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L2-Gain and the Small-Gain Theorem

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Chapter 8

L_2 -Gain and the Small-Gain Theorem

In this chapter, we elaborate on the characterization of finite L_2 -gain for state space systems, continuing on the general theory of dissipative systems in Chap. 3. Within this framework we revisit the Small-gain theorem and its implications for robustness (Sect. 8.2), and extend the small-gain condition to network systems (Sect. 8.3). Furthermore, we provide an alternative characterization of L_2 -gain in terms of response to periodic input functions (Sect. 8.4), and in Sect. 8.5 we end by sketching the close relationships to the theory of (integral-)input-to-state stability.

8.1 L_2 -Gain of State Space Systems

Recall from Chap. 3 the following basic definitions regarding L_2 -gain of a general state space system Σ

$$\Sigma : \begin{cases} \dot{x} = f(x, u), & x \in \mathcal{X}, u \in \mathbb{R}^m \\ y = h(x, u), & y \in \mathbb{R}^p \end{cases} \quad (8.1)$$

with n -dimensional state space manifold $\mathcal{X}$.

Definition 8.1.1 A state space system Σ given by (8.1) has L_2 -gain $\leq \gamma$ if it is dissipative with respect to the supply rate $s(u, y) = \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2$; that is, there exists a storage function $S : \mathcal{X} \rightarrow \mathbb{R}^+$ such that

$$S(x(t_1)) - S(x(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma^2 \|u(t)\|^2 - \|y(t)\|^2) dt \quad (8.2)$$

along all input functions $u(\cdot)$ and resulting state trajectories $x(\cdot)$ and output functions $y(\cdot)$, for all $t_0 \leq t_1$. Equivalently, if S is C^1

$$S_x(x)f(x, u) \leq \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|h(x, u)\|^2, \quad \forall x, u \quad (8.3)$$

The L_2 -gain of Σ is defined as $\gamma(\Sigma) = \inf\{\gamma \mid \Sigma \text{ has } L_2\text{-gain} \leq \gamma\}$. Σ is said to have L_2 -gain $< \gamma$ if there exists $\tilde{\gamma} < \gamma$ such that Σ has L_2 -gain $\leq \tilde{\gamma}$. Finally, Σ is called *inner* if it is conservative with respect to $s(u, y) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|y\|^2$.

A more explicit form of the differential dissipation inequality (8.3) can be obtained for systems that are *affine* in the input u and *without feedthrough* term

$$\Sigma_a : \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x), \end{cases} \quad (8.4)$$

with $g(x)$ an $n \times m$ matrix. In this case, the differential dissipation inequality (8.3) for Σ_a amounts to

$$S_x(x)[f(x) + g(x)u] - \frac{1}{2}\gamma^2\|u\|^2 + \frac{1}{2}\|h(x)\|^2 \leq 0, \quad \forall x, u \quad (8.5)$$

This can be simplified by computing the maximizing u^* (as a function of x) for the left-hand side, i.e.,

$$u^* = \frac{1}{\gamma^2}g^T(x)S_x^T(x), \quad (8.6)$$

and substituting (8.6) into (8.5) to obtain the *Hamilton–Jacobi inequality*

$$S_x(x)f(x) + \frac{1}{2}\frac{1}{\gamma^2}S_x(x)g(x)g^T(x)S_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0, \quad \forall x \in \mathcal{X} \quad (8.7)$$

Thus, Σ_a has L_2 -gain $\leq \gamma$ with a C^1 storage function if and only if there exists a C^1 solution $S \geq 0$ to (8.7). Furthermore, it follows from the theory of *dynamic programming* that if the available storage S_a and required supply S_r (assuming existence) are C^1 , they are actually solutions of the Hamilton–Jacobi (–Bellman) *equality*

$$S_x(x)f(x) + \frac{1}{2}\frac{1}{\gamma^2}S_x(x)g(x)g^T(x)S_x^T(x) + \frac{1}{2}h^T(x)h(x) = 0 \quad (8.8)$$

More information on the structure of the solution set of the Hamilton–Jacobi inequality (8.7) and equality (8.8) will be provided in Chap. 11.

An alternative view on L_2 -gain, directly relating to Proposition 1.2.9 in Chap. 1, is provided by the following definition and proposition.

Definition 8.1.2 Given the state space system Σ given by (8.1) define the *associated Hamiltonian input–output system* Σ^H as

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p}(x, p, u) \\ \Sigma^H : \dot{p} &= -\frac{\partial H}{\partial x}(x, p, u) \\ y_a &= \frac{\partial H}{\partial u}(x, p, u)\end{aligned}\tag{8.9}$$

with state (x, p) , inputs u and outputs y_a , where the Hamiltonian $H(x, p, u)$ is defined as

$$H(x, p, u) := p^T f(x, u) + \frac{1}{2} h^T(x, u) h(x, u)\tag{8.10}$$

Here $p \in T_x^* \mathcal{X} \simeq \mathbb{R}^n$ denotes the co-state vector.

Remark 8.1.3 Geometrically, the state space of Σ^H is given by the cotangent bundle $T^* \mathcal{X}$, with $p \in T_x^* \mathcal{X}$. See [24, 73] for further information. In case of a linear system, $\dot{x} = Ax + Bu$, $y = Cx + Du$ with transfer matrix $K(s) = C(sI - A)^{-1}B + D$, the transfer matrix of the associated input–output Hamiltonian system is given as $K^T(-s)K(s)$.

Proposition 8.1.4 *Consider Σ satisfying $f(0, 0) = 0$, $h(0, 0) = 0$, with input–output map $G_0 : L_{2e}(\mathbb{R}^m) \rightarrow L_{2e}(\mathbb{R}^p)$, which is assumed to be L_2 -stable. Then the input–output map $G_{(0,0)}^H$ of Σ^H for initial state $(x, p) = (0, 0)$ is equal to the composed map*

$$G_{(0,0)}^H(u) = (DG_0(u))^* \circ G_0(u)\tag{8.11}$$

with $DG_0(u)$ the Fréchet derivative of the map G_0 . In particular, it follows that the input–output map G_0 is inner if and only if $G_{(0,0)}^H$ is the identity-map.

Proof Follows directly from the definition of the Fréchet derivative $DG_0(u)$ of G_0 at u . The last statement follows from Proposition 1.2.9 in Chap. 1. $\square$

8.2 The Small-Gain Theorem Revisited

In this section, we provide a state space interpretation of the Small-gain Theorem 2.1.1 for input–output maps in Chap. 2.

Let us consider, as in Chap. 4, the standard feedback configuration $\Sigma_1 \parallel_f \Sigma_2$ of two input-state-output systems

$$\Sigma_i : \begin{aligned} \dot{x}_i &= f_i(x_i, u_i), & x_i &\in \mathcal{X}_i, & u_i &\in U_i \\ y_i &= h_i(x_i, u_i), & y_i &\in Y_i \end{aligned} \quad i = 1, 2\tag{8.12}$$

with $U_1 = Y_2$, $U_2 = Y_1$, cf. Fig. 4.1. Suppose Σ_1 and Σ_2 in (8.12) have L_2 -gain $\leq \gamma_1$, respectively $\leq \gamma_2$. Denote the storage functions of Σ_1 , Σ_2 by S_1 , S_2 , with corresponding dissipation inequalities

$$\begin{aligned} S_1(x_1(t_1)) - S_1(x_1(t_0)) &\leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma_1^2 \|u_1(t)\|^2 - \|y_1(t)\|^2) dt \\ S_2(x_2(t_1)) - S_2(x_2(t_0)) &\leq \frac{1}{2} \int_{t_0}^{t_1} (\gamma_2^2 \|u_2(t)\|^2 - \|y_2(t)\|^2) dt \end{aligned} \quad (8.13)$$

Consider now the feedback interconnection with $e_1 = e_2 = 0$

$$u_1 = -y_2, \quad u_2 = y_1, \quad (8.14)$$

and assume $\gamma_1 \cdot \gamma_2 < 1$. Then we can find an α such that

$$\gamma_1 < \alpha < \frac{1}{\gamma_2} \quad (8.15)$$

Substitute (8.14) into (8.13), multiply the second inequality of (8.13) by α^2 , and add both resulting inequalities, to obtain

$$\begin{aligned} S(x_1(t_1), x_2(t_1)) - S(x_1(t_0), x_2(t_0)) &\leq \\ \frac{1}{2} \int_{t_0}^{t_1} [(\alpha^2 \gamma_2^2 - 1) \|y_1(t)\|^2 + (\gamma_1^2 - \alpha^2) \|y_2(t)\|^2] dt \end{aligned} \quad (8.16)$$

where $S(x_1, x_2) := S_1(x_1) + \alpha^2 S_2(x_2)$. Since α satisfies (8.15) it immediately follows that

$$S(x_1(t_1), x_2(t_1)) - S(x_1(t_0), x_2(t_0)) \leq -\varepsilon \int_{t_0}^{t_1} (\|y_1(t)\|^2 + \|y_2(t)\|^2) dt \quad (8.17)$$

for a certain $\varepsilon > 0$. Thus S is a candidate Lyapunov function for the closed-loop system. In fact, we may immediately apply the reasoning of Lemma 3.2.16, resulting in

Theorem 8.2.1 (Small-gain theorem; state space version) *Suppose Σ_1 and Σ_2 have L_2 -gain $\leq \gamma_1$ and $\leq \gamma_2$, with $\gamma_1 \cdot \gamma_2 < 1$. Suppose $S_1, S_2 \geq 0$ satisfying (8.13) are C^1 and have strict local minima at $x_1^* = 0, x_2^* = 0$, and Σ_1 and Σ_2 are zero-state detectable. Then $x^* = (x_1^*, x_2^*)$ is an asymptotically stable equilibrium of the closed-loop system $\Sigma_1 \parallel_f \Sigma_2$ with $e_1 = e_2 = 0$, which is globally asymptotically stable if additionally S_1, S_2 have global minima at x_1^*, x_2^* and are proper.*

We leave the refinement of Theorem 8.2.1 to positive semidefinite S_1 and S_2 based on Theorem 3.2.19 to the reader (see also [312]). Instead we formulate the following version based on Proposition 3.2.22. For simplicity, assume that $\Sigma_i, i = 1, 2$, are affine systems

$$\Sigma_{ai} : \begin{cases} \dot{x}_i = f_i(x_i) + g_i(x_i)u_i \\ y_i = h_i(x_i) \end{cases} \quad (8.18)$$

Proposition 8.2.2 *Suppose the affine systems Σ_{a1} and Σ_{a2} as in (8.18) have L_2 -gain $\leq \gamma_1$ and $\leq \gamma_2$, with $\gamma_1 \cdot \gamma_2 < 1$. Suppose $S_1, S_2 \geq 0$ satisfying (4.52) are C^1 and*

$S_1(x_1^*) = S_2(x_2^*) = 0$ (that is, S_1 and S_2 are positive semidefinite at x_1^* , respectively, x_2^*). Furthermore, assume that x_i^* is an asymptotically stable equilibrium of $\dot{x}_i = f_i(x_i)$, $i = 1, 2$. Then (x_1^*, x_2^*) is an asymptotically stable equilibrium of the closed-loop system $\Sigma_1 \|_f \Sigma_2$ with $e_1 = 0$, $e_2 = 0$.

Proof The closed-loop system $\Sigma_1 \|_f \Sigma_2$ with $e_1 = 0$, $e_2 = 0$ can be written as

$$\dot{x} = f(x) + g(x)k(x)$$

with $x = (x_1, x_2)$, and

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \quad k = \begin{bmatrix} -h_2 \\ h_1 \end{bmatrix} \quad (8.19)$$

By (8.17)

$$S_x(x_1, x_2) [f(x) + g(x)k(x)] \leq -\varepsilon \|k(x)\|^2, \quad (8.20)$$

while by assumption (x_1^*, x_2^*) is an asymptotically stable equilibrium of $\dot{x} = f(x)$. The statement now follows from Proposition 3.2.22. $\square$

Remark 8.2.3 Contrary to the Small-gain Theorem 2.1.1 for input–output maps in Chap. 2 we can relax the small-gain condition $\gamma_1 \cdot \gamma_2 < 1$ to $\gamma_1 \cdot \gamma_2 \leq 1$, in the sense that for $\gamma_1 \cdot \gamma_2 = 1$ the inequalities (8.17) and (8.20) remain to hold with $\varepsilon = 0$. Hence under appropriate conditions on S_1 , S_2 stability continues to hold if $\gamma_1 \cdot \gamma_2 = 1$.

Remark 8.2.4 Note that the small-gain theorem is equally valid for the *positive* feedback interconnection $u_1 = y_2$, $u_2 = y_1$.

Theorem 8.2.1 and Proposition 8.2.2 have immediate applications to robustness analysis. A simple corollary of Theorem 8.2.1 is the following.

Corollary 8.2.5 Consider a nominal set of differential equations $\dot{x} = f(x)$, $f(0) = 0$, with perturbation model

$$\dot{x} = f(x) + \bar{g}(x)\Delta\bar{h}(x), \quad (8.21)$$

where $\bar{g}(x)$ is a known $n \times \bar{m}$ matrix, $\bar{h} : \mathcal{X} \rightarrow \mathbb{R}^{\bar{p}}$, $\bar{h}(0) = 0$, is a known mapping, and Δ is an unknown $\bar{m} \times \bar{p}$ matrix representing the uncertainty. Suppose the system

$$\bar{\Sigma} : \begin{cases} \dot{x} = f(x) + \bar{g}(x)\bar{u}, & \bar{u} \in \mathbb{R}^{\bar{m}} \\ \bar{y} = \bar{h}(x), & \bar{y} \in \mathbb{R}^{\bar{p}} \end{cases} \quad (8.22)$$

has L_2 -gain $\leq \gamma$ (from $\bar{u}$ to $\bar{y}$), with C^1 storage function having a strict local minimum at 0, or having a local minimum at 0 while 0 is an asymptotically stable equilibrium of $\dot{x} = f(x)$. Then 0 is an asymptotically stable equilibrium of the perturbed system (8.21) for all perturbations Δ having largest singular value less than $\frac{1}{\gamma}$.

Proof Take in Propositions 8.2.1 or 8.2.2 $\Sigma_1 = \bar{\Sigma}$, and Σ_2 equal to the static system corresponding to multiplication by Δ . The L_2 -gain of Σ_2 is the largest singular value of Δ . $\square$

Another direct consequence of the preceding theory concerns *robustness* of finite L_2 -gain.

Corollary 8.2.6 *Consider the perturbed state space system*

$$\Sigma_p : \begin{cases} \dot{x} = [f(x) + \bar{g}(x)\Delta\bar{h}(x)] + g(x)u, & x \in \mathcal{X}, u \in \mathbb{R}^m, \\ y = h(x), & y \in \mathbb{R}^p \end{cases} \quad (8.23)$$

where $\bar{g}$, $\bar{h}$, and Δ are as in Corollary 8.2.5, with $\Delta = 0$ representing the nominal state space system. Suppose there exists a solution $S \geq 0$ to the parametrized Hamilton–Jacobi inequality

$$\begin{aligned} S_x(x)f(x) + \frac{1}{2}\frac{1}{\gamma^2}S_x(x)g(x)g^T(x)S_x^T(x) + \frac{1}{2}h^T(x)h(x) + \\ \frac{1}{2}\frac{1}{\gamma^2}\frac{1}{\varepsilon^2}S_x(x)\bar{g}(x)\bar{g}^T(x)S_x^T(x) + \frac{1}{2}\varepsilon^2\bar{h}^T(x)\bar{h}(x) \leq 0 \end{aligned} \quad (8.24)$$

(with ε a fixed but arbitrary scaling parameter), meaning that the extended system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + \frac{1}{\varepsilon}\bar{g}(x)\bar{u} \\ y &= h(x) \\ \bar{y} &= \varepsilon\bar{h}(x) \end{aligned} \quad (8.25)$$

has L_2 -gain $\leq \gamma$ from $(u, \bar{u})$ to $(y, \bar{y})$. Then the perturbed system Σ_p has L_2 -gain $\leq \gamma$ for all perturbations Δ having largest singular value $\leq \frac{1}{\gamma}$.

Proof For all Δ with largest singular value $\leq \frac{1}{\gamma}$

$$S_x(x)\bar{g}(x)\Delta\bar{h}(x) \leq \frac{1}{2}\frac{1}{\gamma^2}\frac{1}{\varepsilon^2}S_x(x)\bar{g}(x)\bar{g}^T(x)S_x^T(x) + \frac{1}{2}\varepsilon^2\bar{h}^T(x)\bar{h}(x)$$

and thus the expression

$$S_x(x)[f(x) + \bar{g}(x)\Delta\bar{h}(x)] + \frac{1}{2}\frac{1}{\gamma^2}S_x(x)g(x)g^T(x)S_x^T(x) + \frac{1}{2}h^T(x)h(x)$$

is bounded from above by the left-hand side of (8.24), implying that Σ_p has L_2 -gain $\leq \gamma$. $\square$

This last corollary can be extended to *dynamic* perturbations Δ in the following way.

Proposition 8.2.7 *Consider the extended system (8.25), and assume that there exists a solution $S \geq 0$ to (8.24) (implying that (8.25) has L_2 -gain $\leq \gamma$ from $(u, \bar{u})$ to*

$(y, \bar{y})$). Consider another dynamical system Δ with state ξ , inputs $\bar{y}$ and outputs $\bar{u}$, and having L_2 -gain $\leq \frac{1}{\gamma}$ with C^1 storage function $S_\Delta(\xi) \geq 0$. Then the closed-loop system has L_2 -gain $\leq \gamma$ from u to y , with storage function $S(x) + \gamma^2 S_\Delta(\xi)$.

Proof By (8.24)

$$\dot{S} \leq \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2 + \frac{1}{2}\gamma^2\|\bar{u}\|^2 - \frac{1}{2}\|\bar{y}\|^2 \quad (8.26)$$

Furthermore, since Δ has L_2 -gain $\leq \frac{1}{\gamma}$ with storage function S_Δ , i.e.,

$$\dot{S}_\Delta \leq \frac{1}{2}\frac{1}{\gamma^2}\|\bar{y}\|^2 - \frac{1}{2}\|\bar{u}\|^2 \quad (8.27)$$

Premultiplying (8.27) by γ^2 , and adding to (8.26) yields

$$\dot{S} + \gamma^2 \dot{S}_\Delta \leq \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2 \quad (8.28)$$

□

Similar to the developments in Chap. 4, cf. Definition 4.7.1, we can formulate the following state space version of *incremental L_2 -gain*, as already defined for input–output maps in Definition 2.1.5.

Definition 8.2.8 Consider a system (8.1). The system Σ has *incremental L_2 -gain* $\leq \gamma$ if there exists a function, called the *incremental storage function*,

$$S : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \quad (8.29)$$

such that

$$S(x_1(T), x_2(T)) \leq S(x_1(0), x_2(0)) + \frac{1}{2} \int_0^T \gamma^2 \|u_1(t) - u_2(t)\|^2 + \|y_1(t) - y_2(t)\|^2 dt, \quad (8.30)$$

for all $T \geq 0$, and for all pairs of input functions $u_1, u_2 : [0, T] \rightarrow \mathbb{R}^m$ and all pairs of initial conditions $x_1(0), x_2(0)$, with resulting pairs of state and output trajectories $x_1, x_2 : [0, T] \rightarrow \mathcal{X}$, $y_1, y_2 : [0, T] \rightarrow \mathbb{R}^p$.

As before in the context of incremental passivity, cf. Remark 4.7.2, it can be assumed without loss of generality that the storage function $S(x_1, x_2)$ satisfies the symmetry property $S(x_1, x_2) = S(x_2, x_1)$.

The *differential* version of the incremental dissipation inequality (8.29) takes the form

$$S_{x_1}(x_1, x_2)f(x_1, u_1) + S_{x_2}(x_1, x_2)f(x_2, u_2) \leq \frac{1}{2}\gamma^2\|u_1 - u_2\|^2 + \frac{1}{2}\|y_1 - y_2\|^2 \quad (8.31)$$

for all $x_1, x_2, u_1, u_2, y_1 = h(x_1, u_1), y_2 = h(x_2, u_2)$, where $S_{x_1}(x_1, x_2)$ and $S_{x_2}(x_1, x_2)$ denote row vectors of partial derivatives with respect to x_1 , respectively x_2 .

Let us, as above, specialize to systems (8.4), in which case the incremental dissipation inequality (8.31) becomes

$$\begin{aligned} & S_{x_1}(x_1, x_2)f(x_1) + S_{x_1}(x_1, x_2)g(x_1)u_1 + \\ & S_{x_2}(x_1, x_2)f(x_2) + S_{x_1}(x_1, x_2)g(x_2)u_2 \\ & \leq \frac{1}{2}\gamma^2\|u_1 - u_2\|^2 + \frac{1}{2}\|h(x_1) - h(x_2)\|^2 \end{aligned} \quad (8.32)$$

In general, this inequality is hard to solve for the unknown incremental storage function $S(x_1, x_2)$. Assuming¹ that $g(x)$ is independent of x and restricting attention to incremental storage functions of the form $S(x_1, x_2) = \bar{S}(x_1 - x_2)$, implying that $S_{x_1}(x_1, x_2) = -S_{x_2}(x_1, x_2) = \bar{S}_x(x_1 - x_2)$, the inequality (8.32) reduces to

$$\begin{aligned} & \bar{S}_x(x_1 - x_2)[f(x_1) - f(x_2)] + \bar{S}_x(x_1 - x_2)g[u_1 - u_2] \\ & \leq \frac{1}{2}\gamma^2\|u_1 - u_2\|^2 + \frac{1}{2}\|h(x_1) - h(x_2)\|^2 \end{aligned} \quad (8.33)$$

By “completion of the squares” in the difference $u_1 - u_2$, this can be seen to further reduce to the Hamilton–Jacobi inequality

$$\begin{aligned} & \bar{S}_x(x_1 - x_2)[f(x_1) - f(x_2)] + \frac{1}{\gamma^2}\bar{S}_x(x_1 - x_2)gg^T\bar{S}_x^T(x_1 - x_2) \\ & + \frac{1}{2}\|h(x_1) - h(x_2)\|^2 \leq 0 \end{aligned} \quad (8.34)$$

for all x_1, x_2 .

Remark 8.2.9 A trivial example is provided by a *linear* system having L_2 -gain $\leq \gamma$ with quadratic storage function $\frac{1}{2}x^T Qx$. In this case, $S(x_1, x_2) := \frac{1}{2}(x_1 - x_2)^T Q(x_1 - x_2)$ defines an incremental storage function, and hence the system also has incremental L_2 -gain $\leq \gamma$.

8.3 Network Version of the Small-Gain Theorem

The small-gain theorem concerns the stability of the interconnection of two systems in negative or positive feedback interconnection. A *network* version of this can be formulated as follows.

Consider a multiagent system, corresponding to a directed graph $\mathcal{G}$ with N vertices and input-state-output systems $\Sigma_i, i = 1, \dots, N$, associated to these vertices. Furthermore, assume that the edges of the graph are specified by an $N \times N$ *adjacency*

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¹Or assuming the existence of coordinates in which $g(x)$ is constant, which is, under the assumption that $\text{rank}g(x) = m$, equivalent to the Lie brackets of the vector fields $g_1, \dots, g_m$ defined by the columns of $g(x)$ to be zero [233].

matrix [48, 114] $\mathcal{A}$ with elements 0, 1, corresponding to interconnections²

$$u_i = y_j \quad (8.35)$$

if and only if the (i, j) -th element of $\mathcal{A}$ is equal to 1.

Now assume that the systems Σ_i have L_2 -gain $\leq \gamma_i$, $i = 1, \dots, N$. This means that there exist storage functions $S_i : \mathcal{X}_i \rightarrow \mathbb{R}^+$ such that³

$$\dot{S}_i \leq \frac{1}{2}\gamma_i^2 \|u_i\|^2 - \frac{1}{2}\|y_i\|^2, \quad i = 1, \dots, N \quad (8.36)$$

Then define the following $N \times N$ matrix with nonnegative elements

$$\Gamma := \text{diag}(\gamma_1^2, \dots, \gamma_N^2) \mathcal{A} \quad (8.37)$$

The Perron–Frobenius theorem yields the following lemma.

Lemma 8.3.1 ([79]) *Denote by $z > 0$ a (column or row) vector z with all elements positive, and by $z \geq 0$ a vector with nonnegative elements. Furthermore, let I_N denote the $N \times N$ identity matrix.*

Consider an $N \times N$ matrix Γ with all nonnegative elements. Then there exists a vector $\mu > 0$ such that

$$\mu^T (\Gamma - I_N) < 0 \quad (8.38)$$

if and only if

$$r(\Gamma) < 1, \quad (8.39)$$

where $r(\Gamma)$ denotes the spectral radius of Γ .

Proof (If). If $r(\Gamma) < 1$ then by the Perron–Frobenius theorem there exists $\mu > 0$ such that $\mu^T \Gamma < \mu^T$, or equivalently $\mu^T (\Gamma - I_N) < 0$.

(Only if). Conversely, if $r(\Gamma) \geq 1$ then there exists $\nu \geq 0$, $\nu \neq 0$, such that $(\Gamma - I_N)\nu \geq 0$, whence for any $\mu > 0$ we have $\mu^T (\Gamma - I_N)\nu \geq 0$, contradicting $\mu^T (\Gamma - I_N) < 0$. $\square$

Based on this lemma we obtain the following theorem.

Theorem 8.3.2 (Small-gain network theorem) *Consider a directed graph $\mathcal{G}$ with systems Σ_i associated to its vertices, which have L_2 -gains $\leq \gamma_i$ with C^1 storage functions S_i , $i = 1, \dots, N$, and which are interconnected through the adjacency matrix $\mathcal{A}$ defined by (8.35). Consider the matrix Γ given by (8.37). If the spectral*

²The typical situation being that each row in the $\mathcal{A}$ matrix contains only one 1. Multiple occurrence of ones in a row is allowed but will imply an *equality constraint* on the corresponding outputs, leading to algebraic constraints between the state variables.

³The subsequent argumentation directly extends to the case of non-differentiable storage functions, replacing the differential dissipation inequalities by their integral counterparts.

radius $r(\Gamma) < 1$, then there exists $\mu > 0$ such that $\mu^T(\Gamma - I_N) < 0$ and the nonnegative function

$$S(x_1, \dots, x_N) := \sum_{i=1}^N \mu_i S_i(x_i) \quad (8.40)$$

satisfies along trajectories of the interconnected system

$$\dot{S} \leq -\varepsilon_1 \|y_1\|^2 - \varepsilon_2 \|y_2\|^2 \cdots - \varepsilon_N \|y_N\|^2 \quad (8.41)$$

for certain positive constants $\varepsilon_1, \dots, \varepsilon_N$. Hence, if S_i has a strict minimum at x_i^* , $i = 1, \dots, N$, then $x^* = (x_1^*, \dots, x_N^*)$ is a stable equilibrium of the interconnected system, which is asymptotically stable provided the interconnected system is zero-state detectable.

Proof Denote the vector with components $\|y_i\|^2$, $i = 1, \dots, N$, by $\widehat{y}$. Then

$$\begin{aligned} \dot{S} &= \sum_{i=1}^N \mu_i \dot{S}_i(x_i) \leq \frac{1}{2} \sum_{i=1}^N \mu_i (\gamma_i^2 \|u_i\|^2 - \|y_i\|^2) = \frac{1}{2} \mu^T (\Gamma - I_N) \widehat{y} \\ &\leq -\varepsilon_1 \|y_1\|^2 - \varepsilon_2 \|y_2\|^2 \cdots - \varepsilon_N \|y_N\|^2 \end{aligned} \quad (8.42)$$

for certain positive constants $\varepsilon_1, \dots, \varepsilon_N$. □

Example 8.3.3 In case of the feedback interconnection $u_1 = y_2$, $u_2 = y_1$ of two systems Σ_1 , Σ_2 with L_2 -gain $\leq \gamma_1$, respectively, $\leq \gamma_2$, application of Theorem 8.3.2 leads to the consideration of the matrix

$$\Gamma = \begin{bmatrix} 0 & \gamma_1^2 \\ \gamma_2^2 & 0 \end{bmatrix}, \quad (8.43)$$

which has spectral radius < 1 if and only if $\gamma_1 \cdot \gamma_2 < 1$; thus recovering the small-gain condition of Theorem 8.2.1.

8.4 L_2 -Gain as Response to Periodic Input Functions

An interesting interpretation of the L_2 -gain of a nonlinear state space system Σ given by (8.1) in terms of the response to *periodic* input functions was obtained in [140].

Assume that $f(0, 0) = 0$, and that the matrix $A := \frac{\partial f}{\partial x}(0, 0)$ has all its eigenvalues in the open left half plane. Let Σ have L_2 -gain $\leq \gamma$, and consider a *periodic* input function $u_p(\cdot)$ (of period $T > 0$), which is generated by a dynamical system (“*exo-system*”)

$$\Sigma_e : \dot{u} = s(u), \quad s(0) = 0, \quad (8.44)$$

whose linearization has all its eigenvalues on the imaginary axis. Then it follows from center manifold theory (see, e.g., [61]) that the series interconnection of the

exo-system Σ_e with Σ , given as

$$\begin{aligned}\dot{x} &= f(x, u) \\ \dot{u} &= s(u)\end{aligned}\tag{8.45}$$

and having augmented state (x, u) , has a *center manifold*, which is the graph $\mathcal{C} = \{(x, u) \mid x = c(u)\}$ of a mapping $x = c(u)$. Furthermore, for initial conditions close enough to $\mathcal{C}$ the solutions of the composed system (8.45) tend exponentially to $\mathcal{C}$, and thus, since $u_p(\cdot)$ is periodic of period T , the solution $x(\cdot)$ converges to a *steady state* solution $x_p(\cdot)$, with $x_p(t) = c(u_p(t))$, also having period T (see [141]).

It follows that $y_p(t) = h(x_p(t), u_p(t))$ has period T as well, and furthermore by the L_2 -gain dissipation inequality (8.2)

$$\int_{t_0}^{t_0+T} \|y_p(t)\|^2 dt \leq \gamma^2 \int_{t_0}^{t_0+T} \|u_p(t)\|^2 dt \tag{8.46}$$

(since $x_p(t_0 + T) = x_p(t_0)$), for all t_0 .

Defining the *rms* values of any periodic signal z with period T as

$$\|z\|_{rms} := \frac{1}{T} \left(\int_{t_0}^{t_0+T} \|z(t)\|^2 dt \right)^{1/2}, \tag{8.47}$$

the property that Σ has L_2 -gain $\leq \gamma$ therefore implies that

$$\|y_p\|_{rms} \leq \gamma \|u_p\|_{rms}, \tag{8.48}$$

for all periodic input functions u_p which are generated by the exo-system $\dot{u} = s(u)$, $s(0) = 0$, and all initial conditions close enough to $\mathcal{C}$.

8.5 Relationships with IIS- and iIS-Stability

An alternative approach to extending Lyapunov stability theory of autonomous systems $\dot{x} = f(x)$ to systems with inputs $\dot{x} = f(x, u)$ is offered by the theory of (integral-)Input-to-State Stability ((i)ISS). In this section, we will briefly indicate the close relationships of (i)ISS with dissipative systems theory and (a generalized) version of L_2 -gain. We will do so by *not* giving the original definitions of IIS and iIS, but instead by providing their equivalent characterizations in terms of dissipative systems theory.

By itself the close relation between dissipative systems theory on the one hand and stability theory based on the IIS or iIS property on the other is not surprising. Indeed, we have already seen in Chap. 3 how dissipativity implies Lyapunov stability properties of the system for $u = 0$. Furthermore, for $u \neq 0$ the dissipation inequality

may be used for deriving properties of the relationships between input, state and output functions.

Recall that a function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is of class $\mathcal{K}_\infty$, denoted $\alpha \in \mathcal{K}_\infty$, if it is continuous, strictly increasing, unbounded, and satisfies $\alpha(0) = 0$.

Turning the characterization of input-to-state stability (IIS) as given in [323] into a definition we formulate

Definition 8.5.1 ([323]) A system

$$\dot{x} = f(x, u), \quad f(0, 0) = 0, \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \quad (8.49)$$

is ISS if there exist functions $\alpha, \beta \in \mathcal{K}_\infty$ such that the input-state-output system $\dot{x} = f(x, u)$, $y = x$, is dissipative with respect to the supply rate

$$s(u, y) = \beta(\|u\|) - \alpha(\|y\|) \quad (8.50)$$

with a C^1 and radially unbounded storage function S satisfying $S(x) > 0$, $x \neq 0$, $S(0) = 0$.

Note the conceptual difference with dissipative systems theory (apart from minor technical differences like the a priori assumption of strict positivity and radial unboundedness of S). In dissipative systems theory, one starts with a *given* supply rate on the space of inputs and outputs, and derives properties of the system based on this. On the other hand, ISS theory aims at providing stability results for systems with inputs $\dot{x} = f(x, u)$, $f(0, 0) = 0$, and seeks criteria, which can be translated into the *existence* of a supply rate (8.50), where the functions α, β may not be easy to determine explicitly.

We remark that the supply rate (8.50) can be regarded as a generalization of the L_2 -gain supply rate. In fact, by taking α and β to be the quadratic functions $\alpha(r) = \frac{1}{2}r^2$ and $\beta(r) = \frac{1}{2}\gamma^2r^2$ for some constant γ one recovers the L_2 -gain $\leq \gamma$ case for $y = x$. Conversely, by allowing for arbitrary nonlinear coordinate transformations on the space $\mathcal{U} = \mathbb{R}^m$ of inputs and $\mathcal{X} = \mathbb{R}^n$ of states, the dissipation inequality for L_2 -gain (from u to $y = x$) can be seen to transform into (8.50) for certain $\alpha, \beta \in \mathcal{K}_\infty$. See also the discussion in Note 2 in Sect. 2.5 on the related notion of nonlinear L_2 -gain.

With regard to integral input-to-state stability (iISS) case we have the following characterization in terms of dissipative systems theory.

Definition 8.5.2 ([9]) A system (8.49) is iISS if there exists a function $\gamma \in \mathcal{K}_\infty$ and a function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\alpha(0) = 0$, $\alpha(r) > 0$, $r \neq 0$, such that the input-state-output system $\dot{x} = f(x, u)$, $y = x$, is dissipative with respect to the supply rate (8.50) with a C^1 and radially unbounded storage function S satisfying $S(x) > 0$, $x \neq 0$, $S(0) = 0$.

From the above characterization, it is clear that IIS implies iISS, while the converse does not hold. That is, there are systems which are iISS but not IIS. An example is provided by the system

$$\dot{x} = -x + ux \quad (8.51)$$

See, e.g., [9, 319] for a further discussion regarding the differences between ISS and iISS.

Another version of the relationship between iISS and dissipative systems theory is provided in the following proposition.

Proposition 8.5.3 ([9]) *A system 8.49 is iISS if and only if there exists a continuous output function $y = h(x)$ with $h(0) = 0$ for which the resulting system $\dot{x} = f(x, u)$, $y = h(x)$ is zero-detectable and there exists $\beta \in \mathcal{K}_\infty$ and a function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\alpha(0) = 0$, $\alpha(r) > 0$, $r \neq 0$, such that the system is dissipative with respect to the supply rate (8.50) with a C^1 and radially unbounded storage function S satisfying $S(x) > 0$, $x \neq 0$, $S(0) = 0$.*

Note that the “if” part of the last proposition with $\alpha(r) = \frac{1}{2}r^2$ and $\beta(r) = \frac{1}{2}\gamma^2r^2$ implies that a zero-detectable input-state-output system with finite L_2 -gain having a C^1 and radially unbounded storage function S satisfying $S(x) > 0$, $x \neq 0$, $S(0) = 0$, is iISS.

8.6 Notes for Chapter 8

1. Corollary 8.2.6 is a nonlinear generalization of a result given for linear systems in Xie & De Souza [358], and may be found in Shen & Tamura [313].
2. See Angeli [8] for further information regarding incremental L_2 -gain as discussed in Sect. 8.2.
3. Applications of incremental L_2 -gain to model reduction of nonlinear systems can be found in Besselink, van de Wouw & Nijmeijer [38].
4. Section 8.3, in particular Lemma 8.3.1, is largely based on Dashkovskiy, Ito & Wirth [79]. There is a wealth of literature on other network versions of the small-gain theorem; see, e.g., Jiang & Wang [150], Liu, Hill & Jiang [182], Rüffer [261].
5. See Pavlov, van de Wouw & Nijmeijer [254] and the references quoted therein for theory related to Sect. 8.4.
6. The notions of (integral-)Input-to-State Stability were introduced and explored by Sontag and co-workers, see, e.g., Sontag & Wang [323], Sontag [318, 319, 324], Angeli, Sontag & Wang [9] and the references quoted in there. See also the survey Sontag [321], and the account of the relation with dissipative systems theory provided in Isidori [139].
7. The characterization of the set of pairs of functions (β, α) such that the system is dissipative with respect to the supply rate (8.50) is addressed in Sontag & Teel [322].