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Primes Dividing Invariants of CM Picard Curves

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Abstract. We give a bound on the primes dividing the denominators of invariants of Picard curves of genus 3 with complex multiplication. Unlike earlier bounds in genus 2 and 3, our bound is based, not on bad reduction of curves, but on a very explicit type of good reduction. This approach simultaneously yields a simplification of the proof and much sharper bounds. In fact, unlike all previous bounds for genus 3, our bound is sharp enough for use in explicit constructions of Picard curves.

1 Introduction

The Hilbert class polynomial of an imaginary quadratic field $K$ is the polynomial whose roots are the $j$-invariants of the elliptic curves $E$ with endomorphism ring isomorphic to the maximal order $\mathcal{O}_K$. Its roots generate the Hilbert class field and are used for constructing elliptic curves with prescribed order, which are used in cryptography.

The Hilbert class polynomial has integer coefficients, so in order to compute it, it suffices to numerically approximate its coefficients up to the decimal point. These ideas can be generalized to curves of genus $g$ as long as their Jacobians have complex multiplication (CM). The imaginary quadratic field needs to be replaced by a CM field $K$ of degree $2g$, that is, a totally imaginary quadratic extension of a totally real number field of degree $g$, and we consider curves whose Jacobian has endomorphism ring isomorphic to $\mathcal{O}_K$.

Using suitable invariants for curves of genus $g$, this gives rise to class polynomials, whose coefficients are rational, but not necessarily integral. Computational methods for numerically approximating these polynomials are known for $g \leq 3$ [1–3, 8, 9, 13, 16, 18, 25, 27, 28].

For the efficiency of the methods, as well as a proof of their output, and a theoretical understanding of the types of $S$-integers created with such constructions, we need to know the denominators of the coefficients of these polynomials.

For the case $g = 2$, these denominators are now understood, thanks to the work of Bruinier, Goren, Lauter, Viray, and Yang [7, 10, 11, 19]. The denominators in that case are effectively computable products of powers of small primes, which have been used for computing and proving correctness of CM curves of genus two [6].

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In general for \( g = 3 \), it is expected [13, §4] that such a result does not hold. However, for the specific case of hyperelliptic curves \((y^3 = x^8 + \cdots)\), a bound on the primes dividing the denominators of invariants is given in [14]. Its proof also works for Picard curves \((y^3 = x^4 + \cdots)\), except for a technical conjecture in [14], which is a work in progress. The bound in [14] is unfortunately too large to be practical.

All bounds mentioned so far are based on the fact that primes dividing the denominators of invariants are primes of bad reduction. The difficulty that makes the bound of [14] so large is that just knowing bad reduction of the curve does not give much information on the endomorphism structure of the reduction of the Jacobian.

In this paper we propose an alternative set of invariants for Picard curves. With that set of invariants, the primes dividing the denominators are primes of a certain very explicit type of reduction (Lemma 3.1).

We use the embedding of \( \mathcal{O}_K \) into the endomorphism ring of the reduction of the Jacobian (Jac \((C)\) modulo \( p \)) to get a contradiction for large \( p \). We get matrices whose entries are quaternion algebra elements. These quaternion algebra elements are forced to commute when \( p \) gets very large \([10, 14]\). Our explicit type of reduction allows us to prove commutativity directly, so we do not need to assume that \( p \) is very large in our proofs.

This drastically reduces the final bound on \( p \) (Theorem 2.2). In fact, we get a formula for a small set \( S \) of small primes such that the denominators are \( S \)-units (Theorem 9.1). We also conjecture a bound on the exponents, yielding a formula for a denominator that is small enough for practical computation (Conjecture 9.2). We made a SageMath implementation of our bounds (available online [15]) and give numerical examples in Section 10.

2 Picard Curve, Invariants, and Statement of the Main Theorem

In this section we introduce Picard curves, a new set of invariants of Picard curves, and the basic concepts of complex multiplication. We also state the main theorem and give an overview of the proof.

2.1 Picard Curves

Let \( L \) be a field of characteristic not 2 or 3. A Picard curve of genus 3 over \( L \) is a smooth, projective, plane curve given by an equation of the form

\[
C : y^3 = x^4 + ax^2 + bx + c,
\]

where \( a, b, c \in L \). Suppose that \( L \) contains a primitive third root of unity \( \zeta_3 \). The automorphism group of \( C \) then contains \( \rho : (x, y) \mapsto (x, \zeta_3 y) \) of order 3. The push-forward \( \rho_* \) of this automorphism is an automorphism of the Jacobian \( J = \text{Jac}(C) \), and hence is a primitive third root of unity in the endomorphism ring. By abuse of notation, we denote \( \rho_* \in \text{End}(J) \) also by \( \zeta_3 \).

Two Picard curves \( C : y^3 = x^4 + ax^2 + bx + c \) and \( C' : y^3 = x^4 + a'x^2 + b'x + c' \) over a field \( L \) of characteristic not dividing 6 are isomorphic over \( L \) if and only if there
exists a \( \lambda \in L^* \) with
\[
(2.2) \quad \lambda^6 a = a', \quad \lambda^9 b = b', \quad \lambda^{12} c = c'.
\]

### 2.2 Complex Multiplication

Let \( C \) be a Picard curve. We say that \( C \) or its Jacobian \( J = \text{Jac}(C) \) has complex multiplication (CM) if there is a number field \( K \) of degree 6 and an embedding \( \iota : K \to \text{End}(J) \otimes \mathbb{Q} \). Henceforth, assume that this is the case, and let \( K \) and \( \iota \) be as above. We say that \( C \) and \( J \) have CM by the order \( \mathcal{O} = \iota^{-1}(\text{End}(J)) \subset K \).

We say that \( J \) has primitive CM if the embedding \( \iota \) gives an isomorphism from \( K \) to the endomorphism algebra \( \text{End}(J_T) \otimes \mathbb{Q} \) of \( J \) over the algebraic closure of \( L \). If \( \text{char}(L) = 0 \) and \( J \) has CM, then after extending \( L \), the induced representation on the tangent space is isomorphic to a product of embeddings of \( K \) into \( L \). The set \( \Phi \) of such embeddings is called the CM type of the map \( K \to \text{End}(J) \otimes \mathbb{Q} \). It is known that if \( J \) has primitive CM if and only if \( K \) is a CM field and the CM type is primitive, that is, if and only if for every CM subfield \( K_1 \subsetneq K \), the restriction of \( \Phi \) to \( K_1 \) is not a CM type \([17, \S 1.3.5]\).

Note that if \( J \) has primitive CM by \( \mathcal{O} \), then the endomorphism \( \zeta_3 \) of Section 2.1 is (via \( \iota \)) a primitive third root of unity in \( \mathcal{O} \). Conversely, if a smooth curve of genus 3 has primitive CM by an order containing a primitive third root of unity, then the curve is a Picard curve (this is stated in the case of maximal orders \([16, \text{Lemma 1}]\), though the final sentence of the proof is wrong and needs to be replaced by \([12, \text{Lemma 7.3 and the paragraph above it}]\); the proof does not use that the order is maximal, only that the third root of unity is in the order).

This allowed Koike and Weng \([16]\) to construct genus-three Picard curves with CM by orders in fields of the form \( F(\zeta_3) \) for a totally real cubic field \( F \) and a third root of unity \( \zeta_3 \), just as one can construct elliptic curves with CM by orders in imaginary quadratic fields.

### 2.3 Invariants of Picard Curves

A homogeneous Picard curve invariant is a weighted homogeneous polynomial \( I \in \mathbb{Z}[a, b, c] \) where \( a, b, c \) are formal polynomial variables of weights 2, 3, 4, respectively. For a model as in (2.1) of a Picard curve \( C \) over a field \( L \), we get the value \( I(C) \in L \) by evaluating \( I \) in the coefficients of (2.1).

For example, we have the invariant
\[
(2.3) \quad \Delta = -4a^3b^2 + 16a^4c - 27b^4 + 144ab^2c - 128a^2c^2 + 256c^3
\]
of weight 12, and \( \Delta(C) \) is non-zero for all Picard curves \( C \), as it is the discriminant of the right-hand side of (2.1). An absolute Picard curve invariant is a quotient \( j = u/b^2 \), where \( u \in \mathbb{Z}[a, b, c] \) has weight 3\( \ell \). For example, the three rational functions \( j_1 = a^3/b^2 \), \( j_2 = ac/b^2 \), and \( j_3 = c^3/b^4 = j_1^{-1}j_2 \) are absolute Picard curve invariants.

All Picard curves \( C \) with \( a(C) \neq 0 \) can be reconstructed up to twists from the values \( j_1(C) \) and \( j_2(C) \) of the invariants \( j_1 \) and \( j_2 \) as follows. Given a Picard curve \( C \) over a field \( L \), the curve \( D : y^3 = x^4 + j_1(C)x^2 + j_1(C)x + j_1(C)j_2(C) \) is isomorphic to \( C \) over the algebraic closure \( \overline{L} \).
Moreover, the three invariants $j_1$, $j_2$, and $j_3$ generate the ring of all absolute Picard curve invariants. Indeed, an absolute Picard curve invariant is a linear combination of monomials $a^A c^C / b^B$ with $2A + 4C = 3B$, and each such monomial is a non-negative power of $j_2$ times a monomial with $A = 0$ or $C = 0$, which in turn is a power of $j_3$ or $j_1$.

Instead of the quotients $b^2/a^3 = 1/j_1$ and $c/a^2 = j_2/j_1$ used by Koike and Weng [16], we consider $j_1$, $j_2$, $j_3$ because the primes dividing the denominators of our invariants have nice properties that we can use to find good bounds for them (Lemma 3.1 and Proposition 3.3).

**Remark 2.1** If $C$ is a Picard curve of genus 3 over a number field $L$ with primitive CM and $j = u/b^l$ is an absolute Picard curve invariant, then $b(C) \neq 0$ and $j(C) \in L$. This is because if $b(C) = 0$, then the curve $C$ admits a non-constant morphism to an elliptic curve (formula in (3.1)) and hence its Jacobian is not simple, which gives a contradiction with having a primitive CM-type.

In particular, for every sextic CM of order $\emptyset$ with $\zeta_3 \in \emptyset$, the class polynomials

\begin{equation}
H_{\emptyset, 1}(X) = \prod_C (X - j_1(C)),
\end{equation}

\begin{equation}
\tilde{H}_{\emptyset, 2}(X) = \sum_C j_2(C) \prod_{D \neq C} (X - j_1(D)),
\end{equation}

(with sums and products ranging over the isomorphism classes of curves over $\mathbb{C}$ with primitive CM by $\emptyset$; see [9]) are well defined.

### 2.4 Statement of the Main Result and Overview of the Proof

A weak version of our main theorem is as follows.

**Theorem 2.2** Let $C$ be a Picard curve of genus 3 over a number field $L$ with End $(\text{Jac}(C)_L)$ isomorphic to an order $\emptyset$ of a number field $K$ of degree 6. Let $K_+^{\text{d}}$ be the real cubic subfield of $K$. Let $\mu \in \mathbb{Z} + 2\emptyset$ be such that $K_+ = \mathbb{Q}(\mu)$.

Let $j = u/b^l$ be an absolute Picard curve invariant. Let $\mathfrak{p}$ be a prime of $L$ lying over a rational prime $p$. If $\text{ord}_{\mathfrak{p}}(j(C)) < 0$, then

\[
p \leq \text{tr}_{K_+/\mathbb{Q}}(\mu^2) \quad \text{and} \quad p \leq \left(1 + \frac{16}{\pi} |\Delta(\mathfrak{O}_+)|^{1/2}\right)^3 < 196|\Delta(\mathfrak{O}_+)|^{3/2}.
\]

In Sections 3–8 we prove Theorem 2.2. We give a stronger version in Section 9. The stronger version gives an algorithm for computing a set of primes, instead of just a bound on the primes. In Section 9 we also give a conjecture about the powers to which such primes appear in the denominators of the invariants. A SageMath implementation is available online [15]. In Section 10 we give examples that show that the resulting denominator bounds are small enough for practical class polynomial computations.

The first step of the proof of Theorem 2.2 is the explicit type of reduction that is implied by the appearance of a prime $p$ in the invariant $b$. This type of reduction is given in Lemma 3.1. Proposition 3.3 then shows how this type of reduction
makes the reduction of the Jacobian decompose into a product of an elliptic curve $A_1$ and a principally polarized abelian surface $A_2$. The rest of Section 3 is the proof of Proposition 3.3.

Once we know this decomposition of the reduction $\tilde{J}$ of the Jacobian $J = \text{Jac}(C)$, the endomorphisms of $J$ give rise to matrices consisting of homomorphisms between the components $A_1$ and $A_2$ of $\tilde{J}$. This will make $A_2$ decompose further and give our endomorphisms as matrices over the endomorphism ring $R$ of $A_1$ (see Section 4).

An important part of the earlier proofs in genus 2 and 3 [10, 14] is to force these matrices to have entries in a field instead of in the quaternion ring $R$. This was always done by an argument from [10] that uses the fact that elements of small norm of quaternion algebras of large discriminant commute. In order to be able to use this fact, the prime $p$ needs to be very large, which is why bounds based on that type of argument typically are very large (see [19] for an exception that is more complicated and has not yet been generalized to genus 3). In Section 5 we use the explicit endomorphism $\zeta_j = \rho_s$ and the fact that this induces an endomorphism of $A_1$ and $A_2$ to get commutativity. This greatly simplifies our proof and drastically reduces the resulting bounds.

In Section 6 we use primitivity of the CM type, via the tangent space, to show that primes with our type of reduction divide the exponent $n$ of the kernel of the isogeny $\tilde{J} \to A_3^4$. This argument is exactly the same as in [14], hence that section is very short and is basically a reference to [14]. In genus 2 [10] such an argument is not needed; see [14, § 5] for details.

In Section 7 we bound the exponent $n$ mentioned in the previous paragraph. For this, we need to have a well-chosen isogeny in Section 4 on which to base the exponent $n$, and we need to look at what happens with the polarizations (which give rise to positive definite matrices) under our isogenies. This completes the proof of the first inequality of Theorem 2.2.

Section 8 uses geometry of numbers to derive the second inequality from the first.

## 3 Reduction of Picard Curves

In this section we give the explicit type of reduction that follows from a prime dividing the invariant $b$ of a Picard curve. Lemma 3.1 gives the three possible reduction types of the curve, and Proposition 3.3 shows what this implies for the decomposition of the reduction of the Jacobian.

**Lemma 3.1**  Let $C$ be a Picard curve of genus 3 over a number field $L$ and let $p \mid 6$ be a prime of $\mathcal{O}_L$. Let $j = u/b^6$ be an absolute Picard curve invariant.

If $\text{ord}_p(j(C)) < 0$, then after replacing $L$ with an extension and $C$ with an isomorphic curve, we are in one of the following cases.

1. $C : y^3 = x^4 + ax^2 + bx + 1$ with $a, b \in \mathcal{O}_L$ such that $b \equiv 0$ and $a \equiv \pm 2$ modulo $p$. The reduction of this equation (from $\mathcal{O}_L$ to $\mathcal{O}_L/p$) is the singular curve $y^3 = (x^2 \pm 1)^2$ of geometric genus 1.

2. $C : y^3 = x^4 + x^2 + bx + c$ with $b, c \in p$. The reduction of this equation is the singular curve $y^3 = (x^2 + 1)x^2$ of geometric genus 2.
(3) \( C : y^3 = x^4 + ax^2 + bx + 1 \) with \( a, b \in O_L \) such that \( b \equiv 0 \) and \( a \neq \pm 2 \) modulo \( p \).

The reduction of this equation is the smooth curve \( y^3 = x^4 + \bar{a}x^2 + 1 \) of genus 3, where \( \bar{a} = (a \mod p) \).

**Proof** Let \( m_0 = \min\left\{ \frac{1}{3}v(a), \frac{1}{3}v(b), \frac{1}{3}v(c) \right\} \), where \( v = \text{ord}_p \) is the \( p \)-adic valuation. By our assumption that \( p \) divides the denominator of \( j(C) \), this minimum is not attained by \( \frac{1}{3}v(b) \).

If it is attained by \( \frac{1}{3}v(c) \), then we scale the curve over \( \overline{L} \) as in (2.2) so that \( c = 1 \). As the minimum is not attained by \( \frac{1}{3}v(b) \), we get that the reduction is \( y^3 = x^4 + \bar{a}x^2 + 1 \), where the right-hand side has discriminant \( 16(\bar{a} - 2)^2(\bar{a} + 2)^2 \). In particular, if \( \bar{a} \neq \pm 2 \), then we are in case (3).

If \( \bar{a} = \pm 2 \), then the reduction is \( y^3 = (x^2 \pm 1)^2 \). Let \( Y = (x^2 \pm 1)/y \). Then we get \( Y^3 = (x^2 \pm 1) \), that is, the curve \( \overline{C} \) is birational to the elliptic curve \( x^2 = Y^3 \pm 1 \) with \( j \)-invariant 0.

The only remaining case is the case where the minimum is not attained by \( \frac{1}{3}v(c) \). As the minimum is not attained by \( \frac{1}{3}v(b) \), we find that it is only attained by \( \frac{1}{3}v(a) \). Now we scale the curve so that \( a = 1 \). We get that the reduction is \( y^3 = x^4 + x^2 = (x^2 + 1)x^2 \). Let \( Y = y/x \). Then we get \( xY^3 = x^2 + 1 \), which is the hyperelliptic curve \( x^2 - Y^3x = -1 \) of genus 2. In fact, taking \( X = 2x - Y^3 \), we get the hyperelliptic curve \( X^2 = Y^6 - 4 \).

**Example 1** Let \( K = K_+(\zeta_3) \), where \( K_+ = \mathbb{Q}[y]/(y^3 - y^2 - 4y - 1) = \mathbb{Q}(\zeta_{13})_+ \) is totally real abelian of discriminant 132 and conductor 13. Let \( C : y^3 = x^4 - 2 \cdot 7^2 \cdot 13x^2 + 2^3 \cdot 5 \cdot 13 \cdot 47x - 5^2 \cdot 13^2 \cdot 31 \).

The curve \( C \) was computed by Koike and Weng [16, \$6.1(3)], who conjectured that its Jacobian has CM by \( O_K \) of primitive CM type. This curve and its reductions also appear in [5, \$5.2].

We compute

\[
\begin{align*}
    j_1 &= -\frac{7^6 \cdot 13}{2^3 \cdot 5^2 \cdot 47^2}, \\
    j_2 &= \frac{7^2 \cdot 13 \cdot 31}{2^5 \cdot 47^2}, \\
    j_3 &= -\frac{5 \cdot 13^2 \cdot 31^3}{2^{12} \cdot 47^4}.
\end{align*}
\]

We find that the primes in the denominators of \( j_1, j_2, \) and \( j_3 \) are 2, 5, and 47. Lemma 3.1 does not apply to the prime 2 as it divides 6. The prime 5 is of case (2) in Lemma 3.1.

The prime 47 is of case (3) in Lemma 3.1 as follows. Take an integer \( r \equiv 11 \) modulo 47, let \( \alpha = \sqrt{r} \) and \( L = \mathbb{Q}(\alpha) \). Then \( C \) is isomorphic over \( L \) to the curve given by

\[
y^3 = x^4 - \alpha^6 \cdot 2 \cdot 7^2 \cdot 13x^2 + \alpha^9 \cdot 2^3 \cdot 5 \cdot 13 \cdot 47x - \alpha^{12} \cdot 5^2 \cdot 13^2 \cdot 31,
\]

which modulo 47 is \( C : y^3 = x^4 + 19x^2 + 1 \).

**Remark 3.2** We know of no examples of Picard curves with primitive CM by a sextic field that have a reduction as in Lemma 3.1 (1).

**Proposition 3.3** Let \( C \) be a Picard curve of genus 3 over a number field \( L \) containing a primitive third root of unity \( \zeta_3 \). Let \( p \mid 6 \) be a prime of \( L \). Suppose that \( C \) is given (over \( L \)) by an equation as in one of the three cases of Lemma 3.1.
Let $J = \text{Jac}(C)$ be the Jacobian of $C$, let $\mathcal{J}$ be its Néron model over $\mathbb{Z}_p$ and let $\mathcal{J}$ be its reduction modulo $p$. Assume that $J$ has CM or that we are in case 3 of Lemma 3.1. Recall that $\zeta_3 = \rho_\ast$ is a third root of unity in $\text{End}(J)$; it induces endomorphisms of $\mathcal{J}$ and $\mathcal{J}$ that we also denote by $\zeta_3$.

Then there are abelian subvarieties $A_i$ (with inclusion maps $I_i: A_i \to \mathcal{J}$), surjective homomorphisms $s_i: \mathcal{J} \to A_i$ and endomorphisms $e_i \in \text{End}(\mathcal{J})$ for $i \in \{1, 2\}$, and an integer $d \in \{1, 2\}$ such that the following hold for all $i, j \in \{1, 2\}$.

(i) 
$$e_1 + e_2 = [d] \in \text{End}(\mathcal{J}),$$
$$e_2^t = [d] e_1 \in \text{End}(\mathcal{J}),$$
$$e_1 e_2 = e_2 e_1 = 0 \in \text{End}(\mathcal{J}),$$
$$e_i^t = e_i \in \text{End}(\mathcal{J}),$$
where $^t$ denotes the Rosati involution,
$$e_i = I_i s_i \in \text{End}(\mathcal{J}),$$
$$s_i I_i = [d] \in \text{End}(A_i),$$
if $i \neq j$, then $s_i I_j = 0 \in \text{Hom}(A_j, A_i)$.

Here and later, we write simply $f g$ for $f \circ g$ in order to keep the notation clean and concise.

(ii) The abelian variety $A_i$ has dimension $i$ and we have a commutative diagram

$$\mathcal{J} \xrightarrow{(\frac{\mathcal{J}}{\mathcal{J}})} A_1 \times A_2 \xrightarrow{(I_1 I_2)} \mathcal{J} \xrightarrow{(\frac{\mathcal{J}}{\mathcal{J}})} A_1 \times A_2.$$ 

(iii) if $i \neq j$, then we have $s_i \zeta_3 I_j = 0 \in \text{Hom}(A_j, A_i)$.

We prove Proposition 3.3 separately in the smooth case (Lemma 3.1 (3) and in the singular cases (Lemma 3.1 (1) and (2)). The smooth case is the main case and the proof is in Section 3.1. The proofs of the singular cases are in Section 3.2.

### 3.1 The Smooth Case: $y^3 = x^4 + ax^2 + 1$

We now prove Proposition 3.3 in the smooth case (3) of Lemma 3.1, where we will see that it holds with $d = 2$. We consider the Picard curve

$$\overline{C}: y^3 = x^4 + ax^2 + 1.$$ 

The automorphism group $\text{Aut}(\overline{C})$ contains the elements $\sigma: (x, y) \mapsto (-x, y)$ of order 2 and $\rho = \rho_C: (x, y) \mapsto (x, \zeta_3 y)$ of order 3.

As $C$ has good reduction at $p$, we have $\overline{J} = \text{Jac}(\overline{C})$; we associate the $O_{\mathcal{K}, p}$-scheme $\text{Pic}^0(\overline{C})$ with $C$. By [4, Theorem 9.3.7], the special fiber of $\text{Pic}^0(\overline{C})$ is $\text{Jac}(\overline{C})$ and it is smooth. Finally, by [4, Theorem 9.5.1], we have that $\text{Pic}^0(\overline{C})$ is a Néron model $\mathcal{J}$ of the Jacobian of $\overline{C}$. In particular, the special fiber of $\text{Pic}^0(\overline{C})$, i.e., $\text{Jac}(\overline{C})$, is isomorphic to the special fiber of $\mathcal{J}$, that is, $\mathcal{J}$.
The curve $\overline{C}$ is a 2-cover of the elliptic curve $E : v^2 + av = u^3 - 1$ with CM by $\mathbb{Z}[\zeta_3]$. Indeed, we have

\begin{equation}
\phi: \overline{C} \longrightarrow E \\
(x, y) \longmapsto (u, v) = (y, x^2).
\end{equation}

The curve $E$ also has an automorphism $\rho = \rho_E: (u, v) \mapsto (\zeta_3 u, v)$ of order 3, and we have $\rho_E \circ \phi = \phi \circ \rho_{\overline{C}}$, or simply

\begin{equation}
\rho \phi = \phi \rho.
\end{equation}

For every curve morphism $f: D_1 \rightarrow D_2$, we get a push-forward morphism $f_*: \text{Jac}(D_1) \rightarrow \text{Jac}(D_2)$ and a pullback morphism $f^*: \text{Jac}(D_2) \rightarrow \text{Jac}(D_1)$. With this notation, let

$e_1 = \phi^* \phi_*, \quad e_2 = [2] - e_1 \quad \in \text{End} (\overline{J})$.

Let $A_i$ be the image of $e_i$ and let $s_i$ be defined by the commutative diagram

\begin{equation*}
\begin{array}{ccc}
\overline{J} & \xrightarrow{s_i} & A_i \\
\downarrow & & \downarrow \\
\overline{J} & \xrightarrow{e_i} & A_i
\end{array}
\end{equation*}

Let $d = 2$. The equality $e_1 + e_2 = 2$ is the definition of $e_2$. As $\phi$ is a 2-cover, we get $\phi_* \phi^* = [2] \in \text{End}(E)$. In particular, we get $e_1^2 = \phi^* \phi_* \phi^* \phi_* = \phi^*[2] \phi_* = 2e_1$ and $e_2^2 = 4 - 4e_1 + e_1^2 = 2e_2$. We also get $e_1 e_2 = e_1([2] - e_1) = 2e_1 - 2e_1 = 0$ and similarly $e_2 e_1 = 0$.

By Mumford [21, pp. 327–328], if $f: D_1 \rightarrow D_2$ is a non-constant curve morphism and $(\text{Jac}(D_i), \lambda_i)$ is the Jacobian of $D_i$ with its polarization, then

\begin{equation*}
(f_* \phi^*)^\vee = \lambda_1 f^* \lambda_2^{-1} \quad \text{and} \quad \lambda_i^\vee = \lambda_i.
\end{equation*}

Taking duals, we also have $(f^*)^\vee = \lambda_2 f_* \lambda_1^{-1}$. In particular, we get

\begin{align*}
e_1^\dagger &= \lambda_2^{-1}(\phi^* \phi_*)^\vee \phi^* \phi_* = \lambda_2^{-1}(\phi^*)^\vee (\phi^*)^\vee \phi^* \\
\end{align*}

The identities $e_i = I_i s_i$ are the definition of $s_i$. To compute $s_i I_j$, we compose with the surjective map $s_j$ and the injective map $I_j$. If $i = j$, then we get $I_i (s_i I_i) s_i = e_i^2 = 2e_i = I_i [2] s_i$. By surjectivity of $s_j$ and injectivity of $I_i$, this gives $s_i I_i = [2]$. If $i \neq j$, then we get $I_i (s_i I_j) s_j = e_i e_j = 0 = I_i [0] s_j$, hence again by surjectivity and injectivity we get $s_i I_j = 0$. This proves (i).

Commutativity of the diagram follows from $I_i s_i + I_2 s_2 = e_1 + e_2 = 2$ and the formulas for $s_i I_j$. The dimension of $A_1$ is the dimension of $E$, which is 1. The commutativity of the diagram shows that $A_1 \times A_2$ has the same dimension as $I$, hence $A_2$ has dimension 2, which proves (ii).

Finally, we prove (iii). Since $I_i$ is injective and $s_j$ is surjective, it suffices to prove $I_i s_i \zeta_3 I_i s_j = 0$, that is, $e_i \zeta_3 e_j = 0$.

Recall $\zeta_3 = \rho_*$, and by (3.2), we have $\phi_* \rho_* = \rho_* \phi_*$. Hence we get

\begin{align*}
e_1 \zeta_3 e_1 &= \phi^* \phi_* \rho_* \phi^* \phi_* = \phi^* \rho_* \phi_* \phi^* \phi_* = \phi^* \rho_* [2] \phi_* \\
&= \phi^* \phi_* \rho_* [2] = 2e_1 \zeta_3.
\end{align*}
In particular, we get $e_1\zeta_3 e_2 = 2e_1\zeta_3 - 2e_1\zeta_3 = 0$.
We also have $\rho^*\rho_*=1$, so $\zeta_3 = (\rho^*)^{-1}$. Therefore, we also have
\[
e_1\zeta_3 e_1 = \phi^*\phi_*(\rho^*)^{-1}\phi^*\rho_* = \phi^*\phi_*(\rho^*)^{-1}\phi_* = \phi^*[2](\rho^*)^{-1}\phi_* = 2\zeta_3 e_1.
\]
In particular, we get $e_2\zeta_3 e_1 = 2\zeta_3 e_1 - 2\zeta_3 e_1 = 0$. This proves Proposition 3.3 with $d = 2$ in Lemma 3.1 (3).

Remark 3.4 We did not need to write $A_2$ as the Jacobian of an explicit curve for our work. However, for those who are interested, if $\overline{C} : y^3 = x^4 + ax^2 + 1$ with $a \neq 0, -2, 2$ in a field of characteristic not 2 or 3, then a special case by Ritzenthaler and Romagny [23, Theorem 1.1] gives $\overline{J} \sim E \times \text{Jac}(H)$ with $E$ as in Section 3.1 and
\[
H : -ay^2 = (x^2 + 2x - 2) \cdot (x^4 + 4x^3 + (2a^2 - 8)x - a^2 + 4).
\]

3.2 The Singular Cases

In the singular cases, by [14, Theorem 1.1], we already have a bound $p < \frac{1}{3} B^{10}$ under the hypotheses of Theorem 2.2. However, we will see that we can do better.

In this section, we prove Proposition 3.3 in the singular cases of Lemma 3.1 (1) and (2), where we will see that it holds with $d = 1$. In case (g) for $g \in \{1, 2\}$, let $A_g$ be the Jacobian of the smooth model $C_g$ of the curve of geometric genus $g$ listed in Lemma 3.1 (g).

Since the curve $C$ has CM, Proposition 4.2 in [5] applies, so the reduction $\overline{C}$ of a stable model $\mathcal{C}$ of $C$ is tree-like and the reduction $\overline{J}$ of its Jacobian $J = \text{Jac}(C)$ is the polarized product of the Jacobians of the irreducible components of $\overline{C}$.

Then by [5, Corollary 4.3], the reduction of the stable model is a union of either three smooth curves of genus 1 or a smooth curve of genus 1 and a smooth curve of genus 2. By Lemma A.1 (see also Corollary A.2), one of these curves is isomorphic to the curve $C_g$. We conclude that the reduction of the stable model is the union of a copy of $C_g$ and up to two additional smooth curves of total genus $3 - g$. Let $A_g$ be the Jacobian of $C_g$ and let $A_{3-g}$ be the polarized product of the Jacobians of those additional curves, so
\[
\overline{J} = A_1 \times A_2
\]
as principally polarized abelian varieties.

For $i \in \{1, 2\}$, let $I_i$ be the inclusion map of $A_i$ into $\overline{J}$ and let $s_i$ be the projection map of $\overline{J}$ onto $A_i$. Let $e_i = I_i s_i$. Then we get $s_i I_j = 0$ if $i \neq j$ and $s_i I_i = [d]$ with $d = 1$.

As (3.3) is an identity of principally polarized abelian varieties, we get $e_i^1 \equiv e_1$, $e_i^1 \equiv e_2$, and $e_1 + e_2 \equiv [1]$. The identities $e_i^2 = e_i$ and $e_1 e_2 = 2e_1 e_1 = 0$ now follow from the identities in terms of $I_i$ and $s_i$, and the commutativity of the diagram follows from all the given identities. This proves (i) and (ii).

Next we prove (iii). As $I_i$ is an injective map and $s_j$ is a surjective one, it suffices to prove $I_i s_i \overline{\zeta}_3 I_j s_j = 0$, that is, $e_1 \overline{\zeta}_3 e_j = 0$. By the Néron mapping property, the automorphism $\rho$ of $C$ uniquely extends to an automorphism of the stable model. And by the explicit equations in Lemma 3.1, it also extends to an automorphism of order 3 of
$C_g$. Let $\zeta_3$ denote not only $\rho_*$ on $\bar{J}$, but also $\rho_*$ on $A_g$. Then we get $s_g \zeta_3 = \zeta_3 s_g$ and $\zeta_3 I_g = I_g \zeta_3$. So we get

$$e_g \zeta_3 e_g = I_g s_g \zeta_3 I_g s_g = I_g s_g I_g s_g \zeta_3 = I_g s_g \zeta_3 = \zeta_3 s_g e_g = e_g \zeta_3 = \zeta_3 e_g.$$ 

In particular, we get

$$e_g \zeta_3 e_{-g} = e_g \zeta_3 - e_g \zeta_3 = 0, \quad \text{and} \quad e_g \zeta_3 \zeta_3 e_g = \zeta_3 e_g - e_g \zeta_3 e_g = 0.$$

This proves Proposition 3.3 in cases of Lemma 3.1 (1) and (2). Case (3) was done in the previous section.

4 Decomposition and Matrices

If a prime $p$ does not divide 6 and does appear in the denominator of $j(C)$, then Section 3 shows that $\bar{J}$ is isogenous (via the isogeny $F_0$) to a product of abelian varieties $A_1$ and $A_2$ of lower dimension. We also got much information about the isogeny $F_0$, and how it behaves with respect to the third root of unity $\zeta_3 = \rho_* \in \text{End}(\bar{J})$ (see Proposition 3.3).

In this section we show that if $J$ has complex multiplication, then we can use an element $\mu$ of the endomorphism ring of $J$ to decompose $A_2$ further.

Just the fact that $A_2$ is decomposable is not enough. In order to have small and explicit bounds in the end, it is crucial that we can compute the degree of the isogeny $A_2 \to A_1 \times A_1$ in terms of elements of $\mathcal{O}$.

Suppose henceforth that we are in the situation of the hypotheses of the main theorem (Theorem 2.2). In other words, we have $\text{End}(J_T) = \mathcal{O}$ for an order $\mathcal{O}$ in a sextic CM field $K$, we have a totally real element $\mu \in \mathbb{Z} + 2\mathcal{O}\setminus\mathbb{Z}$, an absolute Picard curve invariant $j$, and a prime $p$ of $L$ lying over a rational prime $p$ such that $\text{ord}_p(j(C)) < 0$.

Suppose for now that $p \neq 2, 3$.

We get $\zeta_3 \in \mathcal{O}$ (see Section 2.2) and hence $K = K_+(\zeta_3)$ for the totally real cubic field $K_+$. Of $K$.

Our goal is only to bound $p$, so without loss of generality we assume that all elements of $\text{End}(J_T)$ and the isomorphisms and models of Lemma 3.1 are defined over $L$.

**Remark 4.1** In [14], a $\mu$ is taken with $\mu^2 \in K_+$ totally negative. In our situation, we can switch between totally negative and totally positive $\mu^2$ by replacing $\mu$ by $(2\zeta_3 + 1) \mu$, and the proof remains roughly the same. To make the proof as simple as possible, we will work with totally positive $\mu^2$, that is, totally real $\mu$.

Let $F_0 = (I_1 I_2): A_1 \times A_2 \to \bar{J}$ be the isogeny from Proposition 3.3 (ii), and let $s_1$ and $d$ also be as in that proposition. We get an embedding

$$i_0: \text{End}(\bar{J}) \otimes \mathbb{Q} \to \text{End}(A_1 \times A_2) \otimes \mathbb{Q},$$

$$\alpha \mapsto F_0^{-1} \alpha F_0 = \frac{1}{d} \left( \frac{s_1}{s_2} \right) \circ \alpha \circ (I_1 I_2) = \frac{1}{d} \left( \begin{array}{c} s_1 a I_1 s_1 a I_2 \\ s_2 a I_1 s_2 a I_2 \end{array} \right),$$

sending

$$s_1 a I_1 s_1 a I_2$$

(4.1) \[ \mathbb{Z} + 2\mathcal{O} \subset \mathbb{Z} + d \text{End}(\bar{J}) \to \text{End}(A_1 \times A_2). \]
Write

\[ i_0(\mu) = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \]

where the size of a box reflects the dimension of the domain and codomain of the homomorphism. As \( \mu \in \mathbb{Z} + 2\mathbb{O} \), by (4.1) we get

\[
\begin{align*}
x &= \frac{1}{d} s_1 \mu I_1 \in \text{End}(A_1), \\
y &= \frac{1}{d} s_1 \mu I_2 \in \text{Hom}(A_2, A_1), \\
z &= \frac{1}{d} s_2 \mu I_1 \in \text{Hom}(A_1, A_2), \\
w &= \frac{1}{d} s_2 \mu I_2 \in \text{End}(A_2). \end{align*}
\]

**Lemma 4.2** We have

\[ i_0(2\zeta_3 + 1) = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \]

where \( r_i \in \text{End}(A_i) \) satisfy \( r_i^2 = -3 \).

**Proof** The off-diagonal boxes are zero by the equalities \( s_i \zeta_j I_j = s_i I_j = 0 \) of Proposition 3.3 (i, iii). This gives the shape of the matrix. As its square is \(-3\), we get \( r_i^2 = -3 \).

**Lemma 4.3** The homomorphism

\[
F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & wz \end{pmatrix} : A_1 \times A_1 \times A_1 \to A_1 \times A_2
\]

\[ (P, Q, R) \mapsto (P, z(Q) + wz(R)) \]

is an isogeny.

**Proof** It is necessary and sufficient to prove that the map \( A_1 \times A_1 \to A_2 \) given by \( (Q, R) \mapsto z(Q) + zw(R) \) is an isogeny. But this is analogous to [14, Lemma 3.1], and the proof is identical. We only use that \( \mu \) does not have degree 1 or 2 over \( \mathbb{Q} \).

**Remark 4.4** An alternative choice of isogeny \( F_1 : A_1^3 \to A_1 \times A_2 \) is obtained by replacing \( wz \) by \( z' = \frac{1}{2}(zx + wz) \). Indeed, write \( \mu = i + 2j \) with \( i \in \{0, 1\} \) and \( j \in \mathbb{O} \). Then we get \( \frac{1}{2}(\mu^2 - i) = 2(j^2 - ij) \in 2\mathbb{O} \). As \( z' \) is the lower left entry of \( i_0(\frac{1}{2}(\mu^2 - i)) \), it is in \( \text{Hom}(A_1, A_2) \). Then instead of \( F_1 \) use

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & z' & z' \end{pmatrix} : A_1 \times A_1 \times A_1 \to A_1 \times A_2
\]

\[ (P, Q, R) \mapsto (P, z(Q) + z'(R)). \]
This gives a bound in the end whose valuation at 2 is better, but still non-optimal. As it makes the formulas more complicated, we will not consider it further in this article, but we give this choice as an option in our SageMath implementation.

Let \( \mathcal{R} = \text{End}(A_1) \) and \( \mathcal{B} = \mathcal{R} \otimes \mathbb{Q} \). We get an isogeny \( F = F_0F_1 \) and ring homomorphisms

\[
\iota_1: \text{End}(A_1 \times A_2) \longrightarrow M_{3 \times 3}(\mathcal{B})
\]

\[
f \longmapsto F_1^{-1} f F_1,
\]

and

\[
\iota = \iota_1 \circ \iota_0: \text{End}(\mathcal{F}) \longrightarrow M_{3 \times 3}(\mathcal{B}),
\]

\[
\alpha \longmapsto F^{-1} \alpha F.
\]

Take \( n \in \mathbb{Z}_{\geq 0} \) such that

\[
(4.2) \quad [n] \ker(F_1) = 0.
\]

In (7.2) below, we will take a specific \( n \).

5 Using Commutativity to Get Matrices Over a Field

In this section we use the fact that we have an explicit \( \zeta_3 \) that commutes with \( \mu \) in order to find that the entries of the \( 3 \times 3 \) matrix \( \iota(\mu) \) from Section 4 all lie in the same quadratic field. In the proof of the previous bounds ([10] for \( g = 2 \) and [5, 14] for \( g = 3 \)), we had no such \( \zeta_3 \); the proof that the entries were in a quadratic field was based instead on the fact that "small" elements of large-discriminant quaternion algebras commute, hence that argument worked only for very large primes. Because of our explicit decomposition, our proof is much simpler and our results are much sharper.

We also get various relations between the entries, which we use in Section 7 to bound the entries.

**Lemma 5.1** Let \( \mu \) and \( \iota \) be as in Section 4, let \( \mathcal{R} = \text{End}(A_1) \), and let \( n \) be as in (4.2).

(i) For every \( \alpha \in \mathbb{Z} + 2\mathcal{O} \), the entries of the \( 3 \times 3 \) matrix \( \iota(\alpha) \) are in \( \frac{1}{n}\mathcal{R} \), and the entries of the top row are in \( \mathcal{R} \).

(ii) We have

\[
\iota(\mu) = \begin{pmatrix}
x & a & b \\
1 & 0 & e \\
0 & 1 & f
\end{pmatrix}
\]

with \( x, a, b, ne, nf \in \mathcal{R} \).

(iii) We have

\[
\iota(2\zeta_3 + 1) = \begin{pmatrix}
r_1 & 0 & 0 \\
0 & s & t \\
0 & u & v
\end{pmatrix}
\]

with \( r_1, ns, nt, nu, nv \in \mathcal{R} \), and \( r_1^3 = -3 \).
\textbf{Proof} Let }G: A_1 \times A_2 \to A_1 \times A_1 \times A_1\text{ be the isogeny satisfying }GF_1 = [n]\text{ that exists because of (4.2). Then we have }

\[ F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & wz \\ 0 & w & wz \end{pmatrix} \quad \text{and} \quad F_1^{-1} = \frac{1}{n}G = \frac{1}{n} \begin{pmatrix} n & 0 & 0 \\ 0 & g_1 & g_1w \\ 0 & g_2 & g_2w \end{pmatrix} \]

\text{for some } g_i: A_2 \to A_1 \text{ satisfying } \frac{1}{n}(\frac{g_1}{g_2})(z, wz) = \left( \frac{1}{0}, 1 \right). 

For } i, j \in \{1, 2\}, \text{ the } (i, j)-\text{entry of } t_0(\alpha) \text{ is in } \text{Hom}(A_j, A_1). \text{ Now because of the shape of } F_1 \text{ and } F_1^{-1}, \text{ the matrix } t(\alpha) = F_1^{-1}t_0(\alpha)F_1 \text{ has entries in } \frac{1}{n}R \text{ with the entries of the top row in } R. \text{ This proves (i).}

For (ii), we now only need to compute the lower left } 2 \times 2 \text{ block, so }

\[ t(\mu) = F_1^{-1}t_0(\mu)F_1 = \frac{1}{n} \begin{pmatrix} * & * & * \\ 0 & g_1 & g_1w \\ 0 & g_2 & g_2w \end{pmatrix} \begin{pmatrix} 1 & 0 & * \\ z & w & * \\ 0 & z & * \end{pmatrix} \]

\[ = \begin{pmatrix} * & * & * \\ 1 & 0 & * \\ 0 & 1 & * \end{pmatrix}. \]

For (iii), we note that by Lemma 4.2 we are multiplying block-diagonal matrices as follows:

\[ t(2\zeta + 1) = F_1^{-1}t_0(2\zeta + 1)F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \]

\[ = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}. \]

\[ \blacksquare \]

\textbf{Remark 5.2} Lemma 5.1 (ii) and its proof are analogous to Lemma 3.2 of [14] and its proof.

The following lemma is one of the things that distinguishes our proof from the proofs of earlier denominator bounds. It shows that all entries of the matrices in Lemma 5.1 commute. Contrary to the previous bounds, it shows this without assuming that small elements in large-discriminant quaternion rings commute, and hence without assuming that } p \text{ is large.

\textbf{Lemma 5.3} In the notation of Lemma 5.1, we have \( v = s = r_1 \) and \( u = t = 0. \) Moreover, all of } a, b, e, f, \text{ and all entries of } t(\alpha) \text{ for all } \alpha \in K \text{ are in } \mathbb{Q}(r_1).

\textbf{Proof} As the matrices } t(\mu) \text{ and } t(2\zeta + 1) \text{ commute, we have }

\[ \begin{pmatrix} r_1 & r_1a & r_1b \\ s & t & se + tf \\ u & v & ue + vf \end{pmatrix} = \begin{pmatrix} xr_1 & as + bu & at + bv \\ r_1 & eu & ev \\ 0 & s + fu & t + fv \end{pmatrix}. \]
We immediately read off \( s = r_1 \) and \( u = 0 \), and once we use \( u = 0 \), we also get \( t = 0 \) and \( v = s \). Now \( \iota(2\zeta_3 + 1) \) is \( r_1 \) times the identity matrix, hence the fact that the two matrices commute implies that all entries of the matrices commute with \( r_1 \). As \( r_1 \) is not in \( \mathbb{Q} \), this implies that these entries are in the quadratic field \( \mathbb{Q}(r_1) \). Finally, as \( \mu \) and \( 2\zeta_3 + 1 \) generate the field \( K \), we get that all entries of \( \iota(\alpha) \) are in \( \mathbb{Q}(r_1) \).

In the rest of this section, we express \( b, e, \) and \( f \) in terms of \( x, a, \) and the coefficients of the minimal polynomial of \( \mu \).

As \( \mu \) is a cubic integral over \( \mathbb{Z} \), we have \( \mu^3 - t_1\mu^2 + a_1\mu - N = 0 \), where \( t_1 = \text{tr}_{K_+/\mathbb{Q}}(\mu) \), \( N = N_{K_+/\mathbb{Q}}(\mu) \), and \( a_1 \) are in \( \mathbb{Z} \) and depend only on \( \mu \).

**Lemma 5.4** We have

\[
\begin{align*}
    f &= t_1 - x, \\
    e &= -\left( a_1 + x^2 + a - t_1x \right), \\
    b &= N - \left( x^3 - t_1x^2 + 2xa + a_1x - t_1a \right).
\end{align*}
\]

**Proof** As \( \iota \) is a ring homomorphism, we find that the matrix

\[
M = \iota(\mu)^3 - t_1\iota(\mu)^2 + a_1\iota(\mu) - N\text{Id}_{3 \times 3}
\]

is the zero matrix, where \( \text{Id}_{3 \times 3} \) is the \( 3 \times 3 \) identity matrix.

As the entries of \( \iota(\mu) \) are given explicitly in terms of \( x, a, b, e, f \) in a field \( \mathbb{Q}(r_1) \), we can easily compute \( M \) in terms of these quantities and \( t_1, a_1, N \). The leftmost column is exactly

\[
\begin{pmatrix}
x^3 - t_1x^2 + (2a + a_1)x - t_1a + b - N \\
x^2 - t_1x + a + e + a_1 \\
x + f - t_1
\end{pmatrix},
\]

which proves the result.

---

**6 Tangent Spaces and Primitive CM Types**

As in (4.2), let \( n \in \mathbb{Z}_{>0} \) be such that \( [n] \ker(F_1) = 0 \). In this section, we prove the following proposition implying that in order to bound \( p \), it suffices to find a small \( n \). In (7.2) below, we choose a specific \( n \).

**Proposition 6.1** For \( C \) and \( p \) as in the hypotheses of Theorem 2.2, let \( n \in \mathbb{Z}_{>0} \) be such that \( [n] \ker(F_1) = 0 \). Then \( p \leq 3 \) or \( p \mid n \).

**Proof** Suppose \( p \nmid 6n \). We claim that primtivity of the CM type implies that the matrix \( \iota(2\zeta_3 + 1) \) has two distinct eigenvalues.

Note that having two distinct eigenvalues contradicts the first statement of Lemma 5.3, which was the equality \( \iota(2\zeta_3 + 1) = r_1\text{Id}_{3 \times 3} \). In particular, the result follows once we prove the claim.

The idea behind the claim is as follows. Note that primtivity of the CM type implies that the action of \( 2\zeta_3 + 1 \) on the tangent space of \( J \) has two distinct eigenvalues. If \( p \) does not divide \( 6n \), then these two eigenvalues induce distinct eigenvalues for the action on the tangent space of \( \tilde{J} \) via \( F \) and \( [2n]F^{-1} \). This proves the claim.
In more detail, the proof of the claim is the same as the proof of [14, Proposition 5] with \( \delta = 3 \) and \( \sqrt{-\delta} = 2\zeta_3 + 1 \), except for the following changes.

- We use \( F \) as above and we use \( A_1 \) instead of \( E \), and \( 2n \) instead of \( \eta \). Let \( G = [2n]F^{-1} \).
- Instead of [14, Proposition 4.1], use Lemma 5.3; so in particular the condition \( p > \frac{1}{8} B^{10} \) is not needed.
- The reductions of \( \pm \sqrt{-\delta} \) are distinct as we have \( p \mid 2\delta = 6 \). This also does not need any additional bounds on \( p \).
- The invertibility of \( 2n \) modulo \( p \) follows from our assumption \( p \mid 6n \) and also does not need additional bounds on \( p \).

7 Using the Polarization to Get Bounds

By Proposition 6.1, it now suffices to find a sufficiently well-bounded \( n \in \mathbb{Z}_{>0} \) with \( [n] \ker(F_1) = 0 \). In this section, we do exactly this, using the polarization that \( C \) induces on \( A_1^3 \) via \( F_1 \). The key here is that we constructed \( F_1 \) very explicitly, and that polarizations give rise to positive definite matrices. Compared to [14], our matrices are a bit simpler, since in our situation we are able to prove that the entries are in a field, where [14] needs the bounds in order to prove exactly that.

Let \( \lambda = F'\lambda C F \) be the polarization induced on \( A_1^3 \) by the polarization \( \lambda C \) of \( \mathcal{J} \). We identify \( A_1 \) with its dual via the natural polarization \( \lambda_{A_1} \), which we sometimes leave out from the notation. Then \( \lambda \) can be viewed as an endomorphism of \( A_1^3 \), and the following result gives it as a matrix.

**Lemma 7.1 ([14, Lemma 4.3])** We have

\[
\lambda = \begin{pmatrix}
m & 0 & 0 \\
0 & \alpha & \beta \\
0 & \beta & \gamma
\end{pmatrix}
\]

with \( m, \alpha, \gamma \in \mathbb{Z}_{>0} \), and \( \beta \in \mathcal{R} \) with \( \alpha \gamma - \beta \beta' > 0 \). Moreover, we have \( m \mid 2 \).

**Proof** Recall from just above the statement of the lemma that \( \lambda \) is defined as a homomorphism \( A_1^3 \to (A_1^3)^3 \) by \( \lambda = F'\lambda C F \), and as an endomorphism of \( A_1^3 \) by \( \lambda = (\lambda_{A_1}^{-1} \times \lambda_{A_1}^{-1} \times \lambda_{A_1}^{-1}) F'\lambda C F \). The symmetry of \( \lambda \) now follows from the symmetry of \( \lambda C \) [22, (3), p. 190].

We now prove that the off-diagonal entries of the first row and column of \( \lambda \) are zero. Since \( F = F_0F_1 \), we write

\[
\lambda = \text{diag}(\lambda_{A_1}^{-1}, \lambda_{A_1}^{-1}, \lambda_{A_1}^{-1})F_1'(I_1 I_2)\gamma \lambda C(I_1 I_2)F_1, \quad \text{where } F_1' = \begin{pmatrix}
1 & 0 & 0 \\
0 & z' & 0 \\
0 & z' & w'
\end{pmatrix}
\]

To see that four entries are zero, we only look at the off-diagonal entries of the first row. This suffices by symmetry. By Proposition 3.3 (i), we get \( e_1^* \lambda C e_2 = \lambda C e_1^* e_2 = \)
\( \lambda_C e_1 e_2 = 0 \). As \( e_i = I_1 s_i \) and \( s_i \) is surjective, we get \( I_1^* \lambda_C I_2 = 0 \). Therefore we have

\[
(I_1 I_2)^* \lambda_C (I_1 I_2) = \begin{pmatrix}
* & 0 \\
* & *
\end{pmatrix},
\]

and hence the off-diagonal entries of the first row of \( \lambda \) are zero.

From the final paragraph of Mumford [22, Application III, p. 210], we get that \( \lambda \) is positive definite, hence \( m, \alpha, \gamma, \alpha \gamma - \beta \beta^* > 0 \).

It remains only to prove \( m \mid 2 \). We have \( m = I_1^* \lambda_C I_1 \) since we defined \( m \) to be the first diagonal entry of \( (I_1 I_2)^* \lambda_C (I_1 I_2) \).

Recall that by Proposition 3.3 (i) we have \( e_1 = e_1^t \). This implies \( e_1 = \lambda_C^{-1} e_1^t \lambda_C \) and by \( e_1 = I_1 s_1 \), we get \( I_1 s_1 = \lambda_C^{-1} s_1^t I_1^* \lambda_C \). Therefore, we have \( \lambda_C I_1 s_1 I_1 = s_1^t I_1^* \lambda_C I_1 \), hence \( \lambda_C I_1[d] = s_1^t I_1 \lambda_C I_1 \). Since \( \lambda_C \) is an isomorphism and \( I_1 \) is injective, we get that \( \ker(s_1^t I_1^* \lambda_C I_1) = A_1[d] \). Hence, \( \ker(I_1^* \lambda_C I_1) \subseteq A_1[d] \), and we know that \( m = I_1^* \lambda_C I_1 \) is a positive integer. So we finally get \( m = 1 \) or \( 2 \).

Since \( \mu \in K_+ \), it equals its complex conjugate \( \overline{\mu} \). Moreover, (analogously to [5, Proposition 4.8]), we have for every \( \eta \in K \),

\[
\lambda^{-1} \iota(\eta)^* \lambda = (F^* \lambda_C F)^{-1} (F^{-1} \eta F)^* F^* \lambda_C F \\
= F^{-1} \lambda_C^{-1} \eta \lambda_C F = \iota(\eta)^* = \iota(\overline{\eta}).
\]

Hence \( \iota(\mu)^* \lambda = \lambda \iota(\mu) \), so that

\[
\begin{pmatrix}
mx^* & \alpha \\
ma^* & \beta^* \\
mb^* & \gamma
\end{pmatrix}
= \begin{pmatrix}
mx ma & mb \\
ma \beta & \alpha e + \beta f \\
mb \beta^* e + \gamma f
\end{pmatrix}.
\]

This tells us that

\[
\begin{align*}
(7.1) \quad & \alpha = ma, \text{ hence } a \in \mathbb{Q}_{>0} \cap \mathcal{R} = \mathbb{Z}_{>0}, \\
& \beta^* = \beta = mb, \\
& x = x^*, \text{ hence } x \in \mathbb{Z}, \\
& y = \alpha e + \beta f.
\end{align*}
\]

Combining this with Lemma 5.4, we find explicit expressions for all entries of \( \iota(\mu) \) and \( \lambda \) in terms of \( x, a, m \), and the coefficients of the minimal polynomial of \( \mu \). In particular, these entries are all in \( \mathbb{Z} \).

Let

\[
(7.2) \quad n = \alpha \gamma - \beta \beta^* = m(\alpha \gamma - mb^2) \in m\mathbb{Z}.
\]

Then Lemma 7.1 and the definition of \( \lambda \) give

\[
\begin{pmatrix}
\frac{n}{m} & 0 & 0 \\
0 & \gamma & -\beta \\
0 & -\beta^* & \alpha
\end{pmatrix}
\begin{pmatrix}
F^* \lambda_C F = [n],
\end{pmatrix}
\]
so that in particular the condition \([n] \ker(F) = 0\) from (4.2) is satisfied. We have already expressed \(\alpha, \gamma,\) and \(\beta\) in terms of \(x, a, m,\) and the coefficients of the minimal polynomial of \(\mu.\) As \(m\) is 1 or 2, it suffices to bound \(x\) and \(a\) in order to bound \(n.\)

As the \(3 \times 3\) matrix \(I(\mu^2)\) over \(\mathbb{Q}\) satisfies the (cubic) minimal polynomial of \(\mu^2\) over \(\mathbb{Q},\) we find that its (matrix) trace is the trace of \(\mu^2\) from \(K_+\) to \(\mathbb{Q},\) which is \(t_2 := t_1^2 - 2a_1.\) We get

\[
t_2 = x^2 + 2a + 2e + f^2
\]
\[
= x^2 + 2a + \frac{2}{\alpha} y - \frac{2}{\alpha} \beta f + f^2
\]
\[
= x^2 + 2a + \frac{2}{\alpha} y - \left( \frac{\beta}{\alpha} \right)^2 + \left( \frac{\beta}{\alpha} - f \right)^2
\]
\[
\geq x^2 + 2a.
\]

(7.3)

In particular, we get

\[
|y| \leq \sqrt{t_2} \quad \text{and} \quad 0 < a \leq \frac{1}{2} (t_2 - x^2).
\]

Moreover, by (7.3), we get \(n \leq t_2 a^2 + 2a \leq t_2.\) Then by (7.1), we obtain \(n \leq t_2 a^2 \leq t_2 m^2 a^2 \leq t_3^2\) as \(m \mid 2.\) By Proposition 6.1, we have \(p \leq 3\) or \(p \mid n.\) Hence we get the bound \(p \leq \max\{3, t_3^2\}.

**Lemma 7.2** Let \(\mu\) be a totally real cubic algebraic integer, and let \(t_2\) be the trace of \(\mu^2.\) Then we have \(t_2 \geq 2.\)

**Proof** Let \(a, b, c\) be the images of \(\mu\) under the three embeddings into \(\mathbb{R}.\) Then \(t_2 = a^2 + b^2 + c^2 \in \mathbb{Z}.\) Suppose \(t_2 < 2.\) Then \(t_2 \leq 1\) and \(a^2, b^2, c^2 > 0.\) Hence \(|a|, |b|, |c| \in (0, 1);\) so we get \(|abc| \in (0, 1).\) On the other hand, we have \(|abc| = |N(\mu)| \in \mathbb{Z},\) which is a contradiction. ■

**Proof of the first inequality in Theorem 2.2** As stated above, we proved the inequality \(p \leq \max\{3, t_3^2\}\) under the hypotheses of Theorem 2.2. As Lemma 7.2 gives \(t_3^2 \geq 2^3 > 3,\) we get \(p \leq t_3^2.\) ■

8 **Intrinsic Bounds From Geometry of Numbers**

At the end of Section 7, we finished the proof of the first inequality in Theorem 2.2: \(p \leq \text{tr}_{K_+/\mathbb{Q}}(\mu^2)^3.\) Next we show that there exists an element \(\mu\) for which this right-hand side is explicitly bounded in terms of the discriminant of \(K^+,\) and hence we prove the rest of Theorem 2.2.

Let \(\{\sigma_1, \sigma_2, \sigma_3\}\) be the set of the real embeddings of \(K_+.\) This gives us the map \(\sigma: K_+ \to \mathbb{R}^3\) by sending \(y\) to \((\sigma_i(y))_.\) The order \(\mathbb{Z} + 2\mathcal{O}_+ \subset K_+\) is a lattice of co-volume \(2^{3-1}\Delta(\mathcal{O}_+)^{1/2}\) in \(\mathbb{R}^3.\) Let \(R = 4\pi^{-1/2}\Delta(\mathcal{O}_+)^{1/4} + \varepsilon\) for some \(\varepsilon > 0.\)

We choose a symmetric convex body in \(\mathbb{R}^3:\)

\[
\mathcal{C}_R = \{ x \in \mathbb{R}^3 : |x_1| < 1, x_2^2 + x_3^2 < R^2 \}.
\]
We then have \( \text{vol}(\mathcal{C}_R) = 2\pi R^2 > 32|\Delta(\mathcal{O}_+)|^{1/2} = 2^3 \text{cvol}(\mathcal{Z} + 2\mathcal{O}_+) \). By Minkowski’s first convex body theorem [24, Theorem 10], there is a non-zero \( \mu \in (\mathcal{Z} + 2\mathcal{O}_+) \cap \mathcal{C}_R \). Note that \( \mu \) generates \( \mathcal{K}_+ \). If \( \mu \in \mathcal{Q} \), then \( \mu \in \mathcal{Z} \). But \( |\mu| < 1 \), so we get \( \mu = 0 \), which is a contradiction. Then we get \( \text{tr}_{\mathcal{K}_+/\mathcal{Q}}(\mu^2) = \sum_1 \sigma_i(\mu^2) \leq (1 + R^2) \). Since \( \mu \) is an algebraic integer in \( \mathcal{K}_+ \), we have \( \text{tr}_{\mathcal{K}_+/\mathcal{Q}}(\mu^2) \in \mathcal{Z} \). So when we let \( e \) tend to 0, we get \( t_2 = \text{tr}_{\mathcal{K}_+/\mathcal{Q}}(\mu^2) \leq (1 + \frac{16}{\pi} \Delta(\mathcal{O}_+)|^{1/2}) \).

Since \( p \leq t_2^3 \) and \( |\Delta(\mathcal{O}_+)| \geq 2 \), we get \( p \leq (1 + \frac{16}{\pi} \Delta(\mathcal{O}_+)|^{1/2})^3 < 196\Delta(\mathcal{O}_+)|^{3/2} \). This finishes the proof of Theorem 2.2.

9 Computing the Set of Primes

From the proof of Theorem 2.2, we get much more than just a bound on \( p \) as follows. Take a totally real \( \mu \in \mathcal{Z} + 2\mathcal{O}_+ \). Then list all \( a \) and \( x \) satisfying the bounds of (7.4) and all \( m \in \{1, 2\} \). For each, compute \( n = n(\mu, a, x) \) using (7.1) and (7.2). Then let \( N_\mu \) be the product of the numbers \( n(\mu, a, x) \). Then \( p \) divides \( 6N_\mu \) by Proposition 6.1. This is already much better than just a bound on \( p \).

However, we can do even better. For each \( \mu, a, x, m \), we get a \( \mathcal{Q} \)-algebra homomorphism \( \nu: K \to M_{3\times 3}(\mathcal{Q}(\zeta)) \) (given on generators in Lemma 5.1, coefficients in \( \mathcal{Q}(\zeta) \) by Lemma 5.3, and \( r_1^2 = -3 \)). We know by Lemmas 5.1 and 5.3 that all elements of the image of \( \mathcal{Z} + 2\mathcal{O}_+ \) are matrices with entries in \( \frac{1}{n}\mathcal{Z}[\zeta_3] \), with the entries of the top row in \( \mathcal{Z}[\zeta_3] \). So we compute a \( \mathcal{Z} \)-basis of \( \mathcal{Z} + 2\mathcal{O}_+ \) and throw away all triples \( (x, a, m) \) for which an element of this basis maps to a matrix that does not satisfy the integrality condition. We also throw away all triples \( (x, a, n) \) for which one of \( \alpha, \beta, \gamma \) is non-integral or for which \( y \) or \( n \) is non-positive. By making the set of pairs \( (x, a) \) smaller in this way, the product \( N_\mu \) of the numbers \( n(\mu, a, x) \) becomes much smaller.

We implemented the computation of \( N_\mu \) in SageMath [26] and made the implementation available online [15].

**Theorem 9.1** Let \( C \) be a Picard curve of genus \( 3 \) over a number field \( L \) and suppose that the endomorphism ring \( \text{End}(J) \) of \( J = \text{Jac}(C) \) over the algebraic closure is isomorphic to an order \( \mathcal{O} \) of a number field \( K \) of degree 6.

Let \( K_+ \) be the real cubic subfield of \( K \) and \( \mathcal{O}_+ = K_+ \cap \mathcal{O} \). Let \( \mu \) be a totally real element in \( \mathcal{Z} + 2\mathcal{O}_+ \) such that \( K = \mathcal{Q}_+(\mu)(\zeta_3) \).

Let \( j = u/b^e \) be an absolute Picard curve invariant. Let \( p \) be a prime of \( \mathcal{O}_L \) lying over a rational prime \( p \). If \( \text{ord}_p(j(C)) < 0 \), then \( p \) divides the number \( 6N_\mu \) for \( N_\mu \) as described in the preceding paragraphs.

**Conjecture 9.2** There are constants \( s, e \in \mathbb{Q}_{>0} \) such that the following holds. Let \( j = u/b^e \) be an absolute Picard curve invariant.

Let \( \mathcal{O} \) be an order in a sextic CM field. Let \( \text{CM}_K \) be the set of isomorphism classes of Picard curves \( C \) over \( \overline{\mathbb{Q}} \) of genus \( 3 \) with \( \text{End}(J_{\overline{\mathbb{Q}}}) \) isomorphic to \( \mathcal{O} \). Let \( N_\mu \) be as in Theorem 9.1.

Then for all non-archimedean valuations \( \nu \) of \( \overline{\mathbb{Q}} \), we have

\[
\sum_{C \in \text{CM}_K} \max\{0, \nu(j(C))\} \leq \ell(\nu(s) + e \cdot \nu(N_\mu)).
\]
Remark 9.3 In fact, the examples in Section 10 suggest that when $K/\mathbb{Q}$ is Galois, the constant $e = 1/3$ suffices. The numerology supporting the factor 1/3 is that $K$ has three CM types up to complex conjugation which are all equivalent, so that every curve should be counted three times, but is only counted once in the left-hand side of (9.1).

To prove the conjecture, one would need to retrace our proof, but working over prime-power quotients of $\mathcal{O}_L$ instead of over the field $\mathcal{O}_L/p$. Once the conjecture is proved, our implementation of $N_\mu$, together with an interval-arithmetic-version of Lario–Somoza [18], would give a proven algorithm for computing CM Picard curves and Picard class polynomials. In particular, it would prove the conjectured CM curves of Koike–Weng [16] and Lario–Somoza [18].

10 Examples

Finally, we take a few example curves and compare our bounds with previous bounds, and compare our invariants with previous choices.

Given a Picard curve $C$, let den$_1$ and den$_3$ be the denominators of the absolute invariants $j_1(C) = (a^3/b^2)(C)$ and $j_3(C) = (c^3/b^4)(C)$, respectively. Then we define the absolute denominator $b_{abs}$ of $C$ by

$$b_{abs} = \prod_{p|\text{den}_1\cdot \text{den}_3} p^{\max\{v_p(\text{den}_1)/2, v_p(\text{den}_3)/4\}} = \prod_{p} p^{v_p(b) - \frac{1}{4}\min\{6v_p(a), 4v_p(b), 3v_p(c)\}}.$$

Theorem 9.1 tells us that all primes dividing $b_{abs}$ divide $6N_\mu$. In fact, Conjecture 9.2 implies that $b_{abs}$ divides $sN_\mu^6$.

We define the absolute denominator $a_{abs}$ of the Koike–Weng invariants $j'_1 = b^2/a^3$ and $j'_2 = c/a^2$ in the same way. In other words, let den$_1'$ and den$_3'$ be the denominators of $j'_1(C)$ and $j'_2(C)$, respectively. Then we define

$$a_{abs} = \prod_{p|\text{den}_1'\cdot \text{den}_3'} p^{\max\{v_p(\text{den}_1')/3, v_p(\text{den}_3')/2\}}.$$

Let $\Delta$ be the discriminant invariant (2.3), which has weight 12. We define the invariants

$$i_1 = \frac{a^6}{\Delta}, \quad i_2 = \frac{a^3b^2}{\Delta}, \quad i_3 = \frac{a^4c}{\Delta}, \quad i_4 = \frac{b^4}{\Delta}, \quad i_5 = \frac{c^3}{\Delta},$$

denoted $j_*$ in [14]. Let $\Delta_{abs}$ denote the least common multiple of the denominators of $i_1(C)$, $i_4(C)$, and $i_5(C)$ (equivalently, of all $i_*(C)$).

Let $B = \min\{\text{tr}_{K/\mathbb{Q}}(a\bar{a}) : \alpha \in \mathcal{O}_K \setminus \{0\}, \bar{a} = -a\}$. So, as conjectured [14, Remark 1.6] that primes $p$ of bad reduction are $< \frac{1}{2}B^{10}$ and in the case where $K/\mathbb{Q}$ is cyclic and $C$ has CM by $\mathcal{O}_K$, it follows from [13, Proposition 4.1] that $p$ has exactly one or three prime factors in $\mathcal{O}_K$. The number of such primes below this bound is roughly $\frac{1}{6}B^{10}/\log(\frac{1}{3}B^{10})$ by the prime number theorem and the Chebotarev density theorem. The product of them will therefore have a number of digits that is comparable to $B^{10}$ itself.
Example 1 For the field $K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 - x^2 - 2x + 1)$, let $K = K_+(\zeta_3)$. Then there is exactly one curve with primitive CM by the maximal order $\mathcal{O}_K$ of K. See [16, 6.1.2] and [18, 4.1.2] for a conjectural model. Its invariants are
\begin{itemize}
  \item $i_1 = \frac{7^4}{37}, i_2 = \frac{-7^2}{27}, i_3 = \frac{-7^3}{27},$
  \item $\Delta_{\text{abs}} = 2^{12} \approx 4.1 \cdot 10^3,$
  \item $a_{\text{abs}} = (2^3 \cdot 7^2)^{\frac{1}{2}} \approx 7.3186142004594,$
  \item $j_1 = \frac{-7^2}{27}, j_2 = \frac{7}{27}, b_{\text{abs}} = 2^3 \approx 8,$
  \item $N_{-2a^2+3} = (2^{28} \cdot 7 \cdot 13) \approx (2.4 \cdot 10^{10})^3.$
\end{itemize}

Example 2 For the field $K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 - x^2 - 4x - 1)$, let $K = K_+(\zeta_3)$. Then there is exactly one curve with primitive CM by $\mathcal{O}_K$. See [16, 6.1.3] and [18, 4.1.3] for a conjectural model. Its invariants are
\begin{itemize}
  \item $i_1 = \frac{(7^3 \cdot 13)^2}{(2^5)^3}, i_2 = \frac{-7^6 \cdot 13 \cdot 2^2}{2^7 \cdot 5^3}, i_3 = \frac{-7^3 \cdot 13 \cdot 3}{(2^7 \cdot 5)^3},$
  \item $\Delta_{\text{abs}} = (2^3 \cdot 5)^6 \approx 6.4 \cdot 10^7,$
  \item $j_1 = \frac{-2^3 \cdot 5^4 \cdot 7^2}{27 \cdot (2 \cdot 7)^3}, j_2 = \frac{-2^3 \cdot 7^3}{(2 \cdot 7)^3}, a_{\text{abs}} = (2^3 \cdot 7 \cdot 13)^{\frac{1}{2}} \approx 2.3 \cdot 10^2,$
  \item $j_1 = \frac{-7^4 \cdot 13}{27 \cdot 5^2 \cdot 47}, j_2 = \frac{7^2 \cdot 2^3 \cdot 47}{5^2 \cdot 47}, b_{\text{abs}} = 2^3 \cdot 5 \cdot 47 \approx 1.9 \cdot 10^3,$
  \item $N_{-2a^2 + 2a + 5} = (2^{205} \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 47^2 \cdot 61^2 \cdot 89 \cdot 101 \cdot 139)^3 \approx (6.7 \cdot 10^{23})^3.$
\end{itemize}

Example 3 For the field $K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 + x^2 - 10x - 8)$, let $K = K_+(\zeta_3)$. Then there is exactly one curve with primitive CM by $\mathcal{O}_K$. See [16, 6.1.4] and [18, 4.1.4] for a conjectural model. Its invariants are
\begin{itemize}
  \item $i_1 = \frac{(7^3 \cdot 31 \cdot 73)^2}{(2 \cdot 3^2 \cdot 31)^3}, i_2 = \frac{-2 \cdot 7^6 \cdot 31 \cdot 47 \cdot 73}{23^6}, i_3 = \frac{-7^5 \cdot 31^2 \cdot 47^3 \cdot 11593}{(2 \cdot 3 \cdot 23)^3},$
  \item $\Delta_{\text{abs}} = (2^7 \cdot 23)^6 \approx 2.5 \cdot 10^{15},$
  \item $j_1 = \frac{-2^{19} \cdot 7^2}{3^5 \cdot 7^3 \cdot 11593}, j_2 = \frac{-2^{19} \cdot 7^5 \cdot 11593}{3^5 \cdot 7^3}, a_{\text{abs}} = (2^3 \cdot 7^3 \cdot 31 \cdot 73^3)^{\frac{1}{2}} \approx 3.2 \cdot 10^3,$
  \item $j_1 = \frac{-7^3 \cdot 31 \cdot 73}{2^6 \cdot 47}, j_2 = \frac{7^2 \cdot 31 \cdot 73 \cdot 11593}{2^6 \cdot 47}, b_{\text{abs}} = 2^41 \cdot 47 \approx 9.6 \cdot 10^4,$
  \item $N_{-\alpha^2 + \alpha + 7} = (2^{205} \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 47^2 \cdot 61^2 \cdot 89 \cdot 101 \cdot 139)^3 \approx (7.3 \cdot 10^{61})^3.$
\end{itemize}

Example 4 For the field $K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 - x^2 - 14x - 8)$, let $K = K_+(\zeta_3)$. Then there is exactly one curve with primitive CM by $\mathcal{O}_K$. A conjectural model is given in [16, 6.1.5] and [18, 4.1.5]. Its invariants are
\begin{itemize}
  \item $i_1 = \frac{(7^3 \cdot 43 \cdot 223)^2}{(2^9 \cdot 11 \cdot 47)^3}, i_2 = \frac{-2^3 \cdot 43^2 \cdot 59 \cdot 223^3}{2^9 \cdot 11 \cdot 47^3}, i_3 = \frac{-7^4 \cdot 43 \cdot 223^2 \cdot 419 \cdot 431}{(2 \cdot 3 \cdot 11 \cdot 47)^3},$
  \item $\Delta_{\text{abs}} = (2^5 \cdot 11 \cdot 47)^6 \approx 2.1 \cdot 10^{25},$
  \item $j_1 = \frac{-2^{11} \cdot 11 \cdot 43 \cdot 223^3}{2^9 \cdot 11 \cdot 47^3 \cdot 223^2}, j_2 = \frac{-2^{11} \cdot 11 \cdot 43 \cdot 223^3}{2^9 \cdot 11 \cdot 47^3 \cdot 223^2}, a_{\text{abs}} = (2^3 \cdot 7^3 \cdot 43^2 \cdot 223^3)^{\frac{1}{2}} \approx 3.8 \cdot 10^4,$
  \item $j_1 = \frac{-7^3 \cdot 43 \cdot 223^3}{2^9 \cdot 11 \cdot 47^3 \cdot 223^2}, j_2 = \frac{7^3 \cdot 43 \cdot 223^3 \cdot 419 \cdot 431}{2^9 \cdot 11 \cdot 47^3 \cdot 223^2}, b_{\text{abs}} = 2^7 \cdot 11 \cdot 41 \cdot 47 \approx 3.4 \cdot 10^6,$
  \item $N_{-2a + 1} = (2^{288} \cdot 11^3 \cdot 41 \cdot 43 \cdot 47^2 \cdot 59^3 \cdot 97 \cdot 131 \cdot 173 \cdot 211 \cdot 223 \cdot 269)^3 \approx (4.4 \cdot 10^{124})^3.$
\end{itemize}

Example 5 For the field $K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 - 21x - 28)$, let $K = K_+(\zeta_3)$. Then there is exactly one curve with primitive CM by $\mathcal{O}_K$. A conjectural model is given in [18, 4.2.1.1]. Its invariants are
\begin{itemize}
  \item $i_1 = \frac{-3^3 \cdot 5^2 \cdot 7^4}{2^9}, i_2 = \frac{3^3 \cdot 5^2 \cdot 7^4}{2^9}, i_3 = \frac{3 \cdot 5 \cdot 7 \cdot 2621}{2^9},$
\end{itemize}
\( \Delta_{abs} = (2^8 \cdot 3)^3 \approx 4.5 \cdot 10^8, \quad \frac{1}{8} B^10 \approx 2.1 \cdot 10^{15}, \)

\[ j'_1 = \frac{-5 \cdot 71}{29 \cdot 3^2 \cdot 7 \cdot 73}, \quad j'_2 = \frac{-2621}{23^3 \cdot 3^3 \cdot 7^4}, \quad a_{abs} = \left( 2^3 \cdot 3^6 \cdot 5^6 \cdot 7^2 \right)^{1/2} \approx 1.6 \cdot 10^3, \]

\[ j_1 = \frac{-3 \cdot 5 \cdot 7^3}{29 \cdot 71^2}, \quad j_2 = \frac{3 \cdot 5 \cdot 7 \cdot 2621}{23^3 \cdot 7^4}, \quad b_{abs} = 2^9 \cdot 71 \approx 3.6 \cdot 10^4, \]

\[ N_{2a} = 2^{433} \cdot 35^5 \cdot 71^1 \cdot 31^3 \cdot 47^3 \cdot 59^3 \cdot 61^3 \cdot 71^3 \cdot 173^3 \approx (1.3 \cdot 10^{66})^3. \]

**Example 6** For the field \( K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 - 21x - 35) \), let \( K = K_+(\zeta_3) \). Then there is exactly one curve with primitive CM by \( \mathcal{O}_K \). A conjectural model is given in \([18, 4.2.1.2]\). Its invariants are

\[ i_1 = \frac{-2^{12} \cdot 3^9 \cdot 7^4 \cdot 37^6}{(5 \cdot 11 \cdot 23)^3}, \quad i_2 = \frac{2^6 \cdot 3^2 \cdot 7^7 \cdot 149 \cdot 257^2}{(5^8 \cdot 11^3 \cdot 23^3)^2}, \quad i_3 = \frac{2^8 \cdot 3^7 \cdot 7^3 \cdot 37 \cdot 2683}{(5 \cdot 11 \cdot 23)^2}, \]

\[ \Delta_{abs} = (3 \cdot 5^3 \cdot 11^2 \cdot 23^2)^3 \approx 1.1 \cdot 10^{20}, \quad \frac{1}{8} B^10 \approx 2.1 \cdot 10^{15}, \]

\[ j'_1 = \frac{-5 \cdot 149 \cdot 257^2}{2^8 \cdot 3^5 \cdot 7^2 \cdot 37^2}, \quad j'_2 = \frac{-5 \cdot 2683}{2^8 \cdot 3^5 \cdot 7^2 \cdot 37^2}, \quad a_{abs} = \left( 2^6 \cdot 3^6 \cdot 7^2 \cdot 37^3 \right)^{1/2} \approx 4.9 \cdot 10^4, \]

\[ j_1 = \frac{-2^{12} \cdot 3^9 \cdot 7^2 \cdot 37^3 \cdot 149 \cdot 257^2}{(5 \cdot 11 \cdot 23)^2}, \quad j_2 = \frac{2^8 \cdot 3^2 \cdot 7^3 \cdot 149 \cdot 257^2}{(5 \cdot 11 \cdot 23)^2}, \quad b_{abs} = 5 \cdot 149 \cdot 257 \approx 1.0 \cdot 10^6, \]

\[ N_{-2a^2+4a+28} = 2^{245} \cdot 3^{58} \cdot 53^6 \cdot 78^1 \cdot 111^2 \cdot 23^6 \cdot 71^3 \cdot 149^3 \cdot 257^3 \approx (5.9 \cdot 10^{57})^3. \]

**Example 7** For the field \( K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 - 39x - 26) \), let \( K = K_+(\zeta_3) \). Then there is exactly one curve with primitive CM by \( \mathcal{O}_K \). A conjectural model is given in \([18, 4.2.1.3]\). Its invariants are

\[ i_1 = \frac{-3^7 \cdot 5^2 \cdot 7^6 \cdot 11^6 \cdot 3^2}{(2^9 \cdot 29)^3}, \quad i_2 = \frac{3^7 \cdot 5^2 \cdot 7^6 \cdot 11^6 \cdot 59^2 \cdot 149^2}{(2^9 \cdot 29)^3}, \]

\[ \Delta_{abs} = (2^{12} \cdot 3 \cdot 29^2)^3 \approx 1.1 \cdot 10^{21}, \quad \frac{1}{8} B^10 \approx 1. \cdot 10^{18}, \]

\[ j'_1 = \frac{-2^{11} \cdot 5^2 \cdot 7^2 \cdot 149}{3^7 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13}, \quad j'_2 = \frac{-17 \cdot 17 \cdot 167669}{2^7 \cdot 3^7 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13}, \quad a_{abs} = \left( 2^3 \cdot 3^6 \cdot 5^6 \cdot 7^3 \cdot 11^3 \right)^{1/2} \approx 8.1 \cdot 10^4, \]

\[ j_1 = \frac{-3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}{2^7 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 167669}, \quad j_2 = \frac{3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 167669}{2^7 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 167669}, \quad b_{abs} = 2^7 \cdot 11 \cdot 59 \cdot 149 \approx 1.2 \cdot 10^7, \]

\[ N_{-\frac{1}{2}a^2+\frac{4}{2}a+13} = 2^{2921} \cdot 3^{1000} \cdot 11^{21} \cdot 13^8 \cdot 29^6 \cdot 53^3 \cdot 59^6 \cdot 109^3 \cdot 113^3 \cdot 149^6 \cdot 233^3 \cdot 359^3 \cdot 467^3 \cdot 541^3 \cdot 577^3 \approx (2 \cdot 10^{148})^3. \]

**Example 8** For the field \( K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 - 61x - 183) \), let \( K = K_+(\zeta_3) \). Then there are exactly four curves with primitive CM by \( \mathcal{O}_K \). Conjectural models are given in \([18, 4.3.1]\) (corrected with respect to arXiv version 1). Let \( H_{j_1}, j_2 = H_{\mathcal{O}_K, 2} \) be the polynomials as in (2.4)–(2.5), and let \( H_{j'_1} \) and \( H_{j'_2} \) be defined as in (2.4), but with the invariants \( j'_1 \) and \( i_1 \) instead of \( j_1 \). These polynomials are numerically approximable with the methods of Koike–Weng [16] and Lario–Somoza [18] and satisfy \( H_{j_1}(j_1(C)) = 0 \) and \( j_2(C) = H_{j_1,j_2}(j_1(C))/H_{j'_1}(j_1(C)) \) (see [9]). Then the denominators of these polynomials are

\[ \text{den}(H_{j_1}) = 2^3 \cdot 3^{39} \cdot 11^2 \cdot 23^2 \cdot 41^2 \cdot 53^4 \cdot 89^2 \cdot 113^2 \cdot 149^2 \cdot 191^2 \approx 2.3 \cdot 10^{51}, \]

\[ \text{den}(H_{j'_1}) = 2^5 \cdot 3^{39} \cdot 11^3 \cdot 23^2 \cdot 41^2 \cdot 53^4 \cdot 89^2 \cdot 113^2 \cdot 149^2 \cdot 191^2 \approx 9.9 \cdot 10^{52}, \]

\[ \text{den}(H_{j_2}) = 2^{18} \cdot 7^{12} \cdot 11^6 \cdot 12889^3 \cdot 149 = 20707^3 \approx 7.9 \cdot 10^{53}, \]

\[ \text{den}(H_{j'_2}) = (2^2 \cdot 3^9 \cdot 11^5 \cdot 23^3 \cdot 53^2 \cdot 131^2)^3 \approx 6.7 \cdot 10^{72}, \]

\[ N_{-4a^2+18a+163} = (2^{233} \cdot 3^{148} \cdot 11^2 \cdot 23^6 \cdot 37^3 \cdot 41^3 \cdot 53^3 \cdot 61 \cdot 89 \cdot 113 \cdot 131 \cdot 149 \cdot 191 \cdot 367 \cdot 613 \cdot 643 \cdot 733 \cdot 907)^3 \approx (1.2 \cdot 10^{203})^3. \]
Primes Dividing Invariants of CM Picard Curves

Example 9  For the field \(K_+ = \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(x^3 - x^2 - 22x - 5)\), let \(K = K_+(\zeta_3)\). Then there are exactly four curves with primitive CM by \(\mathcal{O}_K\). See [18, 4.3.2] for conjectural models. In the notation of Example 8, we have

\[
\begin{align*}
den(H_j) &= (2^6 \cdot 3^{15} \cdot 5^4 \cdot 89 \cdot 137 \cdot 149 \cdot 269)^2 \approx 2.5 \cdot 10^{45} \\
den(H_{j_1j_1}) &= (2^6 \cdot 3^{15} \cdot 5^3 \cdot 89 \cdot 137 \cdot 149 \cdot 269)^2 \approx 1 \cdot 10^{44}, \\
den(H_{j_2}) &= 7^{12} \cdot 53^3 \cdot 67 \cdot 107^3 \cdot 179^3 \cdot 3029017^3 \approx 2.7 \cdot 10^{49}, \\
den(H_{i_1}) &= (2^4 \cdot 3^5 \cdot 5^4 \cdot 53 \cdot 59 \cdot 107)^6 \approx 2.9 \cdot 10^{71}, \\
N_{-\frac{1}{2}a^2 + \frac{27}{16}} &= (2^{253} \cdot 3^{355} \cdot 5^{32} \cdot 43^2 \cdot 53^2 \cdot 67 \cdot 89^2 \cdot 107 \cdot 109 \cdot 137 \cdot 149 \cdot 179^2 \cdot 223 \cdot 241 \cdot 263 \cdot 269 \cdot 397 \cdot 643 \cdot 997 \cdot 1087)^3 \approx (1.2 \cdot 10^{224})^3.
\end{align*}
\]

Remark 10.1  Notice that the sizes of the denominators of the absolute invariants \(j_1\) and \(j_2\) of Section 2.3 are similar to the denominators of the Koike–Weng invariants and much smaller than the denominators of the invariants defined by using the discriminant. Theorem 1.3 in [14] suggests a bound for the primes appearing in the discriminant, while we do not have a bound at all for the primes in the denominator of the Koike–Weng invariants. That the primes in the denominators of the Koike–Weng invariants are small, and even smaller than those for our invariants, is a mystery that needs further research.

For our absolute invariants, we have the best bound. Hence among the three kind of invariants, the more suitable ones for constructing Picard curves with CM by a given order \(\mathcal{O}\) are the absolute invariants \(j_1\) and \(j_2\).

A  A Lemma About Components of Bad Reduction

Most of this appendix is an edited copy of an email from Bas Edixhoven to the authors. Lemma A.1 and its proof are well known to many experts, but it seems that neither is written down in the literature. For completeness, as we use it in our proof of the singular case of Proposition 3.3, we state the lemma and provide details of the proof.

Lemma A.1  Let \(R\) be a discrete valuation ring with fraction field \(M\) and residue field \(k\). Let \(X\) be a projective \(R\)-scheme, of dimension 2, flat over \(R\) such that \(X_M\) is smooth, and geometrically connected over \(M\), of genus at least 1. Let \(C\) be an irreducible component of \(X_k\) and assume that \(C\) is geometrically irreducible and birational to a smooth geometrically irreducible projective curve \(C'\) over \(k\) of genus at least 1.

Suppose that there exists an open subscheme \(U\) of \(X\) such that \(U\) is smooth over \(R\), and such that \(U_k\) is non-empty open subset of \(C\).

Let \(M \to M'\) be a finite separable field extension such that \(X_{M'}\) has stable reduction over the integral closure \(R'\) of \(R\) in \(M'\). Then the open subscheme \(U_{R'}\) of \(X_{R'}\) is isomorphic to an open subscheme of the stable model of \(X\). Moreover, the normalization of the special fibre of the stable model of \(X\) contains a copy of \(C'_{k'}\), where \(k'\) is the residue field of \(R'\).

Proof  Let \(X_{R'}\) be the pullback of \(X\) via \(R \to R'\), which is integral by [20, Proposition 4.3.8], and let \(X_{R'}^{\text{stab}}\) be the (unique) stable model of \(X_{M'}\) over \(R'\).

We apply [20, Corollary 8.3.51] to \(X_{R'}\) (see [20, Definition 8.3.39] for a definition of “in the strong sense”). This gives us \(f: X_{R'}^{\text{res}} \to X_{R'}\) birational, with \(X_{R'}^{\text{res}}\) projective
over $R'$ and regular, with $f$ an isomorphism over the open subscheme $U_{R'}$ of $X_{R'}$. Here we use that $U$ is smooth over $R$. Hence $U_{R'}$ is smooth over $R'$; hence $U_{R'}$ is regular.

Liu [20, Theorem 9.3.21] stated that there is a unique minimal regular model $X_{R'}^{\text{res}} \to X_{R'}^{\text{min}}$ of $X_{R'}^{\text{res}}$ that is in fact (see the proof) isomorphic to every relatively minimal model. This morphism is the identity on the generic fibres, and by the construction (Castelnuovo's criterion (Theorem 9.3.8) and Proposition 9.3.19 in [20]), it contracts precisely the irreducible components $E$ of the closed fibre of $X_{R'}^{\text{res}}$ such that $E$ is isomorphic to $\mathbb{P}^1_{k_E}$ with $k_E := H^0(E, O_E)$ (a finite extension of the residue field $k'$ of $R'$ over which $E$ lies) and $E \cdot E = -[k_E : k']$.

Note that the open subscheme $U_{R'}$ of $X_{R'}^{\text{res}}$ is mapped isomorphically to an open subscheme of $X_{R'}^{\text{min}}$, because its closed fibre is an open part of a curve of genus at least 1.

Furthermore, there is a unique morphism to $X_{R'}^{\text{stab}}$ from its minimal desingularization $(X_{R'}^{\text{stab}})^{\text{mindes}}$ and this morphism only contracts $\mathbb{P}^1$'s in the closed fibres of self-intersection $-2$ [20, Corollary 10.3.25]. As the geometric special fibre of $X_{R'}^{\text{stab}}$ has no $\mathbb{P}^1$'s, except with self-intersection $\leq -3$ [20, Definitions 10.3.1–2], we get that the geometric special fibre of $(X_{R'}^{\text{stab}})^{\text{mindes}}$ has no $\mathbb{P}^1$'s except with self-intersection $\leq -2$.

Exactly like $X_{R'}^{\text{res}}$, the surface $(X_{R'}^{\text{stab}})^{\text{mindes}}$ also has a morphism to $X_{R'}^{\text{min}}$ that only contracts curves that are (after field extension) $\mathbb{P}^1$'s of self-intersection $-1$, and as $(X_{R'}^{\text{stab}})^{\text{mindes}}$ has no such $\mathbb{P}^1$'s, this morphism is an isomorphism.

Therefore, through the maps of Figure 1, the open subscheme $U_{R'} \subset X_{R'}$ is mapped isomorphically to an open subscheme of $X_{R'}^{\text{stab}}$.

When we base change $U_{R'}$ to the residue field $k'$, we get an embedding $U_{k'} \to X_{k'}^{\text{stab}}$, where $U_{k'}$ is a base change of a non-empty open smooth part of $C$, hence a base change of a dense part of $C'$. Let $C''$ be the closure of the image of this embedding of curves. Then we get a birational map of curves $C'_{k'} \to C''$, hence an isomorphism to the normalization.

**Corollary A.2** Let $R$ be a discrete valuation ring with maximal ideal $m$, field of fractions $M$, and residue field $k = R/m$ of characteristic not 2 or 3. Let $D$ be a smooth projective, geometrically irreducible curve over $M$ and suppose that $R$ is such that $D$ has stable reduction over $R$.

(i) If $D$ over $M$ is given by $y^3 = x^4 + ax^2 + bx + 1$ with $b, a \pm 2 \in m$, then the stable reduction $\overline{D}$ of $D$ has an irreducible component birational to the elliptic curve $C' : Y^2 = X^3 \pm 1$,
(ii) If \( D \) over \( M \) is given by \( y^3 = x^4 + x^2 + bx + c \) with \( b, c \in m \), then the stable reduction \( \overline{D} \) of \( D \) has an irreducible component birational to the hyperelliptic curve \( C' : Y^2 = X^6 - 4 \).

**Proof** Let \( X \) (respectively, \( C \)) be the plane projective \( R \)-scheme (respectively, \( k \)-scheme) given by the defining polynomial \( f \) of \( D \). Let

\[
U = \text{Spec}(R[x, y, y^{-1}]/(f)).
\]

By Lemma A.1, it now suffices to check that \( C \) is birational to \( C' \), which we did in the proof of Lemma 3.1. \( \square \)

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