Linear differential equations with finite differential Galois group

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\textbf{Abstract}
For a finite irreducible subgroup $H \subset \text{PSL}(C^n)$ and an irreducible, $H$-invariant curve $Z \subset \mathbb{P}(C^n)$ such that $C(Z)^H = C(t)$, a standard differential operator $L_{st} \in C(t)[\frac{d}{dt}]$ is constructed. For $n = 2$ this is essentially Klein’s work. For $n > 2$ an actual calculation of $L_{st}$ is done by computing an evaluation of invariants $C[X_1, \ldots, X_n]^H \to C(t)$ and applying a scalar form of a theorem of E. Compoint in a “Procedure”. Also in some cases where $Z$ is unknown evaluations are produced. This new method is tested for $n = 2$ and for three irreducible subgroups of $\text{SL}_3$. This supplements [18]. The theory developed here relates to and continues classical work of H.A. Schwarz, G. Fano, F. Klein and A. Hurwitz.

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1. Introduction and summary

Let $C$ denote an algebraically closed field of characteristic zero. Let $k$ be $C(z)$ and let $\overline{k}$ denote the algebraic closure of $k$. Both fields are provided with the $C$-linear derivation

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$f \mapsto f'$ with $z' = 1$. The positive but not explicit or constructive answer to the inverse problem of Galois theory is:

*For any finite group $G$ there is a Galois extension $\ell \supset k$ with group $G$.*

Indeed, a proof for the complex case uses analytic tools, in particular the “Riemann Existence Theorem”. The proof for any field $C$ as above is deduced from the complex case. There is an extensive literature on solving the inverse problem *explicitly* for certain finite groups.

A finite Galois extension $\ell \supset k$ can be given as the splitting field of a polynomial $P$ in $k[T]$. Sometimes, a more efficient way is to describe $\ell \supset k$ as the Picard–Vessiot field of a linear differential operator $L$ in $k[\partial]$ with $\partial = \frac{d}{dz}$. From a polynomial $P$ for $\ell \supset k$ one can easily compute a differential operator $L$ for $\ell \supset k$, see [18, §1] and [8, §2]. The other direction is far more complicated (see (ii) below).

Let $\pi$ denote the profinite Galois group of $\overline{k}/k$. There is a well known bijection between the monic differential operators $L \in k[\partial]$ of order $n$, such that all solutions are algebraic over $k$, and the $C$-vector spaces $V \subset \overline{k}$ of dimension $n$ which are stable under $\pi$.

Indeed, one associates to $L$ the $\pi$-stable space $\{ f \in \overline{k} \mid L(f) = 0 \}$ (i.e., the contravariant solution space). On the other hand, let the $\pi$-stable $V \subset \overline{k}$ have basis $b_1, \ldots, b_n$ over $C$. There is a unique operator $L = \partial^n + a_{n-1} \partial^{n-1} + \cdots + a_1 \partial + a_0$ with all $a_i \in \overline{k}$ such that all $L(b_j) = 0$. The uniqueness and the $\pi$-stability of $V$ imply that all $a_i \in k$.

A more abstract way to compare differential equations and Galois extensions $\ell \supset k$ is the following. The category $\text{Diff}_{\overline{k}/k}$ that we study here, has as objects the finite dimensional differential modules $M$ over $k$ which become trivial over the field $\overline{k}$. This condition on $M$ is equivalent to $M$ having a finite differential Galois group. The morphisms in this category are the $k$-linear maps that commute with differentiation.

Let $\text{Repr}_\pi$ denote the category of the (continuous) representations of $\pi$ on finite dimensional $C$-vector spaces. The functor $\text{Diff}_{\overline{k}/k} \to \text{Repr}_\pi$, which associates to a differential module $M$ its (covariant) solution space $\ker(\partial, \overline{k} \otimes_k M)$, is known to be an equivalence of (Tannakian) categories.

The aim of this paper is to make this equivalence of categories explicit for special cases. There are two directions to consider:

(i) Compute a differential operator connected to a given representation of a given finite group and some additional data.

(ii) Describe or construct the Picard–Vessiot field for a given module $M \in \text{Diff}_{\overline{k}/k}$, when $M$ is represented by a differential operator $L$.

We recall some earlier results on (i) and (ii).

*Regarding* (i): The Schwarz’ list (see [18] for a modern version) and Klein’s theorem (e.g., see [1] and [2]) are classical results for the special case of order $n = 2$. We recall the statement of Klein’s theorem:

for each of the irreducible subgroups $G \subset \text{PSL}(C^2)$ (so $G \in \{ D_n, A_4, S_4, A_5 \}$), there is a standard order two differential operator $L_{st}$ having $G$ as projective differential Galois group. It has the universal property that any order two differential operator with projective group $G$ is a “weak pullback” (see Definition 3.6) of $L_{st}$.
In the case $n = 3$ Hurwitz’ paper [12] produces examples. This method was refined in [18]. Klein’s theorem is generalized in, e.g., [2,20–22]. Not much seems to have been done for $n > 3$. Here (Section 3) we treat the general case.

Regarding (ii): This was initiated by J. Kovacic in his paper [14] dealing with $n = 2$. There are many subsequent papers [23,10,11] considering small $n$. For general $n$ there is work of E. Compoint and M.F. Singer [6,7]. The paper [4] discusses the particular case of hypergeometric differential equations.

We now describe the present paper, which is mainly concerned with (i) but also contributes to (ii) by exploiting invariant theory for finite groups and Compoint’s work [6].

Section 2 associates to a differential operator $L \in k[\partial]$ with all solutions in $\bar{k}$, geometric objects: a Picard–Vessiot curve, a Fano curve, Schwarz maps, projective differential Galois groups and an evaluation of invariants.

In Section 3 Klein’s theorem for order two is generalized, resulting in a subtle construction of a standard differential operator $L_{st}$ (Theorem 3.1). The data for this construction are a finite irreducible subgroup $H \subset \text{PSL}(C^n)$, an $H$-invariant irreducible curve $Z \subset \mathbb{P}(C^n)$ such that the normalization of $Z/H$ has genus zero and a variable $z$ with $C(Z/H) = C(z)$. In the construction of $L_{st}$ the group $H$ is replaced by a subgroup $\hat{H} \subset \text{SL}(C^n)$ which is minimal such that $\hat{H} \to H$ is surjective.

The “universal property” of $L_{st}$ is the following:

any differential operator $L$ with projective differential Galois group isomorphic to $H$ and Fano curve isomorphic to $Z$ is a weak pullback of $L_{st}$ (see 3.1 and 3.7). This clarifies and extends the work of [2,20–22].

Section 4. For order $n = 2$ the Fano curve is by definition $\mathbb{P}(C^2)$ and the computation of the standard operators $L_{st}$ is easy and produces the classical operators. For $n > 2$ however, the construction of $L_{st}$ as described in Section 3 does not in an obvious way result in a computation of this operator. A new method for the computation of $L_{st}$ is introduced. We derive a “scalar version” of Compoint’s theorem (see 4.2) which is roughly the following. Let the homogeneous polynomials $f_1, \ldots, f_N$ be generators for the ring of invariants $C[X_1, \ldots, X_n]^H$. An evaluation of the invariants is a suitable homomorphism $ev : C[X_1, \ldots, X_n]^H \to C(t)$ and the Picard–Vessiot field of $L_{st}$ is $K := C(t)[X_1, \ldots, X_n]/(f_1 - ev(f_1), \ldots, f_N - ev(f_N))$.

Our “Procedure” 4.3 computing $L_{st}$ works as follows. A set of homogeneous generators $f_1, \ldots, f_N$ and their relations are taken (if possible) from the literature. The given $H$-invariant irreducible curve $Z \subset \mathbb{P}(C^n)$ with $C(Z)^H = C(t)$ effectively produces an essentially unique evaluation, see 4.6. From the explicit presentation of $K$ one computes the derivation $D$ on $K$ extending $\frac{d}{dt}$. Then one obtains the monic operator $L \in C(t)[\frac{d}{dt}]$ of degree $n$ with kernel $C\mathbf{X}_1 + \cdots + C\mathbf{X}_n$, where $\mathbf{X}_i$ denotes the image of $X_i$ in $K$.

Finally $L_{st}$ is obtained by normalizing $L$ such that its coefficient of $(\frac{d}{dt})^{n-1}$ is zero.

Our Procedure can be seen as the “opposite” of an algorithm, by M. van Hoeij and J.-A. Weil [11], which computes for a given differential operator, the associated evaluation of the invariants $C[X_1, \ldots, X_n]^G \to C(z)$. 
Section 5. For order $n = 2$, we show how to obtain evaluations of the invariants and apply the Procedure to produce the known standard operators. For the group $G_{168} \subset PSL(C^3)$ and the Klein curve $Z \subset \mathbb{P}^2$, a direct computation of the standard operator from its construction in §3 fails. However, evaluation and the Procedure produce the standard operator.

The LIST, copied from [18], contains all possibilities, determined by the Riemann Existence Theorem, of order 3 differential operators over $C(z)$ (up to equivalence) with group $G_{168}$ and singular locus $\{0, 1, \infty\}$. In most of these cases one does not know a stable $Z \subset \mathbb{P}^2$ such that the normalization of $Z/G_{168}$ has genus zero. The methods of [18] produced explicit third order equations for about half of the cases. For the same cases our new method of evaluation and the Procedure produces more easily the standard equations.

In [18] no standard equation for the group $H_{72}^{SL_3}$ was found. Our new methods produce an equation.

In Section 6 standard equations for $A_5 \subset SL_3$ are studied. Moreover, properties in relation with the preimage $A_5^{SL_2} \subset SL_2(C)$ of $A_5 \subset PSL(C^2)$ and the lists of differential operators in [18] are discussed.

2. Objects associated to a differential operator $L$ over $k = C(z)$ with finite differential Galois group

$L$ has the form $d^n z^a + a_{n-1} d^{n-1} z + \cdots + a_0$ with all $a_i \in C(z)$, $d_z = \frac{d}{dz}$ and all solutions are supposed to be algebraic over $C(z)$. Associated to $L$ is:

(1) *The Picard–Vessiot field* $K \subset C(z)$ with its Galois group $G$.

(2) *The (contravariant) solution space* $V \subset K$ of $L$ with the action of $G$ on it. The image of $G \subset GL(V)$ into $PGL(V)$ will be denoted by $G^{proj}$ and is called the *projective differential Galois group*.

(3) *The Picard–Vessiot curve* $X_{pv}$ is the smooth, irreducible, projective curve over $C$ with function field $K$. $G$ acts on $X_{pv}$ and there is an isomorphism $X_{pv}/G \cong \mathbb{P}^1_z$. Here $\mathbb{P}^1_z$ denotes the projective line with function field $C(z)$.

(4) *Evaluation of the invariants*. One considers a $C$-linear homomorphism $\phi : C[X_1, \ldots, X_n] \to K$ which sends the variables $X_1, \ldots, X_n$ to a basis of $V$. The $C$-linear action of $G$ on $C[X_1, \ldots, X_n]$ is defined by the $G$-invariance of $CX_1 + \cdots + CX_n$ and the $G$-equivariance of $\phi$. This makes $G$ into a subgroup of $GL(n, C)$. The homomorphism $\phi$ induces a homomorphism $ev : C[X_1, \ldots, X_n]^G \to K^G = C(z)$ which we will call the *evaluation of the invariants*. Write $C[X_1, \ldots, X_n]^G = C[f_1, \ldots, f_N]$ where $f_1, \ldots, f_N$ are homogeneous generators and $ev$ maps each $f_i$ to an element in $C(z)$.

Now suppose that the action of $G$ on $V$ is known and is irreducible, i.e., no proper linear subspace $\neq (0)$ of $V$ is invariant under $G$. If we define the action of $G$ on $CX_1 + \cdots + CX_n$ such that an equivariant $\phi$ with $\phi(CX_1 + \cdots + CX_n) = V$ exists, then this $\phi$ is unique up to multiplication by a scalar $c \in C^\times$. As a consequence, the *evaluation map is unique* up to changing each $ev(f_i)$ into $c^{\deg f_i} ev(f_i)$ for all $i$. 

(5) The Fano curve. $\mathbb{H} \subset \ker(\phi)$, the “homogeneous kernel”, is the ideal generated by the homogeneous elements in $\ker(\phi)$. For $n = 2$ one has $\mathbb{H} = 0$. For notational reasons we will call $\mathbb{P}(V) = \mathbb{P}^1$ itself the Fano curve in this case.

Suppose that $n > 2$, then $\mathbb{H}$ defines an irreducible curve in $\mathbb{P}^{n-1}$, invariant under the action of $G$. Indeed, $\mathbb{H}$ is the homogeneous ideal induced by the kernel $J$ of the corresponding homomorphism $C[\frac{X_2}{X_1}, \ldots, \frac{X_n}{X_1}] \to K$. It is a curve since $K/C$ has transcendence degree 1. The curve in $\mathbb{P}^{n-1}$ defined by $\mathbb{H}$ will be denoted by $X_{fano}$ and will be called the Fano curve. This curve was indeed considered by Fano in his 1900-paper [9]. We note that $X_{fano}$ can have singularities. From the definition one sees that $C(X_{fano}) = C(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1})$, where $x_1, \ldots, x_n$ is a basis of $V \subset K$.

(6) The Schwarz map. The homomorphism $C[X_1, \ldots, X_n]/\mathbb{H} \to K$ induces a morphism of curves $X_{pv} \to X_{fano}$ which is $G$-equivariant. After dividing by $G$ we obtain a multi-valued map $Schw : \mathbb{P}^1_z = X_{pv}/G \to X_{fano}$ called the Schwarz map. For $n = 2$ it is the well known classical Schwarz map.

After dividing by $G$ we obtain $qSchw : \mathbb{P}^1_z = X_{pv}/G \to X_{fano}/G^{proj}$ which can be called the quotient Schwarz map. We note that $X_{fano}/G^{proj}$ can have singularities. The relation between $X_{pv}$ and $X_{fano}$ is in general not obvious.

**Lemma 2.1.** Suppose that $qSchw : \mathbb{P}^1_z = X_{pv}/G \to X_{fano}/G^{proj}$ is birational. Let $c(G) \subset G$ be the group of the multiples of the identity belonging to $G$. Since $c(G)$ acts trivially on the curve $X_{fano}$, the map $X_{pv} \to X_{fano}$ factors over $X_{pv}/c(G)$. The morphism $X_{pv}/c(G) \to X_{fano}$ is birational.

**Proof.** One has $K = C(X_{pv}) \supset K^{c(G)} \supset C(X_{fano})$. The group $G^{proj} = G/c(G)$ acts faithfully on $K^{c(G)} = C(X_{pv}/c(G))$ and $(K^{c(G)})^{G^{proj}} = C(z)$. Since $C(X_{fano}) = C(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1})$, the group $G^{proj}$ acts faithfully on $C(X_{fano})$. By assumption $C(X_{fano})^{G^{proj}} = C(z)$. Therefore $K^{c(G)} = C(X_{fano})$. □

### 3. Construction of a standard operator and pullbacks

**Theorem 3.1.** Let the following data be given:

(i) a $C$-vector space $V$ with $n := \dim V \geq 2$,

(ii) an irreducible finite subgroup $H \subset \text{PSL}(V)$,

(iii) an irreducible $H$-invariant curve $Z \subset \mathbb{P}(V)$ such that the normalisation of $Z/H$ has genus 0 and

(iv) a variable $z$ such that $C(Z)^H = C(z)$.

The data $V, H, Z, z$ determine a differential operator

$$L_{st} = \left( \frac{d}{dz} \right)^n + a_{n-2} \left( \frac{d}{dz} \right)^{n-2} + \cdots + a_0 \in C(z)[\frac{d}{dz}]$$

such that

(a) The $C$-vector space $W := \{ f \in \overline{C(z)} \mid L_{st}f = 0 \}$ has dimension $n$ (i.e., all solutions are algebraic).
(b) Let $G \subset \text{SL}(W)$ denote the differential Galois group of $L_{st}$. There is a $C$-linear isomorphism $\phi : W \rightarrow V$ such that the projective differential Galois group $G_{\text{proj}}$ is mapped isomorphically to $H$ and the Fano curve $X_{\text{fano}} \subset \mathbb{P}(W)$ of $L_{st}$ is mapped isomorphically to $Z \subset \mathbb{P}(V)$.

**Remarks 3.2.** (0). It is a standard fact that in (b) one has $G \subset \text{SL}(W)$; see, e.g., [17, Exc. 1.35 5(b)-(c)].

(1). The operator $L_{st}$ will be called the standard operator for the data $V, H, Z, z$. For $n = 2$, one has $Z = \mathbb{P}(V)$. Further $Z/H$ is identified with $\mathbb{P}^1_\mathbb{C}$ and so $C(Z/H) = C(z)$.

One knows that the possibilities for $H$ are $D_n, A_4, S_4, A_5$. The variable $z$ is chosen such that $z = 0, 1, \infty$ are the branch of $Z \rightarrow Z/H$. Thus $L_{st}$ depends essentially only on $H$.

(2). In the proof of Theorem 3.1 we will use a group $\tilde{H} \subset \text{SL}(V)$ which maps surjectively to $H$ and is minimal with respect to this property. The kernel of $\tilde{H} \rightarrow H$ has the form \{ $\lambda \cdot 1 | \lambda^m = 1$ \} for a certain divisor $m$ of $n$.

(3). In the construction of $L_{st}$ only the data $V, N, Z, z$ are used. It can be shown that the operator $L_{st}$ is actually determined by the properties (a) and (b) in Theorem 3.1.

(4). The action of $H$ on $Z$ is faithful. Indeed, since $H$ is irreducible and $Z$ is $H$-invariant, $Z$ is not contained in a proper projective subspace of $\mathbb{P}(V)$. By induction on $i$, one finds for $i = 1, \ldots, n$, elements $z_0, \ldots, z_i \in Z$ such that $z_0, \ldots, z_i$ is not contained in a projective subspace of dimension $< i$. Further, for each $j$, one can replace $z_j$ by infinitely many elements $\tilde{z}_j \in Z$ such that $z_0, \ldots, z_{j-1}, \tilde{z}_j, z_{j+1}, \ldots, z_n$ has the same property.

Suppose that $h \in H$ acts as identity on $Z$. Then $h \in \text{PGL}(V)$ has a diagonal matrix with respect to a basis of $V$ corresponding to any sequence $z_0, \ldots, z_{j-1}, \tilde{z}_j, z_{j+1}, \ldots, z_n$. This implies that $h = 1$. 

**Proof.** We start the construction of $L_{st}$. The above data yield inclusions $C(z) = C(Z)^H \subset C(Z) \subset \overline{C(z)}$. The variable $z$ is given in the data and the embedding $C(Z) \subset \overline{C(z)}$ is unique up to an automorphism of $C(Z)$ over $C(z)$, i.e., an element of $H$. We would like to identify $V$ with the solution space in $\overline{C(z)}$ of the standard operator to be constructed. However, $V$ does not lie in $C(Z)$.

One chooses any $\ell \in V$, $\ell \neq 0$. For any $v \in V$ one considers the restriction of the rational function $\frac{v}{\ell}$ on $\mathbb{P}(V)$ to $Z$ (this makes sense because $Z$ is not contained in the hypersurface $\ell = 0$). Write $\frac{v}{\ell}$ for the functions on $Z$ obtained in this way, so $\frac{v}{\ell} \subset C(Z)$. The $C$-vector space $\frac{v}{\ell}$ is not invariant under $H$, or what is the same, it is not invariant under $\pi$. The following lemma is the key ingredient of the construction.

**Lemma 3.3.** There exists an element $f \in \overline{C(z)}$ such that $f \frac{v}{\ell}$ is invariant under $\pi$. The canonical map $\mathbb{P}(V) \rightarrow \mathbb{P}(f \frac{v}{\ell})$, given by $v \mapsto f \cdot \frac{v}{\ell}$ is equivariant for the action of $\pi$.

**Proof.** The group $\tilde{H}$ is supposed to have the properties of Remarks 3.2. For each $\sigma \in H$, one denotes by $\tilde{\sigma}$ an element in $\tilde{H}$ with image $\sigma$. Now $\sigma(\frac{v}{\ell}) = \frac{\ell}{\ell} \cdot \frac{v}{\ell}$. The term $\frac{\ell}{\ell}$ depends in general on the choice of $\tilde{\sigma}$. But $(\frac{\ell}{\ell})^m$ depends only on $\sigma$ and $\sigma \mapsto (\frac{\ell}{\ell})^m$ is
a 1-cocycle. By Hilbert 90, there is an element $f \in C(Z)$ such that \( \frac{\sigma f}{f} \cdot \left(\frac{\ell}{\ell_{\sigma}}\right)^m = 1 \) for all $\sigma \in H$.

For the case $m = 1$ we conclude that $f \cdot \frac{V}{\ell} \subset C(Z)$ is invariant under $H$ (and thus also under $\pi$). For the case $m > 1$ we claim that the equation $T^m - f$ is irreducible over $C(Z)$. Assuming this claim, the field $C(Z)(f_m)$ with $f_m = f$ is a Galois extension of $C(z)$ since for every $\sigma \in H$ one has $\sigma f_m$ is an $m$th power in $C(Z)$. We may embed $C(Z)(f_m)$ into $\overline{C(z)}$ and conclude that $f_m \cdot \frac{V}{\ell}$ is invariant under $\pi$.

Now we prove the claim. If the equation $T^m - f$ is reducible over $C(Z)$, then there exists a proper divisor $d$ of $m$ and an element $g \in C(Z)$ with $g^d = f$. The expression $E(\tilde{\sigma}) := \sigma g \cdot \left(\frac{\ell}{\ell_{\sigma}}\right)^{m/d}$ has the property $E(\tilde{\sigma})^d = 1$. One can consider for each $\sigma \in H$ the elements $\tilde{\sigma} \in \tilde{H}$ such that $E(\tilde{\sigma}) = 1$. This defines a proper subgroup of $\tilde{H}$ which has image $H$. This contradicts the assumptions on $\tilde{H}$.

The last statement of the lemma follows from $\sigma(f \cdot \frac{V}{\ell}) = \frac{\sigma f}{f} \cdot \frac{\ell}{\ell_{\sigma}} \cdot f \cdot \frac{\sigma V}{\ell}$. □

The monic operator $L$ of order $n$ over $\overline{C(z)}$, defined by $\ker(L, \overline{C(z)}) = W := f \cdot \frac{V}{\ell}$ has its coefficients in $C(z)$, since $W$ is invariant under $\pi$. This operator $L$ is not yet unique since we have made choices for $\ell$ and $f$.

The standard operator $L_{st}$ is defined to be the operator of the form $L_{st} = (\frac{d}{dz})^n + 0 \cdot (\frac{d}{dz})^{n-1} + \cdots$, obtained from the above $L$ by a shift $\frac{d}{dz} \mapsto \frac{d}{dz} + a$ for suitable $a = \frac{h'}{h}$ with $h \in \overline{C(z)}^\ast$.

We finish the proof of Theorem 3.1 by stating the following properties:

1. $L_{st}$ does not depend on the choices of $\ell$ and $f$ in Lemma 3.3.
2. The solution space of $L_{st}$ has the form $g \cdot W$ for certain $g \in \overline{C(z)}^\ast$.
3. Let $G \subset \operatorname{SL}(g \cdot W)$ denote the differential Galois group of $L_{st}$. From $g \cdot W = g \cdot f \cdot \frac{V}{\ell}$ one obtains a natural identification of the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(g \cdot W)$ and after this identification one has $G_{proj} = H$ and the Fano curve of $L_{st}$ is $Z$.

Statement (1) follows easily from Lemma 3.4 and Observation 3.5, part (1). Statements (2) and (3) follow from the construction of $L_{st}$. □

**Lemma 3.4.** Let $L_1, L_2$ be monic differential operators over $C(z)$ such that all their solutions are algebraic. Let $V_1, V_2 \subset \overline{C(z)}$ denote the two solution spaces. The following are equivalent:

(a). $L_1$ is obtained from $L_2$ by a shift $\frac{d}{dz} \mapsto \frac{d}{dz} + a$ for some element $a \in C(z)$.

(b). There exists $f \in \overline{C(z)}^\ast$ such that $V_2 = f V_1$.

**Proof.** (a)⇒(b). Let $L_1$ be obtained from $L_2$ by the shift $\frac{d}{dz} \mapsto \frac{d}{dz} + a$. One writes $a = \frac{f'}{f}$ with $f$ in some differential field containing $\overline{C(z)}$. One finds $V_2 = f V_1$. Since $V_1, V_2 \subset \overline{C(z)}$ one actually has $f \in \overline{C(z)}^\ast$.

(b)⇒(a). If $V_2 = f V_1$, then clearly $L_1 = f^{-1} \circ L_2 \circ f$. Since $f^{-1} \circ \frac{d}{dz} \circ f = \frac{d}{dz} + \frac{f'}{f}$, one has that $L_1$ is obtained from $L_2$ by the shift $\frac{d}{dz} \mapsto \frac{d}{dz} + \frac{f'}{f}$. Note that $\frac{f'}{f} \in C(z)$ since $L_1$ and $L_2$ are both defined over $C(z)$. □
Observations 3.5. (1) For general monic differential operators $L_1, L_2$ of order $n$, property (a) of Lemma 3.4 is called projective equivalence. If both $L_1$ and $L_2$ have the form $d^n + 0 \cdot d^{n-1} + \cdots$, then projective equivalence implies equality.

(2) The implication (b)⇒(a) in Lemma 3.4 holds for general differential operators. However (a)⇒(b) is in general false since the equation $f' = af$ with $a \in C(z)$, need not have a solution on $C(z)$.

(3) For differential modules $M_1, M_2$ there is a somewhat different notion of projective equivalence defined by: there is a 1-dimensional module $E$ such that $M_1 \otimes E \cong M_2$.

(4) Projective equivalence of subgroups $G_1, G_2 \subset \text{GL}(V)$ means that $G_1^{\text{proj}} = G_2^{\text{proj}} \subset \text{PGL}(V)$. Projective equivalence of operators implies projective equivalence of their differential Galois groups but the converse is false. □

Definition 3.6. Consider a homomorphism $\phi : C(z)[\frac{d}{dz}] \to C(x)[\frac{d}{dx}]$ of the form: $z \mapsto \phi(z) \in C(x) \setminus C$ and $\frac{d}{dz} \mapsto \frac{1}{\phi(z)}(\frac{d}{dx} + b)$ with $b \in C(x)$. Let $L \in C(z)[\frac{d}{dz}]$. A weak pullback of $L$ is an operator of the form $a \cdot \phi(L)$ with $a \in C(x)^*$. The restriction of $\phi$ to $C(z) \to C(x)$ is called the pullback function.

Proposition 3.7. Let $L \in C(s)[\frac{d}{ds}]$ be an operator of order $n$ such that all solutions are algebraic and let $M \subset C(s)$ denote its solution space. The differential Galois group $G$ of $L$ is a subgroup of $\text{GL}(M)$.

Suppose that $G^{\text{proj}} \subset \text{PSL}(M)$ is irreducible. According to §2, part (5) and (6), $L$ determines some $X_{\text{fano}} \subset \mathbb{P}(M)$ and $C(X_{\text{fano}})^{G^{\text{proj}}}$ is a subfield of $C(s)$. Choose $z$ such that $C(z) = C(X_{\text{fano}})^{G^{\text{proj}}}$.

Then $L$ is a weak pullback of the standard operator $L_{st}$ determined by the data $M$, $H = G^{\text{proj}} \subset \text{PSL}(M)$ and $Z = X_{\text{fano}} \subset \mathbb{P}(M)$ and the variable $z$.

Proof. By construction the standard operator $L_{st}$ has solution space $h \cdot W$ for some $h \in C(z)^*$, where $W = f \frac{M}{m}$ for suitable $f \in C(z)^*$ and $m \in M$, $m \neq 0$. Further, the inclusion $C(z) = C(X_{\text{fano}}/G^{\text{proj}}) \subset C(s)$ determines the pullback function. Using 3.4 and 3.6 one verifies that this pullback function applied to $L_{st}$ produces $L$. □

Remark 3.8. A standard differential equation for given $H \subset \text{PSL}(V)$, Fano curve $Z \subset \mathbb{P}(V)$ and variable $z$ can be a proper pullback of another standard equation. This occurs essentially only when $H$ is a proper subgroup of a finite automorphism group of (the desingularization of) $Z$.

Example 3.9. A calculation of the standard operator $L_{st}$, using the above construction, is possible. One has to compute the $f$ in Lemma 3.3 and one has to compute the derivation on $C(Z)[f]$ in order to compute the monic differential operator $L$ with solution space $f \frac{V}{t} \subset C(Z)[f]$. Further a computation of a generator of $C(Z)^{G^{\text{proj}}}$ is needed. However for the case $n = 2$ the calculation is well known ([1,3]) and rather easy. We illustrate this for the case $H = A_4 \subset \text{PSL}_2$ and its preimage $\tilde{H} = A_4^{\text{SL}_2}$ in $\text{SL}_2$. 
For $Z = \mathbb{P}^1$ we use homogeneous coordinates $x, y$ and the function field is $C(t)$ with $t = \frac{y}{x}$. According to [5], the invariants under the action of $A_{3}^{112}$ are generated by:

$Q_3 = xy(x^4 - y^4)$, $Q_4 = (x^4 + \sqrt{-1}2x^2y^2 + y^4) \cdot (x^4 - \sqrt{-1}2x^2y^2 + y^4)$, $Q_6 = (x^4 + \sqrt{-1}2x^2y^2 + y^4)^3 + (x^4 - \sqrt{-1}2x^2y^2 + y^4)^3$. There is one relation $Q_6^2 - Q_4^3 - 4Q_4^3 = 0$.

The field of the homogeneous invariants of degree zero is generated over $C$ by $\frac{Q_6}{Q_4}$ and $\frac{Q_4}{Q_3}$, and there is one relation $(\frac{Q_6}{Q_4})^2 = 1 + 4\frac{Q_4}{Q_3}$. Hence we can take $z = \frac{Q_6}{Q_4}$ where $x, y$ in this expression is replaced by $x, tx$. This expresses $z$ as rational function in $t$ of degree 12. Thus $\frac{dt}{dx}$ is also known.

Now $V = Cx + Cy$, take $\ell = x$, then $\frac{\ell}{\ell'} = C1 + Ct$. Then $f \in C(t)$ should satisfy $(\frac{\ell}{\ell'})^2 = \frac{f}{\ell'}$. An explicit choice for $f$ turns out to be $\frac{1}{\ell'}$ where $t' := \frac{dt}{dx}$. Then the Picard–Vessiot field is $C(t)[\sqrt{t}]$. The operator that we want to compute has solution space $C\frac{1}{\sqrt{t}} + C\frac{1}{\sqrt{t^3}}$. This leads to the standard operator for case $A_4$. The other standard operators for $n = 2$ can be computed in a similar way. This “classical” calculation fails for $n > 2$ and one needs the new method “evaluation of invariants and Procedure” (see §4,3). This new method will be applied in §5 for another computation of the standard operators for $n = 2$ and for cases with $n > 2$.

**Observation 3.10. The singular points of the standard equations.** Let $L \in C(z)[\frac{d}{dz}]$ be an operator of order $n$ such that its Picard–Vessiot field is a finite extension of $C(z)$. The singular points of $L$, which are not apparent, are the branch points of $X_{pv} \to \mathbb{P}^1_z$. Indeed, suppose that $z = 0$ is not a branch point, then the solutions of $L$ live at any point $p$ above $z = 0$. The fraction field of $\hat{O}_{X_{pv},p}$ can be identified with $C((z))$ and contains $n$ independent solutions of $L$. It follows that the singularity is at most apparent.

In the special case $L_{st}$ and $G = G^{proj} = H$, one can identify $X_{pv}$ with the normalization $\tilde{Z}$ of $Z \subset \mathbb{P}(V)$ and the non apparent singular points are the branch points of $\tilde{Z} \to \mathbb{P}^1_z$. In the general case, the cyclic extension $X_{pv} \to \tilde{Z}$ can be responsible for more singularities of $L_{st}$.

4. Compoint’s theorem and evaluation of invariants

**Notation and assumptions:**

Suppose that the differential equation $y' = Ay$ over $k = C(z)$ has a reductive differential Galois group $G \subset \text{GL}_n(C)$. The differential algebra $R := k[\{X_{i,j}\}, \frac{1}{D}]$ (with $D = \det(X_{i,j})$) is defined by $(X'_{i,j}) = A \cdot (X_{i,j})$.

Let $I$ be a maximal differential ideal in $R$ and $K$ the Picard–Vessiot field obtained as field of fractions of $R/I$.

$\text{GL}_n(C)$ acts on the $C(z)$-algebra $R$ by sending the matrix of variables $(X_{i,j})$ to the matrix $(X_{i,j}) \cdot g$ for any $g \in \text{GL}_n(C)$. Then $G$ is identified with the $g \in \text{GL}_n(C)$ such that $gI = I$.

The algebra of invariants $C[\{X_{i,j}\}]^G$ is generated over $C$ by homogeneous elements $f_1, \ldots, f_N$ (since $G$ is reductive). The natural map $R \to K$ induces a homomorphism $ev_e : C[\{X_{i,j}\}]^G \to C(z)$ which is called the evaluation of the invariants.
Theorem 4.1 (E. Compoint 1998). The ideal $I \subset R$ generated by the elements \{ $f_1 - ev_e(f_1), \ldots, f_N - ev_e(f_N)$ \} is a maximal differential ideal.

The proof of Compoint’s theorem, [6], has been simplified in [3] and Theorem 4.1 is almost identical to the formulation in [3]. Note that although [3] formulates the result for $C = \mathbb{C}$, the argument is completely algebraic hence the result holds for any $C$. We will apply Compoint’s theorem for the case of finite differential Galois groups. Moreover we will need a formulation in terms of differential operators (or scalar differential equations).

Notation and assumptions:

Let $L = \left( \frac{d}{dx} \right)^n + a_{n-1} \left( \frac{d}{dx} \right)^{n-1} + \cdots + a_1 \frac{d}{dx} + a_0$ over $C(z)$ have a finite differential Galois group $G$ and Picard–Vessiot field $K \subset C(z)$.

Consider the homomorphism $\phi : R_0 = C(z)[X_1, \ldots, X_n] \to K$ which sends $X_1, \ldots, X_n$ to a basis of the solution space of $L$ in $K$. $G$ acts $C(z)$-linear on $R_0$ by a $C$-linear action on $CX_1 + \cdots + CX_n$ which coincides with the action of $G$ (or of $\pi$) on the solution space of $L$.

The restriction of $\phi$ to $C[X_1, \ldots, X_n]^G \to C(z)$ is also called the evaluation of the invariants and denoted by $ev$ (see also §2). Write $C[X_1, \ldots, X_n]^G = C[\phi_1, \ldots, \phi_r]$ for certain homogeneous elements $\phi_k$.

Corollary 4.2. The kernel of $\phi : C(z)[X_1, \ldots, X_n] \to K$ is generated by the elements \{ $\phi_1 - ev(\phi_1), \ldots, \phi_r - ev(\phi_r)$ \}.

Proof. Write again $R_0 = C(z)[X_1, \ldots, X_n]$ and $R := C(z)[\{X_i^j\}_{i=1,\ldots,n}^{j=0,\ldots,n-1}]$ where $X_i^j$ denotes formally the $j$th derivative of $X_i$ (all $i, j$). The map $\phi : R_0 \to K$ has a unique extension $\phi_e : R \to K$ defined by $\phi_e(X_i^j) = \phi(X_i)^{(j)}$ (all $i, j$). The restriction of $\phi$ to $R_0^G \to C(z)$ is called $ev$ and the restriction of $\phi_e$ to $R^G \to C(z)$ is called $ev_e$.

By Compoint’s theorem, the ideal $\ker(\phi_e) \subset R$ is generated by the set \{ $F - ev_e(F) \ | \ F \in R^G$ \}. We want to prove that the ideal $\ker(\phi) \subset R_0$ is generated by \{ $F - ev(F) \ | \ F \in R_0^G$ \}. We will construct a $C(z)$-algebra homomorphism $\Psi : R \to R_0$ which has the following properties: $\Psi(r) = r$ for $r \in R_0$; $\Psi(X_i^0) = X_i$; $\phi \circ \Psi = \phi_e$ and $\Psi$ is $G$-equivariant.

Consider an element $\xi \in \ker(\phi)$. Then also $\xi \in \ker(\phi_e)$ and $\xi$ is a finite sum $\sum c(F) \cdot (F - ev_e(F))$ with $F \in R^G$ and $c(F) \in R$. Applying $\Psi$ to this expression yields $\xi = \sum \Psi(c(F)) \cdot (\Psi(F) - \Psi(ev_e(F)))$. Since $\Psi$ is $G$-equivariant $\Psi(F) \in R_0^G$. Moreover $\Psi(ev_e(F)) = ev(\Psi(F))$. This implies that $\xi$ lies in the ideal generated by the \{ $F - ev(F) \ | \ F \in R_0^G$ \} in the ring $R_0$.

Construction of $\Psi$. Define a $C$-linear derivation $E : R_0 \to R_0$ by $E(z) = 1$ and, for $i = 1, \ldots, n$, $E(X_i) \in R_0$ has the property that $\phi E(X_i) = \phi(X_i)'$. We note that $E$ exists since the map $\phi : R_0 \to K$ is surjective. Then $D := \frac{1}{#G} \sum_{g \in G} gEg^{-1} : R_0 \to R_0$ is a $C$-linear derivation with $D(z) = 1$, $\phi(D(X_i)) = \phi(X_i)'$ for all $i$ and $D$ is $G$-equivariant.
Define the $C(z)$-algebra homomorphism $\Psi : R \to R_0$ by $\Psi(X_i^j) = D^j(X_i)$ for all $i, j$. The first two properties of $\Psi$ are obvious. Further $\phi(\Psi(X_i^j)) = \phi(D^j(X_i)) = \phi(X_i)^{(j)}$ (for all $i, j$; the case $j = 1$ given earlier implies the general case) and so $\phi \circ \Psi = \phi_e$. Finally $\Psi$ is $G$-equivariant because $D$ is $G$-equivariant and the actions of $G$ on the vector spaces $CX_1^j + \cdots + CX_n^j$, for $j = 0, \ldots, n - 1$, are identical. \[\square\]

The explicit description of the kernel of $\phi$ given in Corollary 4.2 provides an important step in the computation of a standard operator, as will now be explained.

**Procedure 4.3. Constructing the differential operator from an evaluation.** Let an irreducible finite group $G \subset \text{GL}(C^n)$ be given. The group $G$ acts on $C[X_1, \ldots, X_n]$ by identifying $C^n$ with $\sum CX_j$. Suppose that $C[X_1, \ldots, X_n]^G = C[f_1, \ldots, f_n]$ with known homogeneous elements $f_1, \ldots, f_n$.

Consider a $C$-algebra homomorphism $h : C[X_1, \ldots, X_n]^G \to C(z)$ such that the image of $h$ generates the field $C(z)$ over $C$. We will call such $h$ again an evaluation of the invariants. The aim is to compute a differential operator $L = d_z^n + a_{n-1}d_z^{n-1} + \cdots + a_1d_z + a_0$ over $C(z)$ that induces the group $G$ and such that the evaluation $ev$ defined above Corollary 4.2 is equal to $h$.

The $C(z)$-algebra $R := C(z)[x_1, \ldots, x_n] = C(z)[X_1, \ldots, X_n] / I$, where $I = (f_1 - h(f_1), \ldots, f_n - h(f_n))$, has finite dimension over $C(z)$ (by observing that $C(z)[X_1, \ldots, X_n]$ is finite over $C(z)[f_1, \ldots, f_n]$).

(a). We assume that $R$ is a field. We note that in the opposite case, $R$ cannot be a Picard–Vessiot field for a suitable operator over $C(z)$. The action of $G$ on $C x_1 + \cdots + C x_n$ is induced by the irreducible action of $G$ on $C x_1 + \cdots + C x_n$. Hence either $C x_1 + \cdots + C x_n = (0)$ or it is isomorphic to $C x_1 + \cdots + C x_n$. The assumption that the image of $h$ generates $C(z)$ implies that $C x_1 + \cdots + C x_n \neq 0$, hence it has dimension $n$.

The derivation $\frac{d}{dx}$ has a unique extension to $R$ which we call $\hat{D}$. There is a unique operator $\hat{L} := \hat{D}^n + a_{n-1}\hat{D}^{n-1} + \cdots + a_1\hat{D} + a_0$ (with all $a_i \in R$) having kernel the $n$-dimensional vector space $C x_1 + C x_2 + \cdots + C x_n$. By uniqueness and the $G$-invariance of $C x_1 + C x_2 + \cdots + C x_n$, the operator $\hat{L}$ is $G$-invariant and therefore $a_{n-1}, \ldots, a_0 \in R^G = C(z)$. Then $\hat{L}$ is the differential operator associated to the evaluation $h$ of the invariants.

In order to find $\hat{L}$ one needs to compute the $\hat{D}^j x_i$. This is done as follows.

(b). By assumption $R$ is a finite field extension of $C(z)$ and $I$ is a maximal ideal of $C(z)[X_1, \ldots, X_n]$, which is the coordinate ring of the nonsingular variety $\mathbb{A}^n$ over $C(z)$. The well known Jacobian criterion for smoothness implies that the unit ideal of $C(z)[X_1, \ldots, X_n]$ is generated by $I$ and the determinants $\det \left( \frac{\partial (f_j - h(f_j))}{\partial x_i} \right)_{i=1,\ldots,n}^{j \in J}$, where $J$ ranges over the subsets of $\{1, \ldots, N\}$ with $\# J = n$.

Since the elements $h(f_j)$ belong to $C(z)$ we have $\frac{\partial (f_j - h(f_j))}{\partial x_i} = \frac{\partial f_j}{\partial x_i}$. After renumbering we may suppose that $\text{DET} = \det \left( \frac{\partial f_j}{\partial x_i} \right)_{i=1,\ldots,n}^{j=1,\ldots,n}$ is non zero. Then $df_1 \wedge \cdots \wedge df_n = \text{DET} \cdot dX_1 \wedge \cdots \wedge dX_n$. Thus for $\sigma \in G \subset \text{GL}(C^n)$ one has $\sigma(\text{DET}) = \det(\sigma)^{-1} \cdot \text{DET}$. 


Since $G$ is finite, there exists an integer $m \geq 1$ with $DET^m \in C[X_1, \ldots, X_n]^G$ and $h(DET^m) \in C(z)$. Since $R$ is a field, it follows that $h(DET^m) \neq 0$.

The extension $D$ of $\frac{d}{dz}$ on $R$ lifts to a derivation $D$ on $C(z)[X_1, \ldots, X_n]$ with $D(z) = 1$ and such that $D(I) \subset I$. The lift $D$ is not unique since one can add to each $D(X_i)$ any element in the ideal $I$.

The condition $D(I) \subset I$ with $I = (f_1 - h(f_1), \ldots, f_N - h(f_N))$ can be rewritten as the following explicit formula

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} \cdot D(X_j) \equiv h(f_i)' \mod I, \text{ for } i = 1, \ldots, N.$$

Since we have assumed that $R$ is a field, $h(DET^m) \neq 0$ and this suffices for the computation of the vector $(DX_1, \ldots, DX_n)^t$ satisfying the equation

$$\left( \frac{\partial f_i}{\partial X_j} \right) (DX_1, \ldots, DX_n)^t = (h(f_1)', \ldots, h(f_n)')^t.$$

Then $D(f_i - h(f_i)) \in I$ for all $i = 1, \ldots, N$ and $D(I) \subset I$. One then computes formulas for $D^i$, $i = 0, \ldots, n$. From this one deduces a linear combination $L := d_z^n + a_{n-1}d_z^{n-1} + \cdots + a_1d_z + a_0$ such that $L(x_i) = 0$ for all $i$. This relation is unique since we have assumed that $R$ is a field and we know that $x_1, \ldots, x_n$ are $C$-linearly independent. It follows that $L$ is $G$-invariant and all $a_j \in R^G = C(z)$. We conclude:

$L$ is the differential operator associated to the evaluation $h$. □

Remarks 4.4. (1). We briefly explain why 4.3 is called “Procedure” rather than “Algorithm”. A successful application of the Procedure depends on properties of the evaluation $h$ of the invariants. If the $R$ in 4.3 is known to be a field, then $L$ exists. If $h$ is known to be the evaluation of an operator $L$, then the Procedure computes $L$ up to (projective) equivalence. For some choices of $h$ the operator $L$ does not exist. It can happen, in the case that $R$ is not a field, that $L$ exists but has a differential Galois group which is a proper subgroup of $G$ (see 4.5).

(2). Suppose that the evaluation of the invariants $h$ produces the operator $L$. For the change of $h$ into $h_\lambda$, given by $h_\lambda(f_i) = \lambda^{\deg f_i} h(f_i)$ for all $i$ and fixed $\lambda$ such that a power $\lambda^m \in C(z)^*$. For some integer $m \geq 1$, Procedure computes a new operator, namely $\lambda L \lambda^{-1}$. Thus if $L = d_z^n + a_{n-1}d_z^{n-1} + \cdots + a_1d_z + a_0$, then the new operator is obtained from $L$ by the shift $d_z \mapsto d_z - \frac{a_1}{\lambda}$ (note that $\frac{a_1}{\lambda} \in C(z)$). The evaluations $h$ and $h_\lambda$ are called essentially the same.

Examples 4.5. In some cases, the ideal $I$ of Procedure 4.3 is not a maximal ideal of $C(z)[X_1, \ldots, X_n]$ and therefore $R$ is not field. We consider, as in Example 3.9, $G = A_4^{SL_2}$ and $C[x, y]^G = C[Q_3, Q_4, Q_6]$ with the relation $Q_6^2 - Q_3^4 - 4Q_4^3 = 0$. For the evaluations $h_1 : (Q_3, Q_4, Q_6) \mapsto (z, 0, z^2)$ and $h_2 : (Q_3, Q_4, Q_6) \mapsto (0, z^2, 2z^3)$ the above
ideal $I$ is not maximal. In both cases, $R$ is a product of a number of copies of the field $C(z)$.

Procedure 4.3 applied to the evaluation $h_1$ leads to the first order differential operator $d - \frac{1}{6z}$ instead of a second order operator. This is in accordance with the observation that for suitable $x_0, y_0 \in C^*, x_0 \neq y_0$ one has $Q_3(x_0z^{1/6}, y_0z^{1/6}) = z$, $Q_4(x_0z^{1/6}, y_0z^{1/6}) = 0$, $Q_6(x_0z^{1/6}, y_0z^{1/6}) = z^2$. Moreover the differential Galois group is $C_6$, the cyclic group of order 6, which can be seen as a subgroup of $A_4^{SL_2}$.

The Procedure does not produce an operator for $h_2$. Indeed, $Q_3$ is a product of six linear forms in the two $C$-linearly independent solutions $x, y$ and $h_2(Q_3) = 0$ contradicts this linear independence. \hfill $\Box$

For the existence and construction of an evaluation from a $G$-invariant curve $Z$ with $C(Z)^G = C(z)$ we will use the following lemma and its proof.

**Lemma 4.6.** Let $A$ be a finitely generated graded $C$-algebra. Assume that $A$ is a domain. Let $A_{(0)}$ denote the subfield of the field of fractions $A(0)$ of $A$ consisting of the homogeneous elements of degree 0.

Assume that $A_{(0)} = C(z)$. Then there exists a $C$-algebra homomorphism $h : A \rightarrow C[z]$ such that $h$ induces the identification $A_{(0)} = C(z)$.

**Proof.** Write $A = C[f_1, \ldots, f_r]$ where the $f_1, \ldots, f_r$ are homogeneous elements of degrees $d_1, \ldots, d_r \in \mathbb{Z}_{>0}$. Let $v(i) = (v(i)_1, \ldots, v(i)_r)$ for $i = 1, \ldots, r - 1$ denote free generators of $\{n_i, \ldots, n_r \in \mathbb{Z}^r \mid \sum n_i d_i = 0\}$. We may and will suppose that the matrix $\{v(i)\}_{i,j=1}^{r-1}$ is invertible. Let $m \in \mathbb{Z}_{\neq 0}$ be its determinant.

The elements $\{f_1^{v(i)_1} \cdots f_r^{v(i)_r} \mid i = 1, \ldots, r - 1\}$ generate the field $A_{(0)} = C(z)$ over $C$ and thus we can identify $f_1^{v(i)_1} \cdots f_r^{v(i)_r}$ with some $\alpha_i \in C(z)$. First we define $\tilde{h} : A \rightarrow C(z)$ by $\tilde{h}(f_i) = 1$ and the $\tilde{h}(f_1), \ldots, \tilde{h}(f_{r-1})$ are such that $\tilde{h}(f_1)^{v(i)_1} \cdots \tilde{h}(f_{r-1})^{v(i)_{r-1}} = \alpha_i$ for $i = 1, \ldots, r - 1$. One observes that the $\tilde{h}(f_i)$ are Laurent polynomials in $\alpha_1^{1/m}, \ldots, \alpha_{r-1}^{1/m}$. Thus the expressions $\tilde{h}(f_i)$ have the form $R(z) \cdot (z - a_1)^{n_1/m} \cdots (z - a_s)^{n_s/m}$ with $R \in C(z)$, certain distinct $a_1, \ldots, a_s \in C$ and certain integers $n_i \in \{0, \ldots, m - 1\}$.

The algebraic relations between the $f_1, \ldots, f_r$ are generated by homological relations. Hence for any expression $\lambda \in C(z)$ we can consider the $C$-algebra homomorphism $h$ given as $h(f_i) = \lambda^{d_i} \tilde{h}(f_i)$ for $i = 1, \ldots, r$. Using the shape of the $\tilde{h}(f_i)$ one observes that for suitable $\lambda$ all $\lambda^{d_i} \tilde{h}(f_i) \in C[z]$. Thus the required $h$ exists and can be seen to be unique (up to constants) under the condition that $\sum_{i=1}^r d_i \deg h(f_i)$ is minimal.

We recall that $h$ and $\tilde{h}$ are “essentially the same” according to 4.4. \hfill $\Box$

**Corollary 4.7.** Let be given an irreducible finite group $G \subset SL(V)$ and an irreducible $G$-invariant curve $Z \subset \mathbb{P}(V)$ such that the function field of $Z/G$ is $C(z)$. Lemma 4.6 produces an evaluation of the invariants $h : C[V]^G \rightarrow C[z]$ which induces the identification of the function field of $Z/G$ with $C(z)$.
This evaluation \( h \) is essentially the same as the evaluation of the invariants induced by the standard operator \( L_{st} \) for the data \( G \) and \( Z \) (see §3).

**Proof.** Let \( M \subset C[V] \) be the homogeneous prime ideal of \( Z \subset \mathbb{P}(V) \). Then \( M \cap C[V]^G \) defines the curve \( Z/G \) and the homogeneous algebra of \( Z/G \) is \( A := C[V]^G/(M \cap C[V]^G) \). Now one applies Lemma 4.6 to \( A \). The last statement follows from the unicity of \( h \) up to a change \( h(f_i) \leftrightarrow i^{\text{deg} f_i} h(f_i) \) for \( i = 1, \ldots, r \). \( \square \)

5. Computations with Procedure 4.3

In Sections 5.1–5.3 we present in concrete cases the differential operator obtained from Procedure 4.3. The evaluations used are computed as in the proof of Lemma 4.6.

5.1. Finite subgroups of \( SL_2 \)

For finite subgroups of \( SL_2 \) and their invariants we use the notations and equations from [5]. The standard equations for the subgroups \( D_n, A_4, S_4, A_5 \) of \( PSL_2 \) are classical and well known, see for instance [1,2].

(1) The group \( D_{4n}^{SL_2} \) of order \( 4n \) is generated by \( (0 \; \zeta \; 1), (0 \; -1 \; 1) \) with \( \zeta = e^{2\pi i/2n} \). The semi-invariants are generated by \( f_3 = xy, f_{12} = x^{2n} + y^{2n} \), and \( f_{13} = x^{2n} - y^{2n} \). The invariants have generators

\[
F_1 = f_3 f_{13}, \quad F_2 = f_{12}, \quad F_3 = f_3^2 \text{ and relation } F_1^2 - F_2^2 F_3 + 4 F_3^{n+1} = 0.
\]

This leads to the following evaluations.

If \( n \) is odd, \( A_{(0)} \) is generated by \( F_1, F_2, F_3 \) with relation \( (F_1/F_3^{n+1/2})^2 = F_3/F_3 - 4 \).

Define \( z \) by \( F_3/F_3^{n+1/2} = 2iz \). This gives \( h: (F_1, F_2, F_3) \mapsto (2iz, 2i(z^2 - 1)^{1/2}, 1) \) and \( h: (F_1, F_2, F_3) \mapsto (2i(z^2 - 1)^{(n+1)/2}, 2i(z^2 - 1)^{(n+1)/2}, (z^2 - 1)). \)

For \( 2|n \), generators of \( A_{(0)} \) are \( F_2^2 F_3/F_3^{n+1/2}, F_3^2/F_3^{n+1/2} \) satisfying \( F_1^2/F_3^{n+1/2} = (F_2/F_3^{3/2})^2 - 4 \). Corresponding evaluations are \( h: (F_1, F_2, F_3) \mapsto (2(z^2 - 1)^{1/2}, 2z, 1) \) and \( h: (F_1, F_2, F_3) \mapsto (2(z^2 - 1)^{1+n/2}, 2z(z^2 - 1)^{n/2}, (z^2 - 1)). \)

For all \( n \) the differential operator is \( L = d_z + \frac{3}{16(z^2 - 1)} dz - \frac{1}{4n(z^2 - 1)} \). This becomes, after the transformation \( z \mapsto 2z - 1 \), the standard equation

\[
\left( \frac{d}{dz} \right)^2 + \frac{3}{16z^2} + \frac{3}{16(z-1)^2} = \frac{n^2 + 2}{8n^2 z(z-1)} \quad \text{for } D_n.
\]

(2) For the group \( A_4^{SL_2} \), we continue the discussion from Example 3.9. Generators for the invariants are the homogeneous polynomials \( Q_3, Q_4, Q_6 \) of degrees 6, 8, 12 with relation \( Q_6^2 = Q_4^3 + 4Q_3^3 \). Here \( A_{(0)} = C(Q_3^3, Q_4^3, Q_6^3) \) is \( C(z) \) with \( z = Q_3^3/Q_4^3 \). This leads to the evaluations \( h: (Q_3, Q_4, Q_6) \mapsto (1, (z^2 - 1)^{1/3}, z) \) and \( h: (Q_3, Q_4, Q_6) \mapsto ((z^2 - 1)^2, (z^2 - 1)^3, (z^2 - 1)^4 z) \).
The differential operator is \( d_z^2 + \frac{27z^2 + 101}{144(z^2 - 1)^2} \). This becomes after \( z \mapsto 2z = 1 \) the standard equation

\[
\left( \frac{d}{dz} \right)^2 + \frac{3}{16z^2} + \frac{2}{9(z - 1)^2} - \frac{3}{16z(z - 1)} \quad \text{for } A_4.
\]

(3). The group \( S_4^{SL_2} \) has ring of invariants \( A := C[F_1, F_2, F_3] \) with generators \( F_j \) of degrees 12, 8, 18, respectively. One finds \( A_{((0))} = C[F_3^3/F_3, F_3^2/F_3] \) with relation \( \frac{F_3^3}{F_3} = \frac{F_3^2}{F_3} + 108 \). Put \( \tilde{h} : (F_1, F_2, F_3) \mapsto (1, 3 \cdot 2^{2/3} z, 2 \cdot 3^{3/2} (z^3 - 1)^{1/2}) \) and \( h : (F_1, F_2, F_3) \mapsto (2^2 \cdot 3^3 (z^3 - 1)^3, 2^2 \cdot 3^3 z(z^3 - 1)^2, 2^4 \cdot 3^6 (z^3 - 1)^5) \).

The differential operator is \( d_z^2 + \frac{(7z^3 + 101)z}{144(z^2 - 1)^2} \). The equation has 4 singular points and is a pullback of the standard equation. Note that our choice of evaluation is not ‘minimal’, i.e., the map induced by \( \tilde{h} \) and \( h \) from \( A_{((0))} \) to \( C(z) \) has image \( C(z^3) \). The operator is a pullback of the standard operator

\[
\left( \frac{d}{dz} \right)^2 + \frac{3}{16z^2} + \frac{2}{9(z - 1)^2} - \frac{101}{576(z - 1)} \quad \text{for } S_4.
\]

(4). The group \( A_5^{SL_2} \) has ring of invariants \( A := C[f_9, f_{10}, f_{11}] \), generators of degree 30, 20, 12, respectively, with relation \( f_9^2 + f_{10}^2 - 1728 f_{11}^2 = 0 \). In the present case \( A_{((0))} = C(f_9^2/F_{11}, f_9^3/F_{11}) \) with \( f_9^2/F_{11} = -f_{10}^2/F_{11} + 1728 \). This leads to the evaluations \( \tilde{h} : (f_9, f_{10}, f_{11}) \mapsto (-123^{3/2}(z - 1)^{1/2}, 12 \cdot z^{1/3}, 1) \) and

\[
h : (f_9, f_{10}, f_{11}) \mapsto (12^9 z^{10}(z - 1)^8, -12^6 z^7(z - 1)^5, -12^3 z^4(z - 1)^3).
\]

The differential operator is \( d_z^2 + \frac{864z^2 - 989z + 800}{3600z^2(z - 1)^2} \). After \( z \mapsto 1 - z \), this becomes the standard operator

\[
\left( \frac{d}{dz} \right)^2 + \frac{3}{16z^2} + \frac{2}{9(z - 1)^2} - \frac{611}{3600z(z - 1)} \quad \text{for } A_5.
\]

5.2. \( G = G_{168} \subset SL_3 \)

5.2.1. Computation of the differential equation related to Klein’s quartic

For the unique simple group \( G \subset SL(3, \mathbb{C}) \) of order 168 we use notations and formulas of \([2, \text{ p. 50}].\) Here \( C[X_1, X_2, X_3]^G = C[F_4, F_6, F_{14}, F_{21}]/(\text{rel}) \), where \( F_4, F_6, F_{14}, F_{21} \) are of degrees 4, 6, 14, 21. The Klein quartic \( Z \subset \mathbb{P}^2 \) is given by \( F_4 = 0 \) with \( F_4 := 2(X_1 X_2^3 + X_2 X_3^3 + X_3 X_1^3) \).

Unlike the case of finite irreducible subgroups of \( SL_2 \) (compare Example 3.9), a direct computation of the standard operator for these data with the methods of Section 3 meets difficulties. How to compute \( f \in C(Z)^* \) such that \( \frac{\sigma(f)}{f} = \frac{\sigma X_i}{X_i} \) for all \( \sigma \in G \)? How to compute the derivatives w.r.t. \( d_t = \frac{d}{dt} \) of a basis of the solution space \( W = \langle f, fX_2/X_1, fX_3/X_1 \rangle \)?
We now use the methods of Section 4. The graded algebra of $Z/G$ is

$$\{C[X_1, X_2, X_3]/(F_4)\}^G = C[F_6, F_{14}, F_{21}]/(F_{21}^2 - 4F_{14}^3 - 54F_6^2);$$

the field $A_{(0)} = C(Z/G)$ equals $C(F_6^2/F_6^3, F_6^3/F_6)$ with relation $F_6^2/F_6^3 = (4F_6^3/F_6^4 + 54)^3$. Hence $A_{(0)} = C(t)$ with $t = F_6^3/F_6$. A resulting evaluation is

$$h: (F_4, F_6, F_{14}, F_{21}) \mapsto (0, t^2(4t + 54)^3, t^5(4t + 54)^7, t^7(4t + 54)^{11}).$$

Now Procedure 4.3 leads to an operator $S_0$ with singularities $t = 0, -\frac{27}{2}, \infty$. Its local exponents are $1, 2, 3, 1/3, 1, 1/2, 3/2, -3/7, -5/7, -6/7$.

The change $t = -\frac{27}{2}z$ (hence $d_t = -\frac{2}{27}d_z$) moves the singularities to $0, 1, \infty$, with the same local exponents. The corresponding operator is

$$S_1 := d_z^3 + \frac{1}{z}d_z^2 + \frac{72z^2 + 61z + 56}{252z^2(z - 1)^2} d_z^3 - \frac{6480z^3 + 3945z^2 + 13585z - 5488}{24696z^3(z - 1)^3}.$$

The conjugate $S_2 := z^{-1}(z - 1)^{-1}S_1 z(z - 1)$ has the “classical” local exponents and coincides with the formulas in the literature $[12, 23, 18]$:

$$S_2 = d_z^3 + \frac{7z - 4}{z(z - 1)} d_z^2 + \frac{2592z^2 - 2963z + 560}{252z^2(z - 1)^2} d_z + \frac{72z - 40805}{24696}z^2(z - 1)^2.$$

5.2.2. The Hessian of the Klein quartic

The Hessian is the $G$-invariant curve $Z \subset \mathbb{P}^2$ with equation $F_6 = 0$. The graded algebra of $Z/G$ is $C[F_4, F_{14}, F_{21}]/(F_{21}^2 - 4F_{14}^3 + 8F_{14}F_6^2)$ and $C(Z)^G = C(t)$ with $t = F_6^3/F_6$.

A resulting evaluation is

$$h: (F_4, F_6, F_{14}, F_{21}) \mapsto (t^3(t - 2)^2, 0, t^{11}(t - 2)^7, 2t^{16}(t - 2)^{11}).$$

Procedure 4.3 then yields (after a change of variables) the operator

$$d_z^3 + \frac{3(3z - 2)}{2(z - 1)} d_z^2 + \frac{3(116z - 35)}{112z^2(z - 1)} d_z + \frac{195}{2744z^2(z - 1)}.$$

5.2.3. More third order operators with group $G = G_{168}$

The third order operators over $C(z)$, or more precisely, the differential modules of dimension 3, with singular points $0, 1, \infty$ and differential Galois group $G$ are classified in $[18]$, using the “transcendental” Riemann–Hilbert correspondence. Each case is given by a branch type $[e_0, e_1, e_\infty]$ and a choice of one of the two irreducible characters $\chi_2, \chi_3$ of dimension 3. The LIST is:

- $[2, 3, 7]$, 1 case, $g = 3$; $[2, 4, 7]$, 1 case, $g = 10$; $[2, 7, 7]$, 1 case, $g = 19$;
- $[3, 3, 4]$, 2 cases, $g = 8$; $[3, 3, 7]$, 1 case, $g = 17$; $[3, 4, 4]^*$, 1 case, $g = 15$;
[3, 4, 7], 2 cases, $g = 24$; [3, 7, 7], 2 cases, $g = 33$; [4, 4, 4], 2 cases, $g = 22$; [4, 4, 7], 1 case, $g = 31$; [4, 7, 7]*, 2 cases, $g = 40$; [7, 7, 7], 1 case, $g = 49$.

For many cases in LIST these data lead to a computation of the third order operator. The cases where this fails are indicated by a *.

In general the Fano curve corresponding to an element in LIST is not explicitly known. If one can identify for an item in LIST the $G$-invariant (Fano) curve $Z \subset \mathbb{P}^2$, this results in an evaluation and via Procedure 4.3 in a computation of the desired differential operator. [2, 3, 7], [2, 4, 7] in LIST correspond to $F_4 = 0$ and $F_6 = 0$. [2] considered smooth $G$-invariant $Z \subset \mathbb{P}^2$ with quotient of genus 0 and did not find new examples.

We extend his search and consider the (singular) curves $aF_4^3 + F_6^2 = 0$.

If such a curve $Z = Z_a$ leads to an evaluation $\tilde{h}$ with $\tilde{h}(F_4) = 1$ and $\tilde{h}(F_6) = \lambda$ (so $\lambda^2 = -a$) and $\tilde{h}(F_{14}) = t$, then

$$h(F_{21})^2 = 4t^3 - 44\lambda t^2 + (126\lambda^4 + 68\lambda^2 - 8)t + 54\lambda^7 - 938\lambda^5 + 172\lambda^3 - 8\lambda.$$ 

The discriminant of this polynomial in $t$ equals $-64(27\lambda^2 - 2)^3(\lambda^2 + 2)^4$, so $\lambda = (-2)^{1/2}$ and $\lambda = (27/2)^{1/2}$, or $a = 2, a = -2/27$ are special. Note that if the discriminant is nonzero then the quotient map from $Z_a$ would have at least 5 branch points. Both special cases lead to quotient maps with exactly 3 branch points. In fact $Z_{-2/27}$ is birational to the Klein quartic (of genus 3), and $Z_2$ is birational to the curve given by $F_6 = 0$.

For $\lambda = (-2)^{1/2}$ one finds $\tilde{h}(F_{21}) = 2(-t + 9\sqrt{-2})(t + 7\sqrt{-2})^{1/2}$ and for $\lambda = (2/27)^{1/2}$ we have $\tilde{h}(F_{21}) = \frac{-2\sqrt{3}}{243}(27t + \sqrt{6})(-27t + 35\sqrt{6})^{1/2}$. Using 4.3 the corresponding operators are found. The operators have three singular points and the solutions are generalized hypergeometric functions. We remark that the above “Fricke pencil” $Z_a$ was also studied by M. Kato (see [13, Prop. 2.3]), using Schwarz maps. In a rather different way than ours he found the two special cases as well as the corresponding third order differential operators.

5.2.4. Computing the evaluation for differential operators in LIST

An element in LIST is given by a topological covering of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with group $G = G_{168}$, produced by a triple $g_0, g_1, g_\infty \in G$ satisfying $g_0g_1g_\infty = 1$ and generating $G$. One may hope that from a given triple one can read off a part of an evaluation $h$ of the operator, namely the orders of the functions $h(F_4), h(F_6), h(F_{14}), h(F_{21})$ at the points 0, 1, $\infty$.

In a number of cases knowledge of these orders together with the relation between the four invariants suffices to compute a suitable $h$.

We illustrate this for the item [2, 4, 7] in LIST:

Let $x, y, z$ denote a basis of solutions for the differential equation we try to compute. As $F_4, F_6, F_{14}, F_{21}$ are explicit expressions in $x, y, z$, and one has (by [18, §5.2]) lower bounds $-\frac{1}{2}, -\frac{3}{7}, \frac{8}{7}$ for the local exponents at $t = 0, 1, \infty$, one deduces

$$(h(F_4), h(F_6), h(F_{14}), h(F_{21})) = \left(\frac{f_4}{t^2(t - 1)^3}, \frac{f_6}{t^3(t - 1)^4}, \frac{f_{14} + g_{14}t}{t^4(t - 1)^5}, \frac{f_{21} + 2400}{t^{10}(t - 1)^{15}}\right).$$
for constants $f_4, f_6, f_{14}, g_{14}, f_{21}$ (unique up to an appropriate scaling). The relation between the $F_j$’s yields $(f_4, g_4, f_{14}, g_{14}, f_{21}) = \left( \frac{-7}{4}, \frac{-3}{8}, \frac{-149}{7}, \frac{1}{3}, \frac{1}{8} \right)$.

Evaluations for several items in LIST. The same idea used for $[2, 4, 7]$ above, results in evaluations for various other items in LIST. The next table presents the results. The first row gives the branch type and the rational functions $h(F_4), h(F_6), h(F_{14}), h(F_{21})$. The second row lists the local exponents at $0, 1, \infty$ and the accessory parameter $\mu$ (see [18, §5.1]). The operator is uniquely determined by these data.

- $[2, 3, 7]$ \[ \begin{array}{cccc}
    0 & -3^3 & 2^23^8 & 2^33^{12} \\
    -1 & 0 & 1 & 2 \\
    3 & -3 & -3 & 0 \\
    \frac{7}{7} & \frac{7}{7} & \frac{7}{7} & \frac{7}{7} \\
    \frac{12293}{2496} \\
    (t + 2400) \\
\end{array} \]

- $[2, 4, 7]$ \[ \begin{array}{cccc}
    0 & -3^3 & 2^23^8 & 2^33^{12} \\
    -1 & 0 & 1 & 2 \\
    3 & -3 & -3 & 0 \\
    \frac{7}{7} & \frac{7}{7} & \frac{7}{7} & \frac{7}{7} \\
    \frac{12293}{2496} \\
    (t + 2400) \\
\end{array} \]

- $[2, 7, 7]$ \[ \begin{array}{cccc}
    0 & -3^3 & 2^23^8 & 2^33^{12} \\
    -1 & 0 & 1 & 2 \\
    3 & -3 & -3 & 0 \\
    \frac{7}{7} & \frac{7}{7} & \frac{7}{7} & \frac{7}{7} \\
    \frac{12293}{2496} \\
    (t + 2400) \\
\end{array} \]

- $[3, 3, 7]$ \[ \begin{array}{cccc}
    0 & -3^3 & 2^23^8 & 2^33^{12} \\
    -1 & 0 & 1 & 2 \\
    3 & -3 & -3 & 0 \\
    \frac{7}{7} & \frac{7}{7} & \frac{7}{7} & \frac{7}{7} \\
    \frac{12293}{2496} \\
    (t + 2400) \\
\end{array} \]

- $[3, 3, 7]$ \[ \begin{array}{cccc}
    0 & -3^3 & 2^23^8 & 2^33^{12} \\
    -1 & 0 & 1 & 2 \\
    3 & -3 & -3 & 0 \\
    \frac{7}{7} & \frac{7}{7} & \frac{7}{7} & \frac{7}{7} \\
    \frac{12293}{2496} \\
    (t + 2400) \\
\end{array} \]

- $[4, 4, 7]$ \[ \begin{array}{cccc}
    0 & -3^3 & 2^23^8 & 2^33^{12} \\
    -1 & 0 & 1 & 2 \\
    3 & -3 & -3 & 0 \\
    \frac{7}{7} & \frac{7}{7} & \frac{7}{7} & \frac{7}{7} \\
    \frac{12293}{2496} \\
    (t + 2400) \\
\end{array} \]

- $[7, 7, 7]$ \[ \begin{array}{cccc}
    0 & -3^3 & 2^23^8 & 2^33^{12} \\
    -1 & 0 & 1 & 2 \\
    3 & -3 & -3 & 0 \\
    \frac{7}{7} & \frac{7}{7} & \frac{7}{7} & \frac{7}{7} \\
    \frac{12293}{2496} \\
    (t + 2400) \\
\end{array} \]

Remarks 5.1. (1) For the remaining items in LIST the approach above does not determine $h$ (up to equivalence).

(2) For the case $[3, 4, 4]$ a choice of $h$ leads to a proper subgroup of $G$ (compare [18, §8.2.1 part (6)]).
(3). The items [3, 3, 7] and [3, 7, 7] correspond to weak pullbacks of the “standard” case [2, 3, 7], with pullback functions \( \phi(t) = 4t(t + 1) + 1 \) and \( \phi(t) = \frac{(2t^2 - 36t + 8)^2}{t - 1} \), respectively.

(4). The Fano curve for [2, 4, 7] is given by the equation \(-\frac{7}{54}F_6^3 - \frac{1}{8}F_4F_{14} + F_4^3F_6 = 0\) and it has genus 10. [2, 7, 7] and [4, 4, 7] are weak pullbacks of the “standard” case [2, 4, 7], with pullback functions \( \phi(t) = \frac{(t - 1)^2}{4t(t - 1)} \) and \( \phi(t) = (2t - 1)^2 \), respectively. The Fano curves for these two cases can be obtained from the above one by the same pullbacks.

5.3. \( H = H_{SL_3}^{SL_3} \subset SL_3 \)

The group \( H = H_{SL_3}^{SL_3} \subset SL_3 \) has order 216 and together with its invariants it is described in [5, p. 59] and in [19, Thm. 4]. We use the latter and write

\[
P = xyz, \quad Q = x^3y^3 + x^3z^3 + y^3z^3, \quad S = x^3 + y^3 + z^3,
\]

\[
F_1 = S^2 - 12Q, \quad F_2 = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3), \quad F_3 = S^4 + 216P^3S,
\]

\[
F_4 = (S^2 - 18P^2 - 6PS)^2.
\]

The algebra of invariants is \( C[x, y, z]^H = C[F_1, F_2, F_3, F_4] \) with relation \( 432F_2^2 + 3F_1F_3 - F_1^3 - 4(F_3^3 - 3F_2^2F_3 + 3F_4F_3^2) = 0 \).

Let \( Z \subset \mathbb{P}^2 \) be given by \( F_1 = 0 \). The graded algebra of \( Z/H \) is \( A = C[F_2, F_3, F_4]/(216^2F_2^2 - F_1^3 + 3F_2^2F_3 - 3F_4F_3^2) \). Then \( A_{(0)} = C(Z)^H \) equals \( C(F_2, F_3, F_4) \) and

\[
216^2F_2^2 - F_1^3 + 3F_2^2F_3 - 3F_4F_3^2 = (F_2/F_3)^3 - 3(F_2/F_3)^2 + 3F_4/F_3, \quad \text{so} \quad A_{(0)} = C(F_2/F_3). \]

This yields the evaluation \( (F_1, F_2, F_3, F_4) \mapsto (0, \sqrt[3]{t^3 - 3t^2 + 3t}, 1, t) \) which by scaling simplifies to \( (F_1, F_2, F_3, F_4) \mapsto (0, t^3 - 3t^2 + 3t, 36t(t^2 - 3t + 3), 36t^2(t^2 - 3t + 3)) \). From this, Procedure 4.3 yields the differential operator

\[
d_t^3 + \frac{5t^3 - 15t^2 + 15t - 6}{(t^3 - 3t^2 + 3t)(t - 1)}d_t^2 + \frac{(160t^3 - 480t^2 + 480t - 117)(t - 1)}{48(t^3 - 3t^2 + 3t)^2}dt - \frac{(160t^3 - 480t^2 + 480t - 189)(t - 1)^3}{432(t^3 - 3t^2 + 3t)^3}.
\]

One observes that \( t = 1 \) is an apparent singularity and that \( \infty \) and the three zeros of \( t^3 - 3t^2 + 3t \) are the singular points.

At present no differential equation over \( C(t) \) with three singularities and Galois group \( H \) seems to be known. Another differential equation of order 3 with Galois group \( H \) was found by M. van Hoeij, see [10, § 2].
6. Differential equations for $A_5 \subset \text{SL}_3$

Consider $A_5 \subset \text{SL}_3(\mathbb{Q}((\zeta_5))) \subset \text{SL}_3(\mathbb{C})$ (with $\zeta_5 = e^{2\pi i/5}$) the group with generators

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & \zeta_5 & 0 \\
0 & 0 & \zeta_5^{-1}
\end{pmatrix}, \quad \frac{1}{\sqrt{5}} \begin{pmatrix}
1 & 2 & 2 \\
1 & \zeta_5^2 + \zeta_5^{-2} & \zeta_5 + \zeta_5^{-1} \\
1 & \zeta_5 + \zeta_5^{-1} & \zeta_5^2 + \zeta_5^{-2}
\end{pmatrix}.$$ 

This inclusion corresponds to the irreducible character $\chi_2$ of dimension 3 for $A_5$. The other irreducible character $\chi_3$ of dimension 3 is obtained via the automorphism $\zeta_5 \mapsto \zeta_5^2$ of $\mathbb{Q}(\zeta_5)/\mathbb{Q}$. Denote by $A_5^{\text{SL}_2} \subset \text{SL}_2(\mathbb{C})$ the preimage of $A_5$ under the “symmetric square map” $\text{SL}_2 \to \text{SL}_3$: $A \mapsto \text{sym}^2 A$. The group $A_5^{\text{SL}_2}$ (called the ‘icosahedral group’, of order 120) has two irreducible characters of dimension 2 and their second symmetric powers are the above 3-dimensional characters $\chi_2, \chi_3$ of $A_5$. The following proposition may be known, but we found no proof in the literature. The argument presented here relates to [16, Theorem 2.1]. We also offer a second proof using the solution of an embedding problem.

**Proposition 6.1.** (Comparing differential modules for $A_5$ and $A_5^{\text{SL}_2}$).

(1) Suppose that the 3-dimensional differential module $M$ over $\mathbb{C}(z)$ has differential Galois group $A_5$. Then there is a 2-dimensional differential module $N$ with differential Galois group $A_5^{\text{SL}_2}$ such that $\text{sym}^2 N$ is isomorphic to $M$.

(2) The module $N$ is unique up to tensoring with a 1-dimensional module $D$ such that $D^{\otimes 2}$ is the trivial differential module 1.

**First proof of (1).** The action of $A_5$ on the solution space $W$ of $M$ induces an action on $\text{sym}^2 W$. It has an invariant line (which can be interpreted as an invariant quadratic form on $W$); this corresponds to a 1-dimensional submodule $T$ of $\text{sym}^2(M)$. In terms of a basis of $M$, a nonzero element of $T$ is a nondegenerate quadratic form. This form has a nontrivial zero over $\mathbb{C}(z)$ since the latter is a $C_1$-field. Hence there is a basis $x_1, x_2, x_3$ of $M$ such that $T$ is generated by $x_1x_3 - x_2^2$. Moreover $T$ is the trivial module since $A_5$ is simple. For some $q \in \mathbb{C}(z)^*$ one has $\partial(q(x_1x_3 - x_2^2)) = 0$.

The equation $\frac{1}{q} \partial(q(x_1x_3 - x_2^2)) = 0$ implies that the matrix $A$ of $\partial$ w.r.t. the basis $x_1, x_2, x_3$ of $M$ has the form $$\begin{pmatrix}
a_1 & b_1 & 0 \\
2b_3 & -\frac{q'}{2q} & 2b_1 \\
0 & b_3 & -a_1 - \frac{q'}{q}
\end{pmatrix}.$$ Since $A_5$ is simple, $\det M = 1$.

This implies that the equation $y' = tr(A)y$ has a solution in $\mathbb{C}(z)$. Therefore $q^{-3/2} \in \mathbb{C}(z)$ and thus $q$ is a square.

After changing the basis of $M$ one has $q = 1$. Now $\partial(x_1x_3 - x_2^2) = 0$ implies that the matrix of $\partial$ with respect to the basis $\{x_1, x_2, x_3\}$ has the form $$\begin{pmatrix}
2a & b & 0 \\
2c & 0 & 2b \\
0 & c & -2a
\end{pmatrix}$$ for
certain \(a, b, c\). Consider the 2-dimensional module \(N\) and a basis \(y_1, y_2\) such that the matrix of \(\partial\) is \(
abla = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}\). Then \(\text{sym}^2(N)\) has on basis \(x_1 = y_1^2, x_2 = y_1 y_2, x_3 = y_2^2\) the above matrix. Thus \(\text{sym}^2 N \cong M\). The differential Galois group \(G \subset \text{SL}_2\) of \(N\) has the property that its image \(\text{sym}^2(G)\) in \(\text{SL}_3\) equals \(A_5\). Hence the action of \(G\) on \(\mathbb{P}^1\) is that of \(A_5\) and so \(G = A_5^{\text{SL}_2}\).

**Second proof of (1).** The equivalence of Tannaka categories \(\text{Diff}_{\mathbb{P}^1/k} \to \text{Repr}_\pi\), with \(k = \mathbb{C}(z), \pi = \text{Gal}(\bar{k}/k)\), translates 6.1(1) into solvability of the embedding problem for \(1 \to \{\pm 1\} \to A_5^{\text{SL}_2} \to A_5 \to 1\), namely:

> Any continuous surjective homomorphism \(\pi \to A_5\) lifts to a continuous surjective homomorphism \(\pi \to A_5^{\text{SL}_2}\).

According to [15, Thm. 1.10(a)], this holds for \(k = \mathbb{C}(z)\).

**Proof of (2).** Let the 1-dimensional module \(D\) satisfy \(D^{\otimes 2} = 1\). Then \(\text{sym}^2(D \otimes N) \cong D^{\otimes 2} \otimes \text{sym}^2(N) = M\). This proves one implication for part (2) of 6.1. Using the equivalence of Tannaka categories, part (2) translates into:

> Given (for \(i = 1, 2\)) surjective continuous homomorphisms \(\rho_i: \pi \to A_5^{\text{SL}_2}\) with \(\text{sym}^2(\rho_1) \cong \text{sym}^2(\rho_2)\). Then a continuous homomorphism \(\chi: \pi \to \{\pm 1\} \subset A_5^{\text{SL}_2}\) exists such that \(\rho_1 \cong \rho_2 \circ \chi\).

Let \(\text{can}: \text{SL}_2 \to \text{PSL}_2\) be the canonical map. The assumption on \(\rho_1, \rho_2\) is equivalent to \(\text{can} \circ \rho_1 \cong \text{can} \circ \rho_2\). Since \(A_5 \subset \text{PSL}_2\) is unique up to conjugation, we may suppose \(\text{can} \circ \rho_1(g) = \text{can} \circ \rho_2(g)\) for every \(g \in \pi\). Hence \(\rho_1(g) = \chi(g) \rho_2(g)\) for some continuous homomorphism \(\chi: \pi \to \{\pm 1\}\). \(\square\)

Let \(L_{\text{st}, A_5^{\text{SL}_2}}\) denote the standard second order operator for \(A_5^{\text{SL}_2}\) with local exponents \(1/4, 3/4||1/3, 2/3|| -2/5, -3/5\) (see §5.1 (4)). Define the standard operator \(L_{\text{st}, A_5}\) to be the second symmetric power of \(L_{\text{st}, A_5^{\text{SL}_2}}\).

**Proposition 6.2.** Every third order operator \(L\) over \(C(z)\) with differential Galois group \(A_5\) is equivalent to a weak pullback of \(L_{\text{st}, A_5}\).

**Proof.** This follows from Proposition 6.1 and Klein’s theorem for second order equations with group \(A_5^{\text{SL}_2}\). Indeed, the given \(L\) equals \(\text{sym}^2(L_2)\) for a second order operator \(L_2\) with differential Galois group \(A_5^{\text{SL}_2}\). According to Klein’s theorem \(L_2\) is a weak pullback of the standard operator \(L_{\text{st}, A_5^{\text{SL}_2}}\). Taking symmetric squares the result follows. \(\square\)

**Remarks 6.3.** Differential operators for \(A_5, A_5^{\text{SL}_2}\) and the data of [18]. (1). [18] lists all third order differential operators (up to equivalence) with differential Galois group \(A_5\) and singular points 0, 1, \(\infty\). The branch types are \([2, 3, 5], [2, 5, 5], [3, 3, 5], [3, 5, 5](1), [3, 5, 5](2), [5, 5, 5]\). For each type there are two differential modules; one for each of the
3-dimensional irreducible characters $\chi_2, \chi_3$. The genera for the Picard–Vessiot fields are 0, 4, 5, 9, 9, 13.

One discovers, by comparing the genera, that each $A_5$ case is the second symmetric power of two, three or four second order equations with group $A_5^{SL_2}$ and singularities 0, 1, $\infty$ (also given as a list in [18]). This is explained by Proposition 6.1 and the observation that there are three 1-dimensional modules $D$ with $D \otimes^2 = 1$ and singular points 0, 1, $\infty$. Namely $D = C(z)e$ with $\partial e = ae$ and $a \in \{ \frac{1}{2z}, \frac{1}{2(z-1)}, \frac{1}{2z(z-1)} \}$.

(2). Comparing the local exponents for $A_5^{SL_2}$ and $A_5$ in both lists of [18] one further discovers that only for the two cases of [3,3,5] the operator $L_3$ is a second symmetric power. In all other cases the module $M$ is a sym$^2(N)$ but this does not hold for the operators.

Examples 6.4. Invariants, evaluations and differential operators for $A_5$.

(1). Generators for the ring $C[x, y, z]^{A_5}$ are, according to [5],

$$F_2 = x^2 + yz, \ F_6 = 8x^4yz - 2x^2y^2z^2 - x(y^5 + z^5) + y^3z^3;$$

$$F_{10} = 320x^6y^2z^2 - 160x^4y^3z^3 + 20x^2y^4z^4 + 6y^5z^5 - 4x(y^5 + z^5)(32x^4 - 20x^2yz + 5y^2z^2) + y^{10} + z^{10}; F_{15}.$$

There is one relation (which determines $F_{15}$ up to sign)

$$F_{15}^2 + 1728F_6^3 - F_{10}^3 - 720F_2F_6F_{10} + 80F_2^2F_6F_{10} - 64F_2^3(-F_{10}F_2 + 5F_6^2)^2 = 0.$$

(2). The evaluation for the $A_5$-invariant curve $Z \subset \mathbb{P}^2$ given by $F_2 = 0$.

The graded algebra for $Z/A_5$ is $A = C[F_5, F_{10}, F_{15}]/(F_{15}^2 + 1728F_6^3 - F_{10}^3)$, hence $A_{(0)} = C(Z)^{A_5}$ equals $C(F_5^2, F_6^3, F_{10}^3)$ with $F_5^2 + 1728 - F_6^3 = 0$. This leads to the evaluations $\hat{h} : (F_2, F_6, F_{10}, F_{15}) \mapsto (0, 1, t^{1/3}, (t - 1728)^{1/2})$ and $h : (F_2, F_6, F_{10}, F_{15}) \mapsto (0, t^4(t - 1728)^3, t^7(t - 1728)^5, t^{10}(t - 1728)^8)$.

The third order differential operator deduced from this has three singular points 0, 1728, $\infty$. Scaling moves the singular points to 0, 1, $\infty$ and then conjugation with the function $(t - 1)^{-1/2}t^{-1/3}$ results in an operator $L_c$ with the required local exponents:

$L_c = L_{st,A_5} = \text{sym}^2(L_{st,A_5^{SL_2}})$.

The operator $L_c$ has to be equivalent to one of the two operators for $A_5$ with branch type [2,3,5] in [18]. Explicitly, $L_c$ is equivalent to $L_u$, the one with local data $-1, -1/2, 1/2, -2/3, -1/3, 0|6/5, 9/5, 2|\mu = 43/225$. Below are formulas for $L_c$, for $\tilde{L}_u$ obtained from $L_u$ by $t \mapsto 1 - t$, and for operators $L, L'$ satisfying $\tilde{L}_u L = L' L_c$ (implying $\tilde{L}_u$ and $L_c$ define the same differential module, hence $L_c$ and $L_u$ are equivalent).

$$L_c = d_t^3 + \frac{3(2t - 1)}{t(t - 1)} d_t^2 + \frac{6264t^2 - 6389t + 800}{900t^2(t - 1)^2} d_t + \frac{1728t - 989}{1800t^2(t - 1)^2},$$

$$\tilde{L}_u = d_t^3 + \frac{8t - 4}{t(t - 1)} d_t^2 + \frac{12744t^2 - 13169t + 2000}{900t^2(t - 1)^2} d_t + \frac{7776t^2 - 12683t + 4457}{1800t^2(t - 1)^3}.$$

$$L = L_c; L' = L_c; \text{sym}^2(L_{st,A_5^{SL_2}}).$$
Consider a second order operator $L_2$ with differential Galois group $A_5^{SL_2}$ and Picard–Vessiot field $K^+$. The third order operator $L_3 := \text{sym}^2(L_2)$ has differential Galois group $A_5$ and Picard–Vessiot field $K = (K^+)^Z$, where $Z$ is the center of $A_5^{SL_2}$. The evaluation for $L_2$ is deduced from a homomorphism $h_1 : C(z)[X,Y] \rightarrow K^+$ which sends $X,Y$ to a basis of solutions of $L_2$. The evaluation for $L_3$ is deduced from a homomorphism $h_2 : C(z)[X_1, X_2, X_3] \rightarrow K$ which sends $X_1, X_2, X_3$ to a basis of solutions for $L_3$. We may suppose that $X_1, X_2, X_3$ are mapped to $h_1(X)^2, h_1(XY), -h_1(Y^2)$. It follows that $F_2 = X_1 X_3 + X_2^2$ lies in the kernel of the evaluation $h_2 : C[X_1, X_2, X_3] \rightarrow K$. Hence the evaluation for $L_3$ is induced by an evaluation for $L_2$. \(\square\)

**Examples 6.5. Operators for other evaluations for $A_5 \subset SL_4$.** An evaluation with nonzero image of $F_2$ to 0 is, after scaling, given by $(F_2, F_6, F_{10}, F_{15}) \mapsto (1, a, b, w)$ with $a, b \in C(z)$ not both constant. The assumption $w = 0$ leads to a contradiction. The relation between the invariants therefore implies $w^2 \in C(z)^*$. This leads to $h : (F_2, F_6, F_{10}, F_{15}) \mapsto (w^2, w^6 a, w^{10} b, w^{16})$. The evaluation induces an $A_5$-invariant curve $Z \subset \mathbb{P}^2$ such that the normalisation of $Z/A_5$ has genus 0. We discuss a family of such curves $Z$.

Consider the $A_5$-invariant curve $Z$ given by $-\lambda F_2^3 + F_6 = 0$. Then an evaluation as above with $a = \lambda \in C, b = z$ requires $w^2 \in C[z]$ to be a polynomial of degree 3. The singular points of the third order operator are included in the union of $\{0, \infty\}$ and the zeros of $w^2$. To have at most 3, the discriminant $-4096\lambda^3(\lambda-1)^2(27\lambda-32)^3$ of $w^2$ needs to vanish. This results in the cases:

1. $\lambda = 1$. The curve given by $-F_2^3 + F_6 = 0$ is reducible; the defining polynomial has factors $x$ and $x + y + z$ and a third one defining an irreducible rational curve of degree 4.

The associated operator

$$L = d_t^3 + \frac{3(7t^2 - 147t + 676)}{2(t-4)(3t-37)(t-8)} d_t^2 + \frac{3(149t^2 - 3367t + 13584)}{100(3t-37)(t-8)(t-4)^2} d_t$$

$$- \frac{3(t-29)}{200(3t-37)(t-8)(t-4)^2}$$

factors. The two right hand factors are $L_2 = d_t^3 + \frac{6}{2(t-4)} d_t$ and $L_1 = d_t + \frac{1}{2(t-4)}$. A basis of solutions for $L_2$ is $(t - 6 \pm \sqrt{t^2 - 12t + 32})^{1/10}$, hence the Galois group of $L_2$ is the dihedral group $D_{10}$ (of order 20). One checks that the solution $\sqrt{t-4}$ of $L_1$ is in the Picard–Vessiot field of $L_2$. Hence the Galois group of $L$ is the subgroup $D_{10}$ of $A_5$. 

\[ L = (t^2 - t)d_t^2 + \left( \frac{14t}{5} - \frac{4}{3} \right) d_t + \frac{48t - 49}{60(t - 1)}, \]

\[ L' = (t^2 - t)d_t^2 + \left( \frac{54t}{5} - \frac{16}{3} \right) d_t + \frac{1440t^2 - 1453t + 280}{60(t - 1)}. \]
(2). \( \lambda = 0 \). The curve given by \( F_6 = 0 \) has genus 4 and is a Galois covering of \( \mathbb{P}^1 \) with group \( A_5 \) ramified over the points \( 0, -64, \infty \). The corresponding order three differential operator, with indeed Galois group \( A_5 \), is

\[
d_t^3 + \frac{7t + 256}{2t(t + 64)} d_t^2 + \frac{149t + 1024}{100t^2(t + 64)} d_t - \frac{1}{200t^2(t + 64)}.
\]

(3). \( \lambda = \frac{32}{27} \). We give some details for this interesting example.

The curve \(-\frac{32}{27}F_2^3 + F_6 = 0 \) has genus 0 and has 10 singular points (all over the cyclotomic field \( \mathbb{Q} (\zeta_5) \)). The curve is parametrized by \([-5s^3 : s^6 + 3s : 3s^5 - 1]\) and has function field \( C(s) \). One has \( C(s)^{A_5} = C(z) \) where \( z \) equals

\[
(s^{50} + 2388s^{55} + 326394s^{50} - 8825700s^{45} + 117672975s^{40} + 83075976s^{35} + 380773868s^{30} - 3075976s^{25} + 117672975s^{20} + 8825700s^{15} + 326394s^{10} - 2388s^5 + 1)
\]
\[
/288s^5(s^2 + s - 1)^5(s^4 + 2s^3 + 4s^2 + 3s + 1)^5(s^4 - 3s^3 + 4s^2 - 2s + 1)^5.
\]

The evaluation obtained from this leads, using Procedure 4.3, the operator

\[
L = d_z^3 + \frac{81(567z - 4864)}{2(81z - 1024)(81z - 448)} d_z^2 + \frac{19683(4023z - 46592)}{100(81z - 448)(81z - 1024)^2} d_z
\]
\[
- \frac{531441}{200(81z - 448)(81z - 1024)^2}
\]

with Picard–Vessiot field \( C(s) \) and basis of solutions \( \{-5s^3, s^6 + 3s, 3s^5 - 1\} \). The operator \( L \) is equivalent to the standard one, but is not itself a symmetric square. Indeed, considering the degrees in \( s \), no quadratic relation over \( C \) between the three solutions exists. The branch points of \( C(s) \supset C(z) \) are \( z = \frac{448}{81}, \frac{1024}{81}, \infty \), and the ramification type is \([2, 3, 5]\).

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References


