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Chapter 9

Factorizations of Nonlinear Systems

In this chapter, we apply the L_2 -gain concepts and techniques from Chaps. 3 and 8 to obtain some useful types of representations of nonlinear systems, different from the standard input-state-output representation. In Sect. 9.1 we will derive *stable kernel* and *stable image* representations of nonlinear systems, and we will use them in order to formulate *nonlinear perturbation* models (with L_2 -gain bounded uncertainties) in Sect. 9.2. In Sect. 9.3 we will employ stable kernel representations in order to derive a parametrization of stabilizing controllers, analogous to the Youla–Kucera parametrization in the linear case. Section 9.4 deals with the *factorization* of nonlinear systems into a series interconnection of a minimum phase system and a system which preserves the L_2 -norm. This allows us to control the system based on its minimum phase factor system, using a nonlinear version of the Smith predictor.

9.1 Stable Kernel and Image Representations

A cornerstone of linear robust control theory is the theory of *stable factorizations* of transfer matrices. Let $G(s)$ be a $p \times m$ rational proper transfer matrix. A *stable left*, respectively *right*, factorization of $G(s)$ is $G(s) = D^{-1}(s)N(s)$, respectively, $G(s) = \tilde{N}(s)\tilde{D}^{-1}(s)$, where $D(s)$, $N(s)$, $\tilde{D}(s)$ and $\tilde{N}(s)$ are all *stable* proper rational matrices. These two factorizations can be alternatively interpreted as follows. The relation $y = D^{-1}(s)N(s)u$ for a stable left factorization (with u and y denoting the inputs and outputs in the frequency domain) can be equivalently rewritten as

$$0 = z = [D(s) \quad -N(s)] \begin{bmatrix} y \\ u \end{bmatrix}, \quad (9.1)$$

and thus can be considered as a *stable kernel representation* of the system corresponding to $G(s)$. Indeed, $[D(s) \dot{} - N(s)]$ is a stable transfer matrix with “inputs” y and u , and “outputs” z , while the input–output behavior of the system corresponding to $G(s)$ are all pairs (y, u) which are mapped by $[D(s) \dot{} - N(s)]$ onto $z = 0$.

On the other hand, the relation $y = \tilde{N}(s)\tilde{D}^{-1}(s)u$ for a stable right factorization can be rewritten as

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \tilde{N}(s) \\ \tilde{D}(s) \end{bmatrix} l, \quad (9.2)$$

where l is an arbitrary m -vector of *auxiliary* variables. Thus the input–output behavior of the system corresponding to $G(s)$ are all pairs (y, u) (in the frequency domain) which are in the image of the stable transfer matrix $\begin{bmatrix} \tilde{N}(s) \\ \tilde{D}(s) \end{bmatrix}$. Hence a stable right factorization can be regarded as a *stable image representation*.

Let us generalize this to nonlinear input-state-output systems. For simplicity of exposition we only consider *affine* systems Σ_a , which throughout Sects. 9.1 and 9.2 of this chapter will be simply denoted as

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u, & x \in \mathcal{X}, \quad u \in \mathbb{R}^m \\ y = h(x), & y \in \mathbb{R}^p \end{cases} \quad (9.3)$$

In analogy with Chap. 1 we will say that the state space system Σ is L_2 -stable if all its input–output maps G_{x_0} , $x_0 \in \mathcal{X}$, are L_2 -stable. Recall from Chap. 1 that if Σ has L_2 -gain $\leq \gamma$ for some γ then it is L_2 -stable. Since we will only consider L_q -stability for $q = 2$ the terminology “ L_2 -stable” will be sometimes abbreviated to “stable”.

Definition 9.1.1 A *kernel representation* of Σ given by (6.8) is any system

$$K_\Sigma : \begin{cases} \dot{x} = F(x, u, y), & x \in \mathcal{X}, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p \\ z = G(x, u, y), & z \in \mathbb{R}^q \end{cases} \quad (9.4)$$

such that for every initial condition $x(0) = x_0 \in \mathcal{X}$ and every function $u(\cdot)$ there exists a unique solution $y(\cdot)$ to (9.4) with $z = 0$, which equals the output of Σ for the same initial condition and same input $u(\cdot)$. K_Σ is called a *L_2 -stable kernel representation* of Σ if moreover K_Σ is L_2 -stable (from (u, y) to z).

An *image representation* of Σ is any system

$$I_\Sigma : \begin{cases} \dot{x} = F(x, l), & x \in \mathcal{X}, \quad l \in \mathbb{R}^k \\ \begin{bmatrix} y \\ u \end{bmatrix} = G(x, l) \end{cases} \quad (9.5)$$

such that for every initial condition $x(0) = x_0 \in \mathcal{X}$ and every input function $u(\cdot)$ and resulting output function $y(\cdot)$ of Σ there exists a function $l(\cdot)$ such that the pair (u, y)

produced by I_Σ for the same initial condition x_0 coincides with the input–output pair (u, y) of Σ . I_Σ is an L_2 -stable image representation if I_Σ is L_2 -stable from l to (u, y) .

In case Σ is already itself an L_2 -stable input-state-output system, a stable kernel representation of Σ is simply

$$K_\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u \\ z = y - h(x) \end{cases} \quad (9.6)$$

while a stable image representation of Σ is given by

$$I_\Sigma : \begin{cases} \dot{x} = f(x) + g(x)l \\ y = h(x) \\ u = l \end{cases} \quad (9.7)$$

If Σ is not L_2 -stable then we may proceed as follows. Consider the following two versions of the same Hamilton–Jacobi equation

$$V_x(x)f(x) - \frac{1}{2}V_x(x)g(x)g^T(x)V_x^T(x) + \frac{1}{2}h^T(x)h(x) = 0 \quad (9.8)$$

$$W_x(x)f(x) + \frac{1}{2}W_x(x)g(x)g^T(x)W_x^T(x) - \frac{1}{2}h^T(x)h(x) = 0 \quad (9.9)$$

Suppose there exists a C^1 solution $W \geq 0$ to (9.9). Additionally, assume there exists an $n \times p$ matrix $k(x)$ satisfying

$$W_x(x)k(x) = h^T(x) \quad (9.10)$$

Then the system with “inputs” u and y and “outputs” z given by

$$K_\Sigma : \begin{cases} \dot{x} = [f(x) - k(x)h(x)] + g(x)u + k(x)y \\ z = y - h(x) \end{cases} \quad (9.11)$$

is a *stable kernel representation* of Σ . Indeed, by setting $z = 0$ in (9.11) we recover the original system Σ . Second, from (9.9), (9.10) it follows that

$$\begin{aligned} W_x(x)([f(x) - k(x)h(x)] + g(x)u + k(x)y) = \\ -\frac{1}{2}\|u - g^T(x)W_x^T(x)\|^2 - \frac{1}{2}\|z\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|y\|^2 \leq \\ -\frac{1}{2}\|z\|^2 + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|y\|^2 \end{aligned} \quad (9.12)$$

showing that K_Σ has L_2 -gain ≤ 1 from (u, y) to z . Moreover, from $z = y - h(x)$ it is easily seen that there does not exist $\gamma < 1$ such that the L_2 -gain of K_Σ is

$\leq \gamma$. Therefore, the L_2 -gain of K_Σ is actually *equal* to 1. We summarize this in the following proposition.

Proposition 9.1.2 *Assume there exist solutions $W \geq 0$ to (9.9) and $k(x)$ to (9.10), then K_Σ given by (9.11) is a stable kernel representation of Σ which has L_2 -gain equal to 1.*

Remark 9.1.3 Equation (9.9) is the Hamilton–Jacobi–Bellman equation corresponding to the “optimal control problem in reversed time” or the “optimal filtering problem”

$$\min_u \int_{-\infty}^0 (||u(t)||^2 + ||y(t)||^2) dt \quad , \quad x(-\infty) = 0, \quad x(0) = x, \quad (9.13)$$

and W is the corresponding value function. Conditions for local solvability of (9.9) will be derived in Chap. 11 (see Remark 11.2.4).

Remark 9.1.4 Suppose W has a minimum (or, more generally, a stationary point) at some x_0 with $h(x_0) = 0$. For simplicity set $x_0 = 0$. Then $W_x(0) = 0$ and $h(0) = 0$. It follows by standard arguments (see, e.g., [136, 233]) that we may write, locally about 0,

$$W_x(x) = x^T M(x) \quad , \quad h(x) = C(x)x \quad (9.14)$$

for certain matrices $M(x)$, $C(x)$, with entries smoothly depending on x . If $M(x)$ is invertible then the unique solution $k(x)$ to (9.10) is given as

$$k(x) = M^{-1}(x)C^T(x) \quad (9.15)$$

In Chap. 11, Remark 11.2.4, it will be shown that under minimality assumptions on the system linearized at $x = 0$, the matrix $M(0)$, and thus $M(x)$ for x near 0, is always invertible.

Remark 9.1.5 Substitution of $u = 0$, $y = 0$ in (9.12) yields

$$W_x(x)[f(x) - k(x)h(x)] \leq -\frac{1}{2}h^T(x)h(x) \quad (9.16)$$

Assume that x_0 is an equilibrium, i.e., $f(x_0) = 0$ and $h(x_0) = 0$, and that W has a strict local minimum at x_0 . It follows that the zero-output constrained dynamics of Σ (see Sect. 5.1) is at least stable around x_0 . This property can be regarded as a weak form of zero-state detectability (see Definition 3.2.15). Conversely, if Σ is zero-state detectable then (cf. Lemma 3.2.16) it follows that x_0 is an asymptotically stable equilibrium of $\dot{x} = f(x) - k(x)h(x)$.

Remark 9.1.6 For a linear system Σ , K_Σ equals the normalized left coprime factorization, see, e.g., [212].

Let us now move to *image* representations. In this case, we consider the Hamilton–Jacobi equation (9.8), and the basic proposition is as follows.

Proposition 9.1.7 *Assume there exists a C^1 solution $V \geq 0$ to (9.8), then the system*

$$\begin{aligned} \dot{x} &= [f(x) - g(x)g^T(x)V_x^T(x)] + g(x)s \\ I_\Sigma : \quad y &= h(x) \\ u &= s - g^T(x)V_x^T(x) \end{aligned} \quad (9.17)$$

with auxiliary variables s is a stable image representation of Σ , which has L_2 -gain (from s to (y, u)) equal to 1.

Proof Clearly I_Σ is an image representation. (Eliminate from the last equations the auxiliary variables as $s = u + g^T(x)V_x^T(x)$.) In view of (9.8)

$$\begin{aligned} V_x(x)(f(x) - g(x)g^T(x)V_x^T(x)) + V_x(x)g(x)s &= \\ -\frac{1}{2}V_x(x)g(x)g^T(x)V_x^T(x) - \frac{1}{2}h^T(x)h(x) + V_x(x)g(x)s &= \\ -\frac{1}{2}\|s - g^T(x)V_x^T(x)\|^2 + \frac{1}{2}\|s\|^2 - \frac{1}{2}\|y\|^2 &= \\ \frac{1}{2}\|s\|^2 - \frac{1}{2}\|u\|^2 - \frac{1}{2}\|y\|^2 \end{aligned} \quad (9.18)$$

implying that I_Σ has L_2 -gain = 1. $\square$

Remark 9.1.8 In fact I_Σ is *conservative* (see Definition 3.1.2) with regard to the supply rate $\frac{1}{2}\|s\|^2 - \frac{1}{2}\|u\|^2 - \frac{1}{2}\|y\|^2$.

Remark 9.1.9 (Compare with Remark 9.1.3 and (3.64)) Equation (9.8) is the Hamilton–Jacobi–Bellman equation for the optimal control problem

$$\min_u \int_0^\infty (\|u(t)\|^2 + \|y(t)\|^2) dt, \quad x(0) = x, \quad x(\infty) = 0, \quad (9.19)$$

and conditions for local solvability of (9.8) will be derived in Chap. 10, Sect. 10.2. Clearly, the existence of a solution V requires some kind of stabilizability of Σ .

Remark 9.1.10 For a linear system Σ , I_Σ equals the normalized right coprime factorization (cf. [212]).

Finally assume that C^1 solutions $V \geq 0$, $W \geq 0$ and k to, respectively, (9.8)–(9.10) all exist, and thus that both K_Σ and I_Σ given by (9.11) and (9.17) are well-defined. In this case, we note that a *right inverse* to K_Σ is given by

$$\begin{aligned} \dot{p} &= f(p) - g(p)g^T(p)V_p^T(p) + k(p)\xi \\ K_\Sigma^{-1} : \quad u &= -g^T(p)V_p^T(p) \\ y &= \xi + h(p) \end{aligned} \quad (9.20)$$

Indeed, if $p(0) = x(0)$, then the input–output map (from ξ to z) of $K_\Sigma \circ K_\Sigma^{-1}$ is the identity map. Furthermore, a *left inverse* to I_Σ is given by

$$I_\Sigma^{-1} : \begin{cases} \dot{p} = [f(p) - k(p)h(p)] + g(p)u + k(p)y \\ \xi = u + g^T(p)V_p^T(p) \end{cases} \quad (9.21)$$

since the input–output map of $I_\Sigma^{-1} \circ I_\Sigma$ (from s to ξ) for $p(0) = x(0)$ is again the identity map.

Remark 9.1.11 Note that I_Σ^{-1} is itself a *kernel* representation, and K_Σ^{-1} is itself an *image* representation of the *same* system

$$\tilde{\Sigma} : \begin{cases} \dot{p} = f(p) - g(p)g^T(p)V_p^T(p) + k(p)[y - h(p)] \\ u = -g^T(p)V_p^T(p) \end{cases} \quad (9.22)$$

The system $\tilde{\Sigma}$ is a “*certainty equivalence*” controller: in the *optimal state feedback* $u = -g^T(x)V_x^T(x)$ corresponding to the optimal control problem (9.19) the actual state x is replaced by its *estimate* p , with the dynamics of p being driven by the *observation error* $y - h(p)$. In fact, if Σ is a linear system then $\tilde{\Sigma}$ is exactly the optimal LQG controller (interpreted in a deterministic manner).

As a direct application of the preceding developments, let us consider the control figuration depicted in Fig. 9.1 (customarily called an *observer–controller* configuration), with Σ the plant system, having a stable image representation I_Σ with inverse I_Σ^{-1} given by (9.21). Furthermore, let M denote an input–state–output system with state variables z , defining input–output maps M_{z_0} for every initial state z_0 .

Taking $p(0) = x(0) = x_0$ we obtain the input–output relations

$$u = v - M_{z_0}(I_\Sigma^{-1}(I_\Sigma(s))) = v - M_{z_0}(s) \quad (9.23)$$

Fig. 9.1 Observer–controller configuration

Furthermore, since I_Σ is an image representation of Σ , the input signal u is generated by (cf. (9.17))

$$\begin{aligned}\dot{x} &= [f(x) - g(x)g^T(x)V_x^T(x)] + g(x)s, \quad x(0) = x_0 \\ u &= s - g^T(x)V_x^T(x)\end{aligned}\tag{9.24}$$

Denoting the input–output map (from s to u) of (9.24) by D_{x_0} , we obtain from (9.23)

$$D_{x_0}(s) = v - M_{z_0}(s)\tag{9.25}$$

Hence, whenever M_{z_0} is such that the relation R_{vs} defined by (9.25) is L_2 -stable, we conclude that for $v \in L_2$ also s will be in L_2 , and thus also $u, y \in L_2$. Thus we have obtained a *parametrization* of the stabilizing controllers C in observer–controller configuration. More explicitly, all systems M with the property that $D_{x_0} + M_{z_0}$ has a L_2 -stable inverse define a stabilizing controller C .

9.2 L_2 -Gain Perturbation Models

As in the linear case [212], the stable kernel representation K_Σ and the stable image representation I_Σ both give rise to nonparametric *perturbation models* of Σ .

Recall that given a stable left factorization $G(s) = D^{-1}(s)N(s)$ a *perturbed* transfer matrix is $G_\Delta(s) := [D(s) + \Delta_D(s)]^{-1}[N(s) + \Delta_N(s)]$, with Δ_D, Δ_N stable transfer matrices. This corresponds to a *perturbed stable kernel representation*

$$\begin{aligned}0 &= [D(s) + \Delta_D(s) \dot{} - N(s) - \Delta_N(s)] \begin{bmatrix} y \\ u \end{bmatrix} \\ &= [D(s) \dot{} - N(s)] \begin{bmatrix} y \\ u \end{bmatrix} + [\Delta_D(s) \dot{} - \Delta_N(s)] \begin{bmatrix} y \\ u \end{bmatrix}\end{aligned}\tag{9.26}$$

Analogously, based on the stable kernel representation K_Σ of Σ given by (9.11) we consider the *perturbed stable kernel representation*

$$\begin{aligned}K_{\Sigma, \Delta} : \quad \dot{x} &= (f(x) - k(x)h(x)) + [g(x) \dot{} k(x)] \begin{bmatrix} u \\ y \end{bmatrix} \\ e_p &:= e + w = y - h(x) + w,\end{aligned}\tag{9.27}$$

where w is the output of a nonlinear system Δ with input $\begin{bmatrix} u \\ y \end{bmatrix}$ given as

$$\Delta : \begin{aligned} \dot{\xi} &= \alpha(\xi, u, y) \\ w &= \beta(\xi, u, y) \end{aligned} \quad (9.28)$$

having *finite* L_2 -gain. Setting $e_p = 0$ in (9.27) yields the *perturbed system*

$$\Sigma_\Delta : \begin{aligned} \dot{x} &= f(x) + g(x)u - k(x)w \\ y &= h(x) - w \end{aligned} \quad (9.29)$$

with w being the output of (9.28). Note that the *size* of the perturbation Δ is measured via its L_2 -gain.

Alternatively, recall that given a stable *right* factorization $G(s) = \tilde{N}(s)\tilde{D}^{-1}(s)$ the perturbed transfer matrix $G_{\tilde{\Delta}}(s)$ can be defined as $G_{\tilde{\Delta}}(s) := [\tilde{N}(s) + \Delta_{\tilde{N}}(s)][\tilde{D}(s) + \Delta_{\tilde{D}}(s)]^{-1}$, with $\Delta_{\tilde{N}}, \Delta_{\tilde{D}}$ stable transfer matrices. This corresponds to a *perturbed stable image representation*

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \tilde{N}(s) + \Delta_{\tilde{N}}(s) \\ \tilde{D}(s) + \Delta_{\tilde{D}}(s) \end{bmatrix} l = \begin{bmatrix} \tilde{N}(s) \\ \tilde{D}(s) \end{bmatrix} l + \begin{bmatrix} \Delta_{\tilde{N}}(s) \\ \Delta_{\tilde{D}}(s) \end{bmatrix} l \quad (9.30)$$

Analogously, based on the stable image representation I_Σ given in (9.17) we consider the *perturbed stable image representation*

$$I_{\Sigma, \Delta} : \begin{aligned} \dot{x} &= f(x) - g(x)g^T(x)V_x^T(x) + g(x)s \\ y &= h(x) + w_1 \\ u &= s - g^T(x)V_x^T(x) + w_2, \end{aligned} \quad (9.31)$$

where $w = (w_1, w_2)$ is the output of a nonlinear system $\tilde{\Delta}$ with input s given by

$$\tilde{\Delta} : \begin{aligned} \dot{\xi} &= \tilde{\alpha}(\xi, s) \\ w_1 &= \tilde{\beta}_1(\xi, s) \\ w_2 &= \tilde{\beta}_2(\xi, s) \end{aligned} \quad (9.32)$$

and having *finite* L_2 -gain (from s to w). Elimination of the auxiliary variables s in (9.31) leads to the *perturbed system*

$$\Sigma_{\tilde{\Delta}} : \begin{aligned} \dot{x} &= f(x) + g(x)u - g(x)w_2 \\ y &= h(x) + w_1 \end{aligned} \quad (9.33)$$

with $w = (w_1, w_2)$ the output of the system $\tilde{\Delta}$, with its input s replaced by

$$s = u + g^T(x)V_x^T(x) - w_2 \quad (9.34)$$

Note that again the *size* of the perturbation $\tilde{\Delta}$ is measured by its L_2 -gain (from s to w).

Example 9.2.1 Consider a port-Hamiltonian system (6.1) without dissipation ($R(x) = 0$), given as

$$\Sigma : \begin{aligned} \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) \end{aligned} \quad , \quad J(x) = -J^T(x), \quad (9.35)$$

with internal energy $H \geq 0$. Nonnegative solutions of the Hamilton–Jacobi–Bellman equations (9.8) and (9.9) are both given by $V = H$ and $W = H$. Note furthermore that the solution $k(x)$ to (9.10) is simply $k(x) = g(x)$. Thus based on the stable kernel representation we obtain the perturbed system

$$\Sigma_{\Delta} : \begin{aligned} \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)u - g(x)w \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) - w \end{aligned} \quad (9.36)$$

with w the output of Δ given by (9.28). On the other hand, the perturbed system based on the stable image representation is given as

$$\Sigma_{\tilde{\Delta}} : \begin{aligned} \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)u - g(x)w_2 \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) - w_1 \end{aligned} \quad (9.37)$$

with (w_1, w_2) the output of $\tilde{\Delta}$ in (9.32) driven by the signal $s = u + g^T(x) \frac{\partial H}{\partial x}(x) - w_2$.

9.3 Stable Kernel Representations and Parametrization of Stabilizing Controllers

Given a nonlinear *plant* system together with a *stabilizing controller* system, we shall derive in this section a parametrization of *all* stabilizing controllers. This parametrization generalizes the well-known Youla–Kucera parametrization in the linear case. Key element is the stable kernel representation of the plant and controller system. On the other hand, we need a strengthened notion of closed-loop stability, which in the nonlinear case may be different from the classical one given in Chap. 1 and also different from internal state space (Lyapunov) stability.

Consider a pair of nonlinear state space systems

$$\Sigma_i : \begin{aligned} \dot{x}_i &= f_i(x_i) + g_i(x_i)u_i, & x_i &\in \mathcal{X}_i, \quad u_i \in \mathbb{R}^{m_i} \\ y_i &= h_i(x_i), & y_i &\in \mathbb{R}^{p_i}, \quad i = 1, 2 \end{aligned} \quad (9.38)$$

Assume both systems admit stable kernel representations and suppose for simplicity of exposition that these stable kernel representations are of the affine form as in (9.11), i.e.,

Fig. 9.2 Closed-loop system $\{\Sigma_1, \Sigma_2\}$

$$K_{\Sigma_i} : \begin{cases} \dot{x}_i = [f_i(x_i) - k_i(x_i)h_i(x_i)] + g_i(x_i)u_i + k_i(x_i)y_i \\ z_i = y_i - h_i(x_i), \quad z_i \in \mathbb{R}^{p_i}, \quad i = 1, 2 \end{cases} \quad (9.39)$$

However, k_i need not necessarily be constructed as in (9.10), and thus K_{Σ_i} need not necessarily have L_2 -gain equal to one, but are only required to be L_2 -stable. In fact, the approach works equally well for general stable kernel representations as defined in Definition 9.1.1.

Let $m := m_1 = p_2$ and $p := p_1 = m_2$, and consider the *feedback interconnection* of Σ_1 and Σ_2 . Different from the previous chapters we throughout consider (for simplicity of notation and without loss of generality), the *positive feedback* interconnection. Furthermore, we will *not* consider external inputs e_1, e_2 . Hence we consider throughout the interconnection equations

$$u := u_1 = y_2, \quad y := y_1 = u_2 \quad (9.40)$$

The resulting (autonomous) closed system as given in Fig. 9.2 will be denoted by $\{\Sigma_1, \Sigma_2\}$. We will sometimes compare $\{\Sigma_1, \Sigma_2\}$ with the closed-loop system $\Sigma_1 \parallel_f \Sigma_2$ with external inputs e_1, e_2 as defined before (with the difference that we consider the *positive feedback* interconnection $u_1 = y_2, u_2 = y_1$).

A stable kernel representation of the closed-loop system $\{\Sigma_1, \Sigma_2\}$ (with outputs y and u , and without inputs) is obtained by substituting $u := u_1 = y_2, y := y_1 = u_2$ into (9.39) :

$$K_{\{\Sigma_1, \Sigma_2\}} : \begin{cases} \dot{x}_1 = [f_1(x_1) - k_1(x_1)h_1(x_1)] + g_1(x_1)u + k_1(x_1)y \\ \dot{x}_2 = [f_2(x_2) - k_2(x_2)h_2(x_2)] + g_2(x_2)y + k_2(x_2)u \\ z_1 = y - h_1(x_1) \\ z_2 = u - h_2(x_2) \end{cases} \quad (9.41)$$

Indeed, by setting $z_1 = 0, z_2 = 0$ one recovers the closed-loop system $\{\Sigma_1, \Sigma_2\}$ of Fig. 9.2. On the other hand we may invert in (9.41) the map from y, u to z_1, z_2 by solving y and u as $y = z_1 + h_1(x_1), u = z_2 + h_2(x_2)$ to obtain the *inverse system* of (9.41), denoted as $\{K_{\Sigma_1}, K_{\Sigma_2}\}$, given by

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)h_2(x_2) + g_1(x_1)z_2 + k_1(x_1)z_1 \\
\dot{x}_2 &= f_2(x_2) + g_2(x_2)h_1(x_1) + g_2(x_2)z_1 + k_2(x_2)z_2 \\
\{K_{\Sigma_1}, K_{\Sigma_2}\} : \quad y &= z_1 + h_1(x_1) \\
u &= z_2 + h_2(x_2)
\end{aligned} \tag{9.42}$$

Note that this inverse system can also be regarded as some kind of *closed-loop system* of Σ_1 and Σ_2 , with *external* signals z_1 and z_2 . Indeed, if $k_1 = 0$ and $k_2 = 0$ then (9.42) is exactly the closed-loop system $\Sigma_1 \parallel_f \Sigma_2$ of Fig. 4.1 (for the *positive feedback interconnection*) with $e_1 = z_2, e_2 = z_1$. However if $k_1 \neq 0$ or $k_2 \neq 0$ then $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ will be in general different from $\Sigma_1 \parallel_f \Sigma_2$.

Just as the definition of $\Sigma_1 \parallel_f \Sigma_2$ leads to a notion of closed-loop stability (cf. Definition 1.2.11), the definition of $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ leads to another notion of closed-loop stability:

Definition 9.3.1 The closed-loop system $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ of Σ_1, Σ_2 with stable kernel representations K_{Σ_1} and K_{Σ_2} given by (9.42), is called *strongly (L_2 -) stable* if for every pair of initial conditions $x_i(0) = x_{i0} \in \mathcal{X}_i, i = 1, 2$, and every pair of functions $z_1(\cdot) \in L_2(\mathbb{R}^p), z_2(\cdot) \in L_2(\mathbb{R}^m)$, the solutions $u(\cdot)$ and $y(\cdot)$ to (9.42) are in $L_2(\mathbb{R}^m)$, respectively, L_2^p . (That is, the system (9.42) with inputs z_1, z_2 and outputs y, u is L_2 -stable.) If the above property holds for initial conditions x_{i0} in a *subset* $\bar{\mathcal{X}}_i$ of $\mathcal{X}_i, i = 1, 2$, then $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ is called *strongly stable over* $\bar{\mathcal{X}}_1 \times \bar{\mathcal{X}}_2$.

Remark 9.3.2 It should be stressed that if $k_1 = 0, k_2 = 0$ in (9.42), then $\{K_{\Sigma_1}, K_{\Sigma_2}\} = \Sigma_1 \parallel_f \Sigma_2$ and strong L_2 -stability equals L_2 -stability of $\Sigma_1 \parallel_f \Sigma_2$. For instance, see (6.16), if Σ_1 and Σ_2 are themselves L_2 -stable, then k_1 and k_2 in (9.42) may be taken equal to zero.

A special case arises if one of the systems is the *zero-system* given by the zero input–output map

$$O : u \mapsto y = 0 \tag{9.43}$$

with empty state space $\mathcal{X} = \emptyset$. A stable kernel representation K_O of O is simply

$$K_O : (y, u) \mapsto z = y \tag{9.44}$$

Proposition 9.3.3 Consider a state space system Σ with stable kernel representation

$$\begin{aligned}
K_\Sigma : \quad \dot{x} &= [f(x) - k(x)h(x)] + g(x)u + k(x)y \\
z &= y - h(x)
\end{aligned} \tag{9.45}$$

Then $\{K_\Sigma, K_O\}$ is strongly L_2 -stable if and only if the system

$$\begin{aligned}
\dot{x} &= f(x) + g(x)z_2 + k(x)z_1 \\
w &= h(x)
\end{aligned} \tag{9.46}$$

is L_2 -stable from (z_1, z_2) to w .

Proof Consider (9.42) with $\Sigma_1 = \Sigma$ and $\Sigma_2 = O$. Then (9.42) reduces to

$$\begin{aligned}\dot{x} &= f(x) + g(x)z_2 + k(x)z_1 \\ y &= z_1 + h(x) \\ u &= z_2\end{aligned}\tag{9.47}$$

which is strongly L_2 -stable if and only if (9.46) is L_2 -stable, as follows immediately. $\square$

Corollary 9.3.4 *If $k = 0$ in K_Σ , then $\{K_\Sigma, K_O\}$ is strongly L_2 -stable if and only if Σ is L_2 -stable.*

Now we interpret Σ_1 as the *plant* and Σ_2 as the *controller*. We will say that Σ_2 is *strongly stabilizing* for Σ_1 if $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ is strongly stable.

Based on a strongly stabilizing Σ_2 we wish to parametrize *all* the strongly stabilizing controllers. The key idea is to consider the external signals z_1 and z_2 in (9.39) as input and output signals for *another* state space system

$$Q : \begin{aligned} \dot{\xi} &= \varphi(\xi) + \psi(\xi)z_1 \\ z_2 &= \theta(\xi) \end{aligned} \quad , \quad \xi \in \mathcal{X}_Q, \tag{9.48}$$

which we assume to be L_2 -stable, and thus having a stable kernel representation

$$K_Q : \begin{aligned} \dot{\xi} &= \varphi(\xi) + \psi(\xi)z_1 \\ \omega &= z_2 - \theta(\xi) \end{aligned} \tag{9.49}$$

with external signal ω . Substituting for z_1 and z_2 the expressions from the stable kernel representations K_{Σ_i} in (9.39) (with $u = u_1 = y_2$ and $y = y_1 = u_2$) one obtains

$$K_{\Sigma_2^Q} : \begin{aligned} \dot{\xi} &= \varphi(\xi) - \psi(\xi)h_1(x_1) + \psi(\xi)y \\ \dot{x}_1 &= f_1(x_1) - k_1(x_1)h_1(x_1) + g_1(x_1)u + k_1(x_1)y \\ \dot{x}_2 &= f_2(x_2) - k_2(x_2)h_2(x_2) + g_2(x_2)y + k_2(x_2)u \\ \omega &= u - h_2(x_2) - \theta(\xi) \end{aligned} \tag{9.50}$$

By setting $\omega = 0$, and solving for u , it follows that (9.50) is a stable kernel representation of the following input-state-output system (with input y and output u):

$$\Sigma_2^Q : \begin{aligned} \dot{\xi} &= \varphi(\xi) - \psi(\xi)h_1(x_1) + \psi(\xi)y \\ \dot{x}_1 &= f_1(x_1) - k_1(x_1)h_1(x_1) \\ &\quad + g_1(x_1)h_2(x_2) + g_1(x_1)\theta(\xi) + k_1(x_1)y \\ \dot{x}_2 &= f_2(x_2) - k_2(x_2)h_2(x_2) \\ &\quad + k_2(x_2)h_2(x_2) + k_2(x_2)\theta(\xi) + g_2(x_2)y \\ u &= \theta(\xi) + h_2(x_2) \end{aligned} \tag{9.51}$$

Note that the stable kernel representation $K_{\Sigma_2^Q}$ corresponds to Fig. 9.3.

Fig. 9.3 Stable kernel representation of Σ_2^Q **Fig. 9.4** Closed-loop system for Σ_2^Q

We interpret Σ_2^Q as a *perturbation* of the controller system Σ_2 . Note that the state space of the perturbed controller system Σ_2^Q , defined by Σ_1 and Q , equals $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_Q$.

Theorem 9.3.5 Suppose $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ is strongly L_2 -stable. Then for every L_2 -stable system Q the closed-loop system $\{K_{\Sigma_1}, K_{\Sigma_2^Q}\}$, with state space $\mathcal{X}_1 \times (\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_Q)$, is strongly L_2 -stable over $\text{diag}(\mathcal{X}_1 \times \mathcal{X}_1) \times \mathcal{X}_2 \times \mathcal{X}_Q$.

Proof Consider K_{Σ_1} (with $u_1 = u$, $y_1 = y$) and $K_{\Sigma_2^Q}$. By Definition 9.3.1 strong L_2 -stability follows if for all initial conditions in $\text{diag}(\mathcal{X}_1 \times \mathcal{X}_1) \times \mathcal{X}_2 \times \mathcal{X}_Q$, and all $z_1, \omega \in L_2$ the signals y and u are in L_2 . Thus let $z_1, \omega \in L_2$ and consider initial conditions in $\text{diag}(\mathcal{X}_1 \times \mathcal{X}_1) \times \mathcal{X}_2 \times \mathcal{X}_Q$. Since Q is L_2 -stable we obtain by Corollary 9.3.4 that z_1, z_2 are in L_2 . Since $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ is assumed to be strongly L_2 -stable, this implies that y and u are in L_2 . (Note that the state of the closed-loop system remains in $\text{diag}(\mathcal{X}_1 \times \mathcal{X}_1) \times \mathcal{X}_2 \times \mathcal{X}_Q$, and that $\tilde{z}_1 = z_1$ in Fig. 9.4.) $\square$

Loosely speaking, we may conclude that if Σ_2 is a strongly stabilizing controller of Σ_1 , then also Σ_2^Q will be a strongly stabilizing controller of Σ_1 for every stable system Q . Moreover, we obtain in this way *all* the strongly stabilizing controllers, in the following sense.

Theorem 9.3.6 Suppose $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ is strongly L_2 -stable. Consider a controller Σ^* different from Σ_2

$$\Sigma^* : \begin{aligned} \dot{x}^* &= f^*(x^*) + g^*(x^*)y \\ u &= h^*(x^*) \end{aligned} \quad x^* \in \mathcal{X}^* \quad (9.52)$$

with stable kernel representation

$$K_{\Sigma^*} : \begin{aligned} \dot{z}^* &= f^*(x^*) + g^*(x^*)y + k^*(x^*)u \\ z^* &= u - h^*(x^*) \end{aligned} \quad (9.53)$$

Suppose also $\{K_{\Sigma_1}, K_{\Sigma^*}\}$ is strongly L_2 -stable. Then define K_{Q^*} by composing K_{Σ^*} with $\{K_{\Sigma_1}, K_{\Sigma_2}\}$ given by (9.42), i.e.,

$$K_{Q^*} : \begin{aligned} \dot{x} &= f^*(x^*) + g^*(x^*)(z_1 + h_1(x_1)) + k^*(x^*)(z_2 + h_2(x_2)) \\ \dot{x}_1 &= f_1(x_1) + g_1(x_1)h_2(x_2) + g_1(x_1)z_2 + k_1(x_1)z_1 \\ \dot{x}_2 &= f_2(x_2) + g_2(x_2)h_1(x_1) + g_2(x_2)z_1 + k_2(x_2)z_2 \\ z^* &= z_2 + h_2(x_2) - h^*(x^*) \end{aligned} \quad (9.54)$$

This is a stable kernel representation (set $z^* = 0$ and solve for z_2) of the following system with input z_1 and output z_2 :

$$Q^* : \begin{aligned} \dot{x}^* &= f^*(x^*) + g^*(x^*)(z_1 + h_1(x_1)) + k^*(x^*)h^*(x^*) \\ \dot{x}_1 &= f_1(x_1) + g_1(x_1)h^*(x^*) + k_1(x_1)z_1 \\ \dot{x}_2 &= f_2(x_2) + g_2(x_2)h_1(x_1) + g_2(x_2)z_1 \\ &\quad + k_2(x_2)(h^*(x^*) - h_2(x_2)) \\ z_2 &= h^*(x^*) - h_2(x_2) \end{aligned} \quad (9.55)$$

Note that the state space of Q^* is $\mathcal{X}_{Q^*} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}^*$. Consider as in (9.51) the system $\Sigma_2^{Q^*}$ with state space $\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_{Q^*} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}^*$. Then the input–output map of $\Sigma_2^{Q^*}$ for initial condition $(x_{10}, x_{20}, x_{10}, x_{20}, x_0^*)$ equals the input–output map of Σ^* for initial condition x_0^* , and this holds for all $(x_{10}, x_{20}) \in \mathcal{X}_1 \times \mathcal{X}_2$ and all $x_0^* \in \mathcal{X}^*$.

Proof The input–output map of the kernel representation $K_{\Sigma_2^{Q^*}}$ for initial condition $(x_{10}, x_{20}, x_{10}, x_{20}, x_0^*)$ is given by

$$K_{\Sigma^*}^{x_0^*} \circ \left[K_{\{\Sigma_1, \Sigma_2\}}^{(x_{10}, x_{20})} \right]^{-1} \circ K_{\{\Sigma_1, \Sigma_2\}}^{(x_{10}, x_{20})} = K_{\Sigma^*}^{x_0^*}, \quad (9.56)$$

with the superscripts denoting the initial conditions for the respective input–output maps. $\square$

Remark 9.3.7 In the linear case one recovers the Youla–Kucera parametrization of all stabilizing controllers as follows. Take all initial conditions to be equal to zero. Let Σ_1 be given by the transfer matrix $P(s) = D^{-1}(s)N(s)$, and let Σ_2 be a stabilizing

controller given by $C(s) = X^{-1}(s)Y(s)$, where $D(s)$, $N(s)$, $X(s)$, and $Y(s)$ are stable rational matrices. Equivalently, the systems Σ_1 and Σ_2 are associated with the kernels of $[D(s) \dot{\vdash} -N(s)]$, respectively, $[Y(s) \dot{\vdash} -X(s)]$. Let $Q(s)$ be a stable rational matrix, corresponding to a stable input–output map $z_2 = Q(s)z_1$, or, equivalently, to the kernel of $[I \dot{\vdash} -Q(s)]$. It follows that the set of all linear stabilizing controllers is given by the kernels of

$$(I \dot{\vdash} -Q(s)) \begin{pmatrix} D(s) & -N(s) \\ -Y(s) & X(s) \end{pmatrix} = \begin{pmatrix} D(s) + Q(s)Y(s) & -N(s) - Q(s)X(s) \end{pmatrix} \quad (9.57)$$

or, equivalently, by the transfer matrices $(D(s) + Q(s)Y(s))^{-1}(N(s) + Q(s)X(s))$. Finally, let us consider as a special case of Theorem 9.3.5 the situation that Σ_1 is *itself* already L_2 -stable. Then, as noted before in (6.16), a stable kernel representation K_{Σ_1} of Σ_1 is given as

$$K_{\Sigma_1} : \begin{cases} \dot{x}_1 = f_1(x_1) + g_1(x_1)u_1 \\ z_1 = y_1 - h_1(x_1) \end{cases}$$

Furthermore, in this case the zero-controller $\Sigma_2 = 0$ with stable kernel representation K_O given by (9.44) yields a closed-loop system $\{K_{\Sigma}, K_O\}$ which is by Corollary 9.3.4 strongly L_2 -stable.

Now, consider any L_2 -stable system Q , given by (9.48) with stable kernel representation K_Q as in (9.49). It follows that the stabilizing controller Σ_2^Q is given as (cf. (9.51))

$$\Sigma_2^Q : \begin{cases} \dot{\xi} = \varphi(\xi) + \psi(\xi)(y_1 - h_1(x_1)) \\ u = \theta(\xi) \\ \dot{x}_1 = f_1(x_1) + g_1(x_1)u \end{cases} \quad (9.58)$$

Hence every stabilizing controller Σ_2^Q contains a *model* (or *copy*) of the plant, which can be regarded as a generalization of the concept of *Internal Model Control* (see, e.g., [222]) to the nonlinear setting.

9.4 All-Pass Factorizations

Throughout this section, we consider state space systems Σ satisfying

$$\Sigma : \begin{cases} \dot{x} = f(x, u), & f(0, 0) = 0, & u \in \mathbb{R}^m, & x \in \mathcal{X} \\ y = h(x, u), & h(0, 0) = 0, & y \in \mathbb{R}^p \end{cases} \quad (9.59)$$

Recall, cf. Definitions 3.1.6 and 8.1.1, that a system Σ is *inner* if it is conservative with respect to the L_2 -gain supply rate $s(u, y) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|y\|^2$, that is, there

Fig. 9.5 All-pass factorization $\Sigma = \Theta \circ \bar{\Sigma}$

exists $S : \mathcal{X} \rightarrow \mathbb{R}^+$ such that along every trajectory of (9.59)

$$S(x(t_1)) - S(x(t_0)) = \frac{1}{2} \int_{t_0}^{t_1} (\|u(t)\|^2 - \|y(t)\|^2) dt \quad (9.60)$$

The topic of this section is to *factorize* the nonlinear system Σ as a *series interconnection* $\Theta \circ \bar{\Sigma}$ of an inner system Θ preceded by another nonlinear system $\bar{\Sigma}$, in the sense that for every initial condition of Σ there should exist initial conditions of Θ and $\bar{\Sigma}$ such that the corresponding input–output map of Σ equals the composition of the respective input–output maps of $\bar{\Sigma}$ and Θ (see Fig. 9.5).

We will call this factorization an *all-pass factorization*.¹ An important motivation for this type of factorization is that in view of (9.60) the asymptotic properties of Σ and $\bar{\Sigma}$ are similar, while $\bar{\Sigma}$ may be simpler to control than Σ . In such a case, the control of Σ may be based on the control of $\bar{\Sigma}$, as we will see at the end of this section.

The all-pass factorizations of Σ are based on the following *reversed dissipation inequality* for Σ

$$V(x(t_1)) - V(x(t_0)) + \frac{1}{2} \int_{t_0}^{t_1} \|h(x(t), u(t))\|^2 dt \geq 0 \quad (9.61)$$

in the unknown $V \geq 0$, or its differential version (V assumed to be C^1)

$$V_x(x) f(x, u) + \frac{1}{2} \|h(x, u)\|^2 \geq 0, \quad \text{for all } x, u \quad (9.62)$$

Suppose there exists² $V \geq 0$ satisfying (9.62). Then consider the positive function

$$K_V(x, u) := V_x(x) f(x, u) + \frac{1}{2} \|h(x, u)\|^2 \quad (9.63)$$

¹Called “all-pass” since a single-input single-output linear system is inner if and only if its transfer function has amplitude 1 for all frequencies.

²Many of the subsequent developments continue to hold for any C^1 function V satisfying (9.62) (not necessarily ≥ 0). However, in this case the factor system Θ will not be inner anymore; but only cyclo-conservative with respect to the supply rate $\frac{1}{2} \|\bar{y}\|^2 - \frac{1}{2} \|y\|^2$.

Assumption 9.4.1 There exists a C^1 map $\bar{h} : \mathcal{X} \times \mathbb{R}^m \rightarrow \mathbb{R}^{\bar{p}}$ such that

$$K_V(x, u) = \frac{1}{2} \|\bar{h}(x, u)\|^2 \quad (9.64)$$

Sufficient conditions for the *local* existence of a smooth $\bar{h}$ satisfying (9.64) are provided by the next lemma which follows from an application of Morse's lemma; see the Notes at the end of this chapter for a sketch of the proof.

Lemma 9.4.2 Suppose that K_V is C^2 , and that its Hessian matrix, i.e.,

$$\begin{bmatrix} \frac{\partial^2 K_V}{\partial x^2}(x, u) & \frac{\partial^2 K_V}{\partial x \partial u}(x, u) \\ \frac{\partial^2 K_V}{\partial u \partial x}(x, u) & \frac{\partial^2 K_V}{\partial u^2}(x, u) \end{bmatrix} \quad (9.65)$$

has constant rank $\bar{p}$ on a neighborhood of $(x, u) = (0, 0)$. Then locally near $(0, 0)$ there exists a C^1 mapping $\bar{h} : \mathcal{X} \times \mathbb{R}^m \rightarrow \mathbb{R}^{\bar{p}}$ such that (9.64) holds.

Then define the *factor system*

$$\bar{\Sigma} : \begin{cases} \dot{\bar{x}} = f(\bar{x}, u), & \bar{x} \in \mathcal{X}, u \in \mathbb{R}^m \\ \bar{y} = \bar{h}(\bar{x}, u), & \bar{y} \in \mathbb{R}^{\bar{p}} \end{cases} \quad (9.66)$$

Obviously $\bar{h}(0, 0) = 0$, and thus $\bar{\Sigma}$ is also within the class (8.1).

Furthermore, define the system Θ with inputs $\bar{y}$ and outputs y given in image representation (cf. Definition 9.1.1) as

$$I_{\Theta} : \begin{cases} \dot{\xi} = f(\xi, u) \\ \bar{y} = \bar{h}(\xi, u) \\ y = h(\xi, u) \end{cases} \quad (9.67)$$

It immediately follows from (9.63) and (9.64) that Θ has the property

$$V(\xi(t_1)) - V(\xi(t_0)) = \frac{1}{2} \int_{t_0}^{t_1} (\|\bar{y}(t)\|^2 - \|y(t)\|^2) dt \quad (9.68)$$

along every solution $\xi(\cdot)$. Furthermore, if in (9.59) (9.66) (9.67), $x(t_0) = \bar{x}(t_0) = \xi(t_0)$, then it immediately follows that $x(t) = \bar{x}(t) = \xi(t)$, $t \geq t_0$, so that indeed $\Sigma = \Theta \circ \bar{\Sigma}$.

Now, if the equation $\bar{y} = \bar{h}(\xi, u)$ can be solved for u as $u = \alpha(\xi, \bar{y})$, then the image representation (9.67) can be reduced to a standard input-state-output system

$$\Theta : \begin{cases} \dot{\xi} = f(\xi, \alpha(\xi, \bar{y})) \\ y = h(\xi, \alpha(\xi, \bar{y})) \end{cases} \quad (9.69)$$

defining an *inner* system.

We obtain the following asymptotic relation between Σ and $\bar{\Sigma}$.

Definition 9.4.3 Define for every (x_e, u_e) such that $f(x_e, u_e) = 0$, the *steady-state output amplitude*³ as $\|h(x_e, u_e)\|$.

Proposition 9.4.4 Consider Σ and a factor system $\bar{\Sigma}$. The set of steady-state output amplitudes of Σ and $\bar{\Sigma}$ are equal.

Proof Consider (9.63) and (9.64), and substitute $f(x_e, u_e) = 0$. $\square$

All-pass factorizations of Σ have an immediate interpretation in terms of the Hamiltonian input–output system Σ^H associated to Σ (Definition 8.1.2) given by

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}(x, p, u) \\ \Sigma^H : \dot{p} &= -\frac{\partial H}{\partial x}(x, p, u) \\ y_a &= \frac{\partial H}{\partial u}(x, p, u), \end{aligned} \quad (9.70)$$

where

$$H(x, p, u) = p^T f(x, u) + \frac{1}{2} h^T(x, u) h(x, u) \quad (9.71)$$

Recall from Proposition 8.1.4 that the input–output map $G_{(0,0)}^H$ of Σ^H for initial state $(x, p) = (0, 0)$ is equal to $G_{(0,0)}^H(u) = (DG_0(u))^* \circ G_0(u)$, where $(DG_0(u))^*$ is the adjoint of the Fréchet derivative of the input–output map G_0 of Σ .

Now consider an all-pass factorization $\Sigma = \Theta \circ \bar{\Sigma}$ as above, with Θ inner. Denote the input–output map for zero initial condition of Σ by G_0 , of $\bar{\Sigma}$ by $\bar{G}_0$, and the input–output map of Θ for zero initial condition by A_0 . It follows that

$$\begin{aligned} (DG_0(u))^* \circ G_0(u) &= (D(A_0 \circ \bar{G}_0)(u))^* \circ (A_0 \circ \bar{G}_0)(u) \\ &= (D\bar{G}_0(u))^* (DA_0(\bar{G}_0(u)))^* \circ A_0 \circ \bar{G}_0(u) \\ &= (D\bar{G}_0(u))^* \circ \bar{G}_0(u) \end{aligned} \quad (9.72)$$

The last equality holds since A_0 is inner, and thus, in view of (1.24) in Proposition 1.2.9,

$$(DA_0(\bar{G}_0(u)))^* \circ A_0(\bar{G}_0(u)) = \bar{G}_0(u) \quad (9.73)$$

Hence the input–output map $G_{(0,0)}^H$ of the Hamiltonian input–output system Σ^H associated to Σ is *equal* to the input–output map $\bar{G}_{(0,0)}^H$ of the Hamiltonian input–output system $\bar{\Sigma}^H$ associated to $\bar{\Sigma}$.

This motivates the following state space level interpretation. Consider any C^1 solution V to the dissipation inequality (9.62), with corresponding factor system $\bar{\Sigma}$. Define the following change of coordinates for Σ^H :

³If (x_e, u_e) is an asymptotically stable steady state then $\|h(x_e, u_e)\|$ is the steady-state value of the norm of the output y for a step input with magnitude u_e .

$$\bar{x} = x, \quad \bar{p} = p - V_x^T(x) \quad (9.74)$$

In the new coordinates for $T^*\mathcal{X}$ the Hamiltonian H transforms into

$$\begin{aligned} \bar{H}(\bar{x}, \bar{p}, u) &= (\bar{p} + V_x^T(x))^T f(x, u) + \frac{1}{2} h^T(x, u) h(x, u) \\ &= \bar{p}^T f(x, u) + \frac{1}{2} \bar{h}^T(x, u) \bar{h}(x, u) \end{aligned} \quad (9.75)$$

Therefore, the Hamiltonian input–output system Σ^H given by (9.70) transforms into

$$\begin{aligned} \dot{\bar{x}} &= \frac{\partial \bar{H}}{\partial \bar{p}}(\bar{x}, \bar{p}, u) \\ \dot{\bar{p}} &= -\frac{\partial \bar{H}}{\partial \bar{x}}(\bar{x}, \bar{p}, u) \\ y_a &= \frac{\partial \bar{H}}{\partial u}(\bar{x}, \bar{p}, u) \end{aligned} \quad (9.76)$$

which is exactly the associated Hamiltonian input–output system $\bar{\Sigma}^H$. Thus the Hamiltonian input–output systems associated to Σ and $\bar{\Sigma}$ are *equal*, in accordance with the equality of their input–output maps discussed above. This is summarized in the following proposition.

Proposition 9.4.5 *Consider a system Σ as in (9.59) and a solution V to the reversed dissipation inequality (9.62), together with a resulting all-pass factorization $\Sigma = \Theta \circ \bar{\Sigma}$ as in Fig. 9.5. Then $\Sigma^H = \bar{\Sigma}^H$.*

Remark 9.4.6 In case of a linear system $\Sigma : \dot{x} = Ax + Bu, y = Cx + Du$ the transformation of Σ to $\bar{\Sigma}$ corresponds to a *spectral factorization* of the “spectral density” matrix $(D^T + B^T(-Is - A^T)^{-1}C^T)(C(Is - A)^{-1}B + D)$. In the same spirit, the transformation from a *nonlinear* system Σ to $\bar{\Sigma}$ can be interpreted as a “*nonlinear spectral factorization*” of the associated input–output Hamiltonian system Σ^H .

The reversed dissipation inequality (9.62) is intimately related to the *optimal control problem* of minimizing for every initial condition x_0 of Σ the cost-functional

$$J(x_0, u) = \int_0^\infty \|y(t)\|^2 dt \quad (9.77)$$

along the trajectories of Σ starting at $t = 0$ at x_0 . Define for every $x_0 \in \mathcal{X}$ the *value function* (see Sect. 11.2 for further information)

$$V^*(x_0) = \inf_u \{J(x_0, u) \mid u \text{ admissible, } \lim_{t \rightarrow \infty} x(t) = 0\}, \quad (9.78)$$

where $u : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is called *admissible* if the right-hand side of (9.77) is well-defined.

Assumption 9.4.7 $V^*(x_0)$ exists for every $x_0 \in \mathcal{X}$ (i.e., the system is stabilizable), and V^* is a C^1 function on $\mathcal{X}$.

Proposition 9.4.8 (i) V^* satisfies the reversed dissipation inequality (9.62), and $V^*(0) = 0$.

(ii) Let V satisfy (9.62) and $V(0) = 0$, then $V(x) \leq V^*(x)$ for every $x \in \mathcal{X}$.

Proof (i) Follows directly from the definition of V^* .

(ii) Consider (9.61) with $t_0 = 0$ and $t_1 = T$, and let u be such that $\lim_{T \rightarrow \infty} x(T) = 0$. Then it follows from (9.61) for $T \rightarrow \infty$ that

$$\frac{1}{2} \int_0^\infty \|y(t)\|^2 dt \geq V(x(0)),$$

and thus by definition of V^* we obtain $V^*(x(0)) \geq V(x(0))$. $\square$

Let us denote the factor system $\bar{\Sigma}$ obtained by considering V^* as $\bar{\Sigma}^*$, that is

$$\Sigma^* : \begin{cases} \dot{x} = f(x, u), & f(0, 0) = 0 \\ y^* = h^*(x, u), & h^*(0, 0) = 0 \end{cases} \quad (9.79)$$

with h^* satisfying

$$V_x^*(x) f(x, u) + \frac{1}{2} \|h(x, u)\|^2 = \frac{1}{2} \|h^*(x, u)\|^2 \quad (9.80)$$

The factor system $\bar{\Sigma}^*$ has special properties among the set of all factor systems $\bar{\Sigma}$. First, it is characterized by the following appealing property. Denote the inner system corresponding to $\bar{\Sigma}^*$ by Θ^* , given by the image representation (see (9.67))

$$I_{\Theta^*} : \begin{cases} \dot{\xi} = f(\xi, u) \\ y^* = h^*(\xi, u) \\ y = h(\xi, u) \end{cases} \quad (9.81)$$

Consider any other all-pass factorization $\Sigma = \Theta \circ \bar{\Sigma}$, with $\bar{\Sigma}$ and I_Θ as in (9.66), respectively (9.67). Corresponding to Θ there exists a function $V \geq 0$ satisfying (9.68), and corresponding to Θ^* there is the function V^* satisfying

$$V^*(\xi(t_1)) - V^*(\xi(t_0)) = \frac{1}{2} \int_{t_0}^{t_1} (\|y^*(t)\|^2 - \|y(t)\|^2) dt \quad (9.82)$$

Assume as before that V and V^* are C^1 , and without loss of generality let $V(0) = V^*(0) = 0$. Then it follows that V and V^* are both solutions to the reversed dissipation inequality (9.62), with V^* being the maximal solution (Proposition 9.4.8). Thus, subtracting (9.68) from (9.82), one obtains

$$(V^* - V)(\xi(t_1)) - (V^* - V)(\xi(t_0)) = \frac{1}{2} \int_{t_0}^{t_1} (\|y^*(t)\|^2 - \|\bar{y}(t)\|^2) dt \quad (9.83)$$

Therefore, if we take $t_0 = 0$, $t_1 = T$, $\xi(t_0) = 0$, then

$$\int_0^T \|y^*(t)\|^2 dt \geq \int_0^T \|\bar{y}(t)\|^2 dt, \text{ for all } T \geq 0 \quad (9.84)$$

In this sense, the all-pass factorization $\Sigma = \Theta^* \circ \bar{\Sigma}^*$ is the “*minimal delay factorization*,” since for zero initial conditions the truncated L_2 -norm of the output of the factor $\bar{\Sigma}^*$ is *maximal* among the outputs of all other factor systems $\bar{\Sigma}$; and this holds for *every* input function u .

Remark 9.4.9 In the linear case the above interpretation of $\bar{\Sigma}^*$ amounts to the classical definition of $\bar{\Sigma}^*$ being *minimum phase*.

A very much related characterization of $\bar{\Sigma}^*$ is by the *stability of its zero-output constrained dynamics*. Consider first the case that the optimal control problem of minimizing $J(x_0, u)$ given in (9.77) is *regular*, in the sense that there exists a unique solution $u = u^*(x, p)$ to

$$\frac{\partial}{\partial u} \left[p^T f(x, u) + \frac{1}{2} \|h(x, u)\|^2 \right] = 0, \quad (9.85)$$

which minimizes the expression $p^T f(x, u) + \frac{1}{2} \|h(x, u)\|^2$ as a function of u for every x, p . Then (by dynamic programming) it follows that V^* is the unique solution of the *Hamilton–Jacobi–Bellman* equation

$$V_x(x) f(x, u^*(x, V_x^T(x))) + \frac{1}{2} \|h(x, u^*(x, V_x^T(x)))\|^2 = 0, \quad (9.86)$$

such that $V(0) = 0$ and

$$\dot{x} = f(x, u^*(x, V_x^T(x))) \quad (9.87)$$

is globally asymptotic stable with regard to $x = 0$.

Proposition 9.4.10 *The zero-output constrained dynamics of $\bar{\Sigma}^*$ (see Sect. 5.1) is globally asymptotically stable with regard to the equilibrium $x = 0$.*

Proof Consider (9.80) and (9.86) for $V = V^*$. It follows from the regularity of the optimal control problem that $u = u^*(x, V_x^T(x))$ is the unique input such that $h^*(x, u) = 0$, as follows from (9.86) and (9.80). Thus the zero-output constrained dynamics of $\bar{\Sigma}^*$ is given by the asymptotically stable dynamics (9.87). $\square$

Remark 9.4.11 In the case of an *affine* nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \quad f(0) = 0, \quad u \in \mathbb{R}^m, \quad x \in \mathcal{X} \\ y &= h(x) + d(x)u, \quad h(0) = 0, \quad y \in \mathbb{R}^p \end{aligned} \quad (9.88)$$

the optimal control problem is regular if and only if the $m \times m$ matrix $E(x) := d^T(x)d(x)$ is invertible for all x . Furthermore, in this case the Hamilton–Jacobi–Bellman equation (9.86) reduces to

$$V_x(x)[f(x) - g(x)E^{-1}(x)d^T(x)h(x)] - \frac{1}{2}V_x(x)g(x)E^{-1}(x)g^T(x)V_x^T(x) + \frac{1}{2}h^T(x)[I_p - d(x)E^{-1}(x)d^T(x)]h(x) = 0 \quad (9.89)$$

Furthermore, the map $h^*(x, u)$ is explicitly given as

$$h^*(x, u) = \bar{d}(x)E^{-1}(x)[d^T(x)h(x) + g^T(x)V_x^{*T}(x)] + \bar{d}(x)u, \quad (9.90)$$

where V^* is the unique solution to (9.89) such that

$$\dot{x} = f(x) - g(x)g^T(x)V_x^{*T}(x) \quad (9.91)$$

is asymptotically stable with respect to $x = 0$, and where $\bar{d}(x)$ is any $m \times m$ matrix such that

$$d^T(x)d(x) = \bar{d}^T(x)\bar{d}(x) \quad (9.92)$$

(for example, $\bar{d}(x) = d(x)$).

In case the optimal control problem is *not* regular we proceed as follows. The *zero-output constrained dynamics* (or output-nulling dynamics) for a general system Σ given by (9.59) is defined to be the set of all state trajectories $x(\cdot)$ generated by some input trajectory $u(\cdot)$ such that $y(t) = h(x(t), u(t))$ is identically zero. Under some regularity conditions (see [135, 233]) the zero-output constrained dynamics can be computed as follows. First, we compute the maximal *controlled invariant* output-nulling submanifold $\mathcal{N}^* \subset \mathcal{X}$ (if it exists) as the maximal submanifold $\mathcal{N} \subset \mathcal{X}$, for which there exists a smooth feedback $u = \alpha(x)$, $\alpha(0) = 0$, such that

$$f(x, \alpha(x)) \in T_x\mathcal{N}, \quad h(x, \alpha(x)) = 0, \quad \text{for all } x \in \mathcal{N} \quad (9.93)$$

In general such a smooth feedback $\alpha(x)$ is not unique (even not restricted to $\mathcal{N}$), and (again under some regularity conditions; see [233, Chap. 11]) the whole family of feedbacks $u = \alpha(x)$ satisfying (9.93) for $\mathcal{N} = \mathcal{N}^*$ can be parametrized as $\alpha(x, v)$, with $v \in \mathbb{R}^{\tilde{m}}$, $\tilde{m} \leq m$. Then the zero-output constrained dynamics is generated by the lower dimensional dynamics

$$\tilde{\Sigma}: \quad \dot{\tilde{x}} = \tilde{f}(\tilde{x}, v), \quad \tilde{x} \in \mathcal{N}^*, \quad \tilde{f}(0, 0) = 0, \quad (9.94)$$

where $\tilde{f}(\tilde{x}, v) := f(\tilde{x}, \alpha(\tilde{x}, v))$, and $v \in \mathbb{R}^{\tilde{m}}$ is a remaining input vector.

Definition 9.4.12 Σ in (9.59) is *weakly minimum phase* if its zero-output constrained dynamics $\tilde{\Sigma}$ on $\mathcal{N}^*$ given (9.94) can be rendered *Lyapunov stable* with regard to $x = 0$ by a smooth feedback $v = \tilde{\alpha}(\tilde{x})$. If the zero-output constrained dynamics $\tilde{\Sigma}$

can be rendered locally *asymptotically stable* with respect to $x = 0$, then Σ is called *minimum phase*.

Remark 9.4.13 For linear systems Σ , minimum phase is equivalent to the requirement that the *transmission zeros* of Σ are all in the open left half plane.

Now return to the singular optimal control problem (9.78). Recall Assumption 9.4.7, and consider the factorization (9.80), with the resulting system $\bar{\Sigma}^*$ given by (9.79). For this system, we can again consider the singular optimal control problem (9.77), but now with y replaced by the new output y^* . Equivalently, we can consider the reversed dissipation inequality (9.62) with h replaced by h^* .

Lemma 9.4.14 *The maximal solution to the reversed dissipation inequality (9.62) with $h(x, u)$ replaced by $h^*(x, u)$ is the zero function.*

Proof Clearly $\bar{V} = 0$ satisfies (9.62) with $h(x, u)$ replaced by $h^*(x, u)$. Let now $\bar{V}$ be an arbitrary solution to (9.62) with $h(x, u)$ replaced by $h^*(x, u)$. Then it is immediately verified that $\bar{V} + \bar{V}^*$ is a solution to the original dissipation inequality (9.62) for $h(x, u)$. Since $\bar{V}^*$ is maximal this implies that $\bar{V} \leq 0$. $\square$

An equivalent formulation of Lemma 9.4.14 is the statement that

$$P(x_0) := \inf_{u(\cdot)} \left\{ \frac{1}{2} \int_0^\infty \|y^*(t)\|^2 dt \mid u \text{ admissible s.t. } \lim_{t \rightarrow \infty} x(t) = 0 \right\} \quad (9.95)$$

is zero for all $x_0 \in \mathcal{X}$.

Note that if we would be allowed to replace in (9.95) “inf” by “min” for all $x_0 \in \mathcal{N}^*$ then we could directly conclude that $\bar{\Sigma}^*$ is minimum phase, since in this case there exists an admissible control u such that $\|y^*(t)\| = 0$, $t \geq 0$, and $\lim_{t \rightarrow \infty} x(t) = 0$. Thus the remaining question is how in general $P(x_0) = 0$ for all x_0 implies that $\bar{\Sigma}^*$ is minimum phase.

To answer this question, we take recourse to the *linearization* of $\bar{\Sigma}^*$ and the linear theory for the singular LQ optimal control problem developed in [104, 311]. First, we consider for $\bar{\Sigma}^*$ the regularized cost criterion

$$\frac{1}{2} \int_0^\infty (\|y(t)\|^2 + \epsilon^2 \|u(t)\|^2) dt \quad (9.96)$$

for ϵ small. This is a *regular* optimal control problem, and thus the maximal solution P^ϵ of the regularized reversed dissipation inequality

$$P_x^\epsilon(x) f(x, u) + \frac{1}{2} \|h^*(x, u)\|^2 + \frac{1}{2} \epsilon^2 \|u\|^2 \geq 0 \quad (9.97)$$

is given as the stabilizing solution of the Hamilton–Jacobi–Bellman equation

$$H_{\text{opt}}^\epsilon(x, (P_x^\epsilon(x))^T) = 0, \quad (9.98)$$

Here $H_{\text{opt}}^\epsilon(x, p) := H^\epsilon(x, p, u^*(x, p))$, where $u^*(x, p)$ is the solution to $\frac{\partial H^\epsilon}{\partial u}(x, p, u) = 0$, and

$$H^\epsilon(x, p, u) = p^T f(x, u) + \frac{1}{2} \|h^*(x, u)\|^2 + \frac{1}{2} \epsilon^2 \|u\|^2 \quad (9.99)$$

Since all data $f(x, u), h^*(x, u)$ are *smooth* near 0, also the solution P^ϵ of the Hamilton–Jacobi–Bellmann equation (9.98) is a smooth function near $x = 0$ (cf. Chap. 11). It follows from the interpretation as an optimal control problem that $P^\epsilon \geq 0$, and that P^ϵ is nondecreasing as a function of ϵ . Hence the pointwise limit $P^{\text{lim}}(x) := \lim_{\epsilon \downarrow 0} P^\epsilon(x) \geq 0$ exists. We obtain the following theorem.

Theorem 9.4.15 *Assume that the zero-output constrained dynamics of $\bar{\Sigma}^*$ does not have invariant eigenvalues on the imaginary axis (that is, it is hyperbolic). Furthermore assume that P^{lim} is a C^1 function. Then $\bar{\Sigma}$ is minimum phase.*

Proof Suppose $\bar{\Sigma}$ is not minimum phase. By the assumption of hyperbolicity this means that the zero-output constrained dynamics is *exponentially unstable*. Thus for every smooth feedback $u = \alpha(x)$ such that the output $y^* = h^*(\tilde{x}, \alpha(\tilde{x}))$ is identically zero along the closed-loop dynamics $\dot{\tilde{x}} = f(\tilde{x}, \alpha(\tilde{x}))$, $\tilde{x} \in \mathcal{N}^*$, this closed-loop dynamics is exponentially unstable. Linearizing at $\tilde{x} = 0$, this yields that for any such α , $\exists \lambda \in \mathbb{C}$, $\text{Re } \lambda > 0$, and $\exists v \in \mathbb{C}^n$, $v \neq 0$, such that

$$\begin{aligned} \left[\frac{\partial f}{\partial x}(0, 0) + \frac{\partial f}{\partial u}(0, 0) \frac{\partial \alpha}{\partial x}(0) \right] v &= \lambda v \\ \left[\frac{\partial h^*}{\partial x}(0, 0) + \frac{\partial h^*}{\partial u}(0, 0) \frac{\partial \alpha}{\partial x}(0) \right] v &= 0 \end{aligned} \quad (9.100)$$

However, this implies that the *linear* system

$$\begin{aligned} \dot{x} &= \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial u}(0, 0)u =: Ax + Bu \\ \bar{y} &= \frac{\partial h^*}{\partial x}(0, 0)x + \frac{\partial h^*}{\partial u}(0, 0)u =: Cx + Du \end{aligned} \quad (9.101)$$

is not weakly minimum phase (transmission zeros in the open right half plane). However by *linear* theory [104, 311] this means that there exists a solution $X = X^T \geq 0$, with $X \neq 0$, to the quadratic inequality corresponding to (9.97), i.e.,

$$x^T X (Ax + Bu) + \frac{1}{2} (Cx + Du)^T (Cx + Du) \geq 0 \quad (9.102)$$

By continuity the regularized dissipation inequality (9.97) converges for $\epsilon \downarrow 0$ to

$$P_x^{\text{lim}}(x) f(x, u) + \frac{1}{2} \|h^*(x, u)\|^2 \geq 0 \quad (9.103)$$

and hence $P^{\text{lim}} = P = 0$. On the other hand, by linearization at $x = 0$

Fig. 9.6 Controller $C_{\bar{\Sigma}}$ based on factor system $\bar{\Sigma}$

$$\frac{\partial^2 P^\epsilon}{\partial x^2}(0) \geq X \neq 0 \quad (9.104)$$

for all $\epsilon > 0$, which yields a contradiction. $\square$

Finally, we will discuss the use of the all-pass factorization $\Sigma = \Theta^* \circ \bar{\Sigma}^*$ for *control*, as already alluded to before. Throughout let us assume that the nonlinear system Σ given by (9.59) is *asymptotically stable*⁴ with respect to $x = 0$.

As noted before in Proposition 9.4.4, a useful property of $\bar{\Sigma}^*$ as related to the original system Σ is that $\bar{\Sigma}^*$ and Σ have the same *steady-state output amplitudes*. This has the important consequence that the norm of the steady-state step responses of Σ and $\bar{\Sigma}$ are equal for every constant u . This suggests that output set-point control of Σ may be based on the factor system $\bar{\Sigma}$. Furthermore, since $\bar{\Sigma}^*$ is minimum phase, *inversion* techniques for its control can be used (e.g., by means of nonlinear input–output decoupling [135, 233]). A main open question is then how, once a suitable controller for the factor system $\bar{\Sigma}^*$ has been obtained, this can be translated to a suitable controller for the *original* system Σ . This problem will be solved by a nonlinear version of the structure of the classical *Smith predictor*.⁵ In fact, this allows to convert a controller $C_{\bar{\Sigma}}$ for *any* factor system $\bar{\Sigma}$ (not necessarily $\bar{\Sigma}^*$) to a controller C for the original system Σ .

Thus consider the plant system Σ and an arbitrary factor system $\bar{\Sigma}$. Suppose that we have constructed for $\bar{\Sigma}$ a controller system $C_{\bar{\Sigma}}$, resulting in the closed-loop system as depicted in Fig. 9.6. How do we derive from this configuration a controller for the original system Σ ? We use the following argument stemming from the derivation of the classical Smith predictor for linear systems. We add to the configuration of Fig. 9.6 two additional signal flows which exactly compensate each other, leading to Fig. 9.7. Subsequently, we shift the signal flow of y to the left-hand side of the block diagram, in order to obtain Fig. 9.8. The system within the dotted lines is now seen to be a controller for the original system Σ .

Note that the above construction based on the nonlinear Smith predictor may lead to problems in case Σ is *not* a stable system. Indeed, in this case the transition from

⁴In this case $\bar{\Sigma}^*$ is called an *outer system*: asymptotically stable with asymptotic stable zero-output constrained dynamics.

⁵I thank Gjerrit Meinsma for an illuminating discussion on the derivation of the Smith predictor.

Fig. 9.7 Addition of compensating signal flows

Fig. 9.8 Resulting controller for Σ

Figs. 9.6 to 9.7 may lead to diverging signal behavior, in the sense that although in the absence of disturbances the two added signal flows exactly compensate each other, any disturbance or mismatch may cause instabilities.

9.5 Notes for Chapter 9

1. Sections 9.1 and 9.2 are largely based on Scherpen & van der Schaft [310], van der Schaft [279], as well as Paice & van der Schaft [248, 249]. For the parametrization of stabilizing controllers in observer–controller configuration we refer to Viswanadham & Vidyasagar [345], and Banos [27].
2. Section 9.3 is based on Paice & van der Schaft [247, 249]. In these papers the more general and symmetric result is obtained of parametrizing all the *pairs* of state space systems Σ_1 and Σ_2 which are in stable closed-loop configuration.
3. With regard to the Youla–Kucera parametrization of stabilizing linear controllers of a linear system we refer, e.g., to Francis [103], and Green & Limebeer [117], and the references quoted therein. Apparently, the interpretation given in Remark 9.3.7, due to Paice & van der Schaft [247, 249], is new for linear systems.
4. Section 9.4 is based on van der Schaft & Ball [289]; see also van der Schaft & Ball [288], Ball & van der Schaft [24], which in its turn was motivated by work of Ball and Helton on inner–outer factorization of nonlinear operators, see, e.g., [21], and the theory of linear inner–outer factorization, see, e.g., Francis [103]. The approach taken in van der Schaft & Ball [288] and Ball & van der Schaft [24] emphasizes the “nonlinear spectral factorization” point of view of Proposition 9.4.5 and Remark 9.4.6.
5. The interpretation of the minimum phase factor as a “minimal delay factor,” see Remark 9.4.9, can be found in Robinson [260]; see also Anderson [6].

6. The Morse Lemma (see, e.g., Abraham & Marsden [1]) can be stated as follows. Consider a C^2 function $H : \mathcal{X} \rightarrow \mathbb{R}$, with $H_x(0) = 0$ and $n \times n$ Hessian matrix $H_{xx}(0)$ which is supposed to be *nonsingular*. Then locally near 0 there exist coordinates $z_1, \dots, z_n$ such that

$$H(z) = H(0) + \frac{1}{2}z_1^2 + \dots + \frac{1}{2}z_k^2 - \frac{1}{2}z_{k+1}^2 - \dots - \frac{1}{2}z_n^2, \quad (9.105)$$

with k the number of positive eigenvalues of $H_{xx}(0)$. The following slight generalization can be easily proven (see, e.g., Maas [188]):

Consider a C^2 function $H : \mathbb{R}^n \rightarrow \mathbb{R}$, with $H_x(0) = 0$, and $H_{xx}(x)$ of *constant* rank $\bar{n}$ in a neighborhood of $x = 0$. Then locally near 0 there exist coordinates $z_1, \dots, z_k, z_{k+1}, \dots, z_n$ such that

$$H(z) = H(0) + \frac{1}{2}z_1^2 + \dots + \frac{1}{2}z_k^2 - \frac{1}{2}z_{k+1}^2 - \dots - \frac{1}{2}z_{\bar{n}}^2 \quad (9.106)$$

with, as above, k the number of positive eigenvalues of $H_{xx}(0)$. The application of this result to the factorization in (9.64) is done by considering the function K_V , and assuming that its partial derivatives with regard to x and u are zero at $(x, u) = (0, 0)$, and that its Hessian matrix with regard to x and u has constant rank. Since $K_V \geq 0$ and $K_V(0) = 0$ it then follows that there exists new coordinates z for $\mathcal{X} \times \mathbb{R}^m$ with $z_i = \bar{h}_i(x, u)$, $i = 1, \dots, \bar{p}$, where $\bar{p}$ the rank of the Hessian, such that locally (9.64) holds.

7. The problem of parametrizing the class of stabilizing controllers for a nonlinear plant is also addressed, using related approaches, in Lu [184], Imura & Yoshikawa [133], Fujimoto & Sugie [109, 110].
8. For studies of kernel representations in the context of closed-loop identification of nonlinear systems we refer to Linard, Anderson & De Bruyne [181] and Fujimoto, Anderson & De Bruyne [107].
9. The approach to all-pass factorization in Sect. 9.4 is largely based on Ball, Petersen & van der Schaft [25]; see also Ball & van der Schaft [24] for the regular case.
10. The idea of basing the control of a stable but nonminimum phase system on its minimum phase factor is well known in linear process control, see, e.g., Morari & Zafiriou [222], and Green & Limebeer [117]. The extension of this idea to nonlinear systems has been discussed in Doyle, Allgöwer & Morari [87], and Wright & Kravaris [357]. For a description of the Smith predictor in the linear case see, e.g., Morari & Zafiriou [222] and Meinsma & Zwart [216]. In this last reference also the treatment of the modified Smith predictor in case the linear plant is *not* asymptotically stable can be found.

11. For a study of nonlinear controllability and observability Gramians, their relation to the associated Hamiltonian system Σ^H , and their use for nonlinear Hankel singular values and balancing of nonlinear systems, we refer to, e.g., Scherpen [308], Scherpen & van der Schaft [310], Fujimoto, Scherpen & Gray [108].