University of Groningen

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Published in:
Communications and Control Engineering

DOI:
[10.1007/978-3-319-49992-5_11](https://doi.org/10.1007/978-3-319-49992-5_11)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2017

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

van der Schaft, A. (2017). Hamilton–Jacobi Inequalities. In *Communications and Control Engineering* (pp. 269–302). (Communications and Control Engineering; No. 9783319499918). Springer International Publishing. https://doi.org/10.1007/978-3-319-49992-5_11

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Chapter 11

Hamilton–Jacobi Inequalities

In the previous chapters, we have encountered at various places Hamilton–Jacobi equations, or, more generally, Hamilton–Jacobi inequalities. In this chapter, we take a closer look at conditions for *solvability* of Hamilton–Jacobi inequalities and the structure of their *solution set* using invariant manifold techniques for the corresponding Hamiltonian vector field (Sect. 11.1). In Sect. 11.2 we apply this to the nonlinear optimal control problem. An important theme will be the relation between Hamilton–Jacobi inequalities and the Riccati inequalities obtained by linearization, in particular for dissipativity (Sect. 11.3), and nonlinear $\mathcal{H}_\infty$ control (Sect. 11.4).

11.1 Solvability of Hamilton–Jacobi Inequalities

In Chaps. 3, 4, and 8 we considered differential dissipation inequalities

$$S_x(x) f(x, u) \leq s(u, h(x, u)), \quad \forall x, u \quad (11.1)$$

in an (unknown) storage function S . If the corresponding *pre-Hamiltonian*

$$K(x, p, u) := p^T f(x, u) - s(u, h(x, u)) \quad (11.2)$$

has a maximizing $u^*(x, p)$, that is

$$K(x, p, u) \leq K(x, p, u^*(x, p)), \quad \forall x, p, u, \quad (11.3)$$

then the dissipation inequality (11.1) is equivalent to

$$K(x, S_x^T(x), u^*(x, S_x^T(x))) \leq 0, \quad \forall x \quad (11.4)$$

Defining in this case the *Hamiltonian*

$$H(x, p) := K(x, p, u^*(x, p)) \quad (11.5)$$

we are thus led to *Hamiltonian–Jacobi inequalities*

$$H(x, S_x^T(x)) \leq 0, \quad \forall x \quad (11.6)$$

in the unknown S . Also in Chap. 9, we encountered Hamilton–Jacobi inequalities (in fact *equations*) in the factorization of nonlinear systems, while in Chap. 10 Hamilton–Jacobi inequalities turned out to be key in the study of the (sub-)optimal $\mathcal{H}_\infty$ control problem.

In this section, we will be concerned with deriving conditions for the solvability of Hamilton–Jacobi inequalities, and with the structure of their solution set. Many of the results presented in this section will not be proven here; proofs can be found in the references cited in the Notes at the end of this chapter.

We start on a general level. Consider an n -dimensional manifold M with local coordinates $x = (x_1, \dots, x_n)$. The *cotangent bundle* T^*M is a $2n$ -dimensional manifold, with *natural* local coordinates $(x, p) = (x_1, \dots, x_n, p_1, \dots, p_n)$ defined in the following way. Take any set of local coordinates $x = (x_1, \dots, x_n)$ for M . Let σ be any one-form on M (i.e., $\sigma(q) \in T_q^*M$ is a cotangent vector for every $q \in M$), in the coordinates $x = (x_1, \dots, x_n)$ for M expressed as

$$\sigma = \sigma_1 dx_1 + \sigma_2 dx_2 + \dots + \sigma_n dx_n \quad (11.7)$$

for certain smooth functions $\sigma_1(x), \dots, \sigma_n(x)$. Then, the natural coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$ for T^*M are defined by attaching to $\sigma(q)$ the coordinate values

$$(x_1(q), \dots, x_n(q), \sigma_1(x(q)), \dots, \sigma_n(x(q))), \quad (11.8)$$

with $x_i(\sigma(q)) = x_i(q)$, $p_i(\sigma(q)) = \sigma_i(q)$, $i = 1, \dots, n$

Given the natural coordinates $(x_1, \dots, x_n, p_1, \dots, p_n)$ for T^*M we may locally define the *canonical two-form* ω on T^*M as

$$\omega = \sum_{i=1}^n dp_i \wedge dx_i \quad (11.9)$$

The two-form ω is called the (canonical) symplectic form on the cotangent bundle T^*M .

Definition 11.1.1 An n -dimensional submanifold N of T^*M is *Lagrangian* if ω restricted to N is zero.

Now consider any C^2 function $S : M \rightarrow \mathbb{R}$, and the n -dimensional submanifold $N_S \subset T^*M$, in local coordinates given as

$$N_S = \left\{ (x, p) \in T^*M \mid p_i = \frac{\partial S}{\partial x_i}(x), i = 1, \dots, n \right\} \quad (11.10)$$

It can be immediately checked that N_S is *Lagrangian*. (In fact, this amounts to the property $\frac{\partial^2 S}{\partial x_i \partial x_j}(x) = \frac{\partial^2 S}{\partial x_j \partial x_i}(x)$, $i, j = 1, \dots, n$.) Conversely, defining the canonical projection

$$\pi : T^*M \rightarrow M, \quad (x, p) \mapsto x, \quad (11.11)$$

we obtain by Poincaré’s lemma (see, e.g., [1]):

Proposition 11.1.2 *Let N be a C^{k-1} Lagrangian submanifold of T^*M such that $\pi : N \rightarrow M$ is a C^{k-1} diffeomorphism. Then locally (or globally if M is simply connected) there exists a C^k function $S : M \rightarrow \mathbb{R}$ such that $N = N_S$.*

The property that $\pi : N \rightarrow M$ is a C^{k-1} diffeomorphism will be referred as “*parametrizability of N by the x -coordinates*”. Now take any C^k function $H : T^*M \rightarrow \mathbb{R}$ (not necessarily of the special type as obtained in (11.5)), and consider the Hamilton–Jacobi equality (equation)

$$H(x, S_x^T(x)) = 0 \quad (11.12)$$

in the unknown $S : M \rightarrow \mathbb{R}$. The Hamiltonian vector field X_H on T^*M corresponding to the Hamiltonian H is defined in natural coordinates as

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i}(x, p) \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i}(x, p) \end{aligned} \quad i = 1, \dots, n \quad (11.13)$$

There is a close connection between solutions of the Hamilton–Jacobi equation (11.12) and invariant submanifolds of the Hamiltonian vector field (11.13). Recall (see Chap. 3, Theorem 3.2.8) that a submanifold $N \subset T^*M$ is an *invariant manifold* for X_H if the solutions of (11.13) starting on N remain in N .

Proposition 11.1.3 *Let $S : M \rightarrow \mathbb{R}$, and consider the submanifold $N_S \subset T^*M$ locally given by (11.10). Then*

$$H(x, S_x^T(x)) = \text{constant, for all } x \in M, \quad (11.14)$$

if and only if N_S is an invariant submanifold for X_H .

Note that by subtracting a constant value from H (not changing the Hamiltonian vector field X_H), we may always reduce (11.14) to (11.12). Solutions of (11.12) may thus be obtained by looking for invariant Lagrangian submanifolds of X_H which are parametrizable by the x -coordinates and thus, by Proposition 11.1.2, of the form N_S for some S . Not every n -dimensional invariant submanifold of X_H is Lagrangian, but the following two *special* invariant submanifolds of X_H always are. Consider an equilibrium (x_0, p_0) of X_H , that is

$$\frac{\partial H}{\partial x_i}(x_0, p_0) = \frac{\partial H}{\partial p_i}(x_0, p_0) = 0, \quad i = 1, \dots, n. \quad (11.15)$$

Define $N^- \subset T^*M$ as the set of all points in T^*M converging along the solutions of the vector field X_H for $t \rightarrow \infty$ to (x_0, p_0) , and N^+ as the set of all points converging in *negative* time to (x_0, p_0) . Clearly, N^- and N^+ are invariant sets for X_H .

Proposition 11.1.4 *N^- and N^+ are submanifolds of T^*M , called the stable invariant manifold of X_H , respectively the unstable invariant manifold of X_H . Furthermore ω restricted to N^- and to N^+ is zero. Hence if $\dim N^- = n$ ($\dim N^+ = n$), then N^- (respectively N^+) is a Lagrangian submanifold of T^*M .*

Under additional conditions on the *linearization* of X_H at an equilibrium (x_0, p_0) we can be more explicit. Consider the linearization of X_H at (x_0, p_0) , that is the $2n \times 2n$ matrix

$$\begin{aligned} DX_H(x_0, p_0) &= \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial p}(x_0, p_0) & \frac{\partial^2 H}{\partial p^2}(x_0, p_0) \\ -\frac{\partial^2 H}{\partial x^2}(x_0, p_0) & -\frac{\partial^2 H}{\partial p \partial x}(x_0, p_0) \end{bmatrix} \\ &=: \begin{bmatrix} A & P \\ -Q & -A^T \end{bmatrix} =: \mathcal{H} \end{aligned} \quad (11.16)$$

The matrix $\mathcal{H} = DX_H(x_0, p_0)$ is a *Hamiltonian matrix*, that is

$$\mathcal{H}^T J + J \mathcal{H} = 0, \quad (11.17)$$

with J the linear symplectic form

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \quad (11.18)$$

It follows from (11.17) that if λ is an eigenvalue of $\mathcal{H}$, then so is $-\lambda$, and therefore the set of eigenvalues of $\mathcal{H}$ is symmetric with regard to the imaginary axis.

The equilibrium (x_0, p_0) is called *hyperbolic* if $\mathcal{H} = DX_H(x_0, p_0)$ does not have purely imaginary eigenvalues. This results in

Proposition 11.1.5 *Let (x_0, p_0) be a hyperbolic equilibrium of X_H . Then N^- and N^+ are Lagrangian submanifolds, and N^- (respectively N^+) is tangent at (x_0, p_0) to the stable (respectively unstable) generalized eigenspace of $DX_H(x_0, p_0)$.*

The linearization at (x_0, p_0) of the Hamiltonian vector field X_H given by (11.13) is given by

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ p \end{bmatrix}, \quad (11.19)$$

which is by itself a linear Hamiltonian vector field with respect to the linear symplectic form J (which can be understood as the evaluation of ω at (x_0, p_0)) and the

Hamiltonian

$$p^T Ax + \frac{1}{2} p^T P p + \frac{1}{2} x^T Q x, \quad (11.20)$$

consisting of the *quadratic* part of H . Equivalently, the symmetric matrix

$$\begin{bmatrix} Q & A^T \\ A & P \end{bmatrix} \quad (11.21)$$

is the Hessian of H at (x_0, p_0) .

The Hamilton–Jacobi equation (11.12) for this quadratic Hamiltonian reduces to the *Riccati equation*

$$A^T X + X A + X P X + Q = 0, \quad (11.22)$$

in the unknown symmetric $n \times n$ matrix X (not to be confused with the Hamiltonian vector field X_H). Conversely, the solutions $S(x)$ to (11.12) for a quadratic H given by (11.20) may be restricted to quadratic functions $S(x) = \frac{1}{2} x^T X x$, thus leading to (11.22).

Similarly to Definition 11.1.1, a Lagrangian *subspace* L of $\mathbb{R}^{2n}$ is an n -dimensional subspace such that J restricted to L is zero. If the Lagrangian subspace L is parametrizable by the x -coordinates (meaning that L and $\text{span} \begin{bmatrix} 0 \\ I_n \end{bmatrix}$ are complementary, i.e., $L \oplus \text{span} \begin{bmatrix} 0 \\ I_n \end{bmatrix} = \mathbb{R}^{2n}$), then as in Proposition 11.1.2 there exists $X = X^T$ such that

$$L = \text{span} \begin{bmatrix} I \\ X \end{bmatrix} \quad (11.23)$$

Furthermore

Proposition 11.1.6 *Let (x_0, p_0) be an equilibrium of X_H . Suppose $N \subset T^*M$ is an invariant Lagrangian submanifold of X_H through (x_0, p_0) . Then, the tangent space L to N at (x_0, p_0) is a Lagrangian subspace of $\mathbb{R}^{2n}$ which is an invariant subspace of $DX_H(x_0, p_0)$. In particular, if S is a solution to (11.12) with $\frac{\partial S}{\partial x_i}(x_0) = 0, i = 1, \dots, n$, then the Hessian matrix $X := S_{xx}(x_0)$ is a solution to (11.22).*

Suppose now we have found an invariant Lagrangian submanifold N of X_H through an equilibrium (x_0, p_0) , for example, N^- or N^+ . Then in view of Proposition 11.1.6 the question of parametrizability by the x -coordinates, and thus by Proposition 11.1.2 the existence of a solution S to the Hamilton–Jacobi equality (11.12), can be locally checked by investigating the parametrizability of the Lagrangian subspace L tangent to N at (x_0, p_0) . For this *linear* problem we may invoke the following proposition.

Proposition 11.1.7 *Consider the Hamiltonian matrix $\mathcal{H}$ given by (11.18). Let P be either ≥ 0 or ≤ 0 .*

- (i) If (A, P) is controllable, then every Lagrangian subspace of $\mathbb{R}^{2n}$ which is invariant for $\mathcal{H}$ is complementary to $\text{span} \begin{bmatrix} 0 \\ I_n \end{bmatrix}$ (that is, parametrizable by the x -coordinates).
- (ii) Assume that $\mathcal{H}$ does not have purely imaginary eigenvalues, implying that the stable eigenspace L^- of $\mathcal{H}$, as well as the unstable eigenspace L^+ , are Lagrangian. If (A, P) is stabilizable, then L^- and $\text{span} \begin{bmatrix} 0 \\ I_n \end{bmatrix}$ are complementary. If $(-A, P)$ is stabilizable, then L^+ and $\text{span} \begin{bmatrix} 0 \\ I_n \end{bmatrix}$ are complementary.

With regard to the Hamilton–Jacobi equation (11.12) we derive the following corollary.

Corollary 11.1.8 *Let $H : T^*M \rightarrow \mathbb{R}$ be a Hamiltonian function, with Hamilton–Jacobi equation (11.12), and Hamiltonian vector field X_H , satisfying $\frac{\partial H}{\partial x}(x_0, p_0) = \frac{\partial H}{\partial p}(x_0, p_0) = 0$, and with linearization $DX_H(x_0, p_0) = \mathcal{H}$ given by (11.18). Assume $P = \frac{\partial^2 H}{\partial p^2}(x_0, p_0)$ is either ≥ 0 or ≤ 0 .*

- (i) *If (A, P) is controllable, then every Lagrangian invariant submanifold of X_H is locally near (x_0, p_0) of the form N_S for a certain function $S(x)$ defined for x near x_0 .*
- (ii) *Let (x_0, p_0) be a hyperbolic equilibrium. If (A, P) is stabilizable then locally near (x_0, p_0) the stable invariant manifold N^- is given as N_{S^-} for a certain function $S^-(x)$ defined near x_0 . If $(-A, P)$ is stabilizable then N^+ is given as N_{S^+} for a certain $S^+(x)$ defined near x_0 .*

Let us now assume that the solutions S^- and S^+ to the Hamilton–Jacobi equation (11.12), corresponding to the stable, respectively unstable, invariant manifold, exist globally. It is to be expected that they have special properties among the set of all solutions to the Hamilton–Jacobi equation or even among the set of all solutions to the Hamilton–Jacobi inequality (11.6). In fact, for Hamiltonians H arising from dissipation inequalities we obtain the following results.

Proposition 11.1.9 *Consider a Hamiltonian function H given as (cf. (11.5)) $H(x, p) := K(x, p, u^*(x, p))$, where*

$$K(x, p, u) = p^T f(x, u) - s(u, h(x, u)) \quad (11.24)$$

satisfies (11.3). Additionally, let $f(0, 0) = 0$, $h(0, 0) = 0$, $s(0, 0) = 0$, and suppose $(0, 0)$ is a hyperbolic equilibrium of X_H . Suppose the stable and unstable invariant manifolds N^- and N^+ of X_H through $(0, 0)$ are globally parametrizable by the x -coordinates, leading to solutions S^- and S^+ to (11.12) with $S^-(0) = S^+(0) = 0$. Then every other solution S with $S(0) = 0$ to the dissipation inequality (11.1), or equivalently, to the Hamilton–Jacobi inequality (11.6), satisfies

$$S^-(x) \leq S(x) \leq S^+(x), \quad \forall x \in \mathcal{X} \quad (11.25)$$

Proof S^- satisfies (leaving out for convenience all the “transpose” signs)

$$S_x^-(x) f(x, u^*(x, S_x^-(x))) = s(u^*(x, S_x^-(x)), h(x, u^*(x, S_x^-(x)))), \quad (11.26)$$

and since it corresponds to the stable invariant manifold

$$f(x, u^*(x, S_x^-(x))) \text{ is glob. asymptotically stable w.r.t. } x = 0 \quad (11.27)$$

Let S be an arbitrary solution to the dissipation inequality (11.1). Then

$$S_x(x) f(x, u^*(x, S_x^-(x))) \leq s(u^*(x, S_x^-(x)), h(x, u^*(x, S_x^-(x)))) \quad (11.28)$$

Subtracting (11.26) from (11.28) yields

$$[S_x(x) - S_x^-(x)] f(x, u^*(x, S_x^-(x))) \leq 0 \quad (11.29)$$

and thus by integration along $\dot{x} = f(x, u^*(x, S_x^-(x)))$

$$[S(x(t_1)) - S^-(x(t_1))] \leq [S(x(t_0)) - S^-(x(t_0))] \quad (11.30)$$

Letting $t_1 \rightarrow \infty$, and using (11.27), it follows that $S(x(t_0)) \geq S^-(x(t_0))$ for every initial condition, proving the left-hand side of the inequality (11.25).

For proving the right-hand side we replace S^- by S^+ , noting that by definition of S^+

$$-f(x, u^*(x, S_x^+(x))) \text{ is glob. asymptotically stable w.r.t. } x = 0 \quad (11.31)$$

and therefore letting $t_0 \rightarrow -\infty$ in (11.30), with S^- replaced by S^+ , we obtain $S(x(t_1)) \leq S^+(x(t_1))$ for every $x(t_1)$. $\square$

Remark 11.1.10 It can be also shown that $S^-(x) < S^+(x)$ for all $x \in \mathcal{X}$, $x \neq 0$.

Up to now we did not address the issue of *nonnegativity* of solutions of the Hamilton–Jacobi inequality (11.6), which is especially important if (11.6) arises from the dissipation inequality (11.1). The following sufficient conditions are straightforward.

Proposition 11.1.11 *Consider the dissipation inequality (11.1) and the Hamilton–Jacobi inequality (11.6). Suppose $f(0, 0) = 0$, $h(0, 0) = 0$, and $s(0, 0) = 0$.*

- (i) *If $\dot{x} = f(x, 0)$ is globally asymptotically stable with respect to $x = 0$, then every solution S to (11.1) or (11.6), with $S(0) = 0$, will be nonnegative whenever $s(0, y) \leq 0$ for all y .*
- (ii) *Suppose S^- exists globally, and $s(u^*(x, S_x^-(x)), h(x, u^*(x, S_x^-(x)))) \leq 0$ for all x . Then every solution S to (11.1) or (11.6) with $S(0) = 0$ will be nonnegative.*

Proof (i) Consider $S_x(x)f(x, 0) \leq s(0, h(x, 0)) \leq 0$, and integrate along $\dot{x} = f(x, 0)$ to obtain $S(x(t_1)) \leq S(x(t_0))$. Letting $t_1 \rightarrow \infty$, and thus $S(x(t_1)) \rightarrow S(0) = 0$ yields the result.

(ii) Consider the inequality

$$S_x(x)f(x, u^*(x, S_x^T(x))) \leq s(u^*(x, S_x^T(x)), h(x, u^*(x, S_x^T(x)))) \leq 0,$$

and integrate along the globally asymptotically vector field $\dot{x} = f(x, u^*(x, S_x^T(x)))$. $\square$

Let us now apply the results obtained so far to the Hamilton–Jacobi inequalities encountered in the L_2 -gain analysis of nonlinear systems (Chap. 8) and in nonlinear $\mathcal{H}_\infty$ -control (Chap. 10). (We postpone the treatment of the Hamilton–Jacobi–Bellman equations appearing in Chap. 9 to Sect. 11.2.) For the L_2 -gain case we restrict attention to affine nonlinear systems

$$\Sigma_a : \begin{cases} \dot{x} = f(x) + g(x)u, & f(0) = 0 \\ y = h(x), & h(0) = 0 \end{cases} \quad (11.32)$$

with regard to the L_2 -gain supply rate

$$s(u, y) = \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2 \quad (11.33)$$

The Hamiltonian H now takes the form

$$H(x, p) = p^T f(x) + \frac{1}{2} \frac{1}{\gamma^2} p^T g(x) g^T(x) p + \frac{1}{2} h^T(x) h(x), \quad (11.34)$$

leading to the Hamilton–Jacobi inequality

$$S_x(x)f(x) + \frac{1}{2} \frac{1}{\gamma^2} S_x(x)g(x)g^T(x)S_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0, \quad (11.35)$$

and the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} F & \frac{1}{\gamma^2}GG^T \\ -H^T H & -F^T \end{bmatrix}, \quad F = \frac{\partial f}{\partial x}(0), \quad G = g(0), \quad H = \frac{\partial h}{\partial x}(0) \quad (11.36)$$

Corollary 11.1.8 and Propositions 11.1.9, 11.1.11 all apply in this case, and we obtain:

Corollary 11.1.12 *Assume $\mathcal{H}$ does not have purely imaginary eigenvalues. If (F, G) , respectively $(-F, G)$ is stabilizable, then locally about $x = 0$ there exists a solution S^- , respectively S^+ , to (11.35) with equality, such that $S^-(0) = 0$, $S^+(0) = 0$, and*

$$\begin{aligned}
& f(x) + \frac{1}{\gamma^2} g(x) g^T(x) S_x^-(x) \text{ asymptotically stable} \\
& - \left[f(x) + \frac{1}{\gamma^2} g(x) g^T(x) S_x^+(x) \right] \text{ asymptotically stable}
\end{aligned} \tag{11.37}$$

If S^- and S^+ exist globally, then any other solution S of (11.35) with $S(0) = 0$ satisfies

$$S^-(x) \leq S(x) \leq S^+(x), \quad \forall x \in \mathcal{X} \tag{11.38}$$

In particular, if $S^- \geq 0$, which is the case (see Proposition 11.1.11) if $\dot{x} = f(x)$ is globally asymptotically stable, then $S^- = S_a$ (the available storage, cf. Theorem 3.1.11), and whenever S_r (the required supply from $x^* = 0$) exists, then $S^+ = S_r$ (see Theorem 3.1.16).

Next let us consider the Hamilton–Jacobi inequality (HJ2a) arising in the output feedback $\mathcal{H}_\infty$ control problem, that is (see (10.71))

$$R_x(x) a(x) + \frac{1}{2} \frac{1}{\gamma^2} R_x(x) g(x) g^T(x) R_x^T(x) + \frac{1}{2} h^T(x) h(x) - \frac{1}{2} \gamma^2 c^T(x) c(x) \leq 0$$

corresponding to the dissipation inequality (see (11.103))

$$R_x(x) [a(x) + g(x) d_1] \leq \frac{1}{2} \gamma^2 \|d_1\|^2 + \frac{1}{2} \gamma^2 \|c(x)\|^2 - \frac{1}{2} \|h(x)\|^2 \tag{11.39}$$

The Hamiltonian H for this case is given as

$$H(x, p) = p^T a(x) + \frac{1}{2} \frac{1}{\gamma^2} p^T g(x) g^T(x) p + \frac{1}{2} h^T(x) h(x) - \frac{1}{2} \gamma^2 c^T(x) c(x) \tag{11.40}$$

and leads to the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A & \frac{1}{\gamma^2} G G^T \\ -H^T H + C^T C & -A^T \end{bmatrix}, \tag{11.41}$$

$$A = \frac{\partial a}{\partial x}(0), \quad G = g(0), \quad H = \frac{\partial h}{\partial x}(0), \quad C = \frac{\partial c}{\partial x}(0)$$

In this case only Corollary 11.1.8 and Proposition 11.1.9 apply, while the conditions of Proposition 11.1.11 will not be satisfied. Thus we obtain:

Corollary 11.1.13 Assume $\mathcal{H}$ given by (11.41) does not have purely imaginary eigenvalues. If (F, G) (resp. $(-F, G)$) is stabilizable then locally about $x = 0$ there exists a solution S^- (resp. S^+) to (11.35) with equality, such that $S^-(0) = 0$, $S^+(0) = 0$, and

$$\begin{aligned}
& f(x) + \frac{1}{\gamma^2} g(x) g^T(x) S_x^-(x) \text{ asymptotically stable} \\
& -[f(x) + \frac{1}{\gamma^2} g(x) g^T(x) S_x^+(x)] \text{ asymptotically stable}
\end{aligned} \tag{11.42}$$

If S^- and S^+ exist globally then any other solution S of (11.35) with $S(0) = 0$ satisfies

$$S^-(x) \leq S(x) \leq S^+(x), \quad \forall x \in \mathcal{X} \tag{11.43}$$

In particular, if there exists a solution $S \geq 0$ to (11.35) with $S(0) = 0$, and if S_r (the required supply for ground state $x^* = 0$) for the dissipation inequality (11.39) exists (i.e., is finite) and is C^1 , then $S^+ = S_r \geq 0$.

Proof Only the last statement needs some clarification. If there exists $S \geq 0$ to (11.39), and if S_r exists and is C^1 then S_r is solution of (11.39) and (11.35), and by Theorem 3.1.6 equals the maximal solution. $\square$

Finally, let us consider the Hamilton–Jacobi inequality (HJ1a) of the state feedback $\mathcal{H}_\infty$ control problem, for affine nonlinear systems given as (see (10.3))

$$\begin{aligned}
P_x(x) a(x) + \frac{1}{2} P_x(x) \left[\frac{1}{\gamma^2} g(x) g^T(x) - b(x) b^T(x) \right] P_x^T(x) \\
+ \frac{1}{2} h^T(x) h(x) \leq 0
\end{aligned} \tag{11.44}$$

This Hamilton–Jacobi inequality does *not* correspond to a dissipation inequality. The Hamiltonian

$$H(x, p) = p^T a(x) + \frac{1}{2} p^T \left[\frac{1}{\gamma^2} g(x) g^T(x) - b(x) b^T(x) \right] p + \frac{1}{2} h^T(x) h(x) \tag{11.45}$$

leads to the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A & \frac{1}{\gamma^2} G G^T - B B^T \\ -H^T H & -A^T \end{bmatrix}, \tag{11.46}$$

$$A = \frac{\partial a}{\partial x}(0), \quad G = g(0), \quad B = b(0), \quad H = \frac{\partial h}{\partial x}(0),$$

and we conclude that Corollary 11.1.8 and Propositions 11.1.9, 11.1.11 do not apply. Of course, the general Proposition 11.1.5 may still be invoked, guaranteeing local existence of a solution P to (HJ1a) if $\mathcal{H}$ does not have purely imaginary eigenvalues and its stable (or unstable) eigenspace can be parametrized by the x -coordinates. We will come back to this in Sect. 11.4. The same remarks, of course, can be made about the general Hamilton–Jacobi inequality (HJ1) of the state feedback $\mathcal{H}_\infty$ control problem, that is (see (10.21))

$$K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), u^*(x, P_x^T(x))) \leq 0 \quad (11.47)$$

where (see (10.19))

$$K_\gamma(x, p, d, u) = p^T f(x, u, d) - \frac{1}{2}\gamma^2 \|d\|^2 + \frac{1}{2}\|h(x, u, d)\|^2 \quad (11.48)$$

On the other hand, we will now show that, under some additional conditions, the solution set of (11.47), or, a fortiori, of (11.44), contains a *minimal* element which corresponds to the *stabilizing* solution of (11.47) or (11.44). Let us throughout assume that

$$f(0, 0, 0) = 0, \quad h(0, 0, 0) = 0, \quad (11.49)$$

and that the Hamiltonian $H(x, p)$ corresponding to (11.48), i.e.,

$$H(x, p) := K_\gamma(x, p, d^*(x, p), u^*(x, p)) \quad (11.50)$$

has a Hamiltonian matrix $\mathcal{H} = DX_H(0, 0)$, see (11.16), which does not have purely imaginary eigenvalues. Now consider an arbitrary solution $P \geq 0$ to (11.47) with $P(0) = 0$. Define the feedback

$$\alpha_0(x) := u^*(x, P_x^T(x)) \quad (11.51)$$

It immediately follows that the closed-loop system

$$\Sigma_0 : \begin{cases} \dot{x} = f(x, \alpha_0(x), d) \\ z = h(x, \alpha_0(x), d) \end{cases} \quad (11.52)$$

has L_2 -gain $\leq \gamma$ (from d to z), since

$$K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), \alpha_0(x)) \leq 0 \quad (11.53)$$

Now define the Hamiltonian

$$H_0(x, p) := K_\gamma(x, p, d^*(x, p), \alpha_0(x)) \quad (11.54)$$

This Hamiltonian is of the type as considered in Proposition 11.1.9. Assume that the stabilizing solution $P^1 := S^-$ with $P^1(0) = 0$ to $H_0(x, S_x^T(x)) = 0$ exists globally, and that

$$\dot{x} = f(x, \alpha_0(x), 0) \text{ is globally asymptotically stable} \quad (11.55)$$

Then it follows from Propositions 11.1.9 and 11.1.11(i) that

$$0 \leq P^1 \leq P \quad (11.56)$$

Subsequently, define the feedback

$$\alpha_1(x) := u^*(x, P_x^{1T}(x)) \quad (11.57)$$

Then, by using the second inequality of (10.20)

$$\begin{aligned} 0 &= H_0(x, P_x^{1T}(x)) = K_\gamma(x, P_x^{1T}(x), d^*(x, P_x^{1T}(x)), \alpha_0(x)) \\ &\geq K_\gamma(x, P_x^{1T}(x), d^*(x, P_x^{1T}(x)), u^*(x, P_x^{1T}(x))) \\ &= K_\gamma(x, P_x^{1T}(x), d^*(x, P_x^{1T}(x)), \alpha_1(x)) \end{aligned} \quad (11.58)$$

Hence also the closed-loop system

$$\Sigma_1 : \begin{cases} \dot{x} = f(x, \alpha_1(x), d) \\ z = h(x, \alpha_1(x), d) \end{cases} \quad (11.59)$$

has L_2 -gain $\leq \gamma$, with storage function P^1 . Define the corresponding Hamiltonian

$$H_1(x, p) := K_\gamma(x, p, d^*(x, p), \alpha_1(x)), \quad (11.60)$$

and assume again that the stabilizing solution $P^2 := S^-$ with $P^2(0) = 0$ to $H_1(x, S_x^T(x)) = 0$ exists globally, while

$$\dot{x} = f(x, \alpha_1(x), 0) \text{ is globally asymptotically stable} \quad (11.61)$$

Application of Propositions 11.1.9 and 11.1.11 then yields

$$0 \leq P^2 \leq P^1 \leq P, \quad (11.62)$$

and we subsequently define the feedback

$$\alpha_2(x) := u^*(x, P_x^{2T}(x)) \quad (11.63)$$

Continuing this process we arrive at the following conclusion

Proposition 11.1.14 *Consider the Hamilton–Jacobi inequality (11.47), and assume that (11.49) holds, while $DX_H(0, 0)$ corresponding to (11.50) does not have purely imaginary eigenvalues. Let $P \geq 0$ be any solution to (11.47). Define as above inductively $\alpha_0, \alpha_1, \dots$ and $H_0, H_1, \dots$. Assume that the vector fields $\dot{x} = f(x, \alpha_i(x), 0)$, $i = 0, 1, \dots$ are all globally asymptotically stable, and that the stabilizing solutions $P^1, P^2, \dots$ to $H_i(x, S_x^T(x)) = 0$ with $P^i(0) = 0$ exist globally. Then*

$$P \geq P^1 \geq P^2 \geq \dots \geq 0 \quad (11.64)$$

Assume that the pointwise limit $P^*(x) := \lim_{i \rightarrow \infty} P^i(x)$ is C^1 . Then $P^* \leq P$ is the unique solution to (11.47) with equality, with the property that

$$\dot{x} = f(x, u^*(x, P_x^{*T}(x)), d^*(x, P_x^{*T}(x))) \quad (11.65)$$

is globally asymptotically stable

Proof P^i is the unique solution to

$$K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), u^*(x, P_x^{(i-1)T}(x))) = 0, \quad (11.66)$$

with the property that

$$\dot{x} = f(x, u^*(x, P_x^{(i-1)T}(x)), d^*(x, P_x^T(x))) \quad (11.67)$$

is globally asymptotically stable

Therefore P^* is a solution to (11.47) with equality, such that the vector field in (11.65) is at least *stable*. However by assumption $DX_H(0, 0)$ does not have imaginary eigenvalues, and so necessarily (11.65) holds. $\square$

Note that if the conditions of Proposition 11.1.14 are satisfied for all solutions $P \geq 0$ to (11.47), then P^* , the stabilizing solution of (11.47), is also the *minimal* solution of (11.47).

Remark 11.1.15 With a similar procedure and under similar conditions, one can show that the *anti-stabilizing* solution of (11.47) is also the *maximal* one.

11.2 An Aside on Optimal Control

In this section, we apply some of the techniques developed in the preceding section to the standard optimal control problem as already encountered in Chaps. 3 and 9. It will be seen that this problem is indeed very much related to the dissipation inequalities and Hamilton–Jacobi inequalities treated above.

Let us consider the *infinite horizon optimal control problem*

$$\min_u \int_0^\infty L(x(t), u(t)) dt, \quad (11.68)$$

$$\dot{x} = f(x, u),$$

where $L(x, u)$ is a cost function, which we assume to be non-negative, that is

$$L(x, u) \geq 0, \quad \forall x, u \quad (11.69)$$

We associate with (11.68) the *reversed dissipation inequality*

$$V_x(x) f(x, u) + L(x, u) \geq 0, \quad \forall x, u \quad (11.70)$$

in the unknown V . The basic reason for introducing this inequality is that by the *principle of optimality* the *value function*

$$\bar{V}(x_0) := \min_u \left\{ \int_0^\infty L(x(t), u(t)) dt \mid \dot{x} = f(x, u), x(0) = x_0 \right\} \quad (11.71)$$

satisfies (whenever it exists) for all $t_1 \geq t_0$, and for all $u(\cdot)$

$$\bar{V}(x(t_0)) \leq \bar{V}(x(t_1)) + \int_{t_0}^{t_1} L(x(t), u(t)) dt, \quad (11.72)$$

and thus satisfies, if it is differentiable, the inequality (11.70).

In order to apply the invariant manifold techniques of Sect. 11.1 we will throughout assume that

$$f(0, 0) = 0, L(0, 0) = 0, \frac{\partial L}{\partial x}(0, 0) = 0, \frac{\partial L}{\partial u}(0, 0) = 0 \quad (11.73)$$

Define the linearizations

$$F = \frac{\partial f}{\partial x}(0), G = \frac{\partial f}{\partial u}(0, 0),$$

$$Q = \frac{\partial^2 L}{\partial x^2}(0, 0), R = \frac{\partial^2 L}{\partial u^2}(0, 0), N = \frac{\partial^2 L}{\partial x \partial u}(0, 0)$$

From (11.69) it follows that

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \geq 0, \quad (11.74)$$

and we will moreover assume the *regularity* condition

$$R > 0 \quad (11.75)$$

Defining the pre-Hamiltonian corresponding to (11.70) as

$$K(x, p, u) := p^T f(x, u) + L(x, u), \quad (11.76)$$

then by the Implicit Function theorem the condition (11.75) implies that at least locally near $(0, 0)$ there exists $u^*(x, p)$ such that

$$K(x, p, u) \geq K(x, p, u^*(x, p)) =: H(x, p), \quad \forall x, p, \quad (11.77)$$

but we will assume that $u^*(x, p)$ exists *globally*. Under this assumption the dissipation inequality (11.70) is equivalent to the (reversed) Hamilton–Jacobi inequality

$$H(x, V_x^T(x)) \geq 0, \quad \forall x \in \mathcal{X} \quad (11.78)$$

The Hamiltonian matrix $\mathcal{H} = DX_H(0, 0)$ corresponding to H can be immediately computed as

$$\mathcal{H} = \begin{bmatrix} F - GR^{-1}N^T & -GR^{-1}G^T \\ -Q + NR^{-1}N^T & -(F - GR^{-1}N^T)^T \end{bmatrix} \quad (11.79)$$

Because of (11.74) and (11.75) we see that not only the right upper block of $\mathcal{H}$ is non-positive but also the left-lower block (being a Schur complement of a non-positive matrix). This allows us to state the following strengthened version of Proposition 11.1.7(ii) (see the references in the notes at the end of this chapter).

Proposition 11.2.1 *Consider $\mathcal{H}$ given by (11.79). Assume the pair $(F - GR^{-1}N^T, GR^{-1}G^T)$ is stabilizable, and assume that the purely imaginary eigenvalues of $F - GR^{-1}N^T$ (if existing) are $(Q - NR^{-1}N^T)$ -detectable. Then $\mathcal{H}$ does not have purely imaginary eigenvalues, and the stable eigenspace L^- of $\mathcal{H}$ is Lagrangian and complementary to $\text{span} \begin{bmatrix} 0 \\ I_n \end{bmatrix}$. Hence $L^- = \text{span} \begin{bmatrix} I \\ X^- \end{bmatrix}$ for some symmetric matrix X^- , satisfying $X^- \geq 0$. If the pair $(Q - NR^{-1}N^T, F - GR^{-1}N^T)$ is observable then $X^- > 0$.*

As in Corollary 11.1.8 it follows that under the assumptions of Proposition 11.2.1 the stable invariant manifold N^- of X_H through the equilibrium $(0, 0)$ is given as N_{V^-} for a certain function $V^-(x)$ defined near 0. Let us assume that V^- exists globally. Note that $\frac{\partial V^-}{\partial x}(0) = 0$, $\frac{\partial^2 V^-}{\partial x^2}(0) = X^-$, and without loss of generality $V^-(0) = 0$.

Proposition 11.2.2 *Assume that the stable invariant manifold of X_H through $(0, 0)$ is given as N_{V^-} with $V^-(0) = 0$. Then $V^- \geq 0$, and every other solution V of (11.70) with $V(0) = 0$ satisfies*

$$V(x) \leq V^-(x), \quad \forall x \in \mathcal{X} \quad (11.80)$$

Proof Similar to Proposition 11.1.9 we consider

$$V_x^-(x)f(x, u^*(x, V_x^-(x))) + L(x, u^*(x, V_x^-(x))) = 0 \quad (11.81)$$

where

$$f(x, u^*(x, V_x^-(x))) \text{ is glob. asymptotically stable w.r.t. } x = 0 \quad (11.82)$$

By integration of (11.81) from 0 to T , and letting $T \rightarrow \infty$, using (11.82) and non-negativity of $L(x, u)$ (see (11.69)) it immediately follows that $V^- \geq 0$. Since any solution V to (11.70) satisfies

$$V_x(x)f(x, u^*(x, V_x^-(x))) + L(x, u^*(x, V_x^-(x))) \geq 0 \quad (11.83)$$

we obtain

$$\left[V_x^-(x) - V_x(x) \right] f(x, u^*(x, V_x^-(x))) \leq 0, \quad (11.84)$$

and the result follows from integration from 0 to T , and letting $T \rightarrow \infty$, using again (11.82). $\square$

Remark 11.2.3 Analogously we may prove that every solution V to (11.70) satisfies $V^+(x) \leq V(x)$, with N_{V^+} the *unstable* invariant manifold of X_H .

Remark 11.2.4 Similarly we may look at Hamilton–Jacobi–Bellman equations corresponding to optimal control problems in *reversed* time, where we replace the integral $\int_0^\infty$ by $\int_{-\infty}^0$. In particular, in Sect. 9.1 we studied the Hamilton–Jacobi–Bellman equation

$$W_x(x) f(x) + \frac{1}{2} W_x(x) g(x) g^T(x) W_x^T(x) - \frac{1}{2} h^T(x) h(x) = 0 \quad (11.85)$$

corresponding to a reversed-time optimal control problem. The Hamiltonian matrix in this case is

$$\mathcal{H} = \begin{bmatrix} F & GG^T \\ H^T H & -F^T \end{bmatrix}, \quad (11.86)$$

with F, G as in (11.74), and $H = \frac{\partial h}{\partial x}(0)$. By considering $-\mathcal{H}$ and applying Proposition 11.2.1 it immediately follows that if $(-F, G)$ is stabilizable, and the purely imaginary eigenvalues of F are H -detectable, then $\mathcal{H}$ will not have purely imaginary eigenvalues, and the *unstable* eigenspace L^+ of $\mathcal{H}$ is of the form $\text{span} \begin{bmatrix} I \\ X^+ \end{bmatrix}$ with $X^+ \geq 0$ and $X^+ > 0$ if (H, F) is observable. It follows that at least locally there exists a solution $W \geq 0$ to (11.85), satisfying $W(0) = 0$, $W_x(0) = 0$, $W_{xx}(0) = X^+$.

For the solution of the optimal control problem the following observation is crucial. Let V be any solution to (11.78) *with equality*, that is, the Hamilton–Jacobi(–Bellman) equation

$$H(x, V_x^T(x)) = 0 \quad (11.87)$$

Then by (11.77)

$$R(x, u) := K(x, V_x^T(x), u) \geq 0 \quad (11.88)$$

for all x, u . Since $K(x, V_x^T(x), u) = V_x(x) f(x, u) + L(x, u)$ we obtain by integration

$$\int_0^T L(x(t), u(t)) dt = \int_0^T R(x(t), u(t)) dt + V(x(0)) - V(x(T)) \quad (11.89)$$

for every input function $u(\cdot)$, and for every T .

Proposition 11.2.5 Consider any globally defined solution V to (11.87) with the property that $V(0) = 0$ and the feedback $u = -g^T(x)V_x^T(x)$ is asymptotically stabilizing. Then

$$V(x_0) = \min_u \left\{ \int_0^\infty L(x(t), u(t)) dt \mid \dot{x} = f(x, u), x(0) = x_0, \lim_{t \rightarrow \infty} x(t) = 0 \right\} \quad (11.90)$$

Additionally, since $L(x, u) \geq 0$ this implies that for any such V we have $V \geq 0$. In particular, assume that V^- exists globally, then the above holds for V^- , with optimal stabilizing control given in feedback form as

$$u = u^*(x, V_x^{-T}(x)) \quad (11.91)$$

Proof Since $\lim_{T \rightarrow \infty} x(T) = 0$ the right-hand side of (11.89) is minimized by substituting (11.91). $\square$

On the other hand, it is not clear if (11.91) is also the optimal control corresponding to the original optimal control problem (11.68) *without* the terminal constraint $\lim_{t \rightarrow \infty} x(t) = 0$. Indeed, let V be any other *non-negative* solution to (11.87). Then by considering (11.89) for this V , and noting that $V(x(T)) \geq 0$, while $R(x, u)$ can be rendered zero by choosing $u = u^*(x, V_x^T(x))$, it immediately follows that

$$\min_u \left\{ \int_0^\infty L(x(t), u(t)) dt \mid \dot{x} = f(x, u), x(0) = x_0 \right\} \leq V(x_0) \quad (11.92)$$

Thus if there exists a *non-negative* solution V with $V(0) = 0$ to (11.87) *different* from V^- , that is (see Proposition 11.2.2), $V(x) < V^-(x)$ for some $x \in \mathcal{X}$, then (11.91) will *not* be the optimal control and V^- will *not* be the value function $\bar{V}$. The existence of non-negative solutions V to (11.87) different from V^- can be excluded by imposing a (nonlinear) *detectability* condition.

Proposition 11.2.6 Let V^- exist globally, and let the optimal control problem (11.68) be solvable. Suppose $f(x, u)$, $L(x, u)$ satisfy the following detectability property

$$\lim_{t \rightarrow \infty} L(x(t), u(t)) = 0, \text{ along solutions of } \dot{x}(t) = f(x(t), u(t)), \quad (11.93)$$

$$\text{implies } \lim_{t \rightarrow \infty} x(t) = 0$$

Then the only non-negative solution V to (11.87) with $V(0) = 0$ is V^- . Furthermore, the solution to the optimal control problem (11.68) is given by (11.90) with value function $\bar{V} = V^-$.

Proof If $\min_u \int_0^\infty L(x(t), u(t))dt$ exists, then necessarily along the optimal trajectory $\lim_{t \rightarrow \infty} L(x(t), u(t)) = 0$, and thus by (11.93) $\lim_{t \rightarrow \infty} x(t) = 0$. Thus the optimal control problem (11.68) is the same as the optimal control problem (11.90), which has by Proposition 11.2.5 the solution (11.91) with value function V^- . By the reasoning preceding Proposition 11.2.6, see (11.92), it thus follows that there cannot exist any non-negative solution V to (11.87) with $V(0) = 0$ different from V^- . $\square$

Remark 11.2.7 Note that *without* the detectability condition (11.93) the value function $\bar{V}$ (which in this case may be different from V^-) still satisfies the property $\lim_{t \rightarrow \infty} \bar{V}(x(t)) = 0$ along optimal trajectories.

Finally, let us compare (11.89), which is derived under the *regularity* assumption of existence of $u^*(x, p)$ satisfying (11.77), with the general dissipation inequality (11.70), written in integral form as

$$\int_0^T L(x(t), u(t))dt \geq V(x(0)) - V(x(T)) \quad (11.94)$$

for every input function u . Clearly, the Eq. (11.94) contains much less information than (11.89), but it already suffices to draw the following conclusion, which is also valid if $u^*(x, p)$, and thus $H(x, p)$ in (11.77), cannot be defined. The following proposition was already stated in a more restricted form in Sect. 9.4 as Proposition 9.4.8.

Proposition 11.2.8 *For every x_0 define $V^*(x_0)$ as*

$$\min_u \left\{ \int_0^\infty L(x(t), u(t))dt \mid \dot{x} = f(x, u), x(0) = x_0, \lim_{t \rightarrow \infty} x(t) = 0 \right\} \quad (11.95)$$

and assume that V^ exists for every x_0 . Then*

- (i) $V^* \geq 0$ satisfies the dissipation inequality (11.94), and $V^*(0) = 0$.
- (ii) Let V satisfy (11.94) and $V(0) = 0$, then $V(x) \leq V^*(x)$ for every $x \in \mathcal{X}$.

Proof Consider (11.94) for $u(\cdot)$ such that $\lim_{T \rightarrow 0} x(T) = 0$. Then

$$\int_0^\infty L(x(t), u(t))dt \geq V(x(0)),$$

and by definition of V^* we obtain $V \leq V^*$. $\square$

11.3 Dissipativity of a Nonlinear System and Its Linearization

In this section we will relate the dissipativity of a nonlinear state space system

$$\Sigma : \begin{aligned} \dot{x} &= f(x, u), \quad f(0, 0) = 0 \\ y &= h(x, u), \quad h(0, 0) = 0 \end{aligned} \quad (11.96)$$

to the dissipativity of its *linearization* at $(x, u) = (0, 0)$, and vice versa.

We will do so for supply rates $s(u, y)$ satisfying

$$s(0, 0) = 0, \quad \frac{\partial s}{\partial u}(0, 0) = 0, \quad \frac{\partial s}{\partial y}(0, 0) = 0, \quad (11.97)$$

including the (output or input strict) passivity and L_2 -gain supply rates. Moreover, we throughout assume that there exists a storage function $S \geq 0$ for the dissipation inequality

$$S_x(x) f(x, u) \leq s(u, h(x, u)), \quad (11.98)$$

which is C^2 and has a minimum at $x = 0$ with $S(0) = 0$, implying that

$$S(0) = 0, \quad \frac{\partial S}{\partial x}(0) = 0, \quad X := \frac{\partial^2 S}{\partial x^2}(0) \geq 0 \quad (11.99)$$

Defining

$$\begin{aligned} A &= \frac{\partial f}{\partial x}(0, 0), \quad B = \frac{\partial f}{\partial u}(0, 0), \quad C = \frac{\partial h}{\partial x}(0, 0), \quad D = \frac{\partial h}{\partial u}(0, 0) \\ P &= \frac{\partial^2 s}{\partial u^2}(0, 0), \quad Q = \frac{\partial^2 s}{\partial y^2}(0, 0), \quad R = \frac{\partial^2 s}{\partial u \partial y}(0, 0) \end{aligned} \quad (11.100)$$

it follows by collecting the quadratic terms in (11.98) that

$$\frac{1}{2} \bar{x}^T X (A \bar{x} + B \bar{u}) \leq \frac{1}{2} [\bar{u}^T \quad \bar{y}^T] \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}, \quad \forall \bar{x}, \bar{u} \quad (11.101)$$

where $\bar{y} := C \bar{x} + D \bar{u}$.

This implies that the *linearized system*

$$\bar{\Sigma} : \begin{aligned} \dot{\bar{x}} &= A \bar{x} + B \bar{u} \\ \bar{y} &= C \bar{x} + D \bar{u} \end{aligned} \quad (11.102)$$

is *dissipative* with respect to the quadratic supply rate

$$\bar{s}(\bar{u}, \bar{y}) := \frac{1}{2} \begin{bmatrix} \bar{u}^T & \bar{y}^T \end{bmatrix} \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{y} \end{bmatrix}, \quad (11.103)$$

with storage function $\frac{1}{2} \bar{x}^T X \bar{x} = \frac{1}{2} \bar{x}^T \frac{\partial^2 S}{\partial \bar{x}^2}(0) \bar{x}$. In this sense, dissipativity of the nonlinear system Σ with respect to the supply rate s implies dissipativity of its linearization $\bar{\Sigma}$ with respect to the supply rate $\bar{s}$.

Conversely, let us investigate under which conditions dissipativity of the linearized system $\bar{\Sigma}$ implies dissipativity of the nonlinear system Σ .

Proposition 11.3.1 *Consider the nonlinear system Σ given by (11.96), with linearization $\bar{\Sigma}$ given by (11.102). Consider the supply rate s satisfying (11.97) with quadratic part $\bar{s}$ defined by (11.103) and (11.100). Suppose that $\bar{\Sigma}$ is dissipative with respect to the supply rate $\bar{s}$. Assume $D = 0$ and $P > 0$, and that the Hamiltonian matrix*

$$\mathcal{H} = \begin{bmatrix} A - BP^{-1}RC & BP^{-1}B^T \\ C^T QC & -(A - BP^{-1}RC)^T \end{bmatrix} \quad (11.104)$$

does not have purely imaginary eigenvalues. Also assume that the pair $(A - BP^{-1}RC, BP^{-1}B^T)$ is stabilizable. Then there exists a neighborhood V of $x = 0$ and $U \subset \mathbb{R}^m$ of $u = 0$, and a function $S : V \subset \mathcal{X} \rightarrow \mathbb{R}$ with $S(0) = 0$, $\frac{\partial S}{\partial x}(0) = 0$, such that

$$S_x(x) f(x, u) \leq s(u, h(x, u)), \text{ for all } x \in V, \text{ all } u \in U \quad (11.105)$$

Thus if $S \geq 0$ then Σ is locally dissipative with respect to the supply rate s .

Proof The pre-Hamiltonian corresponding to (11.105) is

$$K(x, p, u) = p^T f(x, u) - s(u, h(x, u)) \quad (11.106)$$

By the Implicit Function theorem and $P > 0$ there exists locally a function $u^*(x, p)$ satisfying (11.3). Furthermore, the resulting Hamiltonian $H(x, p) = K(x, p, u^*(x, p))$ has corresponding Hamiltonian matrix $\mathcal{H}$ given by (11.104). By Corollary 11.1.8 there exists locally near $x = 0$ a function S with $S(0) = 0$, $\frac{\partial S}{\partial x}(0) = 0$ satisfying $H(x, S_x^T(x)) = 0$ (in fact, S corresponds to the stable invariant manifold of X_H). It follows that S satisfies (11.105). $\square$

Remark 11.3.2 For $D \neq 0$ a similar statement can be proved, by replacing the assumption $P > 0$ by $P + RD + D^T R^T + D^T Q D > 0$, and by defining a more complicated Hamiltonian matrix $\mathcal{H}$.

The main drawback of Proposition 11.3.1 is that it does not provide conditions which guarantee that the obtained function S satisfying (11.105) is non-negative on

a neighborhood of the equilibrium $x = 0$, and so is a valid storage function. One possible set of sufficient conditions for non-negativity of S is given in the following corollary.

Corollary 11.3.3 *Consider a nonlinear system Σ satisfying (11.97), with linearization $\bar{\Sigma}$ having $D = 0$ and a supply rate s satisfying (11.97). Suppose $\bar{\Sigma}$ is dissipative with respect to the supply rate $\bar{s}$ given by (11.103). Assume $P > 0$, and assume that the Hamiltonian matrix $\mathcal{H}$ in (11.104) does not have purely imaginary eigenvalues. Also assume that A is asymptotically stable, and that $s(0, y) \leq 0$ for all y . Then there exists a neighborhood V of $x = 0$, and $S : V \subset \mathcal{X} \rightarrow \mathbb{R}$ with $S(0) = 0$, $\frac{\partial S}{\partial x}(0) = 0$, satisfying (11.105) and such that $S(x) \geq 0$, $x \in V$. Thus Σ is locally dissipative on V with respect to the supply rate s .*

Proof The proof of Proposition 11.3.1 yields locally a function S with $S(0) = 0$, $\frac{\partial S}{\partial x}(0) = 0$, and satisfying $H(x, S_x^T(x)) = 0$. By (11.3) it thus follows that $S_x(x)f(x, 0) \leq s(0, h(x)) \leq 0$. Since $A = \frac{\partial f}{\partial x}(0, 0)$ is asymptotically stable, locally near $x = 0$ also $\dot{x} = f(x, 0)$ is asymptotically stable, and $S \geq 0$ follows by integration from 0 to T , and letting $T \rightarrow \infty$, using asymptotic stability. $\square$

For the L_2 -gain supply rate $s(u, y) = \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2$ we obtain the following particularly pleasing corollary.

Corollary 11.3.4 *Consider the nonlinear system Σ given by (11.96), with linearization $\bar{\Sigma}$ having $D = 0$. Let $\gamma > 0$, and suppose that $\bar{\Sigma}$ has L_2 -gain $< \gamma$. Assume that $A = \frac{\partial f}{\partial x}(0, 0)$ is asymptotically stable. Then there exists a neighborhood V of $x = 0$ and U of $u = 0$ such that Σ has locally L_2 -gain $< \gamma$ for $x \in V$ and $u \in U$.*

Proof Since $\bar{\Sigma}$ has L_2 -gain $< \gamma$ the corresponding Hamiltonian matrix $\mathcal{H}$ as in (11.104) does not have purely imaginary eigenvalues (see e.g., [117]). Thus we may apply Corollary 11.3.3 yielding neighborhoods V of $x = 0$ and U of $u = 0$, and a function $S : V \rightarrow \mathbb{R}^+$ such that

$$S_x(x)f(x, u) \leq \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|h(x)\|^2 \quad (11.107)$$

for all $x \in V \subset \mathcal{X}$ and all $u \in U \subset \mathbb{R}^m$. This shows that Σ has locally L_2 -gain $\leq \gamma$. Since the same story can be repeated for some $\tilde{\gamma} < \gamma$ arbitrarily close to γ , it follows that actually Σ has locally L_2 -gain $< \gamma$. $\square$

Remark 11.3.5 For an affine system $\dot{x} = f(x) + g(x)u$, $y = h(x)$ we may always take $U = \mathbb{R}^m$.

Remark 11.3.6 Since $\bar{\Sigma}$ is assumed to have L_2 -gain $< \gamma$ there exists $X = X^T \geq 0$ such that $A^T X + X A + \frac{1}{\gamma^2} X B B^T + C^T C \leq 0$, and thus $A^T X + X A \leq -C^T C$. If (C, A) is detectable, then this actually implies that A is asymptotically stable [356].

Fig. 11.1 Mathematical pendulum

Remark 11.3.7 Since $\bar{\Sigma}$ is assumed to have L_2 -gain $< \gamma$ it can be shown [117] that there exists $X = X^T \geq 0$ satisfying

$$A^T X + XA + C^T C + \frac{1}{\gamma^2} X B B^T X < 0 \quad (11.108)$$

It readily follows that also $S(x) := \frac{1}{2} x^T X x$ will satisfy (11.107), for *different* neighborhoods V and U , however. (In fact, we conjecture that the neighborhoods will be *smaller* than the ones obtained in Corollary 11.3.4; see also the similar discussion in Sect. 11.4.)

Contrary to the L_2 -gain supply rate, the passivity supply rate $s(u, y) = u^T y$ does *not* satisfy the conditions of Proposition 11.3.1, since $P = 0$ in this case. The following physical example shows that, in fact, passivity of the linearized system does *not* imply (local) passivity of the nonlinear system.

Example 11.3.8 Consider a mathematical pendulum with input torque u as depicted in Fig. 11.1 (see also Examples 7.1.12 and 7.2.2). Taking as output the angular velocity $\dot{\varphi}$, the nonlinear system is passive (in fact, lossless) with storage function equal to the total energy

$$S(\varphi, \dot{\varphi}) = \frac{1}{2} m \ell^2 \dot{\varphi}^2 + \ell(1 - \cos \varphi) \quad (11.109)$$

Indeed, $\frac{dS}{dt} = u \dot{\varphi}$. By the discussion preceding Proposition 11.3.1 the system linearized about $\varphi = 0, \dot{\varphi} = 0, u = 0$ is also passive. On the other hand, if we take as output the *horizontal* velocity of the endpoint, that is

$$y = \frac{d}{dt}(\ell \sin \varphi) = \ell \cos \varphi \cdot \dot{\varphi}, \quad (11.110)$$

then the linearized output (about $\varphi = 0$) remains the same, and thus the linearized system is still passive. However, as we will now demonstrate, the nonlinear system is *not* passive anymore. Indeed, the equations of the nonlinear system with output $y = \ell \cos \varphi \cdot \dot{\varphi}$ are given as (take for simplicity $\ell = 1$, $m = 1$, and denote $q = \varphi$)

$$\begin{aligned}\dot{q} &= p \\ \dot{p} &= -\sin q + u \\ y &= p \cos q\end{aligned}\tag{11.111}$$

Suppose $S(q, p)$ is a (locally defined) storage function with respect to the supply rate $s(u, y) = uy$. Then by (4.15) for all q, p close to zero

$$(i) \quad \frac{\partial S}{\partial q} p + \frac{\partial S}{\partial p} \cdot (-\sin q) \leq 0, \quad (ii) \quad \frac{\partial S}{\partial p} = p \cos q\tag{11.112}$$

From (ii) we infer $S(q, p) = \frac{1}{2}p^2 \cos q + F(q)$, for some function F . Substitution in (i) yields

$$(iii) \quad -\frac{1}{2}p^3 \sin q + p \left[\frac{dF}{dq} - \sin q \cos q \right] \leq 0\tag{11.113}$$

For fixed q , the left hand side of (11.113) is a polynomial $ap^3 + bp$. For p small, the linear term dominates, and thus the inequality (11.113) implies that $b = \frac{\partial F}{\partial q} - \sin q \cos q = 0$. Hence, $-\frac{1}{2}p^3 \sin q \leq 0$ for all p, q close to zero, which is a contradiction. We conclude that there are no storage functions, and thus the system is *not* passive.

11.4 $\mathcal{H}_\infty$ Control of a Nonlinear System and Its Linearization

In this section we investigate, in a spirit similar to the previous section, the relations between the (suboptimal) $\mathcal{H}_\infty$ control problem for a nonlinear system with given equilibrium on the one hand, and the same problem for its linearization on the other hand. In particular we show that if the state feedback strictly suboptimal $\mathcal{H}_\infty$ control problem for the linearization is solvable, then the same problem for the nonlinear system is solvable *on a neighborhood* of the equilibrium. Thus solvability of the “linearized problem” implies local solvability of the “nonlinear problem”. Similar results will be shown to hold for the output feedback $\mathcal{H}_\infty$ problem. These results are quite useful since the solvability of the linear $\mathcal{H}_\infty$ problems is relatively easy to check (certainly compared with the nonlinear $\mathcal{H}_\infty$ problems). Also the optimal disturbance level for the linearized problem will provide a useful lower bound for the optimal nonlinear disturbance level.

Let us start with the *state feedback* $\mathcal{H}_\infty$ problem. For simplicity of exposition¹ we focus on affine non-linear systems as in (10.2), that is

$$\begin{aligned} \dot{x} &= a(x) + b(x)u + g(x)d, \quad u \in \mathbb{R}^m, \quad d \in \mathbb{R}^r \\ \Sigma : \quad z &= \begin{bmatrix} h(x) \\ u \end{bmatrix}, \quad x \in \mathcal{X}, \quad z \in \mathbb{R}^s \end{aligned} \quad (11.114)$$

Since we wish to consider the linearization of Σ we assume the existence of an equilibrium $x = 0$, that is,

$$a(0) = 0, \quad h(0) = 0 \quad (11.115)$$

We denote the linearization of Σ about $x = 0$ by

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B\bar{u} + G\bar{d}, \quad \bar{u} \in \mathbb{R}^m, \quad \bar{d} \in \mathbb{R}^r \\ \bar{\Sigma} : \quad \bar{z} &= \begin{bmatrix} H\bar{x} \\ \bar{u} \end{bmatrix}, \quad \bar{x} \in \mathbb{R}^n, \quad \bar{z} \in \mathbb{R}^s \end{aligned} \quad (11.116)$$

with

$$A = \frac{\partial a}{\partial x}(0), \quad B = b(0), \quad G = g(0), \quad H = \frac{\partial h}{\partial x}(0) \quad (11.117)$$

Recall, see (10.3), that the state feedback Hamilton–Jacobi equation (HJ1a) of Σ is given as

$$P_x(x)a(x) + \frac{1}{2}P_x(x) \left[\frac{1}{\gamma^2}g(x)g^T(x) - b(x)b^T(x) \right] P_x^T(x) + \frac{1}{2}h^T(x)h(x) = 0 \quad (11.118)$$

It immediately follows that if $P \geq 0$ is a solution to (11.118) satisfying

$$P(0) = 0 \quad (11.119)$$

(implying also $P_x(0) = 0$), then the Hessian matrix $P_{xx}(0)$ of P at 0

$$X := P_{xx}(0) \geq 0 \quad (11.120)$$

is a solution of the Riccati equation

$$A^T X + XA + X \left(\frac{1}{\gamma^2}GG^T - BB^T \right) X + H^T H = 0 \quad (11.121)$$

Furthermore, if $P \geq 0$ satisfying (11.119) is a solution to the Hamilton–Jacobi *inequality* (HJ1a), then X is a solution to the Riccati *inequality* (11.121), with =

¹However the results to be obtained are directly extendable to more general system classes.

replaced by $\leq$. This Riccati equation (or inequality) is governing the state feedback $\mathcal{H}_\infty$ control problem for the linear system (11.116). In fact we have the following specialization and refinement of Theorem 10.1.1 and Proposition 10.1.2 to the linear case.

Theorem 11.4.1 *Assume (H, A) is detectable. Let $\gamma > 0$. Then there exists a linear feedback*

$$\bar{u} = L\bar{x} \quad (11.122)$$

such that the closed-loop system (11.116, 11.122) has L_2 -gain $\leq \gamma$ and is asymptotically stable (i.e., the linear state feedback suboptimal $\mathcal{H}_\infty$ control problem has been solved), if and only if there exists a solution $X \geq 0$ to (11.121). Furthermore, there exists a linear feedback $\bar{u} = L\bar{x}$ such that the closed-loop system has L_2 -gain strictly less than γ and is asymptotically stable (i.e., the linear state feedback strict suboptimal $\mathcal{H}_\infty$ control problem has been solved), if and only if there exists a solution $X \geq 0$ to (11.121), satisfying additionally

$$\sigma\left(A - BB^T X + \frac{1}{\gamma^2} GG^T X\right) \subset \mathbb{C}^- \quad (11.123)$$

($\mathbb{C}^-$ being the open left half plane). Moreover, if $X \geq 0$ is a solution to (11.121) then the linear feedback $\bar{u} = L\bar{x}$ with

$$L = -B^T X \quad (11.124)$$

solves the linear state feedback suboptimal $\mathcal{H}_\infty$ control problem, respectively the strict suboptimal problem if X satisfies additionally (11.123).

We recall (see e.g., Sect. 10.1) that the Hamiltonian corresponding to (HJ1a) given by (11.118) is

$$H(x, p) = p^T a(x) + \frac{1}{2} p^T \left[\frac{1}{\gamma^2} g(x)g^T(x) - b(x)b^T(x) \right] p + \frac{1}{2} h^T(x)h(x), \quad (11.125)$$

with corresponding Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A & \frac{1}{\gamma^2} GG^T - BB^T \\ -H^T H & -A^T \end{bmatrix} \quad (11.126)$$

Following the approach of Sect. 11.1 we notice that X is a solution to (11.121, 11.123) if and only if the subspace $\text{span} \begin{bmatrix} I \\ X \end{bmatrix}$ is the stable generalized eigenspace of $\mathcal{H}$. Application of Propositions 11.1.5 and 11.1.6 yields the following result.

Theorem 11.4.2 Assume (H, A) is detectable. Let $X \geq 0$ be a solution to (11.121, 11.123). Then there exists a neighborhood W of $x = 0$ and a smooth function $P \geq 0$ defined on W with $P(0) = 0$, $P_x(0) = 0$, such that P is a solution of (HJ1a) given by (11.118) on W .

Proof The local existence of a solution P to (11.118) with $P(0) = 0$, $P_x(0) = 0$, $P_{xx}(0) = X$, immediately follows from Propositions 11.1.5, 11.1.6. Furthermore since by Theorem 11.4.1 $A - BB^T X$ is asymptotically stable the vector field $a - bb^T P_x^T$ is locally asymptotically stable. Rewriting (11.118) as

$$P_x[a - bb^T P_x^T] = -\frac{1}{2} \frac{1}{\gamma^2} P_x g g^T P_x^T - \frac{1}{2} P_x b b^T P_x^T - \frac{1}{2} h^T h,$$

this implies by integration from 0 to T along the asymptotically stable vector field $a - bb^T P_x^T$, letting $T \rightarrow \infty$, that locally about $x = 0$ $P \geq 0$. $\square$

The local existence of a solution $P \geq 0$ to (11.118) yields a local solution to the nonlinear $\mathcal{H}_\infty$ control problem, as formulated in the following corollary.

Corollary 11.4.3 Let $P \geq 0$ defined on a neighborhood W of $x = 0$ be a solution to (11.118). Then with the locally defined feedback

$$u = -b^T(x) P_x^T(x), \quad x \in W, \quad (11.127)$$

the closed-loop system has locally L_2 -gain $\leq \gamma$, in the sense that

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|d(t)\|^2 dt + 2P(x(0)) \quad (11.128)$$

for all $x(0) \in W$, all $T \geq 0$ and all L_2 functions $d(\cdot)$ on $[0, T]$ such that the state space trajectories starting from $x(0)$ do not leave W .

The locally defined feedback (11.127) corresponding to a local solution $P \geq 0$ to (11.118) is not the only feedback resulting in a closed-loop system having locally L_2 -gain $\leq \gamma$. In fact for an arbitrary state feedback $u = \alpha(x)$ we may rewrite $H(x, p)$ given by (11.125) as

$$\begin{aligned} H(x, p) &= p^T[a(x) + b(x)\alpha(x)] + \frac{1}{2} \frac{1}{\gamma^2} p^T g(x) g^T(x) p + \\ &\quad \frac{1}{2} \alpha^T(x) \alpha(x) + \frac{1}{2} h^T(x) h(x) - \frac{1}{2} \|b^T(x) p + \alpha(x)\|^2 \end{aligned} \quad (11.129)$$

Thus if we take the linear feedback

$$\alpha(x) = -B^T X x \quad (11.130)$$

with $X = P_{xx}(0) \geq 0$ being the solution to (11.121), then the zero-th, first and second order terms of $\|b^T(x)p + \alpha(x)\|^2$ are all zero. Thus the Hamiltonian matrix corresponding to $\tilde{H}(x, p) := H(x, p) + \frac{1}{2} \|b^T(x)p + \alpha(x)\|^2$ given as

$$\begin{aligned} \tilde{H}(x, p) = & p^T [a(x) + b(x)\alpha(x)] + \frac{1}{2} \frac{1}{\gamma^2} p^T g(x) g^T(x) p + \\ & \frac{1}{2} \alpha^T(x) \alpha(x) + \frac{1}{2} h^T(x) h(x) \end{aligned} \quad (11.131)$$

is equal to $\mathcal{H}$ in (11.126). Hence there exists locally also a solution $\tilde{P} \geq 0$ to the Hamilton–Jacobi equation

$$\begin{aligned} \tilde{P}_x(x) [a(x) - b(x) B^T X x] + \frac{1}{2} \frac{1}{\gamma^2} \tilde{P}_x(x) g(x) g^T(x) \tilde{P}_x^T(x) \\ + \frac{1}{2} x^T X B B^T X x + \frac{1}{2} h^T(x) h(x) = 0 \end{aligned} \quad (11.132)$$

This implies that the inequality (11.128) also holds for the closed-loop system resulting from the feedback $u = -B^T X x$ if we replace P by $\tilde{P}$, and W by a neighborhood $\tilde{W}$ of $x = 0$ which in general will be *different* from W . Generalizing this observation a little further we obtain

Corollary 11.4.4 *Let $X \geq 0$ be a solution to (11.121, 11.123). Then any feedback $u = \alpha(x)$ with*

$$\alpha(0) = 0, \quad \frac{\partial \alpha}{\partial x}(0) = -B^T X, \quad (11.133)$$

yields a closed-loop system satisfying (11.128) for some neighborhood W of $x = 0$ and some solution $P \geq 0$ to the Hamilton–Jacobi inequality (HJ1a).

Example 11.4.5 Consider the system of Example 10.1.8

$$\dot{x} = u + (\arctan x) d, \quad z = \begin{bmatrix} x \\ u \end{bmatrix} \quad (11.134)$$

Clearly, its linearization at $x = 0$ is not affected by disturbances, and the Riccati equation (11.121) is given as $X^2 = 1$ (independent of γ), yielding the positive solution $X = 1$. The state-feedback Hamilton–Jacobi equation (HJ1a) takes the form

$$\left(\frac{dP}{dx}(x) \right)^2 \left[1 - \frac{1}{\gamma^2} \arctan^2 x \right] = x^2, \quad (11.135)$$

which has for every γ a solution $P_\gamma \geq 0$ on the neighborhood

$$W_\gamma = \{x \in \mathbb{R} \mid |\arctan x| < \gamma\} \quad (11.136)$$

yielding a feedback $u = \alpha_\gamma(x) = -x(1 - \frac{1}{\gamma^2} \arctan^2 x)^{-\frac{1}{2}}$. Note that the solution P_γ as well as the feedback α_γ become unbounded for x approaching the boundary of W_γ . Applying instead of $u = \alpha_\gamma(x)$ its linear part $u = -x$ one obtains the closed-loop system

$$\dot{x} = -x + (\arctan x) d, \quad z = \begin{bmatrix} x \\ -x \end{bmatrix} \quad (11.137)$$

with L_2 -gain Hamilton–Jacobi equation given as

$$-\frac{d\tilde{P}}{dx}(x)x + \frac{1}{2} \frac{1}{\gamma^2} \left(\frac{d\tilde{P}}{dx}(x) \right)^2 \arctan^2 x + \frac{1}{2}x^2 + \frac{1}{2}x^2 = 0, \quad (11.138)$$

having a solution $\tilde{P}_\gamma \geq 0$ on the neighborhood

$$\tilde{W}_\gamma = \{x \in \mathbb{R} \mid |\arctan x| < \frac{1}{2}\sqrt{2} \cdot \gamma\} \quad (11.139)$$

It should be noted that the neighborhood $\tilde{W}_\gamma$ arising from applying the linearized feedback $u = -B^T Xx$ in the above example is *smaller* than the neighborhood W_γ arising from applying the full nonlinear feedback $u = -b^T(x)P_x^T(x)$. This can be conjectured to be true in general; the nonlinear controller will solve the $\mathcal{H}_\infty$ control problem on a larger domain than its linearization. A possible starting point for proving such a conjecture is the observation that by (11.129) $\tilde{H}(x, p) \geq H(x, p)$ for all x, p , and thus every solution of $\tilde{H}(x, P_x^T(x)) \leq 0$ will be also a solution of $H(x, P_x^T(x)) \leq 0$, while the converse is not true. Since the main obstruction for extending a local solution $P \geq 0$ of $H(x, P_x^T(x)) \leq 0$ into a global one seems to be the fact that P can become infinite for finite x (see for instance Example 11.4.5), this suggests that P corresponding to the stable invariant manifold of $H(x, p)$ (which by Sect. 11.1 is the minimal solution to $H(x, P_x^T(x)) \leq 0$!) and the corresponding feedback $u = -b^T(x)P_x^T(x)$ have the largest domain of validity.

Example 11.4.5 also shows that in some cases we need to make a compromise between the achievable level of disturbance attenuation γ , and the domain of validity of the feedback. Indeed comparing with Example 10.1.8 we see that *global* disturbance attenuation is possible for every $\gamma > \frac{\pi}{2}$, while a disturbance attenuation level $0 \leq \gamma \leq \frac{\pi}{2}$ can be only met on the neighborhood W_γ (with W_γ shrinking to the origin for $\gamma \downarrow 0$).

The following example shows that the nonlinear feedback $u = -b^T(x)P_x(x)$ corresponding to a (local) solution $P \geq 0$ to the Hamilton–Jacobi inequality (HJ1a) may have other advantages when compared to its linear part $u = -B^T Xx$.

Example 11.4.6 ([142]) Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= x_1^2 + u + d, \quad z = \begin{bmatrix} x_2 - x_1^2 \\ u \end{bmatrix} \end{aligned} \quad (11.140)$$

The Hamilton–Jacobi inequality (HJ1a) takes the form

$$\frac{\partial P}{\partial x_1} x_1 x_2 + \frac{\partial P}{\partial x_2} x_1^2 + \frac{1}{2} \left(\frac{1}{\gamma^2} - 1 \right) \left(\frac{\partial P}{\partial x_2} \right)^2 + \frac{1}{2} (x_2 - x_1^2)^2 \leq 0 \quad (11.141)$$

This has for $\gamma > 1$ locally positive solutions

$$P(x_1, x_2) = ax_1^2 + bx_2^2 + cx_1^2x_2, \quad (11.142)$$

provided $a > 0$, and b, c are large enough. This yields the feedback

$$u = -\frac{\partial P}{\partial x_2} = -2bx_2 - cx_1^2 \quad (11.143)$$

which, apart from rendering the L_2 -gain of the closed-loop system locally $\leq \gamma$, also locally asymptotically stabilizes the system. On the other hand, the linearized feedback

$$u = -2bx_2 \quad (11.144)$$

does *not* locally asymptotically stabilize the system. In fact it may be proved that there does not exist any linear feedback which renders the system (11.140) locally asymptotically stable. (Note that the linearization of (11.140) does not exactly fit the assumptions of Theorem 11.4.2 and its Corollaries since it is not detectable.)

Next let us consider the *output feedback* case. Again, for simplicity of exposition we focus on affine systems (10.68), that is

$$\begin{aligned} \dot{x} &= a(x) + b(x)u + g(x)d_1, & a(0) &= 0 \\ y &= c(x) + d_2, & c(0) &= 0 \\ z &= \begin{bmatrix} h(x) \\ u \end{bmatrix}, & h(0) &= 0 \end{aligned} \quad (11.145)$$

Apart from the state feedback Hamilton–Jacobi equation (HJ1a) given by (11.118) we also consider, see (10.71), the Hamilton–Jacobi equation (HJ2a) given as

$$R_x(x)a(x) + \frac{1}{2} \frac{1}{\gamma^2} R_x(x)g(x)g^T(x)R_x^T(x) + \frac{1}{2}h^T(x)h(x) - \frac{1}{2}\gamma^2 c^T(x)c(x) = 0 \quad (11.146)$$

By collecting second-order terms it follows that for any solution $R \geq 0$ of (11.146), with $R(0) = 0$ and thus $R_x(0) = 0$, the matrix

$$W := R_{xx}(0) \geq 0 \quad (11.147)$$

is a solution of the Riccati equation

$$A^T W + W A + \frac{1}{\gamma^2} W G G^T W + H^T H - \gamma^2 C^T C = 0, \quad (11.148)$$

where, as before $A = \frac{\partial a}{\partial x}(0)$, $G = g(0)$, $H = \frac{\partial h}{\partial x}(0)$, and additionally

$$C = \frac{\partial c}{\partial x}(0) \quad (11.149)$$

If W is invertible, then $Y := \gamma^2 W^{-1}$ is the solution to the “dual” Riccati equation

$$YA^T + AY + GG^T + \frac{1}{\gamma^2}YH^THY - YC^TCY = 0 \quad (11.150)$$

The Riccati equation (11.150), together with the Riccati equation (11.121), is governing the linear $\mathcal{H}_\infty$ output feedback problem for the system linearized at $x = 0$ given as

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B\bar{u} + G\bar{d}_1 \\ \bar{y} &= C\bar{x} + \bar{d}_2 \\ \bar{z} &= \begin{bmatrix} H\bar{x} \\ \bar{u} \end{bmatrix} \end{aligned} \quad (11.151)$$

Indeed we recall the following basic theorem from linear $\mathcal{H}_\infty$ control theory, see [88, 117].

Theorem 11.4.7 *Consider the linear system (11.151). Assume the triples (A, B, H) and (A, G, C) are stabilizable and detectable. Then there exists a linear dynamic controller such that the closed-loop system has L_2 -gain $< \gamma$ and is asymptotically stable if and only if there exist solutions $X \geq 0$, $Y \geq 0$ to (11.121), respectively (11.150), satisfying additionally (11.123), respectively*

$$\sigma\left(A - YC^TC + \frac{1}{\gamma^2}YH^TH\right) \subset \mathbb{C}^-, \quad (11.152)$$

together with the coupling condition

$$\sigma_{\max}(XY) < \gamma^2 \quad (11.153)$$

where $\sigma_{\max}$ denotes the largest singular value. Furthermore one such controller (called the “central controller”) is given as

$$\begin{aligned} \dot{z} &= \left(A - BB^TX + \frac{1}{\gamma^2}GG^TX\right)z + \left(I - \frac{1}{\gamma^2}YX\right)^{-1}YC^T(y - Cz) \\ u &= -B^TXz \end{aligned} \quad (11.154)$$

We have seen before that the existence of a solution $X \geq 0$ to (11.121, 11.123) implies the local existence of a solution $P \geq 0$ to (HJ1a). Let us now additionally assume that the solution $Y \geq 0$ to (11.150, 11.152) is positive definite, that is, $Y > 0$. Then $W = \gamma^2 Y^{-1} > 0$ satisfies (11.148), while (11.152) can be rewritten as

$$\sigma\left(A + \frac{1}{\gamma^2}GG^TW\right) \subset \mathbb{C}^+ \quad (11.155)$$

This means that $\text{span} \begin{bmatrix} I \\ W \end{bmatrix}$ is the *unstable* generalized eigenspace of the Hamiltonian matrix $\mathcal{H}$ corresponding to (HJ2a) and (11.150), given as

$$\mathcal{H} = \begin{bmatrix} A & \frac{1}{\gamma^2} G G^T \\ \gamma^2 C^T C - H^T H & -A^T \end{bmatrix} \quad (11.156)$$

Application of Propositions 11.1.5 and 11.1.6 (with respect to the unstable invariant manifold) yields

Proposition 11.4.8 *Let $Y > 0$ be a solution to (11.150, 11.152). Then there exists a neighborhood U of $x = 0$ and a smooth function $R > 0$ defined on U with $R(0) = 0$, $R_x(0) = 0$, such that R is a solution of (HJ2a) on U .*

Proof The local existence of a solution R to (HJ2a) with $R(0) = 0$, $R_x(0) = 0$ and $R_{xx}(0) = \gamma^2 Y^{-1}$ follows from Propositions 11.1.5, 11.1.6. Since $R_{xx}(0) = \gamma^2 Y^{-1} > 0$ it follows that $R > 0$ on a neighborhood of $x = 0$. $\square$

Finally, note that the coupling condition $\sigma_{\max}(XY) < \gamma^2$ can be rewritten as

$$X < \gamma^2 Y^{-1} \quad (11.157)$$

Since $P_{xx}(0) = X$ and $R_{xx}(0) = \gamma^2 Y^{-1}$ it follows that for all x near 0

$$P(x) < R(x) \quad (11.158)$$

and so we recover, in a *strict* inequality but *local* form, the weak coupling derived in Sect. 10.2. This is summarized in the following form.

Corollary 11.4.9 *If there exist solutions $X \geq 0$, $Y > 0$ to (11.121), respectively (11.150), satisfying (11.123), respectively (11.152), then there exists a neighborhood U of $x = 0$ and smooth functions $P \geq 0$ and $R > 0$ on U with $P_{xx}(0) = X$ and $R_{xx}(0) = \gamma^2 Y^{-1}$, which are solutions of (HJ1a), respectively (HJ2a), and satisfy (11.158) on U .*

Therefore, loosely speaking, the solvability of the output feedback suboptimal $\mathcal{H}_\infty$ control problem for the linearization (11.151) implies that *locally* all the necessary conditions derived in Sect. 10.2 for solvability of the same problem for the nonlinear system are satisfied.

Furthermore (and more importantly), under the assumptions of Theorem 11.4.7 we can derive a controller which locally solves the *nonlinear* output feedback suboptimal $\mathcal{H}_\infty$ control problem.

Proposition 11.4.10 *Consider the nonlinear system (11.145), together with its linearization (11.151). Assume the triples (A, B, H) and (A, G, C) are stabilizable and detectable. Let $X \geq 0$, $Y > 0$ be solutions to (11.121, 11.123), respectively (11.150,*

(11.152), and satisfying (11.153), leading by Corollary 11.4.9 to local solutions $P \geq 0$ and $R > 0$ of (HJ1a), respectively (HJ2a). Then the nonlinear controller

$$\begin{aligned}\dot{\xi} &= a(\xi) - b(\xi)b^T(\xi)P_\xi^T(\xi) + \frac{1}{\gamma^2}g(\xi)g^T(\xi)P_\xi^T(\xi) \\ &\quad + \gamma^2 \left[R_{\xi\xi}(\xi) - P_{\xi\xi}(\xi) \right]^{-1} \frac{\partial c^T}{\partial \xi}(\xi) [y(t) - c(\xi)] \\ u &= -b^T(\xi)P_\xi^T(\xi), \quad \xi \in \mathcal{X}_c := \mathcal{X}\end{aligned}\tag{11.159}$$

locally solves the output feedback suboptimal $\mathcal{H}_\infty$ control problem for (11.145), in the sense that there exists a neighborhood W of the equilibrium $(0, 0) \in \mathcal{X} \times \mathcal{X}_c$ and a function $S \geq 0$ defined on it, such that the closed-loop system (11.145, 11.159) satisfies

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|d(t)\|^2 dt + 2S(x(0), \xi(0))\tag{11.160}$$

for all $(x(0), \xi(0)) \in W$, and all $T \geq 0$ and $d \in L_2(0, T)$ such that the $(x(t), \xi(t))$ trajectories do not leave W .

Proof The linearization of (11.159) at $\xi = 0$ is precisely the central controller (11.154). Hence the linearization of the closed-loop system (11.145, 11.159) equals the linear closed-loop system (11.151, 11.154), which by Theorem 11.4.7 has L_2 -gain $< \gamma$ and is asymptotically stable. Hence by Corollary 11.3.4 also the nonlinear closed-loop system has locally L_2 -gain $< \gamma$ in the sense of (11.160) for some neighborhood W of $(x, \xi) = (0, 0)$. $\square$

Remark 11.4.11 Note that W may be smaller than the neighborhood $U \times U$ of $(x, \xi) = (0, 0)$, with U as obtained in Corollary 11.4.9.

Similarly to Corollary 11.4.4 we derive from the proof of Proposition 11.4.10 the following corollary.

Corollary 11.4.12 *Under the conditions of Proposition 11.4.10 every nonlinear controller whose linearization equals the “central controller” (11.154) locally solves the output feedback suboptimal $\mathcal{H}_\infty$ control problem.*

The important issue is thus to construct a controller which solves the nonlinear output feedback suboptimal $\mathcal{H}_\infty$ control problem on a domain which is as large as possible. Since the controller (11.159) incorporates in a “maximal way” all the nonlinear characteristics of the nonlinear system under consideration (as compared with the other controllers whose linearization equals the central controller), there is some reason to believe that the controller (11.159) has a large domain of validity.

11.5 Notes for Chapter 11

1. Section 11.1 is an expanded and generalized version of results presented in van der Schaft [272, 275, 278].
2. The Hamilton–Jacobi–Isaacs equation is at the core of dynamic game theory; cf. Basar & Olsder [31].
3. Propositions 11.1.3, 11.1.5 and 11.1.6 can be found in van der Schaft [272].
4. Proposition 11.1.7(i) can be found e.g. in Kucera [175] (Lemma 3.2.1) and part (ii) in Molinari [221], Francis [103].
5. The main idea explored in Proposition 11.1.14 can be found in van der Schaft [275].
6. Section 11.2 is close in spirit to the treatment of the linear quadratic optimal control problem in Anderson & Moore [7], Willems [349].
7. Proposition 11.2.1 can be found in Kucera [174], Francis [103], Kucera [175].
8. For a result related to Proposition 11.2.6 we refer to Byrnes & Martin [57].
9. Using the invariant manifold techniques of Sect. 11.1 it can be also shown that solvability of the *linearized* optimal control problem (for the linearized system and the quadratic part of the cost criterion) implies *local* solvability of the nonlinear optimal control problem, see van der Schaft [274] (compare with Sect. 11.3).
10. Section 11.3 is a generalized version of some results presented in van der Schaft [272, 275] for the L_2 -gain case, and Nijmeijer, Ortega, Ruiz & van der Schaft [234] for the passivity case.
11. The Hamiltonian matrix $\mathcal{H}$ as in (11.104) reduces for the L_2 -gain supply rate $\frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2$ (see Corollary 11.3.4) to

$$\mathcal{H}_\gamma = \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ C^TC & -A^T \end{bmatrix} \quad (11.161)$$

with A assumed to be asymptotically stable.

It follows (see e.g., Green & Limebeer [117], Scherer [304]) that there exists $\gamma^* \geq 0$ such that if

- (a) $\gamma > \gamma^*$, then $\mathcal{H}_\gamma$ does not have purely imaginary eigenvalues.
- (b) $0 \leq \gamma \leq \gamma^*$, then $\mathcal{H}_\gamma$ does have purely imaginary eigenvalues.

In the dynamical systems literature this is known as a *Hamiltonian Hopf bifurcation* of the corresponding Hamiltonian vector field (van der Meer [214]): for γ smaller than γ^* and monotonously increasing the purely imaginary eigenvalues of $\mathcal{H}_\gamma$ become or remain of even multiplicity at $\gamma = \gamma^*$, and for $\gamma > \gamma^*$ split

of into eigenvalues located in the open left-half plane and in the open right-half plane, symmetrically with respect to the imaginary axis.

12. Section 8.4 is largely based on van der Schaft [275, 278]. Example 11.4.6 is due to Isidori & Kang [142].
13. The relations between viscosity solutions of Hamilton–Jacobi equations and the properties of the corresponding Lagrangian submanifolds are explored e.g., in Day [80].
14. For computational schemes for solving Hamilton–Jacobi equations and inequalities we refer to the list of references given in Note 1 in Sect. 10.3 at the end of the previous Chap. 10.