hooke gravity

Choosing $\gamma_1 = \gamma_2 = \gamma_3 = -\varepsilon^4 \kappa$ and taking the limit $\varepsilon \rightarrow \infty$ yields the action

$$S_{\text{Enhanced-BNH}} = 2\kappa \int \left(e^a R_a(B) + t^a R_a(G) - \tau R(Z) - yR(J) - mR(S) + \frac{1}{\ell^2} \epsilon_{ab} \left(m e^a e^b + 2\tau e^a t^b \right) \right), \quad (4.19)$$

which will be referred to as enhanced Bargmann–Newton–Hooke gravity, and has been previously constructed in [13,48].

(b) NR limit of coadjoint $\text{AdS}_3 \oplus \mathfrak{u}(1)^2$ gravity in the chiral basis

The NR contraction (3.5) applied to the coadjoint AdS_3 algebra can be alternatively worked out in the chiral basis by relabelling the relativistic generators (4.6) in the form

$$\mathcal{J}_A^\pm = \{\mathcal{J}_0^\pm \equiv \mathcal{J}^\pm, \mathcal{J}_a^\pm \equiv \mathcal{G}_a^\pm\} \quad \text{and} \quad \ell \mathcal{P}_A^\pm = \{\ell \mathcal{P}_0^\pm \equiv S^\pm, \ell \mathcal{P}_a^\pm \equiv \mathcal{B}_a^\pm\}. \quad (4.20)$$

By adding two $\mathfrak{u}(1)$ generators $Z^\pm$ to the coadjoint AdS_3 algebra, we can define the following contraction for two copies of $\mathfrak{iso}(2, 1) \oplus \mathfrak{u}(1)$:

$$\mathcal{J}^\pm = \frac{1}{2} J^\pm - \varepsilon^4 Z^\pm, \quad (4.21a)$$

$$\mathcal{G}_a^\pm = \frac{\varepsilon}{2} G_a^\pm - \frac{\varepsilon^3}{2} B_a^\pm, \quad (4.21b)$$

$$S^\pm = -\varepsilon^2 S^\pm - \varepsilon^4 Z^\pm, \quad (4.21c)$$

$$Z^\pm = \frac{1}{2} J^\pm + \varepsilon^4 Z^\pm \quad (4.21d)$$

and

$$\mathcal{B}_a^\pm = -\varepsilon G_a^\pm - \varepsilon^3 B_a^\pm. \quad (4.21e)$$

Inverting this change of basis leads to the following expression for the NR generators:

$$J^\pm = \mathcal{J}^\pm + Z^\pm, \quad (4.22a)$$

$$G_a^\pm = \frac{1}{\varepsilon} \left(\mathcal{G}_a^\pm - \frac{1}{2} \mathcal{B}_a^\pm \right), \quad (4.22b)$$

$$S^\pm = -\frac{1}{\varepsilon^2} \left(S^\pm - \frac{1}{2} \mathcal{J}^\pm + \frac{1}{2} Z^\pm \right), \quad (4.22c)$$

$$Z^\pm = -\frac{1}{2\varepsilon^4} (\mathcal{J}^\pm - Z^\pm) \quad (4.22d)$$

and

$$B_a^\pm = -\frac{1}{\varepsilon^3} \left(\mathcal{G}_a^\pm + \frac{1}{2} \mathcal{B}_a^\pm \right), \quad (4.22e)$$

which satisfy the commutation relations

$$[J^\pm, G_a^\pm] = -\varepsilon_a^b G_b^\pm, \quad (4.23a)$$

$$[J^\pm, B_a^\pm] = -\varepsilon_a^b B_b^\pm, \quad (4.23b)$$

$$[S^\pm, G_a^\pm] = -\varepsilon_a^b B_b^\pm, \quad (4.23c)$$

$$[G_a^\pm, B_b^\pm] = \varepsilon_{ab} Z^\pm \quad (4.23d)$$

and

$$[G_a^\pm, G_b^\pm] = \varepsilon_{ab} S^\pm. \quad (4.23e)$$

The enhanced Bargmann–Newton–Hooke symmetry given by the commutation relations (2.10), (3.11) and (4.2) can be recovered from (4.23) by defining

$$J = J^+ + J^-, \quad (4.24a)$$

$$G_a = G_a^+ + G_a^-, \quad (4.24b)$$

$$S = S^+ + S^-, \quad (4.24c)$$

$$B_a = B_a^+ + B_a^-, \quad (4.24d)$$

$$Z = Z^+ + Z^-, \quad (4.24e)$$

$$H = \frac{1}{\ell} (J^+ - J^-), \quad (4.24f)$$

$$P_a = \frac{1}{\ell} (G_a^+ - G_a^-), \quad (4.24g)$$

$$M = \frac{1}{\ell} (S^+ - S^-), \quad (4.24h)$$

$$T_a = \frac{1}{\ell} (B_a^+ - B_a^-) \quad (4.24i)$$

$$\text{and} \quad Y = \frac{1}{\ell} (Z^+ - Z^-). \quad (4.24j)$$

The algebra (4.23) can be alternatively obtained as a finite NR expansion of the $\mathfrak{so}(2, 1)$ algebra

$$[\tilde{J}_A, \tilde{J}_B] = \epsilon_{AB}^C \tilde{J}_C, \quad (4.25)$$

by means of the semigroup $S_E^{(N)}$ for $N=4$ [8]. This, in turn, corresponds to a suitable quotient of an infinite-dimensional expansion of the Lorentz algebra in three dimensions [13]. Moreover, since $\mathfrak{so}(2, 1)$ also defines AdS_2 , we can interpret (4.23) as an expansion of the AdS algebra in 1+1 dimensions. Thus, this result generalizes the fact that the extended Bargmann–Newton–Hooke algebra in three dimensions can be written as two copies of the Bargmann–Newton–Hooke algebra in two dimensions [32,33] or, equivalently, as two copies of the Nappi–Witten algebra [49–51]. Indeed, dividing out (4.23) by the ideal generated by $\{B_a^\pm, Z^\pm\}$ leads to the commutation relations

$$[J^\pm, G_a^\pm] = -\epsilon_a^b G_b^\pm \quad \text{and} \quad [G_a^\pm, G_b^\pm] = \epsilon_{ab} S^\pm. \quad (4.26)$$

Redefining the generators as

$$P_a^\pm = \ell G_a^\pm \quad \text{and} \quad S^\pm = \ell^2 Z^\pm, \quad (4.27)$$

the commutation relations (4.26) take the form

$$[J^\pm, P_a^\pm] = -\epsilon_a^b P_b^\pm \quad \text{and} \quad [P_a^\pm, P_b^\pm] = \epsilon_{ab} Z^\pm. \quad (4.28)$$

This algebra is the universal central extension of the Euclidean algebra $\mathbb{E}_2$ and defines the Nappi–Witten algebra [52,53], which can also be identified with the Maxwell algebra in 1+1 dimensions.⁸ Yet there is another interpretation of the algebra (4.26) that follows from [32,33]. In fact, performing the redefinition

$$\bar{H}^\pm = -\ell J^\pm, \quad \bar{G}^\pm = G_1^\pm, \quad \bar{P}^\pm = \ell G_2^\pm, \quad \bar{M}^\pm = \ell S^\pm, \quad (4.29)$$

the algebra (4.26) can be written as the Bargmann–Newton–Hooke algebra in 1+1 dimensions

$$[\bar{G}^\pm, \bar{H}^\pm] = \bar{P}^\pm, \quad [\bar{P}^\pm, \bar{H}^\pm] = -\frac{1}{\ell^2} \bar{G}^\pm, \quad [\bar{G}^\pm, \bar{P}^\pm] = \bar{M}^\pm. \quad (4.30)$$

Therefore, depending on what interpretation we adopt, the chiral algebra presented in (4.23) can define either an extension of the Nappi–Witten algebra, the Newton–Hooke algebra in two dimensions or even the Maxwell algebra in two dimensions.

Consider now the action for coadjoint AdS_3 gravity (4.13) plus the $\mathfrak{u}(1)^2$ action (4.16) with global factor γ_3 . Going to the chiral basis in the $\mathfrak{u}(1)^2$ sector requires us to define the Abelian fields

$$\chi_\pm = a_1 \pm a_2. \quad (4.31)$$

This leads to the chiral relativistic action (4.13) plus the Abelian CS action

$$\gamma_3 \int \chi_\pm d\chi_\pm, \quad (4.32)$$

which follows from considering two CS theories with connections

$$A^\pm = \Omega_\pm^A \mathcal{J}_A^\pm + E_\pm^A \mathcal{P}_A^\pm + \chi_\pm \mathcal{Z}^\pm, \quad (4.33)$$

⁸In the original definition [54–56], the Nappi–Witten algebra was constructed as a central extension of the Poincaré algebra in 1+1 dimensions, which requires changing the signature of the spatial metric in (4.28). In that case, (4.28) is isomorphic to the Maxwell algebra in two space–time dimensions [57].

where the generator $\mathcal{Z}^\pm$ has been introduced in (4.21).

Defining the NR gauge fields

$$A^\pm = \omega_\pm J^\pm + \omega_\pm^a G_a^\pm + s_\pm \mathcal{S}^\pm + b_\pm^a B_a^\pm + z_\pm Z^\pm, \quad (4.34)$$

and using (4.20), the contraction (4.21) induces the following relationship between relativistic and NR fields:

$$\Omega_\pm^0 = \omega_\pm + \frac{1}{2\varepsilon^2} s_\pm - \frac{1}{2\varepsilon^4} z_\pm, \quad (4.35a)$$

$$\Omega_\pm^a = \frac{1}{\varepsilon} \omega_\pm^a - \frac{1}{\varepsilon^3} b_\pm^a, \quad (4.35b)$$

$$\chi_\pm = \omega_\pm - \frac{1}{2\varepsilon^2} s_\pm + \frac{1}{2\varepsilon^4} z_\pm, \quad (4.35c)$$

$$\frac{1}{\ell} E_\pm^0 = -\frac{1}{\varepsilon^2} s_\pm \quad (4.35d)$$

and

$$\frac{1}{\ell} E_\pm^a = -\frac{1}{2\varepsilon} \omega_\pm^a - \frac{1}{2\varepsilon^3} b_\pm^a. \quad (4.35e)$$

Substituting these expressions into the chiral action formed by (4.13) and (4.32) leads, up to boundary terms, to the following action:

$$\begin{aligned} S[A^\pm] = & \frac{\ell}{2} \int \left[(-\gamma_2 + \gamma_3) \omega_\pm d\omega_\pm + \frac{(-\gamma_1 + \gamma_2)}{\varepsilon^2} \omega_\pm^a F_a^\pm[G^\pm] + \frac{2\gamma_1 - \gamma_2 - \gamma_3}{2\varepsilon^2} (s_\pm d\omega_\pm + \omega_\pm ds_\pm) \right. \\ & - \frac{\gamma_2}{\varepsilon^4} (\omega_\pm^a F_a^\pm[B^\pm] + b_\pm^a F_a^\pm[G^\pm]) + \frac{\gamma_2 + \gamma_3}{2\varepsilon^4} (\omega_\pm dz_\pm + z_\pm d\omega_\pm) \\ & \left. + \frac{4\gamma_1 - \gamma_2 + \gamma_3}{4\varepsilon^4} s_\pm ds_\pm + \frac{3(\gamma_1 - \gamma_2)}{2\varepsilon^4} \epsilon_{ab} s_\pm \omega_\pm^a \omega_\pm^b + O(\varepsilon^{-6}) \right], \end{aligned} \quad (4.36)$$

where we have defined

$$R^{a\pm}[G^\pm] = d\omega_\pm^a + \epsilon_b^a \omega_\pm \omega_\pm^b \quad \text{and} \quad R^{a\pm}[B^\pm] = db_\pm^a + \epsilon_b^a (\omega_\pm b_\pm^b + s_\pm \omega_\pm^b). \quad (4.37)$$

Now we will consider three different choices of the parameters that lead to NR chiral actions.

(i) Abelian CS action

In the case $\gamma_2 \neq \gamma_3$, the limit $\varepsilon \rightarrow \infty$ applied to the relativistic chiral action (4.13) leads to the following simple result:

$$S_{u(1)}^\pm = -\frac{\ell\kappa}{2} \int \omega_\pm d\omega_\pm, \quad (4.38)$$

where in this case we have set $\kappa = \gamma_2 - \gamma_3$. This corresponds to an Abelian CS theory. The action for Galilean gravity (2.21) can be recovered from this action by using (4.12), i.e.

$$S_{\text{Galilei}} = S_{u(1)}^+ - S_{u(1)}^-, \quad (4.39)$$

and the relation

$$\omega_\pm = \omega \pm \frac{1}{\ell} \tau, \quad (4.40)$$

which can be deduced from the change of basis (4.24). This decomposition unveils the minimal symmetry of Galilei gravity (2.21) given by the $u(1)^2$ algebra.

(ii) Nappi–Witten CS action

The choice $\gamma_1 = 0$ and $\gamma_2 = \gamma_3 = \varepsilon^2 \kappa$ leads to the following NR chiral action:

$$S_{\text{NW}}^\pm = \frac{\ell\kappa}{2} \int [\omega_\pm^a R_a^\pm[G^\pm] - \omega_\pm ds_\pm - s_\pm d\omega_\pm]. \quad (4.41)$$

This CS action is invariant under the Nappi–Witten algebra (4.26) and has been studied from different points of view in [49–51]. The action for extended Bargmann–Newton–Hooke gravity

(2.21) follows from the relation (4.12), which in this case has the form

$$S_{\text{Extended-BNH}} = S_{\text{NW}}^+ - S_{\text{NW}}^-, \quad (4.42)$$

using the expression for the chiral fields (4.40) and

$$s_{\pm} = s \pm \frac{1}{\ell} m \quad \text{and} \quad \omega_{\pm}^a = \omega^a \pm \frac{1}{\ell} e^a. \quad (4.43)$$

We should note that the action (4.41) can also be obtained by setting $\gamma_2 = \gamma_3 = 0$, together with the rescaling $\gamma_1 = -\varepsilon^2 \kappa$, in equation (4.13). In particular, this shows that it is possible to define a finite NR limit of the Einstein–Hilbert action without the addition of Abelian fields, leading to a CS action invariant under the Nappi–Witten algebra.

(iii) Enhanced Nappi–Witten CS action

Choosing the parameters as $\gamma_1 = \gamma_2 = \gamma_3 = -\varepsilon^4 \kappa$ and taking the limit $\varepsilon \rightarrow \infty$ yields the chiral action

$$S_{\text{ENW}} = \frac{\ell \kappa}{2} \int [\omega_{\pm}^a R_a^{\pm}[B] + b_{\pm}^a R_a^{\pm}[G] - \omega_{\pm} dz_{\pm} - s_{\pm} ds_{\pm} - z_{\pm} d\omega_{\pm}], \quad (4.44)$$

which is invariant under the enhancement of the Nappi–Witten algebra defined by (4.23), and can be shown to define a CS action for that symmetry. Using the general result (4.12) and the relations (4.40), (4.43) plus

$$z_{\pm} = z \pm \frac{1}{\ell} y \quad \text{and} \quad b_{\pm}^a = b^a \pm \frac{1}{\ell} t^a, \quad (4.45)$$

we find in this case

$$S_{\text{Enhanced-BNH}} = S_{\text{ENW}}^+ - S_{\text{ENW}}^-. \quad (4.46)$$

Thus, the chiral actions (4.46) can be put together to obtain the enhanced Bargmann–Newton–Hooke gravity action (4.19).

It is important to remark that the coadjoint AdS_3 algebra admits a second invariant tensor given by⁹

$$\langle \tilde{J}_A \tilde{S}_B \rangle = \sigma_1 \eta_{AB}, \quad \langle \tilde{P}_A \tilde{T}_B \rangle = \frac{\sigma_1}{\ell^2} \eta_{AB}, \quad \langle \tilde{J}_A \tilde{J}_B \rangle = \sigma_2 \eta_{AB}, \quad \langle \tilde{P}_A \tilde{P}_B \rangle = \frac{\sigma_2}{\ell^2} \eta_{AB}, \quad (4.47)$$

which is non-degenerate for $\sigma_1 \neq 0$. This can be used to define an exotic CS action for three-dimensional coadjoint AdS gravity. In the chiral basis, this action can be obtained by considering

$$S_{\text{CS}}[A] = S[A^+] + S[A^-], \quad (4.48)$$

instead of (4.12). An NR limit of this theory can be constructed along the same lines as have been shown above. These NR actions in the standard (non-chiral) basis have been obtained by means of the Lie algebra expansion method in [13].

5. Conclusion and generalization

In this work, we investigated in which sense some of the NR gravity actions that have appeared in the recent literature could be obtained as the NR limit of a relativistic action with an enhanced Poincaré symmetry. We focused on three-dimensional actions only. To describe these enhanced Poincaré symmetries, a key role was played by the coadjoint Poincaré algebra. Specifically, we found that, for specific choices of the parameters, the CS action based on the coadjoint Poincaré algebra has two finite NR limits: one leads to Galilei gravity and the other one to extended Bargmann gravity. On the other hand, we showed that the CS action based on the coadjoint Poincaré $\oplus \mathfrak{u}(1)^2$ algebra has three NR limits determined by different choices of the parameters: Galilei gravity, extended Bargmann gravity and a third new limit that yields the NR gravity action of [24], which we denominated as enhanced Bargmann gravity.

⁹This expression can be recognized as the coadjoint version of the exotic invariant bilinear form on $\mathfrak{so}(2,2)$ given by (4.18). In the AdS_3 case, the corresponding CS action leads to the exotic variant of three-dimensional gravity studied in [19].

We were able to reproduce the NR algebra underlying the construction of [10,11] by a particular contraction of the coadjoint Poincaré algebra. However, we could not find an NR limit of a three-dimensional coadjoint Poincaré invariant gravity action that leads to an NR gravity theory based on the algebra of [10,11]. Moreover, in [10], it was observed that the same NR algebra could be obtained from a contraction of the direct sum of the Poincaré and Euclidean algebras. It would be interesting to see if there is a relationship between this direct sum and the coadjoint Poincaré algebra we have been considering in this work.

It is natural to speculate about whether the results found in this work are part of a more general construction. At several places in our work, we mentioned that the relevant algebra underlying our constructions could be obtained by a suitable quotient from the infinite-dimensional algebra (A 1) given in appendix A. One could ask oneself what are the relativistic counterparts of more general quotients of this infinite-dimensional algebra. Before doing that, it is instructive to summarize the pattern we have found so far in this paper. We observe that each time a $u(1)^2$ factor is added to the previous algebra a new NR gravity action appears. To see how this goes, we start with the Poincaré algebra that produces the Galilei gravity action.¹⁰ Extending to the Poincaré $\oplus u(1)^2$ algebra leads to the previous result of the Galilei gravity action plus the new extended Bargmann gravity action. In a next step, extending to the coadjoint Poincaré algebra reproduces the two actions we already had constructed. By contrast, a further extension to the coadjoint Poincaré $\oplus u(1)^2$ algebra leads to the previous result plus the new enhanced Bargmann gravity action. Summarizing, we have

$$\begin{aligned} \text{Poincaré} &: \text{Galilei gravity,} \\ \text{Poincaré} \oplus u(1)^2 &: \text{previous plus extended Bargmann gravity,} \\ \text{coadjoint Poincaré} &: \text{previous,} \\ \text{coadjoint Poincaré} \oplus u(1)^2 &: \text{previous plus enhanced Bargmann gravity.} \end{aligned} \quad (5.1)$$

It is interesting to speculate about how the pattern (5.1) extends to larger algebras. Indeed, as shown in [9,13,14], the infinite-dimensional algebra (A 1) can be obtained by considering a sequence of expansions of the Poincaré algebra.

In order to generate relativistic algebras beyond the coadjoint Poincaré algebra, it is instructive to note that the Poincaré algebra itself can be written as an expansion of the form

$$S_E^{(N)} \times \mathfrak{iso}(D-1, 1), \quad (5.2)$$

with $N=0$, while the coadjoint Poincaré algebra corresponds to the $N=1$ case¹¹. Thus, an expansion of this form for the $N=2$ case would triple the number of generators of the Poincaré algebra and is the natural candidate to define a relativistic counterpart of a larger truncation of the infinite-dimensional algebra (A 1).

Regarding three-dimensional gravity actions, note that this mechanism does not provide a relativistic counterpart of the truncations of (A 1) that contain the required central extensions leading to well-defined NR limits. However, as happens in the particular cases explored in this article, for $D=2+1$, these algebras can be conjectured to follow from contractions of direct products of the form

$$\left\{ S_E^{(N)} \times \mathfrak{iso}(2, 1) \right\} \oplus u(1)^2. \quad (5.3)$$

Checking the validity of the scenario sketched above would be a natural continuation of the results presented in this paper.

Finally, given the recent interest in the asymptotic symmetries of gravitational theories, we cannot resist commenting on the boundary dynamics of the NR gravities discussed in this paper. At the relativistic level, it is well known that the asymptotic symmetry of three-dimensional AdS₃ gravity is given by two copies of the Virasoro algebra [58]. In the same way, given the fact

¹⁰We note that the Galilei gravity action itself consists of two $u(1)$ gauge fields realizing a $u(1)^2$ algebra.

¹¹The $S_E^{(N)}$ semigroup has been introduced in [8] to define expansions of Lie algebras.

that the coadjoint AdS_3 algebra is isomorphic to two copies of the Poincaré algebra and that the asymptotic symmetry of three-dimensional flat gravity is given by the $\mathfrak{bms}_3$ algebra [59], it should be possible to find suitable boundary conditions for the gauge fields, such that the asymptotic symmetry of three-dimensional coadjoint AdS_3 gravity is given by the $\mathfrak{bms}_3 \oplus \mathfrak{bms}_3$ algebra. Thus, it would be interesting to investigate the fate of these relativistic asymptotic symmetries when considering NR limits. We hope to address these interesting issues in a future publication.

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Appendix A. Infinite-dimensional algebra

Our starting point is the infinite-dimensional NR expansion of the Poincaré algebra given by [11,14]

$$[B_a^{(m)}, H^{(n)}] = P_a^{(m+n)}, \quad (\text{A } 1a)$$

$$[B_a^{(m)}, P_b^{(n)}] = \delta_{ab} H^{(m+n+1)}, \quad (\text{A } 1b)$$

$$[B_a^{(m)}, B_b^{(n)}] = J_{ab}^{(m+n+1)}, \quad (\text{A } 1c)$$

$$[J_{ab}^{(m)}, J_{cd}^{(n)}] = 4\delta_{[c[b} J_{a]d]}^{(m+n)}, \quad (\text{A } 1d)$$

$$[J_{ab}^{(m)}, B_c^{(n)}] = 2\delta_{c[b} B_{a]}^{(m+n)} \quad (\text{A } 1e)$$

$$\text{and} \quad [J_{ab}^{(m)}, P_c^{(n)}] = 2\delta_{c[b} P_{a]}^{(m+n)}. \quad (\text{A } 1f)$$

The stringy generalization of this kind of infinite algebra has been considered in [12,60], while the (A)dS extension has been found in [13].

The commutation relations of the NR limit of the coadjoint Poincaré algebra (3.6) can be viewed as the quotient of this infinite-dimensional algebra by the ideal generated by

$$\{J_{ab}^{(m \geq 2)}, H^{(m \geq 2)}, B_a^{(n \geq 2)}, P_b^{(n \geq 2)}\}. \quad (\text{A } 2)$$

Furthermore, considering the quotient of the infinite-dimensional algebra (A 1) by the smaller ideal

$$\{J_{ab}^{(m \geq 2)}, H^{(m \geq 2)}, B_a^{(n \geq 1)}, P_b^{(n \geq 1)}\} \quad (\text{A } 3)$$

leads to a $D > 3$ version of the extended Bargmann algebra

$$[G_a, H] = P_a, \quad (\text{A } 4a)$$

$$[G_a, P_b] = \delta_{ab} M, \quad (\text{A } 4b)$$

$$[G_a, G_b] = S_{ab}, \quad (\text{A } 4c)$$

$$[J_{ab}, J_{cd}] = 4\delta_{[a[c}J_{d]b]}, \quad (\text{A } 4d)$$

$$[J_{ab}, S_{cd}] = 4\delta_{[a[c}S_{d]b]}, \quad (\text{A } 4e)$$

$$[J_{ab}, P_c] = 2\delta_{c[b}P_{a]} \quad (\text{A } 4f)$$

and

$$[J_{ab}, G_c] = 2\delta_{c[b}G_{a]}. \quad (\text{A } 4g)$$

In the same way, a $D > 3$ version of the enhanced Bargmann algebra defined by the commutation relations (2.10), (3.11) and (3.26) can be obtained considering the quotient by the ideal

$$\{J_{ab}^{(m \geq 3)}, H^{(m \geq 3)}, B_a^{(n \geq 2)}, P_b^{(n \geq 2)}\}, \quad (\text{A } 5)$$

which enlarges the algebra (A 4) to include the level 2 generators $H^{(2)}$ and $J_{ab}^{(2)}$. Thus, we see that the same infinite-dimensional algebra can be used to generate different finite-dimensional NR symmetries by considering quotients by suitable ideals.

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