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Published in:
Proceedings of the 58th Conference on Decision and Control

DOI:
[10.1109/CDC40024.2019.9029857](https://doi.org/10.1109/CDC40024.2019.9029857)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2019

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Yu, L., Cheng, X., Scherpen, J. M. A., & Xiong, J. (2019). Synchronization Preserving Model Reduction of Multi-Agent Network Systems by Eigenvalue Assignments. In *Proceedings of the 58th Conference on Decision and Control* (pp. 7794-7799). [9029857] IEEE. <https://doi.org/10.1109/CDC40024.2019.9029857>

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Synchronization preserving model reduction of multi-agent network systems by eigenvalue assignments

Lanlin Yu, Xiaodong Cheng, Jacquélien M.A. Scherpen and Junlin Xiong

Abstract—In this paper, structure preserving model reduction problem for multi-agent network systems consisting of diffusively coupled agents is investigated. A new model reduction method based on eigenvalue assignment is derived. Particularly, the spectrum of the reduced *Laplacian matrix* is selected as a subset of the spectrum of the original *Laplacian matrix*. The resulting reduced-order model retains the network protocol of diffusive couplings, and thus the synchronization property is preserved. Moreover, a concise expression for the upper-bound of the $\mathcal{H}_2$ approximation error is presented in the setting of a leader-follower network, and it provides a guideline to select the eigenvalues of the reduced *Laplacian matrix*. The effectiveness of the proposed method is finally illustrated via the application to a spacecraft network, with a comparison of performances with the graph clustering method in [1] and balanced truncation approach in [2].

I. INTRODUCTION

A multi-agent network system describe a collective behavior of multiple interconnected systems. Applications have been found in many areas, including multi-robot systems, power grids, and chemical reaction networks, see e.g., [3]–[5]. Over the past decades, the study of networked dynamic systems has become an active research field within the system and control community, see [6]–[8]. However, dynamic networks with complex interconnection structures are often modeled by high dimensional differential equations, which complicates the analysis and control synthesis. Therefore, reduced-order models that accurately approximate the original systems are of interest. Meanwhile, the reduced-order model is desired to maintain the interconnection structure. The goal of this research is to lower the complexity of multi-agent network systems by reducing the number of vertices.

Synchronization is an important property of complex network systems, which occurs when certain agreements are

reached via exchanging the information among the agents, see [9] for an overview. This property is generally realized on the basis of diffusive couplings among the agents. For the model reduction of synchronized multi-agent networks, it is usually desired to preserve the diffusion protocols in the reduced-order multi-agent network among the agent in order to maintain the synchronization property [2], [10], [11].

In recent years, graph clustering methods (or graph partition) [1], [10]–[14] and a balanced truncation method [2], [15]–[17] have shown a great potential in the structure preserving model reduction problem for multi-agent network systems. The main advantage of graph clustering is that it preserves a interconnection structure and shows an insightful physical interpretation of the reduction process. However, the problem about how to choose the clusters is an NP-hard problem [18], which is the most crucial issue in the graph clustering method, as it determines the quality of the approximation. A special clustering called *almost equitable partition* (AEP) is adopted in [1], [12] to derive a reduced-order multi-agent network that achieves minimal approximation error. However, the method based on the AEP is not applicable to general graphs. In [2], a structure preserving method is developed using generalized balanced truncation for undirected networks. Although a priori approximation error bound is obtained, the resulting reduced-order model can be only guaranteed to be interpreted as a complete graph. Moreover, to preserve synchronization in reduced networks, these existing model reduction methods rely on extra assumptions on agent dynamics, for instance, [2], [10] assume each subsystem to be passive and minimal, and [11] considers dissipative subsystems. This motivates the research of the paper, which investigates a synchronization preserving model reduction problem for more general multi-agent networks without the assumptions required in [1], [2], [10], [11] for each subsystem.

Our goal is to find a reduced-order multi-agent network with synchronization preservation. The key idea is to select a subset of the eigenvalues of the original *Laplacian matrix* to be the eigenvalues of the reduced *Laplacian matrix*. The projection matrix is constructed by using the eigenvectors corresponding to the selected eigenvalues. The resulting reduced-order model preserves synchronization without the additional constraints for the subsystems in [2], [10], [11]. Moreover, a concise upper-bound on the $\mathcal{H}_2$ approximation error is established for leader-follower multi-agent networks. The relation between the AEP-based projection and the proposed projection is also discussed. When an AEP is found in a network, the projection in this paper is equivalent to the

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This work was supported in part by the National Natural Science Foundation of China Under Project 61761136005, National Natural Science Foundation of China Under Project 61773357, the Australian Research Council under grant DP190100887 and DP160104500, the University of Groningen and Chinese Scholarship Council (CSC).

one formed by the AEP as in [1]. For general graphs, the proposed approach is still feasible to obtain a reduced-order multi-agent system preserving the synchronization property. Finally, the effectiveness of the proposed method is illustrated through a multi-spacecraft system. Comparisons are made between this method and the clustering method in [1] and the balanced truncation approach in [2].

Notation: The symbol $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and positive real numbers, respectively. For a matrix A , A^{-1} and A^T stand for the inverse and transpose of A , respectively. $\lambda(A)$ denotes the eigenvalue set of A . The notation $P > 0$ (≥ 0) means that matrix P is positive definite (semi-definite). I_n is the identity matrix of size n and $\mathbf{1}_n$ represents a vector in $\mathbb{R}^n$ of all ones. e_i represents the i -th column of I_n . For matrices A and B , $A \otimes B$ represents the Kronecker product, which satisfies $(A \otimes B)(C \otimes D) = AC \otimes BD$, whenever the products of AC and BD can be formed.

II. PRELIMINARIES & PROBLEM FORMULATION

Some preliminaries on graph theory are presented. The details can be found in [7].

Consider a graph $\mathcal{G}$ that consists of a finite and nonempty node set $\mathcal{V} := \{1, 2, \dots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. For a directed graph, each elements in $\mathcal{E}$ is an ordered pair of $\mathcal{V}$, and if $(i, j) \in \mathcal{E}$, we say that the edge is directed from vertex i to vertex j . The *incidence matrix* $R \in \mathbb{R}^{n \times |\mathcal{E}|}$ of the directed graph has entries 0, ± 1 defined by

$$R_{ij} = \begin{cases} +1 & \text{if edge } j \text{ is directed from vertex } i, \\ -1 & \text{if edge } j \text{ is directed to vertex } i, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

In the case of undirected graphs, each edge is specified by an unordered pair of vertices. If each edge is assigned a positive value (weight), an undirected graph $\mathcal{G}$ is *weighted*. Thereby, the *adjacency matrix* of $\mathcal{G}$, denoted by $\mathcal{A} := [a_{ij}]$, is defined such that $a_{ij} = a_{ji} \in \mathbb{R}_+$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = a_{ji} = 0$ otherwise. Then, the *Laplacian matrix* $L \in \mathbb{R}^{n \times n}$ of the graph $\mathcal{G}$ is defined by

$$L_{ij} = \begin{cases} \sum_{j=1, j \neq i}^n a_{ij} & i = j, \\ -a_{ij} & \text{otherwise.} \end{cases} \quad (2)$$

Alternatively, we can also define the *Laplacian matrix* of an undirected graph using the following formula

$$L = RWR^T, \quad (3)$$

where $W := \text{diag}(w_1, w_2, \dots, w_{|\mathcal{E}|})$ such that w_k indicates the weight associated to the edge k , for each $k = 1, 2, \dots, |\mathcal{E}|$.

Remark 1: Suppose that $\mathcal{G}$ is an undirected connected graph. The associated *Laplacian matrix* L has the properties:

- $L^T = L$ and $L\mathbf{1} = 0$;
- $L_{ij} \leq 0$ if $i \neq j$, and $L_{ij} > 0$ otherwise;
- $L \geq 0$ and has a simple zero eigenvalue.

Consider a network of N agents, and the dynamics of each agent is described by

$$\Sigma_i : \begin{cases} \dot{x}_i = Ax_i + Bu_i, \\ \eta_i = Cx_i, \end{cases} \quad (4)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $\eta_i \in \mathbb{R}^m$ are the states, control inputs and outputs of agent i , $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$. In contrast to [1], [2], [10], A in (4) is not restricted to Hurwitz or stable in this paper.

The agents are assumed to interact with each other over an undirected weighted graph $\mathcal{G}$. Note that the number of inputs u_i to (4) equals the number of outputs η_i , which allows diffusive coupling rule

$$u_i = - \sum_{j=1, j \neq i}^N \omega_{ij}(\eta_i - \eta_j) + \sum_{k=1}^p f_{ik}\tilde{u}_k,$$

where $\omega_{ij} \geq 0$ represents the strength of the coupling between vertices i and j , $\tilde{u}_k \in \mathbb{R}^m$ with $k = \{1, \dots, p\}$ are external inputs, $f_{ik} \in \mathbb{R}$ is the amplification of the k -th input acting on agent i , which is zero when $\tilde{u}_k$ has no effect on agent i .

The total network dynamics can be represented by

$$\Sigma : \dot{x} = (I_N \otimes A - L \otimes BC)x + (F \otimes B)u, \quad (5)$$

where

$$x := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{Nn}, u := \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \in \mathbb{R}^{pm}, \quad (6)$$

and $F \in \mathbb{R}^{N \times p}$ is the collection of f_{ij} , and $L \in \mathbb{R}^{N \times N}$ is the *Laplacian matrix* associated with an undirected weighted graph $\mathcal{G}$.

The definition of synchronization for network system (5) is given as following.

Definition 1: [19] The multi-agent network system (5) is synchronized if when $u = 0$, every solution of (5) satisfies

$$\lim_{t \rightarrow \infty} (x_i(t) - x_j(t)) = 0, \quad \forall i, j \in \{1, 2, \dots, N\},$$

for all initial conditions.

A necessary and sufficient condition for synchronization has been investigated in [19].

Lemma 1: Consider the network system (5) with the underlying graph $\mathcal{G}$ connected, and let the eigenvalues of the *Laplacian matrix* be $\lambda(L) = \{\lambda_1, \dots, \lambda_N\}$, with $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_N$. Then, the network system (5) is synchronized if and only if $A - \lambda_j BC$ is Hurwitz for all $j \in \{2, 3, \dots, N\}$.

This paper aims for a structure preserving model reduction of diffusively-coupled multi-agent networks. It is desired to obtain a reduced-order model that not only approximates the input-output behavior of the original network system with a certain accuracy but also inherits an interconnection structure with diffusively couplings. Meanwhile, synchronization of the reduced multi-agent network is guaranteed. Precisely, we address the studied problem of this paper as follows.

Problem 1: For a given multi-agent network system (5), find a reduced-order model

$$\Sigma_r : \dot{x}_r = (I_r \otimes A - L_r \otimes BC)x_r + (F_r \otimes B)u \quad (7)$$

such that

- 1) $L_r \in \mathbb{R}^{r \times r}$ is a *Laplacian matrix* associated with a weighted graph with fewer vertices, and the set of eigenvalues of the reduced *Laplacian matrix* L_r is a subset of eigenvalues of the original *Laplacian matrix* L , that is, $\lambda(L_r) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$, $\lambda_1 = 0$, $\lambda_i \in \lambda(L) \setminus \{0\}$, $i \in \{2, \dots, r\}$, $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_r$.
- 2) The $\mathcal{H}_2$ approximation error $\|\Sigma - \Sigma_r\|_2^2$ is small.

For the reduced-order model (7), if $\lambda(L_r)$ is a subset of $\lambda(L)$, then we have that $A - \lambda_i BC$ is Hurwitz if $A - \lambda_j BC$ is Hurwitz, λ_i represents the nonzero eigenvalues of L_r , $i = \{2, \dots, r\}$. According to Lemma 1, the reduced multi-agent network (9) is synchronized.

III. PROJECTION BY EIGENVALUE ASSIGNMENTS

In this section, a novel model reduction method based on the eigenvalue decomposition is proposed for the network system (5), which selects $\lambda(L_r)$ as a subset of $\lambda(L)$. Then, how to reinterpret the obtained reduced-order system as a selected directed graph is investigated. Moreover, the relation with the almost equitable partition (AEP) is discussed.

The following sufficient condition is derived for the construction of the desired reduced network system.

Theorem 1: For the original multi-agent network (5), given $1 \leq r < N$, construct the matrix V_r as

$$V_r = [v_1 \ v_2 \ \dots \ v_r], \quad (8)$$

where $v_2, \dots, v_r$ are the eigenvectors corresponding to λ_j of L , $\lambda_j \in \lambda(L) \setminus \{0\}$, $j \in \{2, \dots, N\}$, and

$$v_1 = \mathbf{1}_N - v_2 \dots - v_r.$$

Then, the reduced-order model is given as

$$\dot{x}_r = (I_r \otimes A - L_r \otimes BC)x_r + (F_r \otimes B)u, \quad (9)$$

where

$$L_r = (V_r^T V_r)^{-1} V_r^T L V_r, \quad F_r = (V_r^T V_r)^{-1} V_r^T F.$$

L_r is the *Laplacian matrix* associated with a directed weighted graph. Moreover, $\lambda(L_r) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$, $\lambda_1 = 0$, $\lambda_i \in \lambda(L) \setminus \{0\}$, $\lambda_1 < \lambda_2 \leq \dots \leq \lambda_r$, $i \in \{2, \dots, r\}$.

Proof: First, we prove that the eigenvalues of L_r are as desired, that is, $\lambda(L_r) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$, $\lambda_1 = 0$, $\lambda_i \in \lambda(L) \setminus \{0\}$, $i \in \{2, \dots, r\}$. It follows from

$$V_r = [\mathbf{1}_n - v_2 \dots - v_r \quad v_2 \quad \dots \quad v_r]$$

that

$$L_r \mathbf{1}_r = (V_r^T V_r)^{-1} V_r^T L V_r \mathbf{1}_r = (V_r^T V_r)^{-1} V_r^T L \mathbf{1}_N = 0.$$

Hence, we have that L_r has a zero eigenvalue. In order to prove that $\lambda(L_r) \setminus \{0\} = \{\lambda_2, \dots, \lambda_r\}$, we need to show that $\lambda_i I - L_r$ loses rank. In particular, we show that $(\lambda_i I -$

$L_r)e_i = 0$. The following string of equalities leads to the desired result:

$$\begin{aligned} (\lambda_i I - L_r)e_i &= (\lambda_i I - (V_r^T V_r)^{-1} V_r^T L V_r)e_i \\ &= ((V_r^T V_r)^{-1} V_r^T (\lambda_i I - L))V_r e_i \\ &= ((V_r^T V_r)^{-1} V_r^T (\lambda_i I - L))v_i, \\ &= (V_r^T V_r)^{-1} V_r^T (\lambda_i v_i - L v_i) = 0, \end{aligned}$$

since v_i is the eigenvector corresponding to λ_i of L , $i \in \{2, \dots, r\}$. Thus, L_r has only one zero eigenvalue λ_1 and $\lambda(L_r) \setminus \{0\} = \{\lambda_2, \dots, \lambda_r\}$. Moreover, the eigenvectors corresponding to λ_1 , λ_i , $i \in \{2, \dots, r\}$ of L_r are $\mathbf{1}_r$, e_i , $i \in \{2, \dots, r\}$.

Now, we prove that L_r is a *Laplacian matrix* associated with a directed weighted graph. According to the above proof, we obtain that $L_r U_r = U_r D_r$, where

$$U_r = [\mathbf{1}_r \ e_2 \ \dots \ e_r], \quad D_r = \text{diag}\{0, \lambda_2, \dots, \lambda_r\}.$$

It follows that

$$L_r = U_r D_r U_r^{-1} = [l_1 \ l_2 \ \dots \ l_r], \quad (10)$$

where

$$l_1 = [0 \ -\lambda_2 \ \dots \ -\lambda_r]^T, \quad l_i = \lambda_i e_i, \quad i \in \{2, \dots, r\}.$$

Thus, L_r is a *Laplacian matrix* associated with a directed weighted graph. ■

The following corollary shows the synchronization preservation of the reduced-order model (9).

Corollary 1: Consider the original multi-agent network (5) and the reduced-order model (9). If (5) is synchronized, then reduced network (9) is also synchronized.

The following example is given to illustrate.

Example 1: Consider a star network as shown in Fig. 1a. The agent dynamics is given by (4) with

$$A = \begin{bmatrix} -20 & 5 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (11)$$

which is not passive. The network dynamical can be written as

$$\dot{x} = (I_7 \otimes A - L \otimes BC)x + (F \otimes B)u, \quad (12)$$

where L is the *Laplacian matrix* associate with the star topology as shown in Fig. 1a,

$$L = \begin{bmatrix} \sum_{i=1}^6 \omega_i & -\omega_1 & \dots & -\omega_6 \\ -\omega_1 & \omega_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_6 & 0 & \dots & \omega_6 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

the weights are given as

$$\{\omega_1, \dots, \omega_6\} = \{0.5, 1, 1.5, 2, 3.5, 3\}.$$

The eigenvalues L are

$$\lambda(L) = \{0, 0.5578, 1.1076, 1.6637, 2.3442, 3.2508, 14.0759\}.$$

It can be verified that $A - \lambda_j BC$ is Hurwitz for all $j \in \{2, \dots, 7\}$.

Fig. 1: (a) An undirected weighted star network consisting of 7 vertices, in which vertex 1 is the hub. (b) A reduced star network consisting of 4 vertices, in which vertex 1' is the hub.

We select the eigenvalues of the reduced *Laplacian matrix* as $\lambda(L_r) = \{0, 2.3442, 3.2508, 14.0759\}$. By using the projection in Theorem 1, the reduced-order model (9) can be obtained with

$$L_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2.3442 & 2.3442 & 0 & 0 \\ -3.2508 & 0 & 3.2508 & 0 \\ -14.0759 & 0 & 0 & 14.0759 \end{bmatrix},$$

$$F_r = [0.1429 \quad 0.2636 \quad 0.0890 \quad -0.7582]^T,$$

where L_r is the *Laplacian matrix* associated with the directed weighted star network as shown in 1b, and the weights are given by $\{\hat{w}_1, \hat{w}_2, \hat{w}_3\} = \{2.3442, 3.2508, 14.0759\}$. Thus, the star network structure can be preserved by using the projection proposed in Theorem 1, as the synchronization property.

Remark 2: Note that different choices of the projection matrix V_r leads to different graphs for the reduced-order model. Theorem 1 provides a method for the construction of V_r , which results in a directed weighted star graph. Due to the limitation of space, we will not further discuss the other construction methods of V_r .

The following theorem shows that the proposed method and the graph clustering-based method in [1] are equivalent when the selected clustering is an AEP of the underlying graph.

Theorem 2: For the multi-agent network system (5). If there exists an AEP π^* , then, the reduced-order multi-agent network (9) is equivalent to the reduced-order system obtained by the clustering-based projection according to π^* .

Proof: Let $L_r^{\pi^*}$ denotes the reduced *Laplacian matrix* obtained by the graph clustering-based projection [1], which is given by $L_r^{\pi^*} = (P_{\pi^*}^T P_{\pi^*})^{-1} P_{\pi^*}^T L P_{\pi^*}$, P_{π^*} represents the characteristic matrix of π^* .

According to the Propositions 1 and 2 in [20], we have that $\text{im} P_{\pi^*}$ is L -invariant and the set of eigenvalues of $L_r^{\pi^*}$ is a subset of eigenvalues of L . From the proof of Theorem 1, one obtains that $\text{im} V_r$ is also L -invariant, where V_r is given in (8). Thus, for the same eigenvalues, there always exists a transformation such that $L_r^{\pi^*} = \hat{T}^{-1} L_r \hat{T}$, where L_r is given in (9).

Hence, in terms of AEP, it concludes that the proposed method and the graph clustering-based method [1] are equivalent. ■

For any given partition π , it can not be guaranteed that $\lambda(L_r^\pi)$ is a subset of $\lambda(L)$. According to [12], $A - \lambda_j BC$ being Hurwitz for all nonzero $\lambda_j \in \lambda(L)$ does not imply that $A - \lambda_i BC$ being Hurwitz for all nonzero $\lambda_i \in \lambda(L_r^\pi)$. Thus, synchronization may not be preserved by using a graph clustering-based projection in [1]. However, the $\lambda(L_r)$ obtained by the proposed method is a subset of $\lambda(L)$. It follows from $A - \lambda_j BC$ being Hurwitz that $A - \lambda_i BC$ being Hurwitz for all nonzero $\lambda_i \in \lambda(L_r)$. That is, the synchronization property of the reduced of multi-agent network system can be preserved by the model reduction method proposed in this paper. A further illustration can be given by considering following examples.

Example 2: Consider Example 2 in [1], for the AEP

$$\pi^* = \{\{1, 2, 3, 4\}, \{5, 6\}, \{7\}, \{8\}, \{9, 10\}\},$$

the eigenvalues of the reduced *Laplacian matrix* obtained by graph clustering-based projection in [1] is

$$\lambda(L_r^{\pi^*}) = \{0, 1.0777, 8.3639, 16.0652, 33.4932\},$$

which is a subset of $\lambda(L)$. By selecting $\lambda(L_r)$ as $\lambda(L_r^{\pi^*})$, then the reduced-order system obtained by the proposed method is equivalent to the one obtained by graph clustering-based method [1].

Consider the multi-agent network as shown in Fig. 2 [20]. The eigenvalues of the *Laplacian matrix* is

$$\lambda(L) = \{0, 0.7312, 2.1353, 3.4659, 4.5494, 5.1183\}.$$

There exists only one AEP, which contains all vertices. Thus, the graph clustering-based method [1] is not applicable and the resulting reduced-order multi-agent network may not be synchronized even if the original multi-agent network is synchronized. However, by using the proposed model reduction method, the set of eigenvalues of the reduced *Laplacian matrix* can be selected as a subset of $\lambda(L)$. Thus, synchronization is preserved in the reduced-order network model.

IV. $\mathcal{H}_2$ APPROXIMATION ERROR

In this section, we zoom into a well-studied multi-agent networks, called *leader-follower multi-agent networks*. For more details on leader-follower multi-agent networks, we refer to [1], [21]–[23]. Particularly, the vertex set of $\mathcal{V}$ is partitioned into two subsets, namely,

$$\mathcal{V}_L = \{v_1, v_2, \dots, v_m\}, \text{ and } \mathcal{V}_F = \mathcal{V} \setminus \mathcal{V}_L, \quad (13)$$

which correspond to the leaders and followers such that the matrix F in (5) is formed as

$$F_{ij} = \begin{cases} 1, & \text{if } i = v_j \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

We present a concise expression for the upper-bound of the $\mathcal{H}_2$ approximation error for leader-follower multi-agent networks.

In order to analyze the approximation error between the original multi-agent network system (5) and the reduced-order model (9), we follow the model setting in [1] and choose $y = (W^{\frac{1}{2}} R^T \otimes I_n)x$ as the output matrix of the multi-agent network system (5), where W, R are given in (3). This output can be considered as a measurement of the disagreement between the states of the agents in (5). In this case, we have

$$\begin{aligned} \dot{x} &= (I_N \otimes A - L \otimes BC)x + (F \otimes B)u, \\ y &= (W^{\frac{1}{2}} R^T \otimes I_n)x. \end{aligned} \quad (15)$$

By the PetrovGalerkin projection framework, the reduced-order multi-agent system (9) is modeled as

$$\begin{aligned} \dot{x}_r &= (I_r \otimes A - L_r \otimes BC)x_r + (F_r \otimes B)u, \\ y_r &= (W^{\frac{1}{2}} R^T V_r \otimes I_n)x_r, \end{aligned} \quad (16)$$

From [2], the $\mathcal{H}_2$ -norm the original multi-agent network system (15) exists. Thus, in the following lemma, we derive the expression for the $\mathcal{H}_2$ -norm of the system (15).

Consider the eigenvalue decomposition of L : $U^T L U = \Lambda$, where $\Lambda = \text{diag}\{0, \lambda_2, \dots, \lambda_r, \lambda_{r+1}, \dots, \lambda_N\}$.

Lemma 2: Let $X_j \geq 0$ be the unique solution of the Lyapunov equation

$$X_j(A - \lambda_j BC) + (A - \lambda_j BC)^T X_j + \lambda_j I_n = 0, \quad (17)$$

where $j \in \{2, \dots, N\}$. The $\mathcal{H}_2$ -norm of the original multi-agent network system (15) is given by

$$\|G(s)\|_2^2 = \text{tr}((U^T F F^T U \otimes I) \mathcal{X}), \quad (18)$$

where $\mathcal{X} = \text{blockdiag}\{0, BB^T X_2, \dots, BB^T X_N\}$.

Now, we are ready to study the upper-bound of the $\mathcal{H}_2$ approximation error between the original and reduced-order leader-follower networks.

Theorem 3: Consider the leader-follower multi-agent network (15), and assume that it is synchronized. The $\mathcal{H}_2$ approximation error between the original network (15) and the reduced network (16) is bounded as

$$\|G(s) - G_r(s)\|_2^2 \leq \sum_{\alpha=r+1}^N \text{tr}(BB^T X_\alpha), \quad (19)$$

where $X_\alpha \geq 0$, $\alpha \in \{r+1, \dots, N\}$, is the unique solution of the Lyapunov equation

$$X_\alpha(A - \lambda_\alpha BC) + (A - \lambda_\alpha BC)^T X_\alpha + \lambda_\alpha I_n = 0.$$

Proof: Due to the limitation of space, the proof will be provided in the full version of this paper. ■

Remark 3: Note that the $\mathcal{H}_2$ approximation error bound in (19) provides a guideline for selecting the eigenvalues of the reduced *Laplacian matrix* given in (9).

V. ILLUSTRATIVE EXAMPLE

In this section, we demonstrate the efficiency of the proposed model reduction method.

Fig. 2: Original spacecraft formation network.

Consider the spacecraft formation network as shown in Fig. 2, which also can be found in [16]. The dynamics of each agent (satellite) is given as follows.

$$\begin{aligned} \begin{bmatrix} \dot{r}_i \\ \dot{\ddot{r}}_i \end{bmatrix} &= \begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} r_i \\ \ddot{r}_i \end{bmatrix} + \begin{bmatrix} 0 \\ I_3 \end{bmatrix} u_i, \\ \eta_i &= C \begin{bmatrix} r_i \\ \ddot{r}_i \end{bmatrix}, \end{aligned} \quad (20)$$

where $r_i \in \mathbb{R}^3$ is the position vector of the i -th agent,

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 \times 10^{-6} & 0 \\ 0 & 0 & -10^{-6} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2 \times 10^{-3} & 0 \\ -2 \times 10^{-3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ C &= \begin{bmatrix} 0.6569 & -0.0013 & 0 & 1.9789 & 0 & 0 \\ 0.0013 & 0.6569 & 0 & 0 & 1.9789 & 0 \\ 0 & 0 & 0.6596 & 0 & 0 & 1.9789 \end{bmatrix}. \end{aligned}$$

Thus, the network dynamical can be written as

$$\dot{x} = (I_4 \otimes A - L \otimes BC)x + (F \otimes B)u, \quad (21)$$

where A, B, C are given as in (20), L is the *Laplacian matrix* associate with the topology as shown in Fig. 1. [16],

$$L = \begin{bmatrix} 7 & -1 & -4 & -2 \\ -1 & 6 & -5 & 0 \\ -4 & -5 & 9 & 0 \\ -2 & 0 & 0 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad (22)$$

and $\lambda(L) = \{0, 2.0839, 8, 13.9161\}$. Note that the network (21) is a leader-follower network, in which vertices 1, 2 are leaders. It can be verified that $A - \lambda_j BC$ is Hurwitz for all $\lambda_j \in \lambda(L) \setminus \{0\}$. It means that the original network (21) is synchronized. However, the subsystems in (20) are not passive. Thus, the balanced truncation method [2] is not applicable.

Choosing the output of network (21) as $y = (W^{\frac{1}{2}} R^T \otimes I_4)x$, which represents the measure of the disagreement between the states of the agents in (21). For comparison, we reduce the network vertices to 3 by two methods, namely, the graph clustering method [1] and the proposed method in the paper. Suppose a graph clustering is given by $\pi = \{\{1\}, \{2, 3\}, \{4\}\}$, which is characterized by

$$V_r^\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The reduced *Laplacian matrix* obtained by graph clustering method [1] is given by

$$L_r^\pi = \begin{bmatrix} 7 & -5 & -2 \\ -2.5 & 2.5 & 0 \\ -2 & 0 & 2 \end{bmatrix},$$

Fig. 3: (a) A reduced topology obtained by the graph clustering method in [1]. (b) The reduced topology obtained by the proposed model reduction method.

which is associated with the reduced directed graph as shown in Fig. 3a. Note that $\lambda(L_r^\pi) = \{0, 2.1358, 9.3642\}$. It can be verified that . Thus, the reduced-order system obtained by graph clustering method [1] is not synchronized with respect to the given clustering π , as well as considering the clustering $\{\{1, 2\}, \{3\}, \{4\}\}$. It can be verified that

$$\text{tr}(BB^T X_2) < \text{tr}(BB^T X_3) < \text{tr}(BB^T X_4).$$

According to the error bound given in Theorem 3, we select the eigenvalues of the reduced *Laplacian matrix* as $\lambda(L_r) = \{0, 8, 13.9161\}$. By using the proposed model reduction method, the obtained reduced *Laplacian matrix* is

$$L_r = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 8 & 0 \\ -13.9161 & 0 & 13.9161 \end{bmatrix}, \quad (23)$$

which is associated with the reduced directed graph as shown in Fig. 3b.

The relative $\mathcal{H}_2$ approximation error with respect to the original network system (21) and the reduced-order model obtained by the proposed method is 0.2178. Thus, the reduced-order system obtained by the proposed method in this paper approximates the original network well, as well as preserves the interconnection structure with diffusive couplings and synchronization.

VI. CONCLUSION

We have presented a novel synchronization preserving model reduction method for multi-agent network systems. Based on selected eigenvalues, a projection is constructed to produce a reduced-order network, which retains an interconnection structure of diffusive couplings. For the leader-follower network, a concise expression for the upper-bound of the $\mathcal{H}_2$ approximation error has been derived, which provided a guideline to choose the eigenvalues of the reduced *Laplacian matrix*. The main advantage of this paper is that synchronization of the reduced-order network can be guaranteed without extra assumptions on the subsystems. An illustrative example has shown the effectiveness of the proposed model reduction method. For our future work, the extensions of this method to second-order networks and directed networks are under consideration.

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