L₂-Gain and the Small-Gain Theorem
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Chapter 8

$L_2$-Gain and the Small-Gain Theorem

In this chapter, we elaborate on the characterization of finite $L_2$-gain for state space systems, continuing on the general theory of dissipative systems in Chap. 3. Within this framework we revisit the Small-gain theorem and its implications for robustness (Sect. 8.2), and extend the small-gain condition to network systems (Sect. 8.3). Furthermore, we provide an alternative characterization of $L_2$-gain in terms of response to periodic input functions (Sect. 8.4), and in Sect. 8.5 we end by sketching the close relationships to the theory of (integral-)input-to-state stability.

8.1 $L_2$-Gain of State Space Systems

Recall from Chap. 3 the following basic definitions regarding $L_2$-gain of a general state space system $\Sigma$

$$\begin{align*}
\dot{x} &= f(x, u), \quad x \in \mathcal{X}, \quad u \in \mathbb{R}^m \\
y &= h(x, u), \quad y \in \mathbb{R}^p
\end{align*}$$

(8.1)

with $n$-dimensional state space manifold $\mathcal{X}$.

**Definition 8.1.1** A state space system $\Sigma$ given by (8.1) has $L_2$-gain $\leq \gamma$ if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2} \gamma^2 ||u||^2 - \frac{1}{2} ||y||^2$; that is, there exists a storage function $S : \mathcal{X} \rightarrow \mathbb{R}^+$ such that

$$S(x(t_1)) - S(x(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 ||u(t)||^2 - ||y(t)||^2) dt$$

(8.2)

along all input functions $u(\cdot)$ and resulting state trajectories $x(\cdot)$ and output functions $y(\cdot)$, for all $t_0 \leq t_1$. Equivalently, if $S$ is $C^1$
\[ S_x(x) f(x, u) \leq \frac{1}{2} \gamma^2 ||u||^2 - \frac{1}{2} ||h(x, u)||^2, \quad \forall x, u \] (8.3)

The \(L_2\)-gain of \(\Sigma\) is defined as \(\gamma(\Sigma) = \inf \{ \gamma \mid \Sigma \text{ has } L_2\text{-gain} \leq \gamma \}\). \(\Sigma\) is said to have \(L_2\)-gain \(< \gamma\) if there exists \(\tilde{\gamma} < \gamma\) such that \(\Sigma\) has \(L_2\)-gain \(\leq \tilde{\gamma}\). Finally, \(\Sigma\) is called \(inner\) if it is conservative with respect to \(s(x, y) = \frac{1}{2} ||u||^2 - \frac{1}{2} ||y||^2\).

A more explicit form of the differential dissipation inequality (8.3) can be obtained for systems that are \(affine\) in the input \(u\) and \(without\ feedthrough\ term\)

\[ \dot{x} = f(x) + g(x)u \]

with \(g(x)\) an \(n \times m\) matrix. In this case, the differential dissipation inequality (8.3) for \(\Sigma_a\) amounts to

\[ S_x(x) [f(x) + g(x)u] - \frac{1}{2} \gamma^2 ||u||^2 + \frac{1}{2} ||h(x)||^2 \leq 0, \quad \forall x, u \] (8.5)

This can be simplified by computing the maximizing \(u^*\) (as a function of \(x\)) for the left-hand side, i.e.,

\[ u^* = \frac{1}{\gamma^2} g^T(x) S_x^T(x), \] (8.6)

and substituting (8.6) into (8.5) to obtain the \(Hamilton–Jacobi\ inequality\)

\[ S_x(x) f(x) + \frac{1}{2} \frac{1}{\gamma^2} S_x(x) g(x) g^T(x) S_x^T(x) + \frac{1}{2} h^T(x) h(x) \leq 0, \quad \forall x \in \mathcal{X} \] (8.7)

Thus, \(\Sigma_a\) has \(L_2\)-gain \(\leq \gamma\) with a \(C^1\) storage function if and only if there exists a \(C^1\) solution \(S \succeq 0\) to (8.7). Furthermore, it follows from the theory of \(dynamic\ programming\) that if the available storage \(S_a\) and required supply \(S_r\) (assuming existence) are \(C^1\), they are actually solutions of the \(Hamilton–Jacobi\ (-Bellman)\ equality\)

\[ S_x(x) f(x) + \frac{1}{2} \frac{1}{\gamma^2} S_x(x) g(x) g^T(x) S_x^T(x) + \frac{1}{2} h^T(x) h(x) = 0 \] (8.8)

More information on the structure of the solution set of the \(Hamilton–Jacobi\ inequality\ (8.7) and equality (8.8) will be provided in Chap. 11.

An alternative view on \(L_2\)-gain, directly relating to Proposition 1.2.9 in Chap. 1, is provided by the following definition and proposition.

**Definition 8.1.2** Given the state space system \(\Sigma\) given by (8.1) define the \(associated\ Hamiltonian\ input–output system\ \(\Sigma^H\) as
8.1 $L_2$-Gain of State Space Systems

\[ \dot{x} = \frac{\partial H}{\partial p}(x, p, u) \]
\[ \Sigma^H : \dot{p} = -\frac{\partial H}{\partial x}(x, p, u) \]
\[ y_a = \frac{\partial H}{\partial u}(x, p, u) \]

with state $(x, p)$, inputs $u$ and outputs $y_a$, where the Hamiltonian $H(x, p, u)$ is defined as

\[ H(x, p, u) := p^T f(x, u) + \frac{1}{2} h^T(x, u)h(x, u) \]

(8.10)

Here $p \in T^*_x X \simeq \mathbb{R}^n$ denotes the co-state vector.

Remark 8.1.3 Geometrically, the state space of $\Sigma^H$ is given by the cotangent bundle $T^*_x X$, with $p \in T^*_x X$. See [24, 73] for further information. In case of a linear system, $\dot{x} = Ax + Bu$, $y = Cx + Du$ with transfer matrix $K(s) = C(sI - A)^{-1}B + D$, the transfer matrix of the associated input–output Hamiltonian system is given as $K^T(-s)K(s)$.

Proposition 8.1.4 Consider $\Sigma$ satisfying $f(0, 0) = 0$, $h(0, 0) = 0$, with input–output map $G_0 : L_2^1(\mathbb{R}^m) \to L_2^2(\mathbb{R}^p)$, which is assumed to be $L_2$-stable. Then the input–output map $G^H_{0,0}$ of $\Sigma^H$ for initial state $(x, p) = (0, 0)$ is equal to the composed map

\[ G^H_{0,0}(u) = (DG_0(u))^* \circ G_0(u) \]

(8.11)

with $DG_0(u)$ the Fréchet derivative of the map $G_0$. In particular, it follows that the input–output map $G_0$ is inner if and only if $G^H_{0,0}$ is the identity-map.

Proof Follows directly from the definition of the Fréchet derivative $DG_0(u)$ of $G_0$ at $u$. The last statement follows from Proposition 1.2.9 in Chap. 1. \[ \square \]

8.2 The Small-Gain Theorem Revisited

In this section, we provide a state space interpretation of the Small-gain Theorem 2.1.1 for input–output maps in Chap. 2.

Let us consider, as in Chap. 4, the standard feedback configuration $\Sigma_1 \| f \Sigma_2$ of two input-state-output systems

\[ \Sigma_i : \dot{x}_i = f_i(x_i, u_i), \ x_i \in X_i, \ u_i \in U_i \]
\[ y_i = h_i(x_i, u_i), \ y_i \in Y_i \]

\[ i = 1, 2 \]

(8.12)

with $U_1 = Y_2$, $U_2 = Y_1$, cf. Fig. 4.1. Suppose $\Sigma_1$ and $\Sigma_2$ in (8.12) have $L_2$-gain $\leq \gamma_1$, respectively $\leq \gamma_2$. Denote the storage functions of $\Sigma_1$, $\Sigma_2$ by $S_1$, $S_2$, with corresponding dissipation inequalities.
Consider now the feedback interconnection with \( e_1 = e_2 = 0 \)

\[
\begin{align*}
S_1(x_1(t_1)) - S_1(x_1(t_0)) & \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma_1^2 ||u_1(t)||^2 - ||y_1(t)||^2)dt \\
S_2(x_2(t_1)) - S_2(x_2(t_0)) & \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma_2^2 ||u_2(t)||^2 - ||y_2(t)||^2)dt
\end{align*}
\]  

(8.13)

Substitute (8.14) into (8.13), multiply the second inequality of (8.13) by \( \alpha^2 \), and add both resulting inequalities, to obtain

\[
S(x_1(t_1), x_2(t_1)) - S(x_1(t_0), x_2(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\alpha^2 \gamma_2^2 - 1) ||y_1(t)||^2 + (\gamma_1^2 - \alpha^2) ||y_2(t)||^2)dt
\]  

(8.16)

where \( S(x_1, x_2) := S_1(x_1) + \alpha^2 S_2(x_2) \). Since \( \alpha \) satisfies (8.15) it immediately follows that

\[
S(x_1(t_1), x_2(t_1)) - S(x_1(t_0), x_2(t_0)) \leq -\varepsilon \int_{t_0}^{t_1} ||y_1(t)||^2 + ||y_2(t)||^2)dt
\]  

(8.17)

for a certain \( \varepsilon > 0 \). Thus \( S \) is a candidate Lyapunov function for the closed-loop system. In fact, we may immediately apply the reasoning of Lemma 3.2.16, resulting in

**Theorem 8.2.1** (Small-gain theorem; state space version) *Suppose \( \Sigma_1 \) and \( \Sigma_2 \) have \( L_2 \)-gain \( \leq \gamma_1 \) and \( \leq \gamma_2 \), with \( \gamma_1 \cdot \gamma_2 < 1 \). Suppose \( S_1, S_2 \geq 0 \) satisfying (8.13) are \( C^1 \) and have strict local minima at \( x_1^* = 0, x_2^* = 0 \), and \( \Sigma_1 \) and \( \Sigma_2 \) are zero-state detectable. Then \( x^* = (x_1^*, x_2^*) \) is an asymptotically stable equilibrium of the closed-loop system \( \Sigma_1 \| \Sigma_2 \) with \( e_1 = e_2 = 0 \), which is globally asymptotically stable if additionally \( S_1, S_2 \) have global minima at \( x_1^*, x_2^* \) and are proper.*

We leave the refinement of Theorem 8.2.1 to positive semidefinite \( S_1 \) and \( S_2 \) based on Theorem 3.2.19 to the reader (see also [312]). Instead we formulate the following version based on Proposition 3.2.22. For simplicity, assume that \( \Sigma_i, \ i = 1, 2, \) are affine systems

\[
\Sigma_{ai} : \begin{cases}
\dot{x}_i = f_i(x_i) + g_i(x_i)u_i \\
y_i = h_i(x_i)
\end{cases}
\]  

(8.18)

**Proposition 8.2.2** *Suppose the affine systems \( \Sigma_{a1} \) and \( \Sigma_{a2} \) as in (8.18) have \( L_2 \)-gain \( \leq \gamma_1 \) and \( \leq \gamma_2 \), with \( \gamma_1 \cdot \gamma_2 < 1 \). Suppose \( S_1, S_2 \geq 0 \) satisfying (4.52) are \( C^1 \) and*
8.2 The Small-Gain Theorem Revisited

\[ S_1(x_1^*) = S_2(x_2^*) = 0 \] (that is, \( S_1 \) and \( S_2 \) are positive semidefinite at \( x_1^* \), respectively, \( x_2^* \)). Furthermore, assume that \( x_i^* \) is an asymptotically stable equilibrium of \( \dot{x}_i = f_i(x_i) \), \( i = 1, 2 \). Then \((x_1^*, x_2^*)\) is an asymptotically stable equilibrium of the closed-loop system \( \Sigma_1 \| \Sigma_2 \) with \( e_1 = 0 \), \( e_2 = 0 \).

**Proof** The closed-loop system \( \Sigma_1 \| \Sigma_2 \) with \( e_1 = 0 \), \( e_2 = 0 \) can be written as

\[
\dot{x} = f(x) + g(x)k(x)
\]

with \( x = (x_1, x_2) \), and

\[
f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \quad k = \begin{bmatrix} -h_2 \\ h_1 \end{bmatrix}
\]

(8.19)

By (8.17)

\[
S_x(x_1, x_2) [f(x) + g(x)k(x)] \leq -\varepsilon ||k(x)||^2,
\]

(8.20)

while by assumption \((x_1^*, x_2^*)\) is an asymptotically stable equilibrium of \( \dot{x} = f(x) \). The statement now follows from Proposition 3.2.22.

**Remark 8.2.3** Contrary to the Small-gain Theorem 2.1.1 for input–output maps in Chap. 2 we can relax the small-gain condition \( \gamma_1 \cdot \gamma_2 < 1 \) to \( \gamma_1 \cdot \gamma_2 \leq 1 \), in the sense that for \( \gamma_1 \cdot \gamma_2 = 1 \) the inequalities (8.17) and (8.20) remain to hold with \( \varepsilon = 0 \). Hence under appropriate conditions on \( S_1 \), \( S_2 \) stability continues to hold if \( \gamma_1 \cdot \gamma_2 = 1 \).

**Remark 8.2.4** Note that the small-gain theorem is equally valid for the positive feedback interconnection \( u_1 = y_2, u_2 = y_1 \).

Theorem 8.2.1 and Proposition 8.2.2 have immediate applications to robustness analysis. A simple corollary of Theorem 8.2.1 is the following.

**Corollary 8.2.5** Consider a nominal set of differential equations \( \dot{x} = f(x), \) \( f(0) = 0 \), with perturbation model

\[
\dot{x} = f(x) + \tilde{g}(x)\Delta \tilde{h}(x),
\]

(8.21)

where \( \tilde{g}(x) \) is a known \( n \times \tilde{m} \) matrix, \( \tilde{h} : \mathcal{X} \rightarrow \mathbb{R}^\tilde{p}, \tilde{h}(0) = 0 \), is a known mapping, and \( \Delta \) is an unknown \( \tilde{m} \times \tilde{p} \) matrix representing the uncertainty. Suppose the system

\[
\Sigma : \begin{cases}
\dot{x} = f(x) + \tilde{g}(x)\tilde{u}, \quad \tilde{u} \in \mathbb{R}^m \\
\tilde{y} = \tilde{h}(x), \quad \tilde{y} \in \mathbb{R}^p
\end{cases}
\]

(8.22)

has \( L_2 \)-gain \( \leq \gamma \) (from \( \tilde{u} \) to \( \tilde{y} \)), with \( C^1 \) storage function having a strict local minimum at 0, or having a local minimum at 0 while 0 is an asymptotically stable equilibrium of \( \dot{x} = f(x) \). Then 0 is an asymptotically stable equilibrium of the perturbed system (8.21) for all perturbations \( \Delta \) having largest singular value less than \( \frac{1}{\gamma} \).
Proof Take in Propositions 8.2.1 or 8.2.2 $\Sigma_1 = \bar{\Sigma}$, and $\Sigma_2$ equal to the static system corresponding to multiplication by $\Delta$. The $L_2$-gain of $\Sigma_2$ is the largest singular value of $\Delta$. □

Another direct consequence of the preceding theory concerns robustness of finite $L_2$-gain.

Corollary 8.2.6 Consider the perturbed state space system

$$
\Sigma_p : \dot{x} = [f(x) + \bar{g}(x)\Delta\bar{h}(x)] + g(x)u, \ x \in \mathcal{X}, \ u \in \mathbb{R}^m,
\quad y = h(x), \quad y \in \mathbb{R}^p
$$

(8.23)

where $\bar{g}, \bar{h},$ and $\Delta$ are as in Corollary 8.2.5, with $\Delta = 0$ representing the nominal state space system. Suppose there exists a solution $S \geq 0$ to the parametrized Hamilton–Jacobi inequality

$$
S_x f(x) + \frac{1}{2} S_x g(x) g^T(x) S_x^T(x) + \frac{1}{2} h^T(x) h(x) + \frac{1}{2} \frac{1}{\varepsilon^2} S_x \bar{g}(x) \bar{g}^T(x) S_x^T(x) + \frac{1}{2} \varepsilon^2 \bar{h}^T(x) \bar{h}(x) \leq 0
$$

(8.24)

(with $\varepsilon$ a fixed but arbitrary scaling parameter), meaning that the extended system

$$
\dot{x} = f(x) + g(x)\bar{u} + \frac{1}{\varepsilon} \bar{g}(x)\bar{u} \\
y = h(x) \\
\bar{y} = \varepsilon \bar{h}(x)
$$

(8.25)

has $L_2$-gain $\leq \gamma$ from $(u, \bar{u})$ to $(y, \bar{y})$. Then the perturbed system $\Sigma_p$ has $L_2$-gain $\leq \gamma$ for all perturbations $\Delta$ having largest singular value $\leq \frac{1}{\gamma}$.

Proof For all $\Delta$ with largest singular value $\leq \frac{1}{\gamma}$

$$
S_x f(x) + \frac{1}{2} S_x g(x) g^T(x) S_x^T(x) + \frac{1}{2} h^T(x) h(x) + \frac{1}{2} \frac{1}{\varepsilon^2} S_x \bar{g}(x) \bar{g}^T(x) S_x^T(x) + \frac{1}{2} \varepsilon^2 \bar{h}^T(x) \bar{h}(x)
$$

and thus the expression

$$
S_x [f(x) + \bar{g}(x)\Delta\bar{h}(x)] + \frac{1}{2} S_x g(x) g^T(x) S_x^T(x) + \frac{1}{2} h^T(x) h(x)
$$

is bounded from above by the left-hand side of (8.24), implying that $\Sigma_p$ has $L_2$-gain $\leq \gamma$. □

This last corollary can be extended to dynamic perturbations $\Delta$ in the following way.

Proposition 8.2.7 Consider the extended system (8.25), and assume that there exists a solution $S \geq 0$ to (8.24) (implying that (8.25) has $L_2$-gain $\leq \gamma$ from $(u, \bar{u})$ to
(y, ӯ)). Consider another dynamical system Δ with state ξ, inputs ӯ and outputs ӯ, and having $L_2$-gain $\leq \frac{1}{\gamma}$ with $C^1$ storage function $S_\Delta(\xi) \geq 0$. Then the closed-loop system has $L_2$-gain $\leq \gamma$ from u to y, with storage function $S(x) + \gamma^2 S_\Delta(ξ)$.

**Proof** By (8.24)

$$\dot{S} \leq \frac{1}{2} \gamma^2 \|u\|^2 - \frac{1}{2} \|y\|^2 + \frac{1}{2} \gamma^2 \|\tilde{u}\|^2 - \frac{1}{2} \|\tilde{y}\|^2$$  \hspace{1cm} (8.26)

Furthermore, since Δ has $L_2$-gain $\leq \frac{1}{\gamma}$ with storage function $S_\Delta$, i.e.,

$$\dot{S}_\Delta \leq \frac{1}{2} \frac{1}{\gamma^2} \|\tilde{y}\|^2 - \frac{1}{2} \|\tilde{u}\|^2$$  \hspace{1cm} (8.27)

Premultiplying (8.27) by $\gamma^2$, and adding to (8.26) yields

$$\dot{S} + \gamma^2 \dot{S}_\Delta \leq \frac{1}{2} \gamma^2 \|u\|^2 - \frac{1}{2} \|y\|^2$$  \hspace{1cm} (8.28)

□

Similar to the developments in Chap. 4, cf. Definition 4.7.1, we can formulate the following state space version of *incremental $L_2$-gain*, as already defined for input–output maps in Definition 2.1.5.

**Definition 8.2.8** Consider a system (8.1). The system $\Sigma$ has *incremental $L_2$-gain* $\leq \gamma$ if there exists a function, called the *incremental storage function*,

$$S : X \times X \to \mathbb{R}^+$$  \hspace{1cm} (8.29)

such that

$$S(x_1(T), x_2(T)) \leq S(x_1(0), x_2(0)) + \frac{1}{2} \int_0^T \gamma^2 \|u_1(t) - u_2(t)\|^2 + \|y_1(t) - y_2(t)\|^2 \, dt,$$  \hspace{1cm} (8.30)

for all $T \geq 0$, and for all pairs of input functions $u_1$, $u_2 : [0, T] \to \mathbb{R}^m$ and all pairs of initial conditions $x_1(0)$, $x_2(0)$, with resulting pairs of state and output trajectories $x_1$, $x_2 : [0, T] \to X$, $y_1$, $y_2 : [0, T] \to \mathbb{R}^p$.

As before in the context of incremental passivity, cf. Remark 4.7.2, it can be assumed without loss of generality that the storage function $S(x_1, x_2)$ satisfies the symmetry property $S(x_1, x_2) = S(x_2, x_1)$.

The *differential* version of the incremental dissipation inequality (8.29) takes the form

$$S_{x_1}(x_1, x_2) f(x_1, u_1) + S_{x_2}(x_1, x_2) f(x_2, u_2) \leq \frac{1}{2} \gamma^2 \|u_1 - u_2\|^2 + \frac{1}{2} \|y_1 - y_2\|^2$$  \hspace{1cm} (8.31)
for all $x_1, x_2, u_1, u_2, y_1 = h(x_1, u_1), y_2 = h(x_2, u_2)$, where $S_{x_1}(x_1, x_2)$ and $S_{x_2}(x_1, x_2)$ denote row vectors of partial derivatives with respect to $x_1$, respectively $x_2$.

Let us, as above, specialize to systems (8.4), in which case the incremental dissipation inequality (8.31) becomes

$$S_{x_1}(x_1, x_2)f(x_1) + S_{x_1}(x_1, x_2)g(x_1)u_1 + S_{x_2}(x_1, x_2)f(x_2) + S_{x_2}(x_1, x_2)g(x_2)u_2 \leq \frac{1}{2} \gamma^2 \|u_1 - u_2\|^2 + \frac{1}{2} \|h(x_1) - h(x_2)\|^2 \quad (8.32)$$

In general, this inequality is hard to solve for the unknown incremental storage function $S(x_1, x_2)$. Assuming that $g(x)$ is independent of $x$ and restricting attention to incremental storage functions of the form $S(x_1, x_2) = \tilde{S}(x_1 - x_2)$, implying that $S_{x_1}(x_1, x_2) = -S_{x_2}(x_1, x_2) = \tilde{S}_x(x_1 - x_2)$, the inequality (8.32) reduces to

$$\tilde{S}_x(x_1 - x_2)[f(x_1) - f(x_2)] + \tilde{S}_x(x_1 - x_2)g[u_1 - u_2] \leq \frac{1}{2} \gamma^2 \|u_1 - u_2\|^2 + \frac{1}{2} \|h(x_1) - h(x_2)\|^2 \quad (8.33)$$

By “completion of the squares” in the difference $u_1 - u_2$, this can be seen to further reduce to the Hamilton–Jacobi inequality

$$\tilde{S}_x(x_1 - x_2)[f(x_1) - f(x_2)] + \frac{1}{\gamma^2} \tilde{S}_x(x_1 - x_2)gg^T \tilde{S}_x(x_1 - x_2) + \frac{1}{2} \|h(x_1) - h(x_2)\|^2 \leq 0 \quad (8.34)$$

for all $x_1, x_2$.

**Remark 8.2.9** A trivial example is provided by a linear system having $L_2$-gain $\leq \gamma$ with quadratic storage function $\frac{1}{2}x^T Qx$. In this case, $S(x_1, x_2) := \frac{1}{2}(x_1 - x_2)^T Q(x_1 - x_2)$ defines an incremental storage function, and hence the system also has incremental $L_2$-gain $\leq \gamma$.

### 8.3 Network Version of the Small-Gain Theorem

The small-gain theorem concerns the stability of the interconnection of two systems in negative or positive feedback interconnection. A network version of this can be formulated as follows.

Consider a multiagent system, corresponding to a directed graph $\mathcal{G}$ with $N$ vertices and input-state-output systems $\Sigma_i, i = 1, \ldots, N$, associated to these vertices. Furthermore, assume that the edges of the graph are specified by an $N \times N$ adjacency matrix $A$. Let $\mathcal{E}$ denote the set of edges.

1 Or assuming the existence of coordinates in which $g(x)$ is constant, which is, under the assumption that $\text{rank}g(x) = m$, equivalent to the Lie brackets of the vector fields $g_1, \ldots, g_m$ defined by the columns of $g(x)$ to be zero [233].
matrix \( A \) with elements 0, 1, corresponding to interconnections \(^2\)

\[ u_i = y_j \] (8.35)

if and only if the \((i, j)\)-th element of \( A \) is equal to 1.

Now assume that the systems \( \Sigma_i \) have \( L_2 \)-gain \( \leq \gamma_i \), \( i = 1, \ldots, N \). This means that there exist storage functions \( S_i : X_i \rightarrow \mathbb{R}^+ \) such that \(^3\)

\[ \dot{S}_i \leq \frac{1}{2} \gamma_i^2 \| u_i \|^2 - \frac{1}{2} \| y_i \|^2, \quad i = 1, \ldots, N \] (8.36)

Then define the following \( N \times N \) matrix with nonnegative elements

\[ \Gamma := \text{diag}(\gamma_1^2, \ldots, \gamma_N^2)A \] (8.37)

The Perron–Frobenius theorem yields the following lemma.

**Lemma 8.3.1** ([79]) Denote by \( z > 0 \) a (column or row) vector \( z \) with all elements positive, and by \( z \geq 0 \) a vector with nonnegative elements. Furthermore, let \( I_N \) denote the \( N \times N \) identity matrix.

Consider an \( N \times N \) matrix \( \Gamma \) with all nonnegative elements. Then there exists a vector \( \mu > 0 \) such that

\[ \mu^T (\Gamma - I_N) < 0 \] (8.38)

if and only if

\[ r(\Gamma) < 1, \] (8.39)

where \( r(\Gamma) \) denotes the spectral radius of \( \Gamma \).

**Proof** (If). If \( r(\Gamma) < 1 \) then by the Perron–Frobenius theorem there exists \( \mu > 0 \) such that \( \mu^T \Gamma \leq \mu^T \), or equivalently \( \mu^T (\Gamma - I_N) < 0 \).

(Only if). Conversely, if \( r(\Gamma) \geq 1 \) then there exists \( \nu \geq 0 \), \( \nu \neq 0 \), such that \( (\Gamma - I_N)\nu \geq 0 \), whence for any \( \mu > 0 \) we have \( \mu^T (\Gamma - I_N)\nu \geq 0 \), contradicting \( \mu^T (\Gamma - I_N) < 0 \).

\( \square \)

Based on this lemma we obtain the following theorem.

**Theorem 8.3.2** (Small-gain network theorem) Consider a directed graph \( G \) with systems \( \Sigma_i \) associated to its vertices, which have \( L_2 \)-gains \( \leq \gamma_i \) with \( C^1 \) storage functions \( S_i, i = 1, \ldots, N \), and which are interconnected through the adjacency matrix \( A \) defined by (8.35). Consider the matrix \( \Gamma \) given by (8.37). If the spectral

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\(^2\)The typical situation being that each row in the \( A \) matrix contains only one 1. Multiple occurrence of ones in a row is allowed but will imply an equality constraint on the corresponding outputs, leading to algebraic constraints between the state variables.

\(^3\)The subsequent argumentation directly extends to the case of non-differentiable storage functions, replacing the differential dissipation inequalities by their integral counterparts.
radius \( r(\Gamma) < 1 \), then there exists \( \mu > 0 \) such that \( \mu^T(\Gamma - I_N) < 0 \) and the nonnegative function

\[
S(x_1, \ldots, x_N) := \sum_{i=1}^{N} \mu_i S_i(x_i)
\]

satisfies along trajectories of the interconnected system

\[
\dot{S} \leq -\varepsilon_1 \|y_1\|^2 - \varepsilon_2 \|y_2\|^2 \cdots - \varepsilon_N \|y_N\|^2
\]

for certain positive constants \( \varepsilon_1, \ldots, \varepsilon_N \). Hence, if \( S_i \) has a strict minimum at \( x_i^* \), \( i = 1, \ldots, N \), then \( x^* = (x_1^*, \ldots, x_N^*) \) is a stable equilibrium of the interconnected system, which is asymptotically stable provided the interconnected system is zero-state detectable.

**Proof** Denote the vector with components \( \|y_i\|^2 \), \( i = 1, \ldots, N \), by \( \hat{y} \). Then

\[
\dot{S} = \sum_{i=1}^{N} \mu_i \dot{S}_i(x_i) \leq \frac{1}{2} \sum_{i=1}^{N} \mu_i (\gamma_i^2 \|u_i\|^2 - \|y_i\|^2) = \frac{1}{2} \mu^T(\Gamma - I_N)\hat{y}
\]

for certain positive constants \( \varepsilon_1, \ldots, \varepsilon_N \). \( \square \)

**Example 8.3.3** In case of the feedback interconnection \( u_1 = y_2 \), \( u_2 = y_1 \) of two systems \( \Sigma_1, \Sigma_2 \) with \( L_2 \)-gain \( \leq \gamma_1 \), respectively, \( \leq \gamma_2 \), application of Theorem 8.3.2 leads to the consideration of the matrix

\[
\Gamma = \begin{bmatrix} 0 & \gamma_2^2 \\ \gamma_1 & 0 \end{bmatrix}
\]

which has spectral radius \( < 1 \) if and only if \( \gamma_1 \cdot \gamma_2 < 1 \); thus recovering the small-gain condition of Theorem 8.2.1.

### 8.4 \( L_2 \)-Gain as Response to Periodic Input Functions

An interesting interpretation of the \( L_2 \)-gain of a nonlinear state space system \( \Sigma \) given by (8.1) in terms of the response to periodic input functions was obtained in [140].

Assume that \( f(0,0) = 0 \), and that the matrix \( A := \frac{\partial f}{\partial x}(0,0) \) has all its eigenvalues in the open left half plane. Let \( \Sigma \) have \( L_2 \)-gain \( \leq \gamma \), and consider a periodic input function \( u_p(\cdot) \) (of period \( T > 0 \)), which is generated by a dynamical system (“exo-system”)

\[
\Sigma_e : \dot{u} = s(u), \quad s(0) = 0,
\]

whose linearization has all its eigenvalues on the imaginary axis. Then it follows from center manifold theory (see, e.g., [61]) that the series interconnection of the
exo-system $\Sigma_e$ with $\Sigma$, given as

\[
\begin{align*}
\dot{x} &= f(x, u) \\
\dot{u} &= s(u)
\end{align*}
\] (8.45)

and having augmented state $(x, u)$, has a center manifold, which is the graph $C = \{(x, u) \mid x = c(u)\}$ of a mapping $x = c(u)$. Furthermore, for initial conditions close enough to $C$ the solutions of the composed system (8.45) tend exponentially to $C$, and thus, since $u_p(\cdot)$ is periodic of period $T$, the solution $x(\cdot)$ converges to a steady state solution $x_p(\cdot)$, with $x_p(t) = c(u_p(t))$, also having period $T$ (see [141]).

It follows that $y_p(t) = h(x_p(t), u_p(t))$ has period $T$ as well, and furthermore by the $L_2$-gain dissipation inequality (8.2)

\[
\int_{t_0}^{t_0+T} \|y_p(t)\|^2 dt \leq \gamma^2 \int_{t_0}^{t_0+T} \|u_p(t)\|^2 dt
\] (8.46)

(since $x_p(t_0 + T) = x_p(t_0)$), for all $t_0$.

Defining the rms values of any periodic signal $z$ with period $T$ as

\[
\|z\|_{rms} := \frac{1}{T} \left( \int_{t_0}^{t_0+T} \|z(t)\|^2 dt \right)^{1/2},
\] (8.47)

the property that $\Sigma$ has $L_2$-gain $\leq \gamma$ therefore implies that

\[
\|y_p\|_{rms} \leq \gamma \|u_p\|_{rms},
\] (8.48)

for all periodic input functions $u_p$ which are generated by the exo-system $\dot{u} = s(u)$, $s(0) = 0$, and all initial conditions close enough to $C$.

### 8.5 Relationships with IIS- and iIIS-Stability

An alternative approach to extending Lyapunov stability theory of autonomous systems $\dot{x} = f(x)$ to systems with inputs $\dot{x} = f(x, u)$ is offered by the theory of (integral-)Input-to-State Stability ((i)ISS). In this section, we will briefly indicate the close relationships of (i)ISS with dissipative systems theory and (a generalized) version of $L_2$-gain. We will do so by not giving the original definitions of IIS and iIIS, but instead by providing their equivalent characterizations in terms of dissipative systems theory.

By itself the close relation between dissipative systems theory on the one hand and stability theory based on the IIS or iIIS property on the other is not surprising. Indeed, we have already seen in Chap. 3 how dissipativity implies Lyapunov stability properties of the system for $u = 0$. Furthermore, for $u \neq 0$ the dissipation inequality
may be used for deriving properties of the relationships between input, state and output functions.

Recall that a function \( \alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is of class \( K_{\infty} \), denoted \( \alpha \in K_{\infty} \), if it is continuous, strictly increasing, unbounded, and satisfies \( \alpha(0) = 0 \).

Turning the characterization of input-to-state stability (IIS) as given in [323] into a definition we formulate

**Definition 8.5.1** ([323]) A system

\[
\dot{x} = f(x, u), \quad f(0, 0) = 0, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{8.49}
\]

is ISS if there exist functions \( \alpha, \beta \in K_{\infty} \) such that the input-state-output system \( \dot{x} = f(x, u), y = x \), is dissipative with respect to the supply rate

\[
s(u, y) = \beta(\|u\|) - \alpha(\|y\|) \tag{8.50}
\]

with a \( C^1 \) and radially unbounded storage function \( S \) satisfying \( S(x) > 0, x \neq 0, S(0) = 0 \).

Note the conceptual difference with dissipative systems theory (apart from minor technical differences like the a priori assumption of strict positivity and radial unboundedness of \( S \)). In dissipative systems theory, one starts with a given supply rate on the space of inputs and outputs, and derives properties of the system based on this. On the other hand, ISS theory aims at providing stability results for systems with inputs \( \dot{x} = f(x, u), f(0, 0) = 0 \), and seeks criteria, which can be translated into the existence of a supply rate (8.50), where the functions \( \alpha, \beta \) may not be easy to determine explicitly.

We remark that the supply rate (8.50) can be regarded as a generalization of the \( L_2 \)-gain supply rate. In fact, by taking \( \alpha \) and \( \beta \) to be the quadratic functions \( \alpha(r) = \frac{1}{2} r^2 \) and \( \beta(r) = \frac{1}{2} \gamma^2 r^2 \) for some constant \( \gamma \) one recovers the \( L_2 \)-gain \( \leq \gamma \) case for \( y = x \). Conversely, by allowing for arbitrary nonlinear coordinate transformations on the space \( U = \mathbb{R}^m \) of inputs and \( X = \mathbb{R}^n \) of states, the dissipation inequality for \( L_2 \)-gain (from \( u \) to \( y = x \)) can be seen to transform into (8.50) for certain \( \alpha, \beta \in K_{\infty} \). See also the discussion in Note 2 in Sect. 2.5 on the related notion of nonlinear \( L_2 \)-gain.

With regard to integral input-to-state stability (iIIS) case we have the following characterization in terms of dissipative systems theory.

**Definition 8.5.2** ([9]) A system (8.49) is iISS if there exists a function \( \gamma \in K_{\infty} \) and a function \( \alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \alpha(0) = 0, \alpha(r) > 0, r \neq 0 \), such that the input-state-output system \( \dot{x} = f(x, u), y = x \), is dissipative with respect to the supply rate (8.50) with a \( C^1 \) and radially unbounded storage function \( S \) satisfying \( S(x) > 0, x \neq 0, S(0) = 0 \).

From the above characterization, it is clear that IIS implies iIIS, while the converse does not hold. That is, there are systems which are iIIS but not IIS. An example is provided by the system
\[ \dot{x} = -x + ux \]  
(8.51)

See, e.g., [9, 319] for a further discussion regarding the differences between ISS and iISS.

Another version of the relationship between iIIS and dissipative systems theory is provided in the following proposition.

**Proposition 8.5.3** ([9]) A system 8.49 is iIIS if and only if there exists a continuous output function \( y = h(x) \) with \( h(0) = 0 \) for which the resulting system \( \dot{x} = f(x, u), y = h(x) \) is zero-detectable and there exists \( \beta \in \mathcal{K}_\infty \) and a function \( \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \alpha(0) = 0, \alpha(r) > 0, r \neq 0 \), such that the system is dissipative with respect to the supply rate (8.50) with a \( C^1 \) and radially unbounded storage function \( S \) satisfying \( S(x) > 0, x \neq 0, S(0) = 0 \).

Note that the “if” part of the last proposition with \( \alpha(r) = \frac{1}{2} r^2 \) and \( \beta(r) = \frac{1}{2} \gamma^2 r^2 \) implies that a zero-detectable input-state-output system with finite \( L^2 \)-gain having a \( C^1 \) and radially unbounded storage function \( S \) satisfying \( S(x) > 0, x \neq 0, S(0) = 0 \), is iIIS.

### 8.6 Notes for Chapter 8

1. Corollary 8.2.6 is a nonlinear generalization of a result given for linear systems in Xie & De Souza [358], and may be found in Shen & Tamura [313].
2. See Angeli [8] for further information regarding incremental \( L^2 \)-gain as discussed in Sect. 8.2.
3. Applications of incremental \( L^2 \)-gain to model reduction of nonlinear systems can be found in Besselink, van de Wouw & Nijmeijer [38].
4. Section 8.3, in particular Lemma 8.3.1, is largely based on Dashkovskiy, Ito & Wirth [79]. There is a wealth of literature on other network versions of the small-gain theorem; see, e.g., Jiang & Wang [150], Liu, Hill & Jiang [182], Rüffer [261].
5. See Pavlov, van de Wouw & Nijmeijer [254] and the references quoted therein for theory related to Sect. 8.4.
6. The notions of (integral-)Input-to-State Stability were introduced and explored by Sontag and co-workers, see, e.g., Sontag & Wang [323], Sontag [318, 319, 324], Angeli, Sontag & Wang [9] and the references quoted in there. See also the survey Sontag [321], and the account of the relation with dissipative systems theory provided in Isidori [139].
7. The characterization of the set of pairs of functions \( (\beta, \alpha) \) such that the system is dissipative with respect to the supply rate (8.50) is addressed in Sontag & Teel [322].