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Convergence of linear threshold decision-making dynamics in finite heterogeneous populations[☆]

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ABSTRACT

Linear threshold models have been studied extensively in structured populations; however, less attention is paid to perception differences among the individuals of the population. To focus on this effect, we exclude structure and consider a well-mixed population of heterogeneous agents, each associated with a threshold in the form of a fixed ratio within zero and one that can be unique to this agent. The agents are initialized with a choice of strategy *A* or *B*, and at each time step, one agent becomes active to update; if the ratio of agents playing *A* is higher (resp. lower) than her threshold, she updates to *A* (resp. *B*). We show that for any given initial condition, after a finite number of time steps, the population reaches an equilibrium where no agent's threshold is violated; however, the equilibrium is not necessarily uniquely determined by the initial condition but depends on the agents' activation sequence. We find all those possible equilibria that the dynamics may reach from a given initial condition and show that, in contrast to the case of homogeneous populations, heterogeneity in the agents' thresholds gives rise to several equilibria where both *A*-playing and *B*-playing agents coexist. Perception heterogeneity can, hence, preserve decision diversity, even in the absence of population structure. We also investigate the asymptotic stability of the equilibria and show how to calculate the contagion probability for a population with two thresholds.

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1. Introduction

The spread of social innovations, technological innovations, viral infections and reforms of corporate governance, are examples of a cascading behavior where the adaption of an action by a portion of individuals makes it more likely for the action to spread to the rest of the population (Nowzari, Preciado, & Pappas, 2016). At the individual level, such a behavior can be modeled as follows: I adopt action *A* only if enough of my peers have done so. This is actually the essence of the so called *linear threshold models* originally framed by Granovetter (1978) where each individual has a *threshold* determining whether to adopt a specific action based on the current number of other individuals who have already adopted that action. On the other hand, in a different framework, *evolutionary game theory* (Altman & Hayel, 2010; Kordonis & Papavasiliopoulos, 2016; Madoe & Mocenni, 2015; Nowak, Tarnita,

& Wilson, 2010; Ramazi, Riehl, & Cao, 2018; Theodorakopoulos, Le Boudec, & Baras, 2013; Wiecek, Altman, & Hayel, 2011) postulates an equivalent mechanism to describe such cascading behavior, namely the (*myopic*) *best-response update rule* (Sandholm, 2010), resulting in the *best-response dynamics* (Arcaute, Johari, & Mannor, 2009; Cortés & Martinez, 2015; Etesami & Başar, 2015; Shamma & Arslan, 2005). In this game theoretical representation, individuals interact with each other and earn payoffs determined by their strategies (actions) and utility functions that can be unique to each individual. Then each individual updates her strategy to the one that maximizes her payoff against the average population. In particular, when the payoffs satisfy those of a *coordination game*, the best-response dynamics become exactly equivalent to the linear threshold model. Indeed, a recent study confirms that human does follow the best-response update rule in coordination games (Mäs & Nax, 2016).

Both the linear threshold model and the best-response update rule have been studied in different setups such as when the population is structured (Hauert & Doebeli, 2004; Lelarge, 2012; López-Pintado, 2006; Montanari & Saberi, 2010; Morris, 2000; Sysi-Aho, Saramäki, Kertész, & Kaski, 2005; Young, 2011), the update is noisy (Alós-Ferrer & Netzer, 2010; Alós-Ferrer & Neustadt, 2010; Kreindler & Young, 2013), etc. (Alós-Ferrer, 2003;

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Markou, 2013; Oyama, Sandholm, & Tercieux, 2015; Riehl, Ramazi, & Cao, 2018; Sandholm, 2001). Although these works reveal interesting aspects of linear threshold models, e.g., how the topology of the interaction network affects the spread of a specific action (Montanari & Saberi, 2010), almost all of them consider a homogeneous population, that is when the thresholds (resp. pay-offs) of all individuals are the same. On the other hand, heterogeneous populations do exhibit complex features under similar dynamics. For example, under best-response dynamics, individuals with different perceptions manifest cooperation sustainability (the level-off phenomenon) in anti-coordination games (equivalently anti-threshold models) (Ramazi & Cao, 2015; Ramazi, Hessel, & Cao, 2015), and exhibit perpetual cycles of length at-most-two in their actions. Also, in the experimental work (Mäs & Nax, 2016), deviations from the best-response update rule are reported to be more likely due to individual heterogeneity rather than time or space. These motivate us to investigate the role of heterogeneity in linear threshold models. Indeed, aiming to identify the mere effect of heterogeneity, we exclude any structure and consider a well-mixed heterogeneous population. We assume the agents update their decisions asynchronously since many real-life scenarios can be modeled only by asynchronous decisions, e.g., deciding on which smartphone to buy, movie to watch, and field of research to study. Moreover, the action distribution over individuals evolves deterministically under synchronous updates, whereas under asynchronous updates, it typically evolves stochastically based on the order the individuals update, possibly leading to a more complex behavior.

More specifically, we consider a finite well-mixed population of decision-making individuals having different thresholds. The individuals choose between two options A and B , and update asynchronously based on the linear threshold model. At each time step, an individual becomes active to revise her action based on her threshold and the ratio of A -playing individuals in the population, resulting in population dynamics. We take the distribution of the A -playing individuals among the agents with different thresholds as the state of the system. We show that regardless of the initial state and the activation sequence, the dynamics reach an equilibrium state where one individual becomes a benchmark such that all others with higher tendencies to play A , play A , and all with lower tendencies play B . This distribution of agents' strategies at the equilibrium is similar to that with the anti-coordination best-response dynamics (Ramazi & Cao, 2017). Several results have not been reported in the literature before and are the main contributions of this paper. First, in sharp contrast to the anti-coordination or synchronous setup, the population dynamics may exhibit several possible long-term behaviors for the same initial condition, that is, for the same initial state, the final equilibrium state may vary depending on the activation sequence of the agents. Second, in contrast to homogeneous setups, the dynamics may allow several equilibria where both A -playing and B -playing agents coexist. Therefore, decision diversity can, indeed, be preserved in the long-run, even in a well-mixed population; a feature that is impossible in homogeneous populations. This highlights the key role of (perception) heterogeneity of the individuals in linear threshold models. Third, we investigate the stability of the equilibria and show that clean-cut equilibrium states where individuals with a particular threshold either all play A or all play B , can be asymptotically stable under some conditions on the distribution of the thresholds, yet the other type of equilibrium states, which we refer to as ruffled, are unstable. Fourth, we address the challenging acknowledged problem of finding the contagion probability, that is, the probability of reaching the state where every individual plays A . We show how small increments in the initial number of A -players tremendously enhance the chances of contagion in a population comprising two types of individuals.

We also want to position our contribution in the context of acknowledged results in the highly related fields of *potential games* and *network games*. Our model may be shown to be an ordinal potential game (Monderer & Shapley, 1996; Voornveld, 2000), for which the individuals' actions distribution is known to equilibrate. It is also similar to a network (coordination) game (Panageas & Piliouras, 2016; Ramazi, Riehl, & Cao, 2016), again known to equilibrate, but with self-loops. So the equilibrium convergence result in this paper may not be completely new. However, the exact form of the equilibria, to which the dynamics converge, is not known in either of the two fields. Moreover, both fields use potential functions to prove convergence, whereas we leverage the structure of the equilibria to construct an inductive argument for the proof. This is an insightful method that can possibly be applied to other population dynamics. However, to be in line with the common approach in the literature, we also introduce a new potential function, based on which we provide a second proof for equilibrium convergence and find an upper bound on the convergence time. Finally, our stability analysis clearly stands out from relevant literature in those fields.

2. Linear threshold models

Consider a well-mixed population of n agents who choose one of the options A or B over a time sequence $t = 0, 1, \dots$. Each agent $i \in \{1, \dots, n\}$ has a time-invariant threshold $\tau_i \in (0, 1)$ and her choice (decision) at time t is denoted by $d_i(t)$. At every time step t , an agent i becomes active to revise her choice at $t + 1$. The linear threshold model dictates that agent i chooses A , if and only if a portion greater than her threshold has already chosen A . The choice of B is made if and only if in the population, a portion less than her threshold chooses A . In case when the portion exactly equals the threshold of agent i , she continues with her previous choice, allowing choice robustness for the agents. Namely, the update rule governing agent i 's choice can be written as

$$d_i(t + 1) = \begin{cases} A & x^A(t) > \tau_i \\ d_i(t) & x^A(t) = \tau_i \\ B & x^A(t) < \tau_i \end{cases} \quad (1)$$

where $x^A(t)$ denotes the ratio of agents in the population choosing A at t .

Different from the existing study on homogeneous populations where the evolution of x^A is the subject of interest, in this paper for heterogeneous populations, we consider the challenging problem of how the distribution of the choices over agents with different thresholds changes over time. This can be captured by classifying the heterogeneous agents into different types according to their thresholds. Each agent's threshold lies in one of the intervals $(0, \frac{1}{n}]$, $(\frac{1}{n}, \frac{2}{n}]$, $\dots$, $(\frac{n-1}{n}, 1]$. Denote those intervals that cover at least one agent's threshold by $(\frac{n_l^*-1}{n}, \frac{n_l^*}{n}]$, $\dots$, $(\frac{n_l^*-1}{n}, \frac{n_l^*}{n}]$ where

$$n_j^* < n_{j+1}^* \quad j = 1, \dots, l - 1. \quad (2)$$

Let $\mathcal{L} = \{1, \dots, l\}$. The equality case in (1) never takes place for an agent whose threshold lies in an open interval $(\frac{n_j^*-1}{n}, \frac{n_j^*}{n})$, $j \in \mathcal{L}$, whom we call a *type j agent*, but may take place for an agent whose threshold equals one of the ratios $\frac{n_j^*}{n}$, $j \in \mathcal{L}$, whom we call a *type j^* agent* (see Lemma 9 in the Appendix for how the two types of agents update their choices). This leads to in total $2l$ different possible types of agents. We refer to n_j^* as the *temper* of both type j and j^* agents. Let n_j and $\tilde{n}_j$ denote the number of type j and type j^* agents. The heterogeneity of the population is then characterized by the $2l$ -dimensional vector $(n_1, \tilde{n}_1, \dots, n_l, \tilde{n}_l)$. Let

$n_j^A(t)$ and $\tilde{n}_j^A(t)$ denote the number of type j and type j^* agents whose choices are A at time t . We stack all n_j^A and $\tilde{n}_j^A$ together to get the *population state*

$$x(t) \triangleq (n_1^A(t), \tilde{n}_1^A(t), \dots, n_l^A(t), \tilde{n}_l^A(t)),$$

which lies in the state space

$$\mathcal{X} \triangleq \{(a_1, b_1, \dots, a_l, b_l) \mid a_i, b_i \in \mathbb{Z}_{\geq 0}, \\ a_i \leq n_i, b_i \leq \tilde{n}_i, i = 1, \dots, l\}.$$

Then the total number of agents whose choices are A at time t , denoted by $n^A(t)$, is the sum of all entries in $x(t)$.

The activation mechanism together with update rule (1) governs the dynamics of $x(t)$, which we refer to as the *population dynamics*. The agents update asynchronously, meaning that only one agent at a time gets to revise her strategy. Indeed, each agent can be thought of as having a clock that upon ticking, the agent becomes active and updates her strategy. Now if the clocks are synched (tick at the same time), all agents become active at the same time and update their strategies synchronously. If their clocks are not synched, they become active at different times, and hence, update their strategies asynchronously. We highlight that the agents' activation sequence is not necessarily random. So one may not assume that any arbitrary subsequence of the agents shows up in the activation sequence with probability one. We only require the activation sequence to be *persistent*, that is, every agent becomes active for infinitely many times (Ramazi et al., 2016). This mild assumption guarantees every stationary state to be an equilibrium, because every agent at the stationary state gets the chance to become active, and hence, would switch strategies if the state is not an equilibrium. The main goal of this paper is to determine the asymptotic behavior of $x(t)$ from any given initial condition $x(0)$ and any activation sequence.

3. Equilibrium states

A state $x^* \in \mathcal{X}$ is an *equilibrium state* (of the population dynamics), if whenever the state $x(t)$ equals x^* , it remains there afterwards, regardless of the activation sequence. Our goal is to find the set of all equilibria, denoted by $\mathcal{X}^* \subseteq \mathcal{X}$. For $i = 0, 1, \dots, l$, define the $2l$ -dimensional row vectors

$$s_i \triangleq (n_1, \tilde{n}_1, \dots, n_i, \tilde{n}_i, \underbrace{0, \dots, 0}_{2l-2i}), \\ s_i^* \triangleq (n_1, \tilde{n}_1, \dots, n_i, n_i^* - \sum_{j=1}^{i-1} (n_j + \tilde{n}_j) - n_i, \underbrace{0, \dots, 0}_{2l-2i}),$$

where $s_0 = s_0^* = \mathbf{0}$, the $1 \times 2l$ zero vector. Similar to the definitions in Ramazi and Cao (2017), we refer to the vectors s_r and s_r^* as *clean-cut* and *ruffled*, respectively. All these vectors satisfy two necessary conditions for being an equilibrium state. First, from the update rule (1), we know that an agent plays A (resp. B) at an equilibrium if other agents with a lower (resp. higher) threshold also play A (resp. B). Second, it follows from Lemma 9 that at an equilibrium, for every $j \in \mathcal{L}$, either n^A is no less than n_j^* , and hence, all type j agents play A , or n^A is less than n_j^* , and hence, all type j agents play B . Therefore, s_i and s_i^* , $i = 1, 2, \dots, l$, are candidates for being equilibrium states. We now show that they are the only candidates.

Lemma 1. $\mathcal{X}^* \subseteq \{s_i, s_i^*\}_{i=0}^l$.

Proof. Consider an equilibrium state $x^* \in \mathcal{X}^*$. If $x^* = s_0 = s_0^*$, the result is trivial; otherwise, in view of Lemma 11, $\max\{j \in \mathcal{L} \mid n_j^A =$

$n_j\}$ exists, which we denote by i . Then according to Lemmas 10 and 11, x^* has to take the following form

$$x^* = (n_1, \tilde{n}_1, \dots, n_{i-1}, \tilde{n}_{i-1}, n_i, \tilde{n}_i^A, 0, 0, \dots, 0).$$

Now the equality $n_i^A = n_i$ implies that all type i agents have chosen A , yielding $n^A \geq n_i^*$ in view of Lemma 9. If $n^A > n_i^*$, then Lemma 9 implies that all type i^* agents have chosen A as well, resulting in $x^* = s_i$. If on the other hand, $n^A = n_i^*$, then Lemma 9 implies that the number of type i^* agents who have chosen A are such that the total number of A -players becomes n_i^* , i.e., $\tilde{n}_i^A = n_i^* - \sum_{j=1}^{i-1} (n_j + \tilde{n}_j) - n_i$, resulting in $x^* = s_i^*$. Hence, in either case, $x^* \in \{s_i, s_i^*\}_{i=0}^l$. $\square$

Now we determine those indices of s_i and s_i^* corresponding to the actual equilibrium states by using the sets $\mathcal{E} \subseteq \{0, \dots, l\}$ defined by

$$\mathcal{E} \triangleq \{0, l\} \cup \left\{ i \in \{1, \dots, l-1\} \mid n_i^* \leq \sum_{j=1}^i (n_j + \tilde{n}_j) < n_{i+1}^* \right\} \quad (3)$$

and consequently $\mathcal{E}^* \subseteq \mathcal{L}$ defined by

$$\mathcal{E}^* \triangleq \left\{ i \in \mathcal{L} \mid 0 \leq n_i^* - \sum_{j=1}^{i-1} (n_j + \tilde{n}_j) - n_i < \tilde{n}_i \right\}, \quad (4)$$

where the right hand-side inequality, resulting in $[s_i^*]_{2i} < \tilde{n}_i$, is to distinguish s_i^* from s_i for $i \in \mathcal{E}^*$. Together with the left hand-side inequality that implies $0 \leq [s_i^*]_{2i}$, this guarantees s_i^* , $i \in \mathcal{E}^*$, to belong to the state space $\mathcal{X}$. Now we show that these states are equilibrium states.

Lemma 2. $\{s_r\}_{r \in \mathcal{E}} \cup \{s_r^*\}_{r \in \mathcal{E}^*} \subseteq \mathcal{X}^*$.

Proof. First, we show $\{s_r\}_{r \in \mathcal{E}} \subseteq \mathcal{X}^*$. It can be easily verified that s_0 and s_l are equilibrium states. So let $r \in \mathcal{E} - \{0, l\}$. It suffices to show that under any activation sequence, $x(0) = s_r \Rightarrow x(1) = s_r$. We first observe that given the initial state $x(0) = s_r$, according to the definition of s_r and in view of (3),

$$n^A(0) = \sum_{j=1}^r (n_j + \tilde{n}_j) \geq n_r^* \xrightarrow{(2)} n^A(0) \geq n_i^* \quad \forall i \leq r.$$

Hence, from Lemma 9, for $i = 1, \dots, r$, if a type i or i^* agent is active at $t = 0$, she either chooses A or keeps her choice for time $t = 1$. On the other hand, according to the structure of s_r , the choices of all these agents are already A at $t = 0$. Hence, none of them change their choices at $t = 1$. The same can be shown for all type i and i^* agents, $i = r+1, \dots, l$, proving that s_r is an equilibrium state.

Next, we show $\{s_r^*\}_{r \in \mathcal{E}^*} \subseteq \mathcal{X}^*$. It suffices to show that if $x(0) = s_r^*$, then under any activation sequence, $x(1) = s_r^*$. We first observe that when $x(0) = s_r^*$, according to the definition of s_r^* , $n^A(0) = n_r^*$. Hence, from Lemma 9, if a type r^* agent is active at $t = 0$, she keeps her choice for time $t = 1$, and if a type r agent is active at $t = 0$, she chooses A , which is the same as what she has already chosen at $t = 0$. Hence, none of type r and r^* agents will change their choices at $t = 1$. The same can be shown similarly for the remaining types. $\square$

Lemma 2 implies that the population under study may have several equilibrium states depending on the frequency of the types or equivalently distribution of the thresholds among the agents. Now we show that clean-cut and ruffled equilibria are the only equilibrium states of the system.

Lemma 3. $(\{s_r, s_r^*\}_{r=0}^l - \{s_r\}_{r \in \mathcal{E}} \cup \{s_r^*\}_{r \in \mathcal{E}^*}) \cap \mathcal{X}^* = \emptyset$.

Proof. Let $r \notin \mathcal{E}$, and consider the initial state $x(0) = s_r$. By definition, $n^A(0) = \sum_{j=1}^r (n_j + \tilde{n}_j)$. Hence, in view of (3), $r \notin \mathcal{E}$ implies that one of the following holds: (i) $n^A(0) < n_r^*$ or (ii) $n^A(0) \geq n_{r+1}^*$. Should Case (i) be in force, s_r is not an equilibrium since a type r agent changes her choice from A to B upon activation. Similarly, a type $r+1$ agent will change her choice upon activation if Case (ii) is in force. So s_r is not an equilibrium.

Now let $r \notin \mathcal{E}^*$. According to (4), this yields $\sum_{j=1}^{r-1} (n_j + \tilde{n}_j) + n_r > n_r^*$ or $n_r^* - \sum_{j=1}^{r-1} (n_j + \tilde{n}_j) - n_r = \tilde{n}_r$. The first results in $[s_r^*]_{2r} < 0$, making s_r^* not even a state. The second implies $s_r^* = s_r$, which is investigated in the above. $\square$

The following theorem fully characterizes $\mathcal{X}^*$ and is the main result of this section.

Theorem 1. $\mathcal{X}^* = \{s_r\}_{r \in \mathcal{E}} \cup \{s_r^*\}_{r \in \mathcal{E}^*}$.

Proof. Lemmas 1 and 3 result in $\mathcal{X}^* \subseteq \{s_r\}_{r \in \mathcal{E}} \cup \{s_r^*\}_{r \in \mathcal{E}^*}$. The proof then follows Lemma 2. $\square$

Knowing the equilibrium states, now we are ready to proceed to the long-term behavior of the population dynamics.

4. Convergence analysis

We determine the asymptotic behavior of the state $x(t)$ for any given initial condition $x(0)$.

4.1. Equilibrium convergence

As one of the main results of this paper, we show that the population state reaches one of the clean-cut or ruffled equilibria in finite time.

Proposition 1. *Given any initial condition $x(0)$, there exist some time T and some equilibrium state $x^* \in \mathcal{X}^*$ such that*

$$x(t) = x^* \quad \forall t \geq T.$$

We take an inductive approach for the proof and show via the following lemma that the population eventually takes the form of the states $\{s_i, s_i^*\}_0^l$. Given some $r \in \mathcal{L}$, suppose all type $1, 1^*, \dots, r, r^*$ agents have fixed their strategies to A at some point. Intuitively, if afterwards, a type $r+1$ agent switches her strategy to B , then the total number of A -players will never increase. Hence, all type $(r+1), (r+1)^*, \dots, l, l^*$ agents will eventually fix their strategies to B , resulting in the state s_r . On the other hand, if no type $r+1$ agent switches her strategy to B , then in the long run, either all type $(r+1)$ and $(r+1)^*$ agents fix their strategies to A (and hence, the same inductive argument can be repeated for $r+2$), or all type $(r+1)$ and some type $(r+1)^*$ agents fix their strategies to A , resulting in a state that can be shown to be s_{r+1}^* .

Lemma 4. *For each $p \in \mathcal{L}$, there exists some time t_p such that one of the following takes place for all $t \geq t_p$:*

- (1) for $i = 1, \dots, p$, $n_i^A(t) = n_i$ and $\tilde{n}_i^A(t) = \tilde{n}_i$;
- (2) $x(t) = s_p^*$;
- (3) for $i = p, \dots, l$, $n_i^A(t) = \tilde{n}_i^A(t) = 0$.

Proof. We prove by induction. First we need to show the result for $p = 1$; however, the proof for this case is similar to the general case $p = r \geq 1$. So we proceed as follows. Assume that one of the three cases holds for some $p = r \in \{1, \dots, l-1\}$. It

is straightforward to prove that the lemma then also holds for $p = r+1$ if either of Case 2 or 3 holds for $p = r$ since they already imply that for $i = r+1, \dots, l$,

$$n_i^A(t) = \tilde{n}_i^A(t) = 0 \quad \forall t \geq t_r, \quad (5)$$

which matches the third case of the lemma for $p = r+1$ and $t_{r+1} = t_r$. So consider the situation when Case 1 holds for $p = r$, i.e.,

$$n_i^A(t) = n_i, \quad \tilde{n}_i^A(t) = \tilde{n}_i \quad \forall t \geq t_r \quad \forall i \leq r. \quad (6)$$

In general, one of the following two scenarios happens:

Scenario A: a type $r+1$ agent switches to B at some time T_1+1 , where $T_1 \geq t_r$. Hence, in view of Lemma 9,

$$n^A(T_1) < n_{r+1}^* \stackrel{(2)}{\Rightarrow} n^A(T_1) < n_i^* \quad \forall i \geq r+1.$$

Now we use contradiction to prove the following:

$$n^A(t) < n_i^* \quad \forall t \geq T_1 \quad \forall i \geq r+1. \quad (7)$$

Assume on the contrary that there exists some time when n^A becomes no less than n_i^* for some $i \geq r+1$. Let $T_2 \geq T_1$ denote the first time this happens, i.e.,

$$n^A(t) < n_i^* \quad \forall t \leq T_2 - 1 \quad \forall i \geq r+1, \quad (8)$$

and

$$\exists q \in \{r+1, \dots, l\} : n^A(T_2) = n_q^*.$$

Hence, an agent has changed her choice to A at T_2 . However, none of the type i or i^* agent, $i \geq r+1$, can do so according to (8) and Lemma 9. The same holds for $i \leq r$ in view of (6), a contradiction, implying that (7) holds. Therefore, in view of Lemma 9 and due to the persistent activation assumption, there exists some time $t_{r+1} \geq T_1$ such that

$$n_i^A(t) = \tilde{n}_i^A(t) = 0 \quad \forall t \geq t_{r+1} \quad \forall i \geq r+1.$$

This matches Case 3 of the lemma for $p = r+1$. Hence, the induction statement holds for this scenario.

Scenario B: No type $r+1$ agent switches to B after t_r . If all type $r+1$ agents are already playing B at t_r and never switch to A afterwards, then $n^A(t) < n_{r+1}^*$ for all $t \geq t_r$. Therefore, similar to Scenario A, it can be verified that Case 3 is in force. So consider the case where there is a type $r+1$ agent playing A at some time after t_r , who will not switch to B due to the main assumption of this scenario.

Now, we show by contradiction that

$$n^A(t) \geq n_{r+1}^* \quad \forall t \geq t_r. \quad (9)$$

Assume on the contrary that there exists some time $T_1 \geq t_r$ such that $n^A(T_1) < n_{r+1}^*$. By induction it can be shown that $n^A(t) < n_{r+1}^*$ for all $t \geq T_1$. So when an A -playing type $r+1$ agent is active after T_1 , she updates to B , a contradiction, proving (9). Note that such an agent exists according to the argument made in the beginning of this scenario.

Hence, according to Lemma 9, there exists some time $T_2 \geq t_r$, by which all type $r+1$ agents have updated their choices to A and do not change afterwards, i.e.,

$$n_{r+1}^A(t) = n_{r+1} \quad \forall t \geq T_2. \quad (10)$$

On the other hand, for the type $(r+1)^*$ agents, (9) and Lemma 9 imply that whenever a type $(r+1)^*$ agent is active after t_r , she does not switch to B . So since the number of $(r+1)^*$ agents are finite, there exists some time $T_3 \geq T_2$ after which no $(r+1)^*$ agent switches her strategy. Then the number of A -playing agents of type $(r+1)^*$ does not change after T_3 , resulting in one of the following cases:

Scenario B-1: there exists some time $T_3 \geq T_2$ such that

$$\tilde{n}_{r+1}^A(t) = \tilde{n}_{r+1} \quad \forall t \geq T_3.$$

Take $t_{r+1} = T_3$. Then according to (10) and (6), we arrive at Case 1 of the lemma for $p = r + 1$. Hence, the induction statement holds for this part of Scenario B.

Scenario B-2: there exists some time $T_3 \geq T_2$ such that

$$\tilde{n}_{r+1}^A(t) = c < \tilde{n}_{r+1} \quad \forall t \geq T_3, \quad (11)$$

where $c \in \mathbb{Z}_{\geq 0}$ is constant. First we look at type i and i^* agents, $i = r + 2, \dots, l$. If $n^A(t) \geq n_{r+2}^*$ for all $t \geq T_3$, then in view of (2), $n^A(t) > n_{r+1}^*$ for all $t \geq T_3$. Hence, due to the persistent activation assumption, there exists some time $t_s \geq T_3$ such that $\tilde{n}_{r+1}^A(t_s) = \tilde{n}_{r+1}$, which is impossible due to (11). So there exists some time $T_4 \geq T_3$, such that $n^A(T_4) < n_{r+2}^*$. Then similar to how (7) was proven, via contradiction it can be shown that

$$n^A(t) < n_{r+2}^* \quad \forall t \geq T_4.$$

Hence, in view of Lemma 9, there exists some time $T_5 \geq T_4$ such that

$$n_i^A(t) = \tilde{n}_i^A(t) = 0 \quad \forall t \geq T_5 \quad \forall i \geq r + 2. \quad (12)$$

Now, we can conclude that no agent switches strategies after T_5 according to (6), (10), (11), and (12). So $x(t)$ must be at an equilibrium. On the other hand, the only equilibrium satisfying (6), (10), (11), and (12) is s_{r+1}^* , which matches Case 2 in the lemma for $p = r + 1$.

So the induction statement holds for $r + 1$ in both Scenarios B-1 and B-2. Since it also holds in Scenario A, the proof is complete. $\square$

Proof of Proposition 1. The proof follows from Lemma 4. If Case 1 in Lemma 4 does not take place for $p = 1$, then either Case 2 or Case 3 happens, resulting in $x(t) = s_1^*$ or $x(t) = s_0$, respectively, for all $t \geq t_1$. So consider the situation when Case 1 takes place for $p = 1$. Let r denote the greatest $p \in \mathcal{L}$, for which Case 1 takes place. If $r = l$, then $x(t) = s_l$ for $t \geq t_l$. Otherwise, for $p = r + 1$, either Case 2 or Case 3 takes place, resulting in $x(t) = s_{r+1}^*$ or $x(t) = s_r$, respectively, for $t \geq t_r$. To sum up, there exists some time T and some final state $x^* \in \{s_i\}_{i=1}^l \cup \{s_i^*\}_{i=1}^l$ such that $x(t) = x^*$ for all $t \geq T$. On the other hand, every final state is an equilibrium due to the persistent assumption on the activation sequence. The proof is then complete in view of Lemma 3. $\square$

4.2. Convergence time

Now, we proceed to finding an upper bound for the finite convergence time T mentioned in Proposition 1. Although revealing some key properties of the population dynamics, the inductive approach used to prove convergence does not readily provide the tools for achieving this goal. Instead, and inspired by Ramazi et al. (2016), we provide a potential function that decreases whenever an agent switches strategies. Given an agent $i \in \{1, \dots, n\}$, let $\eta_i = n\tau_i$. Define the function $\phi : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$ as $\phi(t) = \sum_{i=1}^n \phi_i(t)$, where

$$\phi_i(t) = \begin{cases} \lceil \eta_i \rceil - n^A(t) & \text{if agent } i \text{ plays } A \\ -\lfloor \eta_i \rfloor & \text{if agent } i \text{ plays } B \end{cases}.$$

This results in the candidate potential function

$$\phi(t) = -(n^A(t))^2 + \sum_{i: d_i(t)=A} \lceil \eta_i \rceil - \sum_{i: d_i(t)=B} \lfloor \eta_i \rfloor. \quad (13)$$

Lemma 5. If an agent switches strategies at time $t \geq 0$, then $\phi(t + 1) - \phi(t) \leq -2$.

Proof. Let $i \in \{1, \dots, n\}$ denote the agent that switches strategies at time t . Either of the following two cases takes place.

Case 1: agent i switches from A to B at time t . Then

$$\phi_i(t + 1) - \phi_i(t) = n^A(t) - \lfloor \eta_i \rfloor - \lceil \eta_i \rceil.$$

Moreover, every agent $j \neq i$ who plays A at time t will continue doing so at time $t + 1$, yielding

$$\phi_j(t + 1) - \phi_j(t) = (\lceil \eta_j \rceil - n^A(t + 1)) - (\lceil \eta_j \rceil - n^A(t)) = 1.$$

Similarly, for an agent j who plays B at time t we have

$$\phi_j(t + 1) - \phi_j(t) = -\lfloor \eta_j \rfloor + \lfloor \eta_j \rfloor = 0.$$

Therefore,

$$\begin{aligned} \phi(t + 1) - \phi(t) &= \phi_i(t + 1) - \phi_i(t) + \sum_{j \neq i} (\phi_j(t + 1) - \phi_j(t)) \\ &= 2n^A(t) - \lfloor \eta_i \rfloor - \lceil \eta_i \rceil - 1. \end{aligned} \quad (14)$$

Because agent i switches from A to B , it must hold that $n^A(t) < \eta_i$. So if $\eta_i \in \mathbb{N}$, then $n^A(t) \leq \eta_i - 1$. Thus, in view of (14), we have

$$\phi(t + 1) - \phi(t) \leq 2(\eta_i - 1) - 2\eta_i - 1 = -3.$$

If $\eta_i \notin \mathbb{N}$, then $n^A(t) \leq \lfloor \eta_i \rfloor$, resulting in

$$\phi(t + 1) - \phi(t) \leq 2\lfloor \eta_i \rfloor - 2\lfloor \eta_i \rfloor - 1 - 1 = -2.$$

So $\phi(t + 1) - \phi(t) \leq -2$ always holds in this case.

Case 2: agent i switches from B to A at time t . This case can be proven similarly. $\square$

Thus, each time an agent switches strategies, the potential function decreases by at least -2 . So ϕ is monotonically decreasing with respect to the switches. Now, by finding upper and lower-bounds on ϕ , we establish an upper-bound for the number of switches needed for the population to reach an equilibrium.

Theorem 2. After at most $\lceil 5n^2/8 \rceil$ switches, the population reaches an equilibrium.

Proof. Since $\eta_i \in [0, n]$, we have $0 \leq \lceil \eta_i \rceil \leq n$ and $-n \leq -\lfloor \eta_i \rfloor \leq 0$. So in view of (13), the following holds

$$\phi(k) \leq n^A(k)(n - n^A(k)) \leq n^2/4,$$

$$\phi(k) \geq -n^A(k)^2 - (n - n^A(k))n \geq -n^2.$$

Therefore, $\phi(k)$ is at most $n^2/4$, in view of Lemma 5 decreases by at least 2 units per switch, and does not get smaller than $-n^2$. Hence, after at most $\lceil (n^2/4 + n^2)/2 \rceil = \lceil 5n^2/8 \rceil$ switches, the population converges. $\square$

In addition to providing a bound on the convergence time, the potential function results in a new proof for equilibrium convergence. Note that none of these results could be obtained from Ramazi et al. (2016) as the agents here update based on the total number of A -players in the population, including themselves. This is equivalent to having self-loops in the interaction network in Ramazi et al. (2016), which makes the analysis infeasible. Indeed, for anti-coordination games, we have already shown in Ramazi and Cao (2017), that the dynamics do not always reach an equilibrium, implying that the convergence arguments in Ramazi et al. (2016) do not generally apply to networks with self-loops.

4.3. Reachable equilibria from a given initial state

Proposition 1 ensures the population to reach an equilibrium state. However, not all equilibria are reachable from every initial state. Consider an agent of type $r \in \mathcal{L}$. What condition on the

initial state guarantees this agent to play A upon activation at any time in the future? The initial number of A -players exceeding her temper, i.e., $n^A(0) \geq n_r^*$, does not guarantee this, because some other agents with a lower threshold that initially play A might have been active earlier than this agent and switched to B , resulting in n^A becoming less than n_r^* . However, if the initial number of A -players of types $1, 1^*, \dots, r, r^*$ exceeds n_r^* , none of them will ever switch to B , keeping the total number of A -players always greater than n_r^* , regardless of what the other agents do. Denote the maximum of r by k_0 that is formally defined by

$$k_0 \triangleq \max \left\{ r \in \mathcal{L} \cup \{0\} \mid \sum_{i=1}^r n_i^A(0) + \tilde{n}_i^A(0) > n_r^* \right\},$$

where $n_0^* = -1$ and $\sum_{i=a}^b$ is defined to be zero if $a > b$. So we expect the population to eventually reach a state in the form of

$$(n_1, \tilde{n}_1, \dots, n_{k_0}, \tilde{n}_{k_0}, *, \dots, *), \quad (15)$$

where all type j and j^* agents, $j \leq k_0$, have fixed their choices to A . Now, if the total number of these agents exceeds the temper of type $k_0 + 1$, then again we can conclude that eventually all agents of type $k_0 + 1$ and $k_0^* + 1$ fix their choices to A . This iterative process finally results in a maximum type k_1 such that all type $1, 1^*, \dots, k_1, k_1^*$ agents will eventually fix their choices to A :

$$k_1 \triangleq \begin{cases} k_0 & \text{if } \sum_{i=1}^{k_0} n_i + \tilde{n}_i \leq n_{k_0+1}^* \\ w & \text{otherwise} \end{cases} \quad (16)$$

where w is defined to be

$$\max \left\{ s \in \mathcal{L} \mid \sum_{i=1}^s n_i + \tilde{n}_i > n_{s+1}^* \forall j \in \{k_0, \dots, s-1\} \right\}.$$

The following proposition confirms our expectation that all type $1, 1^*, \dots, k_1, k_1^*$ agents will eventually fix their choices to A .

Proposition 2. *If for a given initial condition $x(0)$, $k_1 \geq 1$, then there exists some time $T \geq 0$ such that for $i = 1, \dots, k_1$,*

$$n_i^A(t) = n_i, \quad \tilde{n}_i^A(t) = \tilde{n}_i \quad \forall t \geq T. \quad (17)$$

Proof. First we prove (17) for $i = 1, \dots, k_0$. For this, we first show by contradiction that for $i = 1, \dots, k_0$,

$$n_i^A(t) \geq n_i^A(0), \quad \tilde{n}_i^A(t) \geq \tilde{n}_i^A(0) \quad \forall t \geq 0. \quad (18)$$

Assume on the contrary, there exists some time $t_1 > 0$ such that at least one of the inequalities in (18) fails. Let $t_2 \leq t_1$ be the first time this happens, namely, first of all, for $i = 1, \dots, k_0$:

$$n_i^A(t) \geq n_i^A(0), \quad \tilde{n}_i^A(t) \geq \tilde{n}_i^A(0) \quad \forall t < t_2, \quad (19)$$

and second, there exists some $q \in \{1, \dots, k_0\}$ such that $n_q^A(t_2) < n_q^A(0)$ or $\tilde{n}_q^A(t_2) < \tilde{n}_q^A(0)$. Hence, a type q or q^* agent is active at $t_2 - 1$ and changes her choice from A to B at t_2 . On the other hand,

$$\begin{aligned} n^A(t_2 - 1) &\geq \sum_{j=1}^{k_0} n_j^A(t_2 - 1) + \tilde{n}_j^A(t_2 - 1) \\ &\stackrel{(19)}{\geq} \sum_{j=1}^{k_0} n_j^A(0) + \tilde{n}_j^A(0) > n_{k_0}^* \end{aligned}$$

where the last inequality follows from the definition of k_0 . Hence, in view of (2), $n^A(t_2 - 1) > n_{k_0}^*$. So according to Lemma 9, the active agent at $t_2 - 1$ chooses A , not B , a contrary, implying that (18) is in force.

Next, observe that for all $t \geq 0$,

$$\begin{aligned} n^A(t) &\geq \sum_{j=1}^{k_0} n_j^A(t) + \tilde{n}_j^A(t) \stackrel{(18)}{\geq} \sum_{j=1}^{k_0} n_j^A(0) + \tilde{n}_j^A(0) > n_{k_0}^* \\ &\stackrel{(2)}{\implies} n^A(t) > n_i^* \quad \forall i \leq k_0. \end{aligned}$$

Hence, in view of Lemma 9, for $i = 1, \dots, k_0$, whenever a type i or i^* agent is active, she chooses A . On the other hand, due to the persistent activation assumption, there exists some time $t_{k_0} \geq 0$ such that all type i and i^* agents are active at least once before t_{k_0} , and hence, choose A before t_{k_0} . Therefore, by taking $T = t_{k_0}$, we have proven (17) for $i = 1, \dots, k_0$.

Now if $k_1 = k_0$, the proof of the proposition is complete, so consider the case when $k_1 > k_0$ (the case $k_1 < k_0$ does not take place by the definition of k_1). We use strong induction to show (17) for $i = k_0, k_0 + 1, \dots, k_1$. Eq. (17) was already shown for $i = 1, \dots, k_0$. So assume that it holds for $i = 1, \dots, r$ for some $r \in \{k_0, \dots, k_1 - 1\}$. Namely, there exists some time t_r such that (17) is in force for $T = t_r$ all $i = 1, \dots, r$. Then

$$n^A(t) \geq \sum_{j=1}^r n_j^A(t) + \tilde{n}_j^A(t) = \sum_{j=1}^r n_j + \tilde{n}_j \quad \forall t \geq t_r. \quad (20)$$

On the other hand, from (16), $k_1 > k_0$ implies $k_1 = w$. So $r \in \{k_0, \dots, w - 1\}$. Therefore, based on the definition of w ,

$$\sum_{j=1}^r n_j^A(t) + \tilde{n}_j^A(t) > n_{r+1}^* \stackrel{(20)}{\implies} n^A(t) > n_{r+1}^* \quad \forall t \geq t_r.$$

Then similar to what was shown above for $i = 1, \dots, k_0$, the induction statement can be proven for $i = r + 1$, which completes the proof. $\square$

When the population reaches a state in the form of

$$(n_1, \tilde{n}_1, \dots, n_{k_1}, \tilde{n}_{k_1}, *, \dots, *),$$

still some other types may choose A , but this time it depends on the activation sequence. For example, if the population of all type $1, 1^*, \dots, k_1, k_1^*$ agents, who fix their strategies to A at some time T , plus others who initially play A exceeds the temper of type $k_1 + 1$ agents, i.e., $n_{k_1+1}^*$, then all type $k_1 + 1$ and $(k_1 + 1)^*$ agents may also fix their strategies to A if they update after T and before any of those initially playing A become active and switch to B . The same argument can be made for $k_1 + 2, k_1 + 3, \dots$, resulting in some maximum type k_l such that all type $(k_1 + 1), (k_1 + 1)^*, \dots, k_l, k_l^*$ agents may or may not eventually fix their strategies to A :

$$k_l \triangleq \begin{cases} l & \text{if } k_1 = l \\ w_l & \text{otherwise,} \end{cases}$$

where w_l is defined to be

$$\max \left\{ s \in \{k_1, \dots, l\} \mid \sum_{j=1}^s n_j + \tilde{n}_j + \sum_{j=s+1}^l n_j^A(0) + \tilde{n}_j^A(0) > n_{s+1}^* \right. \\ \left. \forall j \in \{k_1, \dots, s-1\} \right\}.$$

However, then we do not expect the agents of any pair of types $\{(k_l + 1), (k_l + 1)^*\}, \{(k_l + 2), (k_l + 2)^*\}, \dots, \{l, l^*\}$, all play A in the long run. Indeed, we show the somewhat opposite: all type $(k_l + 2), (k_l + 2)^*, \dots, l, l^*$ agents will eventually choose B .

Proposition 3. If for a given initial condition $x(0)$, $k_l \leq l - 2$, then there exists some time T such that for $i = k_l + 2, \dots, l$,

$$n_i^A(t) = \tilde{n}_i^A(t) = 0 \quad \forall t \geq T. \quad (21)$$

Proof. We first show by contradiction that for $i = k_l + 2, \dots, l$,

$$n_i^A(t) \leq n_i^A(0), \quad \tilde{n}_i^A(t) \leq \tilde{n}_i^A(0) \quad \forall t \geq 0. \quad (22)$$

Assume on the contrary, there exists some time step $t_1 > 0$ such that at least one of the inequalities in (22) fails. Let $t_2 \leq t_1$ be the first time this happens, namely, first of all, for $i = k_l + 2, \dots, l$:

$$n_i^A(t) \leq n_i^A(0), \quad \tilde{n}_i^A(t) \leq \tilde{n}_i^A(0) \quad \forall t < t_2, \quad (23)$$

and second, there exists some $q \in \{k_l + 2, \dots, l\}$ such that

$$n_q^A(t_2) > n_q^A(0) \quad \text{or} \quad \tilde{n}_q^A(t_2) > \tilde{n}_q^A(0).$$

Hence, a type q or q^* agent is active at $t_2 - 1$ and changes her choice from B to A at time t_2 . On the other hand,

$$\begin{aligned} n^A(t_2 - 1) &\leq \sum_{i=1}^{k_l} n_i + \tilde{n}_i + \sum_{i=k_l+1}^l n_i^A(t_2 - 1) + \tilde{n}_i^A(t_2 - 1) \\ &\stackrel{(23)}{\leq} \sum_{i=1}^{k_l} n_i + \tilde{n}_i + \sum_{i=k_l+1}^l n_i^A(0) + \tilde{n}_i^A(0) \\ &\leq n_{k_l+1}^*, \end{aligned}$$

where the last inequality follows from the definition of k_l . Thus, in view of (2),

$$n^A(t_2 - 1) < n_{k_l+1}^* \quad \forall i \in \{k_l + 2, \dots, l\}.$$

Hence, $n^A(t_2 - 1) < n_q^*$. Therefore, according to Lemma 9, the active agent at $t_2 - 1$ chooses B , not A , a contrary, implying that (22) is in force. The rest of the proof can be done similarly to that of Proposition 2. $\square$

Propositions 2 and 3 guarantee the population state $x(t)$ to reach, in finite time, a state of the form

$$(n_1, \tilde{n}_1, \dots, n_{k_l}, \tilde{n}_{k_l}, *, \dots, *, \underbrace{0, \dots, 0}_{l-k_l-1}).$$

Namely, all type $1, 1^*, \dots, k_l, k_l^*$ agents will fix their strategies to A , and all type $(k_l + 2), (k_l + 2)^*, \dots, l, l^*$ agents will fix their strategies to B . This confines the set of reachable equilibria in Proposition 1 from $\mathcal{X}^*$ to $\mathcal{X}_{\mathcal{K}}^*$ defined by

$$\mathcal{X}_{\mathcal{K}}^* \triangleq \{s_r\}_{r \in \mathcal{K}} \cup \{s_r^*\}_{r \in \mathcal{K}^*},$$

where $\mathcal{K} = \{k_l, k_l + 1, \dots, k_l\} \cap \mathcal{E}$ and $\mathcal{K}^* = \{k_l + 1, k_l + 2, \dots, k_l + 1\} \cap \mathcal{E}^*$.

Theorem 3. Given any initial condition $x(0)$, there exist some time T and equilibrium state $x^* \in \mathcal{X}_{\mathcal{K}}^*$ such that

$$x(t) = x^* \quad \forall t \geq T. \quad (24)$$

Proof. Proposition 2 confines x^* in Proposition 1 to $\{s_i\}_{i=k_l}^l \cup \{s_i^*\}_{i=k_l+1}^l$. Similarly, Proposition 3 confines x^* to $\{s_i\}_{i=0}^{k_l+1} \cup \{s_i^*\}_{i=1}^{k_l+1}$, yet it can be shown that s_{k_l+1} cannot be reached either. Hence, x^* is confined to $\{s_i\}_{i=k_l}^{k_l} \cup \{s_i^*\}_{i=k_l+1}^{k_l+1}$, which completes the proof. $\square$

5. Stability analysis

From Theorem 3, we know that for any initial condition, the population dynamics reaches an equilibrium state. However, the stability of the equilibrium still remains an open problem that we investigate here. In what follows, by $\|\cdot\|$, we refer to the

one-norm, i.e., $\|a\| = \sum_{j=1}^{2l} |a_j|$, and we denote a ball with radius $r > 0$ and center s in the state space by $\mathcal{U}_r(s) = \{x \in \mathcal{X} \mid \|x - s\| < r\}$. By stability, we refer to the standard definition of Lyapunov stability, but when all possible activation sequences are considered, i.e., an equilibrium state $s \in \mathcal{X}^*$ is stable if for every ball $\mathcal{U}_\epsilon(s)$ there exists a ball $\mathcal{U}_\delta(s)$ such that for every $x(0) \in \mathcal{U}_\delta$, we have $x(t) \in \mathcal{U}_\epsilon$ for all $t \geq 0$ under all activation sequences. Since the state space $\mathcal{X}$ is discrete, it must hold that $\epsilon, \delta > 1$; otherwise, $\mathcal{U}_\delta(s)$ for example, only contains the equilibrium state s , automatically implying $x(t) = s$ for all $t \geq 0$, even if s is unstable. Asymptotic stability is defined correspondingly, namely s is asymptotically stable if it is stable and there exists some $\mathcal{U}_\delta$ such that if $x(0) \in \mathcal{U}_\delta$, then $\lim_{t \rightarrow \infty} \|x(t) - s\| = 0$ under any persistent activation sequence.

First, we focus on the stability of clean-cut equilibria. By definition, we know that $n_i^* \leq \sum_{j=1}^l (n_j + \tilde{n}_j) < n_{i+1}^*$ holds for each clean-cut equilibrium $s_i, i \in \mathcal{E}$. The following result states that the equilibrium is (asymptotically) stable if and only if the inequality is tightened. Define $n_0^* = -2$ and $n_{l+1}^* = n + 2$.

Theorem 4. Suppose that $n_i + \tilde{n}_i \geq 2 \forall i \in \mathcal{L}$. The clean-cut equilibrium state $s_i, i \in \mathcal{E}$, is stable if and only if either of the following two holds:

$$n_i^* + 1 \leq \sum_{j=1}^i (n_j + \tilde{n}_j) < n_{i+1}^* - 1, \quad (25)$$

$$n_i^* + 1 \leq \sum_{j=1}^i (n_j + \tilde{n}_j) = n_{i+1}^* - 1 \text{ and } n_{i+1} = 0, \quad (26)$$

and is asymptotically stable if and only if either of the following two holds:

$$n_i^* + 1 < \sum_{j=1}^i (n_j + \tilde{n}_j) < n_{i+1}^* - 1,$$

$$n_i^* + 1 = \sum_{j=1}^i (n_j + \tilde{n}_j) < n_{i+1}^* - 1 \text{ and } \tilde{n}_i = 0.$$

If the distribution of agents' thresholds is too diverse in a population, it is likely that $n_i^* + 1$ and $n_{i+1}^* - 1$ are close to each other for some $i \in \mathcal{E}$, making it unlikely for (25) or (26) to hold, implying the instability of s_i in view of the theorem. The case when $n_i + \tilde{n}_i = 1$ for some $i \in \mathcal{L}$ results in a similar, yet tedious, theorem that we have omitted here.

For the proof of Theorem 4, we need to investigate the dynamics of $x(t)$ for those initial conditions with distance 1 from s_i that are captured by the following $2l$ -dimensional vectors:

$$b_i(r) = (n_1, \tilde{n}_1, \dots, n_r - 1, \tilde{n}_r, \dots, n_i, \tilde{n}_i, 0, \dots, 0),$$

where $r \in \mathcal{B}_i = \{r \in \{1, \dots, i\} \mid n_r \geq 1\}$,

$$b_i^*(r) = (n_1, \tilde{n}_1, \dots, n_r, \tilde{n}_r - 1, \dots, n_i, \tilde{n}_i, 0, \dots, 0),$$

where $r \in \mathcal{B}_i^* = \{r \in \{1, \dots, i\} \mid \tilde{n}_r \geq 1\}$,

$$c_i(r) = (\underbrace{n_1, \tilde{n}_1, \dots, n_i, \tilde{n}_i}_{2r}, 0, \dots, 0, 1, 0, 0, \dots, 0),$$

where $r \in \mathcal{C}_i = \{r \in \{i, \dots, l - 1\} \mid n_r \geq 1\}$,

$$c_i^*(r) = (\underbrace{n_1, \tilde{n}_1, \dots, n_i, \tilde{n}_i}_{2r}, 0, \dots, 0, 0, 1, 0, \dots, 0),$$

where $r \in \mathcal{C}_i^* = \{r \in \{i, \dots, l - 1\} \mid \tilde{n}_r \geq 1\}$. Then the smallest ball containing the states within a neighborhood of distance 1 of s_i , i.e., $\mathcal{U}_2(s_i) \cap \mathcal{X}$, equals $\mathcal{X}_2(s_i) \triangleq$ defined as

$$\{b_i(r)\}_{r \in \mathcal{B}_i} \cup \{b_i^*(r)\}_{r \in \mathcal{B}_i^*} \cup \{c_i(r)\}_{r \in \mathcal{C}_i} \cup \{c_i^*(r)\}_{r \in \mathcal{C}_i^*},$$

which defines the set that we need to investigate for the stability of s_i . In what follows, by ' $x(0) = b_i(r)$ ', we mean ' $x(0) = b_i(r)$, $r \in \mathcal{B}_i^*$ ', and the same holds for $x(0) = b_i^*(r)$, $c_i(r)$, $c_i^*(r)$. The following lemma describes the behavior of $x(t)$ when starting from any of the states $b_i(r)$ and $b_i^*(r)$.

Lemma 6. Given s_i , $i \in \mathcal{E} - \{0\}$, it holds that

- (1) if $\sum_{j=1}^i (n_j + \tilde{n}_j) - 1 > n_i^*$, then if $x(0) = b_i(r)$ or $x(0) = b_i^*(r)$, there exists some time T such that

$$x(t) = x(0) \quad \forall t \leq T \quad \text{and} \quad x(t) = s_i \quad \forall t > T.$$

- (2) if $\sum_{j=1}^i (n_j + \tilde{n}_j) - 1 = n_i^*$, then

- (a) if $x(0) = b_i(r)$ or $x(0) = b_i^*(r)$, $r \neq i$, there exists some time T such that

$$x(t) = x(0) \quad \forall t \leq T \quad \text{and} \quad x(t) = s_i \quad \forall t > T.$$

- (b) if $x(0) = b_i^*(i)$, then $x(t) = x(0) = s_i^* \quad \forall t$.

- (3) if $\sum_{j=1}^i (n_j + \tilde{n}_j) - 1 < n_i^*$, then

- (a) if $x(0) = b_i(r)$, $r \neq i$, or $x(0) = b_i^*(r)$, $r \neq i$, or if $n_i + \tilde{n}_i \geq 2$ and either $x(0) = b_i(i)$ or $x(0) = b_i^*(i)$, then there exists an activation sequence, under which $x(1) \notin \mathcal{U}_2(s_i)$.

- (b) if $n_i + \tilde{n}_i = 1$, then if $x(0) = b_i(i)$ or $x(0) = b_i^*(i)$,

$$x(t) = x(0) = s_{i-1} \quad \forall t.$$

Proof. First, note that in all cases,

$$n^A(0) = \sum_{j=1}^l n_j^A(0) + \tilde{n}_j^A(0) = \sum_{j=1}^i (n_j + \tilde{n}_j) - 1.$$

Now, we investigate each case separately:

Case (1) We have that $n^A(0) > n_i^*$. Hence, if the active agent at $t = 0$ is a type j or j^* agent, $j = 1, \dots, i$, then in view of Lemma 9, she will update to A at $t = 1$. On the other hand, unless the active agent is a type r agent, she is already playing A at $t = 0$, and hence, does not change her strategy at $t = 1$, resulting in $x(1) = x(0)$. On the other hand, according to the definition of s_i , it holds that $\sum_{j=1}^i (n_j + \tilde{n}_j) < n_{i+1}^*$. Hence, $n^A(0) < n_{i+1}^*$. So if the active agent at $t = 0$ is a type j or j^* agent, $j = i+1, \dots, l$, then in view of Lemma 9, she will update to B at $t = 1$. Since all such agents are already playing B at $t = 0$, none will change their strategies at $t = 1$. Hence, $x(1) = x(0)$ is also in force for this case. Indeed, by induction it can be shown that as long as all B -playing type r agents are inactive, $x(t) = x(0)$. However, because of the persistent assumption on the activation sequence, we know that there exists some time T when a B -playing type r agent becomes active. Then in view of Lemma 9 and $n^A(0) > n_i^*$, she updates her strategy to A in the next time step, yielding $x(T+1) = s_i$. Since s_i is an equilibrium state, it follows that $x(t) = s_i$ for all $t \geq T+1$. The same can be shown for when $x(0) = b_i^*(r)$, which completes the proof of the first case.

Case (2) Part 2-(a) can be proven similarly to Case (1), so we look at Part 2-(b). We have that $n^A(0) = n_i^*$. Hence,

$$\tilde{n}_i^A(0) = n^A(0) - \sum_{j=1}^{i-1} (n_j + \tilde{n}_j) - n_i = n_i^* - \sum_{j=1}^{i-1} (n_j + \tilde{n}_j) - n_i,$$

resulting in $b_i^*(i) = s_i^*$. Moreover, it clearly holds that $0 \leq [s_i^*]_{2i} < \tilde{n}_i$. Thus, $i \in \mathcal{E}^*$. Hence, according to Theorem 1, $s_i^* \in \mathcal{X}^*$, completing the proof.

Case (3) We have that $n^A(0) < n_i^*$. If $x(0) = b_i(r)$ or $b_i^*(r)$ for some $r \leq i$, then

Case 3-(a) either $r \leq i-1$ or both $r = i$ and $n_i + \tilde{n}_i \geq 2$. Then there exists a type i or i^* agent whose choice is A at $t = 0$. Under any activation sequence, that this agent is active at $t = 0$, she will choose B at $t = 1$, resulting in $x(1) \notin \mathcal{U}_2(s_i)$.

Case 3-(b) $r = i$ and $n_i + \tilde{n}_i = 1$. Then $x(0) = s_{i-1}$. We show that $i-1 \in \mathcal{E}$. Since $i \in \mathcal{E}$, we know that

$$n^A(0) = \sum_{j=1}^i (n_j + \tilde{n}_j) - 1 \geq n_i^* - 1,$$

which in view of $n^A(0) < n_i^*$, yields $n^A(0) = n_i^* - 1$. Hence,

$$n_{i-1}^* \leq n^A(0) = \sum_{j=1}^{i-1} (n_j + \tilde{n}_j) = n_i^* - 1 < n_i^*,$$

which implies $i-1 \in \mathcal{E}$ in view of (3). Hence, due to Theorem 1, $s_{i-1} \in \mathcal{X}^*$, completing the proof of this case, and hence the whole. $\square$

The following lemma describes the behavior of $x(t)$ when starting from the states $c_i(r)$ and $c_i^*(r)$.

Lemma 7. Given s_i , $i \in \mathcal{E} - \{l\}$, it holds that

- (1) if $\sum_{j=1}^i (n_j + \tilde{n}_j) + 1 < n_{i+1}^*$, then if $x(0) = c_i(r)$ or $x(0) = c_i^*(r)$, there exists some time T such that

$$x(t) = x(0) \quad \forall t \leq T \quad \text{and} \quad x(t) = s_i \quad \forall t > T.$$

- (2) if $\sum_{j=1}^i (n_j + \tilde{n}_j) + 1 = n_{i+1}^*$, then

- (a) if $x(0) = c_i(r)$, $r \neq i$, or $x(0) = c_i^*(r)$, $r \neq i$, there exists some time T such that

$$x(t) = x(0) \quad \forall t \leq T \quad \text{and} \quad x(t) = s_i \quad \forall t > T.$$

- (b) if $n_{i+1} \geq 1$ and $x(0) = c_i^*(i)$ or if $n_{i+1} \geq 2$ and $x(0) = c_i(i)$, there exists an activation sequence, under which $x(1) \notin \mathcal{U}_2(s_i)$.

- (c) if $n_{i+1} = 0$ and $x(0) = c_i^*(i)$, then $x(t) = x(0) \quad \forall t$.

- (d) if $n_{i+1} = 1$ and $x(0) = c_i(i)$, then $x(t) = x(0) \quad \forall t$.

The main idea of the proof, which we skip here, is that $n^A(0) = \sum_{j=1}^i (n_j + \tilde{n}_j) + 1$, resulting in $n^A(0) < n_{i+1}^*$ and $n^A(0) = n_{i+1}^*$ in Cases 1 and 2, respectively. The case $\sum_{j=1}^i (n_j + \tilde{n}_j) + 1 > n_{i+1}^*$ is not investigated in the lemma since it never takes place. This is because from $i \in \mathcal{E}$, we have that $\sum_{j=1}^i (n_j + \tilde{n}_j) + 1 < n_{i+1}^* + 1$. Knowing the behavior of $x(t)$ for all states with distance 1 from a clean-cut equilibrium state, we now proceed to the proof of the main result.

Proof of Theorem 4. We prove the result for stability; the result for asymptotic stability can be done similarly.

(sufficiency) We show that $\delta = 2$ satisfies the stability condition for any $\mathcal{U}_\epsilon$, $\epsilon \geq 2$. So consider some $x(0) \in \mathcal{U}_2(s_i)$, $x(0) \in \mathcal{X}$, implying $x(0) \in \mathcal{X}_2(s_i)$. Should $x(0)$ be at the equilibrium s_i , it would follow that $x(t) = s_i \in \mathcal{X}_2(s_i)$ for all $t \geq 0$. So consider the case when $x(0) = b_i(r)$, $b_i^*(r)$, $c_i(r)$ or $c_i^*(r)$. Then Lemma 6, Cases 1 and 2, and Lemma 7, Case 1, cover all possibilities under Condition (25), and result in $x(t) \in \mathcal{X}_2(s_i)$ for all $t \geq 0$, implying the stability of s_i .

(necessity) If s_i is stable, then it suffices to show that for any initial condition $x(0)$ in $\mathcal{X}_2(s_i)$, the smallest feasible ball with center s_i , we have $x(t) \in \mathcal{X}_2(s_i)$. We prove by contradiction. Assume on the contrary that either of the following two holds: (i) $n_{i+1}^* - 1 \leq \sum_{j=1}^i (n_j + \tilde{n}_j)$, or (ii) $\sum_{j=1}^i (n_j + \tilde{n}_j) < n_{i+1}^* + 1$. Should (ii) hold, s_i would be unstable in view of Lemma 6, Case 3; unless, $n_i + \tilde{n}_i = 1$. Should (i) hold, s_i would be unstable in view of Lemma 7, Case 2, (and that $n_{i+1}^* - 1 < \sum_{j=1}^i (n_j + \tilde{n}_j)$ never

takes place); unless, one of the following situations corresponding to the sub-cases in the lemma holds: (c) $n_{i+1} = 0$, (d) $n_{i+1} = 1$, $\tilde{n}_{i+1} = 0$, completing the proof. $\square$

Now we focus on the stability of a ruffled equilibrium state, s_i^* , $i \in \mathcal{E}^*$, which we intuitively expect to satisfy a weaker notion of stability since it requires the total number of A-players to be exactly equal to a fixed value, i.e., n_i^* .

Theorem 5. *The equilibrium state s_i^* , $i \in \mathcal{E}^*$, is unstable.*

We only provide the sketch of the proof. The initial conditions with distance 1 from s_i^* are captured by the vectors

$$\begin{aligned}\tilde{b}_i(r) &= s_i^* - \mathbf{1}_{2r-1}, & \tilde{b}_i^*(r) &= s_i^* - \mathbf{1}_{2r} \\ \tilde{c}_i(r) &= s_i^* + \mathbf{1}_{2r+1}, & \tilde{c}_i^*(r) &= s_i^* + \mathbf{1}_{2r+2},\end{aligned}$$

where $\mathbf{1}_i$ is the i th row of the $2l \times 2l$ identity matrix, and r belongs to the sets $\tilde{B}_i, \tilde{B}_i^*, \tilde{C}_i, \tilde{C}_i^*$, defined similar to B_i, B_i^*, C_i, C_i^* , respectively. Since Theorem 5 postulates the instability of s_i^* , it suffices to show the existence of one of $\tilde{b}_i(r)$, $\tilde{b}_i^*(r)$, $\tilde{c}_i(r)$ and $\tilde{c}_i^*(r)$, starting from which, $x(t)$ leaves $\mathcal{U}_2(s_i^*)$, under some activation sequence.

Lemma 8. *Given s_i^* , $i \in \mathcal{E}^*$, if any of the following holds:*

- (1) $i \neq l$, and $x(0) = \tilde{c}_i(i)$ or $x(0) = \tilde{c}_i^*(i)$,
- (2) $i = l$, $\tilde{n}_l \geq 2$, and $x(0) = \tilde{b}_i^*(l)$,
- (3) $i = l$, $\tilde{n}_l = 1$, $n_l \geq 1$, and $x(0) = \tilde{b}_i(l)$,

then there exists an activation sequence, under which $x(1) \notin \mathcal{U}_2(s_i^*)$.

Proof. For $i \in \mathcal{E}^* - \{l\}$, it holds that $[s_i^*]_{2i} < \tilde{n}_i$, implying the existence of a B-playing type i^* agent at time $t = 0$. So starting from any of $\tilde{c}_i(i)$ or $\tilde{c}_i^*(i)$, this agent will switch to A if she is active at $t = 0$ since $n^A(0) = n_i^* + 1$. This results in a state out of $\mathcal{U}_2(s_i^*)$, proving Case 1. The remaining cases can be proven similarly. $\square$

Cases $i = l$, $\tilde{n}_l = 0$ and $i = l$, $\tilde{n}_l = 1$, $n_l = 0$ are not investigated in the lemma since then $i \notin \mathcal{E}^*$. Thus, the lemma covers all s_i^* , $i \in \mathcal{E}^*$, leading to the proof of Theorem 5.

So unlike clean-cut equilibria that may even be asymptotically stable, ruffled equilibrium states are unstable, implying that small perturbations from the equilibrium may lead to moving to other states.

6. Contagion probability

As discussed in Section 4, given the same initial condition, the population dynamics may reach different equilibrium states in the long run, under different activation sequences. Now if we let the activation sequence to be random, that is, a random agent becomes active at each time step, then given an initial condition, each state in $\mathcal{X}_{\mathcal{K}}^*$ may be reached by $x(t)$ with a certain probability. We are interested in the contagion probability, that is the probability of reaching the state s_l where every individual has chosen A. To simplify the analysis, we limit the analysis to a heterogeneous population with just two types: type 1 and 2, and no type 1^* or 2^* , i.e., $\tilde{n}_1 = \tilde{n}_2 = 0$. After all, rarely the threshold of an agent may exactly equal n_i^*/n for some $i = 1, \dots, n$, so this is not a strong assumption. Then we find a recurrence equation for the probability of contagion as follows. Define the recursive function $P(\cdot, \cdot) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ by

$$P(a, b) = \frac{n_1 - a}{n_1 - a + b} P(a + 1, b) + \frac{b}{n_1 - a + b} P(a, b - 1), \quad (27)$$

Table 1

Maximum and expected contagion probabilities for different initial A-players.										
$n^A(0)$	25	26	27	28	29	30	31	32	33	34
max P	0	0.37	1.05	2.35	4.83	9.42	17.72	32.35	57.56	100
Exp[P]	0	0.05	0.16	0.39	0.90	2	4.41	9.81	22.74	100

for $a < n_1$, $b > 0$ and $n_1^* \leq a + b < n_2^*$, with the boundary conditions

$$P(a, b) = \begin{cases} 0 & a + b < n_1^* \\ 1 & a + b \geq n_2^* \\ 0 & n_1^* \leq a + b < n_2^*, a = n_1 \\ 0 & n_1^* \leq a + b < n_2^*, b = 0, n_1 < n_2^* \\ 1 & n_1^* \leq a + b < n_2^*, b = 0, n_1 \geq n_2^*. \end{cases} \quad (28)$$

Proposition 4. *Consider a heterogeneous population of type 1 and 2 agents. Given an initial condition $x(0) = (n_1^A(0), 0, n_2^A(0), 0)$, the contagion probability equals $P(n_1^A(0), n_2^A(0))$.*

Proof. The boundary conditions (28) together with (27) result in a unique solution for $P(a, b)$ for each pair (a, b) . Therefore, it suffices to show that the contagion probability of a population with initial conditions $x(0) = (a, 0, b, 0)$ satisfies the recurrence Eq. (27) and the boundary conditions. The boundary conditions can be easily verified. As for the recurrence equation, we know that the contagion probability for $x(0)$ equals the sum of the probabilities of different possible states taking place at $t = 1$, times the contagion probabilities from those new states $x(1)$. Namely, for $a < n_1$, $b > 0$ and $n_1^* \leq a + b < n_2^*$, one of the following may take place at $t = 0$ and $t = 1$:

(1) The active agent at $t = 0$ is an A-playing type 1 or a B-playing type 2 agent, which occurs with probability $\frac{a+n_2-b}{n}$. Then the active agent does not change her choice, resulting in $n_1^A(1) = n_1^A(0)$, $n_2^A(1) = n_2^A(0)$. So the contagion probability from $x(1)$ equals $P(a, b)$.

(2) The active agent at $t = 0$ is a B-playing type 1 agent, which occurs with probability $\frac{n_1-a}{n}$. Then the active agent chooses A, resulting in $n_1^A(1) = n_1^A(0) + 1$, $n_2^A(1) = n_2^A(0)$. So the contagion probability from $x(1)$ equals $P(a + 1, b)$.

(3) The active agent at $t = 0$ is an A-playing type 2 agent, which occurs with probability $\frac{b}{n}$. Then the active agent chooses B, resulting in $n_1^A(1) = n_1^A(0)$, $n_2^A(1) = n_2^A(0) - 1$. So the contagion probability from $x(1)$ equals $P(a, b - 1)$.

Therefore, we acquire the recurrence equation

$$P(a, b) = \frac{a + n_2 - b}{n} P(a, b) + \frac{n_1 - a}{n} P(a + 1, b) + \frac{b}{n} P(a, b - 1)$$

which is equivalent to (27). $\square$

Example 1. Consider a population of 100 individuals of types 1 and 2 with populations $n_1 = 20$ and $n_2 = 80$ and $n_1^* = 25$ and $n_2^* = 33$ respectively. Using Proposition 4 and by solving the recurrence Eq. (27), we obtain a contagion probability of $3.13e - 8\%$ for an initial condition of 12 A-playing type 1 and 14 B-playing type 2 individuals, i.e., $P(12, 14) = 3.13e - 10$, and that of 6.7% for an initial condition of 19 A-playing type 1 and 14 B-playing type 2 individuals, i.e., $P(19, 14) = .067$. In general, the expected probabilities of contagion based on the number of initial A-players in the population are summarized in Table 1. The contagion probability for low number of initial A-players is quite low, but exponentially grows as the number increases.

7. Concluding remarks

We have performed convergence and stability analysis on a finite well-mixed heterogeneous population where each individual is associated with a possibly unique threshold, and the agents update asynchronously based on the linear threshold model. In addition to the two uniform equilibria of all- A and all- B players, i.e., s_1 and s_0 , the dynamics admit a potentially wide class of mixed equilibria where both A and B -players coexist. Uniform equilibria may represent the most desired or undesired outcome of a population. For example, they may stand for a population of all cooperators or all defectors, provided that A and B stand for cooperation and defection. Mixed equilibria, however, represent the often more-realistic population states with diverse choices of strategies. We have found that the dynamics do converge to some of these mixed equilibria, which can be asymptotically stable. Indeed, our example in the Contagion Probability Section shows that it is quite likely for the population to converge to a mixed equilibrium rather than uniform, although generalizing this statement to all populations requires further theoretical analysis. These results have two main implications: (i) the simple linear threshold dynamics are capable of modeling the often observed diversity of choices in population dynamics if the thresholds are heterogeneous; equivalently, the same can be stated for best response dynamics in the context of coordination games, provided that the payoff matrices are different; (ii) a population structure (network) is unnecessary to achieve any of the existence, stability and convergence features of the mixed equilibria; perception heterogeneity is enough to do so. We are currently working together with theoretical biologists and sociologists to understand better the effect of heterogeneity using real data.

We have also shown that for any given initial condition, after a finite time, the population reaches an equilibrium where no agent violates her threshold. Hence, compared to anticonordinating games, where best-response dynamics either admit and reach a single equilibrium or are empty of equilibria and fluctuate between two states in the long-run (Ramazi & Cao, 2017), coordinating games result in wider yet always settling long-term behaviors.

Although making the analysis tedious, the inclusion of type i^* agents, has revealed their role in the population dynamics. These agents switch to A only if the number of A -players becomes exactly equal to a certain integer. Therefore, perhaps not surprisingly, the only type of equilibrium that would not exist in their absence, i.e., ruffled, is unstable. One may, thus, focus on other types of agents for means of control purposes, particularly robust ones.

We have proved the convergence results in two ways. First, unlike the typical potential function approach towards proving stability and sometimes convergence, we developed an inductive argument to establish equilibrium convergence. This method can be useful when finding a potential function is not trivial and also can provide further intuition on the dynamics. Second, based on a novel potential function, we have both proven stability and provided an upper bound on the number of switches necessary for the population to converge. The potential function is based on the total number of A -players in the population and the agents' thresholds, and may be used in future for other well-mixed populations.

The extension of the results to the case with three or more available strategies, e.g., A , B and C – although interesting – is not trivial. The update rule may no longer be written with respect to just the total number of agents playing a certain strategy, e.g., n^A , but requires the inclusion of two: n^A , n^B . For example, if the total number of A players falls short of some threshold τ_A of an active

agent, then it is unclear to which strategy she will switch; unless, we know n^B (or n^C) and the corresponding threshold τ_B .

Overall, our results highlight the importance of perception heterogeneity in population dynamics, particularly, the interesting long-term behavioral features it brings, even in the absence of any population structure.

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Appendix. Additional lemmas

The proofs of the following lemmas are straightforward.

Lemma 9. *If agent $i \in \{1, \dots, n\}$ is a type j^* agent, $j \in \mathcal{L}$, then the update rule (1) is equivalent to*

$$d_i(t+1) = \begin{cases} A & n^A(t) > n_j^* \\ d_i(t) & n^A(t) = n_j^* \\ B & n^A(t) < n_j^* \end{cases}$$

If agent $i \in \{1, \dots, n\}$ is a type j agent, $j \in \mathcal{L}$, then the update rule (1) is equivalent to

$$d_i(t+1) = \begin{cases} A & n^A(t) \geq n_j^* \\ B & n^A(t) < n_j^* \end{cases}$$

Lemma 10. *At any $x \in \mathcal{X}^*$, it holds that if $n_i^A > 0$ for some $i \in \mathcal{L}$, then $n_j^A = n_j$ and $\tilde{n}_j^A = \tilde{n}_j$ for all $j = 1, 2, \dots, i-1$. Moreover, if $n_i^A = 0$ while $n_i \geq 1$ for some $i \in \mathcal{L}$, then $n_j^A = \tilde{n}_j^A = 0$ for all $j = i, i+1, \dots, l$.*

Lemma 11. *At any $x \in \mathcal{X}^*$, it holds that for every $i \in \mathcal{L}$, either $n_i^A = n_i$ or $n_i^A = 0$.*

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