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Published in:
Journal of Number Theory

DOI:
[10.1016/j.jnt.2019.03.003](https://doi.org/10.1016/j.jnt.2019.03.003)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2019

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Tafazolean, S., & Top, J. (2019). On certain maximal hyperelliptic curves related to Chebyshev polynomials. *Journal of Number Theory*, 203, 276-293. <https://doi.org/10.1016/j.jnt.2019.03.003>

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Journal of Number Theory

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General Section

On certain maximal hyperelliptic curves related to Chebyshev polynomials

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ARTICLE INFO

Article history:

Received 12 February 2018

Received in revised form 4 March 2019

Accepted 26 March 2019

Available online 16 April 2019

Communicated by A. Pal

MSC:

11G20

11M38

14G15

14H25

Keywords:

Finite field

Maximal curves

Hyperelliptic curves

Chebyshev polynomials

Dickson polynomials

ABSTRACT

We study hyperelliptic curves arising from Chebyshev polynomials. The aim of this paper is to characterize the pairs (q, d) such that the hyperelliptic curve C over a finite field $\mathbb{F}_{q^2}$ given by $y^2 = \varphi_d(x)$ is maximal over the finite field $\mathbb{F}_{q^2}$ of cardinality q^2 . Here $\varphi_d(x)$ denotes the Chebyshev polynomial of degree d . The same question is studied for the curves given by $y^2 = (x \pm 2)\varphi_d(x)$, and also for $y^2 = (x^2 - 4)\varphi_d(x)$. Our results generalize some of the statements in [12].

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1. Introduction

Let p be an odd prime number, let q be a power of p , and denote by $\mathbb{F}_{q^2}$ the finite field with q^2 elements. Let $\mathcal{C}$ be a curve (complete, smooth, and geometrically irreducible) of genus $g \geq 0$ over the finite field $\mathbb{F}_{q^2}$. We call the curve $\mathcal{C}$ maximal over $\mathbb{F}_{q^2}$ if the number of rational points of $\mathcal{C}$ over $\mathbb{F}_{q^2}$ attains the upper bound of Hasse-Weil, i.e.,

$$\#\mathcal{C}(\mathbb{F}_{q^2}) = 1 + q^2 + 2gq.$$

Not only have maximal curves several intrinsic geometrical properties, but also they have been investigated in connection with Coding Theory: in some cases the best known linear codes over finite fields of square order are obtained as one-point AG-codes from maximal curves.

In this note we consider hyperelliptic curves given by one of the equations $y^2 = \varphi_d(x)$ or $y^2 = (x \pm 2)\varphi_d(x)$ or $y^2 = (x^2 - 4)\varphi_d(x)$ over $\mathbb{F}_{q^2}$. Here $\varphi_d(x)$ denotes the Chebyshev polynomial of degree d over $\mathbb{F}_p \subset \mathbb{F}_{q^2}$: recall that this is the reduction modulo p of the unique polynomial $\phi(X) \in \mathbb{Z}[X]$ such that

$$x^d + x^{-d} = \phi(x + x^{-1})$$

in $\mathbb{Z}[x, x^{-1}]$.

Remark 1.1. Note that $\varphi_d(x) = D_d(x, 1)$ with D_d the d -th Dickson polynomial of the first kind with parameter 1, defined recursively by

$$D_n(x, 1) = xD_{n-1}(x, 1) - D_{n-2}(x, 1)$$

for $n \geq 2$, and $D_0(x, 1) = 2$ and $D_1(x, 1) = x$. Dickson polynomials are related to the classical Chebyshev polynomials $T_n(x)$, defined for each integer $n \geq 0$ by $T_n(x) = \cos(n \arccos x)$; indeed we have that $D_n(x, 1) = 2T_n(x/2)$. Because of this connection, these Dickson polynomials are also called Chebyshev polynomials (see [15, Page 355]), a convention we follow here.

In Lemma 2.5 we describe the pairs (q, d) such that $\varphi_d(x)$ is a separable polynomial (over $\mathbb{F}_q$). Our main goal is to study the problem, for which pairs (q, d) the curve in question; so, given by one of the equations $y^2 = \varphi(x)$ or $y^2 = (x \pm 2)\varphi_d(x)$ or $y^2 = (x^2 - 4)\varphi_d(x)$, is maximal over $\mathbb{F}_{q^2}$. Throughout, when we write “curve with (affine) equation $y^2 = f(x)$ ” or even $C: y^2 = f(x)$ we mean that we consider the smooth, complete curve birational (over the ground field) to the curve given by the affine equation. We have the following results.

Theorem 1.2. *Let $d > 0$ be an even integer and let q be a prime power with $\gcd(q, d) = 1$. Then the hyperelliptic curve $\mathcal{C}$ given by*

$$y^2 = (x + 2)\varphi_d(x)$$

is maximal over $\mathbb{F}_{q^2}$ if and only if either $q \equiv -1 \pmod{4d}$ or $q \equiv 2d + 1 \pmod{4d}$.

Remark 1.3. The definition of the polynomials φ_d implies that $\varphi_d(-x) = (-1)^d \varphi_d(x)$. As a consequence, for d even and q odd, using a primitive 4-th root of unity $i \in \mathbb{F}_{q^2}$ one obtains an isomorphism over $\mathbb{F}_{q^2}$ given by $(u, v) \mapsto (-u, iv)$ from the curve $\mathcal{C}$ described above, to the curve with equation $y^2 = (x - 2)\varphi_d(x)$. Hence for this curve, the same maximality criteria over $\mathbb{F}_{q^2}$ hold as those described in Theorem 1.2.

Another property that is immediate from the definition of the polynomials $\varphi_d(x)$ is that if $d = a \cdot b$ for positive integers a, b , then $\varphi_d(x) = \varphi_a(\varphi_b(x))$. Applying this in the situation of Theorem 1.2 with $a = d/2$ and $b = 2$, one obtains $\varphi_d(x) = \varphi_{d/2}(x^2 - 2)$. Writing the equation for $\mathcal{C}$ as $y^2 = (x + 2)\varphi_{d/2}(x^2 - 2)$, it already appears in [23, Proposition 3]. In fact this will be used in Section 5.

The analog of Theorem 1.2 for odd d is as follows.

Theorem 1.4. Let $d \geq 1$ be an odd integer and let q be a prime power with $\gcd(q, 2d) = 1$. Then the hyperelliptic curve $\mathcal{C}$ given by

$$y^2 = (x + 2)\varphi_d(x)$$

is maximal over $\mathbb{F}_{q^2}$ if and only if $q \equiv -1 \pmod{2d}$.

For the hyperelliptic curve given by $y^2 = \varphi_d(x)$ our strongest results are obtained in the case that d is even:

Theorem 1.5. Suppose $d > 0$ is an even integer and q is a prime power with $\gcd(q, d) = 1$. Then the following statements are equivalent.

- (i) the hyperelliptic curve $\mathcal{C}_1: y^2 = (x^2 - 4)\varphi_d(x)$ is maximal over $\mathbb{F}_{q^2}$;
- (ii) $q \equiv -1 \pmod{4}$ and the hyperelliptic curve $\mathcal{C}: y^2 = \varphi_d(x)$ is maximal over $\mathbb{F}_{q^2}$;
- (iii) $q \equiv -1 \pmod{2d}$.

For odd $d > 0$ we have the following somewhat weaker result.

Theorem 1.6. Let $d > 0$ be an odd integer and let q be a prime power. Assume that q is coprime to $2d$. If $q \equiv -1 \pmod{4d}$ or $q \equiv 2d + 1 \pmod{4d}$, then the curve $\mathcal{C}: y^2 = \varphi_d(x)$ is maximal over $\mathbb{F}_{q^2}$, and so is the curve $\mathcal{C}_1: y^2 = (x^2 - 4)\varphi_d(x)$. If both $\mathcal{C}$ and $\mathcal{C}_1$ are maximal over $\mathbb{F}_{q^2}$, then either $q \equiv -1 \pmod{4d}$ or $q \equiv 2d + 1 \pmod{4d}$.

Based on considering small cases and experiments using Magma (see also the discussion in Remark 5.4 and the special case based on a result of Kohel and Smith which we discuss in Remark 5.5), we in fact have a stronger expectation for odd $d > 0$:

Conjecture 1.7. *For any prime power q and any odd $d > 0$ with $\gcd(q, 2d) = 1$, the following statements are equivalent.*

- (i) *the hyperelliptic curve $\mathcal{C}_1: y^2 = (x^2 - 4)\varphi_d(x)$ is maximal over $\mathbb{F}_{q^2}$;*
- (ii) *$q \equiv -1 \pmod{4}$ and the hyperelliptic curve $\mathcal{C}: y^2 = \varphi_d(x)$ is maximal over $\mathbb{F}_{q^2}$.*

Clearly if Conjecture 1.7 holds then a more complete and simple criterion follows (using Theorem 1.6 and similar to Theorem 1.5).

In Sections 2 and 3 some necessary background is recalled and a general necessary condition on the characteristic is shown (Proposition 2.3) in order for a hyperelliptic curve with equation $y^2 = xg(x^2)$ over $\mathbb{F}_{q^2}$ (of positive genus) to be maximal. Section 4 contains the proofs of most results announced in this introduction. In Section 5 we prove Theorem 1.5 and discuss Conjecture 1.7. We finish with a small application/illustration of Complex Multiplication theory (Proposition 5.8).

2. Preliminaries

The *zeta function* of a curve $\mathcal{C}$ over a finite field k of cardinality q is a rational function of the form

$$Z(\mathcal{C}/k) = \frac{L(t)}{(1-t)(1-qt)},$$

where $L(t) \in \mathbb{Z}[t]$ is a polynomial of degree $2g = 2 \cdot \text{genus}(\mathcal{C})$ with integral coefficients (see [18, Chapter V]). We call this polynomial the *L-polynomial* of $\mathcal{C}$ over k .

We recall the following fact about maximal curves which can be deduced by extending the argument on p. 182 of [18].

Proposition 2.1. *Suppose q is a square. For a smooth projective curve $\mathcal{C}$ of genus g , defined over $k = \mathbb{F}_q$, the following conditions are equivalent:*

- *$\mathcal{C}$ is maximal over $\mathbb{F}_q$.*
- *$L(t) = (1 + \sqrt{q}t)^{2g}$.*

A common method to construct (explicit) maximal curves is via the following remark which although commonly attributed to J-P. Serre (cf. Lachaud [14]), is implicitly already contained in Tate's seminal paper [22]:

Remark 2.2. Given a non-constant morphism $f: \mathcal{C} \rightarrow \mathcal{D}$ defined over the finite field k , the L -polynomial of $\mathcal{D}$ over k divides the one of $\mathcal{C}$ over k . Hence a subcover $\mathcal{D}$ over $\mathbb{F}_{q^2}$ of a maximal curve $\mathcal{C}$ over $\mathbb{F}_{q^2}$ is also maximal.

Many examples of maximal curves have been found in this way starting from ‘standard’ known ones. In various cases this is done including explicit equations for the

subcover, in other cases by merely identifying appropriate subfields (and the genus of the corresponding curve) of a function field $\mathbb{F}_{q^2}(\mathcal{C})$ of a maximal $\mathcal{C}/\mathbb{F}_{q^2}$. From the abundant literature on this, we mention [7], [1], [5], [2], [19], [20], [21], [9], [8], [3], [11].

In the present paper we work in some sense ‘the other way around’: the curves we study are indeed subcovers $\mathcal{D}$ (by an morphism of degree 2) of curves $\mathcal{C}$ for which maximality properties are precisely known. By identifying the L -polynomial of $\mathcal{C}$ essentially in terms of that of $\mathcal{D}$ in the cases at hand, which is done by ‘understanding’ up to isogeny the Jacobian variety of $\mathcal{C}$ in terms of that of $\mathcal{D}$, we obtain necessary (and not only sufficient) maximality criteria for $\mathcal{D}$.

The following result yields a necessary condition for maximality of a special type of hyperelliptic curves.

Proposition 2.3. *Let $q = p^n$ be the cardinality of a finite field $\mathbb{F}_q$ of characteristic $p > 2$. Suppose $g(x) \in \mathbb{F}_q[x]$ is separable of degree $d \geq 1$, and $g(0) \neq 0$. Let $\mathcal{C}$ be the hyperelliptic curve over $\mathbb{F}_q$ with equation $y^2 = xg(x^2)$.*

If the Jacobian of $\mathcal{C}$ is supersingular, then $p \equiv 3 \pmod{4}$.

As a consequence, if $\mathcal{C}$ is maximal over $\mathbb{F}_{q^2}$, then $p \equiv 3 \pmod{4}$.

Proof. The assumptions imply that $\mathcal{C}$ is a curve of genus $d \geq 1$. Let i be a primitive 4-th root of unity in some extension of $\mathbb{F}_q$. The curve $\mathcal{C}$ admits the automorphism ι given by $\iota(x, y) = (-x, iy)$. The action of ι on the vector space of regular 1-forms on $\mathcal{C}$ is diagonalizable, and has as eigenvalues $\pm i$.

We claim that maximality of $\mathcal{C}$ over $\mathbb{F}_{q^2}$ implies that the characteristic p of $\mathbb{F}_q$ satisfies $p \equiv 3 \pmod{4}$. Indeed, if $p \equiv 1 \pmod{4}$ then take integers a, b such that $p = a^2 + b^2$. As endomorphisms of $\mathcal{J} = \text{Jac}(\mathcal{C})$ this yields a factorization $p = (a + b\iota)(a - b\iota)$. Since multiplication by p is inseparable, at least one of the endomorphisms $a \pm b\iota$ is inseparable as well. However, it is not possible that both are inseparable since that would imply the sum $2a$ to be inseparable as well, which clearly is not the case. This means that after changing the sign of b if necessary, we have that $a + b\iota$ is separable. Hence its kernel $\mathcal{J}[a + b\iota](\overline{\mathbb{F}_q})$ is a nontrivial subgroup of the p -torsion of $\mathcal{J}$, which shows that $\mathcal{J}$ is not supersingular.

Since the p -torsion of the Jacobian of any maximal curve over $\mathbb{F}_{q^2}$ is trivial, $\mathcal{C}$ cannot be maximal over any finite field of characteristic $\equiv 1 \pmod{4}$. So we have $p \equiv 3 \pmod{4}$. $\square$

Remark 2.4. The assumption that a curve $\mathcal{C}$ of genus g is maximal over $\mathbb{F}_{q^2}$ implies that the L -polynomial of $\mathcal{C}$ over $\mathbb{F}_q$ (which has as zeros square roots of the zeros of the L -polynomial of $\mathcal{C}$ over $\mathbb{F}_{q^2}$) must be $(1 + qt^2)^{2g}$. In the situation described in Proposition 2.3 this means that if q is a square, then the quartic twist of $\mathcal{C}$ over $\mathbb{F}_q$ corresponding to the cocycle $F_q \mapsto \iota$ (with F_q the q -th power Frobenius) has L -polynomial $(1 - qt^2)^{2g}$. In case q is not a square, the analogous cocycle results in a twist that has (again) L -polynomial $(1 + qt^2)^{2g}$.

We finish this section with a preliminary result generalizing parts of [6, Theorem 6.1(b)] and [11, Theorem 7.2(a)] (in fact it is based on essentially the same ideas already present in [6]).

Lemma 2.5. *For $d \in \mathbb{Z}_{>0}$ and q a prime power, the Chebyshev polynomial φ_d considered over the finite field $\mathbb{F}_q$ of cardinality q is separable if and only if $\gcd(q, 2d) = 1$ or $d = 1$.*

Proof. Consider the morphism $\alpha: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given (in terms of local coordinates) by $\alpha(x) = x^d + x^{-d}$. One factors $\alpha = \beta \circ \gamma$ with $\gamma: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by $\gamma(x) = x^d$ and $\beta: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by $\beta(x) = x + x^{-1}$. Regarding φ_d as the morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ given by $x \mapsto \varphi_d(x)$, by definition $\alpha = \beta \circ \gamma = \varphi_d \circ \beta$. We study separability of the polynomial φ_d , which means we examine whether the morphism φ_d is separable and moreover has no ramification points over $0 \in \mathbb{P}^1$. To this end, first consider separability (and ramification) of the two morphisms γ and β .

Clearly β is a separable morphism of degree 2, in every characteristic. It is only ramified in ± 1 , and this is one point in characteristic 2 and two points in every other characteristic.

The morphism γ is inseparable precisely when $\gcd(q, d) \neq 1$. If this holds then also $\alpha = \beta \circ \gamma$ is inseparable. As a consequence, so is φ_d since $\alpha = \gamma \circ \varphi_d$ and γ is separable. So

$$\gcd(q, d) \neq 1 \quad \Rightarrow \quad \text{the polynomial } \varphi_d \text{ is inseparable over } \mathbb{F}_q.$$

Next, assume $\gcd(q, d) = 1$ so that γ is separable (over $\mathbb{F}_q$). Then α and hence φ_d are separable morphisms as well. To obtain the ramification points of φ_d in this case, we compute the ramification of $\alpha = \beta \circ \gamma$. First consider the case that q is odd. Then β is only ramified in ± 1 (both points with ramification index $e_{\pm 1} = 2$ and $\beta(\pm 1) = \pm 2$). Moreover γ is only ramified in 0 and in ∞ (both with ramification index d) and $\gamma^{-1}(\pm 1)$ consists of the $2d$ pairwise distinct solutions of $x^{2d} = 1$. Since $\gamma^{-1}(0) = 0$ and $\gamma^{-1}(\infty) = \infty$, the conclusion is that the total map α is ramified only in the following points: $\{0, \infty\}$, each with ramification index d , and in the $2d$ -th roots of unity, each with ramification index 2. Moreover the image of these points under α is $\{\infty, \pm 2\}$. Since $0 \notin \{\infty, \pm 2\}$ and $\alpha = \varphi_d \circ \beta$, one concludes

$$q \text{ is odd and } \gcd(q, d) = 1 \quad \Rightarrow \quad \text{the polynomial } \varphi_d \text{ is separable over } \mathbb{F}_q.$$

Now consider the case $2|q$ and $\gcd(q, d) = 1$. This implies that the map α is separable over $\mathbb{F}_q$. As in the previous case, the ramification of α is easily found using $\alpha = \beta \circ \gamma$. Now β is only ramified at 1, with $\beta(1) = 0$ (ramification index 2). We conclude that α is ramified only in the following points: $\{0, \infty\}$, each with ramification index d , and in the d -th roots of unity, each with ramification index 2. The image of these points under α is $\{\infty, 0\}$. As $\alpha^{-1}(0)$ consists of the d -th roots of unity and only $1 \in \alpha^{-1}(0)$ is a ramification point of β , the decomposition $\alpha = \varphi_d \circ \beta$ shows that whenever $d > 1$ then

$\varphi_d: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is ramified in some points over 0 (namely, in $\zeta + \zeta^{-1}$ with $\zeta \neq 1$ satisfying $\zeta^d = 1$). We showed:

$$2|q \text{ and } \gcd(q, d) = 1 \text{ and } d > 1 \Rightarrow \text{the polynomial } \varphi_d \text{ is inseparable over } \mathbb{F}_q.$$

Since the case $d = 1$ (so $\varphi_d(x) = x$) is trivial, the lemma follows. $\square$

3. The curves $y^2 = x^{2d+1} + x$ and $y^2 = x^{2d} + 1$

Let $d \geq 1$ be an integer, and let q be a prime power such that $\gcd(q, 2d) = 1$. We consider the complete non-singular curve $\mathcal{X}$ over $\mathbb{F}_{q^2}$ birational to the plane affine curve given by

$$y^2 = x^{2d+1} + x.$$

The condition on the pair (q, d) implies that $\mathcal{X}$ has genus d .

The following result is crucial for us (see [19, Theorem 1]).

Theorem 3.1. *The smooth complete hyperelliptic curve $\mathcal{X}$ given by*

$$y^2 = x^{2d+1} + x$$

is maximal over $\mathbb{F}_{q^2}$ if and only if either $q \equiv -1 \pmod{4d}$ or $q \equiv 2d + 1 \pmod{4d}$.

Now let $\mathcal{Y}$ be the complete non-singular curve over $\mathbb{F}_q$ given by $y^2 = x^{2d} + 1$. Note that the condition $\gcd(q, 2d) = 1$ implies that $\mathcal{Y}$ has genus $d - 1$.

One more result which will be used in our proofs is recalled from [20]:

Theorem 3.2. *The smooth complete hyperelliptic curve $\mathcal{Y}: y^2 = x^{2d} + 1$ is maximal over $\mathbb{F}_{q^2}$ if and only if $q \equiv -1 \pmod{2d}$.*

4. Hyperelliptic curves from Chebyshev polynomials

In this section we prove Theorems 1.2, 1.4, 1.6, and we present and prove some preliminary results which will be used in the proof of Theorem 1.5.

Case d even and $v^2 = (u + 2)\varphi_d(u)$

Proof. (of Theorem 1.2). Take $d > 0$ an even integer, and let q be a prime power with $\gcd(d, q) = 1$. We will show that the curve $\mathcal{C}$ with affine equation

$$v^2 = (u + 2)\varphi_d(u)$$

is maximal over $\mathbb{F}_{q^2}$ if and only if the curve $\mathcal{X}$ introduced in Section 3 (with equation $y^2 = x^{2d+1} + x$) is maximal over $\mathbb{F}_{q^2}$. Theorem 1.2 is then a consequence of Theorem 3.1.

The main idea is to decompose the Jacobian variety $\mathcal{J}(\mathcal{X})$ up to isogeny over $\mathbb{F}_{q^2}$. Let $\tau \in \text{Aut}(\mathcal{X})$ be the involution given by $\tau(x, y) = (1/x, y/x^{d+1})$. The quotient of $\mathcal{X}$ by τ is the curve $\mathcal{C} = \mathcal{X} / \langle \tau \rangle$ with equation

$$v^2 = (u + 2)\varphi_d(u);$$

indeed, the functions $u = x + 1/x$ and $v = y(x + 1)x^{-1-d/2}$ generate the subfield of τ -invariants in the function field of $\mathcal{X}$, as is seen as follows. Write $\mathbb{F}_p(x, y)$ for the function field of $\mathcal{X}$ over the prime field $\mathbb{F}_p$ of $\mathbb{F}_q$. We have the inclusions of fields (where the numbers describe the degree of the given extensions)

$$\begin{array}{ccc} \mathbb{F}_p(x, y) & \supset & \mathbb{F}_p(u, v) \\ \cup_2 & & \cup_2 \\ \mathbb{F}_p(x) & \supset & \mathbb{F}_p(u) \end{array}$$

Since $[\mathbb{F}_p(x, y) : \mathbb{F}_p(x, y)^{\langle \tau \rangle}] = 2$ and $u, v \in \mathbb{F}_p(x, y)^{\langle \tau \rangle}$, one has $\mathbb{F}_p(x, y)^{\langle \tau \rangle} = \mathbb{F}_p(u, v)$. Moreover, u, v satisfy

$$v^2 = (x^{2d+1} + x)(x + 2 + x^{-1})x^{-d-1} = (u + 2)\varphi_d(u).$$

We have the basis

$$\{\omega_j := \frac{x^{j-1}dx}{y} \mid 1 \leq j \leq d\}$$

for the space of regular differentials on $\mathcal{X}$. A basis for the differentials invariant under τ is

$$\{\omega_j - \omega_{d-j+1} \mid 1 \leq j \leq d/2\},$$

which also generate the pull-backs of the regular differentials on $\mathcal{C}$ (note that since we assume $\gcd(q, d) = 1$, Lemma 2.5 implies that φ_d is separable over $\mathbb{F}_q$. Also, $\varphi_d(-2) = \varphi_d((-1) + (-1)) = (-1)^d + (-1)^d = 2 \neq 0$, so $\mathcal{C}$ has genus $d/2$).

Let ι be the hyperelliptic involution on $\mathcal{X}$, so $\iota(x, y) = (x, -y)$. The quotient of $\mathcal{X}$ by $\tau\iota$ (this map is an involution defined over the prime field) is the curve $\mathcal{C}_1 = \mathcal{X} / \langle \tau\iota \rangle$ with equation

$$\eta^2 = (\xi - 2)\varphi_d(\xi);$$

indeed, the invariants under ρ in the function field of $\mathcal{X}$ are generated by $\xi := x + x^{-1}$ and $\eta := y(x - 1)x^{-1-d/2}$. These functions satisfy

$$\eta^2 = (x^{2d+1} + x)(x - 2 - x^{-1})x^{-d-1} = (\xi - 2)\varphi_d(\xi).$$

A basis for the differentials invariant under $\tau\iota$ is

$$\{\omega_j + \omega_{d-j+1} \mid 1 \leq j \leq d/2\},$$

which also generate the pull-backs to $\mathcal{X}$ of the regular differentials on $\mathcal{C}_1$.

Fixing a primitive 4-th root of unity $i \in \mathbb{F}_{q^2}$, the map $(u, v) \mapsto (-u, iv)$ yields an isomorphism $\mathcal{C} \cong \mathcal{C}_1$ defined over $\mathbb{F}_{q^2}$. The discussion above shows, with $\sim$ denoting isogeny defined over $\mathbb{F}_{q^2}$, that

$$\mathcal{J}(\mathcal{X}) \sim \mathcal{J}(\mathcal{C}) \times \mathcal{J}(\mathcal{C}_1) \cong \mathcal{J}(\mathcal{C})^2.$$

As a consequence $L_{\mathcal{X}}(t) = L_{\mathcal{C}}(t)^2$ with L denoting an L -polynomial over $\mathbb{F}_{q^2}$. Now Proposition 2.1 implies that the curve $\mathcal{X}$ is maximal if and only if the curve $\mathcal{C}$ is maximal. This completes the proof. $\square$

Remark 4.1. Theorem 1.2 generalizes a part of [12, Proposition 6]. The decomposition up to isogeny of the Jacobian variety $\mathcal{J}(\mathcal{X})$ as a product of Jacobians of quotient curves, can also be obtained using results of Kani and Rosen [10]. There are various examples in the literature illustrating this technique; we refer to [16, § 3.1.1] and [17, p. 36] for situations very similar to the ones discussed in the present paper.

Case d odd and $v^2 = (u + 2)\varphi_d(u)$

Proof. (of Theorem 1.4). This is very similar to the proof of Theorem 1.2. Take $d > 0$ an odd integer, and let q be a prime power with $\gcd(q, 2d) = 1$. We will show that the curve $\mathcal{C}$ with affine equation

$$v^2 = (u + 2)\varphi_d(u)$$

is maximal over $\mathbb{F}_{q^2}$ if and only if the curve $\mathcal{Y}: y^2 = x^{2d} + 1$ is maximal over $\mathbb{F}_{q^2}$. Theorem 1.4 is then a consequence of Theorem 3.2.

Let σ be the involution on $\mathcal{Y}$ defined by $\sigma(x, y) = (1/x, y/x^d)$. The quotient of $\mathcal{Y}$ by σ is the hyperelliptic curve $\mathcal{C}$. Indeed, the functions $u = x + x^{-1}$ and $v = y(1 + x)x^{-(d+1)/2}$ generate the field of functions invariant under σ , and one computes

$$v^2 = y^2 \cdot x^{-(d+1)} \cdot (x + 1)^2 = (x^d + x^{-d})(x + 2 + x^{-1}) = (u + 2)\varphi_d(u).$$

Multiplying σ by the hyperelliptic involution on $\mathcal{Y}$ one obtains another quotient curve which we denote by $\mathcal{C}_1$. The invariant functions under the new involution are generated by $u = x + x^{-1}$ and $w = y(1 - x)x^{-(d+1)/2}$. They satisfy $w^2 = (u - 2)\varphi_d(u)$. The map $(u, w) \mapsto (-u, w)$ defines an isomorphism $\mathcal{C}_1 \cong \mathcal{C}$.

Analogous to the previous proof one concludes $L_Y(t) = L_C(t) \cdot L_{C_1}(t) = L_C(t)^2$, in this case for the L -polynomials over $\mathbb{F}_q$ as well as for those over $\mathbb{F}_{q^2}$. This implies the result. $\square$

Case d odd and $y^2 = \varphi_d(x)$

Proof. (of Theorem 1.6). Let $d \geq 1$ be an odd integer. Take a prime power q such that $\gcd(q, 2d) = 1$. We will consider curves over (the prime field of) $\mathbb{F}_{q^2}$. Recall (see the proof of Theorem 1.2) that the hyperelliptic curve $\mathcal{X}$ with affine equation $y^2 = x^{2d+1} + x$ admits the involution τ defined by $\tau(x, y) = (1/x, y/x^{d+1})$. For odd d , the quotient of $\mathcal{X}$ by τ is the hyperelliptic curve $\mathcal{C}$ with equation

$$y^2 = \varphi_d(x);$$

indeed, a quotient map is given by

$$(x, y) \mapsto (x + 1/x, y/x^{(d+1)/2})$$

(compare [23, Proposition 3]). Now if either $q \equiv -1 \pmod{4d}$ or $q \equiv 2d + 1 \pmod{4d}$, then by Theorem 3.1 the curve $\mathcal{X}$ is maximal over $\mathbb{F}_{q^2}$ which implies that the curve $\mathcal{C}$ is also maximal over $\mathbb{F}_{q^2}$. This proves the first assertion of Theorem 1.6.

To show the remaining parts, we will decompose up to isogeny the Jacobian $\mathcal{J}(\mathcal{X})$ of the curve $\mathcal{X}$. With the basis $\omega_j := x^{j-1}dx/y$ (for $1 \leq j \leq d$) for the regular differentials on $\mathcal{X}$, one checks that a basis for the differentials invariant under τ is

$$\omega_1 - \omega_d, \omega_2 - \omega_{d-1}, \dots, \omega_{(d-1)/2} - \omega_{(d+3)/2},$$

which also generate the pull-backs of the regular differentials on $\mathcal{C}$; note that by Lemma 2.5 the condition $\gcd(q, 2d) = 1$ implies that φ_d is separable over $\mathbb{F}_q$ hence $\mathcal{C}$ has genus $(d-1)/2$.

Let ι be the hyperelliptic involution on $\mathcal{X}$. The quotient of $\mathcal{X}$ by $\tau\iota$ (this automorphism has order 2 and it is defined over the prime field) is the curve $\mathcal{C}_1 = \mathcal{X}/\langle \tau\iota \rangle$ with equation

$$y^2 = (x^2 - 4)\varphi_d(x);$$

indeed, the functions $\xi := x + 1/x$ and $\eta := \frac{y}{x^{(d+1)/2}}(x - \frac{1}{x}) \in \mathbb{F}_q(\mathcal{X})$ are invariant under the action of $\tau\iota$ and $[\mathbb{F}_q(\mathcal{X}) : \mathbb{F}_q(\xi, \eta)] = 2$. Hence ξ, η generate the function field of $\mathcal{C}_1$. We have

$$\eta^2 = \frac{y^2}{x^{d+1}}(x^2 - 2 + x^{-2}) = (x^d + x^{-d}) \left((x + \frac{1}{x})^2 - 4 \right) = (\xi^2 - 4)\varphi_d(\xi).$$

From this, the second assertion in Theorem 1.6 follows: namely, by Theorem 3.1 the congruence condition on q implies that $\mathcal{X}$ is maximal over $\mathbb{F}_{q^2}$. Since $\mathcal{X}$ covers $\mathcal{C}_1$, the same is true for $\mathcal{C}_1$ over $\mathbb{F}_{q^2}$.

Note that $\varphi_d(2) = \varphi_d(1+1) = 1^d + 1^d = 2$ and similarly $\varphi_d(-2) = -2$. Using Lemma 2.5 this implies that in every characteristic coprime to $2d$ the polynomial $(x^2 - 4)\varphi_d(x)$ is separable. A basis for the differentials invariant under τ_ι is $\{\omega_i + \omega_{d-i+1} \mid 1 \leq i \leq (d+1)/2\}$, which also generate the pull-backs of the regular differentials on $\mathcal{C}_1$.

Since the pull-backs of a basis of the regular differentials on $\mathcal{C}$ together with the pull-backs of a similar basis on $\mathcal{C}_1$ yield a basis for the regular differentials on $\mathcal{X}$, one concludes that the Jacobian $\mathcal{J}(\mathcal{X})$ of $\mathcal{X}$ is isogenous to a product

$$\mathcal{J}(\mathcal{C}) \times \mathcal{J}(\mathcal{C}_1),$$

where $\mathcal{J}(\mathcal{C})$ and $\mathcal{J}(\mathcal{C}_1)$ are the Jacobians of the curves $\mathcal{C}$ and $\mathcal{C}_1$, respectively. This implies that $L_{\mathcal{X}}(t) = L_{\mathcal{C}}(t) \cdot L_{\mathcal{C}_1}(t)$ (for L -polynomials over any extension of $\mathbb{F}_q$). Hence if both $\mathcal{C}$ and $\mathcal{C}_1$ are maximal over $\mathbb{F}_{q^2}$ then so is $\mathcal{X}$, which by Theorem 3.1 implies that $q \equiv -1 \pmod{4d}$ or $q \equiv 2d+1 \pmod{4d}$. This finishes the proof. $\square$

Remark 4.2. The special case $d = 3$ of the Theorem 1.6 is a part of [12, Proposition 4]. In fact for $d = 3$ one finds ([12]) that $\mathcal{J}(\mathcal{C})$ is the elliptic curve $\mathcal{E}_1$ with equation $y^2 = x^3 - 3x$ and (up to isogeny) $\mathcal{J}(\mathcal{C}_1)$ is a product $\mathcal{E}_2 \times \mathcal{E}_3$ where $\mathcal{E}_2$ is the elliptic curve with equation $y^2 = x^3 + x$ and $\mathcal{E}_3$ is the one with equation $y^2 = x^3 + 108x$. These two elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ are isogenous over $\mathbb{F}_{q^2}$ (for q any prime power with $\gcd(q, 6) = 1$). So in this case maximality of any one of them over $\mathbb{F}_{q^2}$ is equivalent to $q \equiv 3 \pmod{4}$ and to maximality of any one of the curves $\mathcal{C}$ or $\mathcal{C}_1$ over $\mathbb{F}_{q^2}$. In particular, Conjecture 1.7 holds for $d = 3$.

Case d even and $y^2 = \varphi_d(x)$

A preliminary result relying on an analogous reasoning as above, is the following which will be used in the proof of Theorem 1.5.

Lemma 4.3. *Let $d > 0$ be an even integer and let q be a prime power. Assume $\gcd(q, d) = 1$. The next two statements are equivalent.*

- (i) $q \equiv -1 \pmod{2d}$;
- (ii) the curve $\mathcal{C}$ with affine equation

$$v^2 = \varphi_d(u)$$

is maximal over $\mathbb{F}_{q^2}$, and so is the curve $\mathcal{C}_1$ over $\mathbb{F}_{q^2}$ given by

$$v^2 = (u^2 - 4)\varphi_d(u).$$

Proof. Take $d = 2e$ for some integer $e > 0$ and let q be a prime power with $\gcd(q, d) = 1$. The curve $\mathcal{Y}$ over $\mathbb{F}_q$ with affine equation $y^2 = x^{2d} + 1$ admits the involution σ given by $\sigma(x, y) = (1/x, y/x^d)$. The functions in $\mathbb{F}_q(\mathcal{Y})$ which are invariant under σ are generated by $u = x + 1/x$ and $v = \frac{y}{x^e}$. We have

$$v^2 = x^{-d}(x^{2d} + 1) = \varphi_d(x + \frac{1}{x}) = \varphi_d(u),$$

so the quotient of $\mathcal{Y}$ by σ is the curve $\mathcal{C}$ given by $v^2 = \varphi_d(u)$.

If $q \equiv -1 \pmod{2d}$ then by Theorem 3.2 the curve $\mathcal{Y}$ is maximal over $\mathbb{F}_{q^2}$. Since this curve covers $\mathcal{C}$, it follows from Remark 2.2 that also $\mathcal{C}$ is maximal over $\mathbb{F}_{q^2}$. This shows the first claim in Proposition 4.3.

For the second claim we use the product σ' of σ and the hyperelliptic involution on $\mathcal{Y}$, so $\sigma'(x, y) = (1/x, -y/x^d)$. The invariants in $\mathbb{F}_q(\mathcal{Y})$ under σ' are generated by $u = x + 1/x$ and $w = \frac{y}{x^e}(x - \frac{1}{x})$, and they are related by

$$w^2 = (x^{2d} + 1)x^{-d}(x^2 - 2 + x^{-2}) = (x^d + x^{-d})((x + \frac{1}{x})^2 - 4) = (u^2 - 4)\varphi_d(u).$$

So also $\mathcal{C}_1: w^2 = (u^2 - 4)\varphi_d(u)$ is covered by $\mathcal{Y}$. Hence by Theorem 3.2, if $q \equiv -1 \pmod{2d}$ then $\mathcal{C}_1$ is maximal over $\mathbb{F}_{q^2}$, proving the second claim in Proposition 4.3.

For the last claim, observe that analogous to the other results shown in this section we have that the Jacobian $\mathcal{J}(\mathcal{Y})$ is isogenous over $\mathbb{F}_q$ to the product $\mathcal{J}(\mathcal{C}) \times \mathcal{J}(\mathcal{C}_1)$. Hence the L -polynomial of $\mathcal{Y}$ over any extension of $\mathbb{F}_q$ is the product of the L -polynomials of $\mathcal{C}$ and $\mathcal{C}_1$ (over the same extension). The remaining statement in Proposition 4.3 is an immediate consequence of this. $\square$

5. Relating $y^2 = \varphi_d(x)$ and $y^2 = (x^2 - 4)\varphi_d(x)$

Here we prove Theorem 1.5 and we make some remarks concerning Conjecture 1.7. The following lemma turns out to be useful.

Lemma 5.1. *Let $d \geq 1$ be any integer and let q be a prime power with $\gcd(q, 2d) = 1$. Then the L -polynomial of the elliptic curve over $\mathbb{F}_q$ given by $y^2 = x^3 + x$ divides the L -polynomial of the curve $\mathcal{C}_1$ over $\mathbb{F}_q$ with affine equation $y^2 = (x^2 - 4)\varphi_d(x)$.*

Proof. First consider the case that d is odd. We use the notations from the proof of Theorem 1.6 and we let ζ in some extension of $\mathbb{F}_q$ be a primitive $4d$ -th root of unity. The curve $\mathcal{X}$ admits an automorphism ρ given by $\rho(x, y) = (\zeta^2 x, \zeta y)$. The quotient of $\mathcal{X}$ by the group generated by ρ^4 is the elliptic curve $\mathcal{E}$ given by

$$y^2 = x^3 + x$$

and an explicit quotient map is given by

$$(x, y) \mapsto (x^d, x^{(d-1)/2}y).$$

Note that although the elements of the group generated by ρ^4 may not be defined over $\mathbb{F}_q$, the group is, which explains why the quotient curve and the map to it are defined over $\mathbb{F}_q$. A regular differential on $\mathcal{X}$ invariant under ρ^4 is $\omega_{(d+1)/2} = x^{(d-1)/2}dx/y$; observe that this differential is a pull back of a regular differential on $\mathcal{C}_1$.

As a consequence, the elliptic curve $\mathcal{E}$ is up to isogeny contained in the Jacobian $\mathcal{J}(\mathcal{C}_1)$. This implies the lemma for d odd.

Now assume $d = 2e$ is even. The curve $\mathcal{Y}: y^2 = x^{4e} + 1$ covers the given elliptic curve, with an explicit covering map given by $(x, y) \mapsto (x^{2e}, x^e y)$. Note that $x^{e-1} \frac{dx}{y}$ is a pull-back to $\mathcal{Y}$ of a regular differential on the elliptic curve. The proof of Proposition 4.3 shows that $\mathcal{J}(\mathcal{C}_1)$ is up to isogeny an abelian subvariety of $\mathcal{J}(\mathcal{Y})$, and the regular differentials on $\mathcal{Y}$ coming from $\mathcal{C}_1$ are the ones invariant under the action of the automorphism denoted σ' . As the differential $x^{e-1} \frac{dx}{y}$ is invariant under σ' , it follows that the elliptic curve is up to isogeny contained in $\mathcal{J}(\mathcal{C}_1)$. This implies the lemma for d even. $\square$

Proposition 5.2. *The analogue of Conjecture 1.7 holds in the special case $d \equiv 2 \pmod{4}$.*

Proof. Write $d = 2e$ with e a positive, odd integer and let q be a prime power satisfying $\gcd(q, 2e) = 1$. One decomposes, up to isogeny, the Jacobian $\mathcal{J}(\mathcal{C}_1)$ of the curve $\mathcal{C}_1$ given by $y^2 = (x^2 - 4)\varphi_{2e}(x)$ as follows. Note that $\mathcal{C}_1$ admits the involution α given by $\alpha(x, y) = (-x, y)$. Since $\varphi_{2e}(x) = \varphi_e(\varphi_2(x)) = \varphi_e(x^2 - 2)$, the quotient by α is the curve $\mathcal{D}$ with affine equation $v^2 = (t - 4)\varphi_e(t - 2)$ (with quotient map $(x, y) \mapsto (x^2, y)$). Using the variables v and $u := t - 2$, this equation becomes $v^2 = (u - 2)\varphi_e(u)$.

Using that the curve $\mathcal{D}$ is isomorphic to the one with equation $v^2 = (u + 2)\varphi_e(u)$ (just change the sign of u and use that e is odd), Theorem 1.4 implies that if $\mathcal{C}_1$ is maximal over $\mathbb{F}_{q^2}$, then so is $\mathcal{D}$, and therefore $q \equiv -1 \pmod{2e}$. From Lemma 5.1, the maximality of $\mathcal{C}_1$ over $\mathbb{F}_{q^2}$ implies maximality of the elliptic curve given by $y^2 = x^3 + x$ over $\mathbb{F}_{q^2}$. The latter maximality is equivalent to $q \equiv -1 \pmod{4}$.

Using that e is odd, one concludes that if $\mathcal{C}_1$ is maximal over $\mathbb{F}_{q^2}$, then $q \equiv -1 \pmod{4e}$. Hence Proposition 4.3 implies the implication (i) $\Rightarrow$ (ii) of Conjecture 1.7 in this case.

For the other implication, assume that $\mathcal{C}: y^2 = \varphi_{2e}(x)$ is maximal over $\mathbb{F}_{q^2}$ and that $q \equiv -1 \pmod{4}$. Writing $\varphi_{2e}(x) = \varphi_e(x^2 - 2)$ it is clear that the map $(x, y) \mapsto (x^2 - 2, xy)$ yields a nonconstant morphism from $\mathcal{C}$ to the curve with equation $s^2 = (t + 2)\varphi_e(t)$. Hence the latter curve is maximal over $\mathbb{F}_{q^2}$, which by Theorem 1.6 implies $q \equiv -1 \pmod{2e}$. So again one concludes $q \equiv -1 \pmod{4e}$, and the maximality of $\mathcal{C}_1$ over $\mathbb{F}_{q^2}$ follows from Proposition 4.3. $\square$

Similar ideas allow one to obtain some results in the case $d \equiv 0 \pmod{4}$:

Proposition 5.3. *Suppose the integer $d > 0$ satisfies $4|d$, then the analogue of Conjecture 1.7 holds.*

Proof. With notations as above, write $d = 2e$. The map $(x, y) \mapsto (x^2 - 2, y)$ shows that $\mathcal{C}_1$ covers the curve given by $v^2 = (u - 2)\varphi_e(u)$. Hence as before, by Theorem 1.2 one concludes that if $\mathcal{C}_1$ is maximal over $\mathbb{F}_{q^2}$, then either $q \equiv -1 \pmod{4e}$ or $q \equiv 2e + 1 \pmod{4e}$.

Similarly, the map $(x, y) \mapsto (x^2 - 2, xy)$ shows that $\mathcal{C}_1$ covers the curve with affine equation $w^2 = (u^2 - 4)\varphi_e(u)$. Hence maximality of $\mathcal{C}_1$ over $\mathbb{F}_{q^2}$ implies using Lemma 5.1 that the elliptic curve with equation $y^2 = x^3 + x$ is maximal over $\mathbb{F}_{q^2}$. As a consequence $q \equiv -1 \pmod{4}$.

Combining the congruences for q we conclude that maximality implies $q \equiv -1 \pmod{2d}$. Hence by Proposition 4.3 the curve given by $y^2 = \varphi_d(x)$ is maximal over $\mathbb{F}_{q^2}$, which is what we wanted to show.

Vice versa, is $\mathcal{C}$ with affine equation $y^2 = \varphi_d(x)$ maximal over $\mathbb{F}_{q^2}$ and is moreover $q \equiv -1 \pmod{4}$, then since $(x, y) \mapsto (x^2 - 2, xy)$ shows that $\mathcal{C}$ covers the curve given by $v^2 = (u + 2)\varphi_e(u)$, we conclude using Theorem 1.2 that either $q \equiv -1 \pmod{4e}$ or $q \equiv 2e + 1 \pmod{4e}$. However, the additional condition on q shows that the latter congruence is impossible, so one concludes $q \equiv -1 \pmod{4e}$. But then Proposition 4.3 implies maximality of $\mathcal{C}_1$ over $\mathbb{F}_{q^2}$, which is what we wished to show. $\square$

Evidently, combining Lemmas 4.3 and 5.1, and Propositions 5.2 and 5.3 one obtains a proof of Theorem 1.5.

We will now discuss Conjecture 1.7. To this end, we first describe an attempt to prove the conjecture which unfortunately seems to fail.

Remark 5.4. We continue with the notations introduced in the proofs of Theorem 1.6 and Lemma 5.1; in particular, the integer $d > 0$ is assumed to be odd. A natural way to describe a decomposition of a Jacobian variety such as $\mathcal{J}(\mathcal{X})$ is in terms of suitable endomorphisms of this Jacobian. We refer to the paper of Kani and Rosen [10] which studies the special endomorphisms generated by those coming from automorphisms of the curve.

Consider the action of $1 + \tau$ and of $1 + \rho^4 + \rho^8 + \dots + \rho^{4d-4}$ on $\mathcal{J}(\mathcal{X})$. As endomorphisms on $\mathcal{J}(\mathcal{X})$ these maps are defined over the prime field of $\mathbb{F}_q$. Moreover since $1 + \tau$ acts as 0 on the regular differentials on $\mathcal{X}$ which are pulled back from $\mathcal{C}$, and as multiplication by 2 on the regular differentials pulled back from $\mathcal{C}_1$, it follows that $(1 + \tau)(\mathcal{J}(\mathcal{X}))$ is isogenous to $\mathcal{J}(\mathcal{C}_1)$. An analogous argument shows that

$$(1 + \rho^4 + \dots + \rho^{4d-4})(1 + \tau)(\mathcal{J}(\mathcal{X}))$$

is isogenous to the elliptic curve $\mathcal{E}$. Since $1 + \rho^4 + \dots + \rho^{4d-4}$ acts as multiplication by d on the differential $\omega_{(d+1)/2}$ and as 0 on the differentials $\omega_j + \omega_{d+1-j}$ ($1 \leq j \leq (d-1)/2$), it follows that the abelian variety $\mathcal{A} \subset \mathcal{J}(\mathcal{X})$ defined by

$$\mathcal{A} := (-d + 1 + \rho^4 + \dots + \rho^{4d-4})(1 + \tau)(\mathcal{J}(\mathcal{X}))$$

is defined over the prime field of $\mathbb{F}_q$, and $\dim(\mathcal{A}) = (d-1)/2$, and $\mathcal{J}(\mathcal{C}_1) \sim \mathcal{E} \times \mathcal{A}$ (an isogeny defined over the prime field of $\mathbb{F}_q$). As a result,

$$\mathcal{J}(\mathcal{X}) \sim \mathcal{J}(\mathcal{C}) \times \mathcal{E} \times \mathcal{A}.$$

Suppose that we would know that $\mathcal{A}$ and $\mathcal{J}(\mathcal{C})$ are isogenous over $\mathbb{F}_{q^2}$. Then in particular the L -polynomial $L_{\mathcal{C}}(t)$ divides $L_{\mathcal{C}_1}(t)$ (here we take L -polynomials over $\mathbb{F}_{q^2}$). Clearly, this would imply the case d odd of Conjecture 1.7.

A rather natural idea for showing that indeed the abelian varieties $\mathcal{A}$ and $\mathcal{J}(\mathcal{C})$ are isogenous over $\mathbb{F}_{q^2}$, is to look for endomorphisms in the subalgebra $\mathbb{Z}[\rho, \tau] \subset \text{End}(\mathcal{J}(\mathcal{X}))$ and restrict those to $\mathcal{A}$ or to $(1-\tau)(\mathcal{J}(\mathcal{X})) \sim \mathcal{J}(\mathcal{C})$. Unfortunately, this cannot work, as is seen by the following argument.

Consider the regular differentials on $\mathcal{X}$ that correspond to $\mathcal{A}$ and to $\mathcal{J}(\mathcal{C})$. The action of $\mathbb{Z}[\rho, \tau]$ on the regular differentials on $\mathcal{X}$ has the invariant subspaces V_j spanned by ω_j and ω_{d+1-j} . If $d > 1$ then $\dim(V_1) = 2$ and τ, ρ act on V_1 by the matrices $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \zeta & 0 \\ 0 & -\zeta^{-1} \end{pmatrix}$, respectively. We look for an element in the $\mathbb{Z}$ -algebra generated by these two matrices that sends one of the two lines spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, to the other. However, such an element does not exist.

Remark 5.5. In fact Conjecture 1.7 is true in the case that d is (an odd) prime. Namely, as a special case of Proposition 14 in the paper [13] by Kohel and Smith, one obtains that $\mathcal{J}(\mathcal{X})$ is isogenous to $\mathcal{E} \times \mathcal{J}(\mathcal{C}) \times \mathcal{J}(\mathcal{C})$ with $\mathcal{E}$ the elliptic curve given by $y^2 = x^3 - x$ and $\mathcal{C}: y^2 = \varphi_d(x)$. This means that the L -polynomial of $\mathcal{X}$ over $\mathbb{F}_{q^2}$ is the product of that of $\mathcal{E}$ and two copies of that of $\mathcal{C}$.

As we saw in the proof of Theorem 1.5, the L -polynomial of $\mathcal{X}$ is also the product of that of $\mathcal{C}$ and that of $\mathcal{C}_1: y^2 = (x^2 - 4)\varphi_d(x)$. Combining the two factorizations, one concludes that for $d > 2$ prime, the L -polynomial of $\mathcal{C}_1$ equals the product of that of $\mathcal{C}$ and that of $\mathcal{E}$. This shows Conjecture 1.7 in this case. And so by Tate's classical work [22] we have that $\mathcal{J}(\mathcal{C}_1)$ is isogenous to $\mathcal{E} \times \mathcal{J}(\mathcal{C})$.

A natural approach to proving Conjecture 1.7 would be, to show that also for *composite* odd d one has an isogeny $\mathcal{J}(\mathcal{X}) \sim \mathcal{E} \times \mathcal{J}(\mathcal{C}) \times \mathcal{J}(\mathcal{C})$ defined over $\mathbb{F}_{q^2}$. Although we have not been able to show this, we can in fact prove the weaker statement that these abelian varieties are isogenous over the algebraic closure $\overline{\mathbb{F}_q}$. Indeed, consider the subgroup G of $\text{Aut}(\mathcal{X})$ generated by $r := \rho^4$ and $s := \tau$. Then r has order d and s has order 2. Moreover $srs = r^{-1}$, so G is a dihedral group of order $2d$ (and in the case considered here, d is odd).

Following Paulhus [16, § 3.1.2], who applies Kani-Rosen theory (specifically, [10, Theorem B]) to the subgroups $H_1 = \langle r \rangle$ and $H_j = \langle sr^j \rangle$ ($2 \leq j \leq 2d+1$) of G , and who observes that because d is odd, all groups H_j ($j \neq 1$) are conjugate in G and therefore the quotients $\mathcal{X}/H_j$ are isomorphic, one concludes

$$\mathcal{J}(\mathcal{X}) \times \mathcal{J}(\mathcal{X}/G) \times \mathcal{J}(\mathcal{X}/G) \sim \mathcal{J}(\mathcal{X}/\langle r \rangle) \times \mathcal{J}(\mathcal{X}/\langle s \rangle) \times \mathcal{J}(\mathcal{X}/\langle s \rangle).$$

We analyze the quotient curves appearing here. As we saw in the proof of Theorem 1.6, $\mathcal{X}/\langle s \rangle = \mathcal{X}/\langle \tau \rangle \cong \mathcal{C}$ since d is odd. Moreover, the proof of Lemma 5.1 shows $\mathcal{X}/\langle r \rangle = \mathcal{X}/\langle \rho^4 \rangle \cong \mathcal{E}$, and up to scalars, $x^{(d-1)/2}dx/y$ is the only regular differential on $\mathcal{X}$ invariant under the action of r . As this differential is not fixed by $s = \tau$, no regular differentials fixed by every automorphism in the group G exist. Therefore the genus of $\mathcal{X}/G$ equals 0, so $\mathcal{J}(\mathcal{X}/G) = (0)$. So the displayed isogeny in fact reads

$$\mathcal{J}(\mathcal{X}) \sim \mathcal{E} \times \mathcal{J}(\mathcal{C}) \times \mathcal{J}(\mathcal{C}),$$

which is what we wished to show. Adapting this line of reasoning so that it works over $\mathbb{F}_{q^2}$ as well, would lead to a proof of Conjecture 1.7 but unfortunately, so far we have not been able to do so.

Remark 5.6. Let $d > 0$ be any integer, and let q be a prime power with $\gcd(q, 2d) = 1$. If $3|d$, then $v^2 = \varphi_d(u)$ covers the elliptic curve $v^2 = \varphi_3(u) = u^3 - 3u$ since in this case (see Remark 1.3) we have $\varphi_d(u) = \varphi_3(\varphi_{d/3}(u))$. Hence if in this case the curve given by $v^2 = \varphi_d(u)$ is maximal over $\mathbb{F}_{q^2}$, then the elliptic curve $v^2 = \varphi_3(u)$ is also maximal over $\mathbb{F}_{q^2}$. The latter maximality occurs precisely when $q \equiv -1 \pmod{4}$. As a consequence, for d a multiple of 3 the assumption $q \equiv -1 \pmod{4}$ mentioned in statement (ii) of Conjecture 1.7 can be deleted.

Remark 5.7. In [23], the curve $\mathcal{C}: y^2 = \varphi_d(x)$ is denoted by $\mathcal{C}_0$; one of the results of that paper ([23, Section 3.2]) is that in case $d = \ell$ is an odd prime number, then the endomorphism algebra of $\mathcal{J}(\mathcal{C})$ contains the field $K := \mathbb{Q}(\sqrt{-1}, \zeta_\ell + \zeta_\ell^{-1})$. Note that $[K : \mathbb{Q}] = \ell - 1 = 2g$ where g is the genus of $\mathcal{C}$. Moreover, provided $\ell \neq 5$, regarding $\mathcal{J}(\mathcal{C})$ as an abelian variety in characteristic 0, by [23, Proposition 5] it has no nontrivial abelian subvarieties (over any field extension). This means that $\mathcal{J}(\mathcal{C})$ is a so-called CM abelian variety. The extension $K/\mathbb{Q}$ is Galois (even abelian), with Galois group $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{F}_\ell^\times / \pm 1$; note that this group is cyclic precisely when $\ell \equiv 3 \pmod{4}$.

The CM type corresponding to $\mathcal{J}(\mathcal{C})$ is computed in [23]. One identifies it with the subset $\Phi \subset G$ given by

$$\Phi = \{(0, \pm 1), (1, \pm 2), (0, \pm 3), \dots\}$$

of cardinality $(\ell - 1)/2$.

In [4, Theorem 3.1] it is explained how the slopes of the Newton polygon of Frobenius on a reduction of $\mathcal{C}$ modulo a prime p can be determined from the decomposition group $D \subset G$ at p : the possible slopes are $\#(Dg \cap \Phi)/\#Dg$ with g an element of G . Note that the group D (at any prime p with $\gcd(p, 2\ell) = 1$ which means, at any prime that does not ramify in K) is the cyclic group generated by $((p - 1)/2 \pmod{2}, \pm p \pmod{\ell})$. In particular, taking $p \equiv 1 \pmod{4}$ one has that $D \subset (0) \times \mathbb{F}_\ell^\times / \pm 1$. Hence taking $g = (1, \pm 1) \in G$ one finds $Dg \cap \Phi = \emptyset$. As a result, one of the slopes is 0, implying that the p -rank of $\mathcal{J}(\mathcal{C})$ is positive. In particular, this provides an alternative proof of

Proposition 2.3 for the special case of the polynomial $\varphi_d(x)$ with $d > 1$ odd. Indeed, taking ℓ any prime divisor of d , the equality $\varphi_d(x) = \varphi_\ell(\varphi_{d/\ell}(x))$ implies that the curve with equation $y^2 = \varphi_\ell(x)$ is covered by the curve given by $y^2 = \varphi_d(x)$. Hence if the latter curve is maximal over $\mathbb{F}_{q^2}$ (and $\gcd(q, 2\ell) = 1$), then so is the first, and therefore the characteristic of $\mathbb{F}_{q^2}$ is $\equiv 3 \pmod{4}$.

We illustrate the use of CM theory also in the next result.

Proposition 5.8. *Let q be a prime power with $\gcd(q, 10) = 1$. If the hyperelliptic curve given by $y^2 = x^5 - 5x^3 + 5x$ is maximal over $\mathbb{F}_{q^2}$, then the characteristic of $\mathbb{F}_q$ is either $11 \pmod{20}$ or $19 \pmod{20}$.*

Proof. Note that $x^5 - 5x^3 + 5x = \varphi_5(x)$. We will show the result by using the CM theory described above in Remark 5.7. We therefore use the notations introduced in that remark, for the special case $\ell = 5$.

Let p be the characteristic of $\mathbb{F}_q$. By Proposition 2.3 (and alternatively, by Remark 5.7), maximality of the given curve $\mathcal{C}$ implies that $p \equiv 3 \pmod{4}$. Hence the decomposition group D at $\mathfrak{p}$ in $G = \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{F}_5^\times / \pm 1$ is generated by $(1, \pm p \pmod{5})$.

In case $p \equiv \pm 2 \pmod{5}$, this means

$$D = \{(1, \pm 2), (0, \pm 1)\}.$$

Clearly $D \cdot (0, \pm 2) \cap \Phi = \emptyset$ where $\Phi \subset G$ describes the CM-type of the curve $\mathcal{C}$. As before, this implies that $\mathcal{C}$ cannot be maximal in characteristic p .

So a necessarily condition for maximality in characteristic p is besides $p \equiv 3 \pmod{4}$ that also $p \equiv \pm 1 \pmod{5}$. From this, the result follows. $\square$

Remark 5.9. In the proof above we only used the fact that for a maximal curve, the slopes of Frobenius are all positive. A stronger condition is that in fact they need to be equal to $\frac{1}{2}$. Exploiting that, one obtains similar results for other values of ℓ . For example, with $\ell = 17$ one can exclude characteristic $p \equiv \pm 2 \pmod{17}$ in this way.

We finish this manuscript by briefly mentioning some small cases of Conjecture 1.7.

$d = 1$: here statement (i) asserts the maximality of the elliptic curve given by $y^2 = x^3 - 4x$ over $\mathbb{F}_{q^2}$. This holds precisely when $q \equiv -1 \pmod{4}$. Statement (ii) asserts, besides this congruence condition, also the maximality of the hyperelliptic curve given by $v^2 = u$. Since this maximality holds over any $\mathbb{F}_{q^2}$ (the curve has genus 0), Conjecture 1.7 holds for $d = 1$.

$d \geq 5$: we verified using Magma for all prime powers $q < 100$ and $d \in \{9, 15, 21\}$ Conjecture 1.7 holds. In fact, the experiment shows for these cases, as we saw in Remark 5.5 for the case d is an odd prime, that the curves $\mathcal{C}: y^2 = \varphi_d(x)$

and $\mathcal{C}_1: y^2 = (x^2 - 4)\varphi_d(x)$ over $\mathbb{F}_q$ are related by $\mathcal{J}(\mathcal{C}_1) \sim \mathcal{J}(\mathcal{C}) \times \mathcal{E}$ with $\mathcal{E}: y^2 = x^3 + x$.

Acknowledgment

The first author was supported by FAPESP/SP-Brazil grant 2017/19190-5. The second author thanks Marco Streng and Nurdagül Anbar Meidl for helpful suggestions. We also thank the referee for various comments which helped us improve the exposition.

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