University of Groningen

Nonlinear H^∞ Control

van der Schaft, Arjan

Published in:
Communications and Control Engineering

DOI:
[10.1007/978-3-319-49992-5_10](https://doi.org/10.1007/978-3-319-49992-5_10)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2017

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

van der Schaft, A. (2017). Nonlinear H^∞ Control. In *Communications and Control Engineering* (pp. 241-267). (Communications and Control Engineering; No. 9783319499918). Springer International Publishing. https://doi.org/10.1007/978-3-319-49992-5_10

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Chapter 10

Nonlinear $\mathcal{H}_\infty$ Control

Consider the following standard control configuration, see Fig. 10.1. Let Σ be a nonlinear system on an n -dimensional state space manifold $\mathcal{X}$ of the general form

$$\begin{aligned} \dot{x} &= f(x, u, d) \\ \Sigma : \quad y &= g(x, u, d) \\ z &= h(x, u, d) \end{aligned} \tag{10.1}$$

with two sets of inputs u and d , two sets of outputs y and z , and state x .

Here u stands for the vector of *control* inputs, d are the *exogenous* inputs (disturbances-to-be-rejected or reference signals to-be-tracked), y are the *measured* outputs, and finally z denote the *to-be-controlled* outputs (tracking errors, cost variables). The *optimal $\mathcal{H}_\infty$ control problem*,¹ roughly speaking, is to find a controller C , processing the measurements y and producing the control inputs u , such that in the closed-loop configuration of Fig. 10.2 the L_2 -gain from exogenous inputs d to to-be-controlled outputs z is minimized, while furthermore the closed-loop system is stable.

Instead of directly minimizing the L_2 -gain the more feasible *suboptimal $\mathcal{H}_\infty$ control problem* is addressed. This consists in finding, if possible, for a given disturbance attenuation level γ a controller C such that the closed-loop system has L_2 -gain $\leq \gamma$, and is stable. The solution to the optimal $\mathcal{H}_\infty$ control problem can then be approximated by an iteration of the suboptimal $\mathcal{H}_\infty$ control problem (successively decreasing γ toward the optimal disturbance attenuation level).

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¹The terminology $\mathcal{H}_\infty$ stems from the fact that in the linear case the L_2 -gain of a stable system is equal to the $\mathcal{H}_\infty$ norm of its transfer matrix.

Fig. 10.1 Standard control configuration

Fig. 10.2 Standard closed-loop configuration

10.1 State Feedback $\mathcal{H}_\infty$ Control

In the *state feedback suboptimal $\mathcal{H}_\infty$ control problem* we assume that the whole state is available for feedback; or equivalently $y = x$ in (10.1).

We first study the state feedback problem for nonlinear systems of the special affine form

$$\Sigma : \begin{aligned} \dot{x} &= a(x) + b(x)u + g(x)d, \quad u \in \mathbb{R}^m, d \in \mathbb{R}^r \\ z &= \begin{bmatrix} h(x) \\ u \end{bmatrix} \end{aligned} \quad x \in \mathcal{X}, z \in \mathbb{R}^s \quad (10.2)$$

All data will be assumed to be C^k , with $k \geq 2$. Recall the notation $P_x(x)$ for the row vector $\left(\frac{\partial P}{\partial x_1}(x), \dots, \frac{\partial P}{\partial x_n}(x) \right)$ of partial derivatives of a function $P : \mathcal{X} \rightarrow \mathbb{R}$. We start by stating the main result.

Theorem 10.1.1 *Let $\gamma > 0$. Suppose there exists a C^r ($k \geq r > 1$) solution $P \geq 0$ to the Hamilton–Jacobi inequality (HJ1a) (“a” standing for affine):*

$$\begin{aligned} P_x(x)a(x) + \frac{1}{2}P_x(x) \left[\frac{1}{\gamma^2}g(x)g^T(x) - b(x)b^T(x) \right] P_x^T(x) \\ + \frac{1}{2}h^T(x)h(x) \leq 0 \end{aligned} \quad (\text{HJ1a}) \quad (10.3)$$

Then with the C^{r-1} state feedback

$$u = -b^T(x)P_x^T(x) \quad (10.4)$$

the closed-loop system (10.2), (10.4), that is,

$$\begin{aligned} \dot{x} &= a(x) - b(x)b^T(x)P_x^T(x) + g(x)d \\ z &= \begin{bmatrix} h(x) \\ -b^T(x)P_x^T(x) \end{bmatrix} \end{aligned} \quad (10.5)$$

has L_2 -gain $\leq \gamma$ (from d to z).

Conversely, suppose there exists a C^{r-1} feedback

$$u = \ell(x) \quad (10.6)$$

such that there exists a C^1 storage function $P \geq 0$ for the closed-loop system (10.2), (10.6) with supply rate $\frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|z\|^2$. Then $P \geq 0$ is also a solution of (HJ1a).

Proof Suppose $P \geq 0$ is solution to (HJ1a). Rewrite (HJ1a) as

$$\begin{aligned} P_x(x) [a(x) - b(x)b^T(x)P_x^T(x)] + \frac{1}{2}\frac{1}{\gamma^2}P_x(x)g(x)g^T(x)P_x^T(x) \\ + \frac{1}{2}P_x(x)b(x)b^T(x)P_x^T(x) + \frac{1}{2}h^T(x)h(x) \leq 0 \end{aligned} \quad (10.7)$$

In view of (8.7) this means that P is a storage function for (10.5) with L_2 -gain supply rate $\frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|z\|^2$. Conversely, let $P \geq 0$ be a solution to

$$\begin{aligned} P_x(x) [a(x) + b(x)\ell(x)] + \frac{1}{2}\frac{1}{\gamma^2}P_x(x)g(x)g^T(x)P_x^T(x) \\ + \frac{1}{2}\ell^T(x)\ell(x) + \frac{1}{2}h^T(x)h(x) \leq 0 \end{aligned} \quad (10.8)$$

Then by “completion of the squares”

$$\begin{aligned} &P_x(x) [a(x) - b(x)b^T(x)P_x^T(x)] \\ &= P_x(x) [a(x) + b(x)\ell(x)] - P_x(x)b(x) [b^T(x)P_x^T(x) + \ell(x)] \\ &\leq -\frac{1}{2}\frac{1}{\gamma^2}P_x(x)g(x)g^T(x)P_x^T(x) - \frac{1}{2}h^T(x)h(x) \\ &\quad - \frac{1}{2}\|l(x) + b^T(x)P_x^T(x)\|^2 - \frac{1}{2}P_x(x)b(x)b^T(x)P_x^T(x) \\ &\leq -\frac{1}{2}\frac{1}{\gamma^2}P_x(x)g(x)g^T(x)P_x^T(x) - \frac{1}{2}P_x(x)b(x)b^T(x)P_x^T(x) \\ &\quad - \frac{1}{2}h^T(x)h(x) \end{aligned} \quad (10.9)$$

showing that P is a solution to (10.7), and thus to (10.3). $\square$

Note that we did not say anything so far about the stability of the closed-loop system (10.5) resulting from the state feedback $u = -b^T(x)P_x^T(x)$. However, since $P \geq 0$ is

a storage function for (10.5) we can simply invoke the results of Chap. 3, in particular Propositions 3.2.12 and 3.2.16.

Proposition 10.1.2 *Let $P \geq 0$ be a solution to (HJ1a).*

(i) *Suppose the system*

$$\begin{aligned} \dot{x} &= a(x) \\ z &= \begin{bmatrix} h(x) \\ -b^T(x)P_x^T(x) \end{bmatrix} \end{aligned} \quad (10.10)$$

is zero-state observable, then $P(x) > 0$ for $x \neq 0$.

(ii) *Suppose $P(x) > 0$ for all $x \neq 0$ and $P(0) = 0$, and suppose (10.10) is zero-state detectable. Then $x = 0$ is a locally asymptotically stable equilibrium of*

$$\dot{x} = a(x) - b(x)b^T(x)P_x^T(x) \quad (10.11)$$

If additionally P is proper, then $x = 0$ is a globally asymptotically stable equilibrium of (10.11).

Proof Apply Propositions 3.2.12 and 3.2.16 to (10.5), using (HJ1a) rewritten as (10.7), and note that $a(x) - b(x)b^T(x)P_x^T(x) = a(x)$ whenever $z = 0$. $\square$

Remark 10.1.3 We leave the generalization of Proposition 10.1.2 to positive semi-definite $P \geq 0$, based on Theorem 3.2.17 and Proposition 3.2.19, to the reader.

At this point one may wonder where the Hamilton–Jacobi inequality (HJ1a) comes from, and how similar expressions may be obtained for systems of a more general form than (10.2). An answer to both questions is obtained from the theory of *differential games*. Indeed, one may look at the $\mathcal{H}_\infty$ suboptimal problem as a two-player zero-sum differential game with cost criterion

$$-\frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}\|z\|^2 \quad (10.12)$$

(one player corresponds to the control input u , the other player to the exogenous input d).

The pre-Hamiltonian corresponding to this differential game is given as

$$K_\gamma(x, p, d, u) := p^T[a(x) + b(x)u + g(x)d] - \frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}\|z\|^2 \quad (10.13)$$

with p being the co-state. From the equations $\frac{\partial K_\gamma}{\partial d} = 0$ and $\frac{\partial K_\gamma}{\partial u} = 0$ we determine, respectively,

$$\begin{aligned} d^*(x, p) &= \frac{1}{\gamma^2}g^T(x)p \\ u^*(x, p) &= -b^T(x)p, \end{aligned} \quad (10.14)$$

which have the *saddle point* property

$$\begin{aligned} K_\gamma(x, p, d, u^*(x, p)) &\leq K_\gamma(x, p, d^*(x, p), u^*(x, p)) \\ &\leq K_\gamma(x, p, d^*(x, p), u) \end{aligned} \quad (10.15)$$

for every d, u and every (x, p) . The input u^* may be called the *optimal control*, and d^* the *worst-case* exogenous input (disturbance).

Substitution of (10.14) in $K_\gamma(x, p, d, u)$ leads to the (optimal) Hamiltonian

$$\begin{aligned} H_\gamma(x, p) &= p^T a(x) + \frac{1}{2} p^T \left[\frac{1}{\gamma^2} g(x) g^T(x) - b(x) b^T(x) \right] p \\ &\quad + \frac{1}{2} h^T(x) h(x) \end{aligned} \quad (10.16)$$

The Hamilton–Jacobi inequality (HJ1a) (10.3) is now also expressed as

$$H_\gamma(x, P_x^T(x)) \leq 0, \quad (10.17)$$

which is known in the theory of differential games as the *Hamilton–Jacobi–Isaacs equation*.

The same procedure may be followed for more general equations than (10.2), e.g.,

$$\begin{aligned} \dot{x} &= f(x, u, d) \\ z &= h(x, u, d) \end{aligned} \quad (10.18)$$

by considering the pre-Hamiltonian

$$K_\gamma(x, p, d, u) := p^T f(x, u, d) - \frac{1}{2} \gamma^2 \|d\|^2 + \frac{1}{2} \|z\|^2 \quad (10.19)$$

Suppose, as above, that K_γ has a saddle point $u^*(x, p), d^*(x, p)$, i.e.,

$$\begin{aligned} K_\gamma(x, p, d, u^*(x, p)) &\leq K_\gamma(x, p, d^*(x, p), u^*(x, p)) \\ &\leq K_\gamma(x, p, d^*(x, p), u) \end{aligned} \quad (10.20)$$

for every d, u , and every (x, p) . Then we consider the Hamilton–Jacobi inequality

$$(\mathbf{HJ1}) K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), u^*(x, P_x^T(x))) \leq 0 \quad (10.21)$$

Analogously to Theorem 10.1.1 we obtain

Proposition 10.1.4 *Consider the system (10.18). Let $\gamma > 0$. Assume there exist $u^*(x, p), d^*(x, p)$ satisfying (10.20). Suppose there exists a C^r ($k \geq r > 1$) solution $P \geq 0$ to the Hamilton–Jacobi inequality (HJ1) given by (10.21). Then the C^{r-1} state feedback*

$$u = u^*(x, P_x^T(x)) \quad (10.22)$$

is such that the closed-loop system (10.18), (10.22), i.e.,

$$\begin{aligned}\dot{x} &= f(x, u^*(x, P_x^T(x)), d) \\ z &= h(x, u^*(x, P_x^T(x)), d)\end{aligned}\tag{10.23}$$

has L_2 -gain $\leq \gamma$.

Conversely, suppose there exists a C^{r-1} feedback

$$u = l(x)\tag{10.24}$$

such that there exists a C^1 storage function $P \geq 0$ for the closed-loop system (10.18), (10.24) with supply rate $\frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|z\|^2$. Then $P \geq 0$ is also a solution of (HJ1).

Proof Let $P \geq 0$ satisfy (10.21). Substitute $p = P_x^T(x)$ in the first inequality in (10.20) to obtain

$$\begin{aligned}P_x(x)f(x, u^*(x, P_x^T(x)), d) - \frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}\|h(x, u^*(x, P_x^T(x)), d)\|^2 \\ \leq K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), u^*(x, P_x^T(x))),\end{aligned}\tag{10.25}$$

and thus by (10.21) for all d

$$\begin{aligned}P_x(x)f(x, u^*(x, P_x^T(x)), d) \\ \leq \frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|h(x, u^*(x, P_x^T(x)), d)\|^2,\end{aligned}\tag{10.26}$$

showing that (10.23) has L_2 -gain $\leq \gamma$ with storage function P . Conversely, let $P \geq 0$ be a solution to

$$P_x(x)f(x, l(x), d) \leq \frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|h(x, l(x), d)\|^2\tag{10.27}$$

Then by substituting $p = P_x^T(x)$ and $u = l(x)$ in the *second* inequality in (10.20) we obtain

$$\begin{aligned}K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), u^*(x, P_x^T(x))) \\ \leq K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), l(x))\end{aligned}\tag{10.28}$$

and thus by (10.27) with $d = d^*(x, P_x^T(x))$ we obtain (10.21). $\square$

Remark 10.1.5 The above proof reveals the essence of suboptimal state feedback $\mathcal{H}_\infty$ control. Solvability of the Hamilton–Jacobi inequality (HJ1) together with the first inequality in (10.20) yields the feedback (10.22) for which the closed-loop system (10.23) has L_2 -gain $\leq \gamma$. Conversely, if the second inequality in (10.20) holds then the existence of $u = l(x)$ for which the closed-loop system has L_2 -gain $\leq \gamma$ implies the solvability of (HJ1).

Remark 10.1.6 Note that in case of the affine system equations (10.2) the point $u^*(x, p), d^*(x, p)$ given by (10.14) not only satisfies the saddle point property (10.20) but in fact the stronger property

$$\begin{aligned} K_\gamma(x, p, d, u) &\leq K_\gamma(x, p, d^*(x, p), u) \\ K_\gamma(x, p, d, u^*(x, p)) &\leq K_\gamma(x, p, d, u), \end{aligned} \quad \forall u, d \quad (10.29)$$

We leave it to the reader to formulate the analog of Proposition 10.1.2 for the general case treated in Proposition 10.1.4.

Example 10.1.7 Consider the system

$$\dot{x} = (1 + x^2)u + d, \quad z = \begin{bmatrix} x \\ u \end{bmatrix} \quad (10.30)$$

The Hamilton–Jacobi inequality (HJ1a) reads

$$\left(\frac{dP}{dx}(x) \right)^2 \left((1 + x^2)^2 - \frac{1}{\gamma^2} \right) \geq x^2, \quad (10.31)$$

which has a nonnegative solution for $\gamma > 1$, e.g.,

$$P(x) = \frac{1}{2} \ln \left(1 + x^2 + \sqrt{(1 + x^2)^2 - \frac{1}{\gamma^2}} \right), \quad (10.32)$$

leading to the feedback

$$u = -(1 + x^2) \left((1 + x^2)^2 - \frac{1}{\gamma^2} \right)^{-\frac{1}{2}} x, \quad (10.33)$$

stabilizing the system about $x = 0$.

Example 10.1.8 Consider the system

$$\dot{x} = u + (\arctan x)d, \quad z = \begin{bmatrix} x \\ u \end{bmatrix} \quad (10.34)$$

The Hamilton–Jacobi inequality (HJ1a) reads as

$$\left(\frac{dP}{dx}(x) \right)^2 \left(1 - \frac{\arctan^2 x}{\gamma^2} \right) \geq x^2, \quad (10.35)$$

having solutions $P \geq 0$ for all γ such that

$$|\arctan x| < \gamma, \quad \forall x \in \mathbb{R}, \quad (10.36)$$

that is, for all $\gamma > \frac{\pi}{2}$. The feedback is given as

$$u = -x \left(1 - \frac{\arctan^2 x}{\gamma^2} \right)^{-\frac{1}{2}} \quad (10.37)$$

Example 10.1.9 Consider the system

$$\dot{x} = x(u + d), \quad z = \begin{bmatrix} x \\ u \end{bmatrix} \quad (10.38)$$

The Hamilton–Jacobi inequality (HJ1a) is

$$\left(\frac{dP}{dx}(x) \right)^2 \left(\frac{1}{\gamma^2} - 1 \right) x^2 + x^2 \leq 0 \quad (10.39)$$

which has for $\gamma > 1$ the so-called *viscosity solution* (see the Notes at the end of this chapter)

$$\begin{aligned} P(x) &= \frac{\gamma}{\sqrt{\gamma^2 - 1}} |x| \\ u &= -\frac{\gamma}{\sqrt{\gamma^2 - 1}} |x| \end{aligned} \quad (10.40)$$

Note that P is not differentiable at $x = 0$, however the closed-loop system

$$\dot{x} = -\frac{\gamma}{\sqrt{\gamma^2 - 1}} |x|x + xd \quad (10.41)$$

is asymptotically stable about $x = 0$.

Now let us come back to the L_2 -gain perturbation models formulated in Sect. 9.2, based on stable kernel or stable image representations. Consider an affine nonlinear input-state-output system

$$\begin{aligned} \Sigma : \quad \dot{x} &= f(x) + g(x)u, \quad x \in \mathcal{X}, \quad u \in \mathbb{R}^m \\ y &= h(x), \quad y \in \mathbb{R}^p \end{aligned} \quad (10.42)$$

First, we consider the perturbation Σ_Δ based on the *stable kernel representation* K_Σ , given in (9.29):

$$\Sigma_\Delta : \quad \begin{aligned} \dot{x} &= f(x) + g(x)u - k(x)w \\ y &= h(x) - w \end{aligned} \quad w \in \mathbb{R}^p, \quad (10.43)$$

with w being the output of some nonlinear system Δ with input $\begin{bmatrix} u \\ y \end{bmatrix}$

Fig. 10.3 Perturbed nonlinear system

$$\begin{aligned} \dot{\xi} &= \alpha(\xi, u, y) \\ \Delta : \quad w &= \beta(\xi, u, y) \end{aligned} \quad (10.44)$$

having finite L_2 -gain γ_Δ (see Fig. 10.3). Here the matrix $k(x)$ is determined by Eq. (9.10), that is, $W_x(x)k(x) = h^T(x)$, with W being a nonnegative solution to the “filter” Hamilton–Jacobi equation (9.9).

We now formulate the *robust stabilization problem* as the problem of constructing a controller C processing the measurements $\tilde{y}$ and producing the control u such that the L_2 -gain (from w to z) of Σ_Δ in closed-loop with C is minimized, say equal to γ^* . In the present section, we consider the robust stabilization problem by *state feedback* where we take $\tilde{y} = x$, and in the next section we will consider the *measurement feedback* case where $\tilde{y} = y$.

Once we have solved the robust stabilization problem, it follows from the small-gain theorem that the closed-loop system given in Fig. 10.4 will be closed-loop stable for all perturbations Δ with $\gamma_\Delta \cdot \gamma^* < 1$.

That is, in an input–output map context Theorem 2.1.1 implies that whenever the signals $e_1, e_2 \in L_2$ are such that the signals w and z are in L_{2e} , then actually w and z in L_2 . Furthermore, in a state space context Theorem 8.2.1 will ensure internal stability of the controlled perturbed nonlinear system for $e_1 = e_2 = 0$.

Clearly, the problem of minimizing the L_2 -gain (from w to z) of Σ_Δ by state feedback is a nonlinear $\mathcal{H}_\infty$ optimal control problem. The suboptimal version (for decreasing values of $\gamma > 0$) is addressed, as above, by considering the pre-Hamiltonian corresponding to (10.43):

$$\begin{aligned} K_\gamma(x, p, u, w) &= p^T[f(x) + g(x)u - k(x)w] - \frac{1}{2}\gamma^2\|w\|^2 \\ &\quad + \frac{1}{2}\|u\|^2 + \frac{1}{2}\|h(x) - w\|^2 \end{aligned} \quad (10.45)$$

As in (10.14) we obtain a saddle point $u^*(x, p), w^*(x, p)$ given as

Fig. 10.4 Controlled perturbed nonlinear system (state feedback case)

$$u^* = -g^T(x)p, \quad w^* = \frac{-1}{\gamma^2 - 1}(k^T(x)p + h(x)) \quad (10.46)$$

Necessarily $\gamma \geq 1$, since $y = h(x) + w$ and hence the L_2 -gain from w to z is always ≥ 1 . Substitution of (10.46) into (10.45) leads to the Hamilton–Jacobi inequality (HJ1), written out as

$$P_x(x)[f(x) + (\gamma^2 - 1)^{-1}k(x)h(x)] + \frac{1}{2}\gamma^2(\gamma^2 - 1)^{-1}h^T(x)h(x) + \frac{1}{2}P_x(x)[(\gamma^2 - 1)^{-1}k(x)k^T(x) - g(x)g^T(x)]P_x^T(x) \leq 0 \quad (10.47)$$

Proposition 10.1.10 *Let $\gamma > 1$. Suppose there exists a solution $P \geq 0$ to (10.47). Then the state feedback*

$$u = -g^T(x)P_x^T(x) \quad (10.48)$$

will be such that the controlled system

$$\begin{aligned} \dot{x} &= [f(x) - g(x)g^T(x)P_x^T(x)] - k(x)w \\ z &= \begin{bmatrix} h(x) - w \\ -g^T(x)P_x^T(x) \end{bmatrix} \end{aligned} \quad (10.49)$$

has L_2 -gain $\leq \gamma$ (from w to z).

Furthermore, let Δ have L_2 -gain $\gamma_\Delta < \frac{1}{\gamma}$ with storage function P_Δ . Suppose P and P_Δ have strict local minima at $x^* = 0$, respectively, $\xi^* = 0$, and Σ_Δ and Δ are zero-state detectable. Then (x^*, ξ^*) is a locally asymptotically stable equilibrium of

the closed-loop system (10.49), (10.44), which is globally asymptotically stable if P and P_Δ have global minima at $x^* = 0$ and $\xi^* = 0$ and are proper.

Proof The first part immediately follows from Proposition 10.1.4. The second part follows from the small-gain theorem in the state space setting (Theorem 8.2.1). $\square$

Remark 10.1.11 By taking fixed initial conditions for the controlled system as well as for the perturbation system Δ and considering the resulting input–output maps one may also formulate closed-loop stability statements in the spirit of the small-gain theorem in the input–output mapping setting (Theorem 2.1.1). We leave this to the reader.

Alternatively, we may consider the perturbation $\Sigma_{\tilde{\Delta}}$ based on the *stable image representation* I_Σ of Σ , given in (9.33), i.e.,

$$\Sigma_{\tilde{\Delta}} : \begin{aligned} \dot{x} &= f(x) + g(x)u - g(x)w_2 \\ y &= h(x) + w_1 \end{aligned} \quad (10.50)$$

with (w_1, w_2) the output of some nonlinear system $\tilde{\Delta}$

$$\begin{aligned} \dot{\xi} &= \tilde{\alpha}(\xi, s) \\ \tilde{\Delta} : \quad w_1 &= \tilde{\beta}_1(\xi, s) \\ w_2 &= \tilde{\beta}_2(\xi, s) \end{aligned} \quad (10.51)$$

with $s = u + g^T(x)V_x^T(x) - w_2$, having finite L_2 -gain (from s to (w_1, w_2)). The robust stabilization problem for this perturbation model is to minimize the L_2 -gain (from (w_1, w_2) to s) of

$$\begin{aligned} \dot{x} &= f(x) + g(x)u - g(x)w_2 \\ s &= u + g^T(x)V_x^T(x) - w_2 \end{aligned} \quad (10.52)$$

In the state feedback case this problem admits an easy solution.

Proposition 10.1.12 *The state feedback*

$$u = -g^T(x)V_x^T(x) \quad (10.53)$$

renders the L_2 -gain of (10.52) equal to 1. Furthermore, for every other controller C the L_2 -gain from (w_1, w_2) to s is ≥ 1 .

Proof Since $s = u + g^T(x)V_x^T(x) - w_2$ it is easily seen that the L_2 -gain from w_2 to s is always ≥ 1 . On the other hand, (10.53) clearly renders the L_2 -gain equal to 1. $\square$

The above result has a clear interpretation. Recall (see Remark 9.1.9 and Sect. 11.2) that (10.53) is the feedback that minimizes the cost criterion

$$\int_0^\infty (\|u(t)\|^2 + \|y(t)\|^2) dt \quad (10.54)$$

for $\dot{x} = f(x) + g(x)u$, $y = h(x)$. Proposition 10.1.12 shows that (10.53) also optimizes the robustness margin for the perturbed system $\Sigma_{\tilde{\Delta}}$.

We leave the further translation of the statements of Proposition 10.1.10 to the perturbation model $\Sigma_{\tilde{\Delta}}$ to the reader.

Example 10.1.13 (Example 9.2.1 continued) Consider a port-Hamiltonian system with energy $H \geq 0$ and without energy dissipation

$$\begin{aligned} \dot{x} &= J(x)H_x^T(x) + g(x)u, & J(x) &= -J^T(x) \\ y &= g^T(x)H_x^T(x) \end{aligned} \quad (10.55)$$

In Example 9.2.1, we have derived the perturbed systems Σ_Δ and $\Sigma_{\tilde{\Delta}}$, see (9.36), (9.37). With respect to the perturbed system Σ_Δ the Hamilton–Jacobi inequality (HJ1) takes the form

$$\begin{aligned} &P_x(x)[J(x)H_x^T(x) + (\gamma^2 - 1)^{-1}g(x)g^T(x)H_x^T(x)] \\ &+ \frac{1}{2}[\gamma^2 - 1)^{-1} - 1]P_x(x)g(x)g^T(x)P_x^T(x) \\ &+ \frac{1}{2}\gamma^2(\gamma^2 - 1)^{-1}H_x(x)g(x)g^T(x)H_x^T(x) \leq 0, \end{aligned} \quad (10.56)$$

having for $\gamma > \sqrt{2}$ the positive solution $P(x) = \frac{\gamma^2}{\gamma^2 - 2}H(x)$. Thus by Proposition 10.1.10 the state feedback

$$u = -\frac{\gamma^2}{\gamma^2 - 2}g^T(x)H_x^T(x), \quad \gamma > \sqrt{2}, \quad (10.57)$$

robustly stabilizes Σ_Δ for every perturbation Δ with L_2 -gain $< \frac{1}{\gamma}$. Alternatively, cf. Proposition 10.1.12, the feedback

$$u = -g^T(x)H_x^T(x) \quad (10.58)$$

robustly stabilizes $\Sigma_{\tilde{\Delta}}$ for every perturbation $\tilde{\Delta}$ with L_2 -gain < 1 .

10.2 Output Feedback $\mathcal{H}_\infty$ Control

Consider the nonlinear system (10.1), that is,

$$\begin{aligned}
\dot{x} &= f(x, u, d), \quad x \in \mathcal{X}, \quad u \in \mathbb{R}^m, \quad d \in \mathbb{R}^r \\
\Sigma : \quad y &= g(x, u, d), \quad y \in \mathbb{R}^p \\
z &= h(x, u, d), \quad z \in \mathbb{R}^s
\end{aligned} \tag{10.59}$$

In the *output feedback* suboptimal $\mathcal{H}_\infty$ control problem we want to construct, if possible, for a given attenuation level $\gamma \geq 0$ an output feedback controller

$$\begin{aligned}
C : \quad \dot{\xi} &= \varphi(\xi, y) \\
u &= \alpha(\xi)
\end{aligned} \tag{10.60}$$

such that the closed-loop system has L_2 -gain $\leq \gamma$ (from d to z). Here $\xi = (\xi_1, \dots, \xi_\nu)$ are local coordinates for the state space manifold $\mathcal{X}_c$ of the controller C , and φ and α are mappings whose degree of differentiability in general will depend on the degree of differentiability of the mappings f , g , and h . As before, we assume them to be at least C^2 .

In this section, we will be primarily concerned with finding *necessary* conditions for the solvability of the output feedback suboptimal $\mathcal{H}_\infty$ problem, as well as with the analysis of the *structure* of controllers that solve the problem. In order to do so, let us assume that a controller C as in (10.60) solves the output feedback suboptimal $\mathcal{H}_\infty$ control problem for Σ given by (10.59), for a given disturbance attenuation level γ . Moreover, *assume* that there exists a *differentiable* storage function $S(x, \xi) \geq 0$ for the closed-loop system with respect to the supply rate $\frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|z\|^2$, that is,

$$\begin{aligned}
&S_x(x, \xi)f(x, \alpha(\xi), d) + S_\xi(x, \xi)\varphi(\xi, g(x, \alpha(\xi), d)) \\
&\quad - \frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}\|h(x, \alpha(\xi), d)\|^2 \leq 0, \quad \text{for all } d
\end{aligned} \tag{10.61}$$

Now consider the equation

$$S_\xi(x, \xi) = 0, \tag{10.62}$$

and *assume* this equation has a differentiable solution $\xi = F(x)$. (By the Implicit Function theorem this will locally be the case if the partial Hessian matrix $S_{\xi\xi}(x, \xi)$ is nonsingular for every (x, ξ) satisfying (10.62).) Define

$$P(x) := S(x, F(x)) \tag{10.63}$$

Substitution of $\xi = F(x)$ into (10.61) yields (noting that $P_x(x) = S_x(x, F(x))$ since $S_\xi(x, F(x)) = 0$)

$$P_x(x)f(x, \alpha(F(x)), d) - \frac{1}{2}\gamma^2\|d\|^2 + \frac{1}{2}\|h(x, \alpha(F(x)), d)\|^2 \leq 0 \tag{10.64}$$

for all d . Hence the *state feedback* $u = \alpha(F(x))$ solves the state feedback suboptimal $\mathcal{H}_\infty$ control problem for Σ , with storage function P . Therefore, P is solution of (HJ1).

Thus we have obtained a logical necessary condition for solvability of the output feedback suboptimal $\mathcal{H}_\infty$ control problem, namely solvability of the corresponding state feedback problem.

A second necessary condition is obtained by restricting to the (natural) class of controllers C which produce zero control u for measurements y being identically zero, at least for “zero initial condition”. More specifically, we assume that the controller C satisfies

$$\varphi(0, 0) = 0, \alpha(0) = 0 \quad (10.65)$$

Defining

$$R(x) := S(x, 0), \quad (10.66)$$

substitution of $\xi = 0$ and $y = 0$ in (10.61) then yields

$$(\mathbf{HJ2}) \quad R_x(x) f(x, 0, d) - \frac{1}{2} \gamma^2 \|d\|^2 + \frac{1}{2} \|h(x, 0, d)\|^2 \leq 0 \quad (10.67)$$

for all disturbances d such that the measurements $y = g(x, 0, d)$ remain zero.

Thus a second necessary condition for solvability of the output feedback $\mathcal{H}_\infty$ suboptimal control problem is the existence of a solution $R \geq 0$ to (HJ2). This necessary condition also admits an obvious interpretation; it tells us that if we wish to render Σ dissipative by a controller C satisfying (10.65), then Σ constrained by $u = 0$ and $y = 0$ already has to be dissipative.

If we specialize (as in the previous Sect. 10.1) the equations for Σ and C to

$$\begin{aligned} \dot{x} &= a(x) + b(x)u + g(x)d_1 \\ y &= c(x) + d_2 \\ \Sigma_a : \quad z &= \begin{bmatrix} h(x) \\ u \end{bmatrix}, \end{aligned} \quad (10.68)$$

respectively,

$$\begin{aligned} \dot{\xi} &= k(\xi) + \ell(\xi)y, \quad k(0) = 0 \\ C_a : \quad u &= m(\xi), \quad m(0) = 0, \end{aligned} \quad (10.69)$$

then (10.67) reduces to

$$R_x(x) [a(x) + g(x)d_1] - \frac{1}{2} \gamma^2 (\|d_1\|^2 + \|d_2\|^2) + \frac{1}{2} \|h(x)\|^2 \leq 0 \quad (10.70)$$

for all $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ such that $y = c(x) + d_2$ is zero; implying that $d_2 = -c(x)$. Computing the maximizing disturbance $d_1^* = \frac{1}{\gamma^2} g^T(x) R_x^T(x)$, it follows that (10.70) is equivalent to the Hamilton–Jacobi inequality

$$\text{(HJ2a)} \quad R_x(x)a(x) + \frac{1}{2} \frac{1}{\gamma^2} R_x(x)g(x)g^T(x)R_x^T(x) + \frac{1}{2}h^T(x)h(x) - \frac{1}{2}\gamma^2 c^T(x)c(x) \leq 0, \quad x \in \mathcal{X} \quad (10.71)$$

Remark 10.2.1 In cases different from (10.68) the constraint $y = 0$ may impose constraints on the state space $\mathcal{X}$, in which case we would obtain, contrary to (10.71), a Hamilton–Jacobi inequality defined on a *subset* of $\mathcal{X}$.

Thus we have derived, under mild technical assumptions, two necessary conditions for the solvability of the output feedback suboptimal $\mathcal{H}_\infty$ control problem; namely the existence of a solution $P \geq 0$ to the first Hamilton–Jacobi inequality (HJ1), and the existence of a solution $R \geq 0$ to the second Hamilton–Jacobi inequality (HJ2).

Furthermore, it is clear from the way we have derived the solutions $P \geq 0$ and $R \geq 0$ to these two Hamilton–Jacobi inequalities, that P and R are not unrelated. In fact, as we now will show, the solutions P and R have to satisfy a certain *coupling condition*. The easiest way of obtaining this coupling condition is to consider, as above, $P(x) = S(x, F(x))$ and $R(x) = S(x, 0)$ and to additionally *assume* that S has a minimum at $(0, 0)$, i.e.,

$$S(0, 0) = 0, S_x(0, 0) = 0, S_\xi(0, 0) = 0, S(x, \xi) \geq 0, \quad \forall x, \xi, \quad (10.72)$$

and furthermore that the Hessian matrix of S at $(0, 0)$

$$\begin{bmatrix} S_{xx}(0, 0) & S_{x\xi}(0, 0) \\ S_{\xi x}(0, 0) & S_{\xi\xi}(0, 0) \end{bmatrix} =: \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \quad (10.73)$$

satisfies $S_{22} = S_{\xi\xi}(0, 0) > 0$. By the Implicit Function theorem this will imply the existence of a unique $F(x)$ defined near $x = 0$ such that $S_\xi(x, F(x)) = 0$. It immediately follows that

$$P(0) = 0, P_x(0) = 0, R(0) = 0, R_x(0) = 0, \quad (10.74)$$

and moreover from the definition of P and R it can be seen that

$$P_{xx}(0) = S_{11} - S_{12}S_{22}^{-1}S_{12}^T, \quad R_{xx}(0) = S_{11} \quad (10.75)$$

As a consequence one obtains the “*weak*” coupling condition

$$P_{xx}(0) \leq R_{xx}(0), \quad (10.76)$$

with strict inequality if $S_{12} \neq 0$, in which case $P(x) < R(x)$ for x near 0. In fact, in case of a *linear system* this amounts to the same coupling condition as obtained in linear $\mathcal{H}_\infty$ control theory [117]. See also Corollary 11.4.9 in the next Chap. 11.

A stronger, and more intrinsic, coupling condition can be derived as follows. Suppose there exists a controller C satisfying (10.65) which solves the output feedback

suboptimal $\mathcal{H}_\infty$ control problem with a storage function S for the closed-loop system satisfying $S(0, 0) = 0$ (or, equivalently, S has a *minimum* at $(0, 0)$). Then the closed-loop system satisfies, whenever $x(-T_1) = 0$, $\xi(-T_1) = 0$

$$\begin{aligned} & \frac{1}{2} \int_{-T_1}^0 (\gamma^2 \|d(t)\|^2 - \|z(t)\|^2) dt \\ & + \frac{1}{2} \int_0^{T_2} (\gamma^2 \|d(t)\|^2 - \|z(t)\|^2) dt \geq 0 \end{aligned}$$

for all $T_1 \geq 0$, $T_2 \geq 0$, and all disturbance functions $d(\cdot)$ on $[-T_1, T_2]$. In particular, we obtain for $x(-T_1) = 0$, $\xi(-T_1) = 0$

$$\begin{aligned} & - \int_0^{T_2} \frac{1}{2} (\gamma^2 \|d(t)\|^2 - \|z(t)\|^2) dt \\ & \leq \int_{-T_1}^0 \frac{1}{2} (\gamma^2 \|d(t)\|^2 - \|z(t)\|^2) dt \end{aligned} \quad (10.77)$$

for all $T_1 \geq 0$, $T_2 \geq 0$, and all disturbance functions $d(\cdot)$ on $[-T_1, T_2]$ *such that the measurements $y(\cdot)$ are zero on $[-T_1, 0]$* . Taking in (10.78) the supremum on the left-hand side, and taking the infimum on the right-hand side we obtain

$$\begin{aligned} & \sup_{d(\cdot) \text{ on } [0, T_2], T_2 \geq 0} - \int_0^{T_2} \frac{1}{2} (\gamma^2 \|d(t)\|^2 - \|z(t)\|^2) dt \\ & \leq \inf_{T_1 \geq 0} \int_{-T_1}^0 \frac{1}{2} (\gamma^2 \|d(t)\|^2 - \|z(t)\|^2) dt, \end{aligned} \quad (10.78)$$

where the infimum on the right-hand side is taken over all $T_1 \geq 0$ and all disturbance functions $d(\cdot)$ on $[-T_1, 0]$ such that $y(t) = 0$ for $t \in [-T_1, 0]$. We note that the right-hand side of (10.78) is precisely equal to $S_r(x(0))$; the *required supply* from the state $x^* = 0$ (cf. Theorem 3.1.16), *with regard to all $d(\cdot)$ on $[-T_1, 0]$ such that $y(t) = 0$, $t \in [-T_1, 0]$* . Furthermore, the left-hand side of (10.78) is considered for one *fixed* control strategy $u(\cdot)$ on $[0, T_2]$, namely the one delivered by the output feedback controller C . It follows that for all $T_2 \geq 0$

$$\inf_{u(\cdot)} \sup_{T_2 \geq 0, d(\cdot) \text{ on } [0, T_2]} \frac{1}{2} \int_0^{T_2} (\|z(t)\|^2 - \gamma^2 \|d(t)\|^2) dt \leq S_r(x(0)) \quad (10.79)$$

for all $x(0)$ which can be reached from $x(-T_1) = 0$ by a disturbance function $d(\cdot)$ on $[-T_1, 0]$. Finally, from the theory of (zero-sum) differential games (see, e.g., [325], and the references quoted in the Notes for this chapter it follows that the function $P(x(0))$ determined by the left-hand side of (10.79) is, whenever it is differentiable, a solution of the first Hamilton–Jacobi inequality (in fact, *equality*) (HJ1). Thus we have reached the conclusion that there exists a solution $P \geq 0$ to the first Hamilton–Jacobi inequality (HJ1) such that

$$P(x(0)) \leq S_r(x(0)) \quad (10.80)$$

for all $x(0)$ reachable from $x = 0$ by a disturbance $d(\cdot)$ leaving the measurements y equal to zero. From dissipative systems theory as treated in Chap. 3 it follows that the function $R(x) := S_r(x)$, whenever it is differentiable and defined for all x , is a solution of the second Hamilton–Jacobi inequality (HJ2). Thus we have derived as a *third* necessary condition for solvability of the output feedback suboptimal $\mathcal{H}_\infty$ control problem that there need to exist solutions $P \geq 0$ and $R \geq 0$ to (HJ1), respectively (HJ2), which satisfy the strong *coupling condition*

$$P(x) \leq R(x), \quad \text{for all } x \in \mathcal{X} \quad (10.81)$$

Furthermore, from Chap. 3 (Theorem 3.1.16) it follows that $R := S_r$ is in fact the *maximal* solution to (HJ2), while the theory of differential games indicates that the left-hand side of (10.79) is the *minimal* solution of the first Hamilton–Jacobi inequality (HJ1). (This has been explicitly proven for *linear* systems; see also Sect. 11.1.)

Example 10.2.2 Consider a system

$$\begin{aligned} \Sigma_a : \quad & \dot{x} = f(x) + g(x)u, \quad u, y \in \mathbb{R}^m, \quad f(0) = 0 \\ & y = h(x), \quad x \in \mathcal{X}, \quad h(0) = 0 \end{aligned} \quad (10.82)$$

which is *lossless* (i.e., conservative with respect to the passivity supply rate $u^T y$). Hence there exists a nonnegative storage function (for clarity denoted by $H \geq 0$) such that

$$H_x(x)f(x) = 0, \quad H_x(x)g(x) = h^T(x) \quad (10.83)$$

Furthermore assume that $H(0) = 0$, and that the system $\dot{x} = f(x), y = h(x)$ is *zero-state detectable* (Definition 3.2.15).

Consider the output feedback suboptimal $\mathcal{H}_\infty$ control problem associated with the equations

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + g(x)d_1, \quad x \in \mathcal{X}, \quad u \in \mathbb{R}^m, \quad d_1 \in \mathbb{R}^m \\ y &= h(x) + d_2, \quad y \in \mathbb{R}^m, \quad d_2 \in \mathbb{R}^m \\ z &= \begin{bmatrix} h(x) + d_2 \\ u \end{bmatrix}, \end{aligned} \quad (10.84)$$

where the L_2 -gain from $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ to z is sought to be minimized. Clearly, because of the direct feedthrough term in the z equations, the $\mathcal{H}_\infty$ control problem can never be solved for $\gamma < 1$. The Hamilton–Jacobi inequality (HJ1a) for $\gamma > 1$ takes the form

$$P_x(x)f(x) - \frac{1}{2} \frac{\gamma^2 - 1}{\gamma^2} P_x(x)g(x)g^T(x)P_x^T(x) + \frac{1}{2} \frac{\gamma^2}{\gamma^2 - 1} h^T(x)h(x) \leq 0 \quad (10.85)$$

The stabilizing solution $P \geq 0$ with $P(0) = 0$ to (10.85) is given as

$$P(x) = \frac{\gamma^2}{\gamma^2 - 1} H(x) \quad (10.86)$$

Indeed, by zero-state detectability the vector fields

$$\dot{x} = f(x) - \alpha b(x)b^T(x)H_x^T(x) \quad (10.87)$$

for $\alpha > 0$ are all asymptotically stable. In Sect. 11.1 it will be shown that the stabilizing solution to (10.85) is also the *minimal* solution.

The second Hamilton–Jacobi inequality (HJ2a) takes the following form (see (10.71)):

$$R_x(x)f(x) + \frac{1}{2} \frac{1}{\gamma^2} R_x(x)g(x)g^T(x)R_x^T(x) - \frac{1}{2}\gamma^2 h^T(x)h(x) \leq 0 \quad (10.88)$$

Using again zero-state detectability it is seen that

$$R(x) = \gamma^2 H(x) \quad (10.89)$$

is the antistabilizing solution to (10.88), and thus (see Sect. 11.1) the *maximal* solution. Therefore, the coupling condition tells us that necessarily

$$\frac{\gamma^2}{\gamma^2 - 1} H(x) \leq \gamma^2 H(x) \Leftrightarrow \gamma^2 \geq 2 \quad (10.90)$$

On the other hand, the unity output feedback

$$u = -y \quad (10.91)$$

yields a closed-loop system with L_2 -gain $\leq \sqrt{2}$. Indeed, by (10.91),

$$\begin{aligned} \|d_1\|^2 + \|d_2\|^2 &= \|d_1 + d_2 - y\|^2 - (y - d_2)^T(y - d_1) + \|y\|^2 \\ &= \|d_1 + d_2 - y\|^2 + \|y\|^2 + (y - d_2)^T(d_1 + u) \end{aligned} \quad (10.92)$$

By integrating from 0 to T , and observing that, since the system is lossless, $\int_0^T (y - d_2)^T(d_1 + u)dt = H(x(T)) - H(x(0))$, it follows that

$$\int_0^T \|d(t)\|^2 dt \geq \int_0^T \|y(t)\|^2 dt + H(x(T)) - H(x(0)), \quad (10.93)$$

and thus

$$\int_0^T \|z(t)\|^2 dt \leq 2 \int_0^T \|d(t)\|^2 dt + 2H(x(0)) - 2H(x(T)) \quad (10.94)$$

This implies that the closed-loop system has L_2 -gain $\leq \sqrt{2}$ (with storage function equal to H). We conclude that the unity output feedback (10.91) in fact solves the *optimal* output feedback $\mathcal{H}_\infty$ problem, for the *optimal* disturbance attenuation level $\gamma^* = \sqrt{2}$.

Next we elaborate on the necessary *structure* of a controller C solving the output feedback suboptimal $\mathcal{H}_\infty$ control problem. Consider again the dissipation inequality (10.61) for the closed-loop system. Under quite general conditions the left-hand side of (10.61) as a function of the disturbance d is maximized by a certain $\bar{d}^*(x, \xi)$, in which case it reduces to the Hamilton–Jacobi inequality

$$S_x(x, \xi) f(x, \alpha(\xi), \bar{d}^*(x, \xi)) + S_\xi(x, \xi) \varphi(\xi, g(x, \alpha(\xi), \bar{d}^*(x, \xi)) - \frac{1}{2} \gamma^2 \| \bar{d}^*(x, \xi) \|^2 + \frac{1}{2} \| h(x, \alpha(\xi), \bar{d}^*(x, \xi)) \|^2 \leq 0 \quad (10.95)$$

for $x \in \mathcal{X}$, $\xi \in \mathcal{X}_c$. Assume that (10.96) is satisfied with *equality* (this happens, for example, if S is the *available storage* or *required supply*, cf. Chap. 3 and/or Chap. 11). Assume furthermore, as before, that $S_\xi(x, \xi) = 0$ has a solution $\xi = F(x)$. Finally, assume that $P(x) = S(x, F(x))$ and that

$$\alpha(F(x)) = u^*(x, P_x^T(x)), \quad (10.96)$$

where $u^*(x, p)$ is the minimizing input of the *state feedback* problem, that is, satisfies (10.15). It is easily seen that $\bar{d}^*(x, F(x)) = d^*(x, P_x^T(x))$, with d^* satisfying (10.15). Under these three assumptions, differentiate (10.94) with equality with regard to ξ , and substitute afterward $\xi = F(x)$. Because of the maximizing, respectively minimizing, property of d^* and u^* , differentiating to ξ via d^* and α in $\xi = F(x)$ yields zero. Thus what remains is

$$S_{\xi x}(x, F(x)) f(x, u^*(x, P_x^T(x)), d^*(x, P_x^T(x))) + S_{\xi \xi}(x, F(x)) \varphi(F(x), g(x, u^*(x, P_x^T(x)), d^*(x, P_x^T(x)))) = 0 \quad (10.97)$$

Furthermore, since $S_\xi(x, F(x)) = 0$ for all x , differentiation of this expression with respect to x yields

$$S_{x\xi}(x, F(x)) + S_{\xi\xi}(x, F(x)) F_x(x) = 0 \quad (10.98)$$

Combination of (10.97) and (10.98) leads to

$$S_{\xi\xi}(x, F(x)) \varphi(F(x), g(x, u^*(x, P_x^T(x)), d^*(x, P_x^T(x)))) = S_{\xi\xi}(x, F(x)) F_x(x) f(x, u^*(x, P_x^T(x)), d^*(x, P_x^T(x))) \quad (10.99)$$

Thus, imposing the fourth assumption that the Hessian matrix $S_{\xi\xi}(x, F(x))$ is non-singular, it follows that

$$F_x(x)f(x, u^*(x, P_x^T(x)), d^*(x, P_x^T(x))) = \varphi(F(x), y) \quad (10.100)$$

with

$$y = g(x, u^*(x, P_x^T(x)), d^*(x, P_x^T(x))) \quad (10.101)$$

This constitutes an *invariance principle*. Indeed, whenever $\xi(t_0) = F(x(t_0))$, then during the occurrence of the *worst-case disturbance* $d^*(x(t), P_x^T(x(t)))$, it follows that $F_x(x(t))\dot{x}(t) = \dot{\xi}(t)$, and thus $\xi(t) = F(x(t))$ for $t \geq t_0$. Therefore, the controller C has a *certainty equivalence property*. This is especially clear if F is a diffeomorphism, in which case we may choose coordinates ξ for $\mathcal{X}_c$ such that F is the *identity* mapping. Then C is an *observer*, and $\xi(t)$ is an *estimate* for the actual state $x(t)$ (for the worst-case disturbance), while the control u is given as the state feedback $u^*(x, P_x^T(x))$ with the actual state x replaced by this estimate ξ .

Finally, in the last part of this section we will *transform* the output feedback suboptimal $\mathcal{H}_\infty$ control problem for Σ into an output feedback suboptimal $\mathcal{H}_\infty$ control problem for *another* system $\tilde{\Sigma}$, based on the solvability of the *state feedback* suboptimal $\mathcal{H}_\infty$ control problem. By this we may focus on the inherent difficulties associated with the *output feedback* problem. Furthermore, the transformation will lead us to a *parametrization* of the controllers solving the output feedback problem. We will show that the transformation is very close to the all-pass factorization treated in Sect. 9.4.

We start by considering a general system Σ , as given by Eq. (10.59), and we assume the existence of a solution $P \geq 0$ to the state feedback Hamilton–Jacobi *equality* (HJ1), that is,

$$K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), u^*(x, P_x^T(x))) = 0 \quad (10.102)$$

with d^*, u^* satisfying (10.15), and K_γ given as in (10.19). Now consider the function $K_\gamma(x, P_x^T(x), d, u)$. By (10.102) and (10.20), we have

$$K_\gamma(x, P_x^T(x), d, u^*(x, P_x^T(x))) \leq 0 \leq K_\gamma(x, P_x^T(x), d^*(x, P_x^T(x)), u) \quad (10.103)$$

Now assume that there exist mappings

$$\begin{aligned} r &= r(x, u, d) \\ v &= v(x, u, d) \end{aligned} \quad (10.104)$$

such that

$$K_\gamma(x, P_x^T(x), d, u) = -\frac{1}{2}\gamma^2\|r\|^2 + \frac{1}{2}\|v\|^2 \quad (10.105)$$

If the system equations are given by (10.68), then the factorization (10.105) is in fact easy to obtain by completing the squares, and is given as

$$\begin{aligned} r_1 &= d_1 - \frac{1}{\gamma^2} g^T(x) P_x^T(x), \quad r_2 = d_2 \\ v &= u + b^T(x) P_x^T(x) \end{aligned} \quad (10.106)$$

with $P \geq 0$ satisfying (HJ1a). In general, the factorization is of the same type as the one considered in Sect. 9.4 (the main difference being the *indefinite* sign). *Local* existence of the factorization can be again guaranteed by an application of the Morse Lemma if the Hessian matrix of $K_\gamma(x, P_x^T(x), d, u)$ with respect to d and u is nonsingular (see Lemma 9.4.2 in Chap. 9). Let us additionally assume that the mapping given by (10.104) is *invertible* in the sense that d can be expressed as function of u and r (and x), and u as function of v and d (and x), i.e.,

$$\begin{aligned} d &= d(x, u, r) \\ u &= u(x, v, d) \end{aligned} \quad (10.107)$$

This assumption is trivially satisfied for (10.106), since in this case we may write

$$\begin{aligned} d_1 &= r_1 + \frac{1}{\gamma^2} g^T(x) P_x^T(x), \quad d_2 = r_2 \\ u &= v - b^T(x) P_x^T(x) \end{aligned} \quad (10.108)$$

The system Σ is now factorized as in Fig. 10.5, with $\tilde{\Sigma}$ denoting the transformed system

$$\begin{aligned} \dot{\tilde{x}} &= f(\tilde{x}, u, d(\tilde{x}, u, r)) =: \tilde{f}(\tilde{x}, u, r) \\ \tilde{\Sigma} : y &= g(\tilde{x}, u, d(\tilde{x}, u, r)) =: \tilde{g}(\tilde{x}, u, r) \\ v &= v(\tilde{x}, u, d(\tilde{x}, u, r)) =: \tilde{h}(\tilde{x}, u, r) \end{aligned} \quad (10.109)$$

while the system Θ is given as

$$\begin{aligned} \dot{\theta} &= f(\theta, u(\theta, v, d), d) \\ \Theta : r &= r(\theta, u(\theta, v, d), d) \\ z &= h(\theta, u(\theta, v, d), d) \end{aligned} \quad (10.110)$$

Fig. 10.5 Factorization of Σ

It can be readily seen that if $\tilde{x}(t_0) = \theta(t_0) = x(t_0)$, then also $\tilde{x}(t) = \theta(t) = x(t)$, $t \geq t_0$, and that $z(t)$ and $y(t)$ produced in Fig. 10.5 coincide with $z(t)$ and $y(t)$ produced by Σ for the same $u(t)$ and $d(t)$. Thus Fig. 10.5 indeed constitutes a valid factorization of Σ . Furthermore, the following proposition is immediately derived from (10.105).

Proposition 10.2.3 *Let $P \geq 0$ satisfy (10.102). Consider the factorization of Σ given in Fig. 10.5. Then for all $d(\cdot)$, $v(\cdot)$, and all $t_1 \geq t_0$ and $\theta(t_0)$*

$$P(\theta(t_1)) - P(\theta(t_0)) + \frac{1}{2} \int_{t_0}^{t_1} (\|z(t)\|^2 - \gamma^2 \|d(t)\|^2) dt = \frac{1}{2} \int_{t_0}^{t_1} (\|v(t)\|^2 - \gamma^2 \|r(t)\|^2) dt \quad (10.111)$$

or equivalently,

$$P(\theta(t_1)) - P(\theta(t_0)) = \frac{1}{2} \int_{t_0}^{t_1} (\|v(t)\|^2 + \gamma^2 \|d(t)\|^2) dt - \frac{1}{2} \int_{t_0}^{t_1} (\|z(t)\|^2 + \gamma^2 \|r(t)\|^2) dt \quad (10.112)$$

Thus Θ is inner (Definitions 3.1.6 and 8.1.1) from $\begin{bmatrix} \gamma d \\ v \end{bmatrix}$ to $\begin{bmatrix} z \\ \gamma r \end{bmatrix}$.

A first consequence of Proposition 10.2.3 is that the solution of the output feedback suboptimal $\mathcal{H}_\infty$ control problem for Σ can be reduced, in a certain sense, to the same problem for $\tilde{\Sigma}$:

Proposition 10.2.4 *Let $P \geq 0$ satisfy (10.102). Consider the factorization of Fig. 10.5.*

- (i) *A controller C which solves the output feedback suboptimal $\mathcal{H}_\infty$ control problem for $\tilde{\Sigma}$ also solves the same problem for Σ .*
- (ii) *Suppose the controller C solves the output feedback suboptimal $\mathcal{H}_\infty$ control problem for Σ with a storage function $S(x, \xi)$ satisfying*

$$S(x, \xi) - P(x) \geq 0, \quad x \in \mathcal{X}, \xi \in \mathcal{X}_c \quad (10.113)$$

Then the same controller also solves the output feedback suboptimal $\mathcal{H}_\infty$ control problem for $\tilde{\Sigma}$ with storage function $S(\tilde{x}, \xi) - P(\tilde{x})$.

Proof (i) Rewrite (10.111) as

$$\frac{1}{2} \int_0^T (\|z(t)\|^2 - \gamma^2 \|d(t)\|^2) dt = \frac{1}{2} \int_0^T (\|v(t)\|^2 - \gamma^2 \|r(t)\|^2) dt + P(\theta(0)) - P(\theta(T)) \quad (10.114)$$

If a controller C bounds the first term on the right-hand side by a constant depending on the initial conditions $\tilde{x}(0)$, $\xi(0)$, then the same holds for the left-hand side (since $P(\theta(T)) = P(x(T)) \geq 0$).

(ii) If C solves the output feedback suboptimal $\mathcal{H}_\infty$ control problem for Σ , then $\frac{1}{2} \int_0^T (\|z(t)\|^2 - \gamma^2 \|d(t)\|^2) dt \leq S(x(0), \xi(0))$ and thus by (10.114)

$$\frac{1}{2} \int_0^T (\|v(t)\|^2 - \gamma^2 \|r(t)\|^2) dt \leq S(\tilde{x}(0), \xi(0)) - P(\tilde{x}(0)) \quad (10.115)$$

□

Remark 10.2.5 Note that for part (i) P only needs to satisfy the state feedback Hamilton–Jacobi *inequality* (HJ1).

Since we have already used in the transformation from Σ to $\tilde{\Sigma}$ the knowledge of the existence of a solution $P \geq 0$ to the state feedback Hamilton–Jacobi equation (HJ1) we may *expect* that the output feedback suboptimal $\mathcal{H}_\infty$ control problem for $\tilde{\Sigma}$ will be “easier” than the same problem for Σ . At least the solution of the *state feedback* $\mathcal{H}_\infty$ control problem for $\tilde{\Sigma}$ has become trivial; $u = u^*(x, P_x^T(x))$ and $d^* = d^*(x, P_x^T(x))$ solve the equations

$$\begin{aligned} 0 &= v(x, u, d) \\ 0 &= r(x, u, d), \end{aligned} \quad (10.116)$$

and thus yield a trivial solution to the state feedback $\mathcal{H}_\infty$ control problem for $\tilde{\Sigma}$. In particular, in the affine case, where r and v are given by (10.106), the state feedback $u = -b^T(x)P_x^T(x)$ renders v equal to zero and thus solves the *disturbance decoupling* problem for $\tilde{\Sigma}$.

A second consequence of the factorization in Fig. 10.5 and Proposition 10.2.3 concerns the parametrization of controllers solving the output feedback suboptimal $\mathcal{H}_\infty$ problem. Consider a controller C in closed-loop with the factorization of Fig. 10.6. Denote the system within dotted lines by K . Then it follows from Proposition 10.2.3 that if K has L_2 -gain $\leq \gamma$ (from r to v), then Θ in closed-loop with the “controller” K will also have L_2 -gain $\leq \gamma$ (from d to z). We may also *reverse* this relation. Consider an auxiliary system Q with inputs r and outputs v

$$Q : \begin{aligned} \dot{q} &= f_Q(q, r) \\ v &= h_Q(q, r) \end{aligned} \quad (10.117)$$

and suppose Q has L_2 -gain $\leq \gamma$. Thus, there exists a storage function $S_Q(q) \geq 0$ such that along the trajectories of Q

$$S_Q(q(T)) - S_Q(q(0)) \leq \frac{1}{2} \int_0^T (\gamma^2 \|r(t)\|^2 - \|v(t)\|^2) dt \quad (10.118)$$

Now consider this system Q in conjunction with the system $\tilde{\Sigma}$, i.e.,

Fig. 10.6 Factorized system in closed-loop with controller C

Fig. 10.7 Chain-scattering representation

$$\begin{aligned}
 \dot{q} &= f_Q(q, r) \\
 \dot{\tilde{x}} &= \tilde{f}(\tilde{x}, u, r) \\
 y &= \tilde{g}(\tilde{x}, u, r) \\
 h_Q(q, r) &= \tilde{h}(\tilde{x}, u, r)
 \end{aligned} \tag{10.119}$$

The idea is now to look at (10.119) as a *generalized form of image representation* (with driving variable r) of a *controller* C_Q (producing controls u on the basis of the measurements y). By construction this *implicitly defined* controller C_Q solves the output feedback suboptimal $\mathcal{H}_\infty$ control problem for Σ . Thus for every system Q with L_2 -gain $\leq \gamma$ we obtain in this way a controller C_Q solving the $\mathcal{H}_\infty$ problem.

A more explicit way of describing these controllers C can be obtained by rewriting the configuration of Fig. 10.6 into the form of Fig. 10.7.

Assume that Θ and $\tilde{\Sigma}$ are *invertible* in the sense that Θ admits an input–output representation with inputs r and v and outputs d and z , and that $\tilde{\Sigma}$ admits an input–output representation with inputs y and u and outputs r and v . In this case one speaks

about the *chain-scattering representation*. For more details, we refer to the literature cited in the Notes for this chapter.

10.3 Notes for Chapter 10

1. The computational complexity of obtaining (approximate) solutions to (HJ1) and/or (HJ2) is a major issue for the applicability of the theory. Power series solutions around an equilibrium are described in van der Schaft [275], Isidori and Kang [142], continuing on earlier similar approaches in nonlinear optimal control by Al'brekht [5] and Lukes [187]. See Krener, Aguilar & Hunt [172], Aguilar & Krener [4] for further developments, including the Toolbox by Krener [171]. Various other computational schemes have been proposed and analyzed in the literature; we mention Knobloch, Isidori & Flockerzi [166], McEneaney & Mease [210], Huang & Lin [130], Kram, Kugelman, Pesch & Breitner [169], Beard, Saridis & Wen [33], Beard, Saridis & Wen [34], Beard & McLain [32], Beeler, Tran & Banks [35], Sakamoto [263], Holzhiiter [129], Osinga & Hauser [245], Navasca & Krener [229], Sakamoto & van der Schaft [265], Sakamoto & van der Schaft [266], Sassano & Astolfi [268], Sakamoto [264].
2. For general information concerning the $\mathcal{H}_\infty$ control problem for *linear* systems we refer to, e.g., Francis [103], Green & Limebeer [117], Kwakernaak [178], Scherer [305], Stoorvogel [327]. The state space solution to the linear suboptimal $\mathcal{H}_\infty$ control problem is due to Doyle, Glover, Khargonekar & Francis [88], see also, e.g., Khargonekar, Petersen & Rotea [162], Scherer [304], Tadmor [333]. For the differential game approach to linear $\mathcal{H}_\infty$ control theory see especially Basar & Bernhard [30].
3. The solution to the nonlinear suboptimal $\mathcal{H}_\infty$ control problem as described in Sect. 10.1 was given in van der Schaft [272, 273, 275], Isidori & Astolfi [140], Ball, Helton & Walker [23]. $\mathcal{H}_\infty$ control for general nonlinear systems (not necessarily affine in the inputs and disturbances) is treated in Isidori & Kang [142], Ball, Helton & Walker [23], see also van der Schaft [277]. For earlier work on nonlinear $\mathcal{H}_\infty$ control we refer to Ball & Helton [19, 20].
4. The treatment of the robust stabilization problem in Sect. 10.1 is based on van der Schaft [279]; see also Imura, Maeda, Sugie & Yoshikawa [131], and Astolfi & Guzzella [14], Pavel & Fairman [252] for other developments.
5. The existence of nonnegative solutions to (HJ1) and (HJ2), together with the (weak) coupling condition (10.76), as a necessary condition for the solvability of the nonlinear output feedback $\mathcal{H}_\infty$ control problem was shown in Ball, Helton & Walker [23], van der Schaft [276]. The key idea of deriving (HJ2) for linear systems via the dissipativity of the *constrained* system ($u = 0$, $y = 0$) is due to Khargonekar [161].

6. The invariance property of controllers solving the nonlinear suboptimal $\mathcal{H}_\infty$ problem, as described in Sect. 10.1, is due to Ball, Helton & Walker [23], with the present generalization given in van der Schaft [278].
7. As already shown in Example 10.1.9 it is often necessary, as indicated in Chaps. 3, 8 and 9, to consider *generalized* solutions of the Hamilton–Jacobi inequalities encountered in this chapter; see, e.g., Frankowska [105], James & Baras [145], Ball & Helton [22], Soravia [325], and Yuliar, James & Helton [361], and Day [80] for further information.
8. Much effort has been devoted to finding *sufficient* conditions for solvability of the output feedback suboptimal $\mathcal{H}_\infty$ control problem, but we have decided not to include these important contributions in this book. One line of research is devoted to finding sufficient conditions for the existence of output feedback controllers with dimension equal to the dimension of the plant and having an observer structure (compare with the invariance principle in Sect. 10.2), cf. Isidori & Astolfi [140], Ball, Helton & Walker [23], Isidori [136], Lu & Doyle [185, 186], Isidori & Kang [142]. Another approach is via the theory of differential games (see, e.g., Basar & Olsder [31]), interpreting the output feedback suboptimal $\mathcal{H}_\infty$ control problem as a two-player zero-sum differential game with partial information, see Basar & Bernhard [30], Bernhard [37], Soravia [325]. The resulting “central” controller, however, is in general infinite-dimensional, see Didinsky, Basar & Bernhard [84], van der Schaft [278], Bernhard [37], James & Baras [145]. Under the assumption of a “worst-case certainty equivalence principle” the equations for the central controller are derived in Didinsky, Basar & Bernhard [84], van der Schaft [278], Krener [170], Bernhard [37]. The methods and difficulties in this case are very similar to the ones encountered in nonlinear filtering theory, see, e.g., Mortensen [224], Hijab [121]. Another, more general, approach is to transform the differential game with partial information into an infinite-dimensional differential game with complete information, see, e.g., Helton & James [120]. This “information-state” approach has been detailed in the monograph by Helton & James [119]. For developments exploiting Max-Plus methods see McEneaney [209].
9. Example 10.2.2 is taken from van der Schaft [278].
10. If the system $\tilde{\Sigma}$ as obtained in Fig. 10.5 is stable and minimum phase, then the factorization obtained in Fig. 10.5 is also called an *J-inner-outer factorization* of Σ . A constructive approach to J-inner-outer factorization of Σ (under the assumption of invertibility from d to y) is given in Ball & van der Schaft [24], using the Hamiltonian extension Σ^H of Σ and “nonlinear spectral factorization”; cf. Sect. 9.4. Related work is Baramov & Kimura [29], Pavel & Fairman [253] and Baramov [28]. For further information on the chain-scattering representation, see Kimura [165], Ball & Helton [20], Ball & Verma [26], Ball & Helton [22]. The presentation of the parametrization of $\mathcal{H}_\infty$ controllers given in Sect. 10.2 is inspired by Ji & Gao [148]; see Doyle, Glover, Khargonekar & Francis [88], for

similar ideas in the linear case. For other related works on the parametrization of $\mathcal{H}_\infty$ controllers, see Lu & Doyle [186].

11. For work on nonlinear $\mathcal{H}_\infty$ -filtering we refer to, e.g., Nguang & Fu [231], Berman & Shaked [36], Krener [170].
12. In case the control variables u do not enter the equations for z in an “injective way,” or if the disturbance variables d do not enter the equations for y in a “surjective way,” then we speak about the *singular* $\mathcal{H}_\infty$ control problem; see for the linear case Stoorvogel [327], and the references included therein. For a treatment of the nonlinear state feedback singular $\mathcal{H}_\infty$ control problem we refer to Maas & van der Schaft [189], Maas [188], Baramov [28]. The nonlinear $\mathcal{H}_\infty$ almost disturbance decoupling problem, which can be seen as a special case of the singular $\mathcal{H}_\infty$ control problem has been treated for a special class of systems in Marino, Respondek, van der Schaft & Tomei [196]. An interesting feature of the latter paper is that the solution to the dissipation inequality is *constructed* in an explicit recursive manner, thus avoiding the issue of solvability of multidimensional Hamilton–Jacobi inequalities. The results of [196] are vastly expanded in Isidori [137, 138]. The idea of solving Hamilton–Jacobi inequalities in a recursive manner for multiple cascaded systems is explored in Pan & Basar [250]; see also Dalsmo & Maas [77] for related work.
13. Some applications of nonlinear $\mathcal{H}_\infty$ control theory can be found in Kang, De & Isidori [157], Chen, Lee & Feng [65], Feng & Postlethwaite [96], Astolfi & Lanari [15], Kang [156], Dalsmo & Egeland [75, 76], Kugi & Schlacher [176].
14. The nonlinear suboptimal $\mathcal{H}_\infty$ control problem for $\gamma \geq 0$ is the problem of finding a controller such that the closed-loop system is dissipative with respect to the L_2 -gain supply rate $\frac{1}{2}\gamma^2\|d\|^2 - \frac{1}{2}\|z\|^2$. This suggests to consider the *general dissipative control* problem of finding a controller which renders the closed-loop system dissipative with respect to a given supply rate $s(d, z)$; see Yuliar & James [360], Yuliar, James & Helton [361], Shishkin & Hill [314]. This includes the problem of rendering a system passive as treated in Chap. 5. The problem of considering “finite power” instead of finite L_2 -gain is addressed in Dower & James [86].
15. Using the notion of robust L_2 -gain (cf. Chap. 8, Corollary 8.2.6) one can also consider a “robust” nonlinear $\mathcal{H}_\infty$ control problem, see Shen & Tamura [313], Nguang [232]. A robustness result concerning the solution to the state feedback suboptimal $\mathcal{H}_\infty$ control problem with static perturbations on the inputs is derived in van der Schaft [278] (Proposition 4.7).